EXISTENCE RESULTS FOR SOLUTIONS OF BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

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Abstract: In this paper, we consider boundary value problems for nonlinear differential equations in the Hilbert space $L^2(0,\infty)$ and $L^2(-\infty,\infty)$. Using the Schauder fixed point theorem, the existence results for solutions of the considered boundary value problems are established.

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1 Introduction

We consider the second order nonlinear differential equation

$$-y'' + q(x)y = f(x,y), \quad 0 \leq x < \infty, \tag{1.1}$$

where $y = y(x)$ is a desired solution.

For convenience, let us list some conditions.

(H1) $q(x)$ is real-valued measurable functions on $[0,\infty)$ such that

$$\int_0^b |q(x)|dx < \infty$$

for each finite positive number b. Moreover, the function $q(x)$ is such that all solutions of the second order linear differential equation

$$-y'' + q(x)y = 0, \quad 0 \leq x < \infty, \tag{1.2}$$

belong to $L^2(0,\infty)$, that is Weyl limit circle case holds for the differential expression $Ly = -y'' + q(x)y$ (see Coddington et al [1], Titchmarsh [9]).

(H2) The function $f(x,y)$ is real-valued and continuous in $(x,y) \in [0,\infty) \times \mathbf{R}$ and there exists a function $g_K \in L^2(0,\infty)$ such that

$$|f(x,\tau)| \leq g_K(x). \tag{1.3}$$

where $|\tau| \leq K$.

Let D be the linear manifold of all elements $y \in L^2(0,\infty)$ such that Ly is defined and $Ly \in L^2(0,\infty)$.

Assume $u = u(x)$ and $v = v(x)$ are solutions of (1.2) satisfying the initial conditions

$$u(0) = \beta,\ u'(0) = \alpha\ ;\ v(0) = -\alpha\ , v'(0) = \beta, \tag{1.4}$$

where α, β are arbitrary given real numbers.

We have the following notation

$$[y,z]_x = y(x)z'(x) - z(x)y'(x).$$

Using the Green's formula

$$\int_0^b [(Ly)z - y(Lz)](x)dx = [y,z]_b - [y,z]_0 \tag{1.5}$$

for all $y, z \in D$, we have the limit

$$[y,z]_\infty = \lim_{b\to\infty} [y,z]_b$$

exists and is finite.

We deal with the equation (1.1) whose boundary conditions are

$$\alpha y(0) - \beta y'(0) = 0,\quad \gamma\,[y,u]_\infty + \delta\,[y,v]_\infty = 0, \tag{1.6}$$

where $\alpha, \beta, \gamma,$ and δ are given real numbers satisfying the condition

(H3) $g := \delta(\alpha^2 + \beta^2) \neq 0$.

The way giving boundary condition at infinity is used in Fulton [2], Gasymov et al [3], Guseinov [4], Guseinov et al [5], Guseinov et al [6] and Krein [8].

From $(H3)$ and the constancy of the Wronskian it follows that $W_x(u,v) \neq 0$. Hence, u and v are linearly independent and they form a fundamental system of solutions of (1.2). It follows from the condition $(H1)$ that $u, v \in L^2(0,\infty)$; what is more $u, v \in D$. Consequently for each $y \in D$, the values $[y,u]_\infty$ and $[y,v]_\infty$ exist and are finite.

Now, we define the functions $\varphi_1(x) = u(x)$ and $\varphi_2(x) = \gamma u(x) + \delta v(x)$. φ_1 and φ_2 are linear independent solutions of (1.2), since $W_x(\varphi_1,\varphi_2) = g \neq 0$. From (1.4) and (1.5), φ_1 satisfies the boundary condition at zero, and φ_2 satisfies the boundary condition at infinity.

