On Max-plus Linear Dynamical System Theory: the Observation Problem. ⋆

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Abstract

In this paper, we are interested in the general problem of estimating a linear function of the states for a given Max-Plus linear dynamical system. More precisely, using only the current and past inputs and outputs of the system, we want to construct a sequence that converges in a finite number of steps to the value given by a linear function of the states for all initial conditions of the system. We provide necessary and sufficient conditions to solve this general problem. We also define and study a Max-Plus version of the well-known Luenberger observer, which is a subclass of the general problem that we are interested in, and we also provide necessary and sufficient conditions to solve this particular problem of observer synthesis. Finally, we show that there are important connections between results in the Max-Plus domain and associated results in the standard linear systems theory.

Key words: Max-Plus algebra; Tropical algebra; Observer; Control.

1 Introduction

Max-Plus algebra is an algebraic formalism for modeling a special class of discrete event dynamical systems, namely the Timed Event Graphs (see [2] for the connection between Max-Plus and Discrete Event Systems). Recently, a considerable amount of work has been dedicated in obtaining results for dynamical systems which are linear in this algebra, especially regarding state control (see [5] and the references therein).

In the context of controlling max-plus systems, it is often assumed that all the states are measured (see [6] for example and [12] for an exception). Frequently, to compensate the lack of measurement, an alternative is to use an observer in order to estimate the state variable by using both measured outputs and inputs.

In the max-plus setting, not much research was done in the subject. Indeed, to the best of authors' knowledge, few published papers studied problems related to observability in max-plus setting. To cite some examples, in [17], we have conditions for *structural observability* of Max-Plus systems, which are necessary, but not sufficient, conditions to be able to reconstruct exactly the real state of the system . In [4], we have conditions for computing what the authors call the *Latest Event-Time State*, which is the greatest state that is compatible with the system observations. The authors in [14] propose an observer for a descriptor system, which can model uncertainties in the parameters, that has no inputs. Hence, given that the initial condition of the system $x[0]$ is known, it is possible to find at a given step k all possible values of the state $x[k]$ that could be reached by the system under uncertainties. Although interesting, and indeed the concepts presented there were fundamental for the developments in this paper, the proposed observation problem of that paper does not fit the main objective of this work. In this paper we are interested in reconstructing a linear combination of the states for a system, possibly with inputs, using only the information of the current and past inputs and outputs, assuming that the initial state is unknown.

Alternatively, the works in [8,1,10], for example, have used transfer series methods to devise Luenberger-inspired observers that reconstruct the greatest estimated state $\hat{x}[k]$ that is less than or equal to the real state $x[k]$, whose bounds for the error between the real

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and reconstructed state can be seen in [9]. The work of [3] also considers an observation problem for Max-Plus systems. It considers probabilistic uncertainty in the model, proposing a particle-filter approach for solving the problem. However, it differs from the problem analyzed in this paper because it assumes information (probability density) of the initial state of the system.

Unlike those previously related works found in literature, with the exception of the authors' preliminary work [7], in this paper we are interested in the following problem, to be stated formally in Section 3: using only the current and past values of the system outputs $y[k]$ and inputs $u[k]$, construct a sequence that converges in a finite number of steps to a linear function $s[k] = Wx[k]$, for a given matrix W, for all initial conditions $x[0]$ and all possible inputs $u[k]$ of the system. This approach of computing only a linear combination of the states - instead of the complete information - using the outputs was already explored in the control literature for traditional linear systems since the 60's and was pioneered by Luenberger [15]. The observers are then denominated *functional observers* (see [18] and the references therein).

Estimating $Wx[k]$ directly, instead of first estimating $x[k]$ and then computing $Wx[k]$, as it is considered in this paper, can be very beneficial since it is a weaker problem and usually requires weaker conditions. For example, if W is chosen as a linear state feedback gain matrix F, then this approach can be used to compute $u[k] = Fx[k]$, and thus the control problem would be solved using only output feedback, instead of state feedback. The results in this paper can also be used, for instance, to perform system diagnosis (see [14,8]). Necessary and sufficient conditions for solving this observation problem will be provided, extending the results in the previous paper [7], which presents only sufficient conditions and no comparisons with similar results in the traditional linear systems theory. Furthermore, our derivations in this paper are heavily based on Max-Plus analogues of geometrical control ideas ([19], see also [14] for the specific Max-Plus case).

We consider two variations of the observation problem: *i)* a weaker form, in which we are not interested in the specific way the reconstruction of $Wx[k]$ is done, and *ii)* a stronger form, not presented in [7], in which the desired signal should be obtained using a Luenberger-like observer. In addition, we will show how the Max-Plus results can be compared to their counterparts in the traditional linear dynamical systems theory. For that, we will present also some results in the traditional algebra. Although these results are not contributions of this paper, they will be presented for the sake of comparison. We establish that our results are, up to some point, analogous, but the Max-Plus case becomes more complex since the subtraction operation does not exist.

2 Basic Definitions

We will now make some basic definitions contextualized in the Max-Plus algebra, but sometimes we will also use the same notation for its traditional counterpart in the algebra of real numbers, and we expect that the specific interpretation should be clear from the context. For example, AB can denote the Max-Plus matrix product or the traditional product of real matrices, depending on the context. Further, the matrix I can denote the Max-Plus or the traditional identity matrix, also depending on the context.

More technically *Max-Plus algebra* is the dioid (an idempotent semiring)

$$\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \oplus, \otimes) \tag{1}$$

in which $\oplus$ is the maximum and $\otimes$ is the traditional sum. It has been also called *Tropical Algebra*. The symbol $\otimes$ will be frequently omitted, so ab reads as $a \otimes b = a + b$. We denote the element $-\infty$ by the symbol ε, and it will also be occasionally called "the null element". For two matrices A and B of appropriate dimensions, $A \oplus B$ and $A \otimes B$ will be interpreted as the matrix sum and product, respectively, with $+$ being replaced by $\oplus$ and $\times$ by $\otimes$. Elements in this algebra that have n rows and m columns will be considered to belong to $\mathbb{Z}_{\max}^{n\times m}$, while an element with m rows and 1 column will belong to $\mathbb{Z}_{\max}^{m}$. Thus, all vectors are column vectors. The symbol A^T denotes the transpose of the matrix A. A vector or matrix of appropriate dimensions whose elements are all equal to ε will also be denoted by ε. A matrix without any ε entries is said to be *full*. The symbol I will denote the Max-Plus identity matrix of an appropriate order, that is, a matrix in which each diagonal element is 0 and ε off-diagonal. For a natural number k, the matrix power A^k will be defined recursively as $A^{k+1} = A^k A$, with $A^0 = I$. If λ is a scalar not equal to ε, then λ^{-1} in the Max-Plus algebra results in $-\lambda$ in the traditional algebra.

If $\mathcal{J}$ is a set of natural numbers and x is a vector, $\{x\}_{\mathcal{J}}$ is a sub-vector of x formed only with the entries whose indexes are in $\mathcal{J}$. The *Kleene closure* of a square matrix A is equal to $\bigoplus_{i=0}^{\infty} A^i$. The *spectral radius* of this matrix, $\rho(A)$, is the greatest scalar λ for which there exists a vector $v \neq \varepsilon$ in which $Av = \lambda v$. Generally, even though the entries of the matrix A lie in $\mathbb{Z}$ or are equal to ε, the spectral radius can be a rational number. However, since the units of the problem can be redefined, the entries of the matrix (and thus the spectral radius) can be rescaled so the spectral radius is either an integer or is equal to ε. Thus, hereafter we can assume, without loss of generality, that $\rho(A) \in \mathbb{Z}_{\max}$. In the traditional setting, $\rho(A)$ is the eigenvalue of A with the largest absolute value. For two scalars a and b, we define $a \wedge b = \min(a, b)$. We also denote the point-wise minimum of two matrices A and B as $A \wedge B$.

