

Regular Black Hole Solutions of the Non-minimally Coupled $Y(R)F^2$ Gravity

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Abstract

In this study we investigate regular black hole solutions of the non-minimally coupled $Y(R)F^2$ gravity model. We give two regular black hole solutions and the corresponding non-minimal model for both electrically or magnetically charged cases. We calculate all the energy conditions for these solutions.

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I. INTRODUCTION

To understand the nature of singularities in theories of gravity is a challenging problem. It can be considered that a Quantum Theory of Gravity may solve this problem. As for now, we are very far from the Quantum Theory of Gravity, we can avoid the singularities with regular black hole solutions. The regular black hole solution first were given by Bardeen [1]. Later, this solution of Bardeen was obtained from the field equations of the Einstein-Nonlinear electrodynamics [2]. It is interesting to obtain some regular black hole solutions of the theory which is $f(R)$ minimally coupled to the Non-linear electrodynamics. In recent years, various new regular black holes were proposed and investigated increasingly in the literature [3]-[16] (see for a review [5]). Since the non-minimally coupled $Y(R)$ -Maxwell models [17–25] have some solutions which can explain the rotation curves of galaxies and cosmic acceleration of the universe, then; it is natural to seek regular black hole solutions of the non-minimally coupled electromagnetic fields to gravity. We focus on this subject in this paper.

According to the Penrose-Hawking singularity theorem [4], to arise a singularity inside the horizon of a black hole, the strong energy condition (SEC) has to be satisfied. The regular black holes violate the strong energy condition in the central region inside the black hole. We find various models with the non-minimally coupled $Y(R)$ function for some known regular metric functions. We calculate the energy conditions for the effective energy-momentum tensor of these models. Then we find that they lead to a negative tangential pressure in the central core, and the effective equation of state with negative radial pressure $p_r = -\rho$ is everywhere, which is important for the accelerated expansion phase of the Universe. We see that at least SEC is violated by these solutions in some central regions of the black holes.

II. THE GRAVITATIONAL MODEL WITH $Y(R)F^2$ -TYPE COUPLING

We start with the action with the $Y(R)F^2$ -type non-minimal coupling term [21–23]

$$I[e^a, \omega^a_b, F] = \int_M \left\{ \frac{1}{2\kappa^2} R * 1 - \frac{1}{2} Y(R) F \wedge *F + \lambda_a \wedge T^a \right\}. \quad (1)$$

Here $\{e^a\}$ is the co-frame 1-form, $\{\omega^a{}_b\}$ is the connection 1-form, $F = dA = \frac{1}{2}F_{ab}e^a \wedge e^b$ is the homogeneous electromagnetic field 2-form, λ_a is the Lagrange multiplier 2-form whose variation leads to the torsion-free Levi-Civita connection. Then the connection can be found from $T^a = de^a + \omega^a{}_b \wedge e^b = 0$. In this action, R is the curvature scalar which can be obtained by this operation $\iota_{ba}R^{ab} = R$ from the curvature tensor 2-forms $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ via the interior product ι_a , and $\kappa^2 = 8\pi G$ is universal gravitational coupling constant. We take the space-time metric $g = \eta_{ab}e^a \otimes e^b$ with the signature $(-+++)$. We set the orientated volume element as $*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$.

We obtain gravitational and electromagnetic field equations of the theory by taking infinitesimal variations of the action according to independent variations of $\{e^a\}$, $\{\omega^a{}_b\}$ and $\{A\}$ [21, 22]

$$-\frac{1}{2\kappa^2}R^{bc} \wedge *e_{abc} = \frac{1}{2}Y(\iota_a F \wedge *F - F \wedge \iota_a *F) + \frac{1}{2}Y_R F_{mn} F^{mn} *R_a + \frac{1}{2}D[l^b D(Y_R F_{mn} F^{mn})] \wedge *e_{ab} , \quad (2)$$

$$d(*YF) = 0 , \quad dF = 0 \quad (3)$$

where $Y_R = \frac{dY}{dR}$. The gravitational field equation (2) can be written as

$$\frac{G^a}{\kappa^2} = \tau^a \quad (4)$$

where $G_a = -\frac{1}{2}R^{bc} \wedge *e_{abc} = *R_a - \frac{1}{2}R *e_a$ is the Einstein tensor, and $\tau_a = \tau_{a,b} *e^b$ is the effective energy momentum tensor for this non-minimally coupled model, which is equal to right hand side of (2). The effective energy density, radial pressure, and tangential pressures are found from $\rho = \tau_{0,0}$, $p_r = \tau_{1,1}$, $p_t = \tau_{2,2} = \tau_{3,3}$ using the field equation (4).

