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Abstract

We show that three location models on the Salop circle, involving linear or quadratic transport cost, and asymmetric locations or fixed benefits, are equivalent: they lead to the same demand functions and consumer surplus. The only exception is the case of asymmetric locations with an even number of firms, which has one less degree of freedom. These models are also fully equivalent to a normative representative consumer whose indirect utility is given by the standard Salop consumer surplus. This result provides a further unification of location and representative consumer models.

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1 Introduction

In this paper we compare three types of circular-city Salop models, all of which are generalizations of the original model of Salop (1979). These models contain a different mix of assumptions about transport cost, locations and fixed benefits:

- linear transport cost, symmetric locations, and asymmetric fixed benefit to consumers;
- quadratic transport cost, symmetric locations, and asymmetric fixed benefit to consumers;
- linear transport cost, asymmetric locations, and symmetric fixed benefit to consumers;[1]

It is well-known that the first two models, with linear and quadratic transport costs, both lead to linear demand functions. Thus they are equivalent from this restricted (positive) point of view. But are the implied expressions for consumer surplus comparable in any way? In other words, while the models are equivalent at a descriptive level, are they also equivalent at a normative level? This seems rather unlikely, given that functional forms of transport costs are so different.

On the other hand, it seems at first sight that a model with asymmetric locations leads to demand functions and consumer surplus that are again rather different from those in a model with symmetric locations and asymmetric benefits. Thus the latter model appears to be both descriptively and normatively different.

The first purpose of this paper is to show that these three Salop models are indeed both descriptively and normatively equivalent for a given set of locations. We derive a common expression for consumer surplus and show how these models map bijectively into each other. There is one exception, though: For an even number of firms, the model with asymmetric locations has one degree of freedom less than the other models discussed here.

The main practical lesson one can draw from these equivalences is that the informational content of these three models is the same (up to the caveat just mentioned); and that one can use the formulation which responds better to one's modeling aims. In particular, some formulation may lead to simpler expressions depending on the context in which the model is applied. What will differ, though are the comparative statics with respect to fundamental

[1] With some more notation, the latter case can be extended to asymmetric fixed benefits, too. Our point in the paper is to show how fixed benefits and locations map into each.

market parameters such as benefits or locations: Changing benefits of one single firm, for example, maps into global changes in locations, and vice versa.

On a somewhat deeper level, our result is a useful step towards unifying the "zoo" of available models for applied work. It is often a concern that the results of the modeling exercise might not be robust to how exactly the underlying market model is specified. Here we show when the choice between the three models discussed in this paper does not matter.

The second main point of the paper is that the standard Salop consumer surplus, made up of consumption utility minus "transport cost", represents a valid (quasi-linear) indirect utility function of a representative consumer. Thus, following the concepts discussed in Mas-Colell et al. (1995, ch. 4), consumer surplus portrays a *normative representative* consumer: Not only can aggregate demand be derived from maximizing his utility (this would be a *positive representative consumer*), but this single consumer's surplus can be used for welfare evaluation.[2] Thus apart from having shown that the above three Salop location models are equivalent to each other, they are also all fully equivalent to a specific representative consumer model, providing a further unification of location and representative consumer models.

Anderson et al. (1992, ch. 5) provide a random utility formulation for the Hotelling duopoly model and derive a (direct) utility function from which the standard Hotelling demand can be determined. That is, they construct a positive representative consumer for the Hotelling model. Still, they make no attempt to show that this utility is in any way related to the standard consumer surplus in the Hotelling model, nor that their utility can lead to a valid representation of aggregate welfare. In our paper we show that for the Salop model the standard consumer surplus contains such a valid representation. It would be a simple exercise to show the corresponding result for the Hotelling model following the procedure in our paper.

As a last step we also point out that the Salop model with asymmetric locations and quadratic transport cost is not equivalent to the three models discussed before. The main reason for this is that the quadratic transport cost formulation makes the asymmetric location parameters appear in the slopes of the demand function, so that every firm potentially has a different slope of demand for each of its own and neighbors' prices. This is ruled out in the above models, and thus indicates that their equivalence is not at all obvious.

