Matroidal frameworks for topological Tutte polynomials

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Abstract

We introduce the notion of a delta-matroid perspective. A delta-matroid perspective consists of a triple (M, D, N), where M and N are matroids and D is a delta-matroid such that there are strong maps from M to the upper matroid of D and from the lower matroid of D to N. We describe two Tutte-like polynomials that are naturally associated with delta-matroid perspectives and determine various properties of them. Furthermore, we show when the delta-matroid perspective is read from a graph in a surface our polynomials coincide with B. Bollobás and O. Riordan's ribbon graph polynomial and the more general Krushkal polynomial of graphs in surfaces. This is analogous to the fact that the Tutte polynomial of a graph G coincides with the Tutte polynomial of its cycle matroid. We use this new framework to prove results about the topological graph polynomials that cannot be realised in the setting of cellularly embedded graphs.

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1. Introduction and overview

It is a standard fact that the Tutte polynomial of a graph can be understood as a matroid polynomial. This more general setting allows access to, and understandings of, properties of the graph version of the polynomial that may have otherwise been obscured. Here we show that two recent extensions of the Tutte polynomial from graphs to graphs embedded in surfaces, namely B. Bollobás and O. Riordan's ribbon graph polynomial $R_G(x, y, z)$ of [4, 5] and V. Krushkal's polynomial $K_{G\subset\Sigma}(x, y, a, b)$ of [23], can also be understood as matroidal polynomials and, as with the classical graphs case, this matroidal framework allows access to new results and properties of the topological graph polynomials.

Before we proceed let us understand why a new framework for topological graph polynomials is needed, that this need is driven by application not abstraction, and understand what is required by such a framework. This need is perhaps best explained by considering the deletion-contraction properties of the Tutte polynomial. The Tutte polynomial $T_M(x, y)$ of matroid (or graph) has a

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deletion-contraction relation

$$T_M(x,y) = \begin{cases} yT_{M\backslash e}(x,y) & \text{if } e \text{ is a loop,} \\ xT_{M/e}(x,y) & \text{if } e \text{ is a coloop,} \\ T_{M\backslash e}(x,y) + T_{M/e}(x,y) & \text{otherwise.} \end{cases}$$

Together with its value of 1 on the trivial matroid $U_{0,0}$ (or edgeless graphs) this deletion-contraction relation uniquely defines the Tutte polynomial. Moreover, this relation is very often the key property of the Tutte polynomial that provides its many applications and interpretations in the literature. Returning to the topological setting, while the Bollobás-Riordan polynomial, R_G, and Krushkal polynomial, $K_{G\subset\Sigma}$, have some deletion-contraction relations (see [5, 23]), these relations reduce the computation of the polynomial to that of an infinite class of one vertex graphs in surfaces. This can be problematic, particularly from the perspective of inductive proofs, as the class of one vertex embedded graphs is complicated. The reason why R_G and $K_{G\subset\Sigma}$ do not have a "full" recursive definition (i.e., one that generates a 1-dimensional skein module) is readily explained. For a cellularly embedded graph (for expositional simplicity, we are restricting the domain of $K_{G\subset\Sigma}$ here) and up to normalisation our three polynomials appear in

$$Z_G(W,X,Y,Z) = \sum_{A\subseteq E(G)} \underbrace{\underbrace{\underbrace{W^{c_G(A)}X^{|A|}}_{\text{Tutte}}Y^{b(A)}}_{\text{Boll.-Riord.}}\,Z^{c_{G^*}(E\backslash A)}}_{\text{Krushkal}}. \tag{1}$$

(See Section 6 for definitions of the exponents.) $Z_G(W,X,1,1)$ corresponds to the Tutte polynomial, $Z_G(W,X,Z,1)$ to the Bollobás-Riordan polynomial, and $Z_G(W,X,Y,Z)$ to the Krushkal polynomial. The difficultly is that the parameters $c_G(A)$, $b(A)$, and $c_{G^*}(E\backslash A)$ each demand an incompatible notion of edge contraction. Thus, from the point of view of deletion and contraction, a framework for topological graph polynomials must accommodate each of the three notions of edge contraction required by these parameters in a single operation, and it must do so in a way that keeps some compatibility between them, as is required by the graph in the surface. Where should we look for such a framework? The Tutte polynomial provides a hint: several results about the Tutte polynomial of a graph (for example, its duality formula or interpretations at $(-1,-1)$) require an embedding of that graph in the plane, and so only hold for planar graphs. However they hold for all matroids without any restriction. This indicates that to bring in the topology, we should look to matroids. Indeed it is in matroids that we find the desired framework.

Our matroid framework arises by combining two different extensions of matroids. The first extension is M. Las Vergnas' matroid perspectives from [26], which consist of a pair of matroids (M,M') with a bijection, called a strong map, between them that preserves some structure. The second extension is A. Bouchet's delta-matroids which were introduced in [6]. While a matroid has bases which are equicardinal sets satisfying a symmetric exchange axiom, a delta-matroid has feasible sets which satisfy a symmetric exchange axiom, but do not all have to be of the same size.

Here we introduce the concept of a delta-matroid perspective. This consists of a triple (M,D,M') where M and M' are matroids, D is a delta-matroid and there are strong maps from M to the upper matroid $D_{\max}$ of D and from the lower matroid $D_{\min}$ of D to M'. (All terms and notation used in this introduction will be defined later in the paper.) We determine the natural deletion-contraction invariant associated with delta-matroid perspectives $K_{(M,D,M')}(x,y,a,b)$, and show that this matroidal polynomial coincides with the Krushkal polynomial $K_{G\subset\Sigma}(x,y,a,b)$ of a graph G in a surface

Σ when $D(G)$ is the delta-matroid of G and $(M, D, M') = (B(G^*), D(G), C(G))$. (In the setting of Equation (1), M controls the exponent of Z , D the exponent of Y, and M' the exponent of W.) Similarly we show that the Bollobás-Riordan polynomial $R_G(x, y, z)$ of a ribbon graph G coincides with the natural deletion-contraction invariant $R_{(D,M)}(x, y, z)$ associated with DM-perspectives, which are the subobjects of delta-matroid perspectives consisting of a pair (D, M) where D is a delta-matroid, M is a matroid, and there is a strong map from the lower matroid $D_{\min}$ of D to M. (By 'natural deletion-contraction invariant' we mean that the polynomial is the canonical Tutte polynomial of a Hopf algebra generated by the combinatorial objects and their natural notions of deletion and contraction, see Section 7 for details.) We use our framework to find recursive definitions, duality formulae, and convolution formulae for the topological graph polynomials that cannot be realised without moving to a more general framework.

This paper is structured as follows. Section 2 summarises the matroid and delta-matroid notation and terminology used in this paper. Section 3 introduces delta-matroid perspectives and determines some of their basic properties. Section 4 describes the graphical analogue of a delta-matroid perspective. In Section 5 we introduce the polynomials $R_{(D,M)}(x, y, z)$ and $K_{(M,D,M')}(x, y, a, b)$ and describe several of their properties. We postpone several proofs of these properties until Section 7 where they follow from a more general Hopf algebraic framework. In Section 6 we show that $R_{(D,M)}(x, y, z)$ and $K_{(M,D,M')}(x, y, a, b)$ coincide with the Bollobás-Riordan and Krushkal polynomials. Finally, in Section 7 we discuss Tutte polynomials of Hopf algebras, showing that $R_{(D,M)}(x, y, z)$ and $K_{(M,D,M')}(x, y, a, b)$ are the canonical Tutte polynomials of DM-perspectives and delta-matroid perspectives, respectively. We use this general Hopf algebra framework to deduce various properties of our polynomials.

2. A brief review of matroids and delta-matroids

We start with a brief overview of our matroid and delta-matroid terminology. We denote the complement of $A \subseteq E$ by $A^c := E \backslash A$. As is common practice in the area, we often omit the braces of one element sets, for example writing $A \cup e$ for $A \cup \{e\}$. For sets X and Y, $X \triangle Y := (X \cup Y) \backslash (X \cap Y)$ is their *symmetric difference*.

A *delta-matroid* $D = (E, \mathcal{F})$, introduced by A. Bouchet in [6], consists of a finite set E and a non-empty collection $\mathcal{F}$ of subsets of E that satisfies the *Symmetric Exchange Axiom*. This states that for all $X, Y \in \mathcal{F}$, if there is an element $u \in X \triangle Y$, then there is an element $v \in X \triangle Y$ such that $X \triangle \{u, v\} \in \mathcal{F}$. Elements of $\mathcal{F}$ are called *feasible sets* and E is the *ground set*. We often use $\mathcal{F}(D)$ and $E(D)$ to denote the set of feasible sets and the ground set, respectively, of D.

A *matroid* is a delta-matroid whose feasible sets are equicardinal. (This definition is equivalent to the basis definition of a matroid.) For a matroid $M = (E, \mathcal{B})$, we call the feasible sets in $\mathcal{B}$ *bases* of M. Every subset of every basis is an *independent set*. The *uniform matroid* $U_{k,n}$ is the matroid with a ground set E of n elements whose bases consist of all subsets of E with exactly k elements. For a set $A \subseteq E$, the *rank of* A, written $r(A)$, is the size of the largest intersection of A with a basis of M. That is, $r(A) := \max_{B \in \mathcal{B}} \{|A \cap B|\}$. Matroids can be defined in terms of rank functions: a matroid M is a pair (E, r) where E is the ground set and r is a function from 2^E to the non-negative integers such that for each $A \subseteq E$ and $e, f \in E$ we have $r(\emptyset) = 0$, $r(A \cup e) \in \{r(A), r(A) + 1\}$ and, if $r(A) = r(A \cup e) = r(A \cup f)$ then $r(A \cup \{e, f\}) = r(A)$. A set $A \subseteq E$ is independent precisely when $r(A) = |A|$ and the maximal independent sets are the bases of the matroid. We will work with matroids both in terms of bases, in which case we specify them as $M = (E, \mathcal{B})$, and rank functions, in which case we specify them as $M = (E, r)$.

For a delta-matroid $D = (E, \mathcal{F})$ and element $e \in E$, if e is in every feasible set of D then we say that e is a *coloop* of D. If e is in no feasible set of D, then we say that e is a *loop* of D. If $e \in E$ is not a coloop, then we define D *delete* e, written $D \backslash e$, to be $(E \backslash e, \{F \mid F \in \mathcal{F} \text{ and } F \subseteq E \backslash e\})$. If $e \in E$ is not a loop, then we define D *contract* e, written D/e, to be $(E \setminus e, \{F \setminus e \mid F \in \mathcal{F} \text{ and } e \in F\})$. If $e \in E$ is a loop or coloop, then $D/e = D \backslash e$. If $M = (E, r)$ is a matroid then deletion and contraction can be defined in terms of rank functions by $M \setminus e = (E \setminus e, r|_{E \backslash e})$ and $M/e = (E \setminus e, r')$, where $r'(B) := r(B \cup e) - r(e)$. If D' is a delta-matroid obtained from D by a sequence of edge deletions and edge contractions, then D' is independent of the order of the deletions and contractions used in its construction (see [10]), and so we can define $D \setminus A$ as the result of deleting every element in $A \subseteq E$ in some order. D/A is defined similarly. The *restriction* of D to a subset A of E, written $D|_A$, is equal to $D \setminus (E \setminus A)$. For $A \subseteq E$, the *twist* of $D = (E, \mathcal{F})$ with respect to A is $D * A := (E, \{A \triangle X \mid X \in \mathcal{F}\})$. The *dual* of D is $D^* := D * E$. In the case of matroids, the dual of $M = (E, r)$ equals $M^* = (E, r^*)$ where $r^*(A) := |A| + r(E \backslash A) - r(E)$.

The feasible sets of a delta-matroid $D = (E, \mathcal{F})$ are graded by their cardinality. Let $\mathcal{F}_{\max}(D)$ and $\mathcal{F}_{\min}(D)$ be the set of feasible sets of maximum and minimum cardinality, respectively. If the sets in $\mathcal{F}_{\min}$ (respectively, $\mathcal{F}_{\max}$) are of cardinality m and $k \in \mathbb{Z}$, then $\mathcal{F}_{\min+k}$ (respectively, $\mathcal{F}_{\max+k}$) denotes the set of feasible sets in $\mathcal{F}$ of cardinality $m + k$. The *upper matroid* of D is $D_{\max} := (E, \mathcal{F}_{\max})$ and the *lower matroid* of D is $D_{\min} := (E, \mathcal{F}_{\min})$. Bouchet showed in [8] that these are indeed matroids. Let $r_{\max}$ and $r_{\min}$, respectively, denote the rank functions of upper and lower matroids. The *width*, $w(D)$, of the delta-matroid D is the difference in size between a maximal and minimal feasible set: $w(D) = r_{\max}(D) - r_{\min}(D)$. For $A \subseteq E$ we define

$$\rho_D(A) := \tfrac{1}{2}(r_{\max}(D|_A) + r_{\min}(D|_A)). \tag{2}$$

The functon ρ_D is half-integer-valued, and when D is a matroid ρ_D is exactly its rank function. (Note that ρ here is *not* Bouchet's birank function from [7].)

