On Fibonacci Quaternions

Serpil Halici

Abstract. In this paper, we investigate the Fibonacci and Lucas quaternions. We give the generating functions and Binet formulas for these quaternions. Moreover, we derive some sums formulas for them.

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1. Preliminaries

Quaternions were investigated by Sir William Rowan Hamilton (1805-1865) as an extension to the complex numbers. Until the middle of the 20th century, the practical use of quaternions was minimal in comparison with other methods. Now, there has been an increasing interest in algebra problems on quaternion field since many algebra problems on quaternion field were encountered in some applied science, such as the differential geometry, quantum physics, geostatics, and analysis. A quaternion is a hyper-complex number and is defined by the following equation;

$$q = q_0 + iq_1 + jq_2 + kq_3 = (q_0, q_1, q_2, q_3) \tag{1.1}$$

where q_0, q_1, q_2, and q_3 are real numbers or scalars. Here i, j and k are the standard orthonormal basis in R^3. Then we can write

$$q = q_0 + u = q_0 + iq_1 + jq_2 + kq_3. \tag{1.2}$$

where $u = iq_1 + jq_2 + kq_3$. q_0 is called the scalar part of the quaternion q and u is called the vector part of the quaternion q. The q_0, q_1, q_2, and q_3 are called the components of the quaternion q. The quaternion multiplication is defined by the following rules;

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.3}$$

Note that the rules (1.3) imply $ij = k = -ji, jk = i = -kj$, and $ki = j = -ik$. The set of all quaternions form an associative but non commutative algebra. The conjugate of the quaternion q is denoted by q^* and $q^* = q_0 - u$. For the quaternions p, q, it follows that

$$(pq)^* = q^* p^*, \quad (p^* q)^* = q^* p.$$

The norm of the quaternion q is defined by

$$|q| = N(q) = \sqrt{q^* q}.$$

If a quaternion has a norm equal to one, then it is called as a unit quaternion, that is, we can write

$$|q| = |q^*| = 1 \ , \ q^* q = N^2(q) = 1$$

Thus, we write[8]

$$q^* q q^{-1} = N^2(q) q^{-1} = q^*$$

and

$$q^{-1} = \frac{q^*}{N^2(q)} = \frac{q^*}{|q|^2}.$$

A. F. Horadam[1] defined the *nth* Fibonacci and Lucas quaternions as follows;

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} \tag{1.4}$$

and

$$K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3} \tag{1.5}$$

respectively. Here F_n and L_n are the *nth* Fibonacci and Lucas numbers, respectively. Here the basis i, j, k, are as in the above equation (1.3). Also, Horadam established a few relations for the Fibonacci quaternions;

$$Q_n {Q_n}^* = \sum_{i=0}^{3} {F_{n+i}}^2 = 3F_{2n+3},$$

$$Q_n^2 = 2F_n Q_n - Q_n {Q_n}^*,$$

$$2F_n = Q_n + {Q_n}^*, \ \ Q_n \neq 0.$$

M. N. S. Swamy[6] defined a new quaternion R_n as follows;

$$R_n = M_n + iM_{n+1} + jM_{n+2} + kM_{n+3},$$

where $M_n = M_{n-1} + M_{n-2} \ , \ M_1 = r, \ \ M_2 = r + s.$

M. R. Iyer [5] derived relations connecting the Fibonacci and Lucas quaternions. Furthermore, he listed the relations existing between Fibonacci and Lucas quaternions. A. L. Iakin [3] introduced the concept of a higher order quaternion, and established some identities for these quaternions. In 1993, Horadam [2] examined the quaternion recurrence relations.
Now, in this paper we give the generating functions and Binet formulas for

Fibonacci and Lucas quaternions. Moreover, we obtain some sums formulas for these quaternions.