[2] The latter holds true because a quasi-linear indirect utility is automatically of the Gorman form, which allows this aggregation without having to worry about how welfare of individual consumers is weighted.

2 Example: The Hotelling Model

The Hotelling model (Hotelling, 1929) and its reformulation by d'Aspremont et al. (1979) are widely used building blocks for applied work. Two firms are located at the opposite ends of line of length 1, over which a mass 1 of consumers is uniformly distributed. The location $x \in [0,1]$ of each consumer indicates his ideal variety, consumption of which yields him a utility of v. If he buys from firm 1 at location 0, or firm 2 at location 1, then he suffers a disutility ("linear transport cost") of tx, or $t(1-x)$, respectively, where $t > 0$. D'Aspremont et al. changed this formulation to a quadratic transport tx^2, or $t(1-x)^2$.[3] Given prices p_1 and p_2 for the firms' goods, the market will divide at the indifferent customers y and $\tilde{y}$ given by (utility is assumed to be quasi-linear in money)

$$\begin{aligned} v - p_1 - ty &= v - p_2 - t(1-y) \Longleftrightarrow y = \frac{1}{2} + \frac{p_2 - p_1}{2t} \\ v - p_1 - t\tilde{y}^2 &= v - p_2 - t(1-\tilde{y})^2 \Longleftrightarrow \tilde{y} = \frac{1}{2} + \frac{p_2 - p_1}{2t}. \end{aligned}$$

Thus from a descriptive perspective both models are identical: They lead to the same demand functions. Let us compare consumer surplus:

$$\begin{aligned} CS &= v - yp_1 - (1-y)p_2 - \int_0^y tx\,dx - \int_0^{1-y} tx\,dx, \\ \widetilde{CS} &= v - \tilde{y}p_1 - (1-\tilde{y})p_2 - \int_0^{\tilde{y}} tx^2dx - \int_0^{1-\tilde{y}} tx^2dx. \end{aligned}$$

These terms differ in transport cost. Surely, CS will be quadratic in prices, and $\widetilde{CS}$ of third order? Actually, after some simplifications we obtain

$$\begin{aligned} CS &= v - \frac{p_1 + p_2}{2} + \frac{(p_1 - p_2)^2}{4t} - \frac{1}{4}t, \\ \widetilde{CS} &= v - \frac{p_1 + p_2}{2} + \frac{(p_1 - p_2)^2}{4t} - \frac{1}{12}t, \end{aligned}$$

i.e. the two consumer surplus measures only differ in a constant. Moreover, this consumer surplus is a valid indirect utility function for a representative consumer, as can be shown by applying Roy's identity:

$$-\frac{\partial CS}{\partial p_1} = -\frac{\partial \widetilde{CS}}{\partial p_1} = \frac{1}{2} + \frac{p_2 - p_1}{2t} = y = \tilde{y}.$$

[3] They did this to avoid discontinuities in demand that occur when firms locate closer to each other. This issue will play no role in the present paper.

We have thus shown the following: For fixed locations at the extremes of the Hotelling lines, the Hotelling models with linear and quadratic transport costs are equivalent to each other, both from a positive (demand) and normative (consumer surplus) point of view. Furthermore, the standard expressions for consumer surplus in these discrete choice models can be interpreted equivalently as the indirect utility of a representative consumer. Below we follow the same procedure for the different variants of the Salop model mentioned above.

3 Three Equivalent Salop Models, and a Black Sheep

In the following we derive the demand function and corresponding consumer surplus for each of the three models. We show that this consumer surplus can be interpreted as providing the indirect utility (and thus the preferences) of a representative consumer. All longer proofs have been relegated to an Appendix.

3.1 Linear Transport Cost and Asymmetric Benefits

We first present the Salop model with symmetric locations and asymmetric benefits because it is the simplest to deal with.

A set of $n > 0$ firms $k \in \{1, ..., n\}$ are located symmetrically at locations $(k-1)/n$ on a circle of circumference 1. For notational convenience, we identify firms $k \in \mathbb{Z}$ outside this range with firm $((k-1) \bmod n) + 1$, e.g. firms 0 and $n+1$ are identified with firms n and 1, respectively.

