CAPILLARY-GRAVITY WATER WAVES WITH VORTICITY: STEADY WIND-DRIVEN WAVES AND WAVES WITH A SUBMERGED DIPOLE

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by

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The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

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ABSTRACT

In this thesis, we study two mathematical problems on water waves in the setting of the incompressible Euler equations with vorticity, gravity, and surface tension. We investigate the existence of small-amplitude steady wind-driven water waves in finite depth, using the Crandall–Rabinowitz theorem. As part of the result, elliptic equations with transmission and Wentzell boundary conditions are also examined, and Schauder type estimates on classical solutions are established. The second chapter considers the existence and instability of solitary water waves with a finite dipole in infinite depth. We construct waves of this type using an Implicit Function Theorem argument. Then we establish orbital instability. This is proved using a modification of the classical Grillakis–Shatah–Strauss method.

Chapter 1

Introduction

When we think of fluids, waves come to our mind. They are famously difficult to study mathematically, as the interior dynamics are fundamentally nonlinear and the surface of the water is a priori unknown. There are several ways to create waves, in quiescent water, but the most common is through the presence of wind. Since the pioneering work of Miles [1, 2, 3, 4] in the 1950s and 1960s, it has been understood that vorticity in the atmosphere is crucial to this process. Surface tension also plays an important role in the initial stages of wind-driven water wave development. Waves subject to the effect of gravity and surface tension are called *capillary-gravity* waves.

In this section, we introduce the main equations governing incompressible inviscid fluids and discuss the effect of gravity, surface tension, and vorticity. We will postpone discussing regularity and well-posedness of the model until the later sections.

1.1 Incompressible Euler equations

The study of water waves began with the derivation of equations for incompressible inviscid fluid flows by Euler in 1755 [5]. Euler equations precisely capture an idealized fashion of fluid behavior and have been commonly used in both mathematics and other fields such as weather prediction and exploding supernova [6, 7, 8, 9].

Since these equations are derived in introductory texts of continuum mechanics (see, for example, [10]), we only offer a brief discussion, and focus on the physical phenomena that are most important for the mathematical work in the remainder of the thesis. For each time $t \geq 0$, let $\Omega(t) \subset \mathbb{R}^2$ be the fluid domain and $u = u(t, x) : \Omega(t) \to \mathbb{R}^2$ be the fluid velocity at each point $x \in \Omega(t)$. We first derive an equation for the incompressibility condition. Denote

$$\Omega_T := \{\{t\} \times \Omega(t) : t \in [0, T)\}, \quad \partial\Omega_T := \{\{t\} \times \partial\Omega(t) : t \in [0, T)\}.$$

Let $\boldsymbol{X} = \boldsymbol{X}(t, x) : \Omega_T \to \mathbb{R}^2$ be the Lagrangian flow map defined via the differential equation:

$$\begin{cases} \dot{\boldsymbol{X}} = u(t, \boldsymbol{X}), & \text{in } \Omega_0, \\ \mathcal{V}(\partial\Omega_t) = u \cdot N, & \text{on } \partial\Omega_0, \\ \boldsymbol{X}(0, \cdot) = \mathrm{Id}_{\Omega_0}, \end{cases} \tag{1.1.1}$$

where Id is the identity function on the initial domain $\Omega_0 = \Omega(0)$, $\mathcal{V}$ is the normal velocity of the boundary, and N is the outward unit normal. We use $\dot{\boldsymbol{X}}$ for the time derivative of $\boldsymbol{X}$. The second equation in (1.1.1) is for the particles to remain inside $\Omega(t)$.

Incompressibility refers to the fact that the flow map is measure preserving, that is:

$$|\Omega(t)| := \int_{\Omega(t)} 1 \mathrm{d}x = \int_{\Omega(0)} 1 \mathrm{d}x = |\Omega(0)|.$$

On the other hand, for t sufficiently small, we have

$$|\Omega(t)| = \int_{\Omega(0)} \det J\boldsymbol{X}(t, x) \, \mathrm{d}x,$$

where J is the Jacobian in the spatial variables. Comparing two equations, we see that the flow is incompressible whenever

$$\det J\boldsymbol{X}(t, x) = 1, \quad \text{in } \Omega(0), \forall t \geq 0.$$

Then we compute:

$$\begin{aligned} 0 = \partial_t(\det J\boldsymbol{X}) &= \partial_t((\partial_{x_1}\boldsymbol{X}_1)(\partial_{x_2}\boldsymbol{X}_2) - (\partial_{x_1}\boldsymbol{X}_2)(\partial_{x_2}\boldsymbol{X}_1)) \\ &= (u_{1x_1}(t, \boldsymbol{X}) + u_{2x_2}(t, \boldsymbol{X}))((\partial_{x_1}\boldsymbol{X}_1)(\partial_{x_2}\boldsymbol{X}_2) - (\partial_{x_1}\boldsymbol{X}_2)(\partial_{x_2}\boldsymbol{X}_1)) \\ &= (\nabla \cdot u)(t, \boldsymbol{X})(\det J\boldsymbol{X}). \end{aligned}$$

Therefore, it is equivalent to say that the flow u is *incompressible* if

$$\nabla \cdot u = 0, \quad \text{in } \Omega_T. \tag{1.1.2}$$

This is a special case of Liouville's Theorem (see, for example, [11]).