Residuation theory [2] deals with the conditions for the existence of the greatest element x for the inequality $f(x) \preceq y$, which means that every element of $f(x) \in \mathbb{Z}_{\max}^{n\times m}$ is less than or equal to every corresponding element of $y \in \mathbb{Z}_{\max}^{n\times m}$. The symbol ϕ denote the *(right) residuation operator* and is defined as follows: given the inequality $LP \preceq Q$ with matrices $P \in \mathbb{Z}_{\max}^{n\times m}, Q \in \mathbb{Z}_{\max}^{s\times m}$, there exists the greatest solution $L_{\max} = Q \phi P$, whose entries can be computed by: $(Q\phi P)_{ij} = \bigwedge_{l=1}^{m} (Q_{il} - P_{jl})$, with the definition $\epsilon - \epsilon = \infty$. The operator ϕ has precedence over $\wedge$, thus $A\phi B \wedge C$ reads as $(A\phi B) \wedge C$.

A *semimodule*, over a given dioid, is similar to a vector space over a semiring, that is, a set of elements x together with a scaling operation, *i.e.* $(\lambda, x) \mapsto \lambda x$; and a summation operation; *i.e.* $(x, y) \mapsto x \oplus y$; which preserve some properties in the context of this given dioid (See [13] for a formal definition). In addition, we denote *Im* M as the *image* of M. The image of M is the semimodule generated by the Max-Plus column span of the matrix M, that is, if $M \in \mathbb{Z}_{\max}^{n\times m}$ then *Im* $M = \{Mv \mid v \in \mathbb{Z}_{\max}^{m}\}$.

A *congruence*, over a given dioid, is an equivalence relation on $\mathbb{Z}_{\max}^{n}$ with a semimodule structure formed with pairs (x, x') of vectors which preserve some properties in the context of this given dioid. See [14] for the formal definition. Then, *Ker* M, the *kernel* of M, is the congruence induced by the equality $Mv = Mv'$ in the Max-Plus context, which has a correspondence in the traditional algebra with the statement that $(v - v')$ is in the "null space" of the matrix M; *i.e.* if $M \in \mathbb{Z}_{\max}^{n\times m}$ then *Ker* $M = \{(v, v') \in \mathbb{Z}_{\max}^{m} \times \mathbb{Z}_{\max}^{m} \mid Mv = Mv'\}$. If $\mathcal{Z}$ is a congruence and A a square matrix we define the set $A\mathcal{Z} = \{(Az', Az'')\ ,\ (z', z'') \in \mathcal{Z}\}$, which is, by the way, not a congruence, in general. It is worth to recall that in the traditional setting, *Ker* $M = \{v \in \mathbb{R}^m \mid Mv = 0\}$.

3 Two observation problems

Consider the max-plus linear event-invariant dynamical system $\mathcal{S}(A, B, C, D)$:

$$\begin{aligned} x[k+1] &= Ax[k] \oplus Bu[k]; \\ y[k] &= Cx[k] \oplus Du[k] \end{aligned} \tag{2}$$

in which $x[k] \in \mathbb{Z}_{\max}^{n}, u[k] \in \mathbb{Z}_{\max}^{m}$, $y[k] \in \mathbb{Z}_{max}^{r}$, $A \in \mathbb{Z}_{\max}^{n\times n}, B \in \mathbb{Z}_{\max}^{n\times m}, C \in \mathbb{Z}_{\max}^{r\times n}$ and $D \in \mathbb{Z}_{\max}^{r\times m}$. It is max-plus linear because its equations can be written in a linear way using the max-plus operators $\oplus$ and $\otimes$. It is event-invariant because the matrices A, B, C, D do not depend on the event k.

In the sequel we define the observation problems that we are interested in:

- The *Max-Plus observer problem*, henceforth denoted by $\mathbb{T}\mathcal{O}(\mathcal{S}, W)$ (or simply $\mathbb{T}\mathcal{O}$ when the parameters are clear) can be defined as follows: given a max-plus linear event-invariant dynamical system $\mathcal{S}$ as in Equation (2) and a matrix $W \in \mathbb{Z}_{\max}^{s\times n}$, assuming, at step $k = l$, the knowledge of the inputs $u[k]$, $0 \le k \le l-1$ and of the outputs $y[k]$, $0 \le k \le l-1$ construct a sequence $s[k]$ such that there exists a finite k' in which $s[k] = Wx[k]$ for all $k \ge k'$ and for all initial conditions $x[0]$.
- The *Max-Plus Luenberger observer problem*, henceforth denoted by $\mathbb{T}\mathcal{O}_L(\mathcal{S}, W)$ (or simply $\mathbb{T}\mathcal{O}_L$, when the parameters are clear) has a similar statement to $\mathbb{T}\mathcal{O}(\mathcal{S}, W)$, but with a very specific form of obtaining $Wx[k]$: there must exist matrices P, Q, R, S, T and an integer k' such that the dynamical system

$$z[k] = Pz[k-1] \oplus Qy[k-1] \oplus Ru[k-1] \tag{3}$$

  for *any* initial condition $z[0]$ has the equalities $z[k] = Sx[k]$ and $Tz[k] = TSx[k] = Wx[k]$ for all $k \ge k'$. This means that the observer variable $z[k]$ must converge to a linear function of $x[k]$ ($z[k] = Sx[k]$) and this function must allow the observation of $Wx[k]$ ($Tz[k] = Wx[k]$).

We also define, by straightforward analogy, traditional counterparts to all the aforementioned problems. We will denote by $\mathbb{R}\mathcal{O}$ and $\mathbb{R}\mathcal{O}_L$ the traditional (in the field of $\mathbb{R}$eals) analogous of $\mathbb{T}\mathcal{O}$ and $\mathbb{T}\mathcal{O}_L$ (in Max-Plus, or $\mathbb{T}$ropical Algebra), respectively. In this case, we have state-space equations as in Equation 2, but with $\oplus$ swapped to the traditional $+$ operations and all the matrix multiplications interpreted in the traditional sense. The entries of x, y and u are now in $\mathbb{R}$ instead of $\mathbb{Z}_{\max}$. A non-analogous difference in the definition is that, in $\mathbb{R}\mathcal{O}$ and $\mathbb{R}\mathcal{O}_L$, we now allow k' to be ∞, i.e, we allow asymptotic convergence. Note that, since the entries of the vectors and matrices in the Max-Plus are either integers or ε, there is no sense in considering $k' = \infty$ because there does not exist asymptotic convergence in this case (a sequence of integers either converges in a finite number of steps or does not converges at all). In the field of reals, asymptotic convergence makes sense.

In the sequel we will see that, if the observation problem is solvable, $s[k]$ can be reconstructed as a (Max-plus) non-recursive linear combination of past outputs and inputs. In the traditional setting, the analogous result also holds, and the Luenberger observer is preferred over the non-recursive linear combination of outputs and inputs because the latter can amplify measurement noises, whereas the former can act as a low-pass filter and thus attenuate the sensor noise. We will see that in the Max-Plus case a similar analysis can be done: the linear combination of past outputs may demand a very delayed information (example, information at $k-4$ at the step k), and if there is a perturbation of the output at a given k this error may influence the reconstruction of $s[k]$ in the next events k. The Luenberger observer, on the other hand, only demands current information.

The traditional counterparts of the observation problems have been studied before (see [18] and the references therein) and necessary and sufficient conditions have been derived to solve them. We will compare in this paper how these results compare with the ones obtained in this paper for the Max-Plus case. We start the presentation of the results with the following lemma.

Lemma 1. *There is a solution to $\mathbb{T}\mathcal{O}$ (resp $\mathbb{R}\mathcal{O}$) if there is a solution to $\mathbb{T}\mathcal{O}_L$ (resp $\mathbb{R}\mathcal{O}_L$).* □

This lemma simply states that the Luenberger observation problems are at least as harder to solve as the non-Luenberger ones.

We make the following *assumptions*:

Assumption 1. *for $\mathbb{T}\mathcal{O}$ and $\mathbb{T}\mathcal{O}_L$: We assume, without loss of generality, that no column of B and no row of C is ε. Furthermore, we assume, with a negligible loss of generality, that A is full. Indeed, we can set a reasonably high number h and enforce that $|x_i[k] - x_j[k]| \preceq h$ for all k. This implies that $x[k] = Hx[k]$ for a matrix H whose all entries are $-h$ except the diagonal ones. Thus, we can swap A by AH, and if A has no column full of ε we can see that AH has no ε entries. We can always assume that no column is null, since we can always assume that $x_i[k+1] \geq x_i[k]$ (since $x_i[k]$ represents the time in which the i^{th} state event happened for the k^{th} time) and thus the diagonal entries of A can be chosen to be at least 0. This assumption implies also that A has only one eigenvalue, equal to $\rho(A)$, and that this value is not ε.*

We also assume that $x[0]$ is not the ε vector, which is reasonable because in practical situations, it represents firing times. This way, for all $k > 0$, $x[k] \succeq Ax[k-1]$ is a full vector and all its entries grow with a rate of at least $\rho(A)$. Finally, we assume, without loss of generality, that no row of the matrix W is ε. Otherwise, we want to estimate a signal $s_i[k]$ that is either ε or completely determined by $u[k]$, rendering the observation problem for this variable trivial. □

In summary, the highlights of this paper are as follow:

- The necessary and sufficient conditions for $\mathbb{T}\mathcal{O}$ (Proposition 1) are obtained. They reduce to the solvability of at least one of the collections of infinite one-sided Max-Plus affine equations (Equation 10). Checking whether or not one of the Max-Plus affine equations has a solution can be done in strong polynomial time (Fact 3).

