Equilibrium Bid-Ask Spread of European Derivatives in Dry Markets

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January 3, 2006

Abstract

In the framework of incomplete markets, due to the non-existence of trade at some points in time, and using a partial equilibrium analysis, we show how the bid-ask spread of an European derivative is generated. We also ¯nd conditons for the existence of the spread. These conditions concern the market structure of the maret-makers, which can be a oligolopoly with price competition or a monopoly, as well as the risk-aversion of the demand and supply of the market.

1 Introduction

Financial markets present equilibria characterized by bid-ask spreads, that can only be explained by market imperfections. The literature has been centered in two main imperfections, namely information asymmetries and transaction costs. This papers aims to show that equilibrium bid-ask spreads may be generated by market illiquidity, an alternative market imperfection, even in the absence of information asymmetries and transaction costs.

Among the traditional assumptions on which derivatives' pricing is based, markets are perfect and the underlying asset can be transacted at any point in time. Under the absence of arbitrage opportunities the value of a derivative can be computed as the value of a portfolio on the underlying risky asset and risk-free bonds that exactly replicates its payo®. Such portfolio can be rebalanced in a self-¯nancing way until the maturity of the derivative, by continuously transacting the underlying asset and the bonds. Under these assumptions, the calculated value of the initial portfolio can be shown to be the equilibrium price of the derivative and is unique. In this paper we

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assume that the underlying asset cannot be transacted at every point in time and study the impact of this constraint on the equilibrium pricing of options. The illiquidity implies that markets may become incomplete, in the sense that perfect hedging of the derivative in all states of nature is no longer possible. Hence, a unique price does not exist anymore, opening a range for the possible characterization of the equilibrium bid-ask spread.

The existence of a bid-ask spread may therefore be related to the pricing of derivatives in incomplete markets. The different approaches in the literature to characterize such problem are revisited below.

First, we consider the superreplicating bounds for European derivatives. The nature of such bounds is well characterized in the context of incomplete markets in the papers by El Karoui and Quenez (1991,1995), Edirisinghe, Naik and Uppal (1993) and Karatzas and Kou (1996). A direct application to the case of European option pricing when the market for the underlying is dry can be found in Amaro de Matos and Antão (2001). The superreplication procedure relates to the minimization of the expected losses, as considered by Föllmer and Leukert (1999). Several works, also using superreplication, but with different criteria of pure arbitrage opportunity, got narrower superreplication bounds[1]. In most cases the superreplication bounds produce too broad bounds, and certainly not equilibrium values. An alternative pricing criterion, is the utility indifference pricing, as introduced by Hodges and Neuberger (1989). This criterion, despite having the disadvantage of being utility dependent, has a meaningful economic interpretation. However, as Davis, Panas and Zariphopoulou (1993) suggest, this definition does not allow the determination of equilibrium prices because the reservation write price is always higher than the reservation buy price. Therefore, reservations prices define a range in which the trading of the derivative must take place but they do not determine equilibrium prices. Alternatively, and still in the framework of utility indifference pricing, we may introduce the marginal price as the utility indifference price for an infinitesimal quantity. Marginal pricing have been used in several contexts by Davis (1997), Karatzas and Kou (1996) and Kallsen (2002). The uniqueness of the marginal price is proved by Karatzas and Kou (1996), for portfolio constraints, and by Hugonnier, Kramkov and Schachermayer (2005) when the number of states of Nature is larger than the number of assets. However, this price is not yet an equilibrium price reflecting only the willingness to pay for a marginal (infinitesimal) amount.

An alternative is to price the derivatives as their expected discounted payoff, according to one of the several risk-neutral probability measures. Several criteria have been proposed for the selection of one particular measure. For instance, the minimal martingale measure by Follmer and Schweizer

[1]See the works of Bernardo and Ledoit (2000), Cochrane and Saá-Requejo (2000) and Bondarenko (2003).

(1990), the variance optimal measure by Schweizer (1996) and the minimal entropy measure by Rouge and El Karoui (2000) and Frittelli (2000a),(2000b), who also analyzed the connection between entropy measures and utility-based prices when utilities are exponential. The dependence of the utility-maximization price on the choice of the distance metric, can be found in Henderson (2005) and Henderson et al (2003). Coherent risk measures were studied by Artzner et al (1999), and convex risk measures by Follmer and Schied (2002). Such measures were introduced to axiomatize measures of risk and to generalize the properties of utility-indifference prices.

With the exception of the marginal price, all the methodologies proposed above either establish a range of variation for the value of the derivative or use an ad hoc criterion in order to get the price without any economic insight. In this work we intend to characterize an equilibrium price for our specific incomplete market model. Equilibrium in a financial markets must verify that all agents maximize their utilities and markets clear. Hence, the determination of equilibrium considers both market-makers' and traders' decisions. Using the maximization utility approach, a market demand and market supply for the derivative is constructed. We then introduce the market-makers, who set an ask and a bid price, first in a monopolistic and next in a competitive context.

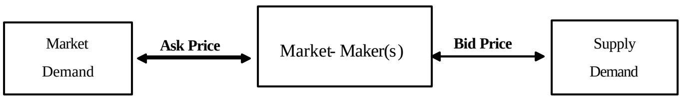

Figure 1: The derivatives' market.

Although a large number of studies on microstructure model of securities markets have been published in the last years, most of the attention has been devoted to stock markets, while only a few papers discuss the microstructure of derivative markets[2]. Moreover, in order to generate a market and a bid-ask spread another imperfections were introduced. We can group them as the inventory approach[3] and the information based approach[4]. In

[2]There is much work on stock bid-ask spreads but the spread of derivatives have been investigated by fewer researchers. Biais, Foucault and Salanié (1998) analyze three different market structures and the ways the associated restrictions lead to differences in prices, bid-ask spreads, trades and risk-sharing. There are also a few empirical studies examine bid-ask spreads in the derivatives markets, such as, George and Longstaff (1993), Chan, Chung and Johnson (1995) and Etling and Miller, jr (2000).

[3]Among others, Stoll (1978) and Amihud and Mendelson (1980) studied bid-ask spreads ans stock inventory. More recently, Lee, Mucklow and Ready (1993), Hasbrouck and Sofianos (1993), Madhavan and Smidt (1993) and Manaster and Mann (1996) also found some evidence on the realationship of bid-ask spreads to market-maker inventory costs.

[4]Some authors discussing the topic: Copeland and Galai (1983), Glosten and Milgrom (1985), Admati and Pfleiderer (1992) and Foster and Viswanathan (1994). More recently, Morrison (2004), Bagnoli et al (2001) and Vayanos (2001).

the former case transaction costs determine the bid-ask spread. The market-maker(s) want(s) to be compensated by the cost of keeping inventory. In the latter case the asymmetry of information leads to the existence of transactions. When there is an order imbalance that moves the market-maker away from his desired inventory position, he adjusts the bid-ask spread to attract orders to move back to his optimal inventory position. On the other hand, information asymmetry models assume that an adverse selection problem exists because the market-maker is at an informational disadvantage to the informed traders. In this case spreads must be kept wide enough to ensure that gains from trading with the uninformed agents exceed the losses associated with trading with informed agents.

Our model assumes neither asymmetric nor optimal inventory strategies and still explains the existence of an equilibrium bid-ask spread. It is related to the failure of continuous hedging portfolio rebalancing hypothesis. We assume, as in Longsta® (2001), that the underlying asset can not be transacted at all points in time. In this paper we provide conditions for a bid-ask spread to exist if the risk-neutral market-maker is a monopolist. However, if there are more than one market-market, and if they compete in prices, there will be no equilibrium if the market-makers are risk neutral.

Our work is organized as follows. In section 2 the demand and supply of the market are derived. In subsection 2.4 some simulations are performed for constant relative risk aversion and constant absolute risk aversion utility functions. Section 3 states the problem of the market-maker(s), presenting ¯rst the monopoly case and then the oligopoly case. Finally, in section 4 we conclude. Our main technical proofs are presented in the appendix.

2 The Model

Consider a discrete-time two-period economy, corresponding to dates $t = 0; 1$ and 2: Due to liquidity constraints transactions are only possible at time $t = 0$ and $t = 2$. At time $t = 2$ there are three possible states of Nature, labelled by $i = 1; 2; 3$: In this economy there are three di®erent assets being transacted. The ¯rst one is a risk free asset with unitary initial value, that provides a certain total return of R per period; the second is a risky asset (the stock) with initial value S_0 and uncertain ¯nal values S_2^i; for $i = 1; 2; 3$: In particular, we number the states in an order such that $S_2^1 > S_2^2 > S_2^3$; notice also that, in order to avoid arbitrage opportunities, we must have $S_2^1 > R^2 S_0 > S_2^3$; ¯nally, the third asset is a European derivative, written on the stock, with expiration date $T = 2$: The possible payo®s at time $t = 2$ are denoted by G_2^i; for $i = 1; 2; 3$; and depend only on the ¯nal state of Nature. We also assume that the payo®s of each considered European derivative are ordered according to the states' labels, in a monotonic way.

Every agent may build a portfolio composed of shares, risk free asset and derivatives. By assumption, each agent can influence neither the market price of the stock, nor the market price of the derivative. Each representative agent maximizes a von Neumann-Morgenstern utility function, $EU(.)$, of the wealth at time $t = 2$. The utility function $U(.)$ is increasing and concave in wealth. These properties imply that the marginal utility is positive, but decreasing, in wealth.

In the following subsection we derive the individual and market demand and supply. In section 2.4 the individual demand and supply for two different types of utility functions are presented.

2.1 Demand

Consider a representative agent that maximizes the expected value of wealth at the terminal date, $t = 2$. The problem that he faces is to choose the number of shares $\Delta_{0,d}$ that he will hold, the amount $B_{0,d}$ invested in the risk-free asset and how many units q_d of the derivative he is going to buy, for a given price of the derivative. Hence, the problem that the representative agent solves is

$$\max_{\Delta_{0,d},B_{0,d},q_d} EU\left[\Delta_{0,d}S_2 + R^2B_{0,d} + q_dG_2\right] = \sum_{i=1} p^iU\left[\Delta_{0,d}S_2^i + R^2B_{0,d} + q_dG_2^i\right]$$

subject to

$$\Delta_{0,d}S_0 + B_{0,d} + q_dP_d \leq y \tag{1}$$
$$\Delta_{0,d}S_2^i + R^2B_{0,d} + q_dG_2^i \geq 0 \quad ; \quad i = 1,2,3 \tag{2}$$

and

$$q_d \geq 0.$$

Proposition 1 Ignoring the positivity constraints of wealth at time 2, presented in equations (2), the optimum values $\Delta_{0,d}^*$ and q_d^* are implicitly given by

$$\begin{aligned} S_0 &= \frac{1}{R^2}\sum_{i=1} \alpha\left(w_d^i\right)S_2^i, \\ P_d &= \frac{1}{R^2}\sum_{i=1} \alpha\left(w_d^i\right)G_2^i \text{ if } q_d^* > 0 \text{ and} \\ P_d &\leq \frac{1}{R^2}\sum_{i=1} \alpha\left(w_d^i\right)G_2^i \text{ if } q_d^* = 0; \end{aligned} \tag{3}$$

with

$$\alpha\left(w_d^i\right) = \frac{p^i U'\left(w_d^i\right)}{\sum_{i=1}^{3} p^i U'\left(w_d^i\right)}:$$

and

$$w_d^i = \Delta_{0;d}^{*}\left(S_2^i - R^2 S_0\right) + R^2 y + q_d^{*}\left(G_2^i - R^2 P_d\right):$$

Proof. As the utility function is increasing in wealth the constraint presented in equation (1) is satisfied in equality. Hence, the problem above can be rewritten as

$$\max_{\Delta_{0;d};q_d} \sum_{i=1} p^i U\left[\Delta_{0;d}\left(S_2^i - R^2 S_0\right) + R^2 y + q_d\left(G_2^i - R^2 P_d\right)\right]$$

subject to

$$\Delta_{0;d}\left(S_2^i - R^2 S_0\right) + R^2 y + q_d\left(G_2^i - R^2 P_d\right) \geq 0 \quad ; \quad i = 1; 2; 3$$

and

$$q_d \geq 0:$$

Ignoring the positivity constraints, the first order conditions are:

$$\begin{cases} \frac{\partial E[U(:)]}{\partial \Delta_{0;d}} = 0 \\ \frac{\partial E[U(:)]}{\partial q_d} \leq 0,\ q_d \geq 0,\ \frac{\partial E[U(:)]}{\partial q_d} q_d = 0 \end{cases}$$

leading to equation (3). The maximum is guaranteed since the second order conditions are satisfied. See appendix A for details. ■

In the next proposition, a necessary and sufficient condition to have a negatively sloped individual demand is established.

Proposition 2 A necessary and sufficient condition for $\phi_d(P_d)$ to be a decreasing function of P_d is that

$$\begin{aligned} 0 \geq\ & \sum_{i=1} p^i\left(S_2^i - R^2 S_0\right)^2 U''\left(w_d^i\right) \sum_{i=1} p^i U'\left(w_d^i\right) \\ & + q\left\{\sum_{i=1} p^i\left(S_2^i - R^2 S_0\right)\left(G_2^i - R^2 P_d\right) U''\left(w_d^i\right) \sum_{i=1} p^i\left(S_2^i - R^2 S_0\right) U''\left(w_d^i\right)\right. \\ & \left. - \sum_{i=1} p^i\left(S_2^i - R^2 S_0\right)^2 U''\left(w_d^i\right) \sum_{i=1} p^i\left(G_2^i - R^2 P_d\right) U''\left(w_d^i\right)\right\} \end{aligned}$$

Proof. See appendix A for details. ■

From the maximization problem above faced by a representative buyer of the derivative, we obtain the optimal amount $q = \phi_d(P_d)$: If this function is monotonic, i.e., if the condition of the above Proposition is satisfied, $\phi_d(P_d)$ may be inverted in order to obtain an individual market demand

$$P_d = \phi_d^{-1}(q):$$

Assuming that there are n equal agents buying the derivative in this economy, the market demand (Q_d) and the inverse market demand can be written as

$$Q_d = nq = n\phi_d(P_d) \Rightarrow P_d = \phi_d^{-1}\left(\frac{Q_d}{n}\right)$$

2.2 Supply

Consider a representative agent that maximizes the expected value of wealth at the terminal date, $t = 2$: The problem that he faces is to choose the number of shares $\Delta_{0;s}$ that he will hold, the amount $B_{0;s}$ invested in the risk-free asset and how many units q_s of the derivative he is going to sell, for a given price of the derivative. Hence, the problem that the representative agent solves is

$$\max_{\Delta_{0;s};B_{0;s};q_s} EU\left[\Delta_{0;s}S_2 + R^2 B_{0;s} - q_s G_2\right] = \sum_{i=1} p^i U\left[\Delta_{0;s}S_2^i + R^2 B_{0;s} - q_s G_2^i\right]$$

subject to

$$\Delta_{0;s}S_0 + B_{0;s} - q_s P_s \le y \qquad (4)$$
$$\Delta_{0;s}S_2^i + R^2 B_{0;s} - q_s G_2^i \ge 0 \quad ; \quad i = 1;2;3 \qquad (5)$$

and

$$q_s \ge 0;$$

where $\Delta_{0;s}$; $B_{0;s}$ and q_s denote, respectively, the number of shares bought/sold, the amount invested in the risk free asset the number of derivatives sold.

Proposition 3 Ignoring the positivity constraints of wealth at time 2, presented in equations (4), the optimum values $\Delta_{0;s}^*$ and q_s^* are implicitly given by

$$
\begin{aligned}
S_0 &= \frac{1}{R^2}\sum_{i=1}^{3} \alpha\left(w_s^i\right) S_2^i; \\
P_s &= \frac{1}{R^2}\sum_{i=1}^{3} \alpha\left(w_s^i\right) G_2^i \text{ if } q_d^* > 0 \text{ and} \qquad (6)\\
P_s &\leq \frac{1}{R^2}\sum_{i=1}^{3} \alpha\left(w_s^i\right) G_2^i \text{ if } q_d^* = 0;
\end{aligned}
$$

with

$$\alpha\left(w_d^i\right) = \frac{p^i U'\left(w_d^i\right)}{\sum_{i=1}^{3} p^i U'\left(w_d^i\right)}.$$

and

$$w_s^i = \Delta_{0,s}^*\left(S_2^i - R^2 S_0\right) + R^2 y - q_s^*\left(G_2^i - R^2 P_d\right).$$

Proof. As the utility function is increasing in wealth the constraint presented in equation (4) is satisfied in equality. Denoting by $\left[B_{0,s}^*(P_s), \Delta_{0,s}^*(P_s)\right]$ the solution of the problem described then

$$\Delta_{0,s}^*(P_s)\, S_0 + B_{0,s}^*(P_s) - qP_s = y.$$

Hence, the problem above can be written as

$$\max_{\Delta_{0,s},q} \sum_{i=1}^{3} p^i U\left[\Delta_{0,s}\left(S_2^i - R^2 S_0\right) + R^2 y - q_s\left(G_2^i - R^2 P_s\right)\right]$$

subject to

$$\Delta_{0,s}\left(S_2^i - R^2 S_0\right) + R^2 y - q_s\left(G_2^i - R^2 P_s\right) \geq 0 \quad ; \quad i = 1, 2, 3$$

and

$$q_s \geq 0.$$

Ignoring the positivity constraints, the first order conditions are:

$$\frac{\partial E[U(\cdot)]}{\partial \Delta_{0,s}} = 0$$

$$\frac{\partial E[U(\cdot)]}{\partial q_s} \leq 0,\ q_s \geq 0,\ \frac{\partial E[U(\cdot)]}{\partial q_s} q_s = 0$$

leading to equation (6). The maximum is guaranteed since the second order conditions are satisfied. See appendix B for details. ■

In the next proposition a necessary and sufficient condition to have a positively sloped individual supply is established.

Proposition 4 A necessary and sufficient condition for $\chi_s(P_s)$ to be a positive function of P_s is that

$$
\begin{aligned}
0 \geq\ & \sum_{i=1} p^i \left(S_2^i - R^2 S_0\right)^2 U''\left(w_s^i\right) \sum_{i=1} p^i U'\left(w_s^i\right) \\
& + q \Bigg\{ \sum_{i=1} p^i \left(S_2^i - R^2 S_0\right)\left(G_2^i - R^2 P_s\right) U''\left(w_s^i\right) \sum_{i=1} p^i \left(S_2^i - R^2 S_0\right) U''\left(w_s^i\right) \\
& - \sum_{i=1} p^i \left(S_2^i - R^2 S_0\right)^2 U''\left(w_s^i\right) \sum_{i=1} p^i \left(G_2^i - R^2 P_s\right) U''\left(w_s^i\right) \Bigg\}
\end{aligned}
$$

Proof. See appendix B for details. ■

From the maximization problem faced by a selling agent presented above, an individual market supply is obtained as

$$q = \chi_s(P_s) \Rightarrow P_s = \chi_s^{-1}(q)$$

Assuming that there are n equal agents in this economy the market supply (Q) and the inverse market supply can be written as

$$Q_s = nq = n\chi_s(P_s) \Rightarrow P_s = \chi_s^{-1}\left(\frac{Q_s}{n}\right)$$

2.3 Arbitrage bounds, reservation prices and fair price

2.3.1 Arbitrage bounds and finite utility

In order to get a finite solution of the problem, we must assure that the price of the derivative is within the superreplication bounds. In this sense, we guarantee that there is no arbitrage opportunities in this market.