III. REGULAR BLACK HOLE SOLUTIONS

We seek regular black hole solutions for the following (1+3)-dimensional spherically symmetric static line element

$$g = -f^2(r)dt^2 + f^{-2}(r)dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2 \quad (5)$$

and consider the electromagnetic tensor F which may have electric and magnetic components

$$F = E(r)dr \wedge dt + B(r)r^2 \sin \theta d\theta \wedge d\phi = E(r)e^1 \wedge e^0 + B(r)e^2 \wedge e^3. \quad (6)$$

The field equations of the model for these ansatz turn out to five equations (three gravitational, two electromagnetic) and four unknown functions (E, B, Y, f) . These equations can be found in [26]. The homogeneous electromagnetic field condition $dF = 0$ determines the magnetic field as

$$B = \frac{q}{r^2} \quad (7)$$

where q is a real integration constant representing magnetic monopole charge. It is impossible to solve these equations without any simplification. Then, we use the following constraint to simplify these equations

$$Y_R(E^2 - B^2) = \frac{1}{\kappa^2}. \quad (8)$$

Under this constraint the number of equations decreases to two

$$f^{2''} - \frac{2}{r^2}(f^2 - 1) = \kappa^2 Y(E^2 + B^2), \quad (9)$$

$$YE = \frac{q_e}{r^2}, \quad (10)$$

where q_e is the electric charge. We note that one can find the constraint (8) by taking differential of the equation (9). To show this, we rewrite the equation (9) using the magnetic field $B = \frac{q}{r^2}$ from (7) and the electric field $E = \frac{q_e}{Yr^2}$ from (10) and find

$$r^4 \left(f^{2''} - \frac{2}{r^2}(f^2 - 1) \right) = \kappa^2 \left(\frac{q_e^2}{Y} + q^2 Y \right). \quad (11)$$

After taking differential of equation (11) we obtain

$$\left(-f^{2'''} - \frac{4}{r}f^{2''} + \frac{2}{r^2}f^{2'} + \frac{4}{r^3}(f^2 - 1) \right) dr = \kappa^2 \left(\frac{q_e^2}{Y^2 r^4} - \frac{q^2}{r^4} \right) dY. \quad (12)$$

We see that left hand side of the equation (12) is equal to differential of the curvature scalar $R = -f^{2''} - \frac{4}{r}f^{2'} - \frac{2}{r^2}(f^2 - 1)$. Using the equations (7), (10) and dR in (12) we obtain the following equation

$$dR = \kappa^2(E^2 - B^2)dY \quad (13)$$

which is differential form of the constraint (8). Thus, it is obvious that we have two equations (9), (10) and three unknowns (f, Y, E). Then, for a given model with a non-minimal function $Y(R)$, we can determine the metric function $f(r)$ from these equations. On the other hand, for a desiring metric we can reach a corresponding model with a non-minimal $Y(R)$ function. In order to be successful in this process, we need to solve r from $R(r)$ and express the function Y depending on R . Then we can determine the corresponding model.