In addition to loops and coloops, delta-matroids have other special elements. An element e of a delta-matroid D is a *ribbon loop* if e is a loop in $D_{\min}$. A ribbon loop e is *orientable* if e is not a loop in $(D * e)_{\min}$, and is *non-orientable* if e is a loop in $(D * e)_{\min}$. Note that it can be determined if e is a (orientable/non-orientable) ribbon loop by looking for its membership in sets of $\mathcal{F}_{\min}$ and $\mathcal{F}_{\min+1}$. The element e is a loop if it is in no feasible set; it is a ribbon loop if it is not in any member of $\mathcal{F}_{\min}$; it is an orientable ribbon loop if it is not in any member of $\mathcal{F}_{\min}$ or $\mathcal{F}_{\min+1}$; and is a non-orientable ribbon loop if it is not in any member of $\mathcal{F}_{\min}$ but is in a member of $\mathcal{F}_{\min+1}$. Thus a loop is a special type of (orientable) ribbon loop, but a (orientable) ribbon loop need not be a loop. (The ribbon loop terminology is from [12, 13] and comes from the correspondence between ribbon graphs and delta-matroids. We comment briefly on this below.)

We assume some familiarity with the basics of embedded graphs and ribbon graphs (see, for example, [15, 19] for background on them). A surface here is a compact, but not necessarily connected, 2-manifold. Recall that a graph is *cellularly embedded* in a surface if its faces are homeomorphic to discs. In general, here graphs embedded in surfaces need not be cellularly embedded. Let $G = (V, E)$ be a graph, ribbon graph, or graph embedded in a surface. Its *cycle matroid* $C(G)$ is $(E, \mathcal{B})$ where $\mathcal{B}$ consists of the edge sets of the maximal spanning forests of G. Its rank function is given by $r(A) = |V| - c(A)$ where $c(A)$ is the number of components of the spanning subgraph (V, A) of G. The *bond matroid* $B(G)$ of G is defined as $C(G)^*$.

If G is a ribbon graph or cellularly embedded graph we can form its geometric dual G^* and consider its cycle and bond matroids, $C(G^*)$ and $B(G^*)$. Note that if G is of genus zero (i.e., plane) then $C(G^*) = C(G)^*$ and so $C(G) = B(G^*)$, but this identity does not hold in general (although $D(G^*)$, defined below, always equals $D(G)^*$). If G is a graph embedded in a surface Σ but is not cellularly embedded then we can still form a dual graph but as there are choices in its construction, it need not have a unique embedding in Σ. However, all these choices result in embeddings of the same graph and so we can define $C(G^*)$ and $B(G^*)$ to be the cycle and bond matroids of any dual of G.

If $G = (V, E)$ is a ribbon graph its delta-matroid $D(G)$ is $(E, \mathcal{F})$ where $\mathcal{F}$ consists of the edge sets of spanning quasi-trees of G, where a ribbon subgraph (V, A) is a *spanning quasi-tree* if and only if (V, A) has exactly $c(G)$ boundary components. If G is a graph embedded in a surface, we define $D(G)$ to be the delta-matroid of the ribbon graph $\hat{G}$ arising as a neighbourhood of G. It is known (see [8, 12]) that $D(G)_{\min} = C(G)$ and $D(G)_{\max} = B(\hat{G}^*)$, in particular, if G is cellularly embedded then $D(G)_{\max} = B(G^*)$. Furthermore, for an edge e, $D(\hat{G})/e = D(\hat{G}/e)$ and $D(\hat{G})\backslash e = D(\hat{G}\backslash e)$, where we use ribbon graph deletion and contraction (note ribbon graph contraction is not compatible with graph contraction). The terminology and notation of the delta-matroids of ribbon graphs used here is from [12].

Graphs in surfaces provide some intuition for working with ribbon loops in delta-matroids. A loop e of a graph G embedded in a surface can have different topological characteristics: it can form a contractible or non-contractible curve in the surface; and a neighbourhood of it is either orientable or non-orientable. We have that e is a loop of G if and only if it is a ribbon loop of $D(G)$; it is contractible if and only if it is a loop of $D(G)$; and a neighbourhood of e is non-orientable if and only if e is a non-orientable ribbon loop in $D(G)$.

3. Matroid perspectives and delta-matroids

A *matroid perspective*, introduced by M. Las Vergnas in [25], consists of a pair of matroids over the same ground set, (M, M'), where $M = (E, r)$ and $M' = (E, r')$, such that for all $A \subseteq B \subseteq E$, we have

$$r(B) - r(A) \geq r'(B) - r'(A). \tag{3}$$

Usually the definition of a matroid perspective only requires that there be a bijection between the ground sets of M and M' (in which case the bijection is a *strong map* or a *morphism of matroids*), but for notational simplicity, and without loss of generality, we define them to be over the same ground set. Two standard reformulations (there are others) of the condition given by (3) are that each circuit of M is a union of circuits of M', or that every flat of M' is a flat of M.

We will make use of the following standard examples of matroid perspectives.

Example 1.

1. For any matroid M over an n element ground set, (M, M), $(U_{n,n}, M)$ and $(M, U_{0,n})$ are matroid perspectives.
2. If (M, N) is a matroid perspective then so are $(M, N) \setminus e := (M \setminus e, N \setminus e)$, $(M, N)/e := (M/e, N/e)$ and $(M, N)^* := (N^*, M^*)$, where e is an element of the ground set.
3. Let G be a graph with a partition $\mathcal{P}$ of its vertex set. Let $G/\mathcal{P}$ denote the graph that is obtained from G by identifying all of the vertices in each block of a partition. Then $(C(G), C(G/\mathcal{P}))$ is a matroid perspective.

4. Let G be a cellularly embedded graph and G^* be its geometric dual. Then $(B(G^*), C(G))$ is a matroid perspective.

To provide a minor-closed setting for the topological Tutte polynomials of Bollobás and Riordan, and Krushkal, our motivating problem from Section 1, we extend the notion of a matroid perspective to delta-matroids.

Definition 2. A *delta-matroid perspective* is a triple (M, D, M') where M and M' are matroids, D is a delta-matroid, M, M' and D are over the same ground set, and such that $(M, D_{\max})$ and $(D_{\min}, M')$ are matroid perspectives.

Example 3. The *trivial* delta-matroid perspective $(U_{0,0}, U_{0,0}, U_{0,0})$ is the unique delta-matroid perspective over the empty set. Up to isomorphism, there are exactly five delta-matroid perspective over a one element set:

$$\begin{array}{lll} S_1 = (U_{1,1}, D_c, U_{1,1}), & S_2 = (U_{0,1}, D_o, U_{0,1}), & S_3 = (U_{1,1}, D_c, U_{0,1}) \\ S_4 = (U_{1,1}, D_n, U_{0,1}), & S_5 = (U_{1,1}, D_o, U_{0,1}). & \end{array}$$

Here $U_{k,n}$ is the uniform matroid and the delta-matroids are $D_c := (\{e\}, \{\{e\}\})$, $D_o := (\{e\}, \{\emptyset\})$, and $D_n := (\{e\}, \{\emptyset, \{e\}\})$.

Proposition 4.

1. *M is a matroid if and only if (M, M, M) is a delta-matroid perspective.*
2. *(M, M') is a matroid perspective if and only if (M, M, M') is a delta-matroid perspective if and only if (M, M', M') is a delta-matroid perspective.*
3. *If (M, D, M') is a delta-matroid perspective then (M, M') is a matroid perspective.*
4. *D is a delta-matroid if and only if $(D_{\max}, D, D_{\min})$ is a delta-matroid perspective.*
5. *(M, D, N) is a delta-matroid perspective if and only if (N^*, D^*, M^*) is.*
6. *Let G be a ribbon graph, then $(B(G^*), D(G), C(G)) = (D(G)_{\max}, D(G), D(G)_{\min})$ is a delta-matroid perspective.*

Proof. Items (1) and (2) are trivial. Item 3 follows since, from [6, 9], $(D_{\max}, D_{\min})$ is a matroid perspective. Item 4 also follows since $(D_{\max}, D_{\min})$ is a matroid perspective. Item 5 is because if (M, D, N) is a delta-matroid perspective then $(M, D_{\max})$ and $(D_{\min}, N)$ are matroid perspectives, and so $((D_{\max})^*, M^*)$ and $(N^*, (D_{\min})^*)$ are. It is easily verified that $(D_{\max})^* = (D^*)_{\min}$ and $(D_{\min})^* = (D^*)_{\max}$, and so $(N^*, (D^*)_{\max})$ and $((D^*)_{\min}, M^*)$ are matroid perspectives. Thus (N^*, D^*, M^*) is a delta-matroid perspective. Item 6 follows from Example 1(4) and the identities $D(G)_{\min} = C(G)$ and $D(G)_{\max} = B(G^*)$. □

We call (N^*, D^*, M^*) the *dual* of (M, D, N), and denote it by $(M, D, N)^*$. The *direct sum* of two delta-matroid perspectives is defined component-wise: $(M, D, N) \oplus (M', D', N') := (M \oplus M', D \oplus D', N \oplus N')$.

In a matroid perspective (M, M'), if e is a loop in M then it is a loop in M', and if e is a coloop in M' then it is a coloop in M. Consequently in a delta-matroid perspective (M, D, M'), if e is a loop in M then it is a loop in M'; if e is a ribbon loop in D then it is a loop in M'; and if e is a coloop in M' then it is a coloop in M.

As with matroid perspectives, we could replace the requirement in Definition 2 that M, D, and M' are over the same set with the weaker requirement that there are bijections between their ground sets, however here we consider them over the same ground set for notational simplicity.

The following theorem says that the set of delta-matroid perspectives is minor-closed. Our motivating problem involves finding a framework for topological graph polynomials that is closed under deletion and contraction and so this is a key result for us.

Theorem 5. *Let (M, D, M') be a delta-matroid perspective over E, and let $e \in E$. Then $(M \setminus e, D \setminus e, M' \setminus e)$ and $(M/e, D/e, M'/e)$ are both delta-matroid perspectives.*

The theorem will follow from a sequence of lemmas, but before we continue let us emphasise that in general $(D/e)_{\min} \neq (D_{\min})/e$ and $(D \setminus e)_{\max} \neq (D_{\max}) \setminus e$, so Theorem 5 does not follow immediately from the fact that the set of matroid perspectives is closed under deletion and contraction.

Our first lemma is from [12].

Lemma 6. *Let $D = (E, \mathcal{F})$ be a delta-matroid, and $e \in E$ be a non-orientable ribbon loop. Let $F \subseteq E \backslash e$. Then $F \in \mathcal{F}(D)_{\min}$ if and only if $F \cup e \in \mathcal{F}(D)_{\min+1}$.*

Lemma 7. *Let $D = (E, \mathcal{F})$ be a delta-matroid, $M = (E, r)$ be a matroid, and $e \in E$. If $(D_{\min}, M)$ is a matroid perspective, then so are $((D \setminus e)_{\min}, M \setminus e)$ and $((D/e)_{\min}, M/e)$.*

Proof. We first prove the claim about deletion, which has the easier proof of the two results. First suppose that e is not a coloop. We start by showing that there is an element of $\mathcal{F}(D)_{\min}$ that does not contain e. Let $X \in \mathcal{F}(D)_{\min}$. If $e \notin X$ we are done, so suppose that $e \in X$. Since e is not a coloop there is some $Y \in \mathcal{F}$ such that $e \notin Y$. Then $e \in X \triangle Y$ and the Symmetric Exchange Axiom gives that there is a $v \in X \triangle Y$ such that $X \triangle \{e, v\} \in \mathcal{F}$. Since $e \in X$ and $X \in \mathcal{F}(D)_{\min}$ it must be that $X \triangle \{e, v\}$ is a set in $\mathcal{F}_{\min}$ not containing e.