The upper bound of arbitrage-free range of variation is given by

$$P^u = \min_{\Delta_0; B_0} \Delta_0 S_0 + B_0$$

subject to

$$\Delta_0 S_2^i + R^2 B_0 \geq G_2^i$$

with $i = 1; 2$ and 3:

The lower bound of arbitrage-free range of variation is given by

$$P^l = \max_{\Delta_0; B_0} \Delta_0 S_0 + B_0$$

subject to

$$\Delta_0 S_2^i + R^2 B_0 \le G_2^i$$

with $i = 1, 2$ and 3.

The upper and lower bounds can be written in a shorter way, if we introduce some simplifying notation. Let π denote the vector of parameters of our model, $\pi \equiv \left(S_0; S_2^1; S_2^2; S_2^3; G_2^1; G_2^2; G_2^3; R\right)$. We further define

$$\begin{aligned}
G^+ &= \left\{\pi : G_2^1 \left(S_2^2 - S_2^3\right) + G_2^2 \left(S_2^3 - S_2^1\right) + G_2^3 \left(S_2^1 - S_2^2\right) \ge 0\right\};\\
G^- &= \left\{\pi : G_2^1 \left(S_2^2 - S_2^3\right) + G_2^2 \left(S_2^3 - S_2^1\right) + G_2^3 \left(S_2^1 - S_2^2\right) \le 0\right\};\\
H^+ &= \left\{\pi : S_2^2 - R^2 S_0 \ge 0\right\};\\
H^- &= \left\{\pi : S_2^2 - R^2 S_0 \le 0\right\}
\end{aligned}$$

and

$$P_{i;j} = \frac{G_2^j \left(S_2^i - R^2 S_0\right) - G_2^i \left(S_2^j - R^2 S_0\right)}{R^2 \left(S_2^i - S_2^j\right)}. \qquad (7)$$

We then have[5]

$$P^u = P_{1;3} \text{ and } P^l = P_{2;3}$$

if $\pi \in G^+ \cap H^+$;

$$P^u = P_{1;3} \text{ and } P^l = P_{1;2}$$

if $\pi \in G^+ \cap H^-$;

$$P^u = P_{2;3} \text{ and } P^l = P_{1;3}$$

if $\pi \in G^- \cap H^+$ and

$$P^u = P_{1;2} \text{ and } P^l = P_{1;3}$$

if $\pi \in G^- \cap H^-$.

We claim that prices above P^u or below P^l will generate an arbitrage opportunity and therefore, infinite utility. The reason is as follows. First consider demand. If the price is below the lower bound, it would be possible to sell a superreplicating portfolio whose current value is higher than the derivative that is being bought. Proceeding in this way, we could assure a positive wealth in all possible states of Nature. Selling an arbitrary number of units of this portfolio and buying an arbitrary number of units of the

[5] See appendix C for the full derivation of the bounds.

derivative would assure unbounded wealth at time $t = 2$. Hence, the utility and the optimal solution would not be finite. In what concerns the individual supply, if the price is higher than the upper bound, it is possible to buy a superreplicating portfolio that is cheaper then the derivative that is being sold. In this way we could assure an unbounded positive wealth as well.

An interesting way of looking at this issue is to go back to the maximization problem of both buyers and sellers of the derivative. In fact, there is no restriction to infinite solutions to these problems. If we would have added the restriction $q < \infty$, the optimal solution would immediately imply that $\phi(P^u) > q > \phi\left(P^l\right)$.

Therefore, the imposition of finite solutions provides an alternative way to characterize the bounds of the no-arbitrage region.

There are two different ways of proceeding, each of them leading to the same constraints on the parameters in order to obtain a finite solution of the problem.

First, we may look at the first order conditions of the representative investor. Alternatively, we may look at the wealth constraints.

The first argument goes like this. Ignoring that the fact that the investor is buying or selling the derivative, the first order conditions can be written as

$$\begin{cases} \sum_{i=1}^{3} \frac{p^i S_2^i}{R^2 S_0} U'\left(w^i\right) = \sum_{i=1}^{3} p^i U'\left(w^i\right) \\ \sum_{i=1}^{3} \frac{p^i G_2^i}{R^2 P_d} U'\left(w^i\right) = \sum_{i=1}^{3} p^i U'\left(w^i\right) \end{cases}$$

If there is a finite solution for the maximization problem then

$$\sum_{i=1} p^i U'\left(w^i\right) \equiv A > 0$$

where w_d^i is evaluated at the optimum values of Δ; $q_d = q_s$ and B. Hence, the first order conditions solve the following system

$$\begin{cases} \sum_{i=1}^{3} \frac{p^i S_2^i}{R^2 S_0} U'\left(w_d^i\right) = A \\ \sum_{i=1}^{3} p^i U'\left(w_d^i\right) = A \\ \sum_{i=1}^{3} \frac{p^i G_2^i}{R^2 P_d} U'\left(w_d^i\right) = A \end{cases}$$

This is a linear system in $U'\left(w_d^1\right)$; $U'\left(w_d^2\right)$ and $U'\left(w_d^3\right)$ whose solution is

$$\begin{cases} U'\left(w_d^1\right) = A \frac{P\left(S_2^2 - S_2^3\right) + G_2^3\left(R^2 S_0 - S_2^2\right) - G_2^2\left(R^2 S_0 - S_2^3\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)} \frac{1}{p_1} \\ U'\left(w_d^2\right) = A \frac{P\left(S_2^3 - S_2^1\right) + G_2^1\left(R^2 S_0 - S_2^3\right) - G_2^3\left(R^2 S_0 - S_2^1\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)} \frac{1}{p_2} \\ U'\left(w_d^3\right) = A \frac{P\left(S_2^1 - S_2^2\right) - G_2^1\left(R^2 S_0 - S_2^2\right) + G_2^2\left(R^2 S_0 - S_2^1\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)} \frac{1}{p_3}. \end{cases}$$

As $U'\left(w_d^1\right)$, $U'\left(w_d^2\right)$ and $U'\left(w_d^3\right)$ are strictly positive, we must impose some constraints on the parameters. These constraints being satisfied imply that $P^u > P > P^l$ for P^u and P^l defined as above for the different regions of π.

Let us now look at the wealth constraints. Each constraint on the final wealth is written as

$$\Delta_0 \left(S_2^i - R^2 S_0\right) + R^2 y + q\left(G_2^i - R^2 P\right) \geq 0,$$

if Δ_0 and q are respectively the number of shares and derivatives transacted at time $t = 0$. Let us now ask the following question. By how many units may Δ_0 and q change, guaranteeing an increase in the final wealth at each state of Nature? Let d_Δ and d_q denote the (finite) variation in the number of stocks and derivatives. The final wealth is thus written as

$$(\Delta_0 + d_\Delta)\left(S_2^i - R^2 S_0\right) + R^2 y + (q + d_q)\left(G_2^i - R^2 P\right).$$

If the above constraint is satisfied, an increase in wealth is equivalent to

$$d_\Delta \left(S_2^i - R^2 S_0\right) + d_q\left(G_2^i - R^2 P\right) \geq 0.$$

Let β_i, for $i = 1, 2$ and 3, be defined as

$$\beta_i = \frac{G_2^i - R^2 P}{S_2^i - R^2 S_0}.$$

We can thus conclude that an increase of wealth in state $i = 1, 2$ and 3 occurs if and only if $d_q\left(\frac{d_\Delta}{d_q} - \beta_i\right) \geq 0$ for $S_2^i - R^2 S_0 > 0$; and $d_q\left(\frac{d_\Delta}{d_q} - \beta_i\right) \leq 0$ for $S_2^i - R^2 S_0 < 0$. Recalling that $S_2^1 > R^2 S_0 > S_2^3$, the conditions above resume to $d_q\left(\frac{d_\Delta}{d_q} - \beta_1\right) \geq 0$, $d_q\left(\frac{d_\Delta}{d_q} - \beta_3\right) \leq 0$ for states 1 and 3, respectively. For state 2, however, the relevant condition depends on whether $S_2^2 - R^2 S_0 > 0$ or $S_2^2 - R^2 S_0 < 0$.

In order to avoid an unbounded optimal solution, we must assure that there are no increments d_Δ and d_q such that the condition $d_\Delta\left(S_2^i - R^2 S_0\right) + d_q\left(G_2^i - R^2 P\right) \geq 0$ is satisfied for all $i \in \{1, 2, 3\}$. As seen from our result above, that depends on the values of β_1, β_2 and β_3.

On one hand, the relation between these values of βs depends on how P compares with the values of $P_{1,2}$, $P_{1,3}$ and $P_{2,3}$, as defined in equation (7), and is given by

	$S_2^2 - R^2 S_0 > 0$	$S_2^2 - R^2 S_0 < 0$
$P \geq P_{2,3}$	$\beta_2 \geq \beta_3$	$\beta_2 \leq \beta_3$
$P \geq P_{1,3}$	$\beta_1 \geq \beta_3$	$\beta_1 \geq \beta_3$
$P \leq P_{1,2}$	$\beta_1 \geq \beta_2$	$\beta_1 \leq \beta_2$

.

On the other hand, the relation between $P_{1;2}$; $P_{1;3}$ and $P_{2;3}$ depends on the parameters ¼: The four possible situations are described in the following Figures. They are a) ¼ 2 $G^+ \setminus H^+$) $P_{1;2} > P_{2;3} > P_{1;3}$; b) ¼ 2 $G^+ \setminus H^i$) $P_{2;3} > P_{1;2} > P_{1;3}$; c) ¼ 2 $G^i \setminus H^i$) $P_{1;3} > P_{1;2} > P_{2;3}$; d) ¼ 2 $G^i \setminus H^+$) $P_{1;3} > P_{2;3} > P_{1;2}$: Each set of three horizontal small arrows, one for each state of Nature, indicates the range of values for $\frac{d¢}{dq}$ such that the constraint, for each of the three states, is respected. Hence, for the range of prices such that the regions identi¯ed by the three arrows have a non-empty intersection, it is possible to ¯nd an unbounded optimal solution, i.e., an arbitrage opportunity.

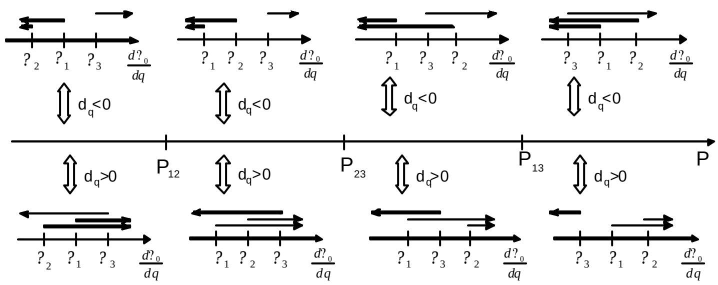

Figure 2: In the considered set of parameters ¼ 2 $G^+ \setminus H^+$, P must belong to $(P_{23}; P_{13})$ in order to avoid an unbounded solution.

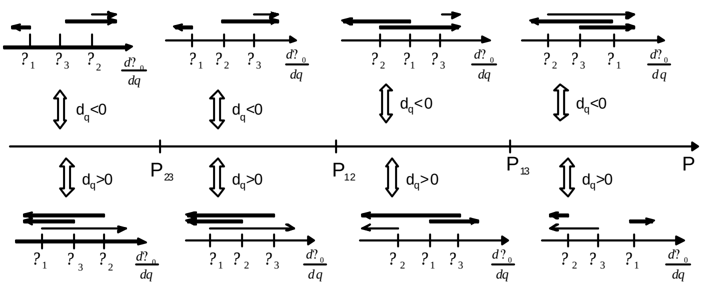

Figure 3: In the considered set of parameters ¼ 2 $G^+ \setminus H^i$, P must belong to $(P_{12}; P_{13})$ in order to avoid an unbounded solution.

Notice that, by using this procedure we end up imposing the same constraints as imposing that prices belong to the arbitrage-free range of variation.

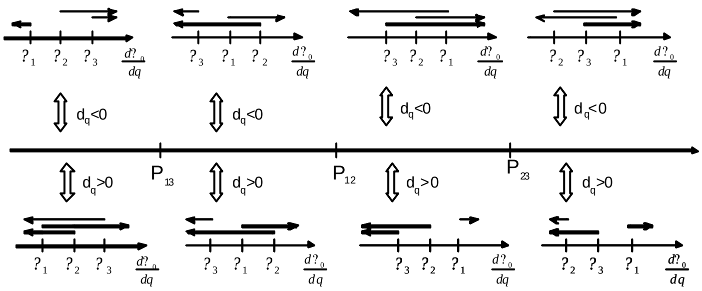

Figure 4: In the considered set of parameters $\pi \in G^{-} \cap H^{-}$, P must belong to $(P_{13}; P_{12})$ in order to avoid an unbounded solution.

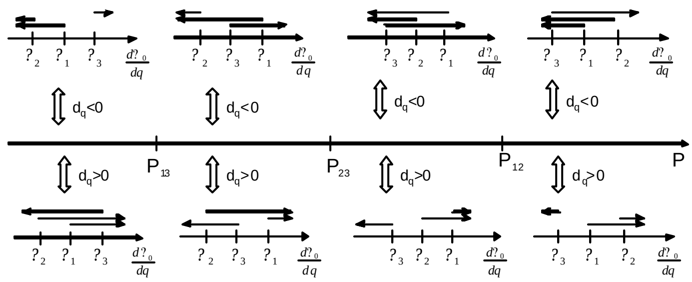

Figure 5: In the considered set of parameters $\pi \in G^{-} \cap H^{+}$, P must belong to $(P_{13}; P_{23})$ in order to avoid an unbounded solution.

2.3.2 Reservation prices and the fair price

In this section we characterize the behavior of investors when the price of the derivative is actuarially fair.[6] In particular, we establish under what conditions the investors prefer either to buy or to sell the derivative. We also provide conditions for the investors to prefer either to buy or to sell the underlying asset. In both cases the conditions do not depend on the preferences. Additionally, we establish the relation between the actuarially fair price and the reservation price of the derivative, where the reservation price is defined as the price such that the optimal transacted quantity is zero. The results are as follows.

If both the derivative and underlying asset have actuarially fair values, the investor transacts neither the derivative nor the risky asset. In this case the investor prefers to assure a risk-free wealth at maturity.

Alternatively, if the price of the underlying asset is not actuarially fair, we have two possibilities. First, if the asset is undervalued, the agent will buy it, i.e., $S_0 < \frac{1}{R^2}\sum_{i=1}^{3} S_2^i \Rightarrow \Delta^* > 0$; second, if the asset is overvalued, the agent will sell it, i.e., $S_0 > \frac{1}{R^2}\sum_{i=1}^{3} S_2^i \Rightarrow \Delta^* < 0$. Furthermore, in this situation the investors buy or sell the derivative, depending on the payoff structure, as characterized in the following lemma.

Lemma 5 Let the price of the derivative P be actuarially fair, i.e., $P = P^1 \equiv \frac{1}{R^2}\sum_{i=1}^{3} p^i G_2^i$. Also, let P^1 belong to the arbitrage-free range of variation. Then the sign of Δ^* and q^* are characterized in the Table below

	$G_2^1 > G_2^2 \geq G_2^3$ or $G_2^1 \geq G_2^2 > G_2^3$	$G_2^1 < G_2^2 \leq G_2^3$ or $G_2^1 \leq G_2^2 < G_2^3$
$\sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) > 0$	$\Delta^* > 0;\ q^* < 0$	$\Delta^* > 0;\ q^* > 0$
$\sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) = 0$	$\Delta^* = 0;\ q^* = 0$	$\Delta^* = 0;\ q^* = 0$
$\sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) < 0$	$\Delta^* < 0;\ q^* > 0$	$\Delta^* < 0;\ q^* < 0$

Proof. As the utility function is strictly concave in Δ and q then

$$\sum_{i=1}^{3} p^i U\left[\Delta_0\left(S_2^i - R^2 S_0\right) + q\left(G_2^i - R^2 P\right) + R^2 y\right]$$
$$\leq U\left[\Delta_0 \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) + q\sum_{i=1}^{3} p^i \left(G_2^i - R^2 P\right) + R^2 y\right]$$

■

[6] We say that the price of the derivative is actuarially fair if it equals the expected payoff at maturity discounted at the risk-free rate.

Proof. If $P = \frac{1}{R^2}\sum_{i=1}^{3} G_2^i$ and $S_0 = \frac{1}{R^2}\sum_{i=1}^{3} S_2^i$ we have

$$\sum_{i=1} U\left[\delta_0\left(S_2^i - R^2 S_0\right) + q\left(G_2^i - R^2 P\right) + R^2 y\right] \le U\left[R^2 y\right]$$

Hence, in order to maximize the expected utility, the best strategy is $\delta^* = q^* = 0$ leading to the maximal expected utility $U\left[R^2 y\right]$.

In the remainder of this proof we shall concentrate in the case where $P = \frac{1}{R^2}\sum_{i=1}^{3} G_2^i$ and $S_0 \neq \frac{1}{R^2}\sum_{i=1}^{3} S_2^i$. Regarding the sign of the optimal δ_0, notice that

$$\sum_{i=1} p^i U\left[\delta_0\left(S_2^i - R^2 S_0\right) + q\left(G_2^i - R^2 P\right) + R^2 y\right] \le U\left[\delta_0 \sum_{i=1} p^i\left(S_2^i - R^2 S_0\right) + R^2 y\right].$$

In order to guarantee a level of expected utility above $U\left[R^2 y\right]$, which is the utility with $\delta = q = 0$, we must assure that $\delta_0 > 0$ if $\sum_{i=1}^{3}\left(S_2^i - R^2 S_0\right) > 0$ and $\delta_0 < 0$ if $\sum_{i=1}^{3}\left(S_2^i - R^2 S_0\right) < 0$.

Regarding the sign of the optimal q, notice that

$$P = P^1 \Leftrightarrow p_2\left(G_2^2 - R^2 P\right) = -p_1\left(G_2^1 - R^2 P\right) - p_3\left(G_2^3 - R^2 P\right).$$

implying that the first order condition

$$\sum_{i=1} p_i\left(G_2^i - R^2 P\right) U'\left(w^i\right) = 0$$

may be rewritten as

$$p_1\left(G_2^1 - R^2 P\right)\left[U'\left(w^1\right) - U'\left(w^2\right)\right] = -p_3\left(G_2^3 - R^2 P\right)\left[U'\left(w^3\right) - U'\left(w^2\right)\right], \qquad (8)$$

with $w^i = \delta_0\left(S_2^i - R^2 S_0\right) + R^2 y + q\left(G_2^i - R^2 P\right)$, $i = 1, 2$ and 3. Using the assumption about the order of the payoffs and the fairness of P,

$$\begin{aligned} R^2 P &\in \left\{\min\left(G_2^1, G_2^3\right), \max\left(G_2^1, G_2^3\right)\right\} \\ &\Leftrightarrow \frac{\left(G_2^1 - R^2 P\right)}{\left(G_2^3 - R^2 P\right)} < 0 \\ &\Rightarrow \frac{\left[U'\left(w^1\right) - U'\left(w^2\right)\right]}{\left[U'(w^3) - U'(w^2)\right]} < 0 \end{aligned}$$

which implies that

$$w^1 > w^2 \text{ and } w^3 > w^2 \qquad \text{(Case 1)}$$

or

$$w^1 < w^2 \text{ and } w^3 < w^2. \qquad \text{(Case 2)}$$

Note that $w^i = \Delta_0 \left(S_2^i - R^2 S_0\right) + R^2 y + q\left(G_2^i - R^2 P\right)$, implying that

$$w^i > w^j \Leftrightarrow \Delta_0 \left(S_2^j - S_2^i\right) < q\left(G_2^i - G_2^j\right). \qquad (9)$$

We now consider the four possible situations, analyzing in each one the two cases mentioned above.