A. Regular Black Hole Solution-1

The field equations of this non-minimal $Y(R)F^2$ model (9), (10) accept the following regular black hole solution

$$f^2(r) = 1 - \frac{2m}{r} \left(1 - \frac{1}{(1 + a^3 r^3)^{1/3}} \right) \quad (14)$$

with the magnetic field ($E = 0$) and the non-minimal function

$$B(r) = \frac{q}{r^2} \quad (15)$$

$$Y(r) = \frac{8ma^6 r^7}{\kappa^2 q^2 (1 + a^3 r^3)^{7/3}} \quad (16)$$

where we have defined a new constant $a = \frac{2m}{q^2}$. The metric function (14) can be found in [11] as an electrically charged solution of Einstein-Non-linear Electrodynamics. In this notation, we calculate the curvature scalar R and the invariant 4-form $R_{ab} \wedge *R^{ab}$ for the metric function (14)

$$R(r) = \frac{8ma^3}{(1 + a^3 r^3)^{7/3}} \quad (17)$$

$$R_{ab} \wedge *R^{ab} = \left(\frac{24m^2}{r^6} - \frac{48m^2}{r^6 p} - \frac{32m^2 a^6}{p^7} + \frac{24m^2}{r^6 p^2} + \frac{40m^2 a^6}{p^8} + \frac{32m^2 a^{12} r^6}{p^{14}} \right)$$

$$-\frac{16m^2a^3}{r^3p^4} + \frac{16m^2a^3}{r^3p^5}) * 1 \quad (18)$$

where we have defined $p = (1 + a^3r^3)^{1/3}$. When we check the limits

$$\lim_{r \rightarrow 0} R = 8ma^3 = \frac{64m^4}{q^6} \quad (19)$$

$$\lim_{r \rightarrow 0} R_{ab} \wedge *R^{ab} = \frac{16}{3}m^2a^6 * 1 = \frac{2^{10}m^8}{3q^{12}} * 1, \quad (20)$$

they are regular at the center of black hole. It needs to solve r from (17) to rewrite the non-minimal function Y in terms of R

$$r(R) = \frac{1}{a} \left(\left(\frac{8ma^3}{R} \right)^{3/7} - 1 \right)^{1/3}. \quad (21)$$

When we substitute the inverse function (21) in the non-minimal function (16) we obtain

$$Y(R) = \frac{[(8ma^3)^{3/7} - R^{3/7}]^{7/3}}{\kappa^2 q^2 a^4} \quad (22)$$

and the corresponding model is written as:

$$L = \frac{1}{2\kappa^2} R * 1 - \frac{[(8ma^3)^{3/7} - R^{3/7}]^{7/3}}{2\kappa^2 q^2 a^4} F \wedge *F + \lambda_a \wedge T^a. \quad (23)$$

After the duality transformation $B \rightarrow -YE$, $q \rightarrow -q_e$ and $Y \rightarrow \frac{1}{Y}$; which is given in [26], we reach the field equations of the model for the electromagnetic tensor F with only electric component ($B = 0$, $E \neq 0$). As a consequence of this transformation, the same metric function (14) determines the electric field and the non-minimal function of this model as follows

$$Y(r) = \frac{\kappa^2 q_e^2 (1 + a^3 r^3)^{7/3}}{8ma^6 r^7} \quad (24)$$

$$E(r) = \frac{4q_e}{\kappa^2 r^2} \left(1 + \frac{1}{a^3 r^3} \right)^{-7/3} = \frac{q_e}{Y(r)r^2} \quad (25)$$

with $a = \frac{2m}{q_e}$. We see that the electric field is regular at the center of black hole, $\lim_{r \rightarrow 0} E(r) = 0$. We can rewrite the non-minimal function (24) in terms of R as

$$Y(R) = \frac{\kappa^2 q_e^2 a^4}{[(8ma^3)^{3/7} - R^{3/7}]^{7/3}} \quad (26)$$

and we write the corresponding Lagrangian of the model

$$L = \frac{1}{2\kappa^2} R * 1 - \frac{\kappa^2 q_e^2 a^4}{2[(8ma^3)^{3/7} - R^{3/7}]^{7/3}} F \wedge *F + \lambda_a \wedge T^a \quad (27)$$

via the duality transformation.

The same metric function (14) and an electric field different from (25) only up to a scale factor was obtained from a different theory with Einstein-nonlinear electrodynamics in [11].