Next, the incompressible Euler equations are derived from several conservation laws. Let $\Omega_0' \subset \Omega_0$ and $\Omega'(t)$ be its image under the flow map. If $\rho = \rho(t,x) > 0$ is the density of the particles at x, the total fluid mass on $\Omega'(t)$ is given by

$$m(t) := \int_{\Omega'(t)} \rho(t,x)\,\mathrm{d}x.$$

Supposing that the mass of the total fluid is conserved, which we often anticipate when working with liquids, we can compute

$$\begin{aligned} 0 = m'(t) &= \int_{\Omega'(t)} \partial_t \rho\,\mathrm{d}x + \int_{\partial\Omega'(t)} \rho\, u_n\,\mathrm{d}S \\ &= \int_{\Omega'(t)} \partial_t \rho\,\mathrm{d}x + \int_{\partial\Omega'(t)} \rho u \cdot N\,\mathrm{d}S = \int_{\Omega'(t)} \left[\partial_t \rho + \nabla \cdot (\rho u)\right]\,\mathrm{d}x, \end{aligned}$$

where N is the outward unit normal to $\partial\Omega'(t)$ and $\mathrm{d}S$ is the surface measure on $\partial\Omega'(t)$. This follows from the Divergence Theorem and the fact that the fluid particles must remain in Ω_t, so the outward normal velocity $u \cdot N$ has to coincide with the normal velocity at the boundary $\mathcal{V}(\partial\Omega'(t))$. Since the above expression holds for any fixed time and fluid region $\Omega'(t) \subset \Omega(t)$, we conclude that

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad \text{in } \Omega(t). \tag{1.1.3}$$

Next, we derive the equation for conservative of momentum. Suppose that there are two types of forces: body forces, which act on the fluid's center of mass, and stress forces, which act on the surface of the particle. We assume that the only body force is gravity, and it is given by $-g\rho$, where $g > 0$ is the gravitational constant. For an inviscid fluid, the stress force is a gradient of the pressure force acting across the interface in the inward normal direction (see, for example, [12]). This force strength per unit surface area is equal to the pressure $p = p(t,x) : \Omega(t) \to \mathbb{R}$.

Let $P(t)$ be the total momentum on $\Omega'(t)$ defined by

$$P(t) = \int_{\Omega'(t)} \rho u\,\mathrm{d}x.$$

By Newton's Second Law, $P'(t)$ is equal to the force acting on it. Then

$$P'(t) = -\int_{\partial\Omega'(t)} pN\,\mathrm{d}S + \int_{\Omega'(t)} -g\rho\,\mathrm{d}x = -\int_{\Omega'(t)} (\nabla p + g\rho)\,\mathrm{d}x.$$

On the other hand, by Liouville's Theorem, we can write

$$P'(t) = \int_{\Omega'(t)} [\partial_t(\rho u) + (u \cdot \nabla)(\rho u)] \, \mathrm{d}x.$$

Thus, we obtain

$$\int_{\Omega'(t)} [\partial_t(\rho u) + (u \cdot \nabla)(\rho u) + \nabla p + g\rho] \, \mathrm{d}x = 0,$$

which by similar arguments as above, implies

$$\partial_t(\rho u) + (u \cdot \nabla)(\rho u) + \nabla p + g\rho = 0, \quad \text{in } \Omega(t).$$

Combing with equations (1.1.2)–(1.1.3), we arrive at the incompressible Euler equations

$$\begin{cases} \rho(\partial_t + u \cdot \nabla)u + \nabla p + g\rho = 0, & \\ (\partial_t + u \cdot \nabla)\rho = 0, & \text{in } \Omega_T. \\ \nabla \cdot u = 0, & \end{cases} \tag{1.1.4}$$

Since the quadratic nonlinear term $(u \cdot \nabla)u$ is the advection of the velocity field by itself, the analysis of the equations is more difficult. The operator $\partial_t + u \cdot \nabla$ is called the *material derivative* or *advective derivative* associated with the flow.