1. We first consider the case $G_2^1 \geq G_2^2 \geq G_2^3$ and $\sum_{i=1}^3 p^i \left(S_2^i - R^2 S_0\right) > 0$.

   (a) If $G_2^1 > G_2^2 > G_2^3$, in case 1 we must have

   $$w^3 > w^2 \Leftrightarrow \Delta_0 \left(S_2^2 - S_2^3\right) < q\left(G_2^3 - G_2^2\right)$$

   As $\Delta > 0$, q must be strictly negative. The same procedure applies if case 2 is considered, leading to

   $$w^1 < w^2 \Leftrightarrow \Delta_0 \left(S_2^2 - S_2^1\right) > q\left(G_2^1 - G_2^2\right) \qquad (10)$$

   and a strictly negative value for q.

   (b) $G_2^1 > G_2^2 = G_2^3$ is incompatible with case 1. However, case 2 applies.

   (c) $G_2^1 = G_2^2 > G_2^3$ is incompatible with case 2 but case 1 applies.

2. We next consider the case $G_2^1 \leq G_2^2 \leq G_2^3$ and $\sum_{i=1}^3 p^i \left(S_2^i - R^2 S_0\right) > 0$.

   (a) If $G_2^1 < G_2^2 < G_2^3$, in case 1 we must assure $\Delta_0 \left(S_2^2 - S_2^1\right) > q\left(G_2^1 - G_2^2\right)$. As $\Delta > 0$, q must be strictly positive. A similar reasoning applies in case 2. Using the relation in (10), $\Delta > 0 \Rightarrow q > 0$.

   (b) $G_2^1 < G_2^2 = G_2^3$ is incompatible with case 1. However, case 2 applies.

   (c) $G_2^1 = G_2^2 < G_2^3$ is incompatible with case 2 but case 1 applies.

3. We now consider the case $G_2^1 \geq G_2^2 \geq G_2^3$ and $\sum_{i=1}^3 p^i \left(S_2^i - R^2 S_0\right) < 0$.

   (a) If $G_2^1 > G_2^2 > G_2^3$ in case 1 we must have

   $$w^1 > w^2 \Leftrightarrow \Delta_0 \left(S_2^2 - S_2^1\right) < q\left(G_2^1 - G_2^2\right);$$

and $\Delta < 0 \Rightarrow q > 0$. A similar procedure applies in case 2, where we take

$$w^3 < w^2 \Leftrightarrow \Delta_0 \left(S_2^2 - S_2^3\right) > q\left(G_2^3 - G_2^2\right);$$

and $\Delta < 0 \Rightarrow q > 0$.

(b) $G_2^1 > G_2^2 = G_2^3$ is incompatible with case 1. However, case 2 applies.

(c) $G_2^1 = G_2^2 > G_2^3$ is incompatible with case 2 but case 1 applies.

4. We finally consider the case $G_2^1 \leq G_2^2 \leq G_2^3$ and $\sum_{i=1}^3 p^i \left(S_2^i - R^2 S_0\right) < 0$:

   (a) If $G_2^1 > G_2^2 > G_2^3$ in case 1 we must have $\Delta_0 \left(S_2^2 - S_2^1\right) < q\left(G_2^1 - G_2^2\right)$. As $\Delta < 0$; q must be strictly negative. A similar reasoning applies in case 2. We must assure that $\Delta_0 \left(S_2^2 - S_2^3\right) > q\left(G_2^3 - G_2^2\right)$: As $\Delta < 0$, q must be strictly negative.

   (b) $G_2^1 < G_2^2 = G_2^3$ is incompatible with case 1. However, case 2 applies.

   (c) $G_2^1 = G_2^2 < G_2^3$ is incompatible with case 2 but case 1 applies.

■

We now turn to the relation between the actuarially fair price of the derivative, P^{1}; and its reservation price $\hat{P}$: The reservation price of the derivative is defined as the price such that the optimal transacted quantity for a given investor, is zero. In fact, and as opposed to the exogenous P^{1}; the reservation price $\hat{P}$ depends on the investor's utility.

We now present conditions that relate $\hat{P}$ to P^{1}.

Proposition 6 Let P^{1} belong to the arbitrage-free range of variation and let the optimal transacted quantity of the derivative decrease with price. Then,

	$G_2^1 > G_2^2 \geq G_2^3$ or $G_2^1 \geq G_2^2 > G_2^3$	$G_2^1 < G_2^2 \leq G_2^3$ or $G_2^1 \leq G_2^2 < G_2^3$
$\sum_{i=1}^3 p^i \left(S_2^i - R^2 S_0\right) > 0$	$\hat{P} < P^{1}$	$\hat{P} > P^{1}$
$\sum_{i=1}^3 p^i \left(S_2^i - R^2 S_0\right) = 0$	$\hat{P} = P^{1}$	$\hat{P} = P^{1}$
$\sum_{i=1}^3 p^i \left(S_2^i - R^2 S_0\right) < 0$	$\hat{P} > P^{1}$	$\hat{P} < P^{1}$

Proof. From theorem 5 we know that when $P = P^{1}$; the sign of the optimal transacted quantity of the derivative is well defined and does not depend on the risk aversion of the investors. Hence, $\hat{P}$ will be larger that P^{1} if the quantity demand is positive for $P = P^{1}$ and $\hat{P}$ will be lower that P^{1} if the quantity demand is negative for $P = P^{1}$: ■

Remark 7 Notice that, although $\hat{P}$ depends on the utility of the investors, the fact that $\hat{P} \leq P^{\infty}$ or $\hat{P} \geq P^{\infty}$ depends only on the parameters of the economy, not on the investors' preferences.

Remark 8 Notice that, if $\hat{P}$ belongs to the arbitrage-free range, P^{∞} must be outside the range defined by $\left(\inf \hat{P}, \sup \hat{P}\right)$, where the infimum and the supremum above are taken over the class of all admissible utility functions.

2.4 Illustrations

We now consider the cases of a Constant Absolute Risk Aversion (CARA) utility function

$$u(w) = -e^{-\delta w}$$

with $\delta > 0$, and of the Constant Relative Risk Aversion (CRRA) utility function

$$u(w) = \frac{w^{1-\gamma} - 1}{1-\gamma}$$

with $\gamma > 0$.

Notice that, when $\gamma = 0$, the utility function becomes $u(w) = w - 1$, which is a utility function characterizing a risk neutral agent. Moreover, when $\gamma \to \infty$ the utility function becomes $u(w) = \ln(w)$.

2.4.1 Explicit solution for demand and supply

For these utility functions it is possible to have an explicit solution for the demand and supply of the derivative. The derivation is presented in appendix D.1. Let

$$\mu_1 = \frac{P\left(S_2^2 - S_2^3\right) + G_2^3\left(R^2 S_0 - S_2^2\right) - G_2^2\left(R^2 S_0 - S_2^3\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)} \frac{1}{p_1}$$

$$\mu_2 = \frac{P\left(S_2^3 - S_2^1\right) + G_2^1\left(R^2 S_0 - S_2^3\right) - G_2^3\left(R^2 S_0 - S_2^1\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)} \frac{1}{p_2}$$

$$\mu_3 = \frac{P\left(S_2^1 - S_2^2\right) - G_2^1\left(R^2 S_0 - S_2^2\right) + G_2^2\left(R^2 S_0 - S_2^1\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)} \frac{1}{p_3}$$

If a CARA utility function is considered then

$$\begin{cases} q_d = q & , q > 0 \\ q_s = -q & , q < 0. \end{cases}$$

where

$$q = -\frac{1}{\delta} \frac{\left(S_2^2 - S_2^3\right)\ln \mu_1 + \left(S_2^3 - S_2^1\right)\ln \mu_2 + \left(S_2^1 - S_2^2\right)\ln \mu_3}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) + G_2^3\left(S_2^1 - S_2^2\right)}$$

If a CRRA utility function is considered then

$$\begin{cases} q_d = q & ; q > 0 \\ q_s = -q & ; q < 0. \end{cases}$$

where

$$q = -R^2 y \frac{\mu_1^{-\frac{1}{\gamma}}\left(S_2^3 - S_2^2\right) + \mu_2^{-\frac{1}{\gamma}}\left(S_2^1 - S_2^3\right) + \mu_3^{-\frac{1}{\gamma}}\left(S_2^2 - S_2^1\right)}{\begin{array}{l} \mu_1^{-\frac{1}{\gamma}}\left[-\left(S_2^2 - R^2 S_0\right)\left(G_2^3 - R^2 P\right) + \left(S_2^3 - R^2 S_0\right)\left(G_2^2 - R^2 P\right)\right] \\ +\mu_2^{-\frac{1}{\gamma}}\left[-\left(S_2^1 - R^2 S_0\right)\left(G_2^3 - R^2 P\right) + \left(S_2^3 - R^2 S_0\right)\left(G_2^1 - R^2 P\right)\right] \\ +\mu_3^{-\frac{1}{\gamma}}\left[-\left(S_2^2 - R^2 S_0\right)\left(G_2^1 - R^2 P\right) + \left(S_2^1 - R^2 S_0\right)\left(G_2^2 - R^2 P\right)\right] \end{array}} \tag{11}$$

2.4.2 Properties of individual demand and supply

In what follows we present some properties of the individual demand and supply. If $u(w)$ is a CARA utility function then[7]

1. The individual demand for the derivative is a decreasing function of the price, i.e.,
$$\frac{\partial q_d}{\partial P_d} \leq 0.$$
2. The individual supply for the derivative is an increasing function of the price, i.e.,
$$\frac{\partial q_d}{\partial P_s} \geq 0.$$
3. The optimal number of options multiplying by δ is constant. Hence, the demand and/or supply will shift downwards when the coefficient of absolute relative aversion increases, i.e.,
$$\frac{\partial q_d}{\partial \delta} \leq 0 \text{ and } \frac{\partial q_d}{\partial \delta} \leq 0$$
4. The price such that the optimal number of derivatives is equal to zero is independent of the δ.
5. The optimal number of shares is independent of the initial wealth.

If $u(w)$ is a CARA utility function then[8]

[7]See appendix D.2.1 for details.
[8]See appendix D.2.7 for details.

1. Numerically, the individual demand for the derivative can be shown to be a decreasing function of the price, i.e.

$$\frac{\partial q_d}{\partial P_d} \leq 0$$

2. Numerically, the individual supply for the derivative can be shown to be an increasing function of the price, i.e.

$$\frac{\partial q_s}{\partial P_s} \geq 0$$

3. The demand and supply are increasing functions of the exogenous wealth, i.e.,

$$\frac{\partial q_d}{\partial y} > 0 \text{ and } \frac{\partial q_s}{\partial y} > 0.$$

4. The reservation price is an increasing function of γ if $G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) < 0$. Otherwise, the reservation price is an increasing function of γ.

We perform two simulations to illustrate property 4. In the first simulation we choose parameters' values such that $G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) < 0$. In the second simulation, we have $G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) > 0$.

For the first case, the values of the parameters are described in the following table:

S_0	S_2^1	S_2^2	S_2^3	G_2^1	G_2^2	G_2^3	p_1	p_2	p_3	R	Y
10.5	14	10	8	4	2	0.5	0.1	0.6	0.3	1	500

Note that $E[G] = 1,75$ and the arbitrage-free range of variation for the value of the derivative is $(1,958, 2,25)$. Moreover, $G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) = -2$. The reservation prices for a CRRA utility function are presented in the following table

γ	0,1	3	10
P	2.1279	2.1493	2.1519

If a CARA utility function is presented the reservation price is independent of δ and equals 2.1532.

For the second case the parameters are

S_0	S_2^1	S_2^2	S_2^3	G_2^1	G_2^2	G_2^3	p_1	p_2	p_3	R	Y
10,5	15	10	8	0.5	2	4	0.1	0.6	0.3	1	500

In this case E [G] = 2;45 and the arbitrage-free range of variation is (1:85; 2:75) : Moreover, $G_2^1 \left(S_2^2 ¡ S_2^3\right) + G_2^2 \left(S_2^3 ¡ S_2^1\right) ¡ G_2^3 \left(S_2^2 ¡ S_2^1\right) = 7$: The reservation prices for a CRRA utility function are as follows

°	0; 1	3	10
P	2:2133	2:1705	2:1662

If a CARA utility function is presented the reservation price equals 2:1643:

In what follows we present some simulations to obtain the individual demand and supply. The initial value of the risky asset is S_0 = 10: At time t = 2, the risky asset can assume three di®erent values, $S_{1;2}$ = 12 ; $S_{2;2}$ = 10:5 and $S_{3;2}$ = 9: A European option with strike K = 11 and maturity of two periods is considered. Hence, its payo®s at time two are $G_{1;2}$ = 0 ; $G_{2;2}$ = 0:5 and $G_{3;2}$ = 2: We consider that the rate of return of the risk free asset is zero and a initial wealth of 500; i.e., y = 500: The probability of occurrence of the ¯rst, second and third state of nature are, respectively, given by 0:2, 0:3 and 0:5. Using pure arbitrage arguments we ¯nd that the arbitrage-free range of variation for the value of the put options is $\left(1; \frac{4}{3}\right)$: For each price within this range, the ¯gures below show for a CRRA utility function with a curvature ° = 0:1; a) the demand and supply curves; b) transacted amount of underlying asset; c) transacted amount of risk-free asset, and d) the utility level attained for each di®erent price.

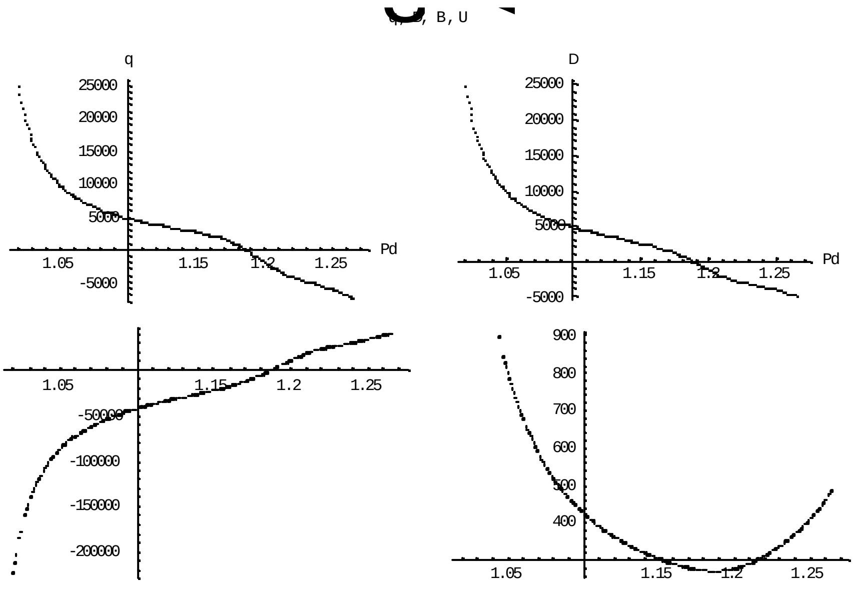

Figure 6: The optimum number of units of the derivative, number of shares, amount invested in the risk free asset and the the value of the utility for di®erent values of the price of the derivative are presented.

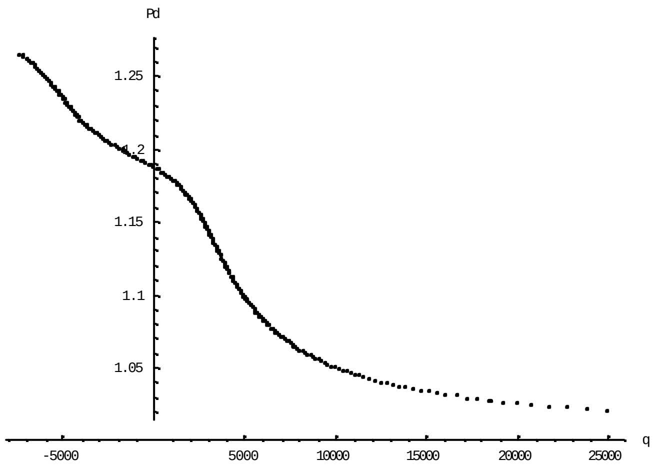

Figure 7: Individual demand and supply of the derivative in the same graph. For prices above 1:18 the agent is selling the derivative. However, if the price is below that threshold the agent is buying the derivative.

This last ¯gure is obtained by inverting the ¯rst graph above and illustrates the individual demand and supply of the derivative. For prices above 1:18 the agent is selling ($q < 0$) the derivative. However, if the price is below that threshold, the agent is buying the derivative ($q < 0$). Finally, a CARA utility function would present the same basic features of the curves above.

3 Market-Makers

In the previous section we presented the optimization problem of illiquid traders and derive the demand and supply of derivatives as a function of exogenous endowments. In this section we present the problem faced by market-makers, given the demand and supply functions for derivatives. In fact, the optimal strategy of a ¯nancial institution transacting such contracts depends on wether there are competitors or not. First, we are going to consider a monopolistic market-maker, Then, we allow competition between market-makers[9]. Given optimal prices and quantities the market-maker(s) must also choose an hedging strategy, constituted of stocks and bonds.

[9]One important features of exchanges trading options is the use of specialits by the American Stock Exchange in place of competing market-makers on the Chicago Board Options Exchange. The option specialist has access to a greater amount of information than other traders and, therefore, can mantain a monopolistic position. For instance, on many exchanges, only the specialist has information about the orders at the opening of the market. The access to this information allows him to extract some monopolistic pro¯ts. In contrast, at the CBOE each market-maker is required to compete with others market-makers. CBOE requires that each transaction be executed at the highest bid and lowest ask prices emerging from the group of market-makers participating in the process.

3.1 Monopolistic Market-Maker

The monopolist market-maker's problem consists of choosing the bid and ask prices, together with a hedging strategy, so as to maximize his or her expected utility. Equivalently, the problem can also be solved choosing the optimal quantities to transact (sell and buy) and the optimal hedging strategies. This equivalence follows from the assumption that the monopolistic market-maker must satisfy all market demand and supply at the ask and bid prices that he sets. Let $Q_A (Q_B)$ be the number of European derivatives that the market-maker is selling (buying), Δ be the number of shares and B be the amount invested in the risk-free asset. In what follows we allow the optimal quantity sold (Q_A) by the market-maker to be different from the optimal quantity bought (Q_B).

If the market-maker is risk neutral he faces the following problem.

$$\max_{Q_B; Q_A; \Delta; B} E[\pi] = \sum_{i=1} p_i \left[Q_B G_2^i - Q_A G_2^i + \Delta S_2^i + R^2 B \right]$$

subject to

$$\Delta S_0 + B - Q_A P_d(Q_A) + Q_B P_s(Q_B) \leq y \tag{12}$$

$$Q_B G_2^i - Q_A G_2^i + \Delta S_2^i + R^2 B \geq 0 \quad ; \quad i = 1; 2; 3 \quad \text{and} \tag{13}$$

$$Q_B \geq 0; Q_A \geq 0 \tag{14}$$

Several assumptions concerning the market demand and supply are made.