- If the problem $\mathbb{T}\mathcal{O}$ is solvable, it is solvable using a (Max-plus) linear moving filter of the past $u[k]$ and $y[k]$ (Proposition 1, Equation 11). The same holds, analogously, for $\mathbb{R}\mathcal{O}$.

- The necessary and sufficient conditions for $\mathbb{R}\mathcal{O}$ depend solely on the system matrix A and output matrix C. This is a classical result: for traditional linear systems, observability is independent of the input. In contrast, the necessary and sufficient conditions for solving $\mathbb{T}\mathcal{O}$ *do* depend also on the control matrices B and D. This happens because there is no subtraction in the Max-Plus case, since we are working with semirings.

- Necessary and sufficient conditions for $\mathbb{T}\mathcal{O}_L$ (Proposition 2) are obtained. They reduce to the solvability of at least one of the collections of non-linear Max-Plus equations with a spectral radius constraint (Equation 13). We show how we can derive from this condition a solely sufficient but easy-to-verify condition.

- If $\mathbb{R}\mathcal{O}$ has solution, so does $\mathbb{R}\mathcal{O}_L$. Thus, in the light of Lemma 1, $\mathbb{R}\mathcal{O}$ and $\mathbb{R}\mathcal{O}_L$ are equivalent (see Fact 5). In the Max-Plus case, $\mathbb{T}\mathcal{O}$ may have a solution whereas $\mathbb{T}\mathcal{O}_L$ does not (Proposition 3).

4 Solving $\mathbb{T}\mathcal{O}$

4.1 Preliminary definition and results

First of all, before discussing the proposed problems, we will present some definition and basic results.

Definition 1. (t^{th} Max-Plus observability matrix) *the* t^{th} Max-Plus observability matrix *(or simply observability matrix, when the context is clear), $\hat{O}_t(A,C)$ is defined recursively as $\hat{O}_{t+1}(A,C) = \left(C^T \;\; (\hat{O}_t(A,C)A)^T\right)^T$ being $\hat{O}_{-1}(A,C)$ the empty matrix.* □

The observability matrices itselves are not as important as their kernels, $Ker\ \hat{O}_t(A,C)$. This t^{th} kernel has all the pairs of states $x'[k], x''[k]$ of the system in Equation 2, with $u[k] = \varepsilon$, which are indistinguishable after observing t outputs: $y[k]$ to $y[k+t-1]$. An analogous observation holds for the traditional counterparts of the t^{th} observation matrix.

In the traditional setting, *the* observability matrix of a linear dynamical system of n states is the traditional counterpart of $\hat{O}_{n-1}(A,C)$. Thanks to the Cayley-Hamilton theorem - which states that A^n is a linear combination of smaller powers of A - we can conclude that $Ker\ \hat{O}_{n-1}(A,C) = Ker\ \hat{O}_k(A,C)$ for all $k \geq n-1$. Since when discussing observability we are mainly interested in the kernel of the observability matrix, and not in the matrix itself, this implies that there is no sense, in the traditional setting, in considering observability matrices for $k \geq n-1$.

In the Max-Plus setting the analysis is a bit more complex. Indeed, for general matrices A, there is a weaker form of Cayley-Hamilton theorem, (see [16]), but it is not as useful because we cannot, in general, infer from it that A^n is a Max-Plus linear combination of past powers of i. Moreover, there are matrices A for which

A^t can *never* be written as a Max-Plus linear combination of past powers of A (take, for instance, any diagonal matrix with different diagonal entries). This implies that we can have a sequence of decreasing Kernels $Ker\ \hat{O}_{t+1}(A,C) \subsetneq Ker\ \hat{O}_t(A,C)$ that does not stabilizes for a finite t.

Fortunately, for a very wide and useful class of matrices there is a Cayley-Hamilton-like result.

Theorem 1. *(See [2]) Let A be a full square matrix. Then, there exist two natural numbers $\tau(A)$, denoting the* coupling time [1] *and $\sigma(A)$, denoting the cyclicity, for which $A^{\tau+k\sigma} = \rho(A)^{k\sigma} A^{\tau}$ for all k.* □

The previous theorem implies that, if A is full, $Ker\ \hat{O}_{\tau+\sigma-1}(A,C) = Ker\ \hat{O}_k(A,C)$ for any $k \geq \tau+\sigma-1$. The major problem lies in that the constant $\tau(A)$ can be very large even for matrices of small dimensions, since it depends on the entries of the matrix A.

Example : Consider the matrix

$$A = \begin{pmatrix} 0 & -p \\ 0 & -1 \end{pmatrix} \qquad (4)$$

for a natural number $p \geq 0$. It is easy to verify that

$$A^k = \left\{ \begin{pmatrix} 0 & -p \\ 0 & -k \end{pmatrix} \text{ if } 0 < k \leq p; \quad \begin{pmatrix} 0 & -p \\ 0 & -p \end{pmatrix} \text{ if } k > p. \right.$$

Thus $\rho(A) = 0$, $\tau(A) = p$ and $\sigma(A) = 1$. Consequently, even for 2×2 matrices, the coupling time can be very large (take, for instance, $p = 100$). □

The observability matrix is related to another important concept of geometrical control: of (A,C)-conditioned invariant spaces. As was done in [14], we generalize this concept to Max-Plus in a straightforward way.

Definition 2. (Max-Plus (A,C)-conditioned invariant congruence, see [14]). *A congruence $\mathcal{H}$ is said to be* Max-Plus (A,C)-conditioned invariant *((A,C)-MPCI, henceforth) if $A(\mathcal{H} \cap Ker\ C) \subseteq \mathcal{H}$, the operations being interpreted in Max-Plus sense.* □

A congruence $\mathcal{H} = Ker\ H$ is (A,C)-MPCI if, considering system 2 with input $u[k] = \varepsilon$, for any two states $x'[k], x''[k]$ inside the congruence (i.e $Hx'[k] = Hx''[k]$), if they produce the same outputs (i.e $Cx'[k] = Cx''[k]$) then the pair $x''[k+1], x'[k+1]$ is also inside the congruence, (i.e $HAx'[k] = HAx''[k]$).

The following results are important.

[1] Since τ is a number of events, not a duration of time, perhaps a better name is *coupling event*

Theorem 2. *(see [14]) If $\mathcal{H} = Ker\ H$ is (A,C)-MPCI then there exist matrices M, N such that $MH \oplus NC = HA$.* □

Theorem 3. *(in analogy with the results in [19]) $Ker\ \hat{O}_\infty(A,C)$ is an (A,C)-MPCI invariant congruence. Indeed, it is the largest (A,C)-MPCI invariant congruence inside $Ker\ C$.* □

The previous results have straightforward traditional analogues (see [19]), i.e, if the null space $\mathcal{H}$ is (A,C) conditioned invariant then there exist matrices M and N such that $MH + NC = HA$ and $Ker\ \hat{O}_\infty(A,C)$ is the largest (A,C) -conditioned invariant null space contained inside $Ker\ C$. As discussed before, owing to the Cayley-Hamilton theorem, $Ker\ \hat{O}_\infty(A,C)$ can be replaced, in the traditional case, by $Ker\ \hat{O}_{n-1}(A,C)$. In the Max-Plus case, thanks to Theorem 1, we can state that, for full matrices, $Ker\ \hat{O}_\infty(A,C) = Ker\ \hat{O}_{\tau+\sigma-1}(A,C)$.

