Continuous versus Discrete Time Forest Management Models with Carbon Sequestration Benefits

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Abstract

Forest literature uses both continous and discrete time models to study forest management problems, and when carbon sequestration benefits are considered, the results obtained in both approaches are not always equivalent. This issue is relevant from a policy point of view if credits are to be allocated to forest owners within the implementation of the Kyoto Protocol. This note explores the impact of different carbon sequestration accounting methods on both settings. It studies the specific conditions for optimal rotation period and the value of a marginal unit of bare land on a one stand model and compare them with the long run optimal stationary steady state of a forest vintage model.

1 Introduction

The implementation of the Kyoto Protocol and the important role that may be played by forests in the global carbon cycle to limit the impact of GHGs emissions, has brought the consideration of carbon sequestration benefits to the center of recent developments in the context of forestry literature.

Two alternative frameworks to model forest management problems have been typically considered: the one stand optimal rotation model *à la* Faustmann, based on a continuous time framework and the multi-vintage age class model leading to the fully regulated or normal forest, using a discrete time model.

For the multiple vintage forest model, Salo and Tahvonen [1], [2] and [3], were able to provide a full proof for the optimality of the long run equilibrium of a normal forest based on the Faustmann rotation period, that is, the one that maximizes present value from timber production. The results are based on a strictly concave utility function for timber harvesting and positive discounting, as well as on the possibility of allocating part of the land to other uses. Following their approach, Costa-Duarte, Cunha-e-Sá and Rosa [9] extend the results on the existence of optimal stationary steady-states to the case where optimal use of land also considers the benefits from carbon sequestration. In fact, when introducing carbon sequestration benefits in a multi-vintage model, the long run equilibrium is again the normal forest based on the Faustmann rotation period.

The approach based on a one stand continuous time rotation model has also been considered in the literature. In particular, Van Kooten, Binkley and Delcourt [4] modeled a scheme to allocate carbon credits, under which the carbon credit cash flows are a function of the annual change in the forest carbon stock (carbon flow regime), Spring, Kennedy, and Nally [6] study the effect of carbon sequestration, fire frequency and water scarcity in tree harvest decision, while in Cunha-e-Sá and Rosa [5] different carbon accounting methods are introduced in the model of the private forester with constant and rising carbon prices.

The results obtained in both approaches are not easily comparable not only because the assumptions differ significantly, but also due to the continuous versus time discrete analytical setting.

This note studies the impact of carbon sequestration benefits on both settings. The one stand model is formalized both as a discrete and continuous time one, to make results more easily comparable.

In order to compare forest carbon sequestration with avoided emissions the IPCC Special Report on Land Use, Land-Use Change and Forestry [7] considers different accounting methods to apply to forest or land use change investment projects, namely, the stock change method, the average stock method and the tonne yearly crediting. Different versions of these three carbon sequestration accounting methods are used in this note to model carbon credit regimes and to study their impact on the models referred above.

The remainder of the paper is organized as follows. In section 2, three formalizations of forest models are presented, the one-stand continuous and the one-stand discrete time models and the multi-vintage model. In section 3, 4 and 5 we introduce carbon sequestration benefits in the models, using carbon flow, tonne year crediting and average storage accounting methods, respectively. Finally, conclusions are presented.

2 Forest Models

2.1 One-Stand Continuous Time Model

The one stand model is an extension of the Faustmann rotation model where the main assumptions are a continuous timber growth function, constant timber prices, constant technology and a constant discount rate, r. The main results are summarized in Aronsson and Lofgren [8] ,Theorem 1- "*A Forest stand should be harvested when the change in its value with respect to time is equal to the interest on the value of the stand plus the interest on the value of the bare land*".

Let us assume that the timber price is constant and given by P, the biomass content in timber per unit of land is a continuous function depending on the age trees s, and is given by $f(s), s =]0, n[$, where n represents the age class where timber growth is maximum and e^{-rt} is the continuous time discount factor.

By extending the typical one stand multiple rotation model to the presence of carbon sequestration benefits, the optimal rotation period is the one that maximizes the present value of net benefits from timber production and carbon sequestration, as follows:

$$V(T) = \underset{T}{Max} \left[Pf(T)\frac{e^{-rT}}{1-e^{-rT}} + S(T)\frac{1}{1-e^{-rT}} \right] \tag{1}$$

where $S(T)$, represents the present value of net carbon sequestration benefits during a rotation period.

The value of $S(T)$ depends not only on the accounting method considered, in this note, the carbon flow method, the tonne-year crediting and the average carbon storage, respectively, but also, on the specific payment schedule, namely, whether the payments are due continuously or only at harvest time and how payments are defined.

For the moment, let us assume $S(T) = 0$. The forest owner's problem consists of choosing the optimal rotation period T, for all possible age trees.The optimal rotation period is given by the usual first order condition:

$$\frac{f'(T)}{f(T)} = \frac{r}{1-e^{-rT}}$$

and the value of a unit of bare land given by:

$$V(T) = \frac{Pf(T)e^{-rT}}{1-e^{-rT}} \tag{2}$$

2.2 One stand Discrete Time Model

Let us now formalize a similar one stand model as a discrete time model. We assume that the timber price is constant and given by P, and f_s represents the biomass content in timber per unit of land with trees of age class s, where $0 \leq f_1 \leq .... \leq f_n$. As in [9], n represents the age class where timber growth is maximum and b is the discrete time discount factor.

Under these assumptions, the one stand multiple rotation model can be rewriten as a dynamic programing model. Let $V_s(f_s)$ represent the value of one stand with trees of age s

$$V_s(f_s) = Max\{Pf_s + S_s + V_0(f_0); b(Pf_{s+1} + S_{s+1} + V_0(f_0))\} \tag{3}$$

that is, the maximum between the value of the timber plus carbon benefits that the forest owner receives if he decides to cut, and the maximum amount that he could get if he decides to delay the cutting time for one period.

Here again, the carbon benefits received by the forest owner in a one stand model will depend not only on the accounting method but also on the specific payment procedures, that is, whether the payments are due in an yearly basis or only at harvest. In all cases presented in the next section, S_s represents the value received by a forest owner with trees of age s. For the moment, let us assume $S_s = 0$, for all s.

The forest owner's dynamic programing problem consists of choosing the optimal cutting time m, from all possible age trees.

Given (4) and $b < 1$, at n it is always optimal to cut, meaning that no tree of that age will remain.

$$V_n(f_n) = Max\{Pf_n + V_0(f_0); b(Pf_n + V_0(f_0))\} \tag{4}$$

Therefore, the optimal cutting time (also, the optimal rotation period) must satisfy $1 < m < n$, and at m it has to be the case that the forest owner has neither incentive to postpone harvest nor to cut earlier. Thus, it has to be the case that given

$$V_m(f_m) = Max\,\{Pf_m + V_0(f_0); b(Pf_{m+1} + V_0(f_0))\}$$

and

$$V_{m-1}(f_{m-1}) = Max\,\{Pf_{m-1} + V_0(f_0); b(Pf_m + V_0(f_0))\}$$

the two following conditions (5) and (6a) must hold simultaneously at m,

$$Pf_m + V_0(f_0) \geq b(Pf_{m+1} + V_0(f_0) \tag{5}$$

$$Pf_{m-1} + V_0(f_0) \leq b(Pf_m + V_0(f_0)) \tag{6a}$$

Let us assume that m is unique. Then, if it is optimal to cut at m, it is optimal to delay harvest for all age trees $s < m$, and the maximum value of a unit of bare land is given by:

$$V_0(f_0) = Max\,\{V_0(f_0); b^m(f_m + V_0(f_0))\} = \frac{b^m P f_m}{1 - b^m} \tag{7}$$

which is the same as the one resulting from the optimal solution of the multi-vintage model (20), and is also the same solution that we obtain in the usual one stand continuous time model (2).