Assumption 1: The supply and the demand functions are, respectively, increasing and decreasing in the transacted quantities,

$$\frac{dP_s(Q_B)}{dQ_B} > 0$$

$$\frac{dP_d(Q_A)}{dQ_A} < 0.$$

Assumption 2: The function

$$\sum_{i=1} p_i \left[Q_B \left(G_2^i - R^2 P_s(Q_B) \right) - Q_A \left(G_2^i - R^2 P_d(Q_A) \right) + \Delta \left(S_2^i - R^2 S_0 \right) + R^2 y \right]$$

is concave in $Q_A; Q_B$ and Δ[10].

These rules induce a strong competition between market-makers.

[10]See appendix C.4 for details on this assumption. This function is simply the objective function of the monopolist, incorporating the first restriction, shown later to be always binding.

Just for notation let $\tilde{X}^k_2 \equiv X^k_2/R^2$. Also, let the sets K and $B_k(Q_A;Q_B;\Delta;B)$ be defined as follows.

$$K \equiv \left\{ m : \text{sign}\left(\sum_{i=1} p_i \tilde{S}^i_2 - S_0\right) = -\text{sign}\left(\tilde{S}^m_2 - S_0\right) \right\}:$$

and

$$B_k(Q_A;Q_B;\Delta;B) = \left\{ \begin{array}{ll} (Q_A;Q_B;\Delta;B) : & \Delta S_0 + B + Q_A P_d(Q_A) - Q_B P_s(Q_B) = y; \\ & Q_B G^i_2 - Q_A G^i_2 + \Delta S^i_2 + R^2 B \geq 0; i \neq k \text{ and} \\ & Q_B G^k_2 - Q_A G^k_2 + \Delta S^k_2 + R^2 B = 0 \end{array} \right\}:$$

For simplicity of notation let us introduce the constant

$$\sigma^k \equiv \frac{\sum_{i=1}^3 p_i \tilde{S}^i_2 - S_0}{\tilde{S}^k_2 - S_0};$$

the function of Q[11]

$$\Phi^k(Q) = \frac{\sum_{i=1}^3 p_i \tilde{G}^i_2 - \frac{d[QP(Q)]}{dQ}}{\tilde{G}^k_2 - \frac{d[QP(Q)]}{dQ}}$$

and the value

$$\Psi(Q^*) = \left.\frac{d[QP(Q)]}{dQ}\right|_{Q^*}:$$

The existence of the bid-ask spread is characterized in the following theorem.

Theorem 9 (Equilibrium Condition) Under assumptions 1 and 2, and in the presence of a risk-neutral monopolist market-maker, a sufficient condition for the existence of an equilibrium with strictly positive quantities $\{Q^*_B;Q^*_A\}$; is characterized by Q^*_A and Q^*_B satisfying either

$$\Phi^k(Q^*_B) = \Phi^k(Q^*_A) = \sigma^k \qquad (15)$$

with $(Q^*_A;Q^*_B;\Delta^*;B^*) \in B_k(Q_A;Q_B;\Delta;B)$ and $k \in K$ or

$$\left\{ \begin{array}{l} \sum_{i=1}^3 p_i \left(\tilde{G}^i_2 - \Psi(Q^*_B)\right) - \lambda_k \left[\tilde{G}^k_2 - \Psi(Q^*_B)\right] - \lambda_j \left[\tilde{G}^j_2 - \Psi(Q^*_B)\right] = 0 \\ \sum_{i=1}^3 p_i \left(\tilde{G}^i_2 - \Psi(Q^*_A)\right) - \lambda_k \left[\tilde{G}^k_2 - \Psi(Q^*_A)\right] - \lambda_j \left[\tilde{G}^j_2 - \Psi(Q^*_A)\right] = 0 \\ \sum_{i=1}^3 p_i \left(\tilde{S}^i_2 - S_0\right) - \lambda_k \left[\tilde{S}^k_2 - S_0\right] - \lambda_j \left[\tilde{S}^j_2 - S_0\right] = 0 \end{array} \right. \qquad (16)$$

[11] Note that, if we consider the market supply, Φ^k is evaluated at $Q = Q_B$. Alternatively, if we consider the market demand, Φ^k is evaluated at $Q = Q_A$.

with $\lambda_k < 0$; $\lambda_j < 0$; $k \neq j$ with at least one of k and j in K and $(Q_A^*; Q_B^*; \delta^*; B^*) \in B_k(Q_A; Q_B; \delta; B) \cap B_j(Q_A; Q_B; \delta; B)$: Moreover, we can assure that the above conditions are also sufficient to generate a bid-ask spread, i.e., $P_d(Q_A^*) > P_s(Q_B^*)$:

Proof. Taking into account that, at the optimum, $\delta S_0 + B + Q_A P_d - Q_B P_s = y$; the problem of the monopolistic risk-neutral market-maker can be rewritten as

$$\max_{Q_B; Q_A; \delta} \sum_{i=1}^{3} p_i \left\{ Q_B \left[G_2^i - P_s(Q_B) \right] - Q_A \left[G_2^i - P_d(Q_A) \right] + \delta \left(S_2^i - S_0 \right) + y \right\}$$

subject to

$$-Q_B \left[G_2^i - P_s(Q_B) \right] + Q_A \left[G_2^i - P_d(Q_A) \right] - \delta \left(S_2^i - S_0 \right) - y \le 0$$

for $i = 1; 2$ and 3 and

$$Q_B \ge 0; Q_A \ge 0:$$

Since the objective function is quasiconcave and the constraint set is convex[12], we can assure[13] the existence of a solution for the problem. Moreover, we can assure[14] that the solution of the problem is given by the Kuhn-Tucker solutions, provided that all the constraints that hold in equality are independent.

The Lagrangean of the problem is given by

$$\begin{aligned} L = & \sum_{i=1}^{3} p_i \left\{ Q_B \left[G_2^i - P_s(Q_B) \right] - Q_A \left[G_2^i - P_d(Q_A) \right] + \delta \left(S_2^i - S_0 \right) + R^2 y \right\} \\ & - \sum_{i=1}^{3} \lambda_i \left\{ Q_B \left[G_2^i - P_s(Q_B) \right] - Q_A \left[G_2^i - P_d(Q_A) \right] + \delta \left(S_2^i - S_0 \right) + R^2 y \right\} \end{aligned}$$

The first order Kuhn-Tucker conditions are

$$\begin{cases} \frac{dL}{dQ_B} \le 0 & Q_B \ge 0 & \frac{dL}{dQ_B} Q_B = 0 \\ \frac{dL}{dQ_A} \le 0 & Q_A \ge 0 & \frac{dL}{dQ_A} Q_A = 0 \\ \frac{dL}{d\delta} = 0 & & \\ \frac{dL}{d\lambda_i} \le 0 & \lambda_i \le 0 & \frac{dL}{d\lambda_i} \lambda_i = 0 \end{cases}$$

---

[12] See appendix C.4.

[13] See theorem MK4, in Mas-Colell et al. (1995), page 962.

[14] and once again using a theorem of Mas-Colell et al. (1995), theorem MK2, page 959,

for $i = 1, 2$ and 3. If the solution is characterized by $Q_A^* > 0$ and $Q_B^* > 0$, the first order conditions at the optimum are

$$\begin{cases} \frac{dL}{dQ_B} = 0 \\ \frac{dL}{dQ_A} = 0 \\ \frac{dL}{d\Delta} = 0 \end{cases} \Leftrightarrow \begin{cases} \sum_{i=1}^{3} p_i \left[G_2^i - \phi(Q_B^*)\right] - \sum_{i=1}^{3} \lambda_i \left[G_2^i - \phi(Q_B^*)\right] = 0 \\ \sum_{i=1}^{3} p_i \left[G_2^i - \phi(Q_A^*)\right] - \sum_{i=1}^{3} \lambda_i \left[G_2^i - \phi(Q_A^*)\right] = 0 \\ \sum_{i=1}^{3} p_i \left(S_2^i - S_0\right) - \sum_{i=1}^{3} \lambda_i \left(S_2^i - S_0\right) = 0 \end{cases}$$

Notice that it is not possible to have all $\lambda_i < 0, i = 1, 2, 3$ at the optimal point characterizing the solution. If $\lambda_i < 0, i = 1, 2, 3$, all constraints would be binding and the value of the objective function would be zero. Moreover, it is not possible to have all $\lambda_i = 0, i = 1, 2, 3$, unless $\sum_{i=1}^{3} p_i \left[S_2^i - S_0\right] = 0$. Hence, either there is only one value of i such that $\lambda_i = 0$, or there is only one value of i such that $\lambda_i < 0$. In the latter case, the first order conditions presented above result in equation (15) with $\lambda_k = \sigma^k < 0 \Rightarrow k \in K$. In the former case, the first order conditions lead to equation (16).

The result of existence of a bid-ask spread under the above conditions is discussed in appendix E. ■

In what follows, we present a necessary and sufficient condition in order to have equation (15) fulfilled. Notice that

$$\Psi^k(Q_B = 0) = \frac{\sum_{i=1}^{3} p_i G_2^i - \hat{P}_s}{G_2^k - \hat{P}_s} \text{ and } \Psi^k(Q_A = 0) = \frac{\sum_{i=1}^{3} p_i G_2^i - \hat{P}_d}{G_2^k - \hat{P}_d}$$

where $\hat{P}_s$ and $\hat{P}_d$ are the supply and demand reservation prices, respectively.

Corollary 10 A necessary and sufficient condition to have equation (15) satisfied is that there are reservation prices $\hat{P}_d$ and $\hat{P}_s$ such that

$$\Psi^k(Q_A = 0) \leq \sigma^k \leq \Psi^k(Q_B = 0) < 0 \quad \text{(Case I)}$$

if $\sum_{i=1}^{3} p_i G_2^i - G_2^k \leq 0$; and

$$0 > \Psi^k(Q_A = 0) \geq \sigma^k \geq \Psi^k(Q_B = 0) \quad \text{(Case II)}$$

if $\sum_{i=1}^{3} p_i G_2^i - G_2^k \geq 0$. Moreover, necessary conditions to have equations (Case I) and (Case II) satisfied are

$$\sum_{i=1} p_i G_2^i < \hat{P}_d \leq \hat{P}_s < G_2^k. \quad \text{(Case I)}$$

$$\sum_{i=1} p_i G_2^i > \hat{P}_d \geq \hat{P}_s > G_2^k. \quad \text{(Case II)}$$

Proof. In order to study the behaviour of the function $\Pi^k(Q)$ we take the derivative

$$\frac{d\Pi^k(Q)}{dQ} = \frac{\frac{d^2[QP(Q)]}{dQ^2}\left[\sum_{i=1}^{3} p_i \bar{G}_2^i - \bar{G}_2^k\right]}{\left[\bar{G}_2^k - \frac{d[QP(Q)]}{dQ}\right]^2}.$$

The sign of $\frac{d^2[Q_B P_s(Q_B)]}{dQ_B^2}$ and $\frac{d^2[Q_A P_d(Q_A)]}{dQ_A^2}$ is well defined by assumption 2. Therefore, we identify the regions where $\Pi^k(Q_B)$ is decreasing in Q_B, and $\Pi^k(Q_A)$ is increasing in Q_A,

	$\sum_{i=1}^{3} p_i \bar{G}_2^i - \bar{G}_2^k \geq 0$	$\sum_{i=1}^{3} p_i \bar{G}_2^i - \bar{G}_2^k \leq 0$
$\frac{d\Pi^k(Q_B)}{dQ_B}$	≥ 0	≤ 0
$\frac{d\Pi^k(Q_A)}{dQ_A}$	≤ 0	≥ 0

Equation (15) reads $\Pi^k(Q_A^*) = \Pi^k(Q_B^*) = \sigma^k$. Now consider the case $\sum_{i=1}^{3} p_i \bar{G}_2^i - \bar{G}_2^k \geq 0$. In that region $\Pi^k(Q_A^*) = \Pi^k(Q_B^*) \Rightarrow \Pi^k(Q_A = 0) \geq \Pi^k(Q_B = 0)$, and the full equation (15) is satisfied iff

$$\Pi^k(Q_A = 0) \geq \sigma^k \geq \Pi^k(Q_B = 0).$$

As $\Pi^k(Q = 0)$ is an increasing function on P, then $\hat{P}_d \geq \hat{P}_s$.

In what follows we present the relation between $\hat{P}_d$, $\hat{P}_s$, $\sum_{i=1}^{3} p_i \bar{G}_2^i$ and $\bar{G}_2^k$ at the optimum of the monopolistic market-maker.

Suppose that $\hat{P}_d$ and $\hat{P}_s$ belong to the arbitrage-free range of variation for the value of a European derivative. It is not possible to have both $\hat{P}_d$ and $\hat{P}_s$ above $\sum_{i=1}^{3} p_i \bar{G}_2^i$ or below $\bar{G}_2^k$, otherwise $\Pi^k(Q_A = 0) > 0$ and $\Pi^k(Q_B = 0) > 0$, which is incompatible with $\sigma^k \leq 0$. Moreover, it is not possible to have $\hat{P}_d > \sum_{i=1}^{3} p_i \bar{G}_2^i$ and $\bar{G}_2^k \leq \hat{P}_s < \sum_{i=1}^{3} p_i \bar{G}_2^i$, because from theorem (6) we know that $\sum_{i=1}^{3} p_i \bar{G}_2^i \in [\hat{P}_s, \hat{P}_d]$. Finally, the situation where $\hat{P}_s < \bar{G}_2^k$ is also not possible because in that case $\Pi^k(Q_B = 0) > 0$, contradicting the fact that $\sigma^k \geq \Pi^k(Q_B = 0)$ and $\sigma^k \leq 0$. Hence, the case that remains is

$$\sum_{i=1}^{3} p_i \bar{G}_2^i > \hat{P}_d \geq \hat{P}_s > \bar{G}_2^k.$$

Proceeding in the same way for the case $\sum_{i=1}^{3} p_i \bar{G}_2^i - \bar{G}_2^k \leq 0$ we find out that

$$\sum_{i=1}^{3} p_i \bar{G}_2^i < \hat{P}_s \leq \hat{P}_d < \bar{G}_2^k.$$

■

There are two important remarks regarding the equilibrium condition above. First, in order to have an equilibrium bid-ask spread, we must assure that $\hat{P}_d > \hat{P}_s$: Hence, the risk aversion characterizing each representative agent of the demand side has to be different from the risk aversion characterizing each representative agent of the supply side; second, in equilibrium the market-maker gains in one side of the market but loses in the other side of the market.

We illustrate the second point as follows. Consider case II presented above. In equilibrium, the market-maker gains, in terms of expected profit, with the bought units of the derivative (because $P_s < \sum_{i=1}^{3} p_i \phi_2^{1i}$), but loses in terms of expected profit, with the sold units (because $P_d < \sum_{i=1}^{3} p_i \phi_2^{1i}$). The interpretation of this equilibrium results from the fact that (i) in equilibrium, at least one of the wealth constraints is binding and (ii) the interval defined by the the demand and supply reservation prices does not contain the expected value of the derivative's payoff, $\sum_{i=1}^{3} p_i \phi_2^{1i}$. The intuition is as follows. The latter fact implies that a market-maker in both sides of the market (selling and buying the derivative) has necesarily positive utility on one side, and negative expected utility on the other side. A market-maker may thus choose to be only on the side of the market that provides a positive expected utility. However, in order to maximize the expected utility, the market-maker may find an incentive to enter on the other side of the market, just to relax the binding restrictions. This would happen only if the resulting negative expected utility would be more than compensated by the improvement of the positive expected utility on the other side of the market. Our result reflects the fact that a bid-ask spread exists only when the market-maker faces one such incentive.

3.2 Competition Between Market-Makers

In this section, our model is extended to consider the presence of several market-makers. In an oligopoly, the payoffs for one market-maker depends on its own actions, as well as on the actions of the other market-makers. The strategic interactions between the market-makers will determine the equilibrium.

Here, individual market-makers simultaneously determine their bid and ask prices, the number of shares and the amount invested in the risk-free asset, behaving in their own interest in an noncooperative game. The objective is to compute the Nash equilibrium of this game. A Nash equilibrium in prices is a vector of prices such that each market-maker maximizes its expected payoffs given the other market-maker's decisions.

Let M be the number of market-makers in this market. Market-maker i; $i = 1; \ldots; M$; has expected utility $U_i\left(P_{i;d}; P_{i;s}; P_{-i;d}; P_{-i;s}; \Delta_i; \Delta_{-i}\right)$; where $P_{i;d}$, $P_{i;s}$ and Δ_i are, respectively, the ask price, the bid price and the number of units of the underlying asset hold by market-maker i: The values

$P_{-i;d}, P_{-i;s}$ and Δ_{-i} correspond to the components of the vector of the analogous variables relative to the remaining $M-1$ market-makers, i.e., $(P_{-i;d}, P_{-i;s}, \Delta_{-i}) = (P_{j;d}, P_{j;s}, \Delta_j)_{j\in\{1,\ldots,M\}, j\neq i}$.

Formally, the expected utility of market-maker j corresponds to its expected profit, given by

$$\begin{aligned}\Pi_j(P_{j;d}, P_{j;s}, P_{-j;d}, P_{-j;s}, \Delta_j, \Delta_{-j}) = \sum_{i=1} p^i \Big[Q_{j;B}(P_{j;s}, P_{-j;s})\left(G_2^i - R^2 P_{j;s}\right) \\ - Q_{j;A}(P_{j;d}, P_{-j;d})\left(G_2^i - R^2 P_{j;d}\right) + \Delta_j \left(S_2^i - R^2 S_0\right) + R^2 y\Big].\end{aligned}$$

For given prices, the optimal amount transacted solve the problem

$$U_i(P_{j;d}, P_{j;s}, P_{-j;d}, P_{-j;s}) = \max_{\Delta_j} \Pi_j(P_{j;d}, P_{j;s}, P_{-j;d}, P_{-j;s}, \Delta_j, \Delta_{-j})$$

leading to an optimal amount $\Delta_j^* = \Delta_j(P_{j;d}, P_{j;s}, P_{-j;d}, P_{-j;s})$ and characterizing the expected utility function to be maximized on prices. In fact, for that given amount Δ_j^*, prices are set optimally as the solution of

$$\max_{P_{j;s}, P_{j;d}} U_i(P_{j;d}, P_{j;s}, P_{-j;d}, P_{-j;s})$$

subject to

$$\begin{aligned}0 \leq\; & Q_{j;B}(P_{j;s}, P_{-j;s})\left(G_2^i - R^2 P_{j;s}\right) - \\ & Q_{j;A}(P_{j;d}, P_{-j;d})\left(G_2^i - R^2 P_{j;d}\right) + \Delta_j^*\left(S_2^i - R^2 S_0\right) + R^2 y,\end{aligned}$$

for $i = 1, 2, 3$; and where $Q_{j;B}(P_{j;s}, P_{-j;s})$ and $Q_{j;A}(P_{j;d}, P_{-j;d})$ are, respectively, the demand and supply functions faced by firm j.