1.2 Boundary conditions

The Euler equations in the previous section are for particles in the interior of the fluid. This section provides conditions on $\partial\Omega_t$. Let $(x_1(t), x_2(t))$ be some point on $\partial\Omega(t)$. We assume that the interface can be written as the graph of a function $\eta = \eta(t, x_1)$. Then

$$x_2(t) = \eta(t, x_1(t)) \tag{1.2.1}$$

must hold for all $t \geq 0$. Since $(x_1'(t), x_2'(t)) = (u_1(t, x_1, x_2), u_2(t, x_1, x_2))$, differentiating (1.2.1) gives *kinematic boundary condition*

$$\partial_t \eta = -u_1 \eta' + u_2, \quad \text{on } \partial\Omega(t). \tag{1.2.2}$$

Here we denote prime for the derivative in x_1.

In addition to equation (1.2.2), there must be a balance of the pressure force at the fluid surface.

This is expressed through the *dynamic boundary condition*:

$$p = \alpha^2 \kappa(\eta), \quad \text{on } \partial\Omega(t), \tag{1.2.3}$$

where $\alpha^2 > 0$ is the coefficient of surface tension and

$$\kappa = \kappa(x_1) := -\frac{\eta''}{(1 + (\eta')^2)^{3/2}}$$

is the mean curvature of the surface. Equation (1.2.3) follows from the Young–Laplace law, which states that the jump in the pressure across a fluid interface is proportional to its curvature. Equation (1.2.3) is given under the assumption that the region above the fluid surface is at constant pressure 0.

The curvature is derived as following. Let $f(x_1, x_2) := x_2 - \eta(x_1)$, which vanishes on the surface. The unit outward normal vector is given by

$$N = \frac{\nabla f}{|\nabla f|} = \frac{(1, -\eta'(x_1))}{(1 + (\eta')^2)^{1/2}},$$

so the curvature is computed:

$$\nabla \cdot N = -\frac{\eta''}{(1 + (\eta')^2)^{3/2}} = \kappa.$$

Equations (1.1.4)–(1.2.3) are governing equations for gravity-capillary waves (see, for example, [13]).

1.3 Traveling water wave

A *steady* or *traveling wave* is a solution to a time-dependent problem that evolves by translating at a fixed velocity without altering its shape. Traveling waves are found throughout nature; they include ripples propagating along the surface of a pond, ignition fronts in combustion theory [14], seismic waves [15], and even tsunamis [16]. They are therefore the subject of intensive studies in a myriad of applied scientific fields, as well as a classical object of interest to mathematicians. Despite these considerable efforts, many fundamental aspects of traveling waves remain poorly understood.

Both historically and currently, traveling waves have been especially important to fluid mechanics and oceanography. Indeed, the concept of a steady wave originates with the observation by Russell of the famous Great Wave of Translation moving across the Glasgow–Edinburgh canal in 1844 [17]. This inspired a tremendous amount of research into the mathematical properties of such waves. In fact, it was not clear initially that steady waves were possible, nor that they were stable enough to be observed in the field. Some of the earliest work in this direction was due to Cauchy [18], Euler [19], Laplace [20], and Stokes [21]. Even now, traveling water wave is an extremely active and rapidly developing field with strong interdisciplinary connections.

Mathematically, to say a two-dimensional wave is traveling means that the vector field has the form

$$u(t, x) = \hat{u}(x - tc)$$

for some profile $\hat{u}$, wave speed $c = (c_1, 0) \in \mathbb{R}^2$, and likewise for the pressure, and density. The transformation $x \mapsto x - ct$ is called the *moving frame*. It follows that

$$\partial_t u = (-c \cdot \nabla)\hat{u}, \quad \partial_t \rho = (-c \cdot \nabla)\hat{\rho}.$$

In order for these coordinates to be appropriate, we will require that the domain $\Omega(t)$ is invariant under the moving frame. Abusing notation, we identify $\hat{u}$ with u and $\Omega(0)$ with $\Omega(t)$, the incompressible Euler equations (1.1.4) now become the steady equations

$$\begin{cases} \rho(u - c) \cdot \nabla u + \nabla p + g\rho = 0, \\ (u - c) \cdot \nabla \rho = 0, & \text{in } \Omega. \\ \nabla \cdot u = 0, \end{cases}$$

As the dynamic boundary condition (1.2.3) contains no time derivatives, its expression remains unchanged in the moving coordinates.

When the flow is steady at speed $(c, 0)$, we have the *Bernoulli condition* [22, 23], which states that

$$E := p + \frac{\rho}{2}((u_1 - c)^2 + u_2^2) + g\rho x_2 = p + \frac{1}{2}|\nabla\psi|^2 + g\rho x_2$$

is constant along the streamlines, which are the integral curves of $(u_1 - c, u_2)$.