Substituting (7) in (5) and (6a), and rearranging terms we have:

$$Pf_m(\frac{1 - b^{m+1}}{1 - b^m}) > bPf_{m+1} => \mathbf{f}_m > \mathbf{f}_{m+1}(\frac{b - b^{m+1}}{1 - b^{m+1}}) \tag{8}$$

$$Pf_m(\frac{b - b^m}{1 - b^m}) > Pf_{m-1} => \mathbf{f}_m > \mathbf{f}_{m-1}(\frac{1 - b^m}{b - b^m}) \tag{9}$$

Conditions (8) and (9) are together sufficient conditions for optimality. Adding both inequalities and rearranging terms leads to a necessary condition that provides interesting insights:

$$\frac{1}{b} \geq \frac{f_{m+1} - f_m}{f_m - f_{m-1}} \tag{10}$$

2.3 The Multi-vintage Model

The model used follows closely the multiple vintage forest model developed in Salo and Tahvonen [3], which can be summarized as follows. The model assumes multi vintages forest land, where $s = 1, ..., n$ represents the age of trees, $x_{s,t}$ the area of forest land allocated to the age class s in period t, f_s the biomass content in timber per unit of land with trees of age class s, and $0 \leq f_1 \leq .... \leq f_n$. Land allocation must satisfy

$$0 \leq y_t = 1 - \sum_{s=1}^{n} x_{s,t} \tag{11}$$

that is, total land area equals 1, and y_t is the area of land allocated to an alternative use (agriculture or urban use).

Let us denote by $U(c_t) = \int D(c)dc$ the social utility from timber consumption, where $D(.)$ is the inverse demand for timber, and assume $U(.)$ is a continuous, twice differentiable, increasing and strictly concave function. Also, $W(y_t) = \int Q(y)dy$, where $W(.)$ is a continuous, twice differentiable, increasing and concave function. Finally, S_t represents current carbon sequestration benefits at t and depends on the way the benefits from carbon sequestration are accounted for.[1]

Thus, the problem of optimal forest harvesting with carbon sequestration benefits and allocation of land is obtained by maximizing the present value of social utility from the use of land as follows:

$$v(x_{1,0}, ...x_{n,0}) = \underset{\{x_{s,t+1}, s=1,...n, t=0,...\}}{Max} \sum_{t=0}^{\infty} b^t \left[U\left(c_t\right) + S_t + W\left(y_t\right)\right] \tag{12}$$

subject to

$$c_t = \sum_{s=1}^{n-1} f_s \left(x_{s,t} - x_{s+1,t+1}\right) + f_n x_{n,t} \tag{13}$$

[1] This definition of S_t implicitly assumes that payments to each individual stand owner are scheduled on a yearly basis.

$$y_t = 1 - \sum_{s=1}^{n} x_{s,t} \tag{14}$$

$$x_{s+1,t+1} \leq x_{s,t}, s = 1, ...n - 1 \tag{15}$$

$$\sum_{s=1}^{n} x_{s,t+1} \leq 1 \tag{16}$$

$$x_{s,t} \geq 0, s = 1, ..., n \tag{17}$$

for all $t = 0, 1...$, where S_t is given by (53), (67), or (108), respectively, depending on the particular carbon benefits accounting method used. Finally, the initial land distribution satisfies

$$x_{s,0} \geq 0, s = 1, ..., n, \sum_{s=1}^{n} x_{s,0} \leq 1 \tag{18}$$

Therefore, given the discount factor b, the problem is to choose the next period state, that is, the land allocation between different vintages and competing uses of land for all $t = 1, ...$

The necessary conditions for optimal solutions can be obtained from the following Lagrangian problem. For (12-18) it can be stated as :

$$L = \sum_{t=0}^{\infty} b^t \left[U\left(c_t\right) + S_t + W\left(y_t\right) \right] + \lambda_t \left(1 - \sum_{s=1}^{n} x_{s,t+1} \right) + \sum_{s=1}^{n-1} \left[p_{s,t} \left(x_{s,t} - x_{s+1,t+1} \right) \right] \tag{19}$$

where $p_{s,t}$ and λ_t are the Lagrangian multipliers. While $p_{s,t}$ can be interpreted as the value of marginal changes in forest land area of vintage s at the beginning of period $t + 1$, λ_t represents the value of marginal changes in land allocation between forest and alternative uses.

Salo and Tahvonen [3] provide a full proof on the long-run optimality of the normal forest steady-state for the above problem. When $S_t = 0$, the steady state optimal condition is given by:

$$W'(y_\infty)\frac{b}{1-b} - \frac{b^m f_m}{1-b^m} U'(\frac{(1-y_\infty) f_m}{m}) = 0 \tag{20}$$

where m is the Faustmann rotation period satisfying the condition:

$$\frac{b^m f_m}{1-b^m} \geq \frac{b^s f_s}{1-b^s} \quad \text{for } s = 1, ..., n \tag{21}$$

3 Carbon flow regime

The carbon flow regime considers that an increase in the forest standing biomass corresponds to an increase in the carbon stock, and that harvest reduces the carbon stock. Notice that once carbon has been sequestered, no further carbon benefits will be obtained. Thus, in this case, what is relevant when modeling carbon sequestration benefits in a standing forest is the change in the per period carbon uptake. Finally, carbon released at harvest depends on the final use of timber and to take into account different uses of timber we introduce a parameter θ which measures the fraction of timber that is harvested but goes into long-term storage in structures and landfills.

Two alternative payment procedures will be considered for this accounting method, whether the carbon net payments are evaluated and due every period or only at harvest time.

3.1 Continuous time

Case1-*Payments are due continuously and are equal to the increase in the value of carbon sequestration net benefits.*

In the continuous time one stand model, $S(T)$ represents the present value of net carbon sequestration benefits for one rotation period. Here $S(T^c)$, is given by:

$$S(T^c) = P_c\beta \int_0^{T^c} f^{'}(s)e^{-rs}ds - P_c\beta(1-\theta)f(T^c)e^{-rT^c} \tag{22}$$

We derive the first-order condition of (22) with respect to T^c, as follows:

$$\frac{f'(T^c)}{f(T^c)} = \frac{r}{1-e^{-rT^c}} + \frac{\frac{r}{1-e^{-rT^c}} P_c\beta \left[\int_0^{T^c} f'(s)e^{-rs}ds - (1-\theta)f(T^c)\right] - P_c\beta\theta f'(T^c)}{Pf(T^c)} \tag{23}$$

When $\theta = 0$, that is, when no carbon is released at harvest, (23) can be restated as:

$$\frac{f'(T^c)}{f(T^c)} = \frac{r}{1-e^{-rT^c}} + \frac{\frac{r}{1-e^{-rT^c}} P_c\beta \int_0^{T^c} f(s)e^{-rs}ds - P_c\beta f(T^c)}{Pf(T^c)} \tag{24}$$

Since $f(t)$ is increasing, if it is also strictly concave, the term in square brackets, which can be denoted by carbon balance, is always negative.[2] Therefore, the optimal rotation period may increase relatively to the case without carbon sequestration benefits, $T^c > T$.

When $\theta = 1$, and all carbon is released at harvest, (23) can be restated as:

$$\frac{(P + P_c\beta)f'(T^c)}{Pf(T^c)} = \frac{r}{1-e^{-rT^c}} + \frac{\frac{r}{1-e^{-rT^c}} P_c\beta \int_0^{T^c} f'(s)e^{-rs}ds}{Pf(T^c)} \tag{25}$$

Here, both sides of the equality increase and the final result on the optimal rotation period is undeterminate. Numerical examples suggest that rotation period still increases.