2. Some Properties of Fibonacci Quaternions

We consider the quaternions defined in the equations (1.4) and (1.5). These quaternions can be written as

$$Q_n = F_n + u \ ; \quad \mathrm{u} = \mathrm{iF}_{n+1} + j\mathrm{F}_{n+2} + k\mathrm{F}_{n+3} \tag{2.1}$$

and

$$Q_n{}' = L_n + v \ ; \quad \mathrm{v} = \mathrm{iL}_{n+1} + j\mathrm{L}_{n+2} + k\mathrm{L}_{n+3}. \tag{2.2}$$

Note that, for $n \geq 0$

$$Q_{n+2} = Q_{n+1} + Q_n \ ; \ \mathrm{Q}'_{n+2} = \mathrm{Q}'_{n+1} + \mathrm{Q}'_n \tag{2.3}$$

can be written. So, the Fibonacci and Lucas quaternions are the second order linear recurrence sequence. Then, if we define the sets H and H' as follows

$$H = \{Q_n : Q_n = (F_n, F_{n+1}, F_{n+2}, F_{n+3})\,;\ F_n \text{is nth Fibonacci number}\}$$

and

$$H' = \left\{P_n : P_n = \begin{pmatrix} w & -z \\ \bar{z} & \bar{w} \end{pmatrix};\ \mathrm{w, z} \in C\right\}$$

then there is a isomorphism between H and H' such that

$$Q_n = (F_n, F_{n+1}, F_{n+2}, F_{n+3}) \rightarrow P_n = \begin{pmatrix} F_n + iF_{n+1} & -F_{n+2} - iF_{n+3} \\ F_{n+2} - iF_{n+3} & F_n - iF_{n+1} \end{pmatrix}.$$

Thus, we can write

$$P_n = F_n E + F_{n+1} I + F_{n+2} J + F_{n+3} K, \tag{2.4}$$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$.

Since $det(P_n) \neq 0$, there is the inverse of matrix P_n and it is in H'. Using the relationship, we will give some properties of the Fibonacci and Lucas quaternions.

3. Main Results

The Binet formula is the explicit formula to obtain the *nth* Fibonacci and Lucas numbers. In any case, Binet formula can be employed to drive a myriad of Fibonacci identities. M. Iyer[5] derived relations connecting the Fibonacci quaternions and Lucas quaternions with the Fibonacci and Lucas numbers. It is well known that for the Fibonacci and Lucas numbers, Binet formulas are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n$$

respectively, where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

Then, we can give the following theorem.

Theorem 3.1. *(Binet's Formulas) For $n \geq 0$, the Binet formulas for the Fibonacci and Lucas quaternions are follows;*

$$Q_n = \frac{1}{\sqrt{5}} \left(\underline{\alpha}\, \alpha^n - \underline{\beta}\, \beta^n \right), \tag{3.1}$$

and

$$Q_n{}' = \left(\underline{\alpha}\, \alpha^n + \underline{\beta}\, \beta^n \right) \tag{3.2}$$

respectively, where $\underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$.

Proof. The characteristic equation of recurrence relation (2.3) is

$$t^2 - t - 1 = 0. \tag{3.3}$$

The roots of this equation are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Using recurrence relation and the initial values $Q_0 = (0, 1, 1, 2)$, $Q_1 = (1, 1, 2, 3)$ the Binet's formula for Q_n is obtained as follows;

$$Q_n = A\alpha^n + B\beta^n = \frac{1}{\sqrt{5}} \left(\underline{\alpha}\, \alpha^n - \underline{\beta}\, \beta^n \right). \tag{3.4}$$

Similarly, we can get

$$Q_n{}' = \left(\underline{\alpha}\, \alpha^n + \underline{\beta}\, \beta^n \right).$$

Thus, the proof is completed. □

The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

is called the generating function for the sequence $\{a_0, a_1, a_2, ...\}$. Generating functions provide a powerful tool for solving linear recurrence relations with constant coefficients. It is well known that[7], the generating function of the Fibonacci sequence $\{F_n\}_{n \geq o}$ is

$$f(x) = \frac{x}{1 - x - x^2}.$$

Theorem 3.2. *The generating function for the Fibonacci quaternion Q_n is*

$$G(x,t) = \frac{t + i + j(t+1) + k(t+2)}{1 - t - t^2}. \tag{3.5}$$

Proof. If the generating function of sequence H is $G(x,t) = \sum_{n=0}^{\infty} Q_n(x) t^n$, then using the equations $tG(x,t)$ and $t^2 G(x,t)$,

$$G(x,t) = \frac{Q_0 + (Q_1 - Q_0)t}{1 - t - t^2}. \tag{3.6}$$

is obtained. So, we write

$$G(x,t) = \frac{t + i + j(t+1) + k(t+2)}{1 - t - t^2}.$$

□

Theorem 3.3. *For $m, n \in Z$ the generating function of the quaternion Q_{m+n} is*

$$\sum_{n=0}^{\infty} Q_{m+n} x^n = \frac{Q_m + Q_{m-1} x}{1 - x - x^2}. \tag{3.7}$$