We also assume that a mass 1 of consumers is located uniformly around the circle. Each consumer's location describes his ideal good, and he suffers a disutility or "transport cost" td from buying the good at a firm at distance $d \geq 0$ along the circle, where the parameter $t > 0$ measures the strength of preferences.

A consumer buying from firm k obtains surplus (before transport cost) of

$$w_k = \beta_k - p_k,$$

where $\beta_k > 0$ is an idiosyncratic benefit from consuming firm k's good, and p_k is firm k's price.

Here and in the following we will assume that prices p_k are low enough as compared to fixed surplus β_k so that all consumers are willing to buy from some firm. In applications this will usually be guaranteed by enough competition between neighboring firms. Furthermore, we assume that asymmetries are weak enough so that no firm is excluded from the market; this implies that each firm competes directly only with its two neighbors.[4]

In order to find firm k's demand, we follow the standard steps of first deriving the locations of the consumers that are indifferent between its offer and that of its neighbors, and then determining the mass of consumers between these indifferent consumers. Denote by x_k the location of the indifferent consumer between firms k and $k+1$ relative to the location of firm k, i.e. $x_k \in [0, 1/n]$. Then his location is given by the indifference condition

$$w_k - tx_k = w_{k+1} - t\left(\frac{1}{n} - x_k\right),$$

and consequently

$$x_k = \frac{1}{2n} + \frac{w_k - w_{k+1}}{2t}.$$

The demand q_k of firm k is given by a mass of x_k consumers on its right and $\left(\frac{1}{n} - x_{k-1}\right)$ on its left, or

$$q_k = x_k + \left(\frac{1}{n} - x_{k-1}\right) = \frac{1}{n} + \frac{2w_k - w_{k+1} - w_{k-1}}{2t} \tag{1}$$

The consumer surplus in the Salop model is given by the sum of consumption benefits minus aggregate transport cost, as

$$CS = \sum_{k=1}^{n}\left(w_k q_k - \int_0^{x_k} txdx - \int_0^{1/n - x_{k-1}} txdx\right) \tag{2}$$

We now restate this consumer surplus in a rather simpler form:

Proposition 1 *Consumer surplus in the n-firm Salop model with linear transport cost, symmetric locations and asymmetric fixed benefits can be stated as*

$$CS = \sum_{k=1}^{n}\left(\frac{s_k}{n} + \frac{(s_k - s_{k+1})^2}{4t}\right),$$

where

$$s_k = \beta_k - p_k - \frac{t}{4n}.$$

[4] See Hoernig (2014) for a generalized Hotelling model where each firms directly competes with all other firms in the market.

Interpreting this consumer surplus as a quasi-linear indirect utility function of a representative consumer, as in (let $p = (p_1, ..., p_n)$ be the vector of prices)

$$v(p, w) = w + CS,$$

we can apply Roy's Lemma:

$$\begin{aligned} q_k &= -\frac{\partial v(p,w)}{\partial p_k} \Big/ \frac{\partial v(p,w)}{\partial w} = -\frac{\partial CS}{\partial p_k} \\ &= -\frac{\partial}{\partial p_k}\left(\frac{s_k}{n} + \frac{(s_k - s_{k+1})^2}{4t} + \frac{(s_k - s_{k-1})^2}{4t}\right) \\ &= \frac{1}{n} + \frac{2w_k - w_{k+1} - w_{k-1}}{2t} \end{aligned}$$

This faithfully reproduces the demand function (1). Since an indirect utility function contains the same information about a consumer as a direct utility function over consumption bundles, consumer surplus (2) represents the preferences of a *normative* representative consumer, as defined by Mas-Colell et al. (1995)

3.2 Quadratic Transport Costs and Asymmetric Benefits

Now we change the definition of transport from linear (td) to quadratic, τd^2 with $\tau > 0$. As we will see immediately, for the purpose of comparison it is very useful to write the transport cost parameter as $\tau = nt$. Otherwise the model remains identical to the above linear specification. The indifferent consumer between firms k and $k+1$ is given by

$$w_k - \tau x_k^2 = w_{k+1} - \tau (1/n - x_k)^2,$$

i.e.