In this case, the value of a unit of bare land is given by:

$$V(T^c) = (P + P_c\beta\theta)f(T^c)\frac{e^{-rT^c}}{1-e^{-rT^c}} + P_c\beta(\int_0^{T^c} f^{'}(s)e^{-rs}ds - f(T^c)e^{-rT^c})\frac{1}{1-e^{-rT^c}} \tag{26}$$

The optimal rotation period may increase due to financial gains from payment schedules. The value of land increases due to long run carbon storage and to the increased finantial gains.

Case2-*Payments are due at harvest and are equal to the value of carbon sequestration net benefits at harvest time*

Assuming now that the payments due to the forest owner are undertaken and evaluated at harvest, instead of being delivered in a yearly basis, the results are different. Here $S(T^c)$, representing the present value of net carbon sequestration benefits for one rotation period, is given by:

$$S(T^c) = (P_c\beta \int_0^{T^c} f^{'}(s)ds - P_c\beta(1-\theta)f(T^c))e^{-rT^c} = P_c\beta\theta f(T^c)e^{-rT^c} \tag{27}$$

[2] see Theorem 5 in Aronsson et al, [8]

In this case, the first-order condition for $0 < \theta < 1$ becomes:

$$\frac{(P + P_c\beta\theta)f'(T^c)}{(P + P_c\beta\theta)f(T^c)} = \frac{r}{1 - e^{-rT^c}} \tag{28}$$

which holds for $T^c = T$.

The value of a unit of bare land is:

$$V(T^c) = (P + P_c\beta\theta)f(T^c)\frac{e^{-rT^c}}{1 - e^{-rT^c}} \tag{29}$$

The optimal rotation period is the same as without carbon benefits and the value of land is increased only if $\theta > 0$.

3.2 Discrete time

Case 1- *Payments due every year and are equal to the yearly increase in the value of carbon sequestration benefits.*

Here S_s given by :

$$S_s = \sum_{i=1}^{s} b^{i-s}P_c\beta(f_i - \frac{f_{i-1}}{b}) - P_c\beta(1-\theta)f_s \tag{30}$$

Let us define $V_s(f_s)$ for S_s given by (30), as follows:

$$V_s^c(f_s) = Max\left\{\begin{array}{c} Pf_s + P_c\beta\sum_{i=1}^{s} b^{i-s}(f_i - \frac{f_{i-1}}{b}) - P_c\beta(1-\theta)f_s + V_0^c(f_0); \\ b(Pf_{s+1} + P_c\beta\sum_{i=1}^{s+1} b^{i-1-s}(f_i - \frac{f_{i-1}}{b}) - P_c\beta(1-\theta)f_{s+1} + V_0^c(f_0)) \end{array}\right\} \tag{31}$$

Let

$$\begin{aligned} V_n(f_n) \quad &= \quad Max\{Pf_n + P_c\beta\sum_{i=1}^{n} b^{i-n}(f_i - \frac{f_{i-1}}{b}) - P_c\beta(1-\theta)f_n + V_0^c(f_0); \\ &\quad\quad b(Pf_n + P_c\beta\sum_{i=1}^{n} b^{i-1-n}(f_i - \frac{f_{i-1}}{b}) - P_c\beta(1-\theta)f_n + V_0^c(f_0))\} \end{aligned}$$

Given that $b < 1$, and that at n no additional carbon intakes will take place, it is always optimal to cut.

Let m^c be the optimal rotation period for the carbon flow accounting method case, then for $0 \leq \theta \leq 1$, m^c is given by:

$$Pf_{m^c} + P_c\beta\theta f_{m^c} > b(Pf_{m^c+1} + P_c\beta\theta f_{m^c+1}) - (1-b)V_0^c(f_0) \tag{32}$$

$$P f_{m^c-1} + P_c\beta\theta f_{m^c-1} < b(P f_{m^c} + P_c\beta\theta f_{m^c}) - (1-b)V_0^c(f_0) \qquad (33)$$

If it is optimal to cut at m^c, it is optimal to delay harvest for all age trees $s < m^c$. Therefore, the maximum value of a unit of bare land is given by

$$V_0^c(f_0) = Max\left\{V_0^c(f_0); b^{m^c}(P f_{m^c} + P_c\beta\sum_{i=1}^{m^c} b^{i-m^c}(f_i - \frac{f_{i-1}}{b}) - P_c\beta(1-\theta)f_{m^c} + V_0^c(f_0))\right\}$$

$$= \frac{b^{m^c} P f_{m^c}}{1-b^{m^c}} + \frac{b^{m^c} P_c\beta\sum_{i=1}^{m} b^{i-m^c}(f_i - \frac{f_{i-1}}{b})}{1-b^{m^c}} - \frac{b^{m^c} P_c\beta(1-\theta)f_{m^c}}{1-b^{m^c}} =$$

$$= \frac{b^{m^c} P f_{m^c}}{1-b^{m^c}} + \frac{P_c\beta\sum_{i=1}^{m^c} b^{i}(f_i - \frac{f_{i-1}}{b})}{1-b^{m^c}} - \frac{b^{m^c} P_c\beta(1-\theta)f_{m^c}}{1-b^{m^c}} =$$

$$= \frac{P b^{m^c} f_{m^c}}{1-b^{m^c}} + \frac{P_c\beta\theta b^{m^c} f_{m^c}}{1-b^{m^c}} = (P + P_c\beta\theta)\frac{b^{m^c} f_{m^c}}{1-b^{m^c}} \qquad (34)$$

as all the other terms cancel out. Taking equation (34) and substituting in (32) and (33) we have:

$$(P + P_c\beta\theta)f_{m^c} > b(P + P_c\beta\theta)f_{m^c+1} - (1-b)(P + P_c\beta\theta)\frac{b^{m^c} f_{m^c}}{1-b^{m^c}} \qquad (35)$$

$$(P + P_c\beta\theta)f_{m^c-1} < b(P + P_c\beta\theta)f_{m^c} - (1-b)(P + P_c\beta\theta)\frac{b^{m^c} f_{m^c}}{1-b^{m^c}} \qquad (36)$$

Rearranging terms:

$$(P + P_c\beta\theta)f_{m^c}(\frac{1-b^{m^c+1}}{1-b^{m^c}}) > b(P + P_c\beta\theta)f_{m^c+1} => \mathbf{f}_m^c > \mathbf{f}_{m^c+1}(\frac{b-b^{m^c+1}}{1-b^{m^c+1}}) \qquad (37)$$

$$(P + P_c\beta\theta)f_{m^c}(\frac{b-b^{m^c}}{1-b^{m^c}}) > (P + P_c\beta\theta)f_{m^c-1} => \mathbf{f}_m > \mathbf{f}_{m-1}(\frac{1-b^{m^c}}{b-b^{m^c}}) \qquad (38)$$

We conclude that the sufficient conditions for optimality, (37)and (38), are the same as (8) and (9), therefore $m^c = m$. If it is optimal to cut at m without carbon benefits it is also optimal to cut at when they are accounted for.

Just by comparing (34) with (7) we may conclude that $m^c = m$. In fact, in this case, the value of a unit of bare land increases the present value of timber biomass by a constant amount of $(P_c \beta \theta)$.

Alternatively, by rearranging (32) and (33), adding up both inequalities and collecting terms, we obtain:

$$\frac{f_{m^c+1} - f_{m^c}}{f_{m^c} - f_{m^c-1}} < \frac{1}{b} \left[\frac{P + \theta P_c \beta}{P + \theta P_c \beta} \right] \tag{39}$$

and the necessary condition (39) is the same condition as (10).