In the traditional linear setting, we do not need to consider the input or the input matrices B and D in the observation problem. This, as it will be clear soon, is due to the fact that in the traditional setting, which is based on field of reals, we can *subtract*. Since this is not true in the Max-Plus setting, we need to introduce definitions concerning inputs.

Definition 3. (t^{th} Max-Plus controllability matrix) *the* t^{th} Max-Plus controllability matrix, *$\hat{K}_t(A,B)$ (or simply controllability matrix when the context is clear) is defined recursively as $\hat{K}_{t+1}(A,B) = (A\hat{K}_t(A,B)\ B)$, being $\hat{K}_{-1}(A,B)$ the empty matrix.* □

Note that a similar discussion to the one that was done for observability matrix, but with images instead of kernels, can be made for controllability matrices. The controllability matrix can be used to write in a compact way the value of $x[k+t]$ as a function of $x[k]$ and the outputs from $u[k]$ to $u[k+t-1]$. For that, we need the following definition.

Definition 4. (Extended vectors) *given a sequence $z[k]$ and two naturals $0 \leq k_1 \leq k_2$, we define $z[k_1 : k_2]$ recursively as the vector $(z[k_1]^T\ z[k_1+1 : k_2]^T)^T$, being $z[k_1 : k_1-1]$ the empty matrix.* □

And then, we can easily deduce the following.

Fact 1. *Considering the dynamical system in Equation 2, it is easy to see that* $x[k+t] = A^t x[k] \oplus \hat{K}_{t-1}(A,B) u[k : k+t-1]$. □

A similar result as Fact 1 holds for outputs. For that, we need another definition.

Definition 5. (k^{th} mixed matrix) *the* k^{th} mixed matrix, *$\hat{M}_k(A,B,C,D)$ is defined recursively as*

$$\hat{M}_{t+1}(A,B,C,D) = \begin{pmatrix} D & \varepsilon \\ \hat{O}_t B & \hat{M}_t(A,B,C,D) \end{pmatrix} \qquad (5)$$

being $M_{-1}(A,B,C,D)$ the empty matrix. □

And then:
Fact 2. *It is easy to see that, considering the dynamical system in Equation 2, that $y[k:k+t] = \hat{O}_t(A,C)x[k] \oplus \hat{M}_t(A,B,C,D)u[k:k+t]$.* □

4.2 Necessary and sufficient conditions for $\mathbb{T}\mathcal{O}$

We will present necessary and sufficient conditions for solving $\mathbb{T}\mathcal{O}$. We will also discuss on how these derivations compare to the classical ones for $\mathbb{R}\mathcal{O}$.

We can derive the following result.
Proposition 1. *Let $\hat{O}_t = \hat{O}_t(A,C)$, $\hat{K}_t = \hat{K}_t(A,B)$ and $\hat{M}_t = \hat{M}_t(A,B,C,D)$. The problem $\mathbb{T}\mathcal{O}(\mathcal{S},W)$ has a solution if and only if there exists a finite t such that*

$$\text{Ker} \begin{pmatrix} \hat{O}_t & \hat{M}_t \\ \varepsilon & I \end{pmatrix} \subseteq \text{Ker}\ W(A^{t+1}\ \hat{K}_t). \tag{6}$$

Furthermore, k' can be taken to be equal to $t+1$, i.e, convergence happens in at most $t+1$ steps.

Proof. The proof is an extension of classical geometrical control arguments (see [19]).

Necessity: Suppose $k' = t+1$ for a finite t, that is, convergence to $Wx[k]$ happened in at most $t+1$ steps. Consider two different initial conditions $x'[0]$ and $x''[0]$ and $y'[0:t]$ and $y''[0:t]$ their respective output vectors, both with the same control inputs $u[0:t]$. Then, if we have $y'[0:t] = y''[0:t]$, in order to retrieve $Wx[t+1]$ we must have $Wx'[t+1] = Wx''[t+1]$. Otherwise they will have the same inputs and outputs but different $Wx[t+1]$, and then it is impossible, using only past inputs and outputs, to recover $Wx[t+1]$. In the light of Fact 2, we can write $y'[0:t] = y''[0:t]$ as

$$\hat{O}_t x'[0] \oplus \hat{M}_t u[0:t] = \hat{O}_t x''[0] \oplus \hat{M}_t u[0:t]. \tag{7}$$

We can rewrite this equation in a more convenient form: create two variables $u'[0:t]$ and $u''[0:t]$ and rewrite Equation 7 as

$$\begin{aligned} \hat{O}_t x'[0] \oplus \hat{M}_t u'[0:t] &= \hat{O}_t x''[0] \oplus \hat{M}_t u''[0:t]; \\ u'[0:t] &= u''[0:t]. \end{aligned} \tag{8}$$

Thus, if $y'[0:t] = y''[0:t]$ with the same control inputs, which is equivalent to Equation 8, we need to have the same $Wx[t+1]$, thus $Wx'[t+1] = Wx''[t+1]$ and, in the light of Fact 1

$$W(A^{t+1}x'[0] \oplus \hat{K}_t u'[0:t]) = W(A^{t+1}x''[0] \oplus \hat{K}_t u''[0:t]). \tag{9}$$

Consequently, Equation 8 implies Equation 9. This statement is equivalent to Equation 6.

Sufficiency: The condition $Ker\ G \subseteq Ker\ H$ is equivalent to the existence of a matrix L such that $LG = H$ (See [14].). Thus, Equation 6 implies that there exists matrices L_y and L_u such that

$$L_y \hat{O}_t = WA^{t+1}\ ,\ L_y \hat{M}_t \oplus L_u = W\hat{K}_t. \tag{10}$$

Let $k \geq t+1$. Post-multiplying the first equation in Equation 10 by $x[k-(t+1)]$, the second equation by $u[k-(t+1):k-1]$, summing up both equations and using Facts 1 and 2, we conclude that

$$Wx[k] = L_y y[k-(t+1):k-1] \oplus L_u u[k-(t+1):k-1]. \tag{11}$$

And then we can, for $k \geq t+1$, retrieve the values of $Wx[k]$ using only the past outputs $y[k-(t+1):k-1]$ and inputs $u[k-(t+1):k-1]$. □

The result derived in Proposition 1 can be compared to our previous result in [7]. In that paper, it was proposed sufficient conditions - also a set of Max-Plus one-sided affine equations with increasing complexity- for solving the observation problem $\mathbb{T}\mathcal{O}$. It turns out that these equations are a special case of Equation 10. Furthermore, in that paper only sufficient conditions were presented, whereas in the present one we also established the necessity.

The following lemma is clear, since Equation 6 is a necessary and sufficient condition for finding a solution in $t+1$ steps and, if we can find a solution in $t+1$ steps, we can also find in $t'+1$ steps for $t' \geq t$.
Lemma 2. *If the condition in 6 (or equivalently Equation 10) holds for a t, it holds for any $t' \geq t$.* □

4.3 Connection with Classical Results

We will now discuss the connection between the result obtained in Proposition 1 and the result for the traditional counterpart of $\mathbb{R}\mathcal{O}$. Indeed, exactly the same reasoning can be made for the traditional counterpart problem, and, additionally, in traditional algebra simplifications can be made. The traditional counterpart of Equation 10, which is equivalent to the traditional counterpart of Equation 6, can be simplified since the second equation, $L_y \hat{M}_{t-1} + L_u = W\hat{K}_{t-1}$, is trivially satisfied as we can always choose $L_u = W\hat{K}_{t-1} - L_y \hat{M}_y$. Note that we can only do this because we can *subtract*, something which is impossible in the Max-Plus case.

With this simplification, the traditional counterpart of Equation 6 reduces to the condition $Ker\ \hat{O}_{t-1} \subseteq WA^t$.

This implies that the observability problem in the traditional case is *independent* of the input matrices B and D, a result which is well known in the control literature.

Such simplification is not always possible in Max-Plus algebra, since subtraction does not exists, and then the second equation in 10 may render the problem unsolvable if $L_y\hat{M}_{t-1} \preceq W\hat{K}_{t-1}$ is not true. This implies that, in the Max-Plus setting, *the control matrices influence the observation problem.*

It is easy to check whether Equation 10 has, for a given t, a solution or not and, in the case of a positive answer, obtain it. This is thanks to the following fact.