Now as e is not a coloop, $\mathcal{F}(D \setminus e) = \{F \mid F \in \mathcal{F} \text{ and } F \subseteq E \setminus e\}$. Using the fact that e is not in some element of $\mathcal{F}_{\min}$, we see $\mathcal{F}(D \setminus e)_{\min} = \{F \mid F \in \mathcal{F}(D)_{\min} \text{ and } e \notin F\}$, but this is exactly $\mathcal{F}((D_{\min}) \setminus e)$. Thus $(D \setminus e)_{\min} = (D_{\min}) \setminus e$, and

$$((D \setminus e)_{\min}, M \setminus e) = (D_{\min} \setminus e, M \setminus e) = (D_{\min}, M) \setminus e. \tag{4}$$

Since $(D_{\min}, M)$ is a matroid perspective so is $(D_{\min}, M) \setminus e = ((D \setminus e)_{\min}, M \setminus e)$, as required.

Next suppose that e is a coloop. Then e is in every element of $\mathcal{F}$, and $\mathcal{F}(D \setminus e)$ is obtained by removing e from each element of $\mathcal{F}$. It follows that $(D \setminus e)_{\min} = (D_{\min}) \setminus e$. Equation (4) then applies in this case and so, as above, $((D \setminus e)_{\min}, M \setminus e)$ is a matroid perspective.

We now prove the claim about contraction. We consider three cases: when e is not a ribbon loop, an orientable ribbon loop, and a non-orientable ribbon loop.

First suppose that e is not a ribbon loop in D. Then e is not a loop in $D_{\min}$ so is not a loop in D. Thus $\mathcal{F}(D/e) = \{F \setminus e \mid F \in \mathcal{F}(D) \text{ and } e \in F\}$ from which it is easily seen that $\mathcal{F}(D/e)_{\min} = \{F \setminus e \mid F \in \mathcal{F}(D)_{\min} \text{ and } e \in F\}$. Since these are also the feasible sets of $(D_{\min})/e$ it follows that $(D/e)_{\min} = (D_{\min})/e$. Then

$$((D/e)_{\min}, M/e) = (D_{\min}/e, M/e) = (D_{\min}, M)/e. \tag{5}$$

Since $(D_{\min}, M)$ is a matroid perspective so is $(D_{\min}, M)/e = ((D/e)_{\min}, M/e)$, as required. Note that we have just shown that to prove $((D/e)_{\min}, M/e)$ is a matroid perspective it suffices to show that Equation (5) holds. We will use this observation below.

Next suppose that e is a non-orientable ribbon loop. Then e is not in any element of $\mathcal{F}(D)_{\min}$ but is in some element of $\mathcal{F}(D)_{\min+1}$. This means e is not a loop so $\mathcal{F}(D/e) = \{F \setminus e \mid F \in \mathcal{F}(D) \text{ and } e \in F\}$. As e is not in any element of $\mathcal{F}(D)_{\min}$ but is in some element of $\mathcal{F}(D)_{\min+1}$, we see that $\mathcal{F}(D/e)_{\min}$ is obtained by removing e from the feasible sets of $\mathcal{F}(D)_{\min+1}$ that contain e. By Lemma 6, for each $F \subseteq E \setminus e$, we have $F \in \mathcal{F}(D)_{\min}$ if and only if $F \cup e \in \mathcal{F}(D)_{\min+1}$, and so it follows that $\mathcal{F}(D/e)_{\min} = \mathcal{F}(D)_{\min}$. Since e is a loop in $D_{\min}$, it follows that $\mathcal{F}(D)_{\min} = \mathcal{F}(D_{\min}/e)$ and so $\mathcal{F}(D/e)_{\min} = \mathcal{F}(D_{\min}/e)$, giving $(D/e)_{\min} = D_{\min}/e$. Equation (5) then holds and it follows that $((D/e)_{\min}, M/e)$ is a matroid perspective.

For the final case suppose that e is an orientable ribbon loop. There are two sub-cases given by if e is a loop of D or not. If e is a loop of D then $\mathcal{F}(D/e) = \mathcal{F}(D)$, and so $(D/e)_{\min} = D_{\min}/e$, Equation (5) holds, and $((D/e)_{\min}, M/e)$ is a matroid perspective.

Now suppose that e is not a loop of D. Then e is not in any elements of $\mathcal{F}(D)_{\min}$ or $\mathcal{F}(D)_{\min+1}$, but e is in some element of $\mathcal{F}(D)$. We will show that e is in some element of $\mathcal{F}(D)_{\min+2}$. For this let $X \in \mathcal{F}(D)_{\min}$, and $Y \in \mathcal{F}$ be such that $e \in Y$. Then $e \in X \triangle Y$, and so by the Symmetric Exchange Axiom there is some $v \in X$ such that $X \triangle \{e, v\} \in \mathcal{F}$ since $e \notin X$ and e is not in any elements of $\mathcal{F}(D)_{\min}$ or $\mathcal{F}(D)_{\min+1}$, it follows that $e \in X \triangle \{e, v\} \in \mathcal{F}(D)_{\min+2}$, as required. In fact we have shown that

$$X \in \mathcal{F}(D)_{\min} \implies X \triangle \{e, v\} \in \mathcal{F}(D)_{\min+2}, \quad \text{for some } v \in E. \tag{6}$$

As e is not a loop, $\mathcal{F}(D/e) = \{F \setminus e \mid F \in \mathcal{F}(D) \text{ and } e \in F\}$. Since e is not in any elements of $\mathcal{F}(D)_{\min}$ or $\mathcal{F}(D)_{\min+1}$, but is in an element of $\mathcal{F}(D)_{\min+2}$ it follows that

$$\mathcal{F}(D/e)_{\min} = \{F \setminus e \mid F \in \mathcal{F}(D)_{\min+2} \text{ and } e \in F\}. \tag{7}$$

Next, since e is a loop in $D_{\min}$, we have that

$$\mathcal{F}(D_{\min}/e) = \mathcal{F}(D_{\min}). \tag{8}$$

Now consider the matroid $(D * e)_{\min}$. For its bases, since e is not in any member of $\mathcal{F}(D)_{\min}$ or $\mathcal{F}(D)_{\min+1}$, we have

$$\mathcal{F}(D * e)_{\min} = \{F \setminus e \mid F \in \mathcal{F}(D)_{\min+2} \text{ and } e \in F\} \cup \{F \cup e \mid F \in \mathcal{F}(D)_{\min}\}. \tag{9}$$

It follows from Equations (7)–(9) that $(D * e)_{\min}$ is a matroid with the properties that e forms an independent set, $r((D/e)_{\min}) = r((D * e)_{\min})$, $((D * e)_{\min}) \setminus e = (D/e)_{\min}$, and $((D * e)_{\min})/e = D_{\min}/e$. Thus, following the terminology of Section 7.3 of [28] and using its Lemma 7.3.3 and Proposition 7.3.6, $D_{\min}/e$ is a quotient of $(D/e)_{\min}$, and so $((D/e)_{\min}, D_{\min}/e)$ is a matroid perspective. Since $(D_{\min}, M)$ is a matroid perspective, so is $(D_{\min}/e, M/e)$, and by transitivity so is $((D/e)_{\min}, M/e)$, as required. This completes the proof that $((D/e)_{\min}, M/e)$ is a matroid perspective and the proof of the theorem. $\square$

Lemma 8. *Let $D = (E, \mathcal{F})$ be a delta-matroid, $M = (E, r)$ be a matroid, and $e \in E$. If $(M, D_{\max})$ is a matroid perspective, then so are $(M \setminus e, (D \setminus e)_{\max})$ and $(M/e, (D/e)_{\max})$.*

Proof. First observe that the following hold: $(D_{\max})^* = (D^*)_{\min}$, $D^*/e = (D \setminus e)^*$, $D^* \setminus e = (D/e)^*$, and (M, N) is a matroid perspective if and only if (N^*, M^*) is.

Suppose that $(M, D_{\max})$ is a matroid perspective. By duality, so is $((D_{\max})^*, M^*)$. As $(D_{\max})^* = (D^*)_{\min}$, we have that $((D^*)_{\min}, M^*)$ is a matroid perspective. An application of Lemma 7 and the above identities give

$$((D^*/e)_{\min}, M^*/e) = (((D \setminus e)^*)_{\min}, (M \setminus e)^*) = (((D \setminus e)_{\max})^*, (M \setminus e)^*) = (M \setminus e, (D \setminus e)_{\max})^*$$

is a matroid perspective. The deletion result follows. Starting with $((D^* \setminus e)_{\min}, M^* \setminus e)$ and arguing similarly gives that $(M/e, (D/e)_{\max})$ is a matroid perspective. □

Proof of Theorem 5. The result follows immediately from Lemmas 7 and 8. □

With Theorem 5 we can define deletion and contraction for delta-matroid perspectives.

Definition 9. Let (M, D, M') be a delta-matroid perspective over E, and let $e \in E$. Then (M, D, M') *delete* e, denoted $(M, D, M') \setminus e$, is the delta-matroid perspective $(M \setminus e, D \setminus e, M' \setminus e)$. Similarly, (M, D, M') *contract* e, denoted $(M, D, M')/e$, is the delta-matroid perspective $(M/e, D/e, M'/e)$.

Note that the order of the deletion and contraction of distinct elements does not matter. If $A \subseteq E$, then $(M, D, M') \setminus A$ denotes the result of deleting each element of A, and $(M, D, M')/A$ denotes the result of contracting each element of A.

It is important to remember that there are two different types of deletion and contraction being used in Definition 9; those of matroids and those of delta-matroids. This is particularly important in Section 6 when considering topological Tutte polynomials as it means that in general $(D(G)_{\max}, D(G), D(G)_{\min})/e$ does not equal $(D(G/e)_{\max}, D(G/e), D(G/e)_{\min})$.

Two natural objects arise by considering pairs of elements of a delta-matroid perspective:

Definition 10. Let $M = (E, r)$ be a matroids, and $D = (E, \mathcal{F})$ be a delta-matroid. Then

- the pair (D, M) is a *DM-perspective* if $(D_{\min}, M)$ is a matroid perspective, and
- the pair (M, D) is an *MD-perspective* if $(M, D_{\max})$ is a matroid perspective.

DM-perspectives will prove to be important later when studying the Bollobás-Riordan polynomial. For the moment we only note that by Lemma 7 we can define deletion and contraction for DM-perspectives by $(D, M) \setminus e := (D \setminus e, M \setminus e)$ and $(D, M)/e := (D/e, M/e)$, and the set of DM-perspectives is closed under these operations. An analogous comment holds for MD-perspectives.

4. Graphical analogues

The point of view of matroids as being an extension of graphs is well-known and serves to guide development in both areas. In this section we describe the graphical analogues of delta-matroid perspectives and DM-perspectives.

A *vertex partitioned graph in a surface* $(G \subset \Sigma, \mathcal{P})$ consists of a graph $G = (V, E)$ embedded in a surface Σ, and a partition $\mathcal{P}$ of its vertex set V. Every vertex partitioned graph in a surface $(G \subset \Sigma, \mathcal{P})$ has three graphs associated with it, one of which is embedded. The original graph in the surface $G \subset \Sigma$, which we can take a dual graph G^* and take the delta-matroid of; its underlying abstract graph G; and the abstract graph $G/\mathcal{P}$ obtained from G by identifying all the elements of each block of the partition $\mathcal{P}$ to a single vertex.

Note that a vertex partitioned graph in a surface can be viewed as a graph in a pseudo-surface by identifying all of the vertices in a block of the partition to a pinch point. However, it is more convenient for our purposes to work with vertex partitions.

Proposition 11. *Let $(G \subset \Sigma, \mathcal{P})$ be a vertex partitioned graph in a surface. Then*

$$\mathbf{P}(G \subset \Sigma, \mathcal{P}) := (B(G^*), D(G), C(G/\mathcal{P}))$$

is a delta-matroid perspective.

Before proving Proposition 11 we highlight an easily overlooked subtlety. The matroid $B(G^*)$ is formed using the dual of $G \subset \Sigma$ which may not be cellularly embedded. The delta-matroid $D(G)$ is read from a ribbon graph formed by a regular neighbourhood of G. Denote this by $\hat{G}$ for now. Then $D(G)_{\max} = B(\hat{G}^*)$ but this need not equal $B(G^*)$, as G and $\hat{G}$ could be embedded in different surfaces.

In this section it will be convenient to view $G \subset \Sigma$ as the complex $\mathcal{V} \cup \mathcal{E} \cup \mathcal{R}$ that arises from a neighbourhood of G. Here $\mathcal{V}$ is a set of discs arising from neighbourhoods of vertices, $\mathcal{V} = N(V)$; $\mathcal{E}$ is a set of discs arising from neighbourhoods of edges $\mathcal{E} = N(E) \setminus N(V)$, and $\mathcal{R}$ arises from what remains, $\mathcal{R} = \Sigma \setminus (N(V) \cup N(E))$. Thus $\Sigma = \mathcal{V} \cup \mathcal{E} \cup \mathcal{R}$. This is exactly a band decomposition when G is cellularly embedded in Σ. It is clear how to move between the two descriptions $G \subset \Sigma$ and $\mathcal{V} \cup \mathcal{E} \cup \mathcal{R}$ of the same object.