$$x_k = \frac{1}{2n} + \frac{n(w_k - w_{k+1})}{2\tau} = \frac{1}{2n} + \frac{w_k - w_{k+1}}{2t}.$$

As above, firm k's demand is determined as

$$q_k = x_k + \left(\frac{1}{n} - x_{k-1}\right) = \frac{1}{n} + \frac{2w_k - w_{k+1} - w_{k-1}}{2t}.$$

We see that the expressions for the indifferent consumers and demands are identical to those we found above.

The definition of consumer surplus now must take into account that transport costs are quadratic:

$$CS = \sum_{k=1}^{n} \left(w_k q_k - \int_0^{x_k} \tau x^2 dx - \int_0^{1/n - x_{k-1}} \tau x^2 dx \right)$$

This expression contains cubic terms and therefore it seems at first sight that it cannot possibly coincide with consumer surplus (2) in the linear case. We show that it actually only differs by a constant:[5]

Proposition 2 *Consumer surplus in the n-firm Salop model with quadratic transport cost, symmetric locations and asymmetric fixed benefits can be stated as*

$$CS = \sum_{k=1}^{n} \left(\frac{\tilde{s}_k}{n} + \frac{(\tilde{s}_k - \tilde{s}_{k+1})^2}{4t} \right),$$

where

$$\tilde{s}_k = \beta_k - p_k - \frac{t}{12n}.$$

Again Roy's lemma applies:[6]

$$\begin{aligned} -\frac{\partial CS}{\partial p_k} &= \frac{1}{n} + \frac{\tilde{s}_k - \tilde{s}_{k+1}}{2t} + \frac{\tilde{s}_k - \tilde{s}_{k-1}}{4t} \\ &= \frac{1}{n} + \frac{2w_k - w_{k+1} - w_{k-1}}{2t} = q_k. \end{aligned}$$

Thus, contrary to what one might expect, the Salop models with linear and quadratic transport costs are perfectly equivalent to each other. The only necessary change is a shift in fixed consumer surplus so that

$$\tilde{s}_k = s_k + \left(\frac{t}{4n} - \frac{t}{12n} \right) = s_k + \frac{t}{6n},$$

implying the same constant difference in aggregate consumer surplus of $t/6n$.

[5] One can actually show that this equivalence holds in similar form for higher powers in the transport cost function: Consumer surplus for transport costs td^m and τd^{m+1} are equivalent whenever m is an odd integer (i.e., the terms of power $m+1$ cancel).

[6] Now that we know that consumer surplus also in this case is just a quadratic and not a cubic function of individual consumers' surplus, this is to be expected.

3.3 Linear Transport Cost and Asymmetric Locations

In this section we set out the Salop model with asymmetric locations. In order to focus on how location asymmetry maps into asymmetric benefits under symmetric locations, we assume that consumers' fixed benefits are symmetric.

Firm k is located at $(k-1)/n + \delta_k$, where the $|\delta_k|$ are small enough so that the order of firms on the circle does not change and that consumers still choose between their two neighboring firms. The individual consumer's surplus from choosing firm k is given by $v_k = v - p_k$. We define the indifferent consumer's location as relative to $(k-1)/n$ as above, so that this time it is given by

$$v_k - t(x_k - \delta_k) = v_{k+1} - t\left(\frac{1}{n} + \delta_{k+1} - x_k\right),$$

or

$$x_k = \frac{1}{2n} + \frac{\delta_k + \delta_{k+1}}{2} + \frac{v_k - v_{k+1}}{2t}.$$

The demand of firm k is given by

$$q_k = x_k + \left(\frac{1}{n} - x_{k-1}\right) = \frac{1}{n} + \frac{\delta_{k+1} - \delta_{k-1}}{2} + \frac{2v_k - v_{k-1} - v_{k+1}}{2t}.$$

Both expressions indicate that immediately mapping them into (1) is more difficult than in the previous case.