The optimal rotation period is the same with or without carbon benefits and the value of land is increased only if $\theta > 0$.[3]

[3] These results are different from the ones obtained with the continuous time version. They will be equivalent if S_s is defined as

$$S_s = \sum_{i=1}^{s} b^{i-s} P_c \beta (f_i - f_{i-1}) - P_c \beta (1 - \theta) f_s \tag{40}$$

Here we assume that the benefits from carbon sequestration are payed in an yearly basis, but the discount is applied to the discrete change in the the production value of timber at each time period. In this case, $V_s(f_s)$ for S_s given by (40) can be stated as

$$V_s^c(f_s) = Max \left\{ \begin{array}{c} P f_s + P_c \beta \sum_{i=1}^{s} b^{i-s} (f_i - f_{i-1}) - P_c \beta (1-\theta) f_s + V_0^c(f_0); \\ b(P f_{s+1} + P_c \beta \sum_{i=1}^{s+1} b^{i-1-s} (f_i - f_{i-1}) - P_c \beta (1-\theta) f_{s+1} + V_0^c(f_0)) \end{array} \right\} \tag{41}$$

For $0 \leq \theta \leq 1$, we have

$$P f_{m^c} - b P_c \beta (f_{m^c+1} - f_{m^c}) - P_c \beta (1-\theta)(f_{m^c} - b f_{m^c+1}) \geq b P f_{m^c+1} - (1-b) V_0^c(f_0) \tag{42}$$

$$P f_{m^c-1} - b P_c \beta (f_{m^c} - f_{m^c-1}) - P_c \beta (1-\theta)(f_{m^c-1} - b f_{m^c}) \leq b P f_{m^c} - (1-b) V_0^c(f_0) \tag{43}$$

Adding both inequalities and collecting similar terms, we obtain:

$$\frac{f_{m^c+1} - f_{m^c}}{f_{m^c} - f_{m^c-1}} \leq \frac{1}{b} \left[\frac{P + \theta P_c \beta - P_c \beta (1-b)}{P + \theta P_c \beta} \right] < \frac{1}{b} \tag{44}$$

where the term in square brackets is lower than one, implying that there may exist a $m^c > m$ for which the optimal cutting time is delayed.

Taking the derivative of the term inside the square brackets with respect to θ, we obtain:

$$\frac{\partial \left[\frac{P + \theta P_c \beta - P_c \beta (1-b)}{P + \theta P_c \beta} \right]}{\partial \theta} = \frac{P_c^2 \beta^2 (1-b)}{(P + \theta P_c \beta)^2} > 0 \tag{45}$$

Therefore, the term in square brackets increases with $\theta = \rfloor 0, 1 \lfloor$

$$\frac{P - P_c \beta (1-b)}{P} < \frac{P + P_c \beta b}{P + P_c \beta} \tag{46}$$

The delay is larger the lower is θ, as costs of carbon release upon harvest increase.

Case 2 *Payments are due at harvest and are equal to the value of carbon sequestration net benefits at harvest time.*

Here S_s is given by:

$$S_s = \sum_{i=1}^{s} P_c\beta(f_i - f_{i-1}) - P_c\beta(1-\theta)f_s = P_c\beta\theta f_s \tag{48}$$

For S_s given by (48), $V_s^c(f_s)$ can be stated as:

$$V_s^c(f_s) = Max\left\{Pf_s + P_c\beta\theta f_s + V_0^c(f_0); b(Pf_{s+1} + P_c\beta\theta f_{s+1} + V_0^c(f_0))\right\} \tag{49}$$

When $\theta = 0$, that is, when no carbon is released at harvest, (49) is the same as in the $S_s = 0$ case, and the equivalence result between the two models holds. In this case, the carbon flow accounting method has no impact, neither in forest management nor in land allocation.

For $0 < \theta \leq 1$, m^c must satisfy simultaneously:

$$Pf_{m^c} + P_c\beta\theta f_{m^c} + V_0^c(f_0) \geq b(Pf_{m^c+1} + P_c\beta\theta f_{m^c+1} + V_0^c(f_0) \tag{50}$$

and

$$Pf_{m^c-1} + P_c\beta\theta f_{m^c-1} + V_0^c(f_0) \leq b(Pf_{m^c} + P_c\beta\theta f_{m^c} + V_0^c(f_0)) \tag{51a}$$

If m^c is unique, it is optimal to delay harvest for all age trees $s < m^c$. Therefore, the maximum value of a unit of bare land is given by:

$$V_0^c(f_0) = Max\left\{V_0^c(f_0); b^{m^c}(Pf_{m^c} + P_c\beta\theta f_{m^c} + V_0^c(f_0))\right\} = (P + P_c\beta\theta)\frac{b^{m^c} f_{m^c}}{1 - b^{m^c}} \tag{52}$$

Alternatively, if m^c is unique, it is optimal to delay cutting for all age trees $s < m^c$. Therefore, the maximum value of a unit of bare land is given by:

$$\begin{aligned} V_0^c(f_0) &= Max\left\{V_0^c(f_0); b^{m^c}(Pf_{m^c} + P_c\beta\sum_{i=1}^{m^c} b^{i-m^c}(f_i - f_{i-1}) - P_c\beta(1-\theta)f_{m^c} + V_0^c(f_0))\right\} \\ &= (P + P_c\beta\theta)\frac{b^{m^c} f_{m^c}}{1 - b^{m^c}} + \frac{P_c\beta\sum_{i=1}^{m^c-1} b^i(1-b)(f_i - f_{i-1})}{1 - b^{m^c}} \end{aligned} \tag{47}$$

Here, as in (26) even for $\theta = 0$, the crediting regime has a positive impact on both forest management and the value of forest land.

Also, rearranging (50) and (51a), adding both inequalities and collecting similar terms, we obtain again necessary condition (39), meaning that ($m^c = m$), for $0 \leq \theta \leq 1$.

The optimal rotation period is the same as without carbon benefits and the value of land is increased only if $\theta > 0$.

3.3 Multi vintage model

Under similar assumptions, the current net benefits from carbon sequestration at any period t, S_t, for the multi-vintage model, can be represented as follows:

$$S_t = P_c \beta f_1 x_{1,t} + \sum_{s=2}^{n} P_c \beta (f_s - \frac{f_{s-1}}{b}) x_{s,t} - P_c \beta (1-\theta) c_t \tag{53}$$

where the first two terms represent the value of the carbon stock increase in forest standing biomass, in period t, for all the area of forest land, and the last term represents the value of the decrease in the carbon stock due to timber harvesting.

This model formalizes the social planner's perspective and it is equivalent to a situation where in the stationary steady state payments are done yearly to forest owners of all vintages age classes.

Assuming that m is unique, for a stationary state, we have that $p_{s,t} = p_{s,\infty}$, $c_t = c_\infty$, $y_t = y_\infty$, $\lambda_t = 0$, and $x_{m,t} = x_\infty$, where $c_\infty, y_{,\infty}$, x_∞, and $p_{s,\infty}$, for $s = 1, ..., n-1$, are constant. From Costa-Duarte et al [9]:

$$p_s = W'(y_\infty) \sum_{i=0}^{s-1} b^{-i} - f_s \left[U'(C_\infty) + \beta p_c \theta \right], \; s = 1, ..., n \tag{54}$$

for $s = 1, ..., n,$.