Proof of Proposition 11. $D(G)_{\min} = C(G)$, so $(D(G)_{\min}, C(G/\mathcal{P}))$ is a matroid perspective by Example 1(3). For $(B(G^*), D(G)_{\max})$, view $G \subset \Sigma$ as the complex $\mathcal{V} \cup \mathcal{E} \cup \mathcal{R}$. Let $\hat{G} \subset \hat{\Sigma}$ be the cellularly embedded graph that arises by capping off all the holes of $\mathcal{V} \cup \mathcal{E}$ with a set of discs $\hat{\mathcal{R}}$. Then $D(G)_{\max} = B(\hat{G}^*)$. The abstract graph G^* has vertex set $\mathcal{R}$ and edge set defined by $\mathcal{E}$. The abstract graph $\hat{G}^*$ has vertex set $\hat{\mathcal{R}}$ and edge set defined by $\mathcal{E}$. Since each element of $\hat{\mathcal{R}}$ corresponds to a set of discs in $\mathcal{R}$ we see that G^* can be obtained from $\hat{G}^*$ by identifying vertices. By Example 1(3), $(C(\hat{G}^*), C(G^*))$ is a matroid perspective. Duality gives that $(B(G^*), B(\hat{G}^*)) = (B(G^*), D(G)_{\max})$ is also a matroid perspective. □

We next show that delta-matroid perspective operations are compatible with operations on vertex partitioned graph in a surfaces. Deletion and contraction for vertex partitioned graphs in a surface were defined in [22]. For deletion, $(G \subset \Sigma, \mathcal{P}) \setminus e$ is obtained by removing the edge e from G. For contracting an edge $e = (u, v)$, view $G \subset \Sigma$ as the complex $\mathcal{V} \cup \mathcal{E} \cup \mathcal{R}$, delete $N(e) \cup N(v) \cup N(u)$, attach a disc to each of the (one or two) boundary components this creates. These new discs form vertices. For the partition, identify the block(s) of the partition containing u and v, remove u and v from the block of the partition created, and add the new vertices to this block of the partition.

We introduce the concept of the dual of a vertex partitioned graph in a surface. Given $(G \subset \Sigma, \mathcal{P})^*$, view it as a complex $\mathcal{V} \cup \mathcal{E} \cup \mathcal{R}$ where $\mathcal{P}$ is a partition of $\mathcal{V}$. Form a complex $\mathcal{V}^* \cup \mathcal{E} \cup \mathcal{R}^*$ in the following way.

Let $\mathcal{P}_i$, for $i = 1, \ldots, |\mathcal{P}|$, denote the blocks of the partition $\mathcal{P}$ of $\mathcal{V}$. For each $\mathcal{P}_i$, delete the $|\mathcal{P}_i|$ elements of $\mathcal{V}$ it contains from the complex $\mathcal{V} \cup \mathcal{E} \cup \mathcal{R}$. The resulting complex is a surface with $|\mathcal{P}_i|$ holes. Next take a disk D_i with $|\mathcal{P}_i| - 1$ holes and identify each boundary component of D_i with a distinct boundary component of the complex $(\mathcal{V} \cup \mathcal{E} \cup \mathcal{R}) \setminus \mathcal{P}_i$. Let $\mathcal{R}^* = \{D_1, \ldots, D_{|\mathcal{P}|}\}$.

Next, for each element R_j of $\mathcal{R}$, where $j = 1, \dots, |\mathcal{R}|$, deleting R_j from $\mathcal{V} \cup \mathcal{E} \cup \mathcal{R}^*$ (or $\mathcal{V} \cup \mathcal{E} \cup \mathcal{R}$) will create k_j holes, for some k_j. Fill each hole with a distinct disc and let $\{D_{j,1}, \dots, D_{j,k_j}\}$ be the set of discs added. Then let $\mathcal{V}^* = \{D_{j,q_j} \mid j = 1, \dots, |\mathcal{R}|, q_j = 1, \dots k_j\}$, and let $\mathcal{P}^*$ be the partition of $\mathcal{V}^*$ given by the blocks $\{D_{j,1}, \dots, D_{j,k_j}\}$, for $j = 1, \dots, |\mathcal{R}|$ (so each block of the partition corresponds to an element of $\mathcal{R}$).

The vertex partitioned graph in a surface corresponding to $\mathcal{V}^* \cup \mathcal{E} \cup \mathcal{R}^*$ and the partition $\mathcal{P}^*$ is called the *dual* of $(G \subset \Sigma, \mathcal{P})$ and is denoted $(G \subset \Sigma, \mathcal{P})^*$.

Note that this dual operation is not involutory since $(G \subset \Sigma, \mathcal{P})^{**}$ may not be embedded in Σ, although $(G \subset \Sigma, \mathcal{P})^{***} = (G \subset \Sigma, \mathcal{P})^*$. The only difference between $(G \subset \Sigma, \mathcal{P})^{**}$ and $(G \subset \Sigma, \mathcal{P})$ is that the latter may have handles on its surface that can be removed without altering the number of faces or the incidences of edges and faces. It follows that $\mathbf{P}((G \subset \Sigma, \mathcal{P})^{**}) = \mathbf{P}(G \subset \Sigma, \mathcal{P})$.

Proposition 12. *Let $(G \subset \Sigma, \mathcal{P})$ be a vertex partitioned graph in a surface and $e \in E(G)$. Then the following hold.*

1. $\mathbf{P}(G \subset \Sigma, \mathcal{P})^* = \mathbf{P}\left((G \subset \Sigma, \mathcal{P})^*\right)$,
2. $\mathbf{P}(G \subset \Sigma, \mathcal{P}) \setminus e = \mathbf{P}\left((G \subset \Sigma, \mathcal{P}) \setminus e\right)$,
3. $\mathbf{P}(G \subset \Sigma, \mathcal{P})/e = \mathbf{P}\left((G \subset \Sigma, \mathcal{P})/e\right)$.

Proof. For the duality result suppose that $(G \subset \Sigma, \mathcal{P})^* = (H \subset \Sigma', \mathcal{P}^*)$. It is clear from the construction of the dual that $H^* \cong G/\mathcal{P}$, $H/\mathcal{P}^* = G^*$, and $\hat{H} = \hat{G}^*$, where $\hat{H}$ and $\hat{G}$ denote the underlying ribbon graphs of the two embedded graphs. Then $\mathbf{P}\left((G \subset \Sigma, \mathcal{P})^*\right) = (B(H^*), D(H), C(H/\mathcal{P}^*)) = (C(G/\mathcal{P})^*, D(\hat{G}^*), C(\hat{G}^*)) = (C(G/\mathcal{P})^*, D(\hat{G})^*, B(\hat{G}^*)^*) = \mathbf{P}(G \subset \Sigma, \mathcal{P})^*$.

For the deletion result observe that $(G \backslash e)^* = G^*/e$ (where the contraction acting on G^* is graph contraction), $(G \backslash e)/\mathcal{P} = G/\mathcal{P} \backslash e$, and $\widehat{G \setminus e} = \hat{G} \backslash e$, where $\widehat{G \setminus e}$ and $\hat{G}$ denote the underlying ribbon graphs of the two embedded graphs. Then $\mathbf{P}\left((G \subset \Sigma, \mathcal{P}) \setminus e\right) = (B((G \setminus e)^*), D(\widehat{G \setminus e}), C((G \setminus e)/\mathcal{P})) = (C(G^*/e)^*, D(\hat{G} \backslash e), C(G/\mathcal{P} \backslash e)) = (C(G^*)^* \backslash e, D(\hat{G}) \backslash e, C(G/\mathcal{P}) \backslash e)) = \mathbf{P}(G \subset \Sigma, \mathcal{P}) \backslash e$.

The contraction result follows in a similar way to the deletion result (just switch the roles of deletion and contraction in the above argument). Alternatively, it can be deduced by applying the deletion result to $\mathbf{P}\left(((G \subset \Sigma, \mathcal{P})/e)^{**}\right)$, and applying the duality result. (It does not matter that the duality here is not involutory since the delta-matroid perspectives are unchanged by taking the two duals.) We omit the routine details of the argument. □

The graphical analogue of a DM-perspective is a *vertex partitioned ribbon graph*, $(G, \mathcal{P})$. This consists of a ribbon graph $G = (V, E)$ and a partition $\mathcal{P}$ of its vertex set V. By restricting the results above on $\mathbf{P}(G \subset \Sigma, \mathcal{P})$ to the second two components we obtain the following.

Corollary 13. *Let $(G, \mathcal{P})$ be a vertex partitioned ribbon graph, $e \in E(G)$, and let $\mathbf{DM}(G) := (D(G), C(G/\mathcal{P}))$. Then*

1. $\mathbf{DM}(G) = (D(G), C(G/\mathcal{P}))$ *is a DM-perspective,*
2. $\mathbf{DM}(G)^* = \mathbf{DM}\left(G^*\right)$,
3. $\mathbf{DM}(G) \setminus e = \mathbf{DM}\left(G \setminus e\right)$,
4. $\mathbf{DM}(G)/e = \mathbf{DM}\left(G/e\right)$.

5. The Tutte polynomial of delta-matroid perspectives

In this section we introduce analogues of the Tutte polynomial for delta-matroid perspectives and for DM-perspectives. We will see in Section 6 that these are the matroidal versions of Krushkal's, and Bollobás and Riordan's polynomials of graphs in surfaces we were looking for. We will also see that the matroidal polynomials share key features with the classical Tutte polynomial of a graph or matroid that topological polynomials do not. We will postpone the proofs of several results in this section until Section 7 where we will deduce them from the theory of Tutte polynomials of Hopf algebras.

Matroids, matroid perspectives and delta-matroids arise as special cases of delta-matroid perspectives. The Tutte polynomial of a matroid $M = (E, r)$ is

$$T_M(x, y) := \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \tag{10}$$

Las Vergnas, in [25], defined the *Tutte polynomial of the matroid perspective* (M, M'), where $M = (E, r)$ and $M' = (E, r')$, as

$$T_{(M,M')}(x, y, z) := \sum_{A \subseteq E} (x-1)^{r'(E)-r'(A)}(y-1)^{|A|-r(A)} z^{(r(E)-r(A))-(r'(E)-r'(A))}. \tag{11}$$

From [12, 22], the Tutte polynomial of a delta-matroid $D = (E, \mathcal{F})$ is the *2-variable Bollobás-Riordan polynomial*

$$\tilde{R}_D(x, y) := \sum_{A \subseteq E} (x-1)^{\rho(E)-\rho(A)}(y-1)^{|A|-\rho(A)}, \tag{12}$$

where ρ is given by Equation (2).

The following definition provides the natural (in the sense discussed in Section 7) extension of the Tutte polynomial to delta-matroid perspectives.

Definition 14. Let (M, D, M') be a delta-matroid perspective, where $M = (E, r)$, $M' = (E, r')$, and $\rho = \rho_D$. Then we define $K_{(M,D,M')} \in \mathbb{Z}[x, y, a^{1/2}, b^{1/2}]$ by

$$K_{(M,D,M')}(x, y, a, b) := \sum_{A \subseteq E} x^{r'(E)-r'(A)} y^{|A|-r(A)} a^{\rho(A)-r'(A)} b^{r(A)-\rho(A)}. \tag{13}$$

We will also be particularly interested in DM-perspectives and their natural Tutte polynomial.

Definition 15. Let (D, M) be a DM-perspective, where $M = (E, r)$, and $\rho = \rho_D$. Then we define $R_{(D,M)} \in \mathbb{Z}[x, y^{1/2}, z^{1/2}]$ by

$$R_{(D,M)}(x, y, z) := \sum_{A \subseteq E} x^{r(E)-r(A)} y^{|A|-\rho(A)} z^{\rho(A)-r(A)}. \tag{14}$$

Because we are motivated by the Krushkal and Bollobás-Riordan polynomials we will not pay particular attention to MD-perspectives, which are dual to DM-perspectives, although a discussion of their polynomials can be found in Remark 38.

Matroids, matroid perspectives and delta-matroids can all be regarded as delta-matroid perspectives, so we would expect that the Tutte polynomials of these objects can be obtained from the Tutte polynomial of delta-matroid perspectives. The following proposition shows that this is indeed the case.