Consumer surplus is again defined as the sum of benefits minus transport costs, taking into account now the specific locations of individual firms:

$$CS = \sum_{k=1}^{n}\left(v_k q_k - \int_{\delta_k}^{x_k} t(x - \delta_k)\, dx - \int_{-\delta_k}^{1/n - x_{k-1}} t(x + \delta_k)\, dx\right).$$

In the Appendix, we show the following:

Proposition 3 *Consumer surplus in the n-firm Salop model with linear transport cost, asymmetric locations and symmetric fixed benefits can be stated as*

$$CS = \sum_{k=1}^{n}\left(\frac{\hat{s}_k}{n} + \frac{(\hat{s}_k - \hat{s}_{k+1})^2}{4t}\right),$$

where

$$\hat{s}_k = v + \gamma_k - p_k - \frac{t}{4n},$$

and ($\bar{\delta} \equiv \frac{1}{n}\sum_{j=1}^{n}\delta_j$)

$$\gamma_k \equiv t\left(\sum_{j=1}^{n-1}\frac{n-2j}{n}t\delta_{k+j} - \sum_{j=1}^{n}\left(\delta_j - \bar{\delta}\right)^2\right)$$

satisfies $2\gamma_k - \gamma_{k+1} - \gamma_{k-1} = t\left(\delta_{k+1} - \delta_{k-1}\right)$.

Thus interpreting the term $v+\gamma_k$ as an asymmetric surplus parameter allows us to map this model with asymmetric locations into to the two previous models with symmetric locations but asymmetric benefits.

We will now show inductively how and when this mapping can be inverted to a mapping of a model of asymmetric benefits to asymmetric locations. First take odd $n \geq 3$, let $\delta_1 = 0$,[7] and $\delta_k = \delta_{k-2} + \left(2\gamma_k - \gamma_{k+1} - \gamma_{k-1}\right)/t$ for all $k = 3, ..., 2n+1$ (intuitively, the δ_k are determined in two rotations around the circle). It is easy to see that this set of indices covers all of $k = 1, ..., n$ once and implies a specific value for δ_{2n+1} which must be equal to δ_1:

$$\delta_{2n+1} = \frac{1}{t}\sum_{i=1}^{n}\left(2\gamma_{2i+1} - \gamma_{2i+2} - \gamma_{2i}\right) = \frac{1}{t}\left(\sum_{i=1}^{n}2\gamma_i - \sum_{i=1}^{n}\gamma_i - \sum_{i=1}^{n}\gamma_i\right) = 0.$$

Thus for odd n the asymmetric location model maps directly back into an asymmetric benefit model.

We will now see that for even $n \geq 2$ the situation is slightly more complicated. A first observation is that since the condition determining the γ_k only involves the term $(\delta_{k+1} - \delta_{k-1})$, the δ_k with even and odd indices are determined independently of each other, since already after one rotation we reach the δ_k we started with. Thus we define $\delta_1 = \delta_2 = 0$,[8] let $\delta_k = \delta_{k-2} + \left(2\gamma_k - \gamma_{k+1} - \gamma_{k-1}\right)/t$ for all $i = 3, ..., n+2$, and in the end need to verify the conditions $\delta_{n+1} = \delta_1$ and $\delta_{n+2} = \delta_2$. We have

$$\delta_{n+1} = \frac{1}{t}\sum_{i=1}^{n/2}\left(2\gamma_{2i+1} - \gamma_{2i+2} - \gamma_{2i}\right) = \frac{2}{t}\left(\sum_{j \text{ odd}}\gamma_j - \sum_{i \text{ even}}\gamma_i\right).$$

Thus for $\delta_{n+1} = \delta_1$ to hold we must have

$$\sum_{j \text{ odd}}\gamma_j - \sum_{i \text{ even}}\gamma_i = 0 \tag{3}$$

[7] This starting value is actually arbitrary and has no effect on demand and consumer surplus.

[8] Again, δ_1 could have any value, and δ_2 could differ from δ_1. This latter difference must be small enough, however, so that consumers still choose between their two neighboring firms.

(the same condition follows from $\delta_{n+2} = \delta_2$). This means that for even n the asymmetric location model can be mapped into the asymmetric benefit model if and only if condition (3) holds.