Fact 3. *(see [2]) There exists a solution L for a Max-Plus one-sided affine equation of the form $LV = U$ if and only if $(U\phi V)V = U$, and furthermore $L = U\phi V$ is the greatest solution. Additionally, if $V \in \mathbb{Z}_{\max}^{a\times b}$ and $U \in \mathbb{Z}_{\max}^{c\times b}$, this equality can be checked in $O(abc)$ time.* □

Thus, for a given t, the solvability of Equation 10 can be checked in polynomial time. The only problem is that, in principle, we need to check for *all* t's and then, in general, the obtained condition can never be used to decide that the problem is unsolvable (it works only as a *sufficient* condition). A natural question then arises: *Is there a maximum finite t' for which we need to check?*. Or, equivalently: *Is there a t' such that if there is no solution for $t = t'$ we can guarantee that there is no solution for all $t \geq t'$?*

We will answer this question first for the traditional problem $\mathbb{R}\mathcal{O}$. In this setting an analogous result in Proposition 1 holds. As mentioned before, in this case the condition in Equation 6 reduces to *Ker* $\hat{O}_{t-1} \subseteq$ *Ker* WA^t. Considering Lemma 2 (which is also true in the traditional setting), the weaker condition happens when t goes to ∞. In this case, thanks to the Cayley-Hamilton theorem, *Ker* $\hat{O}_{t-1}$ reduces to *Ker* $\hat{O}_{n-1}$, in which n is the size of the square matrix A. Hence the Kernel cannot change from $t \geq n-1$ in comparison to *Ker* $\hat{O}_{n-1}$, which only has powers of A below n.

We will also simplify the term *Ker* WA^t. In order to do that, we need the following lemma.

Lemma 3. *Let J be a matrix with only non-stable eigenvalues ($|\lambda| \geq 1$). Then $\lim_{k\to\infty} WJ^kx = 0 \Rightarrow Wx = 0$.*

Proof. Suppose $J \in \mathbb{R}^{n\times n}$. Since J is invertible, apply Cayley-Hamilton theorem in J^{-k} to conclude that $J^{-nk} = \sum_{i=0}^{n-1} \alpha_i[k]J^{-ik}$ for constants $\alpha_i[k]$. Since all the eigenvalues of J^{-k} for $k > 0$ are inside or on the border of the unit circle, these coefficients $\alpha_i[k]$ all converge to a finite value as $k \to 0$. This is true because they are the coefficients of the characteristic polynomial of J^{-k}, and hence, thanks to the Vieta's formulas, they are sum of products of the eigenvalues of J^{-k}, i.e, the eigenvalues of J to the $(-k)^{th}$ power, which all are finite as $k \to \infty$. Thus, multiplying the equation for J^{-nk} by J^{nk} we see that $I = \sum_{i=0}^{n-1} \alpha_i[k]J^{(n-i)k}$ Now, since $\lim_{k\to\infty} WJ^{k(n-i)}x = 0$ for all $n > i \geq 0$ and $\lim_{k\to\infty} \alpha_i[k]$ is finite, we have that $\lim_{k\to\infty} W(\alpha_i[k]J^{(n-i)k})x = 0$. Summing up these statements and applying the derived equation for I, we conclude that $Wx = 0$, as we wished to prove. □

We also need the important definition.

Definition 6. (Stable annihilator matrix) *Consider the matrix $A \in \mathbb{R}^{n\times n}$ with Jordan canonical form*

$$A = Q\begin{pmatrix} J_s & 0 \\ 0 & J_u \end{pmatrix}Q^{-1}$$

in which J_s is the Jordan block related to stable eigenvalues ($|\lambda| < 1$) and J_u the remaining ones ($|\lambda| \geq 1$). The stable annihilator matrix *is the matrix $\Pi(A)$ obtained from A by replacing J_s by the zero matrix and J_u by the identity matrix I.* □

And then, we can use the Jordan decomposition of A to obtain the following corollary of Lemma 3.

Corollary 1. *(of Lemma 3) It holds that $\lim_{t\to\infty} WA^tx = 0 \Rightarrow W\Pi(A)x = 0$ or, equivalently, as $t \to \infty$, Ker $WA^t \subseteq$ Ker $W\Pi(A)$.* □

With Corollary 1, we see that as $t \to \infty$, the condition *Ker* $\hat{O}_{t-1} \subseteq$ *Ker* WA^t implies that *Ker* $\hat{O}_{n-1} \subseteq$ *Ker* $W\Pi(A)$. Consequently, the latter therefore is a *necessary* condition for solving $\mathbb{R}\mathcal{O}$. It is also sufficient, because it implies that there exists L_y such that $L_y\hat{O}_{n-1} = W\Pi(A)$. Post-multiplying by A^t and using the fact that $L_y\hat{O}_{n-1}A^t = \hat{L}_y\hat{O}_{n-1}$ for a matrix $\hat{L}_y$ (thanks to the Cayley-Hamilton theorem), taking the limit to $t \to \infty$ in both sides and noting that $\lim_{t\to\infty}(A^t - A^t\Pi(A)) = 0$, we see that it also implies that, as $t \to \infty$, the condition *Ker* $\hat{O}_{t-1} \subseteq$ *Ker* WA^t holds.

Thus, we arrive at a very nice condition for solving $\mathbb{R}\mathcal{O}$.

Fact 4. $\mathbb{R}\mathcal{O}(\mathcal{S}, W)$ *is solvable if and only if*

$$\textit{Ker}\ \hat{O}_{n-1}(A, C) \subseteq \textit{Ker}\ W\Pi(A) \tag{12}$$

which is a weaker form of traditional detectability (indeed, if $W = I$ we recover the traditional meaning of detectability). □

For the Max-Plus case, $\mathbb{T}\mathcal{O}$, the situation is not so easy. There is no straightforward simplification, as it was done for $\mathbb{R}\mathcal{O}$, for the condition 6 as $t \to \infty$. Indeed, for $t \to \infty$, we have an infinite system of equations which is therefore impossible to be checked. For a finite t, even for small systems, it may be the case that the observation problem $\mathbb{T}\mathcal{O}$ has a solution but a very large t must be considered.

Example : Consider the problem $\mathbb{T}\mathcal{O}(\mathcal{S}, W)$ with matrices A as in Equation 4, $B = \varepsilon$, $C = (0\ 0)$, $D = \varepsilon$ and $W = (0\ \varepsilon)$. In the case that B and D are null matrices, Equation 6 reduces to $Ker\ \hat{O}_{t-1} \subseteq Ker\ WA^t$. The Kernel of $\hat{O}_{t-1}$ can be seen to be, using the expression for A^k in Equation 5 , spanned by the row vectors $(0\ \ -k)$ for $0 \leq k \leq t-1$ if $t-1 \leq p$, and spanned by the row vectors $(0\ \ -k)$ for $0 \leq k \leq p$ if $t-1 > p$. Thus, for $t-1 > p$ the kernel of $\hat{O}_{t-1}$ does not change.

Now, we can calculate that $WA^t = (0\ \ -p)$ for all t. It is easy to see, then, that it is impossible for $Ker\ \hat{O}_{t-1}$ to be contained in $Ker\ WA^t$ for $t-1 < p$, and for $t-1 \geq p$ it is possible, and thus the observation problem has solution. This example shows that, even for an inputless small system, the observation problem may be solved only after a high number of steps t (in the example take, for instance, $p = 100$). □

It remains an open question how to obtain the greatest t' that we need to consider condition in 6. That is, a t' such that if there is no solution for this t' we can declare that there is no solution for any $t \geq t'$ either.

5 Solving $\mathbb{T}\mathcal{O}_L$

5.1 Necessary and sufficient condition for $\mathbb{T}\mathcal{O}_L$

We will now study the Luenberger problem $\mathbb{T}\mathcal{O}_L(\mathcal{S}, W)$.