Proposition 16. *The following identities hold.*

$$T_M(x,y) = K_{(M,M,M)}(x-1,y-1,a,b) \tag{15}$$
$$T_{M'}(x,y) = K_{(M,D,M')}(x-1,y-1,y-1,y-1) \tag{16}$$
$$T_{(M,M')}(x,y,z) = z^{r(E)-r'(E)} K_{(M,M,M')}(x-1,y-1,z^{-1},b) \tag{17}$$
$$T_{(M,M')}(x,y,z) = z^{r(E)-r'(E)} K_{(M,M',M')}(x-1,y-1,a,z^{-1}) \tag{18}$$
$$T_{(M,M')}(x,y,z) = z^{r(E)-r'(E)} K_{(M,D,M')}(x-1,y-1,z^{-1},z^{-1}) \tag{19}$$
$$\tilde{R}_D(x+1,y+1) = x^{w(D)/2} K_{(D_{\max},D,D_{\min})}(x,y,x^{-1},y) \tag{20}$$
$$R_{(D,M')}(x,y,z) = K_{(M,D,M')}(x,y,z,y) \tag{21}$$

Proof. Each of the identities can be established by writing down the state sums definitions of the polynomials given in (10)–(14), recalling for a matroid M that ρ_M is exactly its rank function, and comparing the exponents. Since this is a routine verification we omit the details. □

As with the Tutte polynomial of a matroid, and unlike the Krushkal and Bollobás-Riordan polynomials, $K_{(M,D,M')}(x,y,a,b)$ can be defined recursively through deletion-contraction relations and its value of 1 on the trivial delta-matroid perspective.

Theorem 17. *The polynomial $K_{(M,D,M')}(x,y,a,b)$ is uniquely defined by the following deletion-contraction relations together with its value of 1 on delta-matroid perspectives over the empty set.*

1. *If e is a coloop in M' (and hence a coloop in M) then*

$$K_{(M,D,M')}(x,y,a,b) = x\, K_{(M,D,M')\backslash e}(x,y,a,b) + K_{(M,D,M')/e}(x,y,a,b).$$

2. *If e is a loop in M (and hence a loop in M') then*

$$K_{(M,D,M')}(x,y,a,b) = K_{(M,D,M')\backslash e}(x,y,a,b) + y\, K_{(M,D,M')/e}(x,y,a,b).$$

3. *If e is not a loop in M, not a ribbon loop in D, and a loop in M' then*

$$K_{(M,D,M')}(x,y,a,b) = K_{(M,D,M')\backslash e}(x,y,a,b) + a\, K_{(M,D,M')/e}(x,y,a,b).$$

4. *If e is not a loop in M, an orientable ribbon loop in D, and hence a loop in M' then*

$$K_{(M,D,M')}(x,y,a,b) = K_{(M,D,M')\backslash e}(x,y,a,b) + b\, K_{(M,D,M')/e}(x,y,a,b).$$

5. *If e is not a loop in M, a non-orientable ribbon loop in D, and hence a loop in M' then*

$$K_{(M,D,M')}(x,y,a,b) = K_{(M,D,M')\backslash e}(x,y,a,b) + \sqrt{ab}\, K_{(M,D,M')/e}(x,y,a,b).$$

6. *If e is not a loop or coloop in M' then*

$$K_{(M,D,M')}(x,y,a,b) = K_{(M,D,M')\backslash e}(x,y,a,b) + K_{(M,D,M')/e}(x,y,a,b).$$

We will postpone the proof of Theorem 17 until Section 7.

Corollary 18. *The polynomial $R_{(D,M)}(x,y,z)$ is uniquely defined by the following deletion-contraction relations together with its value of 1 on DM-perspectives over the empty set.*

1. *If e is a coloop in M then*

$$R_{(D,M)}(x,y,z) = x\,R_{(D,M)\backslash e}(x,y,z) + R_{(M,D)/e}(x,y,z).$$

2. *If e is not a ribbon loop in D and a loop in M then*

$$R_{(D,M)}(x,y,z) = R_{(D,M)\backslash e}(x,y,z) + z\,R_{(D,M)/e}(x,y,z).$$

3. *If e is an orientable ribbon loop in D and a loop in M then*

$$R_{(D,M)}(x,y,z) = R_{(D,M)\backslash e}(x,y,z) + y\,R_{(D,M)/e}(x,y,z).$$

4. *If e is a non-orientable ribbon loop in D and a loop in M then*

$$R_{(D,M)}(x,y,z) = R_{(D,M)\backslash e}(x,y,z) + \sqrt{yz}\,R_{(D,M)/e}(x,y,z).$$

5. *If is not a loop or coloop in M then*

$$R_{(D,M)}(x,y,z) = R_{(D,M)\backslash e}(x,y,z) + R_{(D,M)/e}(x,y,z).$$

Proof. Suppose that (D,M) is a DM-perspective over E and $|E| = n$. Since $(U_{n,n}, D_{\max})$ is a matroid perspective, $(U_{n,n}, D, M)$ is a delta-matroid perspective. By Equation (21), $R_{(D,M')}(x,y,z) = K_{(U_{n,n},D,M')}(x,y,z,y)$, and the result follows by an application of Theorem 17, noting that $U_{n,n}$ has no loops so Case 2 of Theorem 17 cannot happen. □

A key property of the Tutte polynomial is its duality relation $T_M(x,y) = T_{M^*}(y,x)$ from [14, 31]. The polynomial $K_{(M,D,M')}$ satisfies such an identity.

Theorem 19. *Let (M,D,M') be a delta-matroid perspective, r be the rank function of M and r' be the rank function of M'. Then*

$$K_{(M,D,M')^*}(x,y,a,b) = a^{r(M)-\rho(D)} b^{\rho(D)-r'(M')} K_{(M,D,M')}(y,x,b^{-1},a^{-1}).$$

We will postpone the proof of Theorem 19 until Section 7.

The duality relation in Theorem 19 specialises, via Proposition 16, to give the duality relations $T_M(x,y) = T_{M^*}(y,x)$, $T_{(M,M')}(x,y,z) = z^{r(E)-r'(E)} T_{(M,M')^*}(y,x,1/z)$ from [26], and $\widetilde{R}_D(x,y) = \widetilde{R}_{D^*}(y,x)$ from [12]. See also Remark 22 for the application to topological Tutte polynomials.

We conclude this section with convolution formulae for $K_{(M,D,M')}$ and $R_{(D,M)}$. The *convolution formula* for the Tutte polynomial appeared in W. Kook, V. Reiner and D. Stanton's paper [21], and implicitly in G. Etienne and M. Las Vergnas' paper [18]. It holds for various extensions of the Tutte polynomial (for example, [3, 20, 22, 24, 32]). The convolution formula expresses the Tutte polynomial of a graph or matroid M in terms of 1-variable specialisations:

$$T_M(x,y) = \sum_{A\subseteq E(M)} T_{M\backslash A^c}(0,y)\cdot T_{M/A}(x,0).$$

More generally, the following hold.

Theorem 20.

1. *Let (M, D, M') be a delta-matroid perspective over E. Then*

$$K_{(M,D,M')}(x,y,a,b) = \sum_{A\subseteq E} K_{(M,D,M')\backslash A^c}(-1,y,a,b) \cdot K_{(M,D,M')/A}(x,-1,-1,-1).$$

2. *Let (D, M) be a DM-perspective over E. Then*

$$R_{(D,M)}(x,y,z) = \sum_{A\subseteq E} R_{(D,M)\backslash A^c}(-1,y,z) \cdot R_{(D,M)/A}(x,-1,-1).$$

In these identities the negative root of 1 should be taken when computing the contraction terms.

Note if D is an even delta-matroid then no roots will need to be evaluated and the rule for choosing the negative root can be dropped. We postpone the proof of Theorem 20 until Section 7.

6. Connections with topological Tutte polynomials

Our reason for introducing delta-matroid perspectives was to provide a matroidal framework for topological graph polynomials. Introduced in [4, 5], the *Bollobás-Riordan polynomial* of a ribbon graph $G = (V, E)$ is defined by

$$R_G(x,y,z) := \sum_{A\subseteq E} (x-1)^{r(G)-r(A)} y^{|A|-r(A)} z^{\gamma(A)}.$$

Here r is the rank function of $C(G)$ and $\gamma(A)$ is the Euler genus (which is twice its genus if it is orientable and its genus if it is not) of the spanning ribbon subgraph (V, A) of G. By Euler's formula, $|V| - |A| + b(A) = 2c(A) - \gamma(A)$, where $b(A)$ is the number of boundary components of (V, A).

For a graph $G = (V, E)$ embedded in a surface Σ (but not necessarily cellularly embedded) the *Krushkal polynomial*, introduced by S. Krushkal in [23] for graphs in orientable surfaces, and extended by C. Butler in [11] to graphs in non-orientable surfaces, is defined by

$$K_G(x,y,a,b) := \sum_{A\subseteq E} x^{r(G)-r(A)} y^{\kappa(A)} a^{\frac{1}{2}s(A)} b^{\frac{1}{2}s^{\perp}(A)}. \tag{22}$$

Here, using $N(X)$ to denote a regular neighbourhood of a subset X of Σ, we have $s(A) := \gamma(N(V \cup A))$, $s^{\perp}(A) := \gamma(\Sigma\backslash N(V \cup A))$, and

$$\kappa(A) := \#\text{cpts}(\Sigma\backslash N(V \cup A)) - \#\text{cpts}(\Sigma), \tag{23}$$

where "#cpts" is shorthand for "the number of connected components of". Note that we use the form of the exponent of y from the proof of Lemma 4.1 of [2] rather than the homological definition given in [23].

Theorem 21.

1. *Let G be a ribbon graph and $(D, M) = (D(G), C(G))$. Then*

$$R_{(D,M)}(x,y,z) = R_G(x+1, y, \sqrt{z/y}),$$

and

$$R_G(x,y,z) = R_{(D,M)}(x-1, y, yz^2).$$

2. *Let $G=(V,E)$ be a graph embedded in a surface Σ, and $(M,D,M')=(B(G^*),D(G),C(G))$. Then*
$$K_{(M,D,M')}(x,y,a,b)=b^{\gamma(\Sigma)/2}K_{G\subset\Sigma}(x,y,a,b^{-1}),$$
and
$$K_{G\subset\Sigma}(x,y,a,b)=b^{\gamma(\Sigma)/2}K_{(M,D,M')}(x,y,a,b^{-1}).$$

We will prove Theorem 21 later in this section by showing it is a special case of a more general result.

The significance of Theorem 21 is that it provides a framework for the topological graph polynomials in which they satisfy key properties of the classical Tutte polynomial, such as a full deletion-contraction definition, duality relation, and convolution formula (by Theorems 17–20). They do not have these properties (more precisely, they only have partial versions of these properties) in their original framework of graphs in surfaces.

Remark 22. It is worth highlighting how duality interacts with $R_G(x,y,z)$. In their 2002 paper [5], Bollobás and Riordan showed that R_G satisfies a 1-variable duality formula. A 2-variable duality relation was subsequently found independently by I. Moffatt in [27], and by J. Ellis-Monaghan and I. Sarmiento in [17], but no full 3-variable duality formula was found. The DM-perspective framework explains why: DM-perspectives are not closed under duality. Moreover the 2-variable formula exists since the relevant specialisation of R_G is entirely determined by the delta-matroid, as in [12], and delta-matroids are closed under duality. Bollobás and Riordan concluded their paper with the remark "it is possible that there is a generalization of the ribbon graph polynomial behaving well with respect to duals". The matroidal framework tells us that $K_{(M,D,M')}$ is exactly this generalisation (see also [23] for the case of ribbon graphs).

By the results in Section 4 we see that for a graphical analysis of $K_{(M,D,M')}$ it is natural to work with vertex partitioned graphs in surfaces rather than graphs in surfaces. Accordingly we consider the extensions of $K_{G\subset\Sigma}$ and R_G to this setting from [22]. The *Krushkal Polynomial* of a vertex partitioned graph in a surface is
$$K_{(G\subset\Sigma,\mathcal{P})}(x,y,a,b):=\sum_{A\subseteq E(G)}x^{r(G/\mathcal{P})-r(A/\mathcal{P})}y^{\kappa(A)}a^{\rho(A)-r(A/\mathcal{P})}b^{|A|-\rho(A)-\kappa(A)}.$$
Here $\kappa(A)$ is defined as in Equation (23) and
$$\rho(A):=\tfrac{1}{2}\left(|A|+|V|-b(A)\right), \tag{24}$$
where $b(A)$ denotes the number of boundary components of $N(V\cup A)$, and $r(A/\mathcal{P}):=r_{G/\mathcal{P}}(A)$. (We note that, from [22], the ρ in Equations (2) and (24) agree, that is, $\rho_{D(G)}(A)=\rho_G(A)$. Its appearance in the two contexts should therefore cause no confusion.)