From the previous discussion follows:

Proposition 4 *For odd $n \geq 3$, the asymmetric location Salop model is equivalent to the asymmetric benefit Salop model, while for even $n \geq 2$ the asymmetric location Salop model is only equivalent to the asymmetric benefit Salop model if condition (3) holds.*

In other words, for even n the asymmetric benefit model has one more degree of freedom and thus allows for slightly less restrictive demand patterns. This is easily shown for $n = 2$: In this case the values of δ_1 and δ_2 have no influence on demand (remember that the index 0 is identified with $n = 2$):

$$q_1 = \frac{1}{n} + \frac{\delta_2 - \delta_0}{2} + \frac{2v_1 - v_0 - v_2}{2t} = \frac{1}{n} + \frac{p_2 - p_1}{t};$$

on the other hand, with asymmetric fixed benefits we obtain

$$q_1 = \frac{1}{n} + \frac{2w_k - w_{k+1} - w_{k-1}}{2t} = \frac{1}{n} + \frac{\beta_1 - \beta_2}{t} + \frac{p_2 - p_1}{2t}.$$

Thus with asymmetric benefits there is scope for one additional asymmetry in market shares.

Our formulation of consumer surplus makes it easy to show that also in the case of asymmetric locations Roy's lemma applies:

$$\begin{aligned} -\frac{\partial CS}{\partial p_k} &= \frac{1}{n} + \frac{2\gamma_k - \gamma_{k+1} - \gamma_{k-1}}{2t} + \frac{2v_k - v_{k+1} - v_{k-1}}{2t} \\ &= \frac{1}{n} + \frac{\delta_{k+1} - \delta_{k-1}}{2} + \frac{2v_k - v_{k+1} - v_{k-1}}{2t} = q_k, \end{aligned}$$

so as before aggregate demand in this model can be derived from the utility-maximizing choice of a representative consumer.

3.4 Non-Equivalence with Quadratic Transport Cost and Asymmetric Locations

Here we point out that if one joins the assumptions of quadratic transport cost and asymmetric locations then one obtains a model that is not equivalent to those discussed above. Thus the equivalence results from above are non-trivial in the sense that it is easy to find a model, using the same assumptions, that is different at a fundamental level.

We adopt the notation of the previous sections, so that the indifferent consumer now is defined by

$$v_k - \tau \left(x_k - \delta_k\right)^2 = v_{k+1} - \tau \left(\frac{1}{n} + \delta_{k+1} - x_k\right)^2,$$

or, with $\tau = nt$,

$$x_k = \frac{1}{2n} + \frac{\delta_{k+1} + \delta_k}{2} + \frac{v_k - v_{k+1}}{2t\left(1 + n\left(\delta_{k+1} - \delta_k\right)\right)}.$$

Demand becomes

$$\begin{aligned} q_k &= x_k + \left(\frac{1}{n} - x_{k-1}\right) = \frac{1}{n} + \frac{\delta_{k+1} + \delta_{k-1}}{2} \\ &+ \frac{v_k - v_{k+1}}{2t\left(1 + n\left(\delta_{k+1} - \delta_k\right)\right)} + \frac{v_k - v_{k-1}}{2t\left(1 + n\left(\delta_k - \delta_{k-1}\right)\right)}. \end{aligned}$$

The latter expression reveals a fundamental difference from the above three models: The slope of demand with respect to the price of firm k and its neighbors depends directly on the location parameters, while in the previous three models these slopes were all constant with absolute value $1/2t$. Thus in the present model demand elasticity depends directly on the relative location of firms.

A variant of this model, with three firms and only one degree of freedom in locations, has appeared in Brito and Pereira (2010). Our treatment shows that this model is not equivalent to formulations of the Salop model with either symmetric locations or linear transport cost, which implies that predictions derived from these models can differ.