From $s = m$, and given that, $p_{m,\infty} = 0$, with some more algebra, we can write the following steady-state condition:

$$W'(y_\infty) \frac{b}{1-b} - \frac{b^m f_m}{1-b^m} U'(\frac{(1-y_\infty) f_m}{m}) - \frac{b^m f_m}{1-b^m} \beta p_c \theta = 0 \tag{55}$$

The rotation period and the value of a unit of bare forest land in the long run optimal stationary steady state of a forest vintage model is equivalent to both cases of the discrete single-stand rotation model. The optimal rotation period is the same as without carbon benefits and the value of land is increased only if $\theta > 0$. The one stand continuous time model has additional financial gains derived from the fact that payments are due continuously.

4 Tonne-year crediting

The tonne-year accounting method consists of crediting a forestry project with a fraction of its total yearly GHG benefit. This fraction is based on the stock of carbon stored each year, which is then converted, using (E_f), to its equivalent amount of preventing effect.[4]

The aim of this accounting method is to provide an yearly revenue for the forest owner. Therefore, only this case will be considered.

4.1 Continuous time

Under these assumptions, $S(T^t)$, the present value of the continuous net benefits from carbon sequestration during a rotation period, can be represented by:

$$S(T^t) = P_c \beta E_f \int_0^{T^t} f(s) e^{-rs} ds \tag{56}$$

where we consider that the payments due to the forest owner are undertaken continuously. The first-order condition for the forest owner problem, with $S(T^t)$ given by (56), can be stated as follows:

$$\frac{f'(T^t)}{f(T^t)} = \frac{r}{1 - e^{-rT^t}} + \frac{\frac{r}{1-e^{-rT^t}} P_c \beta E_f \int_0^{T^t} f(s) e^{-rs} ds - P_c \beta E_f f(T^t)}{P f(T^t)} \tag{57}$$

[4] Here, we consider E_f constant. This assumption is consistent with Moura-Costa and Wilson' [10] approach, and also with Fearnside et al. [11], if in this last case we assume that the equivalence factor measures only the benefit of storing carbon in the forest for one additional year. To be fully consistent with Fearnside et al. [11], the equivalence factor should be different for each age class s, that is, $E_f\ (s)$.

Similarly, using the result in Aronsson et al [8], we conclude that the carbon balance is negative. Therefore, the optimal rotation is increased $T^t > T$.[5]

In this case the maximum value of a unit of bare land is given by:

$$V(T^t) = \left[Pf(T^t)\frac{e^{-rT^t}}{1-e^{-rT^t}} + (P_c\beta E_f \int_0^{T^t} f(s)e^{-rs}ds)\frac{1}{1-e^{-rT^t}} \right] \tag{58}$$

The optimal rotation period is increased and also the value of land due to the accounted carbon benefits.

4.2 Discrete time

Under the above assumptions, the current net benefits from carbon sequestration for a stand of tree with age s, S_s can be represented by:

$$S_s = P_c\beta E_f \sum_{i=1}^{s-1} b^{i-s} f_i \tag{59}$$

Let us define $V_s(f_s)$ for S_s given by (59) as follows:

$$V_s^t(f_s) = Max \left\{ \begin{array}{c} Pf_s + P_c\beta E_f \sum_{i=1}^{s-1} b^{i-s} f_i + V_0^t(f_0); \\ b(Pf_{s+1} + P_c\beta E_f \sum_{i=1}^{s} b^{i-1-s} f_i + V_0^t(f_0)) \end{array} \right\} \tag{60}$$

Let m^t be the optimal rotation period in the ton-year accounting method case.[6] In this case, m^t must satisfy simultaneously:

$$Pf_{m^t} - P_c\beta E_f f_{m^t} \geq bPf_{m^t+1} - (1-b)V_0^t(f_0) \tag{61}$$

and

$$Pf_{m^t-1} - P_c\beta E_f f_{m^t-1} \leq bPf_{m^t} - (1-b)V_0^t(f_0) \tag{62}$$

Again, if m^t is unique, it is optimal to delay cutting for all age trees $s < m^t$. Therefore, the maximum value of a unit of bare land is given by:

$$V_0^t(f_0) = Max \left\{ V_0^t(f_0); b^{m^t}(Pf_{m^t} + P_c\beta E_f \sum_{i=1}^{m^t-1} b^{i-m^t} f_i + V_0^t(f_0)) \right\} =$$

[5] With this accounting method, and if carbon sequestration benefits are high compared to timber value, it may be optimal to never cut or to cut trees at an age greater than n.

[6] With this accounting method, and if carbon sequestration benefits are high compared to timber value, it may be optimal to never cut or to cut trees at an age greater than n.

$$= \frac{Pb^{m^t} f_{m^t}}{1 - b^{m^t}} + \frac{P_c \beta E_f \sum_{i=1}^{m^t - 1} b^i f_i}{1 - b^{m^t}} \tag{63}$$

Equation (63) is equivalent to the optimal solution of the multi-vintage model (69), and similar to (58) of the continuous time model..

Taking (63) and substituting in (61) and (62), and rearranging terms, we have the following sufficient conditions:

$$Pf_{m^t} \geq Pf_{m^t+1}\left(\frac{b - b^{m^t+1}}{1 - b^{m^t+1}}\right) + P_c \beta E_f f_{m^t}\left(\frac{1 - b^{m^t}}{1 - b^{m^t+1}}\right) - \left(\frac{1 - b}{1 - b^{m^t+1}}\right)\left(P_c \beta E_f \sum_{i=1}^{m^t - 1} b^i f_i\right) \tag{64}$$

$$Pf_{m^t} \geq Pf_{m^t-1}\left(\frac{1 - b^{m^t}}{b - b^{m^t}}\right) - P_c \beta E_f f_{m^t-1}\left(\frac{1 - b^{m^t}}{b - b^{m^t}}\right) + \left(\frac{1 - b}{b - b^{m^t}}\right)\left(P_c \beta E_f \sum_{i=1}^{m^t - 1} b^i f_i\right) \tag{65}$$

It is not possible to infer directly from (64) and (65) if the optimal cutting time is delayed or advanced. Alternatively, adding up(61) and (62), and rearranging terms we obtain the following necessary condition for optimality:

$$\frac{f_{m^t+1} - f_{m^t}}{f_{m^t} - f_{m^t-1}} \leq \frac{1}{b}\left[\frac{P - P_c \beta E_f}{P}\right] < \frac{1}{b} \tag{66}$$

Here, the term in square brackets in (66) is positive, and smaller than one. Therefore, it provides a clear signal that it may be optimal to postpone harvest, $m^t > m$.

Finally, from (63), if $m^t > m$, the first term is lower than at m, but there may exists an additional term that compensate for the loss in the first term. If there exists a $m^t > m$ for which

$$\frac{Pb^{m^t} f_{m^t}}{1 - b^{m^t}} + \frac{P_c \beta E_f \sum_{i=1}^{m^t - 1} b^i f_i}{1 - b^{m^t}} > \frac{Pb^m f_m}{1 - b^m} + \frac{P_c \beta E_f \sum_{i=1}^{m-1} b^i f_i}{1 - b^m}$$

the optimality at $m^t > m$ is guaranteed.

The optimal rotation period may increase and also the value of land due to the accounted carbon benefits.