The *Bollobás-Riordan polynomial* of a vertex partitioned ribbon graph $(G,\mathcal{P})$ is
$$R_{(G,\mathcal{P})}(x,y,z):=\sum_{A\subseteq E(G)}(x-1)^{r(G/\mathcal{P})-r(A/\mathcal{P})}y^{|A|-r(A/\mathcal{P})}z^{2(\rho(A)-r(A/\mathcal{P}))}.$$

Theorem 23. *Let $(G\subset\Sigma,\mathcal{P})$ be a vertex partitioned graph in a surface Σ and $(M,D,M')=(B(G^*),D(G),C(G/\mathcal{P}))$. Then*
$$K_{(G\subset\Sigma,\mathcal{P})}(x,y,a,b)=K_{(M,D,M')}(x,y,a,b). \tag{25}$$

Proof. Since the rank function for a graph is the rank function of its cycle matroid, $r(A/\mathcal{P})$ and $r_{C(G/\mathcal{P})}(A)$ coincide. Observe that $N(V \cup E)$ and $N(V \cup A)$ can be viewed as ribbon graphs. It was shown in [22] in the proof of Theorem 8 that the function ρ from Equation (24) agrees with the function ρ from Equation (2) when $D = D(G)$ for a ribbon graph $G = N(V \cup E)$.

Next $r_{B(G^*)}(A) = r_{(C(G^*))^*}(A) = |A| + r_{C(G^*)}(A^c) - r_{C(G^*)}(E) = |A| + c_{G^*}(A^c) - c_{G^*}(E)$. We can compute $c_{G^*}(A^c)$ by observing that G^* and $(\Sigma\backslash N(V \cup E)) \cup N(E)$ have the same number of components, and then that the spanning subgraph $(V(G^*), A^c)$ of G^* and $(\Sigma\backslash N(V\cup E))\cup N(A^c) = \Sigma\backslash N(V\cup A)$ also do. Since $\Sigma\backslash N(V\cup\emptyset)$ and Σ have the same number of components, it then follows that $r_{B(G^*)}(A) = \#\text{cpts}(\Sigma\backslash N(V \cup A)) - \#\text{cpts}(\Sigma) = \kappa(A)$.

The theorem then follows by using these three identities to match up the exponents in each side of Equation (25). $\square$

Corollary 24. *Let $(G,\mathcal{P})$ be a vertex partitioned ribbon graph and $(D,M) = (D(G), C(G/\mathcal{P}))$. Then*

$$R_{(G,\mathcal{P})}(x+1, y, \sqrt{z/y}) = R_{(D,M)}(x, y, z).$$

Proof. By Equation (21) and Theorem 23 we have

$$R_{(D(G),C(G/\mathcal{P}))}(x,y,z) = K_{(B(G^*),D(G),C(G/\mathcal{P}))}(x,y,z,y) = K_{(G\subset\Sigma,\mathcal{P})}(x,y,z,y).$$

It was shown in [22] that $R_{(G,\mathcal{P})}(x,y,z) = K_{(G\subset\Sigma,\mathcal{P})}(x-1,y,yz^2,y)$, and so $K_{(G\subset\Sigma,\mathcal{P})}(x,y,z,y) = R_{(G,\mathcal{P})}(x,y,\sqrt{z/y})$. $\square$

Proposition 12 gives that the operations of duality, deletion, and contraction for delta-matroid perspectives and vertex partitioned graphs in surfaces are compatible with each each other. Consequently the results of Section 5 descend to the topological setting, giving results about $R_{(G,\mathcal{P})}$ and $K_{(G\subset\Sigma,\mathcal{P})}$. For brevity, we will not provide a statement for each of these results for the topological graph polynomials, but will record that the results can be deduced in the following.

Corollary 25. *Theorems 17, 19, and 20, and Corollary 18 specialise to give identities for the topological Tutte polynomials $R_{(G,\mathcal{P})}$ and $K_{(G\subset\Sigma,\mathcal{P})}$.*

We will now prove Theorem 21, which gave the matroidal analogues of the Bollobás-Riordan and Krushkal polynomials.

Proof of Theorem 21. Let $G \subset \Sigma$ be a graph in a surface. Let $\mathcal{P} = \{\{v\} \mid v \in V\}$, so $G/\mathcal{P} = G$. Then, from [22], $b^{\gamma(\Sigma)/2}K_{G\subset\Sigma}(x,y,a,1/b) = K_{(G\subset\Sigma,\mathcal{P})}(x,y,a,b)$, which by Theorem 23 equals $K_{(B(G^*),D(G),C(G))}(x,y,a,b)$.

Next, if G is a ribbon graph, write it as a cellularly embedded graph $G \subset \Sigma$. As G is cellularly embedded we have $\gamma(G) = \gamma(\Sigma)$. Then

$$\begin{aligned} R_{(D(G),D(G)_{\min})}(x,y,z) &= K_{(B(G^*),D(G),C(G))}(x,y,z,y) \\ &= y^{\gamma(G)/2}K_{G\subset\Sigma}(x,y,z,1/y) \\ &= y^{\gamma(G)/2}R_G(x+1,y,\sqrt{z/y}), \end{aligned}$$

where the last equality uses the writing of the Bollobás-Riordan polynomial in terms of the Krushkal polynomial from [11, 23]: $R_G(x,y,z) = y^{\frac{1}{2}\gamma(G)}K_{G\subset\Sigma}(x-1,y,yz^2,y^{-1})$. $\square$

Remark 26. Theorem 21 leads us to the surprising conclusion that, contrary to what has previously been understood, the Bollobás-Riordan polynomial is not a polynomial of cellularly embedded graphs (i.e., of ribbon graphs), but rather a polynomial of (not necessarily cellularly embedded) graphs in surfaces. This conclusion follows naturally from the exposition of the Bollobás-Riordan polynomial R_G presented here. We have shown that R_G should be understood as a polynomial of DM-perspectives, and so of vertex partitioned ribbon graphs. However, a more familiar setting for R_G can be obtained by observing that a DM-perspective contains exactly as much information as its dual. Therefore, we may view R_G as being associated with the dual of vertex partitioned ribbon graphs, i.e., graphs embedded in surfaces. This observation is notable because graphs embedded (but not necessarily cellularly) in surfaces are a much more familiar object in the community than vertex partitioned ribbon graphs.

The two standard examples of a matroid perspectives from graphs are given in Examples 1(3) and 1(4), namely $(B(G^*), C(G))$ and $(C(G), C(G/\mathcal{P}))$. Of course we can consider the Tutte polynomials of these matroid perspectives, $T_{(B(G^*),C(G))}(x,y,z)$ and $T_{(C(G),C(G/\mathcal{P}))}(x,y,z)$. The polynomial $T_{(B(G^*),C(G))}(x,y,z)$ is a polynomial of cellularly embedded graphs. It was considered in [26] and is known as the *Las Vergnas polynomial* of $G \subset \Sigma$ and denoted $L_{G\subset\Sigma}(x,y,z)$. It was shown in [2] (see also [11, 16]) that it can be obtained as a specialisation of the Krushkal polynomial: $L_{G\subset\Sigma}(x,y,z) = z^{\frac{1}{2}(s(E)-s^{\perp}(E))}K_{G\subset\Sigma}(x-1,y-1,z^{-1},z)$. The other natural graphic polynomial, $T_{(C(G),C(G/\mathcal{P}))}(x,y,z)$, is also related to the Krushkal polynomial:

Corollary 27.

1. *Let $(G \subset \Sigma, \mathcal{P})$ be a vertex partitioned graph in a surface Σ of genus zero. Then*

$$K_{(G\subset\Sigma,\mathcal{P})}(x,y,a,b) = a^{r(G/\mathcal{P})-r(G)}T_{(C(G),C(G/\mathcal{P}))}(x+1,y+1,1/a).$$

2. *Let $(G, \mathcal{P})$ be a vertex partitioned ribbon graph of genus zero. Then*

$$R_{(G,\mathcal{P})}(x,y,z) = (yz^2)^{r(G/\mathcal{P})-r(G)}T_{(C(G),C(G/\mathcal{P}))}(x,y+1,1/(yz^2)).$$

Proof. First observe since the graphs in question are in surfaces of genus zero, $B(G^*) = (C(G^*))^* = C(G)$ and $D(G) = C(G)$. Then for the Krushkal polynomial apply Theorem 23 to write $K_{G\subset\Sigma}$ as $K_{(C(G),C(G),C(G/\mathcal{P}))}$ and apply Proposition 16(17). For the Bollobás-Riordan polynomial apply Corollary 24 to write $R_{(G,\mathcal{P})}$ as $R_{(C(G),C(G/\mathcal{P}))}$ then apply Equations (21) and (17). □

Remark 28. The interactions with embedded graphs in this section explain why we need to look to delta-matroid perspectives for a framework for topological graph polynomials, rather than just triples containing arbitrary matroids and delta-matroids (i.e., why we need the matroid perspective conditions in Definitions 2 and 10). If we start with a graph in a surface and look at which vertex partitioned graphs in surfaces can arise via deletion and contraction (these operations are described in [22]) we find that exactly those that arise have $(B(G^*), D(G), C(G/\mathcal{P}))$ as a delta-matroid perspective. Without the matroid perspective conditions we are considering too coarse an object to properly represent the topological graphs. A similar comment holds for DM-perspectives.

7. Hopf algebraic constructions of the polyomials

In [22], T. Krajewski, I. Moffatt and A. Tanasa described how Hopf algebras have a natural Tutte polynomial associated with them. By using the fact that (suitable) sets of combinatorial objects equipped with a notion of deletion and a notion of contraction (such as matroids, graphs, matroid perspectives, ribbon graphs, etc.) admit natural Hopf algebra structures, this results in a canonical way to construct "Tutte polynomials" of various combinatorial classes. In [22] it was shown that various graph polynomials from the literature arise as canonical Tutte polynomials and are therefore unified by this algebraic framework. In this section we show that the polynomials $K_{(M,D,M')}$ and $R_{(D,M)}$ are the canonical Tutte polynomials associated with delta-matroid perspectives and with DM-perspectives, respectively. We will use this fact to deduce properties of $K_{(M,D,M')}$ and $R_{(D,M)}$ stated in Section 6 from the general theory presented in [22].

For the convenience of the reader, we begin with a brief review of the Tutte polynomials of Hopf algebras from [22], but working only in the generality needed here. It is not possible for the exposition in this section to be fully self-contained, but we aim to provide enough background for understanding the Hopf algebra framework as it is applied here. We use e^c to denote the set $E \setminus e$, where E is given by context.

Throughout, for each $i \in \mathbb{N}_0$, $\mathcal{H}_i$ will be a free module of finite rank over a unital ring $\mathbb{K}$, and $\mathcal{H}$ will be the graded module $\mathcal{H} := \bigoplus_{i\geq 0} \mathcal{H}_i$. (Since we work with free modules of finite rank the reader may assume that $\mathbb{K}$ is a field and work with vector spaces and their bases without losing the essence of the ideas.) For the application here, we require $|\mathcal{H}_0| = 1$ and assume this to be the case. In practice, we specify each $\mathcal{H}_i$ by a basis and work with $\mathcal{H}$ as the module of finite formal $\mathbb{K}$-linear combinations of basis elements. Throughout, "$\otimes$" denotes the tensor product of modules, and "$\oplus$" denotes the direct sum of modules.

The module $\mathcal{H}$ becomes a *commutative algebra* if we equip it with a module morphism $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, called a *product*, such that $(m \otimes \mathrm{id}) \circ m = (\mathrm{id} \otimes m) \circ m$. It is a *graded* algebra if $m : \mathcal{H}_i \otimes \mathcal{H}_j \to \mathcal{H}_{i+j}$ for all i, j. Coalgebras arise by "reversing the directions of the mappings" in the definition of an algebra. The module $\mathcal{H}$ becomes a *co-commutative coalgebra* if we equip it with a module morphism $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, called a *coproduct*, such that $\Delta \circ (\Delta \otimes \mathrm{id}) = \Delta \circ (\mathrm{id} \otimes \Delta)$. It is a *graded* coalgebra if in addition $\Delta : \mathcal{H}_k \to \bigoplus_{i+j=k} \mathcal{H}_i \otimes \mathcal{H}_j$, for all k.

$\mathcal{H}$ is a *Hopf algebra* if it is equipped with a product m making it into a commutative algebra, a coproduct Δ making it into a co-commutative coalgebra, a unit η, a counit ε, and an anti-homomorphism $S : \mathcal{H} \to \mathcal{H}$. Furthermore, m, Δ, η, ε, and S should satisfy various "compatibility relations". For brevity, and since an examination of their technical details is not needed here, we exclude the lengthy definitions of the unit, counit, and compatibility relations. These can be found in any standard Hopf algebra text such as [1]. A Hopf algebra is *graded* if it is graded as both an algebra and coalgebra, and *connected* if $|\mathcal{H}_0| = 1$.