Proposition 2. *Let $\hat{K}_{t-1} = \hat{K}_{t-1}(A, B)$ and $\mathcal{J} \subseteq \{1, 2, ..., s\}$ be the set of all j such that for all k the j^{th} row of P^k is not ε. Then, the problem $\mathbb{T}\mathcal{O}_L(\mathcal{S}, W)$ has a solution if and only if there exist a finite t and matrices P, Q, R, S and T solving the following system of conditions*

$$\begin{array}{rll} (i) & : SA^{t+1} & = PSA^t \oplus QCA^t; \\ (ii) & : WA^t & = TSA^t; \\ (iii) & : SA\hat{K}_{t-1} & = PS\hat{K}_{t-1} \oplus QC\hat{K}_{t-1}; \\ (iv) & : W\hat{K}_{t-1} & = TS\hat{K}_{t-1}; \\ (v) & : SB & = QD \oplus R; \\ (vi) & : \rho(P) & < \rho(A); \\ (vii) & : & \text{For all } j \in \mathcal{J}, \text{ there must exists } i(j) \text{ such that the } j^{th} \text{ row in } P^iQ \text{ is not null.} \end{array} \tag{13}$$

Proof. **Sufficiency**: Let $k \geq t+1$. Post-multiply Equation 13-(i) by $x[k-(t+1)]$, Equation 13-(iii) by $u[k-(t+1) : k-2]$ and Equation 13-(v) by $u[k-1]$. Sum all these statements and use Fact 1 to conclude that

$$Sx[k] = P(Sx[k-1]) \oplus Qy[k-1] \oplus Ru[k-1] \tag{14}$$

which is of the form in Equation 3 with $z[k] = Sx[k]$ for $k \geq t+1$. Consider, again, $k \geq t+1$. Now, post-multiply Equation 13-(ii) by $x[k-t]$ and Equation 13-(iv) by $u[k-t : k-1]$. Sum these statements, and use again Fact 1 to conclude that $TSx[k] = Tz[k] = Wx[k]$ for $k \geq t+1$. This implies that, for $k \geq t+1$ and with the initial condition $z[0] = Sx[0]$, Equation 3 is such that $Tz[k]$ converges to $Wx[k]$.

We will now establish that the initial condition $z[0]$ can be any vector other than the vector that guarantees convergence, $z[0] = Sx[0]$. This implies convergence for all $z[0]$. Indeed, we will see that Equation 13-(vi,vii) guarantee that the system state $z[k]$ is, eventually, *independent* of $z[0]$.

To establish that, note that Equation 13-(vii) guarantees that there exists a map $i : \mathcal{J} \mapsto \mathbb{N}$ such that the j^{th} row in $P^{i(j)}Q$ is not null. Now, let $k \geq \max_{j \in \mathcal{J}} i(j)$ and write, expanding Equation 3

$$z[k] = P^k z[0] \oplus \bigoplus_{j \in \mathcal{J}} P^{i(j)} Q y[k - i(j)] \oplus f \tag{15}$$

in which f is the remaining term in the expansion that is *independent* of $z[0]$. Now, since $y[k]$ grows with a rate at least equal to $\rho(A)$ (see Assumption 1) , each of the entries of each one of the signals $y[k-i(j)]$ grow with this rate. Furthermore, for each row $j \in \mathcal{J}$ the j^{th} row of $P^{i(j)}Q$ is not null, and therefore we conclude that the signal $g[k] = \bigoplus_{j \in \mathcal{J}} P^{i(j)}Qy[k-i(j)]$ is such that all its entries in $\mathcal{J}$ grow with a rate of at least $\rho(A)$. To conclude, due to the Equation 13-(vi), there exists a finite $\hat{k} \geq \max_{j \in \mathcal{J}} i(j)$ such that eventually $\{g[k]\}_{\mathcal{J}} \succeq \{P^k z[0]\}_{\mathcal{J}}$ for all $k \geq \hat{k}$ and $z[0]$. Consequently, for $k \geq \hat{k}$

$$\{z[k]\}_{\mathcal{J}} = \{P^k z[0] \oplus g[k] \oplus f\}_{\mathcal{J}} = \{g[k] \oplus f\}_{\mathcal{J}} \tag{16}$$

which is independent of $z[0]$. Then for the entries in $\mathcal{J}$ the signal $z[k]$ is, in steady state, independent of $z[0]$. For the entries not in $\mathcal{J}$, that is, in $\overline{\mathcal{J}}$, this is also true since $\{P^k z[0]\}_{\overline{\mathcal{J}}} = \varepsilon$ for at least a $k = \tilde{k}$. Since $P^{\tilde{k}+s} = (P^{\tilde{k}})P^s$, we have that $\{P^k z[0]\}_{\overline{\mathcal{J}}} = \varepsilon$ for any $k \geq \tilde{k}$. Consequently, it is also true that the steady-state signal will be independent from $z[0]$ in the entries $j \in \overline{\mathcal{J}}$. Thus, as k grows the initial condition $z[0]$ is immaterial for the state $z[k]$.

Necessity: **Condition (i,iii,v):** Suppose there is a t such that the dynamical system in Equation 3 is such that $z[k] = Sx[k]$ and $Tz[k] = TSx[k] = Wx[k]$ for all $k \geq t$. Hence, for $k \geq t$, $z[k] = SA^k x[0] \oplus S\hat{K}_{k-1}u[0 : k-1]$. In special, we can take $k = t$ and $k = t+1$ and plug into Equation 3 to conclude that

$$\begin{aligned} & SA^{t+1}x[0] \oplus S\hat{K}_t u[0{:}t] = PS^t x[0] \oplus PS\hat{K}_{t-1}u[0{:}t-1] \\ & \oplus QCA^t x[0] \oplus QC\hat{K}_{t-1}u[0{:}t-1] \oplus QDu[t] \oplus Ru[t]. \end{aligned} \tag{17}$$

Equation 17 must hold *for all* $x[0]$ and vector $u[0:t]$. Take $u[0:t]=\varepsilon$ and $x[0]$ the columns of the identity matrix of same dimension of $x[0]$ to conclude the necessity of Equation 13-(i). Take $x[0]=\varepsilon$, $u[t]=\varepsilon$ and $u[0:t-1]$ the columns of the identity matrix of the same dimension of $u[0:t-1]$ to conclude the necessity of Equation 13-(iii). Finally, take $x[0]=\varepsilon$, $u[0:t-1]=\varepsilon$ and $u[t]$ the columns of the identity matrix of the same dimension of $u[t]$ to conclude the necessity of Equation 13-(v).

Condition (ii,iv): For Equation 13-(ii,iv), it suffices to consider that the equation $TSx[k]=Wx[k]$ must hold for $k\geq t$. Take $k=t$. Then, substitute $x[t]=A^t x[0]\oplus \hat{K}_{t-1}u[0:t-1]$. Noticing that it must hold *for all* $x[0]$ and $u[0:t-1]$ and using the same strategy of considering the columns of the identity matrix as before we can conclude the necessity of 13-(ii,iv).

It remains to establish the necessity of Equation 13-(vi,vii). For that, consider a trajectory $x[k]$ generated with an initial condition $x[0]$ and $u[k]=\varepsilon$ for all k, so $y[k]=CA^k x[0]$. Then, we can write, expanding Equation 3

$$z[k]=P^k z[0]\oplus\bigoplus_{i=0}^{k}P^iQCA^{k-i}x[0]. \tag{18}$$

Condition (vii): Suppose then Equation 13-(vii) does not hold. Then, there exists a $\hat{j}\in\mathcal{J}$ such that the $\hat{j}^{th}$ entry of $\bigoplus_{i=0}^{k}P^iQCA^{k-i}x[0]$ is ε for all k. These statements, together with Equation 18, imply that for any k the $\hat{j}^{th}$ entry of $z[k]$ is equal to the $\hat{j}^{th}$ entry of $P^k z[0]$, which is not ε (by the definition of $\mathcal{J}$) as long as we choose $z[0]$ without ε entries. This implies that this entry does not depend on $x[0]$ and can be made arbitrarily large by choosing $z[0]$ arbitrarily large. Thus, the required convergence $z[k]=Sx[k]$ in that entry can never be achieved for all $z[0]$. This implies the necessity of Equation 13-(vii).

Condition (vi): The necessity of Equation 13-(vi) comes with a similar argument by studying Equation 18. Suppose Equation 13-(vi) it does not hold, then there exists a entry of P^k which grows with a rate of at least $\rho(A)$. Suppose this entry lies in the j^{th} row. Thus, we can choose $z[0]$ sufficiently large so $\{P^kz[0]\}_j\geq\{\bigoplus_{i=0}^{k}P^iQCA^{k-i}x[0]\}_j$, because the right side of this inequality grows with a rate of at most $\rho(A)$. So $\{z[k]\}_j=\{P^kz[0]\}_j$. This implies that it is impossible to have the equality $\{z[k]\}_j=\{Sx[k]\}_j$ for any $z[0]$ because $z[0]$ can be sufficiently large and independent of $x[0]$. ☐

5.2 *Equivalence between observation problems*

We will now establish an interesting result. From Lemma 1, we know that $\mathbb{TO}_L$ is at least as hard to solve as $\mathbb{TO}$. We will now establish that $\mathbb{TO}_L$ is strictly harder to solve.