4.3 Multi-vintage model

In this case, S_t can be defined as follows:

$$S_t = P_c(E_f \beta \sum_{s=1}^{n-1} f_s x_{s+1,t+1}) \tag{67}$$

where the term in brackets represents the equivalent amount of emissions avoided in year t due to the amount of carbon stored in forest during year t.[7]

Assuming again that m is unique, for a stationary state, we have that $p_{s,t} = p_{s,\infty}$, $c_t = c_\infty$, $y_t = y_\infty$, $\lambda_t = 0$, and $x_{m,t} = x_\infty$, where $c_\infty, y_{,\infty}$, x_∞, and $p_{s,\infty}$, for $s = 1, ..., n-1$, are constant. From From Costa-Duarte et al[9]:

$$p_s = W'(y_\infty) \sum_{j=0}^{s-1} b^{-j} - f_s U'(C_\infty) - b^{-s} \beta p_c E_f \left(\sum_{i=1}^{m^t-1} b^i f_i - \sum_{i=s}^{m^t-1} b^i f_i \right) \tag{68}$$

Again with some more algebra, we can write for $s = m^t$ given that, $p_{m^t,\infty} = 0$,the optimal steady state is defined by:

$$W'(y_\infty) \frac{b}{1-b} - \frac{b^{m^t} f_{m^t}}{1-b^{m^t}} U'(\frac{(1-y_\infty) f_{m^t}}{m^t}) - \frac{\beta p_c E_f}{1-b^{m^t}} \sum_{i=1}^{m^t-1} b^i f_i = 0 \tag{69}$$

The rotation period and the value of a unit of bare forest land in the long run optimal stationary steady state of a forest vintage model are equivalent to both the discrete and continuous single-stand rotation models. The optimal rotation period is increased when carbon benefits are accounted for according to this method.

5 Average Carbon Storage

The average storage accounting method consists of crediting a forestry project with the amount of carbon benefits that the land allocated to forest generates, on average, at the end of each rotation. For the one stand model, different alternatives can be considered. Either the payments are only due at harvest

[7] By considering $f_s x_{s+1,t+1}$, this formalization excludes from benefits' accounting all possible harvesting of younger age classes, in period t.

time, or are undertaken every year, and in each case, the payment can be either based on the effective amount of carbon sequestered by the forest during a rotation period, or on an estimated average (constant amount).

5.1 Continuous time

Case1-*Payment at harvest are equal to the value of the average amount of carbon sequestered by the forest during a rotation period*

The first case considered assumes that the forest owner is payed at harvest time the current value of the average amount of carbon sequestered by the forest during the rotation period. In this case, the present value of carbon sequestration benefits in one rotation period is given by:

$$S(T^a) = P_c\beta\frac{\int_0^{T^a} f(s)ds)}{T^a}e^{-rT^a} \tag{70}$$

The corresponding first-order condition is given by:

$$\frac{f'(T^a)}{f(T^a)} = \frac{r}{1-e^{-rT^a}} + \frac{1}{T^a}\frac{\frac{r}{1-e^{-rT^a}}P_c\beta\int_0^{T^a} f(s)ds - P_c\beta f(T^a) + \frac{P_c\beta\int_0^{T^a} f(s)ds}{T^a}}{Pf(T^a)} \tag{71}$$

where the sign of the second term on the right-hand side is undeterminate. Therefore, we cannot say if optimal rotation period the increases or decreases.

In this case the maximum value of a unit of bare land is given by:

$$V(T^a) = Pf(T^a)\frac{e^{-rT^a}}{1-e^{-rT^a}} + (P_c\beta\frac{(\int_0^{T^a} f(s)ds)}{T^a})\frac{e^{-rT^a}}{1-e^{-rT^a}} \tag{72}$$

The optimal rotation period changes but the sign is undeterminate. The value of land increases due to the accounted carbon benefits.

Case 2- *Continuous payments* are *based on the average value of the amount of carbon sequestered*

Alternatively, we may consider that the payments due to the forest owner are also based on the average value of the amount of carbon sequestered by the forest during a rotation period, but are undertaken in a continuous basis. In this case, the present value of the carbon sequestration benefits in one rotation period is given by:

$$S(T^a) = \int_0^{T^a} P_c\beta(\frac{\int_0^{T^a} f(s)ds}{T^{a2}})e^{-rt}dt \tag{73}$$

and the corresponding first-order condition is:

$$\frac{(Pf'(T^a)}{Pf(T^a)} = \frac{r}{(1-e^{-rT})} - \frac{P_c\beta\left[\int_0^{T^a} \frac{(f(T^a)}{T^{a2}} e^{-r(t-T^a)}dt - \int_0^{T^a} (\frac{2\int_0^{T^a} f(s)ds}{T^{a3}})e^{-r(t-T^a)}dt\right]}{Pf(T^a)} -$$

$$-\frac{P_c\beta\left[\frac{\int_0^{T^a} f(s)ds}{T^{a2}} - \frac{r}{(1-e^{-rT})}\int_0^{T^a} \frac{\int_0^{T^a} f(s)ds}{T^{a2}} e^{-rt}dt\right]}{Pf(T^a)} = 0 \tag{74}$$

Again, the sign of the additional term is undeterminate.

In this case the maximum value of a unit of bare land is given by:

$$V(T^a) = Pf(T^a)\frac{e^{-rT^a}}{1-e^{-rT^a}} + \int_0^{T^a} P_c\beta(\frac{\int_0^{T^a} f(s)ds}{T^{a2}})e^{-rt}dt\frac{1}{1-e^{-rT^a}} \tag{75}$$

$$= Pf(T^a)\frac{e^{-rT^a}}{1-e^{-rT^a}} + \frac{P_c\beta(\frac{\int_0^{T^a} f(s)ds}{T^{a2}})}{r} \tag{76}$$

The optimal rotation period is changed but the sign is undeterminate. The value of land increases due to the accounted carbon benefits, the increase is equal to a perpetuity equal to the value of average optimal storage.

Case 3-*Constant payment at harvest*

Another case consists of considering that the payments are due at harvest, but based on a average amount of carbon that is assumed to be sequestered during a rotation period, $k = \left(\frac{\int_0^{T^e} f(s)ds}{T^e}\right)$. Here:

$$S(T^a) = e^{-rT^a} P_c\beta k$$

The corresponding first-order condition is given by

$$\frac{f'(T^a)}{f(T^a)} = \frac{r}{1-e^{-rT^a}} - \frac{P_c\beta k}{Pf(T^a)} + \frac{rT^a}{1-e^{-rT^a}}\frac{P_c\beta k}{Pf(T^a)} \tag{77}$$

As $\frac{r}{1-e^{-rT^a}} > \frac{1}{T^a}$ by inspection, we conclude the additional terms are negative and that $T^a < T$, i.e. optimal rotation period decreases.

In this case the maximum value of a unit of bare land is given by:

$$V(T^a) = Pf(T^a)\frac{e^{-rT^a}}{1-e^{-rT^a}} + (P_c\beta k)\frac{e^{-rT^a}}{1-e^{-rT^a}} \tag{78}$$

The optimal rotation period is shortened. The value of land increases due to the accounted carbon benefits.

Case 4- *Continuous constant payments*

Finally, if the payments due to the forest owner are either a continuous lump-sum value or a perpetuity, represented by:

$$S(T^a) = \int_0^{T^a} P_c\beta k e^{-rs} ds$$

the corresponding first-order condition is given by:

$$\frac{f'(T^a)}{f(T^a)} = \frac{r}{1-e^{-rT^a}} \tag{79}$$

As, $T^a = T$, the equivalence holds.

In this case the maximum value of a unit of bare land is given by:

$$V(T^a) = Pf(T^a)\frac{e^{-rT^a}}{1-e^{-rT^a}} + \int_0^{\infty} P_c\beta k e^{-rt} dt = Pf(T^a)\frac{e^{-rT^a}}{1-e^{-rT^a}} + \frac{P_c\beta k}{r} \tag{80}$$

The optimal rotation period is the same as without carbon benefits. The value of land increases by the value of the perpetuity.