Financial products are generally accepted as being homogeneous goods[15]. As a consequence, options traded be different intermediaries are perfect substitutes in the investor's utility function, and investors will transact with the intermediary setting the best price. The point is that in financial markets the best quotes can be easily found. In particular, automated trading systems facilitate the disclosure of the best price. Hence, the homogeneous good assumption gives rise to discontinuity of the demand and supply curves. Typically, market-maker j is viewed as facing the demand curve $Q_{j;A}(P_{j;d}, P_{-j;d})$, which is a function of the ask-price that all market-makers quote. Supposing that $P_{i;d} = P_{k;d}$, for all $i, k \neq j$,

$$Q_{j;A}(P_{j;d}, P_{-j;d}) = \begin{cases} \chi_A(P_{j;d}, P_{i;d}) & ; \text{ if } P_{j;d} > P_{i;d}, \text{ for all } i \\ \frac{1}{M} Q_A(P_{j;d}) & ; \text{ if } P_{j;d} = P_{i;d}, \text{ for all } i \\ Q_A(P_{j;d}) & ; \text{ if } P_{j;d} < P_{i;d}, \text{ for all } i. \end{cases} \quad (17)$$

[15] There are, however, some exceptions to this view. Menyah and Paudyal (2000) argue that the "order flow on the LSE like Nasdaq does not necessarily go to the dealer with the most competitive quotes because of preferencing and internalisation by brokers". In such a case, orders may be satisfied by market-makers not posting the best quoted prices.

where $\chi(P_{j,d}, P_{i,d}) \in \left[0, Q_A(P_{j,d}) - \frac{M-1}{M} Q_A(P_{i,d})\right]$.

The last line reflects the fact that, if market-maker j quotes a lower price than the competitors, he will face all the market demand. The second line is trivially equivalent to a fair ratio among the market-makers. The first line, however, is more subtle. If market-maker j quotes a higher price than his competitors, several situations are possible. As the competitors of market-maker j have wealth constraints, it may happen that they are not able to sell additional units at the lower price to former customers of market-maker j. In that case $Q_{j,A}(P_{j,d}, P_{-j,d}) = Q_A(P_{j,d}) - \frac{M-1}{M} Q_A(P_{i,d})$. However, it also may be possible that, even having binding constraints[16], competitors are able to sell additional units to former customers of market-maker j. In this case, the demand faced by market-maker j when he quotes a higher price will be zero, i.e., $Q_{j,A}(P_{j,d}, P_{-j,d}) = 0$. An intermediary solution is also possible, where the competitors are not able to sell for all the former customers of market-maker j. Hence, we can conclude that when market-maker j sets a price above his competitors, his demand will belong to the range $\left[0, Q_A(P_{j,d}) - \frac{M-1}{M} Q_A(P_{i,d})\right]$. Figure 8 illustrates this discontinuity in the market demand.

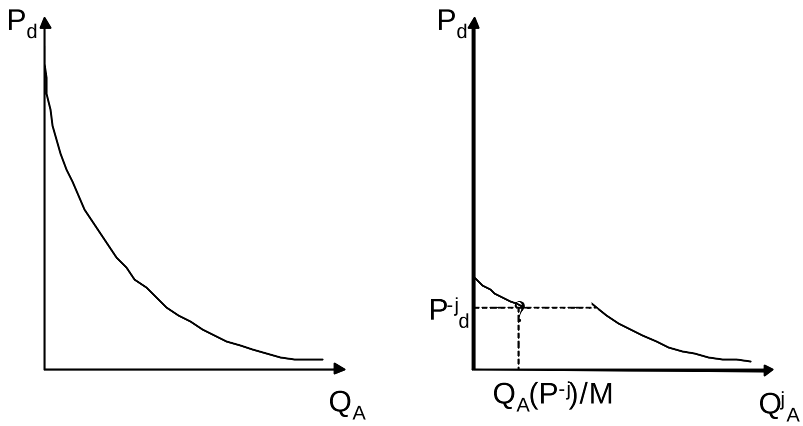

Figure 8: The market demand and the demand faced by market-maker j, for the case where the wealth restrictions of the competitors do not allow to sell the derivative to any of his former customers.

On the other hand, market-maker j is viewed as facing the following

[16] This is the case if $\sum_{i=1}^{3} p^i G_2^i - R^2 P_{i,d} < 0$ and $G_2^k - R^2 P_{i,d} < 0$, where k identifies the binding restriction.

supply curve

$$Q_{j,B}(P_{j,s}, P_{-j,s}) = \begin{cases} Q_B(P_{j,s}) & ; \text{ if } P_{j,s} > P_{i,s}, \text{ for all } i \\ \frac{1}{M} Q_B(P_{j,s}) & ; \text{ if } P_{j,s} = P_{i,s}, \text{ for all } i \\ \chi_B(P_{j,s}, P_{i,s}) & ; \text{ if } P_{j,s} < P_{i,s}, \text{ for all } i. \end{cases} \tag{18}$$

where $\chi_B(P_{j,s}, P_{i,s}) \in \left[0, Q_B(P_{j,s}) - \frac{M-1}{M} Q_B(P_{-j,s})\right]$. See Figure 9 for an illustration.

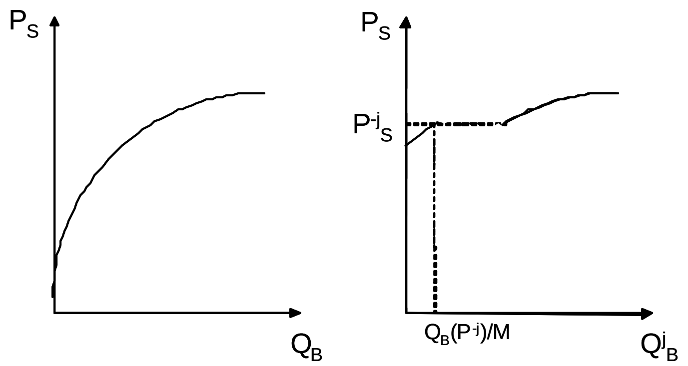

Figure 9: The market supply and the supply faced by market-maker j, for the case where the wealth restrictions of the competitors do not allow to buy the derivative to any of his former suppliers.

Definition 11 A vector of ask and bid prices $(P^*_d, P^*_s, \Delta^*) = \left(P^*_{i,d}, P^*_{i,s}, \Delta^*_i\right)_{i=1,...,M}$ is an equilibrium iff, for all i and all possible prices, $(P_d, P_s) = (P_{i,d}, P_{i,s})_{i=1,...,M}$

$$U_i\left(P^*_{i,d}, P^*_{i,s}, P^*_{-i,d}, P^*_{-i,s}\right) \geq U_i(P_{i,d}, P_{i,s}, P_{-i,d}, P_{-i,s}).$$

In other words, a set of prices is a Nash equilibrium if market-makers have no incentive to set different prices to obtain higher utility.

The equilibrium result known as Bertrand Paradox establishes that price competition between two or more identical firms with no constraints leads to price equal to marginal cost, and firms make no profit. As market-makers are perfect competitors in prices, it is usual in the literature to accept that market-makers earn zero profits. However, that is not the case here, since there are positive wealth constraints. Hence, we must investigate further the existence of a pure Nash equilibrium of this game.

For each firm j we define two reaction functions as the optimal prices (demand and supply) as a function of the prices quoted by other firms,

$$\begin{aligned} P^*_{j,s} &= \Upsilon_{j,s}(P_{-j,s}, P_{j,d}, P_{-j,d}), \\ P^*_{j,d} &= \Upsilon_{j,d}(P_{-j,d}, P_{j,s}, P_{-j,s}). \end{aligned}$$

The symmetric Nash equilibrium of the game is the set of prices that solve the following system

$$\begin{cases} P^*_{j,s} = \Upsilon_{j,s}\left(P^*_{-j,s}, P^*_{j,d}, P^*_{-j,d}\right), & \forall j \in \{1,\dots,M\} \\ P^*_{j,d} = \Upsilon_{j,d}\left(P^*_{-j,d}, P^*_{j,s}, P^*_{-j,s}\right), & \forall j \in \{1,\dots,M\} \end{cases}.$$

Theorem 12 Under the assumptions of the model, if all market-makers are risk-neutral, there is no pure symmetric Nash equilibrium of the game.

Proof. Let (P^*_d, P^*_s) be an equilibrium' candidate. We prove that there is always a profitable deviation and therefore (P^*_d, P^*_s) cannot be an equilibrium. The proof examines the case of zero, one, two or all binding constraints.

If no wealth constraint is binding, a profitable deviation is easily identified. If $\left(\sum_{i=1}^{3} p_i S_2^{1i} - S_0\right) > 0\,(<0)$ each market-maker could increase the expected profit by increasing (decreasing) Δ. Therefore, in order to eliminate all the deviations concerning Δ we must analyze the cases with one, two or three binding constraints.

A solution with three binding constraints would not be an equilibrium of the game. In that case, the expected profit would be zero and a profitable deviation would be, for instance, $P_d = \hat{P}_d$, $P_s = \hat{P}_s$ and $\Delta = 0$, with an expected profit of $R^2 y$.

Now let us turn to the more complex case of one binding constraint (say, constraint k). On one hand, if $\left(\sum_{i=1}^{3} p_i S_2^{1i} - S_0\right) > 0$ the constraint that is binding is the one such that $S_2^{1k} - S_0 < 0$, because if that is not the case that would be possible to increase utility increasing Δ. On the other hand, for the same reasons, if $\left(\sum_{i=1}^{3} p_i S_2^{1i} - S_0\right) < 0$ the constraint that is binding is the one such that $S_2^{1k} - S_0 > 0$.

In order to check for all the profitable deviations for market-maker j assume that all other market-makers are playing the hypothetical equilibrium. If market-maker j decides to slightly increase the price that he is charging to the demand $(P_{j,d})$, and if the wealth constraints of his competitors do not allow them to sell more units of the derivative, then the impact in the expected profit is

$$-\frac{\partial Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)}{\partial P_{j,d}}\left[\sum_{i=1}^{3} p^i G_2^{1i} - P_{j,d}\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right). \qquad (19)$$

However, if the market-maker j decides to decrease $P_{j,d}$, he will face all the market demand. Hence, the impact in the expected profit is

$$-d_{Q_{j,A}}\left[\sum_{i=1}^{3} p^i G_2^{1i} - \left(P^*_{j,d} - \varepsilon\right)\right], \qquad (20)$$

where $d_{Q_{j,A}}$ denotes the variation in the quantity sold by firm j. Note that, and ignoring the positivity constraints of wealth at time $T = 2$, in order to be profitable to slightly increase the price, we must assure that equation (19) is positive. Moreover, it would be profitable to slightly decrease the price, increasing the quantity sold, if equation (20) is positive.

Additionally, we must take care of the wealth constraints. As already mentioned, because the expected profit and the wealth constraints are linear in the quantity of stocks bought/sold (Δ), at least one of the wealth constraints is binding. Let this constraint be denoted by k. In an analogous way to the case just described of the expected profit, if the market-maker decides to slightly increase the price that he is charging to the demand ($P_{j,d}$) then the impact in the wealth constraint that is binding is

$$-\frac{\partial Q_{j,A}\left(P_{j,d};P^{*}_{-j,d}\right)}{\partial P_{j,d}}\left[\tilde{G}^{k}_{2} - P_{j,d}\right] + Q_{j,A}\left(P_{j,d};P^{*}_{-j,d}\right). \tag{21}$$

However, if the market-maker j decides to slightly decrease $P_{j,d}$, he will face all the market demand. Hence, the impact in the wealth constraint that is binding is

$$-d_{Q_{j,A}}\left[\tilde{G}^{k}_{2} - \left(P^{*}_{j,d} - \varepsilon\right)\right], \tag{22}$$

where $d_{Q_{j,A}}$ denotes the variation in the quantity sold by firm j, as before.

All the possibilities concerning the sign of equations (19), (20), (21) and (22) are presented in the next table.

	Eq. (19)	Eq. (20)	Eq. (21)	Eq. (22)
Case I	≤ 0	> 0	≤ 0	> 0
Case II	≤ 0	> 0	> 0	> 0
Case III	≤ 0	> 0	> 0	< 0
Case IV	≤ 0	> 0	> 0	$= 0$
Case V	> 0	≥ 0	> 0	≥ 0
Case VI	> 0	≥ 0	> 0	≤ 0
Case VII	> 0	> 0	≤ 0	> 0
Case VIII	> 0	$= 0$	≤ 0	> 0
Case IX	> 0	≤ 0	> 0	≥ 0
Case X	> 0	≤ 0	> 0	≤ 0
Case XI	> 0	< 0	< 0	> 0
Case XII	> 0	< 0	$= 0$	> 0

If cases I and II were considered the market-maker j could slightly decrease the price that he is charging, increasing the quantity that he is selling, which would result in an increase of the expected value of the wealth. In case III the market-maker can find a profitable deviation by changing $P_{j,d}$ and Δ_j. See appendix F. In case IV, if market-maker j decides to increase the price the quantity demand will be zero. The reason is that as equation (20) and equation (22) are nonnegative the other market-makers can increase their expected wealth by selling to the investors that used to buy from market-maker j. Hence, we can not find a profitable deviation changing $P_{j,d}$. In what follows we will find a profitable equilibrium changing $P_{j,s}$. We begin by noticing that, as $\sum_{i=1}^{3} p^i \hat{G}^{i}_{2}$ does not belong to the interval defined by the reservation price, then $\sum_{i=1}^{3} p^i \hat{G}^{i}_{2}$ does not belong to the interval defined $P_{-j,s}$ and $P_{-j,d}$, with $P_{-j,d} > P_{-j,d}$. Hence, the situation described in this case, $\sum_{i=1}^{3} p^i \hat{G}^{i}_{2} < P_{-j,d} = \hat{G}^{k}_{2}$, implies $\sum_{i=1}^{3} p^i \hat{G}^{i}_{2} < P_{-j,s} \leq \hat{G}^{k}_{2}$. Notice that the impact of decreasing the price $P_{j,s}$ in the expected wealth is

$$-\left\{\frac{\partial Q_{j,B}\left(P_{j,s}, P^{*}_{-j,s}\right)}{\partial P_{j,s}}\left[\sum_{i=1}^{3} p^i \hat{G}^{i}_{2} - P_{j,s}\right] - Q_{j,B}\left(P_{j,s}, P^{*}_{-j,s}\right)\Delta\right\} > 0$$

Moreover, the impact on the constraint is

$$-\left\{\frac{\partial Q_{j,B}\left(P_{j,s}, P^{*}_{-j,s}\right)}{\partial P_{j,s}}\left[\hat{G}^{k}_{2} - P_{j,s}\right] - Q_{j,B}\left(P_{j,s}, P^{*}_{-j,s}\right)\Delta\right\} \gtrless 0$$

If the impact on the constraint is positive a profitable deviation for market-maker j is slightly decrease the price $P_{j,s}$. However, if that is not the case it is possible to find a profitable deviation changing $P_{j,s}$ and Δ_j. See appendix F.

Case V is analogous to case IV. In what concerns cases VI, IX and X, the market-maker j must increase the price that he is charging, decreasing the quantity that he is selling and increase the expected value of the wealth. Case VII is equal to case I and II. Case VIII is not an admissible possibility because by equation (19) and (21) we conclude that $\sum_{i=1}^{3} p^i \hat{G}^{i}_{2} > \hat{G}^{k}_{2}$, whether by equations (20) and (22) we conclude $\sum_{i=1}^{3} p^i \hat{G}^{i}_{2} < \hat{G}^{k}_{2}$, that is a contradiction. The remaining cases are presented in the appendix.

Another possibility is that there is two constraints binding. Let them be denoted by m and n. In appendix F, all the possibilities concerning the relation between P_d, P_s, $\hat{G}^{n}_{2}$ and $\hat{G}^{m}_{2}$, are presented in Figure 10. Futhermore, a profitable deviation for each case is identified. ■

Hence, if price competition between market-makers is introduced in our model, there would not exist a pure and symmetric Nash Equilibrium of the game.

4 Conclusion

In this paper we have considered a simple economy where markets are incomplete due to the inexistence of transactions of the underlying at some points in time.

Although our two-period economy may be seen as simple, our main contributions are quite robust since they do not depend on the type of utility functions considered. They may be summarized as follows.

First, we have characterized the investment decisions in the risky assets, when the derivative is fairly priced.

Second, we ¯nd that if the fair price is in the no-arbitrage region, then it is either above the reservation ask price or below the reservation bid price. The implication is that, for a risk-neutral, monopolistic market-maker to transact in both sides of the market, a loss in one side is necessary to justify the gain in the other side.

Third, su±cient conditions for an equilibrium to exist under a risk-neutral, monopolistic market-maker are presented.

Finally and interestingly, the imperfection considered here (dry markets) su±ces to provide conditions assuring the existence of a bid-ask spread under a monopolistic market-maker, although one such equilibrium can be shown not to exist when competition in prices is introduced.

Furthermore, for some speci¯c and standard utility functions we can show several additional results.

First, demand and supply curves have the desired behaviour.

Second, the reservation prices do not depend on the agents' initial wealth level.

Third, the reservation prices may depend on the agents' risk aversion. In the case of a CARA utility function the reservation price does not depend on the absolute risk aversion coe±cient. However, in the case of a CRRA utility function, the reservation price does depend on the relative risk aversion coe±cient.

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A Some Proofs on the Demand

Individual Demand: Second Order Conditions

In this appendix we present the second order conditions of the problem presented in section 2.1 and show that, using the concavity of the utility function, they are always respected. The second order conditions are:

$$\begin{cases} \frac{\partial^2 E[U(.)]}{\partial \phi_0^2} \le 0 \\ \frac{\partial^2 E[U(.)]}{\partial q^2} \le 0 \\ \frac{\partial^2 E[U(.)]}{\partial \phi_0^2}\frac{\partial^2 E[U(.)]}{\partial q^2} - \left[\frac{\partial^2 E[U(.)]}{\partial q \partial \phi_0}\right]^2 > 0 \end{cases},$$

$$\begin{cases} \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)^2 U''\left(w_d^i\right) \le 0 \\ \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_d\right)^2 U''\left(w_d^i\right) \le 0 \\ \left\{\sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)^2 U''\left(w_d^i\right)\right\}\left\{\sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_d\right)^2 U''\left(w_d^i\right)\right\} \\ \quad - \left[\sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)\left(G_2^i - R^2 P_d\right) U''\left(w_d^i\right)\right]^2 > 0 \end{cases}$$

$$\begin{cases} \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)^2 U''\left(w_d^i\right) \le 0 \\ \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_d\right)^2 U''\left(w_d^i\right) \le 0 \\ \sum_{i \neq j} p^i p^j U''\left(w_d^j\right) U''\left(w_d^i\right)\left[\left(S_2^i - R^2 S_0\right)\left(G_2^j - R^2 P_d\right) + \left(G_2^i - R^2 P_d\right)\left(S_2^j - R^2 S_0\right)\right]^2 \ge 0 \end{cases}$$

As, $U''\left(w_d^i\right) \ge 0$ the second order conditions are always satisfied.