The following example of a Hopf algebra of matroids is well-known (see, for example, [29]).

Example 29. The set of finite formal $\mathbb{Z}$-linear combinations of isomorphism classes of matroids forms a graded connected Hopf algebra. The grading is given by the cardinality of the ground set, multiplication is given by the direct sum of matroids, and a coproduct is given by $\Delta(M) = \sum_{A\subseteq E(M)} M\backslash A^c \otimes M/A$. Note that it also forms a (different) Hopf algebra if the contraction in the coproduct is replaced with a deletion, or the deletion with a contraction.

That the objects in Example 29 are matroids is not key to the Hopf algebra structure. What is key is that there is a set of combinatorial objects with some notion of 'edges', some notion of 'deletion', and some notion of 'contraction'. Many combinatorial objects have such notions and give rise to a Hopf algebra. The following definition from [22] was made to capture this essential structure.

Definition 30. A *minors system* consists of the following.

1. A graded set $\mathcal{S} = \bigcup_{n\geq 0} \mathcal{S}_n$ of finite combinatorial objects such that each $S \in \mathcal{S}_n$ has a finite set $E(S)$ of exactly n sub-objects associated with it, and such that there is a unique element $1 \in \mathcal{S}_0$.
2. Two *minor operations* $\setminus, /$ called *deletion* and *contraction*, respectively, that associate elements $S \setminus e$ and S/e, respectively, to each pair $(S \in \mathcal{S}_n, e \in E(S))$, where $E(S \setminus e) = E(S/e) = E(S) \setminus e$, and such that for $e \neq f$

$$(S \setminus e) \setminus f = (S \setminus f) \setminus e, \quad (S/e) \setminus f = (S \setminus f)/e, \quad (S/e)/f = (S/f)/e.$$

As an example, matroids with their usual deletion and contraction from a minors system.

From [22], minors systems have a natural Hopf algebra structure:

Proposition 31. *The module $\mathcal{H}$ of formal $\mathbb{Z}$-linear combinations of elements of a minors system $\mathcal{S}$ forms a coalgebra with counit under*

$$\Delta(S) = \sum_{A\subseteq E(S)} (S \setminus A^c) \otimes (S/A), \qquad \varepsilon(S) = \begin{cases} 1 & \text{if } S \in \mathcal{S}_0 \\ 0 & \text{otherwise.} \end{cases}$$

If, in addition, the vector space forms an algebra with multiplication m and unit η such that $\eta(1) \in \mathcal{S}_0$; for all $S_1, S_2 \in \mathcal{S}$

$$E(m(S_1 \otimes S_2)) = E(S_1) \sqcup E(S_2),$$

and for each $A_i \subseteq E(S_i)$

$$m(S_1 \setminus A_1 \otimes S_2 \setminus A_2) = m(S_1 \otimes S_2) \setminus (A_1 \sqcup A_2), \quad m(S_1/A_1 \otimes S_2/A_2) = m(S_1 \otimes S_2)/(A_1 \sqcup A_2),$$

then it is a graded connected Hopf algebra.

The set of isomorphism classes of delta-matroid perspectives with deletion and contraction forms a minors system where the grading is given by the cardinality of the ground set. Proposition 31 then gives the following.

Proposition 32. *The set of finite formal $\mathbb{Z}$–linear combinations of isomorphism classes of delta-matroid perspectives forms a graded connected Hopf algebra where the grading is given by the cardinality of the ground set, multiplication is given by direct sum and coproduct is given by*

$$\Delta((M, D, M')) = \sum_{A\subseteq E} (M, D, M')\backslash A^c \otimes (M, D, M')/A,$$

where E is the ground set of (M, D, M). We will refer to this Hopf algebra as the Hopf algebra of delta-matroid perspectives*.*

We can now move on to canonical Tutte polynomials. Suppose that $\mathcal{H} = \bigoplus_{i\geq 0} \mathcal{H}_i$ is a graded connected Hopf algebra. If f and g are mappings from $\mathcal{H}$ into some algebra with product m, then their *convolution product*, $f * g$, is the mapping from $\mathcal{H}$ defined by $f * g := m \circ (f \otimes g) \circ \Delta$. Let $\{S_i\}_{i\in I}$ be a basis for $\mathcal{H}_1$ and, for each $i \in I$, let $\delta_i : \mathcal{H} \to \mathbb{K}$ be the linear extension of

$$\delta_i(S) := \begin{cases} 1 & \text{if } S = S_i \\ 0 & \text{otherwise} \end{cases}. \tag{26}$$

Let $\{x_j\}_{j\in J}$ and $\{y_j\}_{j\in J}$ sets of indeterminates, and $a_i \in \mathbb{K}[\{x_j\}_{j\in J}]$, and $b_i \in \mathbb{K}[\{y_j\}_{j\in J}]$, for each $i \in I$. Define

$$\delta_{\mathbf{a}} := \sum_{i\in I} a_i\delta_i \quad \text{and} \quad \delta_{\mathbf{b}} := \sum_{i\in I} b_i\delta_i.$$

We can consider the $*$-exponentials of these mappings:

$$\exp_*(\delta_{\mathbf{a}}) = \sum_{m\geq 0} \frac{{\delta_{\mathbf{a}}}^{*m}}{m!} = \varepsilon + \delta_{\mathbf{a}} + \frac{1}{2}(\delta_{\mathbf{a}} * \delta_{\mathbf{a}}) + \cdots,$$

The $*$-exponential of $\delta_{\mathbf{b}}$, $\exp_*(\delta_{\mathbf{b}})$, is defined similarly.

The *Tutte polynomial* of $\mathcal{H}$, is defined by

$$\alpha(\mathbf{a}, \mathbf{b}) := \exp_*(\delta_{\mathbf{a}}) * \exp_*(\delta_{\mathbf{b}}) \in \mathbb{K}[\{x_j, y_j\}_{j\in J}]. \tag{27}$$

We call $\alpha(\mathbf{a}, \mathbf{b})$ the *canonical Tutte polynomial* of an object if it is the Tutte polynomial of the Hopf algebra associated with a minors system of those objects with their natural deletion and contraction operations.

The point here is that a minors system gives rise to a graded connected Hopf algebra and therefore has a canonical Tutte polynomial. Many classes of combinatorial object form minors systems and therefore have a canonical Tutte polynomial. For example, $T_M(x, y)$ is the canonical Tutte polynomial of matroids, and $T_{M,M'}(x, y, z)$ the canonical Tutte polynomial of matroid perspectives.

Identifying a canonical Tutte polynomial can be tricky, and to do so we make use of the following theorem from [22].

Theorem 33. *Let $\mathcal{H}$ be a combinatorial Hopf algebra of a minors system $\mathcal{S}$. Suppose that a set I indexes the elements of $\mathcal{H}_1$, and that the functions δ_i are defined by Equation (26). Suppose also that for each j in some indexing set J there is a function $r_j : \mathcal{H} \to \mathbb{Q}$ such that*

$$r_j(S) = r_j(S/e) + m_{ij} \qquad \textit{when } \delta_i(S\backslash e^c) = 1, \tag{28}$$

where $S \in \mathcal{H}$, $e \in S$ and $m_{ij} \in \mathbb{Q}$; and such that $r_j(S) = 0$ when $S \in \mathcal{H}_0$.

For a set of indeterminates $\{x_j\}_{j\in J}$ define

$$\delta_{\mathbf{a}} := \sum_{i\in I} a_i\delta_i \quad \textit{where} \quad a_i := \prod_{j\in J} x_j^{m_{ij}}, \tag{29}$$

and if $\delta_{\mathbf{b}} := \sum_{i\in I} b_i\delta_i$ with $b_i := \prod_{j\in J} y_j^{m_{ij}}$, the Tutte polynomial of $\mathcal{H}$ satisfies

$$\alpha(\mathbf{a}, \mathbf{b})(S) = \prod_{j\in J} y_j^{r_j(S)} \sum_{A\subseteq E(S)} \prod_{j\in J} \left(\frac{x_j}{y_j}\right)^{r_j(A)}, \tag{30}$$

where $r_i(A) := (S\backslash A^c)$.

It is informative to see an application of Theorem 33 in a simple case before tackling a more involved one.

Example 34. For the Hopf algebra of matroids from Example 29, $\{U_{1,1}, U_{0,1}\}$ forms a basis of $\mathcal{H}_1$. Let δ_c take the value 1 on $U_{1,1}$, and 0 otherwise: and let δ_l take the value 1 on $U_{0,1}$, and 0 otherwise. Take taking $r_1(M) := r(M)$ to be the rank function of a matroid, and $r_2 := |E(M)| - r(M)$ to be nullity. Then $\delta_{\mathbf{a}} = x_1\delta_c + x_2\delta_l$ and $\delta_{\mathbf{b}} = y_1\delta_c + y_2\delta_l$. Theorem 33 then gives $\alpha(\mathbf{a}, \mathbf{b})(M) = x_1^{r(M)} y_2^{|E(M)|-r(M)} T_M\left(y_1/x_1 + 1, x_2/y_2 + 1\right)$, the classical Tutte polynomial of a matroid.

Theorem 35. *The polynomial $K_{(M,D,M')}(x, y, a, b)$ is the canonical Tutte polynomial of the Hopf algebra of delta-matroid perspectives. In particular,*

$$\alpha(\mathbf{a}, \mathbf{b})(M, D, M') = K_{(M,D,M')}(x, y, a, b) \tag{31}$$

where $\mathbf{a} = (a_1, \dots, a_5) = (1, y, a, \sqrt{ab}, b)$, $\mathbf{b} = (b_1, \dots, b_5) = (x, 1, 1, 1, 1)$, and $S_1, \dots, S_5$ are as in Example 3.

Proof. We will obtain the result as an application of Theorem 33. For this let (M, D, M') be a delta-matroid perspective over E with r the rank function of M and r' the rank function of M'. Let r_1, r_2, r_3, and r_4 be defined by

$$r_1((M, D, M')) := r'(M'), \qquad r_2((M, D, M')) := |E| - r(M),$$
$$r_3((M, D, M')) := \rho(D) - r'(M'), \qquad r_4((M, D, M')) := r(M) - \rho(D).$$

To apply Theorem 33 we need to determine the values of m_{ij} such that $r_j((M, D, M')) = r_j((M, D, M')/e) + m_{ij}$ when $\delta_i((M, D, M')\backslash e^c) = 1$. That is we need to relate $r_j((M, D, M'))$ and $r_j((M, D, M')/e)$ with cases determined by when $(M, D, M')\backslash e^c$ is each of $S_1, \dots, S_5$. Each r_j is given in terms of r, r' and ρ so we start by determining how these are changed by contraction.

Two standard properties of matroids are that

$$r(M) = \begin{cases} r(M/e) & \text{if } e \text{ is a loop in } M, \\ r(M/e) + 1 & \text{otherwise} \end{cases} \tag{32}$$

and that e is a loop in M if and only if $M\backslash e^c \cong U_{0,1}$. A similar statement holds for r'. In [22] it was shown that

$$\rho(D) = \begin{cases} \rho(D/e) + 1 & \text{if } e \text{ is a not a ribbon loop}, \\ \rho(D/e) & \text{if } e \text{ is an orientable a ribbon loop}, \\ \rho(D/e) + \frac{1}{2} & \text{if } e \text{ is a non-orientable a ribbon loop}, \end{cases} \tag{33}$$

and also that e is not a ribbon loop (is an orientable ribbon loop, is a non orientable ribbon loop) if and only if $D\backslash e^c$ is isomorphic to D_c (D_o, D_n, respectively).

We then see that $\delta_1((M, D, M')\backslash e^c) = 1$ if and only if e is not a loop in M or M' and e is not a ribbon loop in D. Applying Equations (32) and (33) then gives that in this case $r_1((M, D, M')) = r_1((M, D, M')/e) + 1$, so $m_{11} = 1$, and $r_j((M, D, M')) = r_j((M, D, M')/e)$ for $j = 2, 3, 4$, giving $m_{12} = m_{13} = m_{14} = 0$.

Proceeding similarly for the cases when $\delta_i((M, D, M')\backslash e^c) = 1$ for $i = 2, 3, 4, 5$ gives $m_{22} = m_{33} = m_{54} = 1$, $m_{43} = m_{44} = \frac{1}{2}$, and that all other m_{ij} are zero.