5.2 Discrete time

Case1-*Payment at harvest are equal to value of the average amount of carbon sequestered by the forest during a rotation period*

The first case considered assumes that the forest owner is payed at harvest time the average value of the amount of carbon sequestered by the forest during a rotation period. This average is calculated using a constant weight, which corresponds to the average amount of the carbon stock stored, which is applied to every class s. Thus, S_s can be defined as follows:

$$S_s = (P_c\beta\frac{\sum_{i=1}^{s-1} f_i}{s}) \tag{81}$$

Let us define $V_s(f_s)$ for S_s given by (81):

$$V_s^a(f_s) = Max\left\{Pf_s + P_c\beta\frac{\sum_{i=1}^{s-1} f_i}{s} + V_0^a(f_0); b(Pf_{s+1} + P_c\beta\frac{\sum_{i=1}^{s} f_i}{s+1} + V_0^a(f_0))\right\} \tag{82}$$

Let m^a represent the optimal harvest time. In this case, the sufficient conditions for optimality are

$$Pf_{m^a} + P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^a} + V_0^a(f_0) \geq b(Pf_{m^a+1} + P_c\beta\frac{\sum_{i=1}^{m^a} f_i}{m^a+1} + V_0^a(f_0)) \tag{83}$$

and

$$Pf_{m^a-1} + P_c\beta\frac{\sum_{i=1}^{m^a-2} f_i}{m^a-1} + V_0^a(f_0) \leq b(Pf_{m^a} + P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^a} + V_0^a(f_0)) \tag{84}$$

Again, if m^a is unique, it is optimal to delay cutting for all age trees $s < m^a$. Therefore, the maximum value of a unit of bare land is given by:

$$V_0^a(f_0) = Max\left\{V_0^a(f_0); b^{m^a}(Pf_{m^a} + P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^a} + V_0^a(f_0))\right\}$$

$$= \frac{Pb^{m^a}f_{m^a}}{1-b^{m^a}} + \frac{b^{m^a}P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^a}}{1-b^{m^a}} \tag{85}$$

Taking equation (85), substituting in (83) and (84), and rearranging terms, we have the following sufficient conditions:

$$Pf_{m^a} \geq Pf_{m^a+1}\frac{b-b^{m^a+1}}{1-b^{m^a+1}} + (P_c\beta\frac{\sum_{i=1}^{m^a} f_i}{m^a+1})\frac{b-b^{m^a+1}}{1-b^{m^a+1}} - P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^a} \tag{86}$$

and

$$Pf_{m^a} \geq Pf_{m^a-1}\frac{1-b^{m^a}}{b-b^{m^a}} + (P_c\beta\frac{\sum_{i=1}^{m^a-2} f_i}{m^a-1})\frac{1-b^{m^a}}{b-b^{m^a}} - P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^a} \quad (87)$$

The sign of the additional terms in (86) and (87) is undeterminate. Therefore, in this case it is not possible to determine if the rotation period increases or decreases. Alternatively, by adding and rearranging equations (83) and (84) we obtain the following necessary condition:

$$\frac{Pf_{m^a} - Pf_{m^a-1}}{Pf_{m^a+1} - Pf_{m^a}} \geq b + \frac{b(P_c\beta\frac{\sum_{i=1}^{m^a} f_i}{m^a+1} - P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^a}) + (P_c\beta\frac{\sum_{i=1}^{m^a-2} f_i}{m^a-1} - P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^a}))}{Pf_{m^a+1} - Pf_{m^a}} \quad (88)$$

Again, the sign of the additional term on the right is undeterminate and it is not possible to determine whether rotation period is increased or decreased.

The optimal rotation period may change but the sign of the change is undeterminate. The value of land increases due to the accounted carbon benefits.

Case 2- *Yearly payments are based on the average value of the amount of carbon sequestered until harvest*

Alternatively, we may consider that the payments due to the forest owner are still based on the average value of the amount of carbon sequestered by the forest during a rotation period, but are undertaken in a yearly basis. In this case, S_s can be defined as follows:

$$S_s = \sum_{i=1}^{s} b^{i-s}(P_c\beta\frac{\sum_{i=1}^{s-1} f_i}{s^2}) \quad (89)$$

Let us now define $V_s(f_s)$ for S_s given by (89), as follows:

$$V_s^a(f_s) = Max\left\{\begin{array}{c} Pf_s + \sum_{i=1}^{s} b^{i-s}(P_c\beta\frac{\sum_{i=1}^{s-1} f_i}{s^2}) + V_0^a(f_0); \\ b(Pf_{s+1} + \sum_{i=1}^{s+1} b^{i-1-s}(P_c\beta\frac{\sum_{i=1}^{s} f_i}{s+1}) + V_0^a(f_0)) \end{array}\right\} \quad (90)$$

Let m^a be the optimal rotation period. Therefore, in this case, we have:

$$Pf_{m^a} + \sum_{i=1}^{m^a} b^{i-m^a}(P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{(m^a)^2}) \geq b(Pf_{m^a+1} + \sum_{i=1}^{m^a+1} b^{i-1-m^a}(P_c\beta\frac{\sum_{i=1}^{m^a} f_i}{(m^a+1)^2})) - (1-b)V_0^a(f_0) \quad (91)$$

$$Pf_{m^a-1} + \sum_{i=1}^{m^a-1} b^{i-m^a+1}(P_c\beta\frac{\sum_{i=1}^{m^a-2} f_i}{(m^a-1)^2}) \leq b(Pf_{m^a} + \sum_{i=1}^{m^a} b^{i-m^a}(P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{(m^a)^2})) - (1-b)V_0^a(f_0) \quad (92)$$

Independently of the value of m^a, it is optimal to delay cutting for all age trees $s < m^a$. Therefore, the maximum value of a unit of bare land is given by:

$$V_0^a(f_0) = Max\left\{V_0^a(f_0); b^{m^a}(Pf_{m^a} + \sum_{i=1}^{m^a} b^{i-m^a}(P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{(m^a)^2}) + V_0^a(f_0))\right\}$$

$$= \frac{b^{m^a}Pf_{m^a}}{1-b^{m^a}} + \frac{\sum_{i=1}^{m^a} b^i(P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^{a2}})}{1-b^{m^a}} =$$

$$= \frac{b^{m^a}Pf_{m^a}}{1-b^{m^a}} + \frac{b(P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^{a2}})}{1-b} \quad (93)$$

Taking equation (93), substituting in (91) and (92) and rearranging terms, we have the sufficient conditions:

$$Pf_{m^a} \geq Pf_{m^a+1}\frac{b-b^{m+1}}{1-b^{m+1}} + \sum_{i=1}^{m+1} b^{i-1-m}(P_c\beta\frac{\sum_{i=1}^{m} f_i}{(m+1)^2})) + \frac{b-b^{-m}}{1-b^m}\sum_{i=1}^{m^a} b^i(P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^{a2}}) \quad (94)$$

$$Pf_{m^a} \geq Pf_{m^a-1}\frac{1-b^m}{b-b^m} + \sum_{i=1}^{m-1} b^{i-m+1}(P_c\beta\frac{\sum_{i=1}^{m-2} f_i}{(m-1)^2}) - \frac{1-b^{-m+1}}{1-b^m}\sum_{i=1}^{m^a} b^i(P_c\beta\frac{\sum_{i=1}^{m^a-1} f_i}{m^{a2}}) \quad (95)$$

The sign of the additional terms in (94) and (95) is undeterminate.Ttherefore, in this case, it is not possible to establish if rotation period is increased or decreased.

The optimal rotation period may change but the sign is undeterminate. The value of land increases due to the accounted carbon benefits the increase is equal to the perpetuity equal to the value of average optimal storage.