By a variation of constants formula, the general solution of the nonhomogeneous equation

$$-y'' + q(x)y = h(x),\quad 0 \leq x < \infty, \tag{1.7}$$

is $y(x) = c_1\varphi_1(x) + c_2\varphi_2(x) - \frac{1}{g}\int_0^x [\varphi_1(s)\varphi_2(x) - \varphi_2(s)\varphi_1(x)]\,h(s)ds,$ where c_1, c_2 are arbitrary given real numbers. Then the nonhomogeneous boundary value problem $(1.7), (1.6)$ has a solution $y \in L^2(0,\infty)$ given by the formula

$$y(x) = \int_0^\infty G(x,s)h(s)ds, \quad 0 \le x < \infty \ ,$$

where

$$G(x,s) = -\frac{1}{g} \begin{cases} \varphi_1(x)\varphi_2(s) & 0 \le x \le s < \infty, \\ \varphi_1(s)\varphi_2(x) & 0 \le s \le x < \infty. \end{cases}$$

Since $\varphi_1, \varphi_2 \in L^2(0,\infty)$, we obtain

$$\int_0^\infty \int_0^\infty |G(x,s)|^2 dxds < \infty. \tag{1.8}$$

Hence, the nonlinear boundary value problem (1.1), (1.6) is equivalent to the nonlinear integral equation

$$y(x) = \int_0^\infty G(x,s)f(s,y(s))ds, \quad 0 \le x < \infty.$$

Then investigating the existence of solutions of the nonlinear BVP (1.1), (1.6) is equivalent to investigating fixed points of the operator $A : L^2(0,\infty) \to L^2(0,\infty)$ by the formula

$$Ay(x) = \int_0^\infty G(x,s)f(s,y(s))ds, \quad 0 \le x < \infty, \tag{1.9}$$

where $y \in L^2(0,\infty)$.

2 Existence of solutions on half-line

In this section we will use the Schauder Fixed Point Theorem to show the existence of solutions of the BVP (1.1), (1.6).

Theorem 1. **(Schauder Fixed Point Theorem)** *Let $\mathcal{B}$ be a Banach space and $\mathcal{S}$ a nonempty bounded, convex, and closed subset of $\mathcal{B}$. Assume $A : \mathcal{B} \to \mathcal{B}$ is a completely continuous operator. If the operator A leaves the set $\mathcal{S}$ invariant then A has at least one fixed point in $\mathcal{S}$.*

Let's state the theorem used in Lemma 3.

Theorem 2. **(Yosida** [10] , **Fréchet-Kolmogorov)** *Let $\mathcal{S}$ be the real line, $\mathcal{B}$ the $\sigma-$ring of Baire subsets B of $\mathcal{S}$ and $m(B) = \int_B dx$ the ordinary Lebesgue measure of B. Then a subset K of $L^p(\mathcal{S},\mathcal{B},m)$, $1 \le p < \infty$, is strongly pre-compact iff it satisfies the conditions:*

i) $\sup_{x\in K} \|x\| = \sup_{x\in K} \left(\int_{\mathcal{S}} |x(s)|^p \, ds\right)^{1/p} < \infty$,

ii) $\lim_{t\to 0} \int_{\mathcal{S}} |x(t+s) - x(s)|^2 ds = 0$ *uniformly in* $x \in K$,

iii) $\lim_{\alpha\to\infty} \int_{s>\alpha} |x(s)|^p ds = 0$ *uniformly in* $x \in K$.

Lemma 3. *Under the conditions* $(H1)$, $(H2)$, *and* $(H3)$ *the operator* A *defined in* (1.9) *is completely continuous.*

Proof. We must show that the operator A is continuous and compact operator. Firstly, we want to show that when $\varepsilon > 0$ and $y_0 \in L^2(0,\infty)$, there exists $\delta > 0$ such that

$$y \in L^2(0,\infty) \text{ and } \|y - y_0\| < \delta \text{ implies } \|Ay - Ay_0\| < \varepsilon. \quad (2.1)$$

It can be easily seen that the inequality

$$|Ay(x) - Ay_0(x)|^2 \le M \int_0^\infty |f(s,y(s)) - f(s,y_0(s)|^2 ds,$$

where

$$M = \int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds.$$

It is known (see Krasnosel'skii [7]) that the operator F defined by $Fy(x) = f(x,y(x))$ is continuous in $L^2(0,\infty)$. Therefore for the given ε, we can find a $\delta > 0$ such that

$$\|y - y_0\| < \delta \text{ implies } \int_0^\infty |f(s,y(s)) - f(s,y_0(s)|^2 ds < \frac{\varepsilon^2}{M}.$$

Hence, we obtain desired result (2.1), that is, the operator A is continuous.