An application of Theorem 33 gives $\mathbf{a} = (a_1, \dots, a_5) = (x_1, x_2, x_3, \sqrt{x_3x_4}, x_4)$, $\mathbf{b} = (b_1, \dots, b_5) = (y_1, y_2, y_3, \sqrt{y_3y_4}, y_4)$ and

$$\alpha(\mathbf{a}, \mathbf{b})(M, D, M') = y_1^{r'(M')} y_2^{|E|-r(M)} y_3^{\rho(D)-r'(M')} y_4^{r(M)-\rho(D)} \sum_{A\subseteq E} \left(\frac{x_1}{y_1}\right)^{r'(A)} \left(\frac{x_2}{y_2}\right)^{|A|-r(A)} \left(\frac{x_3}{y_3}\right)^{\rho(A)-r'(A)} \left(\frac{x_4}{y_4}\right)^{r(A)-\rho(A)} \qquad (34)$$

The theorem then follows by setting $y_1 = x$, $x_2 = y$, $x_3 = a$, $x_4 = b$, and all other x_j and y_j to 1. □

Remark 36. The argument in the proof of Theorem 35 can easily be adapted to show that $R_{(D,M')}$ is the canonical Tutte polynomial of a Hopf algebra of DM-perspectives. This Hopf algebra is defined in a way analogous to the Hopf algebra of delta-matroid perspectives. For the construction of the canonical Tutte polynomial we can take a basis of the graded dimension one subspace to be $S_1 = (D_c, U_{1,1})$, $S_2 = (D_o, U_{0,1})$, $S_3 = (D_n, U_{0,1})$, and $S_4 = (D_c, U_{0,1})$; and set $r_1(D, M) = r(M)$, $r_2(D, M) = |E| - \rho(D)$, $r_3(D, M) = \rho(D) - r(M)$. Proceeding as in the proof of Theorem 35 will give that $\alpha(\mathbf{a}, \mathbf{b})(D, M) = R_{(D,M)}(x, y, z)$ where $\mathbf{a} = (1, y, \sqrt{yz}, z)$, $\mathbf{b} = (x, 1, 1, 1)$. Corollaries 18 and 20(2) can be deduced from this result.

Remark 37. The framework of canonical Tutte polynomials explains the distinction between the matroidal frameworks for the polynomials presented here, and a result of L. Traldi from [30] where it was shown that $R_G(x, y, z)$ can be obtained by information in the transition matroid of G, and a result of C. Chun, I. Moffatt, S. Noble and R. Rueckriemen in [12] where $R_G(x, y, z)$ was shown to be a special case of a delta-matroid polynomial. In the present setting these polynomials arise canonically from the Hopf algebras of delta-matroid perspectives and DM-perspectives, but they are not the polynomials associated with Hopf algebras of transition matroids or delta-matroids (e.g., the 2-variable specialisation $x^{\gamma(G)/2} R_G(x + 1, y, 1/\sqrt{xy})$ arises canonically from the delta-matroid framework).

Proof of Theorem 17. An application of Theorem 13 of [22] shows that every canonical Tutte polynomial $\alpha(\mathbf{a}, \mathbf{b})$ arising from Theorem 33 has a deletion-contraction definition. In particular, By Theorem 35, the result of [22] gives that $K_{(M,D,M')}(x, y, a, b)$ is uniquely defined by

$$K_{(M,D,M')} = \begin{cases} \delta_{\mathbf{b}}((M, D, M')/e^c) \cdot K_{(M,D,M')\backslash e} + \delta_{\mathbf{a}}((M, D, M')\backslash e^c) \cdot K_{(M,D,M')/e} & \text{if } E \neq \emptyset \\ 1 & \text{if } E = \emptyset \end{cases}$$

where $\delta_{\mathbf{a}}$ and $\delta_{\mathbf{b}}$ are as in Theorem 35. The non-trivial part of this deletion-contraction relation has twenty five cases corresponding to $(M, D, M')/e^c$ and $(M, D, M')\backslash e^c$ being isomorphic to each of $S_1, \dots, S_5$. We need to reduce these twenty five cases to the six of the theorem statement.

For this first observe that since $\mathbf{b} = (x, 1, 1, 1, 1)$, we have $\delta_{\mathbf{b}}((M, D, M')/e^c) = 1$ unless $(M, D, M')/e^c \cong S_1$ in which case it equals x. This reduces the twenty five cases to ten. We can reduce the five cases in which $\delta_{\mathbf{b}}((M, D, M')/e^c) = x$ by observing that if $(M, D, M')/e^c \cong S_1 = (U_{1,1}, D_c, U_{1,1})$ then $M'/e^c \cong U_{1,1}$ and therefore e is a coloop of M'. But then $M'\backslash e^c \cong U_{1,1}$ and so it is necessarily the case that $(M, D, M')\backslash e^c \cong S_1 = (U_{1,1}, D_c, U_{1,1})$, since this is the only delta-matroid perspective over one element with $U_{1,1}$ in the last position. Thus if $(M, D, M')/e^c$ is isomorphic to S_1 then so is $(M, D, M')\backslash e^c$, so the five cases in which $\delta_{\mathbf{b}}((M, D, M')/e^c) = x$ reduces to Case 1 of the theorem.

We have five cases remaining: when $(M,D,M')/e^c \ncong S_1$ and $(M,D,M')\backslash e^c \cong S_i$, for each of $i = 1,\dots,5$. We match these up with Cases 2–6 of the theorem statement.

If $(M,D,M')\backslash e^c \cong S_2 = (U_{0,1}, D_o, U_{0,1})$ then e must be a loop in M. Since S_2 is the only delta-matroid perspective with $U_{0,1}$ in the first position, if e is a loop in M then $(M,D,M')\backslash e^c \cong S_2$. This reduces to Case 2 of the theorem statement.

The remaining cases then each follow similarly by using that e is not a coloop in M', e is a loop if and only if $M'\backslash e^c \cong U_{0,1}$, and, from [22], that e is not a ribbon loop (is an orientable ribbon loop, is a non orientable ribbon loop) if and only if $D\backslash e^c$ is isomorphic to D_c (D_o, D_n respectively). We have also used the facts that since $(M, D_{\max})$, $(D_{\max}, D_{\min})$, $(D_{\min}, M')$ are matroid perspectives then if e is a loop in M it is a loop in $D_{\min}$ and M' so is not a ribbon loop, and if e is a coloop in M' then it is a coloop in M. □

Proof of Theorem 19. By Theorem 35 we can write $K_{(M,D,M')}(x,y,a,b) = \alpha(\mathbf{a},\mathbf{b})(M,D,M')$ where $\mathbf{a} = (a_1,\dots,a_5)$ and $\mathbf{b} = (b_1,\dots,b_5)$ (we will avoid the specialisation of the a_i and b_i for the moment). Theorem 17 of [22] gives that

$$\alpha(\mathbf{a},\mathbf{b})((M,D,M')) = \alpha(\mathbf{b}^*,\mathbf{a}^*)((M,D,M')^*), \tag{35}$$

where $\mathbf{a}^* = (a_1^*,\dots,a_5^*)$, each a_i^* is defined by $\delta_{\mathbf{a}^*} = \sum_{i=1}^5 a_i^*\delta_i = \delta_{\mathbf{a}} \circ *$ where $* : (M,D,M') \mapsto (M,D,M')^*$ is the duality map, and where $\mathbf{b}^*$ is defined similarly. Now $S_1^* = S_2$, $S_3^* = S_5$, and $S_4^* = S_4$, so $a_1^* = \delta_{\mathbf{a}} \circ *(S_1) = \delta_{\mathbf{a}}(S_2) = a_2$. Similar computations for the remaining a_i^* give $\mathbf{a}^* = (a_2,a_1,a_5,a_4,a_3)$ and $\mathbf{b}^* = (b_2,b_1,b_5,b_4,b_3)$. Finally, taking $\mathbf{a} = (1,y,a,\sqrt{ab},b)$, $\mathbf{b} = (x,1,1,1,1)$ gives $\mathbf{a}^* = (y,1,b,\sqrt{ab},a)$, $\mathbf{b}^* = (1,x,1,1,1)$, then Equations (35) and (34) give

$$\begin{aligned}
K_{(M,D,M')^*}(x,y,a,b) &= \alpha(\mathbf{a},\mathbf{b})(M,D,M')^*) \\
&= \alpha(\mathbf{b}^*,\mathbf{a}^*)(M,D,M') \\
&= a^{r(M)-\rho(D)} b^{\rho(D)-r'(M')} K_{(M,D,M')}(y,x,b^{-1},a^{-1}).
\end{aligned}$$

□

Proof of Theorem 20. From Theorem 35, and the definition of $\alpha(\mathbf{a},\mathbf{b})$ given in Equation (27), and the definition of the convolution product $*$, we can write

$$\begin{aligned}
K_{(M,D,M')}(x,y,a,b) &= \alpha(\mathbf{a},\mathbf{b})(M,D,M') \\
&= [\exp_*(\delta_{\mathbf{a}}) * \exp_*(\delta_{\mathbf{b}})](M,D,M') \\
&= [\exp_*(\delta_{\mathbf{a}}) * \exp_*(\delta_{\mathbf{c}}) * \exp_*(\delta_{-\mathbf{c}}) * \exp_*(\delta_{\mathbf{b}})](M,D,M') \\
&= [\exp_*(\delta_{\mathbf{a}}) * \exp_*(\delta_{\mathbf{c}})] \otimes [\exp_*(\delta_{-\mathbf{c}}) * \exp_*(\delta_{\mathbf{b}})](\Delta((M,D,M'))) \\
&= \sum_{A\subseteq E} \alpha(\mathbf{a},\mathbf{c})((M,D,M') \setminus A^c) \cdot \alpha(-\mathbf{c},\mathbf{b})((M,D,M')/A^c) \\
&= \sum_{A\subseteq E} K_{(M,D,M')\backslash A^c}(-1,y,a,b) \cdot K_{(M,D,M')/A}(x,-1,-1,-1),
\end{aligned}$$

where $\mathbf{a} = (1,y,a,\sqrt{ab},b)$, $\mathbf{b} = (x,1,1,1,1)$, $\mathbf{c} = (-1,1,1,1,1)$, and we choose the negative root when evaluating $K_{(M,D,M')/A}(x,-1,-1,-1)$.

The second result of the Theorem follows by applying its first item to $K_{(D_{\max},D,M)}(x,y,z,y)$ then applying Equation (21). □

Remark 38. In this paper we discussed a Tutte polynomial of DM-perspectives, but not MD-perspectives. Our reason for favouring one over the other is the connection with the Bollobás-Riordan polynomial given in Theorem 21. Our work here adapts to the setting of MD-perspectives. The Hopf algebra of MD-perspectives is defined in a way analogous to the Hopf algebra of delta-matroid perspectives. For the construction of its canonical Tutte polynomial we can take a basis of the graded dimension one subspace to be $S_1 = (U_{0,1}, D_o)$, $S_2 = (U_{1,1}, D_c)$, $S_3 = (U_{1,1}, D_n)$, and $S_4 = (U_{1,1}, D_o)$; and set $r_1(M, D) = |E| - r(M)$, $r_2(M, D) = \rho(D)$, $r_3(M, D) = r(M) - \rho(D)$. Proceeding as in the proof of Theorem 35 will give that

$$\alpha(\mathbf{a}, \mathbf{b})(M, D) = y_1^{|E|-r(M)} y_2^{\rho(D)} y_3^{r(M)-\rho(D)} \sum_{A \subseteq E} \left(\frac{x_1}{y_1}\right)^{|A|-r(A)} \left(\frac{x_2}{y_2}\right)^{\rho(A)} \left(\frac{x_3}{y_3}\right)^{r(A)-\rho(A)},$$

where $\mathbf{a} = (x_1, x_2, \sqrt{x_2 x_3}, x_3)$ and $\mathbf{b} = (y_1, y_2, \sqrt{y_2 y_3}, y_3)$. We choose the specialisation $\mathbf{a} = (x, 1, 1, 1)$ and $\mathbf{b} = (1, y, \sqrt{yz}, z)$, and define $R_{(M,D)}(x, y, z) = \alpha(\mathbf{a}, \mathbf{b})(M, D)$ for these values of $\mathbf{a}$ and $\mathbf{b}$. Thus we have

$$R_{(M,D)}(x, y, z) = \sum_{A \subseteq E} x^{|A|-r(A)} y^{\rho(D)-\rho(A)} z^{(r(M)-\rho(D))-(r(A)-\rho(A))}.$$

This polynomial is a specialisation of our delta-matroid perspective polynomial

$$R_{(M,D)}(x, y, z) = y^{\rho(D)-r'(M')} z^{r(M)-\rho(D)} K_{(M,D,M')}(y, x, y^{-1}, z^{-1}).$$

However, this polynomial of MD-perspectives does not contain any information that the DM-perspective does not, since the two polynomials are related by duality:

$$R_{(M,D)}(x, y, z) = R_{(D^*,M^*)}(x, y, z).$$

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