Proof. Using the Binet formula of Q_n, we can write the following equation;

$$\sum_{n=0}^{\infty} Q_{m+n} x^n = \sum_{n=0}^{\infty} \left(\frac{\underline{\alpha}\, \alpha^{m+n} - \underline{\beta}\, \beta^{m+n}}{\alpha - \beta} \right) x^n,$$

$$\sum_{n=0}^{\infty} Q_{m+n} x^n = \frac{1}{\alpha - \beta} \left(\frac{\underline{\alpha}\, \alpha^m}{1 - \alpha x} - \frac{\underline{\beta}\, \beta^m}{1 - \beta x} \right) = \frac{Q_m + Q_{m-1} x}{1 - x - x^2}$$

which is desired. □

Now, let us define the following matrix as

$$Q = \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}. \tag{3.8}$$

This matrix can be called as the Fibonacci quaternion matrix. Then, we can give the next theorem by the Fibonacci quaternion matrix.

Theorem 3.4. *(Cassini Identity) For $n \geq 1$, we have the following formula;*

$$Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n(2Q_1 - 3k) \tag{3.9}$$

Proof.

$$Q = \begin{pmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{pmatrix}, det(Q) = Q_0Q_2 - {Q_1}^2 = -(2Q_1 - 3k)$$

$$Q^2 = Q_2 = \begin{pmatrix} Q_3 & Q_2 \\ Q_2 & Q_1 \end{pmatrix}, det(Q^2) = Q_1Q_3 - {Q_2}^2 = (2Q_1 - 3k)$$

By the aid of quaternion multiplication, we can compute Q^n as follows;

$$Q^n = (F_{n-1} + iF_n + jF_{n+1} + kF_{n+2})(F_{n+1} + iF_{n+2} + jF_{n+3} + kF_{n+4}) \\ -(F_n + iF_{n+1} + jF_{n+2} + kF_{n+3})^2$$

and

$$Q^n = Q_n = (-1)^n(2Q_1 - 3k)$$

which is desired. □

Corollary 1. *For the Fibonacci quaternion Q_n, we have*

$$Q^{-n} = Q_{-n} = (-1)^n(-F_n + iF_{n+1} - jF_{n+2} + kF_{n+3}), \tag{3.10}$$

$$Q^{-n} + conjugate(Q^{-n}) = 2(-1)^{n+1}F_n \tag{3.11}$$

and

$$\frac{1}{2}\left(Q^n + Q^{-n}\right) = \begin{cases} F_n + jF_{n+2}; & \text{n odd} \\ iF_{n+1} + kF_{n+3}; & \text{n even} \end{cases} \tag{3.12}$$

Proof. The proof can be easily seen by the Fibonacci quaternion matrix. □

Now, we will give without proof the following corollary.

Corollary 2. *For the Fibonacci quaternion Q_n, we have*

$$a) \sum_{i=0}^{n} Q_i = Q_{n+2} - Q_1,$$

$$b) \sum_{i=0}^{n} Q_{2i} = Q_{2n+1} - (1, 0, 1, 1),$$

$$c) \sum_{i=0}^{n-1} Q_{2i+1} = Q_{2n} - Q_0.$$

Theorem 3.5. *For $n \geq 0$, we have the following sums formulas;*

$$\sum_{i=0}^{n} \binom{n}{i} Q_i = Q_{2n}, \tag{3.13}$$

and

$$\sum_{i=0}^{n} \binom{n}{i} Q_i (-1)^i = (-1)^n Q_{-n}. \tag{3.14}$$

Proof. From Binet formula

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} \left(\frac{\underline{\alpha}\, \alpha^i - \underline{\beta}\, \beta^i}{\alpha - \beta} \right) &= \frac{\underline{\alpha}}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} \alpha^i - \frac{\underline{\beta}}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} \beta^i \\ &= \frac{\underline{\alpha}}{\alpha - \beta} [(1+\alpha)^n] - \frac{\underline{\beta}}{\alpha - \beta} [(1+\beta)^n] = Q_{2n} \end{aligned}$$

can be written. Thus, the proof is completed. □

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Serpil Halici
Department of Mathematics, Faculty of Arts and Sciences
Sakarya University, 54187 Sakarya/TURKEY

e-mail: `shalici@sakarya.edu.tr`