Now, we must show that $A(Y)$ is a pre-compact set in $L^2(0,\infty)$ where $\|y\| \le c$ for all $y \in Y$. For this purpose, we will use Theorem 2.

For all $y \in Y$, from (1.8) and (1.3) we have

$$\|Ay\|^2 \le M \int_0^\infty g_c^2(s) ds < \infty. \quad (2.2)$$

Further, for all $y \in Y$, we get

$$\int_0^\infty |Ay(t+x) - Ay(x)|^2 dx \le \int_0^\infty \int_0^\infty |G(t+x,s) - G(x,s)|^2 dx ds \int_0^\infty |f(s,y(s)|^2 ds$$

$$\le \int_0^\infty \int_0^\infty |G(t+x,s) - G(x,s)|^2 dx ds \int_0^\infty g_c^2(s) ds.$$

From (1.8), $\int_0^\infty |Ay(t+x) - Ay(x)|^2 dx$ converges uniformly to zero as $t \to 0$.

We also have, for all $y \in Y$,

$$\int_\alpha^\infty |Ay(x)|^2 dx \le \int_\alpha^\infty \int_0^\infty |G(x,s)|^2 dx ds \int_0^\infty |f(s,y(s)|^2 ds \le M \int_0^\infty g_c^2(s) ds.$$

Again by (1.8), $\int_\alpha^\infty |Ay(x)|^2 dx$ converges uniformly to zero as $\alpha \to \infty$.

Thus, $A(Y)$ is a strongly pre-compact set in $L^2(0,\infty)$. This completes the proof of Lemma 3.

Theorem 4. *Assume conditions* $(H1)$, $(H2)$, *and* $(H3)$ *are satisfied. In addition, let there exist a number* $R > 0$ *such that*

$$M\{\sup_{y \in \mathcal{S}} \int_0^\infty |g_R(s)|^2 ds\} \le R^2, \quad (2.3)$$

where $M = \int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds$ *and* $\mathcal{S} = \{y \in L^2(0,\infty) : \|y\| \le R\}$. *Then the BVP* (1.1), (1.6) *has at least one solution* $y \in L^2(0,\infty)$ *with*

$$\int_0^\infty |y(x)|^2 dx \le R^2.$$

Proof. By Lemma 3, the operator A is completely continuous. Further, it is obvious that the set $\mathcal{S}$ is bounded, convex, and closed. By (2.2) and (2.3), A maps the set $\mathcal{S}$ into itself, and thus the proof is completed.

3 Boundary value problems on the whole axis

Consider the equation

$$-y'' + q(x)y = f(x, y), \quad -\infty < x < \infty. \tag{3.1}$$

For convenience, let us list some conditions.

(C1) $q(x)$ is real-valued measurable functions on $(-\infty, \infty)$ such that

$$\int_a^b |q(x)|dx < \infty$$

for each finite real numbers a and b with $a < b$. Moreover, the function $q(x)$ is such that all solutions of the second order linear differential equation

$$-y'' + q(x)y = 0, \quad -\infty < x < \infty, \tag{3.2}$$

belong to $L^2(-\infty, \infty)$.

(C2) The function $f(x, y)$ is real-valued and continuous in $(x, y) \in \mathbf{R} \times \mathbf{R}$ and there exists a function $g_K \in L^2(-\infty, \infty)$ such that

$$|f(x, \tau)| \leq g_K(x).$$

where $|\tau| \leq K$.

Let D be the linear manifold of all elements $y \in L^2(-\infty, \infty)$ such that Ly is defined and $Ly \in L^2(-\infty, \infty)$.