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Appendix: Omitted Proofs

Proof of Proposition 1:

In the following we transform the expression (2) of Salop consumer surplus in various steps. First we need to simplify the transport cost terms (remember the naming convention that indices k outside the range $1, ..., n$ are mapped back into it in the obvious manner):

$$\begin{aligned}
CS &= \sum_{k=1}^{n}\left(w_k q_k - \int_0^{x_k} tx dx - \int_0^{1/n - x_{k-1}} tx dx\right) \\
&= \sum_{k=1}^{n}\left(w_k q_k - \frac{t}{2}x_k^2 - \frac{t}{2}\left(\frac{1}{n} - x_{k-1}\right)^2\right) \\
&= \sum_{k=1}^{n}\left(\frac{w_k}{n} + \frac{w_k^2 - w_k w_{k+1}}{t} - \frac{t}{2}\left(\frac{1}{2n} + \frac{w_k - w_{k+1}}{2t}\right)^2\right. \\
&\qquad \left. - \frac{t}{2}\left(\frac{1}{2n} + \frac{w_{k+1} - w_k}{2t}\right)^2\right) \\
&= \sum_{k=1}^{n}\left(\frac{w_k}{n} - \frac{t}{4n^2} + \frac{3}{4t}w_k^2 - \frac{1}{2t}w_k w_{k+1} - \frac{1}{4t}w_{k+1}^2\right) \\
&= \sum_{k=1}^{n}\frac{w_k - t/4n}{n} + \sum_{k=1}^{n}\frac{(w_k - w_{k+1})^2}{4t}
\end{aligned}$$

Note that in the last step we freely used $\sum_{k=1}^{n} w_k^2 = \sum_{k=1}^{n} w_{k+1}^2$. Letting $s_k = w_k - t/4n$ then leads to the result in the Proposition. ■

Proof of Proposition 2:

We now simplify the expressions for consumer surplus under quadratic transport cost, shifting terms containing $k-1$ to k:

$$\begin{aligned}
CS &= \sum_{k=1}^{n}\left(w_k q_k - \int_0^{x_k}\tau x^2 dx - \int_0^{1/n-x_{k-1}}\tau x^2 dx\right)\\
&= \sum_{k=1}^{n}\left(w_k q_k - \frac{nt}{3}x_k^3 - \frac{nt}{3}\left(\frac{1}{n}-x_{k-1}\right)^3\right)\\
&= \sum_{k=1}^{n}\left(\frac{w_k}{n} + \frac{w_k^2 - w_k w_{k+1}}{t} - \frac{nt}{3}\left(\frac{1}{2n}+\frac{w_k - w_{k+1}}{2t}\right)^3\right.\\
&\qquad \left. -\frac{nt}{3}\left(\frac{1}{2n}+\frac{w_{k+1}-w_k}{2t}\right)^3\right)\\
&= \sum_{k=1}^{n}\left(\frac{w_k}{n} - \frac{t}{12n^2} + \frac{3}{4t}w_k^2 - \frac{1}{2t}w_k w_{k+1} - \frac{1}{4t}w_{k+1}^2\right)\\
&= \sum_{k=1}^{n}\frac{w_k - t/12n}{n} + \sum_{k=1}^{n}\frac{(w_k - w_{k+1})^2}{4t}
\end{aligned}$$

Thus defining $\tilde{s}_k = w_k - \frac{t}{12n} = w_k - \frac{\tau}{12n^2}$ leads to the above result. ■

Proof of Proposition 3:

Consumer surplus can be reformulated as, again shifting the last term:

$$\begin{aligned}
CS &= \sum_{k=1}^{n}\left(v_k q_k - \int_{\delta_k}^{x_k} t\,(x-\delta_k)\,dx - \int_{-\delta_k}^{1/n-x_{k-1}} t\,(x+\delta_k)\,dx\right)\\
&= \sum_{k=1}^{n}\left(v_k q_k - \frac{t}{2}(x_k-\delta_k)^2 - \frac{t}{2}\left(\frac{1}{n}-x_k+\delta_{k+1}\right)^2\right)\\
&= \sum_{k=1}^{n}\left(\frac{v_k}{n} + v_k\left(\frac{\delta_{k+1}-\delta_{k-1}}{2}+\frac{2v_k - v_{k-1}-v_{k+1}}{2t}\right)\right.\\
&\qquad -\frac{t}{2}\left(\frac{1}{2n}+\frac{\delta_k+\delta_{k+1}}{2}+\frac{v_k - v_{k+1}}{2t}-\delta_k\right)^2\\
&\qquad \left. -\frac{t}{2}\left(\frac{1}{n}-\left(\frac{1}{2n}+\frac{\delta_k+\delta_{k+1}}{2}+\frac{v_k-v_{k+1}}{2t}\right)+\delta_{k+1}\right)^2\right)\\
&= \sum_{k=1}^{n}\left(\frac{v_k}{n} - \frac{t}{4n^2}\right.\\
&\qquad \left. + \frac{1}{2t}\left(v_k^2 - v_k v_{k+1} + (t\delta_{k+1} - t\delta_{k-1})\,v_k - \frac{1}{2}(t\delta_k - t\delta_{k+1})^2\right)\right)
\end{aligned}$$

Writing $\hat{s}_k = v_k + \gamma_k - \frac{t}{4n}$ for as-yet-to-be determined terms γ_k, we obtain

$$\begin{aligned} CS &= \sum_{k=1}^{n} \left(\frac{\hat{s}_k}{n} + \frac{(\hat{s}_k - \hat{s}_{k+1})^2}{4t} \right) \\ &= \sum_{k=1}^{n} \left(\frac{v_k}{n} - \frac{t}{4n^2} + \frac{\gamma_k}{n} \right. \\ &\quad \left. + \frac{1}{2t} \left(v_k^2 - v_k v_{k+1} + \left(2\gamma_k - \gamma_{k+1} - \gamma_{k-1}\right) v_k + \frac{1}{2} \left(\gamma_k - \gamma_{k+1}\right)^2 \right) \right). \end{aligned}$$

Matching the coefficients on the v_k and the constant, we obtain the conditions

$$\begin{aligned} 2\gamma_k - \gamma_{k+1} - \gamma_{k-1} &= t\delta_{k+1} - t\delta_{k-1} \quad \forall\ k \in \{1, ..., n\}, \\ \sum_{k=1}^{n} \left(\frac{\gamma_k}{n} + \frac{1}{4t} \left(\gamma_k - \gamma_{k+1}\right)^2 \right) &= -\frac{1}{4t} \sum_{k=1}^{n} \left(t\delta_k - t\delta_{k+1}\right)^2 \end{aligned}$$

It is straightforward (though cumbersome) to verify that the solutions to the first set of conditions are given by

$$\gamma_k = \sum_{j=1}^{n-1} \frac{n-2j}{n} t\delta_{k+j} + K + Ck,$$

for some constants $K, C \in \mathbb{R}$.[9] First, the identification of index k with index $k+n$ implies that $C = 0$. We now determine the constant K from the remaining condition. From the expression for γ_k it follows that $\gamma_k - \gamma_{k+1} = t\delta_{k+1} + t\delta_k - 2t\bar{\delta}$, where $\bar{\delta} \equiv \frac{1}{n} \sum_{j=1}^{n} \delta_j$, and

$$\sum_{k=1}^{n} \frac{\gamma_k}{n} = K + \sum_{j=1}^{n-1} \frac{n-2j}{n} t\bar{\delta} = K.$$

Then we obtain, shifting indices from $k+1$ to k in the last step,

$$\begin{aligned} K &= -\frac{1}{4t} \sum_{k=1}^{n} \left(\left(\gamma_k - \gamma_{k+1}\right)^2 + \left(t\delta_k - t\delta_{k+1}\right)^2 \right) \\ &= -t \sum_{k=1}^{n} \left(\delta_k - \bar{\delta}\right)^2. \end{aligned}$$

Thus the final expression for γ_k becomes

$$\gamma_k = t \left(\sum_{j=1}^{n-1} \frac{n-2j}{n} t\delta_{k+j} - \sum_{j=1}^{n} \left(\delta_j - \bar{\delta}\right)^2 \right). \quad \blacksquare$$

[9] The roots of the characteristic equation of this second-degree difference equation are both equal to 1, so this is indeed the general solution.

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