It is important to check all the energy conditions for this solution. The conditions which are calculated as below have to be equal or greater than zero for rising a singularity in General Relativity. But in the regular black holes at least the strong energy condition has to be violated. To show this we firstly calculate the energy density $\rho(r)$, the radial and tangential pressures p_r, p_t for the metric function (14) and the magnetic field (15). We note that all the following results also can be obtained from the solutions with the electric field (25) which has the electric charge $q_e = q$

$$\rho(r) = \frac{16m^4 q^2}{\kappa^2 (q^6 + 8m^3 r^3)^{4/3}} = -p_r(r) \quad p_t(r) = \frac{16q^2 m^4 (8m^3 r^3 - q^6)}{\kappa^2 (q^6 + 8m^3 r^3)^{7/3}}. \quad (28)$$

We find the following energy conditions using the energy density $\rho(r)$, the radial and tangential pressures p_r, p_t

$$DEC_1 = \rho \geq 0, \quad (29)$$

$$NEC_1 = WEC_1 = \rho + p_r = 0, \quad (30)$$

$$NEC_2 = WEC_2 = \rho + p_t = \frac{2^8 m^7 q^2 r^3}{\kappa^2 (q^6 + 8m^3 r^3)^{7/3}}, \quad (31)$$

$$SEC = \rho + p_r + 2p_t = \frac{32m^4 q^2 (8m^3 r^3 - q^6)}{\kappa^2 (q^6 + 8m^3 r^3)^{7/3}}, \quad (32)$$

$$DEC_2 = \rho - p_r = 2\rho, \quad (33)$$

$$DEC_3 = \rho - p_t = \frac{32m^4 q^8}{\kappa^2 (q^6 + 8m^3 r^3)^{7/3}}. \quad (34)$$

Thus, we see that all the energy conditions are satisfied in the region $r \geq \frac{q^2}{2m}$ for the electrically or magnetically charged solutions. But, only the SEC is violated in the central region $r < \frac{q^2}{2m}$.

B. Regular Black Hole Solution-2

Secondly, we take the metric function from [6] and [16] with the electric charge q_e

$$f^2(r) = 1 - \frac{2m}{r} e^{-\frac{q_e^2}{2mr}}. \quad (35)$$

The Ricci scalar and $R_{ab} \wedge *R^{ab}$ are calculated using the metric function (35) as

$$R(r) = \frac{q_e^4}{2mr^5} e^{-\frac{q_e^2}{2mr}} \quad (36)$$

$$R_{ab} \wedge *R^{ab} = \left(\frac{24m^2}{r^6} - \frac{24mq_e^2}{r^7} + \frac{12q_e^4}{r^8} - \frac{2q_e^6}{mr^9} + \frac{q_e^8}{8m^2r^{10}} \right) e^{-\frac{q_e^2}{mr}} *1. \quad (37)$$

When we check their limits

$$\lim_{r \rightarrow 0} R = 0, \quad \lim_{r \rightarrow 0} R_{ab} \wedge *R^{ab} = 0 \quad (38)$$

we see that they are regular at the center of black hole. We find the solution of these differential equations (9) and (10) for this regular metric function (35) as follows

$$Y(r) = \frac{2\kappa^2 mr}{(8mr - q_e^2) e^{-\frac{q_e^2}{2mr}}} \quad (39)$$

$$E(r) = \frac{q_e}{r^2 Y(r)} = \frac{q_e (8mr - q_e^2) e^{-\frac{q_e^2}{2mr}}}{2m\kappa^2 r^3}. \quad (40)$$

While this electric field is regular at the center, it has the following asymptotic behavior

$$E(r) = \frac{4q_e}{\kappa^2 r^2} - \frac{5q_e^3}{2m\kappa^2 r^3} + \frac{3q_e^5}{4m^2 \kappa^2 r^4} + O\left(\frac{1}{r^5}\right). \quad (41)$$

The inverse function of $R(r)$ in (36) can be found in terms of Lambert function [27] as

$$r(R) = -\frac{q_e^2}{10m} W^{-1} \left[-\frac{1}{10} \left(\frac{2q_e^6 R}{m^4} \right)^{1/5} \right]. \quad (42)$$