Case 3- *Constant payment at harvest time*

Finally, we may consider that, instead, the estimated average (or a constant amount) is due at harvest time. In this case, the carbon sequestration benefits are given by:

$$S_s = P_c \beta k \tag{96}$$

Let us now define $V_s(f_s)$ for S_s given by (96)

$$V_s^a(f_s) = Max\left\{Pf_s + P_c\beta k + V_0^a(f_0); b(Pf_{s+1} + P_c\beta k + V_0^a(f_0))\right\} \tag{97}$$

Let m^a be the optimal rotation period in this case. Therefore, we have the following sufficient conditions

$$Pf_{m^a} + P_c\beta k + V_0^a(f_0) \geq b(Pf_{m^a+1} + P_c\beta k + V_0^a(f_0)) \tag{98}$$

and

$$Pf_{m^a-1} + P_c\beta k + V_0^a(f_0) \leq b(Pf_{m^a} + P_c\beta k + V_0^a(f_0)) \tag{99}$$

Again, if it is optimal to cut at m^a, it is optimal to delay cutting for all age trees $s < m^a$. Therefore, the maximum value of a unit of bare land is given by:

$$V_0^a(f_0) = Max\left\{V_0^a(f_0); b^{m^a}(Pf_{m^a} + P_c\beta k + V_0^a(f_0))\right\} =$$

$$= \frac{b^{m^a} Pf_{m^a}}{1 - b^{m^a}} + \frac{b^{m^a} P_c\beta k}{1 - b^{m^a}} \tag{100}$$

Taking equation (100), and substituting in (98) and (99) and rearranging terms, we have:

$$Pf_{m^a} \geq Pf_{m^a+1}\frac{b - b^{m^a+1}}{1 - b^{m^a+1}} - P_c\beta k\frac{1 - b}{1 - b^{m^a+1}}) \tag{101}$$

and

$$Pf_{m^a} \geq Pf_{m^a-1}\frac{1 - b^{m^a}}{b - b^{m^a}} + P_c\beta k\frac{1 - b}{b - b^{m^a}} \tag{102}$$

By inspection, we may conclude that $m^a < m$. However, by adding and rearranging (98) and (99), we obtain (10), that is the same necessary condition as without carbon benefits.

The optimal cutting time may decrease comparing with the forest without carbon benefits . If $k = \sum_{i=1}^{m} b^{i-m}(\frac{\sum_{s=1}^{m-1} f_s}{m^2})$, the results are equivalent to Case 4 and to the multi-vintage steady state solution (110).

Case 4 - *Yearly payment of a constant average estimate of sequestered carbon*

Another case consists of considering that the payments are only due at harvest, but are based on the amount of carbon that is estimated to be sequestered during a rotation period, k. In this case, the carbon sequestration benefits are given by:

$$S_s = P_c \beta \sum_{i=1}^{s} b^{i-s} k \tag{103}$$

Let us now define $V_s(f_s)$ for S_s given by (103):

$$V_s^a(f_s) = Max \left\{ Pf_s + P_c \beta k \sum_{i=1}^{s} b^{i-s} + V_0^a(f_0); b(Pf_{s+1} + P_c \beta k \sum_{i=1}^{s+1} b^{i-s-1} + V_0^a(f_0)) \right\} \tag{104}$$

Let m^a represent the optimal harvest time. In this case, we have

$$Pf_{m^a} + V_0^a(f_0) \geq b(Pf_{m^a+1} + P_c \beta k + V_0^a(f_0)) \tag{105}$$

and

$$Pf_{m^a-1} + V_0^a(f_0) \leq b(Pf_{m^a} + P_c \beta k + V_0^a(f_0)) \tag{106}$$

Again, if it is optimal to cut at m^a, it is optimal to delay cutting for all age trees $s < m^a$. Therefore, the maximum value of a unit of bare land is given by:

$$V_0^a(f_0) = \{Max V_0^a(f_0); b^{m^a}(Pf_{m^a} + P_c \beta k \sum_{i=1}^{m^a} b^{i-m^a} + V_0^a(f_0)\} =$$

$$= \frac{b^{m^a} P f_{m^a}}{1 - b^{m^a}} + \frac{b P_c \beta k \sum_{i=0}^{m^a - 1} b^i}{1 - b^{m^a}} =$$

$$= \frac{b^{m^a} P f_{m^a}}{1 - b^{m^a}} + \frac{b P_c \beta k}{1 - b} \tag{107}$$

By rearranging (105) and (106), we obtain conditions (8) and (9) that, together, are sufficient conditions for optimality in the case without carbon, and $m^a = m$.

The optimal rotation period is Faustmann rotation. The value of land increases due to the accounted carbon benefits. The increase is equal to a perpetuity equal to the constant payment. If $k = \frac{\sum_{s=1}^{m-1} f_s}{m^2}$, (107) is the same as (110).

5.3 Multi-vintage model

In this case, S_t can be defined as follows:

$$S_t = P_c \beta \frac{\sum_{s=1}^{m-1} f_s}{m^2} \sum_{s=1}^{n} x_{s,t} \tag{108}$$

Assuming again that m is unique, for a stationary state, we have that $p_{s,t} = p_{s,\infty}$, $c_t = c_\infty$, $y_t = y_\infty$, $\lambda_t = 0$, and $x_{m,t} = x_\infty$, where c_∞, y_∞, x_∞, and $p_{s,\infty}$, for $s = 1, ..., n-1$, are constant. From [9]:

$$p_s = W'(y_\infty) \sum_{j=0}^{s-1} b^{-j} - f_s U'(C_\infty) - \sum_{j=0}^{s-1} b^{-j} D \tag{109}$$

for $s = 1, ..., n,$,

With some more algebra, we can write for $s = m$, given $p_{m,\infty} = 0$

$$W'(y_\infty) \frac{b}{1-b} - \frac{b^m f_m}{1 - b^m} U'(\frac{(1 - y_\infty) f_m}{m}) - \frac{b}{1-b} D = 0 \tag{110}$$

The rotation period and the value of a unit of bare forest land in the long run optimal stationary steady state of a forest vintage model are equivalent to both the discrete and continuous single-stand rotation model when payments are made on a yearly basis. The optimal rotation period may decrease when carbon benefits are accounted for.

6 Conclusion

Typically, the one stand model setup consists of a multiple rotation model *à la* Faustmann, in continuous time. In this note, we introduce carbon sequestration benefits and solve both for the continuous and the corresponding discrete time problems, and compare the results also with the multi-vintage case.

In general, we conclude that the results are very sensitive to the carbon accounting method chosen as well as to the payment schedule used.

In the carbon flow accounting method, two cases are considered. Either payments of carbon sequestration benefits are undertaken in a yearly basis or only at harvest time. In both cases, the equivalence of results can be established in the sense defined before, that is, both in what concerns the optimal rotation period and the value of bare land in the steady-state. In order to make it comparable with the typical continuous version of the one stand model, we assume that the benefits from carbon sequestration are payed in an yearly basis, but the discount is applied to the discrete change in the production value of timber at each time period. In this case, no equivalence result holds, and it is always optimal to postpone harvest. The cost of carbon release is what is driving this result, and the larger the costs (lower θ) the greater the incentive to cut later.

In the case of the tonne-year crediting regime, the optimal rotation period is postponed in all cases.

Finally, in the average carbon storage method, different alternatives are considered. Either the payments are undertaken in a yearly basis or at harvest time. In both, we consider two cases: the payment is based on the effective sequestered carbon average value, or it is based on an estimated average of the sequestered carbon. In this last case, the equivalence results holds for the cases of continuous payments.

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