Proof of Proposition 2

Let $F(q_d; \alpha; P_d)$ and $G(q_d; \alpha; P_d)$ denote the first order conditions, for a positive q_d; of the problem that must be solved to find the market demand, i.e.,

$$\begin{aligned} F(q_d; \alpha; P_d) &= \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) U'\left(w_d^i\right) \\ G(q_d; \alpha; P_d) &= \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_d\right) U'\left(w_d^i\right) \end{aligned}$$

Using the implicit function theorem we know

$$
\begin{aligned}
\frac{dq_d}{dP_d} &= -\frac{\frac{\partial F}{\partial \delta}\frac{\partial G}{\partial P_d} - \frac{\partial G}{\partial \delta}\frac{\partial F}{\partial P_d}}{\frac{\partial F}{\partial \delta}\frac{\partial G}{\partial q_d} - \frac{\partial F}{\partial q_d}\frac{\partial G}{\partial \delta}} \\
&= -\frac{\begin{array}{c}\left[\sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)^2 U''\left(w_d^i\right)\right] \times \\ \times \left[-R^2 \sum_{i=1}^{3} p^i U'\left(w_d^i\right) - R^2 q_d \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_d\right) U''\left(w_d^i\right)\right] \\ - \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_d\right)\left(S_2^i - R^2 S_0\right) U''\left(w_d^i\right) \times \\ \times R^2 q_d \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) U''\left(w_d^i\right)\end{array}}{\frac{\partial F}{\partial \delta}\frac{\partial G}{\partial q_d} - \frac{\partial F}{\partial q_d}\frac{\partial G}{\partial \delta}} \\
&= \frac{\begin{array}{c} R^2 \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)^2 U''\left(w_d^i\right) \cdot \sum_{i=1}^{3} p^i U'\left(w_d^i\right) \\ + R^2 q \left\{\sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)\left(G_2^i - R^2 P_d\right) U''\left(w_d^i\right) \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) U''\left(w_d^i\right) \right. \\ \left. - \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)^2 U''\left(w_d^i\right) \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_d\right) U''\left(w_d^i\right)\right\}\end{array}}{\frac{\partial F}{\partial \delta}\frac{\partial G}{\partial q_d} - \frac{\partial F}{\partial q_d}\frac{\partial G}{\partial \delta}}
\end{aligned}
$$

where

$$
\frac{\partial F}{\partial \delta}\frac{\partial G}{\partial q_d} - \frac{\partial F}{\partial q_d}\frac{\partial G}{\partial \delta} = \frac{\partial^2 E\left[U(.)\right]}{\partial \delta_0^2}\frac{\partial^2 E\left[U(.)\right]}{\partial q^2} - \left(\frac{\partial^2 E\left[U(.)\right]}{\partial q \partial \delta_0}\right)^2.
$$

As, using the second order conditions, we have

$$
\frac{\partial F}{\partial \delta}\frac{\partial G}{\partial q_d} - \frac{\partial F}{\partial q_d}\frac{\partial G}{\partial \delta} > 0.
$$

the sign of $\frac{dq_d}{dP_d}$ depends on the sign of the numerator.

B Some Proofs on the Supply

Individual Supply: Second Order Conditions

In this appendix we present the second order conditions of the problem presented in section 2.2 and show that, using the concavity of the utility function, they are always respected. The second order conditions are:

$$
\begin{cases}
\frac{\partial^2 E[U(.)]}{\partial \delta_0^2} \leq 0 \\
\frac{\partial^2 E[U(.)]}{\partial q^2} \leq 0 \\
\frac{\partial^2 E[U(.)]}{\partial \delta_0^2}\frac{\partial^2 E[U(.)]}{\partial q^2} - \left[\frac{\partial^2 E[U(.)]}{\partial q \partial \delta_0}\right]^2 > 0
\end{cases}
$$

Using the same procedure as in appendix (A) we can conclude that the second order conditions are always satisfied.

Proof of Proposition 4

Let $F(q_s, \Delta, P_s)$ and $G(q_s, \Delta, P_s)$ denote the first order conditions, for a positive q_s, of the problem that must be solved to find the market demand, i.e.,

$$F(q_s, \Delta, P_s) = \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0 \right) U'\left(w_s^i \right)$$

$$G(q_s, \Delta, P_s) = \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_s \right) U'\left(w_s^i \right)$$

Using the implicit function theorem we know

$$\frac{dq_s}{dP_s} = -\frac{\frac{\partial F}{\partial \Delta}\frac{\partial G}{\partial P_s} - \frac{\partial G}{\partial \Delta}\frac{\partial F}{\partial P_s}}{\frac{\partial F}{\partial \Delta}\frac{\partial G}{\partial q_s} - \frac{\partial F}{\partial q_s}\frac{\partial G}{\partial \Delta}}$$

$$= -\frac{\begin{array}{c} -\sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0 \right)^2 U''\left(w_s^i \right) \times \\ \times \left[-R^2 \sum_{i=1}^{3} p^i U'\left(w_s^i \right) - R^2 q_s \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_s \right) U''\left(w_s^i \right) \right] \\ + \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_s \right)\left(S_2^i - R^2 S_0 \right) \times \\ \times U''\left(w_s^i \right) R^2 q_s \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0 \right) U''\left(w_s^i \right) \end{array}}{SOC_s^{\Delta, q_s}}$$

$$= -\frac{\begin{array}{c} R^2 \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0 \right)^2 U''\left(w_d^i \right) \ast \sum_{i=1}^{3} p^i U'\left(w_d^i \right) \\ +qR^2 \left\{ \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0 \right)\left(G_2^i - R^2 P_d \right) U''\left(w_d^i \right) \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0 \right) U''\left(w_d^i \right) \right. \\ \left. - \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0 \right)^2 U''\left(w_d^i \right) \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_d \right) U''\left(w_d^i \right) \right\} \end{array}}{SOC_s^{\Delta, q_s}}$$

where SOC_s^{Δ, q_s} is the second order condition. As SOC_s^{Δ, q_s} is always positive the sign of $\frac{\partial q}{\partial P}$ depends on the sign of the numerator.

C Arbitrage bounds

The upper bound is given by

$$P^u = \min_{\Delta_0, B_0} \Delta_0 S_0 + B_0$$

subject to

$$\Delta_0 S_2^i + R^2 B_0 \geq G_2^i$$

with $i = 1, 2$ and 3.

In the optimum two of the wealth constraints will be binding. Remember that, by assumption, $S_2^1 > S_2^2 > S_2^3$. Three possibilities must be considered.

1. The constraints binding are the first and the second. In that case the solution would be given by

$$\Delta_0 = \frac{-G_2^1 + G_2^2}{S_2^2 - S_2^1};$$
$$B_0 = -\frac{S_2^1 G_2^2 - G_2^1 S_2^2}{\left(S_2^2 - S_2^1\right) R^2}.$$

The third constraint will be respected if and only if

$$G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) \leq 0.$$

If that is the case the upper bound will be given by

$$P^u = \frac{G_2^1\left(S_2^2 - R^2 S_0\right) - G_2^2\left(S_2^1 - R^2 S_0\right)}{R^2\left(S_2^2 - S_2^1\right)}$$

2. The constraints binding are the first and the third. In that case the solution would be given by

$$\Delta_0 = \frac{-G_2^3 + G_2^1}{S_2^1 - S_2^3};$$
$$B_0 = \frac{-S_2^3 G_2^1 + G_2^3 S_2^1}{\left(S_2^1 - S_2^3\right) R^2}.$$

The second constraint will be respected if and only if

$$G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) \geq 0.$$

If that is the case the upper bound will be given by

$$P^u = \frac{G_2^3\left(S_2^1 - R^2 S_0\right) - G_2^1\left(S_2^3 - R^2 S_0\right)}{R^2\left(S_2^1 - S_2^3\right)}$$

3. The constraints binding are the second and the third. In that case the solution would be given by

$$\Delta_0 = \frac{-G_2^3 + G_2^2}{S_2^2 - S_2^3};$$
$$B_0 = -\frac{S_2^3 G_2^2 - G_2^3 S_2^2}{\left(S_2^2 - S_2^3\right) R^2}.$$

The second constraint will be respected if and only if

$$G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) \leq 0.$$

If that is the case the upper bound will be given by

$$P^u = \frac{G_2^3\left(S_2^2 - R^2 S_0\right) - G_2^2\left(S_2^3 - R^2 S_0\right)}{R^2\left(S_2^2 - S_2^3\right)}$$

Note that if

$$G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) \geq 0$$

the upper bound will be the one described in situation 2. However, if

$$G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) \leq 0$$

there are two possible solutions. The solution described in situation 1 has a higher value than the one described in situation 3 if

$$\frac{G_2^2\left(S_2^1 - R^2S_0\right) - G_2^1\left(S_2^2 - R^2S_0\right)}{R^2\left(S_2^1 - S_2^2\right)} \geq \frac{G_2^3\left(S_2^2 - R^2S_0\right) - G_2^2\left(S_2^3 - R^2S_0\right)}{R^2\left(S_2^2 - S_2^3\right)}$$

$$\Leftrightarrow$$

$$G_2^1\left(S_2^2 - S_2^3\right)\left(S_2^2 - R^2S_0\right) + G_2^2\left(S_2^3 - S_2^1\right)\left(S_2^2 - R^2S_0\right) - G_2^3\left(S_2^2 - S_2^1\right)\left(S_2^2 - R^2S_0\right) \geq 0$$

Hence, using the definition

$$\begin{aligned} G^+ &= \left\{\pi : G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) \geq 0\right\}; \\ G^- &= \left\{\pi : G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right) \leq 0\right\}; \end{aligned}$$

$$H^+ = \left\{\pi : S_2^2 - R^2S_0 \geq 0\right\}$$

and

$$H^- = \left\{\pi : S_2^2 - R^2S_0 \leq 0\right\}$$

we can write the upper bound

$$P^u = \frac{G_2^3\left(S_2^1 - R^2S_0\right) - G_2^1\left(S_2^3 - R^2S_0\right)}{R^2\left(S_2^1 - S_2^3\right)}$$

if $\pi \in G^+$;

$$P^u = \frac{G_2^3\left(S_2^2 - R^2S_0\right) - G_2^2\left(S_2^3 - R^2S_0\right)}{R^2\left(S_2^2 - S_2^3\right)}$$

if $\pi \in G^- \cap H^+$ and

$$P^u = \frac{G_2^1\left(S_2^2 - R^2S_0\right) - G_2^2\left(S_2^1 - R^2S_0\right)}{R^2\left(S_2^2 - S_2^1\right)}$$

if $\pi \in G^{-} \cap H^{-}$:

The lower bound is given by

$$P^{l} = \max_{\Delta_0; B_0} \Delta_0 S_0 + B_0$$

subject to

$$\Delta_0 S_2^i + R^2 B_0 \leq G_2^i$$

with $i = 1, 2$ and 3.

Proceeding in the same way we find out that

$$P^{l} = \frac{G_2^3 \left(S_2^1 - R^2 S_0\right) - G_2^1 \left(S_2^3 - R^2 S_0\right)}{R^2 \left(S_2^1 - S_2^3\right)}$$

if $\pi \in G^{-}$;

$$P^{u} = \frac{G_2^3 \left(S_2^2 - R^2 S_0\right) - G_2^2 \left(S_2^3 - R^2 S_0\right)}{R^2 \left(S_2^2 - S_2^3\right)}$$

if $\pi \in G^{+} \cap H^{+}$ and

$$P^{u} = \frac{G_2^1 \left(S_2^2 - R^2 S_0\right) - G_2^2 \left(S_2^1 - R^2 S_0\right)}{R^2 \left(S_2^2 - S_2^1\right)}$$

if $\pi \in G^{+} \cap H^{-}$.

D Illustrations

D.1 Explicit Solution for the individual demand and supply functions

A solution for the demand and supply of the derivative for a CARA and a CRRA utility functions can be explicitly obtained. The procedure is the following:

Ignoring that the fact that the investor is buying or selling the derivative, the first order conditions can be written as

$$\begin{cases} \sum_{i=1}^{3} \frac{p^i S_2^i}{R^2 S_0} U'\left(w^i\right) = \sum_{i=1}^{3} p^i U'\left(w^i\right) \\ \sum_{i=1}^{3} \frac{p^i G_2^i}{R^2 P_d} U'\left(w^i\right) = \sum_{i=1}^{3} p^i U'\left(w^i\right) \end{cases}$$

If there is a finite solution for the maximization problem then

$$\sum_{i=1}^{3} p^i U'\left(w^i\right) \equiv A > 0$$

where w^i is evaluated at the optimum values of Δ; $q_d = q_s$ and B. Hence, the first order conditions solve the following system

$$\begin{cases} \sum_{i=1}^{3} \frac{p^i S_2^i}{R^2 S_0} U'\left(w^i\right) = A \\ \sum_{i=1}^{3} p^i U'\left(w^i\right) = A \\ \sum_{i=1}^{3} \frac{p^i G_2^i}{R^2 P_d} U'\left(w^i\right) = A \end{cases}$$

This is a linear system in $U'\left(w^1\right)$; $U'\left(w^2\right)$ and $U'\left(w^3\right)$ whose solution is

$$\begin{cases} U'\left(w^1\right) = A\frac{P\left(S_2^2 - S_2^3\right) + G_2^3\left(R^2S_0 - S_2^2\right) - G_2^2\left(R^2S_0 - S_2^3\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)}\frac{1}{p_1} \\ U'\left(w^2\right) = A\frac{P\left(S_2^3 - S_2^1\right) + G_2^1\left(R^2S_0 - S_2^3\right) - G_2^3\left(R^2S_0 - S_2^1\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)}\frac{1}{p_2} \\ U'\left(w^3\right) = A\frac{P\left(S_2^1 - S_2^2\right) - G_2^1\left(R^2S_0 - S_2^2\right) + G_2^2\left(R^2S_0 - S_2^1\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)}\frac{1}{p3}. \end{cases}$$

Let

$$\begin{aligned} \mu_1 &= \frac{P\left(S_2^2 - S_2^3\right) + G_2^3\left(R^2S_0 - S_2^2\right) - G_2^2\left(R^2S_0 - S_2^3\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)}\frac{1}{p_1} \\ \mu_2 &= \frac{P\left(S_2^3 - S_2^1\right) + G_2^1\left(R^2S_0 - S_2^3\right) - G_2^3\left(R^2S_0 - S_2^1\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)}\frac{1}{p_2} \\ \mu_3 &= \frac{P\left(S_2^1 - S_2^2\right) - G_2^1\left(R^2S_0 - S_2^2\right) + G_2^2\left(R^2S_0 - S_2^1\right)}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)}\frac{1}{p3} \end{aligned}$$

and denote by $[U'(.)]^{-1}$ be the inverse function of the marginal utility. Hence

$$U'\left(w^i\right) = A\mu_i \Rightarrow w^i = \left[U'(A\mu_i)\right]^{-1}.$$

Moreover, as $w^i = \Delta_0\left(S_2^i - R^2S_0\right) + R^2y + q\left(G_2^i - R^2P\right)$ the following system is obtained

$$\begin{aligned} \Delta_0\left(S_2^1 - R^2S_0\right) + q\left(G_2^1 - R^2P\right) - [U'(A\mu_1)]^{-1} &= -R^2y \\ \Delta_0\left(S_2^2 - R^2S_0\right) + q\left(G_2^2 - R^2P\right) - [U'(A\mu_2)]^{-1} &= -R^2y \\ \Delta_0\left(S_2^3 - R^2S_0\right) + q\left(G_2^3 - R^2P\right) - [U'(A\mu_3)]^{-1} &= -R^2y \end{aligned} \tag{23}$$

The system presented above is a system of three equations and three variables $(q; \Delta_0; A)$, with a unique solution for the CARA and CRRA utility functions. First, we are going to consider the CARA utility functions and then the CRRA utility functions.

If a CARA utility is considered then $U'\left(w^i\right) = -\delta e^{-\delta w^i}$. Therefore,

$$U'\left(w^i\right) = A\mu_i \Rightarrow w^i = \left[U'(A\mu_i)\right]^{-1} = -\frac{1}{\delta}\ln\left(-\frac{A\mu_i}{\delta}\right)$$

Hence, system (23) can be written as

$$
\begin{aligned}
\Delta_0\left(S_2^1 - R^2 S_0\right) + q\left(G_2^1 - R^2 P\right) + \frac{1}{\delta}\ln\left(-\frac{A\mu_1}{\delta}\right) &= -R^2 y\\
\Delta_0\left(S_2^2 - R^2 S_0\right) + q\left(G_2^2 - R^2 P\right) + \frac{1}{\delta}\ln\left(-\frac{A\mu_2}{\delta}\right) &= -R^2 y\\
\Delta_0\left(S_2^3 - R^2 S_0\right) + q\left(G_2^3 - R^2 P\right) + \frac{1}{\delta}\ln\left(-\frac{A\mu_3}{\delta}\right) &= -R^2 y
\end{aligned}
$$

Solving for q we obtain the individual demand/supply for the derivative, i.e.,

$$
q = -\frac{1}{\delta}\frac{\left(S_2^2 - S_2^3\right)\ln\mu_1 + \left(S_2^3 - S_2^1\right)\ln\mu_2 + \left(S_2^1 - S_2^2\right)\ln\mu_3}{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) + G_2^3\left(S_2^1 - S_2^2\right)}
$$

If the CRRA utility is considered then $U'\left(w^i\right) = \left(w^i\right)^{-\gamma}$. Therefore,

$$
U'\left(w^i\right) = A\mu_i \Rightarrow w^i = \left[U'\left(A\mu_i\right)\right]^{-1} = \left(A\mu_i\right)^{-\frac{1}{\gamma}}
$$

Hence, system (23) becomes

$$
\begin{aligned}
\Delta_0\left(S_2^1 - R^2 S_0\right) + q\left(G_2^1 - R^2 P\right) - \left(A\mu_1\right)^{-\frac{1}{\gamma}} &= -R^2 y\\
\Delta_0\left(S_2^2 - R^2 S_0\right) + q\left(G_2^2 - R^2 P\right) - \left(A\mu_2\right)^{-\frac{1}{\gamma}} &= -R^2 y\\
\Delta_0\left(S_2^3 - R^2 S_0\right) + q\left(G_2^3 - R^2 P\right) - \left(A\mu_3\right)^{-\frac{1}{\gamma}} &= -R^2 y
\end{aligned}
$$

Solving for q

$$
q = \frac{\mu_1^{-\frac{1}{\gamma}}\left(S_2^3 - S_2^2\right) + \mu_2^{-\frac{1}{\gamma}}\left(S_2^1 - S_2^3\right) + \mu_3^{-\frac{1}{\gamma}}\left(S_2^1 - S_2^3\right)}{\begin{array}{l}\mu_1^{-\frac{1}{\gamma}}\left[-\left(S_2^2 - R^2 S_0\right)\left(G_2^3 - R^2 P\right) + \left(S_2^3 - R^2 S_0\right)\left(G_2^2 - R^2 P\right)\right]\\ + \mu_2^{-\frac{1}{\gamma}}\left[-\left(S_2^1 - R^2 S_0\right)\left(G_2^3 - R^2 P\right) + \left(S_2^3 - R^2 S_0\right)\left(G_2^1 - R^2 P\right)\right]\\ + \mu_3^{-\frac{1}{\gamma}}\left[-\left(S_2^2 - R^2 S_0\right)\left(G_2^1 - R^2 P\right) + \left(S_2^1 - R^2 S_0\right)\left(G_2^2 - R^2 P\right)\right]\end{array}}
$$

D.2 Properties of the individual demand and supply

D.2.1 CARA utlity functions

D.2.2 Property 1

For a CARA utility function the first order conditions are given by:

$$
\begin{cases}
\sum_{i=1}^{3} p^i\left(S_2^i - R^2 S_0\right) e^{-\delta\left[\Delta_{0;d} S_2^i + q_d G_2^i\right]} = 0\\
\sum_{i=1}^{3} p^i\left(G_2^i - R^2 P_d\right) e^{-\delta\left[\Delta_{0;d} S_2^i + q_d G_2^i\right]} \leq 0
\end{cases}
$$