Proposition 3. *There exists a system $\mathcal{S}$ and a matrix W for which the problem $\mathbb{TO}(\mathcal{S},W)$ has a solution but $\mathbb{TO}_L(\mathcal{S},W)$ does not.*

Proof. We will prove this result presenting an example. Consider the problem in which $\mathcal{S}$ is given by

$$x[k+1]=\begin{pmatrix}0&0\\0&0\end{pmatrix}x[k]\oplus u[k];\quad y[k]=(0\ \ \varepsilon)x[k]$$

and W is equal to $(0\ \ 0)$. The problem $\mathbb{TO}(\mathcal{S},W)$ has a solution because Equation 6 holds for $t=1$. We will now show that $\mathbb{TO}_L(\mathcal{S},W)$ is not solvable.

Consider the necessary and sufficient conditions in Equation 13, presented in Proposition 2. We will prove that the conditions in Equation 13 can not hold simultaneously, and thus $\mathbb{TO}_L(\mathcal{S},W)$ does not have a solution. Indeed, Equation 13-(iii) has, for all $t>0$, the subequation $SAB=PSB\oplus QCB$. Since $B=I$, this reduces to $SA=PS\oplus QC$. For $t=0$, this same equation appears in Equation 13-(i). So, for any t, the equation $SA=PS\oplus QC$ must have a solution.

Let $\hat{p}^T$ be a left-eigenvector of P associated with the *greatest* eigenvalue γ, that is, $\gamma=\rho(P)$. Pre-multiply $SA=PS\oplus QC$ by $\hat{p}^T$ and let $p^T=\hat{p}^TS$ and $q=\hat{p}^TQ$. Then, if $SA=PS\oplus QC$ has a solution, so does $p^TA=\gamma p^T\oplus qC$ for p,q,γ. In our case, this equation reduces to, considering $p^T=(p_1\ p_2)$, the scalar equations $(a):\ p_1\oplus p_2=\gamma p_1\oplus q$ and $(b)\ :\ p_1\oplus p_2=\gamma p_2$. Now, considering Equation 13-(vi), we have that the greatest eigenvalue γ is smaller than $\rho(A)$. Thus, the scalar equations (a) and (b) must have a solution for p_1,p_2,q,γ, with $\gamma<\rho(A)=0$. It is easy to see that the equation is only solvable with $p_1=p_2=q=\varepsilon$. In particular, this implies then that $p^T=\varepsilon$.

To conclude, note that we can assume, without loss of generality, that no row of S is null. Indeed, in Equation 13, it can be seen that if it has a solution with a S with a null row, it also have a solution for the S obtained removing the same row. Thus, since $\hat{p}^T$ is an eigenvector, and thus not null, and S has no null row, $p^T=\hat{p}^TS$ cannot be the ε vector, which is a contradiction. And the proof is complete. ☐

Proposition 3 is quite relevant because of the following fact:

Fact 5. *The problems $\mathbb{RO}(\mathcal{S},W)$ and $\mathbb{RO}_L(\mathcal{S},W)$ are equivalent.* ☐

That is, in the traditional case, both the observer and Luenberger observer problems are equivalent, whereas in the Max-Plus case this is not true (Proposition 3).

To establish the result in Fact 5, owing to Lemma 1, we simply need to establish that if $\mathbb{R}\mathcal{O}$ has a solution, so does $\mathbb{R}\mathcal{O}_L$.

An analogous of Proposition 2, with an analogous proof, can be derived for $\mathbb{R}\mathcal{O}_L$. In this case, the necessary and sufficient conditions are analogous to the ones in Equation (13), but with (a) the operators interpreted in the traditional algebra, (b) with $\rho(A)$ replaced by 1 in (vi) and (c) without (vii).

It will be established now that $S = (\hat{O}_{n-1}^T\ (I-\Pi(A))^T)^T$ solves the traditional analogue of Equation 13-(i,iii,vi) for $t = 0$, that is, with this S there exist matrices P, Q solving all these three conditions. For that, we will need to use the traditional analogue of Theorems 2 and 3, that is, there exist matrices $\overline{P}, \overline{Q}$ such that $\hat{O}_{n-1}A = \overline{P}\hat{O}_{n-1} + \overline{Q}C$. It turns out that an even stronger result holds.

Lemma 4. *In the equation $\hat{O}_{n-1}A = \overline{P}\hat{O}_{n-1} + \overline{Q}C$, the matrix $\overline{P}$ can be chosen so $\rho(\overline{P}) < 1$.*

Proof. Let $A \in \mathbb{R}^{n\times n}$. For the solution of $\hat{O}_{n-1}A = \overline{P}\hat{O}_{n-1} + \overline{Q}C$, it is clear that we can choose an arbitrary $\overline{Q} = (Q_0^T\ Q_1^T\ \dots\ Q_{n-1}^T)^T$ and

$$\overline{P} = \begin{pmatrix} -Q_0 & I & 0 & \dots & 0 \\ -Q_1 & 0 & I & \dots & 0 \\ -Q_2 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -Q_{n-1}+\alpha_0 I & \alpha_1 I & \alpha_2 I & \dots & \alpha_{n-1}I \end{pmatrix} \quad (19)$$

in which $A^n = \sum_{i=0}^{n-1}\alpha_i A^i$ due to Cayley-Hamilton Theorem. It is a well known fact that, in this configuration, we can choose $\overline{Q}$ so $\rho(\overline{P}) < 1$. This is true because, considering the matrices

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} \end{pmatrix} \quad (20)$$

and $\tilde{C} = (1\ \ 0\ \ 0\ \ \dots\ \ 0)$, the pair $(\tilde{A}, \tilde{C})$ is clearly observable. Thus, there exists $\tilde{Q} = (q_1\ q_2\ \dots\ q_n)^T$ so $\rho(\tilde{A} - \tilde{Q}\tilde{C}) < 1$ (see [19]). Let $\otimes$ denote the Kronecker product of matrices and choose $\overline{Q} = \tilde{Q} \otimes I$. As $\overline{P} = (\tilde{A} - \tilde{Q}\tilde{C}) \otimes I$, we can see that $\rho(\overline{P}) < 1$, because the eigenvalues of the Kronecker product $U \otimes V$ are all possible combinations of products of eigenvalues of U and V (see [11]). ☐

Lemma 4 establishes that $\hat{O}_{n-1}A = \overline{P}\hat{O}_{n-1} + \overline{Q}C$ can be solved with $\rho(\overline{P}) < 1$. We will now derive, from Lemma 4, a related result.

Lemma 5. *Let $S = (\hat{O}_{n-1}^T\ (I-\Pi(A))^T)^T$. So, in the equation $SA = PS + QC$, the matrix P can be chosen so $\rho(P) < 1$.*

Proof. Considering the matrices $\overline{P}$ and $\overline{Q}$ in Lemma 4, choose

$$P = \begin{pmatrix} \overline{P} & 0 \\ 0 & A(I-\Pi(A)) \end{pmatrix},\ Q = \begin{pmatrix} \overline{Q} \\ 0 \end{pmatrix}. \quad (21)$$

We can see by straightforward verification that $SA = PS + QC$ and that $\rho(P) < 1$, since both $\rho(\overline{P}) < 1$ and $\rho(A(I-\Pi(A))) < 1$ hold. ☐

Lemma 5 establishes that, with $S = (\hat{O}_{n-1}^T\ (I - \Pi(A))^T)^T$, we can solve $SA = PS + QC$ with $\rho(P) < 1$.

We will now construct a solution to the traditional analogue of Equation 13. Equation 13-(v) is trivial to solve in the traditional algebra since we can subtract and R does not appear in other equations: $R = SB - QD$. Now, we can solve Equation 13-(i,iii,iv,vi) with $t = 0$ (so Equations 13-(iii,iv) are trivialized), $S = (\hat{O}_{n-1}^T\ (I-\Pi(A))^T)^T$ and the P, Q induced in Lemma 5. It remains to show that, if we consider these parameters t, S, P, Q, there is a corresponding T that solves Equation 13-(ii).