Assume $u = u(x)$ and $v = v(x)$ are solutions of (3.2) satisfying the initial conditions

$$u(0) = \beta, \; u'(0) = \alpha \; ; \; v(0) = -\alpha \; , v'(0) = \beta, \tag{3.3}$$

where α, β are arbitrary given real numbers.

Using the Green's formula

$$\int_a^b [(Ly)z - y(Lz)](x)dx = [y, z]_b - [y, z]_a \tag{3.4}$$

for all $y, z \in D$, we have the limit

$$[y, z]_{-\infty} = \lim_{a \to -\infty} [y, z]_a \, , \quad [y, z]_{\infty} = \lim_{b \to \infty} [y, z]_b$$

exist and are finite.

We deal with the equation (3.1) whose boundary conditions are

$$\alpha \, [y, u]_{-\infty} + \beta \, [y, v]_{-\infty} = 0, \quad \gamma \, [y, u]_{\infty} + \delta \, [y, v]_{\infty} = 0, \tag{3.5}$$

where $\alpha, \beta, \gamma,$ and δ are given real numbers satisfying the condition

(C3) $g := \delta(\alpha^2 + \beta^2) \neq 0.$

It follows from the condition $(C1)$ that $u, v \in L^2(-\infty, \infty)$; moreover, $u, v \in D$. Hence for each $y \in D$, the values $[y, u]_{\pm\infty}$ and $[y, v]_{\pm\infty}$ exist and are finite.

Now, we define the functions $\varphi_1(x) = u(x)$ and $\varphi_2(x) = \gamma u(x)+\delta v(x)$. From (3.3) and (3.4), φ_1 satisfies the boundary condition at $-\infty$, and φ_2 satisfies the boundary condition at ∞.

The general solution of the nonhomogeneous equation

$$-y'' + q(x)y = h(x), \quad -\infty < x < \infty, \tag{3.6}$$

is $y(x) = c_1\varphi_1(x) + c_2\varphi_2(x) - \frac{1}{g}\int_{-\infty}^{x} [\varphi_1(s)\varphi_2(x) - \varphi_2(s)\varphi_1(x)]\, h(s)ds$, where c_1, c_2 are arbitrary given real numbers. Then the nonhomogeneous boundary value problem $(3.6), (3.5)$ has a solution $y \in L^2(-\infty,\infty)$ given by the formula

$$y(x) = \int_{-\infty}^{\infty} G(x,s)h(s)ds, \quad -\infty < x < \infty,$$

where

$$G(x,s) = -\frac{1}{g}\begin{cases} \varphi_1(x)\varphi_2(s) & -\infty < x \le s < \infty, \\ \varphi_1(s)\varphi_2(x) & -\infty < s \le x < \infty. \end{cases}$$

Since $\varphi_1, \varphi_2 \in L^2(-\infty,\infty)$, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,s)|^2 dxds < \infty.$$

Hence, the nonlinear boundary value problem (3.1), (3.5) is equivalent to the nonlinear integral equation

$$y(x) = \int_{-\infty}^{\infty} G(x,s)f(s,y(s))ds, \quad -\infty < x < \infty.$$

Then investigating the existence of solutions of the nonlinear BVP (3.1), (3.5) is equivalent to investigating fixed points of the operator $A : L^2(-\infty,\infty) \to L^2(-\infty,\infty)$ by the formula

$$Ay(x) = \int_{-\infty}^{\infty} G(x,s)f(s,y(s))ds, \quad -\infty < x < \infty,$$

where $y \in L^2(-\infty,\infty)$.

Next reasoning as in the previous section we can prove the following theorem.

Theorem 5. *Assume conditions $(C1)$, $(C2)$, and $(C3)$ are satisfied. In addition, let there exist a number $R > 0$ such that*

$$M\{\sup_{y\in\mathcal{S}} \int_{-\infty}^{\infty} |g_R(s)|^2 ds\} \le R^2,$$

where $M = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,s)|^2 dxds$ and $\mathcal{S} = \{y \in L^2(-\infty,\infty) : \|y\| \le R\}$. Then the BVP (3.1), (3.5) has at least one solution $y \in L^2(-\infty,\infty)$ with $\|y\| \le R$.

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