In order to prove that $\frac{\partial q_d}{\partial P_d} < 0$ notice that, as $U''\left(w_d^i\right) = -\delta U'\left(w_d^i\right)$, we have

$$\sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) U''\left(w_d^i\right) = \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_d\right) U''\left(w_d^i\right) = 0.$$

Using proposition 2

$$\frac{\partial q_d}{\partial P_d} = \frac{R^2 \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)^2 U''\left(w_d^i\right) * \sum_{i=1}^{3} p^i U'\left(w_d^i\right)}{\frac{\partial F}{\partial \Delta}\frac{\partial G}{\partial q_d} - \frac{\partial F}{\partial q_d}\frac{\partial G}{\partial \Delta}} < 0$$

D.2.3 Property 2

For a CARA utility function the first order conditions presented in equations (6) can be written as

$$\begin{cases} \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) \exp\left\{-\delta\left[\Delta^*_{0,s}\left(S_2 - R^2 S_0\right) + R^2 y - Q\left(G_2^i - R^2 P_s\right)\right]\right\} = 0 \\ \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_s\right) \exp\left\{-\delta\left[\Delta^*_{0,s}\left(S_2 - R^2 S_0\right) + R^2 y - Q\left(G_2^i - R^2 P_s\right)\right]\right\} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) \exp\left\{-\delta\left[\Delta^*_{0,s} S_2^i - Q G_2^i\right]\right\} = 0 \\ \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_s\right) \exp\left\{-\delta\left[\Delta^*_{0,s} S_2^i - Q G_2^i\right]\right\} = 0 \end{cases}$$

Solving for the second equation for P_s

$$P_s = \frac{\sum_{i=1}^{3} p^i G_2^i \exp\left\{-\delta\left[\Delta^*_{0,s} S_2^i - Q G_2^i\right]\right\}}{R^2 \sum_{i=1}^{3} p^i \exp\left\{-\delta\left[\Delta^*_{0,s} S_2^i - Q G_2^i\right]\right\}}$$

Notice that $U''\left(w_s^i\right) = -\delta U'\left(w_s^i\right)$, we have $\sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right) U''\left(w_s^i\right) = \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_s\right) U''\left(w_s^i\right) = 0.$ Using proposition 4

$$\frac{\partial q_s}{\partial P_s} = -\frac{R^2 \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)^2 U''\left(w_s^i\right) * \sum_{i=1}^{3} p^i U'\left(w_s^i\right)}{F_{\Delta,Q}} > 0$$

D.2.4 Property 3

In order to prove that $\frac{d\phi_d(P_d)}{d\delta} \leq 0$ notice that the first and second conditions of the optimization problem can be written in terms of $\Lambda = \delta\Delta$ and $\Gamma = \delta q$; eliminating δ and q from the first and second order conditions. Hence, we can only find the optimal values of Λ and Γ (Λ^* and Γ^*). The optimal values of the number of shares bought/sold and the number of options bought/sold is given by

$$\Delta = \frac{\Lambda^*}{\delta}$$

$$q = \frac{\Gamma^*}{\delta}.$$

D.2.5 Property 4

This property follows from the fact that δq is constant.

D.2.6 Property 5

Moreover, note that the first and second conditions are independent of y. Hence, the optimal values will also be independent of y.

D.2.7 CRRA utility functions

For a CRRA utility function the first order conditions presented in equations (3) can be written as

$$\begin{cases} \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)\left[{}^{*}_{0,d}\left(S_2 - R^2 S_0\right) + R^2 y + Q\left(G_2^i - R^2 P_d\right)\right]^{-\gamma} = 0 \\ \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_d\right)\left[{}^{*}_{0,d}\left(S_2 - R^2 S_0\right) + R^2 y + Q\left(G_2^i - R^2 P_d\right)\right]^{-\gamma} = 0 \end{cases}$$

For a CRRA utility function the first order conditions presented in equations (6) can be written as

$$\begin{cases} \sum_{i=1}^{3} p^i \left(S_2^i - R^2 S_0\right)\left[{}^{*}_{0,s}\left(S_2 - R^2 S_0\right) + R^2 y - Q\left(G_2^i - R^2 P_s\right)\right]^{-\gamma} = 0 \\ \sum_{i=1}^{3} p^i \left(G_2^i - R^2 P_s\right)\left[{}^{*}_{0,s}\left(S_2 - R^2 S_0\right) + R^2 y - Q\left(G_2^i - R^2 P_s\right)\right]^{-\gamma} = 0 \end{cases}$$

D.2.8 Property 1 and 2

See Section 2.4.

D.2.9 Property 3

The property is straightforward using the optimal quantity defined in equation (11).

D.2.10 Property 4

Let the function $\chi(P;\gamma)$ be defined as[17]

$$\chi(P) = \mu_1^{-\frac{1}{\gamma}}\left(S_2^3 - S_2^2\right) + \mu_2^{-\frac{1}{\gamma}}\left(S_2^1 - S_2^3\right) + \mu_3^{-\frac{1}{\gamma}}\left(S_2^2 - S_2^1\right)$$

[17] Note that each μ_i is a function of P.

Using the optimal quantity, defined in equation (11), the reservation price is the price $\hat{P}$ such that

$$\psi\left(\hat{P};\gamma\right) = 0. \qquad (24)$$

Using the implicit function theorem we have

$$\begin{aligned}
\frac{d\hat{P}}{d\gamma} &= -\frac{\frac{\partial \psi(\hat{P};\gamma)}{\partial \gamma}}{\frac{\psi(\hat{P};\gamma)}{\partial \hat{P}}} \\
&= -\frac{G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)}{\gamma} \\
&\quad \frac{\ln(\mu_1)\,\mu_1^{-\frac{1}{\gamma}}\left(S_2^3 - S_2^2\right) + \ln(\mu_2)\,\mu_2^{-\frac{1}{\gamma}}\left(S_2^1 - S_2^3\right) + \ln(\mu_3)\,\mu_3^{-\frac{1}{\gamma}}\left(S_2^2 - S_2^1\right)}{\mu_1^{-\frac{1}{\gamma}-1}\left(S_2^3 - S_2^2\right)^2 + \mu_2^{-\frac{1}{\gamma}-1}\left(S_2^1 - S_2^3\right)^2 + \mu_3^{-\frac{1}{\gamma}-1}\left(S_2^2 - S_2^1\right)^2}.
\end{aligned}$$

As the denominator of the second fraction is always positive we have to check the sign of the numerator in order to define the sign of $\frac{d\hat{P}}{d\gamma}$.

Using equation (24) we can write $\mu_2^{-\frac{1}{\gamma}}$ as an weighted average of $\mu_1^{-\frac{1}{\gamma}}$ and $\mu_3^{-\frac{1}{\gamma}}$, i.e.,

$$\mu_2^{-\frac{1}{\gamma}} = \frac{S_2^2 - S_2^3}{S_2^1 - S_2^3}\mu_1^{-\frac{1}{\gamma}} + \frac{S_2^3 - S_2^1}{S_2^1 - S_2^3}\mu_3^{-\frac{1}{\gamma}}.$$

Morover, as $p_1\mu_1 + p_2\mu_2 + p_3\mu_3 = 1$; we must have one of the following situations:

$$\begin{aligned}
\mu_1^{-\frac{1}{\gamma}} &> \mu_2^{-\frac{1}{\gamma}} > \mu_3^{-\frac{1}{\gamma}} \quad , \quad \mu_1 < \mu_2 < \mu_3 \quad , \quad \mu_1 < 1; \mu_3 > 1 \quad \text{(case A)} \\
\mu_1^{-\frac{1}{\gamma}} &< \mu_2^{-\frac{1}{\gamma}} < \mu_3^{-\frac{1}{\gamma}} \quad , \quad \mu_1 > \mu_2 > \mu_3 \quad , \quad \mu_1 > 1; \mu_3 < 1. \quad \text{(case B)}
\end{aligned}$$

Additionally, as $\ln(.)$ is a concave function we have

$$\ln(\mu_2) \geq \ln(\mu_1)\frac{S_2^2 - S_2^3}{S_2^1 - S_2^3} + \ln(\mu_3)\frac{S_2^2 - S_2^1}{S_2^1 - S_2^3}$$

Hence,

$$\begin{aligned}
& \ln(\mu_1)\,\mu_1^{-\frac{1}{\gamma}}\left(S_2^3 - S_2^2\right) + \ln(\mu_2)\,\mu_2^{-\frac{1}{\gamma}}\left(S_2^1 - S_2^3\right) + \ln(\mu_3)\,\mu_3^{-\frac{1}{\gamma}}\left(S_2^2 - S_2^1\right) \\
\geq\; & \ln(\mu_1)\,\mu_1^{-\frac{1}{\gamma}}\left(S_2^3 - S_2^2\right) + \left[\ln(\mu_1)\frac{S_2^2 - S_2^3}{S_2^1 - S_2^3} + \ln(\mu_3)\frac{S_2^1 - S_2^2}{S_2^1 - S_2^3}\right]\mu_2^{-\frac{1}{\gamma}} \\
& + \ln(\mu_3)\,\mu_3^{-\frac{1}{\gamma}}\left(S_2^2 - S_2^1\right) \\
=\; & \ln(\mu_1)\left(S_2^2 - S_2^3\right)\left[\mu_2^{-\frac{1}{\gamma}} - \mu_1^{-\frac{1}{\gamma}}\right] + \ln(\mu_3)\left(S_2^1 - S_2^2\right)\left[\mu_2^{-\frac{1}{\gamma}} - \mu_3^{-\frac{1}{\gamma}}\right].
\end{aligned}$$

If case A is considered

$$\underbrace{\ln(\mu_1)}_{<0}\underbrace{\left(S_2^2 - S_2^3\right)}_{>0}\underbrace{\left[\mu_2^{-\frac{1}{\gamma}} - \mu_1^{-\frac{1}{\gamma}}\right]}_{<0} + \underbrace{\ln(\mu_3)}_{>0}\underbrace{\left(S_2^1 - S_2^2\right)}_{>0}\underbrace{\left[\mu_2^{-\frac{1}{\gamma}} - \mu_3^{-\frac{1}{\gamma}}\right]}_{>0} > 0;$$

If case B is considered

$$\underbrace{\ln(\mu_1)}_{>0}\underbrace{\left(S_2^2 - S_2^3\right)}_{>0}\underbrace{\left[\mu_2^{-\frac{1}{\gamma}} - \mu_1^{-\frac{1}{\gamma}}\right]}_{>0} + \underbrace{\ln(\mu_3)}_{<0}\underbrace{\left(S_2^1 - S_2^2\right)}_{>0}\underbrace{\left[\mu_2^{-\frac{1}{\gamma}} - \mu_3^{-\frac{1}{\gamma}}\right]}_{<0} > 0.$$

resulting in

$$\ln(\mu_1)\,\mu_1^{-\frac{1}{\gamma}}\left(S_2^3 - S_2^2\right) + \ln(\mu_2)\,\mu_2^{-\frac{1}{\gamma}}\left(S_2^1 - S_2^3\right) + \ln(\mu_3)\,\mu_3^{-\frac{1}{\gamma}}\left(S_2^2 - S_2^1\right) \geq 0.$$

It follows that

$$\text{sign}\left(\frac{d\hat{P}}{d\gamma}\right) = -\text{sign}\left[G_2^1\left(S_2^2 - S_2^3\right) + G_2^2\left(S_2^3 - S_2^1\right) - G_2^3\left(S_2^2 - S_2^1\right)\right].$$

E Monopolistic Market Maker

Conditions on Assumption 2

A function $f : A \to \mathbb{R}$ is concave if and only if for every $x \in A$; the Hessian matrix $D^2 f(x)$ is negative semidefinite. For the function considered, the Hessian matrix is

$$\begin{bmatrix} -\frac{d^2[Q_B P_s(Q_B)]}{dQ_B^2} & 0 & 0 \\ 0 & \frac{d^2[Q_A P_d(Q_A)]}{dQ_A^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, this matrix negative semidefinite if and only if $Q_B P_s(Q_B)$ is a convex function in Q_B and $Q_A P_d(Q_A)$ is a concave function in Q_A; i.e.,

$$\frac{d^2[Q_B P_s(Q_B)]}{dQ_B^2} = 2\frac{dP_s(Q_B)}{dQ_B} + \frac{\partial^2 P_s(Q_B)}{\partial Q_B^2} \geq 0$$

$$\frac{d^2[Q_A P_d(Q_A)]}{dQ_A^2} = 2\frac{dP_d(Q_A)}{dQ_A} + \frac{\partial^2 P_d(Q_A)}{\partial Q_A^2} \leq 0.$$

Proof of theorem 9

E.0.11 Convexity of the constraint set

In order to check for the convexity of the constraint set consider two possible elements of the constraint set. Let them be $\left(Q_B^1; Q_A^1; \Delta^1\right)$ and $\left(Q_B^2; Q_A^2; \Delta^2\right)$. Then, for each constraint $i = 1; 2$ and 3 we have

$$-Q_B^1\left(G_2^i - R^2 P_s\left(Q_B^1\right)\right) + Q_A^1\left(G_2^i - R^2 P_d\left(Q_A^1\right)\right) - \Delta^1\left(S_2^i - R^2 S_0\right) - R^2 y \le 0 \tag{25}$$

and

$$-Q_B^2\left(G_2^i - R^2 P_s\left(Q_B^2\right)\right) + Q_A^2\left(G_2^i - R^2 P_d\left(Q_A^2\right)\right) - \Delta^2\left(S_2^i - R^2 S_0\right) - R^2 y \le 0 \tag{26}$$

If the element

$$\left(Q_B^3; Q_A^3; \Delta^3\right) = \left(\lambda Q_B^1 + (1-\lambda) Q_B^2; \lambda Q_A^1 + (1-\lambda) Q_A^2; \lambda \Delta^1 + (1-\lambda)\Delta^2\right)$$

respects the three constraints then the constraint set is convex. In order to check for that characteristic of the constraint set, multiply equation (25) by λ and equation (26) by $1-\lambda$. The following equation is obtained

$$\begin{aligned} &-Q_B^3 G_2^i + Q_A^3 G_2^i - \Delta^3\left(S_2^i - R^2 S_0\right) - R^2 y + \\ &-R^2\left(\lambda Q_A^1 P_d\left(Q_A^1\right) + (1-\lambda) Q_A^2 P_d\left(Q_A^2\right)\right) + \\ &+R^2\left(\lambda Q_B^1 P_s\left(Q_B^1\right) + (1-\lambda) Q_B^2 P_s\left(Q_B^2\right)\right) \le 0 \end{aligned}$$

As $Q_A P_d(Q_A)$ is a concave function

$$\begin{aligned} R^2\left(\lambda Q_A^1 P_d\left(Q_A^1\right) + (1-\lambda) Q_A^1 P_d\left(Q_A^1\right)\right) &\le R^2 Q_A^3 P_d\left(Q_A^3\right) \\ -R^2\left(\lambda Q_A^1 P_d\left(Q_A^1\right) + (1-\lambda) Q_A^1 P_d\left(Q_A^1\right)\right) &\ge -R^2 Q_A^3 P_d\left(Q_A^3\right). \end{aligned}$$

Moreover, as $Q_B P_s(Q_B)$ is a convex function

$$R^2\left[\lambda Q_B^1 P_s\left(Q_B^1\right) + (1-\lambda) Q_B^2 P_s\left(Q_B^2\right)\right] \ge R^2 Q_B^3 P_s\left(Q_B^3\right)$$

we have

$$-Q_B^3\left[G_2^i - R^2 P_s\left(Q_B^3\right)\right] + Q_A^3\left[G_2^i - R^2 P_d\left(Q_A^2\right)\right] - \Delta^3\left(S_2^i - R^2 S_0\right) - R^2 y \le 0.$$

E.0.12 Existence of the Bid-Ask Spread

In what follows the proof of the second part of theorem 9, that concerns the existence of a bid-ask spread if there is an equilibrium with strictly positive quantities $\{Q_B^*; Q_A^*\}$, is presented.

First, consider the optimal condition expressed in equality (15) is considered. Then, the optimal condition expressed in equality (16) is also considered.

In the case pesented in equations (15) the proof is done by contradiction. Suppose that $P_d(Q_A^*) \leq P_s(Q_B^*)$. Consider equation (16). Three cases must be considered concerning the relation between $P_d(Q_A^*), Q_A^* \frac{dP_d(Q_A^*)}{dQ_A}$, $P_s(Q_B^*), Q_B^* \frac{dP_s(Q_B^*)}{dQ_B}$ and $\hat{G}_2^k$. First, consider that $\hat{G}_2^k > P_s(Q_B^*) + Q_B^* \frac{dP_s(Q_B^*)}{dQ_B}$. Using equation (15), we obtain

$$\hat{G}_2^k - P_d(Q_B^*) - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} \geq \hat{G}_2^k - P_s(Q_B^*) - Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} > 0,$$

$$\sum_{i=1} p_i \hat{G}_2^i - P_d(Q_A^*) - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} \leq \sum_{i=1} p_i \hat{G}_2^i - P_s(Q_B^*) - Q_B^* \frac{dP_s(Q_B^*)}{dQ_B},$$

$$P_s(Q_B^*) - P_d(Q_A^*) \leq Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} - Q_B^* \frac{dP_s(Q_B^*)}{dQ_B}$$

As, by assumption 1,

$$Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} - Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} < 0$$

then,

$$P_s(Q_B^*) - P_d(Q_A^*) < 0;$$

contradicting $P_d(Q_A^*) \leq P_s(Q_B^*)$.

In the second case, consider that $P_d(Q_A^*) < P_s(Q_B^*)$ and $P_d(Q_A^*) + Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} < \hat{G}_2^k < P_s(Q_B^*) + Q_B^* \frac{dP_s(Q_B^*)}{dQ_B}$. From equation (15) we obtain

$$\hat{G}_2^k - P_s(Q_B^*) Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} \leq \hat{G}_2^k - P_d(Q_A^*) - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A}, \quad (27)$$

$$\sum_{i=1} p_i \hat{G}_2^i - P_s(Q_B^*) - Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} \geq \sum_{i=1} p_i \hat{G}_2^i - P_d(Q_A^*) - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A},$$

$$P_d(Q_A^*) - P_s(Q_B^*) \geq Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A}$$

As, by assumption 1,

$$Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} > 0$$

then,

$$P_d(Q_A^*) - P_s(Q_B^*) > 0;$$

contradicting $P_d(Q_A^*) \leq P_s(Q_B^*)$.

At last, consider that $\hat{G}_2^k < P_d(Q_A^*) + Q_A^* \frac{dP_d(Q_A^*)}{dQ_A}$. In this case, from equation(15) we obtain the same relation that is displayed in equation (27), for the case $P_d(Q_A^*) < G_2^k < P_s(Q_B^*)$. Therefore, by contraction we prove that for the tangency solution $P_d(Q_A^*) > P_s(Q_B^*)$.

Now, consider the optimal conditions expressed in equations (16). Subtracting the second equation from the first one we have

$$(1 - \lambda_k - \lambda_j)\left[\pi(Q_A^*) - \pi(Q_B^*)\right] = 0.$$

As $1 - \lambda_k - \lambda_j > 0$, we must assure $\pi(Q_A^*) - \pi(Q_B^*) = 0$. Then, using assumption 1, and noting that $\pi(Q_A^*) - \pi(Q_B^*) = 0$ is equivalent to

$$P_d(Q_A^*) - P_s(Q_B^*) = -Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} + Q_B^* \frac{dP_s(Q_B^*)}{dQ_B}$$

we have $P_d(Q_A^*) - P_s(Q_B^*) > 0$.