The additional condition that guarantees this is the solvability of $\mathbb{R}\mathcal{O}$. Indeed, suppose $\mathbb{R}\mathcal{O}$ has a solution. From Fact 4, $Ker\ \hat{O}_{n-1} \subseteq Ker\ W\Pi(A)$. Thus, there exists a matrix L such that $L\hat{O}_{n-1} = W\Pi(A)$. Choose $T = (L\ W)$. Note that $TS = L\hat{O}_{n-1} + W(I - \Pi(A)) = W\Pi(A) + W(I - \Pi(A)) = W$. With this choice of T and S, we have that 13-(ii) can also be solved for $t = 0$, establishing Fact 5.

Someone could argue why we can not use the same strategy to solve Equation 13 (for $\mathbb{T}\mathcal{O}_L$) that we used in Equation 13 (for $\mathbb{R}\mathcal{O}_L$)? Is it possible to take $S = \hat{O}_p$, for a p, in the Max-Plus case and derive an analogous of Lemma 4? That is, can we solve $\hat{O}_pA = P\hat{O}_p \oplus QC$ with $\rho(P) < \rho(A)$? Unfortunately, we cannot proceed because we do not have subtractions in the Max-Plus case, which is fundamental in the construction in Equation 19.

5.3 Simplification strategy

For a fixed t, Equation 13 is a nonlinear Max-Plus equation with spectral radius constraints, which is, in general, hard to solve. However, if we fix the matrix S (for instance, inspired in the last developments for the traditional case, take $S = \hat{O}_p$ for a p such that $Ker\ \hat{O}_\infty =$

Ker $\hat{O}_p$), it can actually be solved (or shown to have no solution) very easily. For that, we need the following lemma, which is a slight extension of the well known result in Fact 3.

Lemma 6. *There is a solution L for a Max-Plus one-sided affine equation of the form $LV = U$ with the constraint $L \preceq L_{\max}$ if and only if $(U \phi V \wedge L_{\max})V = U$. Furthermore $L = (V \phi U \wedge L_{\max})$ is the greatest solution.*

Proof. It follows immediately as a corollary of Fact 3 by noting that $LV = U$ and $L \preceq L_{\max}$ can be rewritten as the one-sided Max-Plus affine system of equations $LV = U, L \oplus F = L_{\max}$ for the unknown variables L, F. Applying Fact 3 in this new equation, the result becomes clear. □

We then replace the condition of $\rho(P) < \rho(A)$ with the constraint $P \preceq P_{\max}$, in which $P_{\max}$ is a matrix for which all the entries are $h < \rho(A)$ for a fixed h. Note that $P \preceq P_{\max}$ implies that $\rho(P) < \rho(A)$, but the converse is not true (that is, there may exist matrices P such that $\rho(P) < \rho(A)$ but $P \npreceq P_{\max}$), so this condition is stronger. Nevertheless, Equation 13 with S and t fixed and this modified condition Equation 13-(vi) is a Max-Plus one-sided affine equation with bound constraints (Q,R and T have no bound constraints, and thus we can say that $Q \preceq Q_{\max}$, $R \preceq R_{\max}$ $T \preceq T_{\max}$, in which $Q_{\max}, R_{\max}, T_{\max}$ are matrices with very large entries). Thus, we can write Equation 13 as $LV = U$, $L \leq L_{max}$, in which $L = (P\ Q\ T\ R)$. We can then try to solve it using the result in Lemma 6. Note that, due to the fact that we find the greatest solution of the constrained affine equation in the strategy described in Lemma 6, Equation 13-(vii) will be satisfied, if possible, because the matrix will be as less sparse as possible.

6 Example

Consider the dynamical system $\mathcal{S}$

$$x[k+1] = \begin{pmatrix} 0 & -10 & -10 & 1 \\ 0 & 0 & -10 & 2 \\ -10 & 0 & 0 & 3 \\ -10 & -10 & 0 & 4 \end{pmatrix} x[k] \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u[k] \qquad (22)$$

with $y[k] = (\varepsilon\,\varepsilon\,\varepsilon\,0)x[k]$ and $W = (0\,\varepsilon\,\varepsilon\,0)$. The problem $\mathbb{TO}(\mathcal{S}, W)$ has a solution, because Equation 10 has a solution with $t = 3$ (and for $t < 3$ it is not solvable): $L_y = (16\ \ 12\ \ 8\ \ 4)$ and $L_u = (12\ \ 8\ \ 4\ \ 0)$. Thus, the sequence in Equation (11) can be used to recover $s[k] = Wx[k]$. Note that we use the delayed informations $y[k-1], y[k-2], y[k-3]$ and $y[k-4]$ in this approach.

In addition, $\mathbb{TO}_L(\mathcal{S}, W)$ has a solution as well. Using $S = I$ and considering a stronger condition on the spectral radius bound, $P_{ij} \leq 0 < \rho(A) = 4$, we can use the simplification strategy proposed in Subsection 5.3. For $t = 0$ we have a solution to the modified Equation 13 with these considerations, which generate the following Luenberger observer

$$z[k+1] = \begin{pmatrix} 0 & -10 & -10 & 0 \\ 0 & 0 & -10 & 0 \\ -10 & 0 & 0 & 0 \\ -10 & -10 & 0 & 0 \end{pmatrix} z[k] \oplus \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} y[k] \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u[k]$$

in which $s[k] = Tz[k] = (0\ \ \varepsilon\ \ \varepsilon\ \ 0)z[k]$ eventually converges to $Wx[k]$.

We will now compare the two solutions. For that, we will use an input signal $u[k]$ generated by the expression $u[k] = \sum_{i=0}^{k} \hat{u}[i]$ in which $\hat{u}[i]$ are random integer variables distributed uniformly on $[0, 8]$, so the mean is $\rho(A) = 4$ and $u[k]$ grows, on average, with the same rate as $\rho(A)$. We used a random initial condition for $x[0]$ and $z[0] = (0\ \ 0\ \ 0\ \ 0)^T$. We then computed the error in the two approaches between the desired signal $s[k] = Wx[k]$ and the constructed sequences $s_{lc}[k]$ (linear combination from Equation 13) and $s_{lue}[k]$ (Luenberger observer) for k ranging from 0 to 40. If we measure the signal perfectly, eventually both approaches converge to the real signal $s[k]$. To test the influence of errors in the measurement, at $k = 0, 10, 20, 30$ we generated a random error in the output $y[k]$ (used by boath approaches) by adding to it an integer random number between 0 and 10. The results, the difference between the real $s[k]$ and the sequences $s_{lc}[k]$ and $s_{lue}[k]$, can be seen in Figure 1. We note that the Luenberger approach is better at rejecting the perturbation. This happens because the approach based in Equation (11) uses delayed information, up to $y[k-4]$, so when there is an error in the measurement at the event k, it can influence negatively the sequence $s_{lc}[k]$ in the next four events.

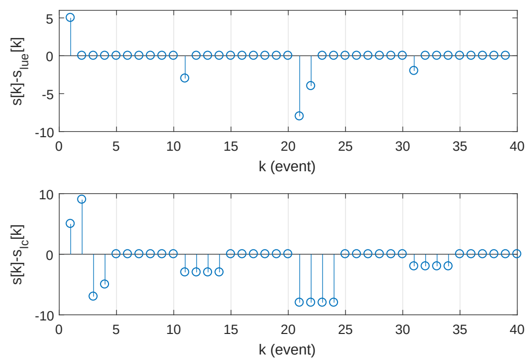

Fig. 1. Error between the real $s[k] = Wx[k]$ and reconstructions for both approaches.

7 Conclusion

In this paper, we were interested in Max-Plus observation problems, which were formulated in a general way as a problem of estimating a desired linear function of the state of the system. We have provided necessary and sufficient conditions for solving the problems, as well as comparisons with the analogous problem that appears in classical linear system theory. We showed that, due to the nonexistence of subtraction in the Max-Plus case, the observation problems are more difficult to solve and that we need to carefully take into account the input behaviour to solve the problems. For future works, it would be interesting to investigate how to quickly compute a maximum t that we need to check for a solution in Equation 6 (and Equation 13), and also develop an efficient method for solving the system of conditions in Equation 13.

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