Then we rewrite the non-minimal function of this model as

$$Y(R) = -\frac{10^5 \kappa^2 m^4 W^5 \left[-\frac{1}{10} \left(\frac{2q_e^6 R}{m^4} \right)^{1/5} \right]}{2q_e^6 R \left(4 + 5W \left[-\frac{1}{10} \left(\frac{2q_e^6 R}{m^4} \right)^{1/5} \right] \right)}. \quad (43)$$

From the duality transformation $YE \rightarrow -B$, $q_e \rightarrow -q$, $Y \rightarrow \frac{1}{Y}$ [26], we can find the magnetic solution for this same metric (35), which corresponds to dual solution of the electrically charged solution (39), (40). Then the resulting magnetic field

$$B(r) = \frac{q}{r^2} \quad (44)$$

and the corresponding non-minimal function

$$Y(R) = -\frac{2q^6 R \left(4 + 5W \left[-\frac{1}{10} \left(\frac{2q^6 R}{m^4} \right)^{1/5} \right] \right)}{10^5 \kappa^2 m^4 W^5 \left[-\frac{1}{10} \left(\frac{2q^6 R}{m^4} \right)^{1/5} \right]} \quad (45)$$

constructs the dual solution. We calculate the energy density and pressures for the metric function (35) with the magnetic charge q and the the magnetic field (44). We note that all the following results also can be obtained from the solutions with the electric field (40) which has the electric charge $q_e = q$

$$\rho(r) = \frac{q^2 e^{\frac{-q^2}{2mr}}}{\kappa^2 r^4} = -p_r(r) \quad p_t(r) = \rho(r) - \frac{q^4 e^{\frac{-q^2}{2mr}}}{4\kappa^2 m r^5}. \quad (46)$$

Now we calculate all the energy conditions for the metric function (35)

$$DEC_1 = \rho \geq 0, \quad (47)$$

$$NEC_1 = WEC_1 = \rho + p_r = 0, \quad (48)$$

$$NEC_2 = WEC_2 = \rho + p_t = \frac{q^2 e^{\frac{-q^2}{2mr}}}{4m\kappa^2 r^5} (8mr - q^2), \quad (49)$$

$$SEC = \rho + p_r + 2p_t = \frac{q^2 e^{\frac{-q^2}{2mr}}}{2m\kappa^2 r^5} (4mr - q^2), \quad (50)$$

$$DEC_2 = \rho - p_r = 2\rho, \quad (51)$$

$$DEC_3 = \rho - p_t = \frac{q^4 e^{\frac{-q^2}{2mr}}}{4\kappa^2 m r^5}. \quad (52)$$

We found that all the energy conditions are satisfied in the region $r \geq \frac{q^2}{4m}$ for these electrically ($q_e = q$) or magnetically charged solutions. But, in the region $\frac{q^2}{8m} \leq r < \frac{q^2}{4m}$ only the SEC is violated by this solution. Furthermore, in the central region $r < \frac{q^2}{8m}$ the conditions NEC_2, WEC_2 together with SEC are violated.

IV. CONCLUSION

We have investigated various regular black hole solutions of the non-minimally coupled $Y(R)F^2$ theory. We found electrically charged or magnetically charged (after the duality transformation) regular black hole solutions which can be obtained from the non-minimal model with some specific non-minimal functions $Y(R)$. We calculated all the energy conditions for these solutions using the effective energy-momentum tensor that comes from the non-minimally coupled $Y(R)F^2$ term.

The first regular black hole solution violates only the strong energy condition in a central region $r < \frac{q^2}{2m}$, inside the event horizon. This solution is in agreement with the singularity theorem of General Relativity [28]. But the second regular black hole solution violates the weak energy condition together with the strong energy condition in the region $r < \frac{q^2}{8m}$, while it satisfies all the energy conditions in the outer region $r \geq \frac{q^2}{4m}$ for the electric or magnetic fields. The same energy conditions of the second regular black hole with an electric field are also found in [16] for a different theory which is $f(R)$ minimally coupled to the Non-linear electrodynamics.

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