F Competition Between Market-Makers

Proof of theorem (12)

Proof. Case III

In this case we must consider two possible situations:

Situation 1:

$$\left| \frac{-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left[\sum_{i=1}^3 p^i G_2^i - P_{j;d}\right] + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left(G_2^k - P_{j;d}\right) + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)} \right| < \left| \frac{\sum_{i=1}^3 p^i S_2^i - S_0}{S_2^k - S_0} \right|$$

Deviation: Increase $P_{j;d}$ and decrease Δ_0 if $S_2^k - S_0 > 0$ or increase Δ_0 if $S_2^k - S_0 < 0$

Situation 2:

$$\left| \frac{-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left[\sum_{i=1}^3 p^i G_2^i - P_{j;d}\right] + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left(G_2^k - P_{j;d}\right) + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)} \right| \geq \left| \frac{\sum_{i=1}^3 p^i S_2^i - S_0}{S_2^k - S_0} \right|$$

As

$$\left| \frac{\sum_{i=1}^3 p^i G_2^i - P_{j;d}}{G_2^k - (P_{j;d} - \varepsilon)} \right| > \left| \frac{-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left[\sum_{i=1}^3 p^i G_2^i - P_{j;d}\right] + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left(G_2^k - P_{j;d}\right) + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)} \right|$$

we have,

$$\left| \frac{\sum_{i=1}^3 p^i G_2^i - P_{j;d}}{G_2^k - P_{j;d}} \right| > \left| \frac{\sum_{i=1}^3 p^i S_2^i - S_0}{S_2^k - S_0} \right|$$

Deviation: Decrease $P_{j;d}$ $\rightarrow$ increase $Q^*_{j;A}$ and increase Δ_0 if $\bar S^k_2 - S_0 > 0$ or decrease Δ_0 if $\bar S^k_2 - S_0 < 0$

Case IV

Notice that the impact of slightly decrease the price $P_{j;s}$ in the expected wealth is

$$-\left\{\frac{\partial Q_{j;B}\left(P_{j;s};P^*_{-j;s}\right)}{\partial P_{j;s}}\left[\sum_{i=1}^{3} p^i\left(\bar G^i_2 - P_{j;s}\right) - Q_{j;B}\left(P_{j;s};P^*_{-j;s}\right)\right]\right\} > 0$$

Moreover, the impact on the constraint is

$$-\left\{\frac{\partial Q_{j;B}\left(P_{j;s};P^*_{-j;s}\right)}{\partial P_{j;s}}\left[\left(\bar G^k_2 - P_{j;s}\right) - Q_{j;B}\left(P_{j;s};P^*_{-j;s}\right)\right]\right\} < 0$$

Moreover,

$$\sum_{i=1}^{3} p^i\bar G^i_2 - P_{j;s} \le 0 \text{ and } \bar G^k_2 - P_{j;s} \ge 0$$

Proceeding in an analogous way as in case III we can find a profitable deviation changing $P_{j;s}$ and Δ_j.

Case XI

In this case two possibilities must be considered

Situation 1:

$$\left|\frac{-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left[\sum_{i=1}^{3} p^i\bar G^i_2 - P_{j;d}\right] + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left[\bar G^k_2 - P_{j;d}\right] + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}\right| > \left|\frac{\sum_{i=1}^{3} p^i\bar S^i_2 - S_0}{\bar S^k_2 - S_0}\right|$$

Deviation: Increase $P^*_{j;d}$ and increase Δ_0 if $\bar S^k_2 - S_0 > 0$ or decrease Δ_0 if $\bar S^k_2 - S_0 < 0$

Situation 2:

$$\left|\frac{-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left[\sum_{i=1}^{3} p^i\bar G^i_2 - P_{j;d}\right] + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left[\bar G^k_2 - P_{j;d}\right] + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}\right| \le \left|\frac{\sum_{i=1}^{3} p^i\bar S^i_2 - S_0}{\bar S^k_2 - S_0}\right|$$

Note that

$$-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left[\bar G^k_2 - P_{j;d}\right] + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right) < 0 \Rightarrow \bar G^k_2 - P_{j;d} < 0.$$

As,

$$\sum_{i=1}^{3} p^i \acute{G}^i_2 - P^*_{j;d} > 0$$

In this case we can check that

$$\left|\frac{-dQ_{j;A}\left[\sum_{i=1}^{3} p^i \acute{G}^i_2 - P_{j;d}\right]}{-dQ_{j;A}\left[\acute{G}^k_2 - P_{j;d}\right]}\right| < \left|\frac{-\frac{\partial Q_{j;A}(P_{j;d};P^*_{-j;d})}{\partial P_{j;d}}\left[\sum_{i=1}^{3} p^i \acute{G}^i_2 - P_{j;d}\right] + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{-\frac{\partial Q_{j;A}(P_{j;d};P^*_{-j;d})}{\partial P_{j;d}}\left[\acute{G}^k_2 - P_{j;d}\right] + Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}\right|$$

hence, as

$$\left|\frac{-dQ_{j;A}\left[\sum_{i=1}^{3} p^i \acute{G}^i_2 - P_{j;d}\right]}{-dQ_{j;A}\left[\acute{G}^k_2 - P_{j;d}\right]}\right| < \left|\frac{\sum_{i=1}^{3} p^i \acute{S}^i_2 - S_0}{\acute{S}^k_2 - S_0}\right|$$

we have the following deviation

Deviation: Decrease $P_{j;d} \to$ increase $Q^*_{j;A}$ and decrease Δ_0 if $\acute{S}^k_2 - S_0 > 0$ or increase Δ_0 if $\acute{S}^k_2 - S_0 < 0$:

Case XII

In this case note that as

$$-\frac{\partial Q_{j;A}\left(P_{j;d};P^*_{-j;d}\right)}{\partial P_{j;d}}\left[\acute{G}^k_2 - (P_{j;d} - \varepsilon)\right] + Q_{j;A}\left[P_{j;d};P^*_{-j;d}\right] < 0$$

case XI applies.

Now, consider the case when two constraints are binding. Let them be constraint m and n: All the possibilities concerning the relation between P_d; P_s, $\acute{G}^n_2$ and $\acute{G}^m_2$; are presented in Figure 10.

If an agent decides to increase P_s or decrease P_d, the positive alteration in quantities must be such that

$$dQ_{j;B}\left[\acute{G}^m_2 - (P_{j;s} + \varepsilon)\right] - dQ_{j;A}\left[\acute{G}^m_2 - (P_{j;d} - \varepsilon)\right] \geq 0$$
$$dQ_{j;B}\left[\acute{G}^n_2 - (P_{j;s} + \varepsilon)\right] - dQ_{j;A}\left[\acute{G}^n_2 - (P_{j;d} - \varepsilon)\right] \geq 0$$

The alteration in the utility is

$$dQ_{j;B}\left(\sum_{i=1}^{3} p^i \acute{G}^i_2 - (P_{j;s} + \varepsilon)\right) - dQ_{j;A}\left(\sum_{i=1}^{3} p^i \acute{G}^i_2 - (P_{j;d} - \varepsilon)\right) > 0$$

and is positive or equal to zero if

$$\begin{cases} \frac{dQ_{j;B}}{dQ_{j;A}} > \frac{\sum_{i=1}^{3} p^i \acute{G}^i_2 - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i \acute{G}^i_2 - (P_{j;s} + \varepsilon)} & ; \quad P_{j;d} \leq \sum_{i=1}^{3} p^i \acute{G}^i_2 \\ \forall Q_{j;B}; \forall Q_{j;A} & ; \quad P_{j;s} < \sum_{i=1}^{3} p^i \acute{G}^i_2 < P_{j;d} \\ \frac{dQ_{j;B}}{dQ_{j;A}} < \frac{\sum_{i=1}^{3} p^i \acute{G}^i_2 - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i \acute{G}^i_2 - (P_{j;s} + \varepsilon)} & ; \quad \sum_{i=1}^{3} p^i \acute{G}^i_2 < P_{j;s} \end{cases}$$

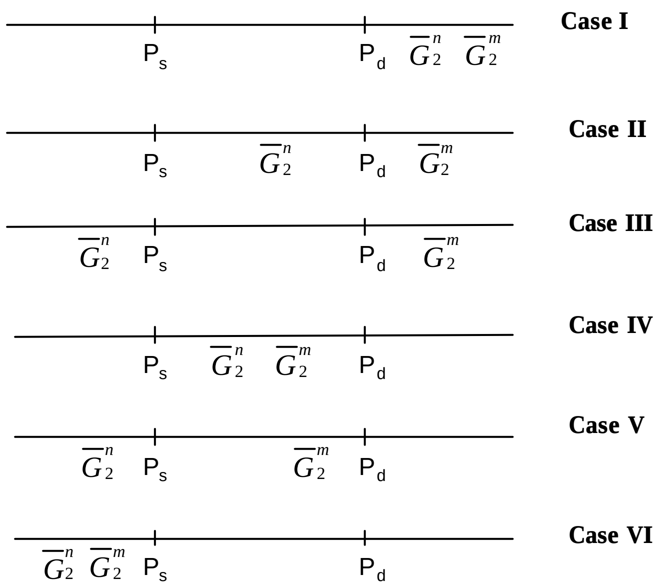

Figure 10: The possibilities concerning the relation between P_d; P_s, $\bar{G}_2^n$ and $\bar{G}_2^m$.

The constraints will be respected if

$$dQ_{j;B}\left(\bar{G}_2^k - (P_{j;s}+\varepsilon)\right) - dQ_{j;A}\left(\bar{G}_2^k - (P_{j;d}-\varepsilon)\right) > 0$$

$$\begin{cases} \frac{dQ_{j;B}}{dQ_{j;A}} > \frac{\bar{G}_2^k - (P_{j;d}-\varepsilon)}{\bar{G}_2^k - (P_{j;s}+\varepsilon)} & P_{j;d} \leq \bar{G}_2^k \\ \forall Q_{j;B}, \forall Q_{j;A} & ; \quad P_{j;s} < \bar{G}_2^k < P_{j;d} \\ \frac{dQ_{j;B}}{dQ_{j;A}} < \frac{\bar{G}_2^k - (P_{j;d}-\varepsilon)}{\bar{G}_2^k - (P_{j;s}+\varepsilon)} & \bar{G}_2^k < P_{j;s} \end{cases}$$

Now, for each case presented in Figure 10 a profitable deviation will be presented.

Case I

In order to have the wealth constraints respected we must have

$$\frac{dQ_{j;B}}{dQ_{j;A}} > \max\left\{\frac{\bar{G}_2^n - (P_{j;d}-\varepsilon)}{\bar{G}_2^n - (P_{j;s}+\varepsilon)}, \frac{\bar{G}_2^m - (P_{j;d}-\varepsilon)}{\bar{G}_2^m - (P_{j;s}+\varepsilon)}\right\}$$

Situation A:

$$\frac{dQ_{j;B}}{dQ_{j;A}} > \max\left\{\frac{\bar{G}_2^n - (P_{j;d}-\varepsilon)}{\bar{G}_2^n - (P_{j;s}+\varepsilon)}, \frac{\bar{G}_2^m - (P_{j;d}-\varepsilon)}{\bar{G}_2^m - (P_{j;s}+\varepsilon)}, \frac{\sum_{i=1}^{3} p^i \bar{G}_2^i - (P_{j;d}-\varepsilon)}{\sum_{i=1}^{3} p^i \bar{G}_2^i - (P_{j;s}+\varepsilon)}\right\}$$

Situation B:

$$\frac{dQ_{j;B}}{dQ_{j;A}} > \max\left\{\frac{G_2^n - (P_{j;d} - \varepsilon)}{G_2^n - (P_{j;s} + \varepsilon)}, \frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)}\right\}$$

Situation C:

$$\max\left\{\frac{G_2^n - (P_{j;d} - \varepsilon)}{G_2^n - (P_{j;s} + \varepsilon)}, \frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)}\right\} < \frac{dQ_{j;B}}{dQ_{j;A}} < \frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)}$$

which is verified.

Case II

In order to have the wealth constraints respected we must have

$$\frac{dQ_{j;B}}{dQ_{j;A}} > \frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)}.$$

Situation A:

In order to have an increase in utility

$$\frac{dQ_{j;B}}{dQ_{j;A}} > \frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)}$$

.Hence, any deviation:

$$\frac{dQ_{j;B}}{dQ_{j;A}} > \max\left\{\frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)}, \frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)}\right\}.$$

Situation B:

$$\frac{dQ_{j;B}}{dQ_{j;A}} > \frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)}.$$

Situation C:

In order to have an increase in utility

$$\frac{dQ_{j;B}}{dQ_{j;A}} < \frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)}$$

.Hence, as

$$\frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)} > \frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)}$$

any $\frac{dQ_{j;B}}{dQ_{j;A}}$ such that

$$\frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)} < \frac{dQ_{j;B}}{dQ_{j;A}} < \frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)}$$

will increase utility.

Case III

In order to have the wealth constraints respected we must have

$$\frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)} < \frac{dQ_{j;B}}{dQ_{j;A}} < \frac{G_2^n - (P_{j;d} - \varepsilon)}{G_2^n - (P_{j;s} + \varepsilon)}.$$

It is easy to check that there is a non-empty set for $\frac{dQ_{j;B}}{dQ_{j;A}}$.

Situation A:

In order to have an increase in utility

$$\frac{dQ_{j;B}}{dQ_{j;A}} > \frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)}$$

.Hence, as

$$\frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)} < \frac{G_2^n - (P_{j;d} - \varepsilon)}{G_2^n - (P_{j;s} + \varepsilon)}$$

any $\frac{dQ_{j;B}}{dQ_{j;A}}$ such that

$$\max\left\{\frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)}; \frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)}\right\} < \frac{dQ_{j;B}}{dQ_{j;A}} < \frac{G_2^n - (P_{j;d} - \varepsilon)}{G_2^n - (P_{j;s} + \varepsilon)}$$

Situation B:

$$\frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)} < \frac{dQ_{j;B}}{dQ_{j;A}} < \frac{G_2^n - (P_{j;d} - \varepsilon)}{G_2^n - (P_{j;s} + \varepsilon)}.$$

Situation C:

In order to have an increase in utility

$$\frac{dQ_{j;B}}{dQ_{j;A}} < \frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)}$$

.Hence, as

$$\frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)} > \frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)}$$

any $\frac{dQ_{j;B}}{dQ_{j;A}}$ such that

$$\frac{G_2^m - (P_{j;d} - \varepsilon)}{G_2^m - (P_{j;s} + \varepsilon)} < \frac{dQ_{j;B}}{dQ_{j;A}} < \min\left\{\frac{G_2^n - (P_{j;d} - \varepsilon)}{G_2^n - (P_{j;s} + \varepsilon)}; \frac{\sum_{i=1}^{3} p^i G_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i G_2^i - (P_{j;s} + \varepsilon)}\right\}$$

will increase utility.

Case IV

Any possible $\frac{dQ_{j;B}}{dQ_{j;A}}$ will respect the wealth constraints. Hence, it is possible to find a deviation that increases utility.

Case V

In order to have the wealth constraints respected we must have

$$\frac{dQ_{j;B}}{dQ_{j;A}} < \frac{\hat{G}_2^n - (P_{j;d} - \varepsilon)}{\hat{G}_2^n - (P_{j;s} + \varepsilon)}.$$

Situation A:

In order to have an increase in utility

$$\frac{dQ_{j;B}}{dQ_{j;A}} > \frac{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;s} + \varepsilon)}$$

.Hence, as

$$\frac{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;s} + \varepsilon)} < \frac{\hat{G}_2^n - (P_{j;d} - \varepsilon)}{\hat{G}_2^n - (P_{j;s} + \varepsilon)}$$

any $\frac{dQ_{j;B}}{dQ_{j;A}}$ such that

$$\frac{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;s} + \varepsilon)} < \frac{dQ_{j;B}}{dQ_{j;A}} < \frac{\hat{G}_2^n - (P_{j;d} - \varepsilon)}{\hat{G}_2^n - (P_{j;s} + \varepsilon)}$$

Situation B:

$$\frac{dQ_{j;B}}{dQ_{j;A}} < \frac{\hat{G}_2^n - (P_{j;d} - \varepsilon)}{\hat{G}_2^n - (P_{j;s} + \varepsilon)}.$$

Situation C:

In order to also have an increase in utility

$$\frac{dQ_{j;B}}{dQ_{j;A}} < \min \left\{ \frac{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;s} + \varepsilon)}; \frac{\hat{G}_2^n - (P_{j;d} - \varepsilon)}{\hat{G}_2^n - (P_{j;s} + \varepsilon)} \right\}$$

Case VI

In order to have the wealth constraints respected we must have

$$\frac{dQ_{j;B}}{dQ_{j;A}} < \min \left\{ \frac{\hat{G}_2^n - (P_{j;d} - \varepsilon)}{\hat{G}_2^n - (P_{j;s} + \varepsilon)}; \frac{\hat{G}_2^m - (P_{j;d} - \varepsilon)}{\hat{G}_2^m - (P_{j;s} + \varepsilon)} \right\}$$

Situation A:

In order to increase utility

$$\frac{dQ_{j;B}}{dQ_{j;A}} > \frac{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;s} + \varepsilon)}$$

hence, as

$$\frac{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;s} + \varepsilon)} < \min\left\{\frac{\hat{G}_2^n - (P_{j;d} - \varepsilon)}{\hat{G}_2^n - (P_{j;s} + \varepsilon)}; \frac{\hat{G}_2^m - (P_{j;d} - \varepsilon)}{\hat{G}_2^m - (P_{j;s} + \varepsilon)}\right\}$$

a deviation is

$$\frac{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;s} + \varepsilon)} < \frac{dQ_{j;B}}{dQ_{j;A}} < \min\left\{\frac{\hat{G}_2^n - (P_{j;d} - \varepsilon)}{\hat{G}_2^n - (P_{j;s} + \varepsilon)}; \frac{\hat{G}_2^m - (P_{j;d} - \varepsilon)}{\hat{G}_2^m - (P_{j;s} + \varepsilon)}\right\}$$

Situation B:

$$\frac{dQ_{j;B}}{dQ_{j;A}} < \min\left\{\frac{\hat{G}_2^n - (P_{j;d} - \varepsilon)}{\hat{G}_2^n - (P_{j;s} + \varepsilon)}; \frac{\hat{G}_2^m - (P_{j;d} - \varepsilon)}{\hat{G}_2^m - (P_{j;s} + \varepsilon)}\right\}$$

Situation C:

$$\frac{dQ_{j;B}}{dQ_{j;A}} < \min\left\{\frac{\hat{G}_2^n - (P_{j;d} - \varepsilon)}{\hat{G}_2^n - (P_{j;s} + \varepsilon)}; \frac{\hat{G}_2^m - (P_{j;d} - \varepsilon)}{\hat{G}_2^m - (P_{j;s} + \varepsilon)}; \frac{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;d} - \varepsilon)}{\sum_{i=1}^{3} p^i \hat{G}_2^i - (P_{j;s} + \varepsilon)}\right\}$$

which is verified. ■