1.4 Vorticity

The *vorticity* ω of a two-dimensional velocity field $u = (u_1, u_2)$ is the scalar-valued function:

$$\omega := \operatorname{rot} u = \partial_{x_1} u_2 - \partial_{x_2} u_1. \tag{1.4.1}$$

It measures the rotation of the fluid particle. The vast majority of the literature on water waves is concerned with the case of so-called *irrotational flow*, where the vorticity is assumed to be identically 0. The most compelling reason to work with irrotational flows is mathematical convenience. In particular, this assumption implies that the velocity field is the gradient of a harmonic function, called the *velocity potential* ϕ. This enables us to use many powerful results on harmonic functions and complex analysis. Examples of such tools are the use of conformal variables to fix the domain. Therefore, the velocity field can be determined by solving the Laplace equation given the behavior on the boundary.

However, there are several situations where consideration of rotational waves is needed. The classical Kelvin–Helmholtz theorem implies that the circulation around any smooth simple curve remains zero as long as it is only affected by conservative forces such as gravity [24]. Nonetheless, effects such as wind and temperature can induce rotation.

One of the earliest works of rotational waves is Gerstner in 1809 [25], who gave the first and explicit formula for solutions to the Euler equations in infinite depth under the effect of gravity and constant vorticity. Much later, in 1934 Dubreil–Jacotin [26] developed a non-conformal coordinate transformation to construct small-amplitude periodic water waves with vorticity. Then Ter-Krikorov [27, 28] presented a rigorous proof of small-amplitude rotational solitary waves. The transformation by Dubreil–Jacotin was again used by Constantin and Strauss to examine large waves using global bifurcation theory [29]. This was done under the assumption of a relation between the strength of the vorticity and the volumetric mass flux. Recently, Wheeler showed the existence of large-amplitude rotational solitary water waves [30, 31, 32] using a new global bifurcation technique. Moreover, gravity-capillary waves with compactly supported vorticity on infinite depth were constructed by Shatah, Walsh, and Zeng [33]. In this work, the vorticity comes from either a point vortex, the simplest possible compactly supported vorticity, or a vortex patch. We will discuss about point vortices in Section 1.5.

For a steady incompressible two-dimensional flow u, there exists a *stream function* ψ such that

$$u = \nabla^{\perp}\psi, \tag{1.4.2}$$

where $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$. The level sets of ψ are called the *streamlines* and contain information about the flow, since $\nabla\psi$ is orthogonal to the velocity field at each point. It is straightforward from the definitions (1.4.1) and (1.4.2) that

$$\Delta\psi = \omega, \tag{1.4.3}$$

which means that ψ is determined by ω only up to a harmonic function. Taking the curl of the incompressible Euler equations (1.1.4) and assuming constant density (which is the case in the two projects of this thesis), we can see that

$$\omega_t + \nabla \cdot (u\omega) = 0. \tag{1.4.4}$$

This is called the *vorticity transport equation*.

1.5 Point vortices and dipole

When a submerged object, such as a submarine, moves through water, it can "shed" vortices in the sense that complicated vortical structures develop in its wake. A simple model for this phenomenon is to imagine that the velocity of the water about the object has a dipole structure. This means that there are two point vortices with equal but opposite strength that move in parallel. This serves as a model for the propagation of vortices shed by the flow over a thin body. Mathematically, the vorticity for a *point vortex* is represented by a δ-measure. In particular, we have

$$\omega_{\text{point vortex}}(t,x) = \epsilon \delta_{\bar{x}(t)}(x),$$

where ϵ is the vortex strength, and $\bar{x}(t)$ is the vortex center at time t. Note that the above expression is not a solution to the equation (1.4.4) as it requires further weakening Euler equations as in Kirchhoff–Helmholtz [34, 50].

The vorticity for the finite dipole consisting of two point vortices is in the form:

$$\omega_{\text{finite dipole}}(t,x) = \epsilon_1 \delta_{\bar{x}(t)}(x) + \epsilon_2 \delta_{\bar{y}(t)}(x),$$

where ϵ_1 and ϵ_2 are the vortex strengths, and $\bar{x}(t)$ and $\bar{y}(t)$ are the vortex centers. Another type of highly localized vorticity is the point dipole. A *point dipole* occurs when the distance between two point vortices tends to 0. That is

$$\omega_{\text{point dipole}}(t,x) = \xi(t) \cdot \nabla \delta_{\bar{x}(t)},$$

where $\xi = \xi(t) := \epsilon \widetilde{\xi}$, $\epsilon = \epsilon(t)$, and $\widetilde{\xi} = \widetilde{\xi}(t) = \left(\widetilde{\xi}_1(t), \widetilde{\xi}_2(t)\right) \in \mathbb{R}^2$.

The study of point vortices and dipoles began with Helmholtz [34], who introduced the intuition that vortices are like particles. Since then, there has been extensive research on this subject both in physics and mathematics. For instance, Kelvin [35] gave the conservation of circulation in an inviscid incompressible fluid subject to certain forces. In 1876, Kirchhoff [36] was the first to derive governing equations for point vortices in the two-dimensional plane. Later, Thomson [37] investigated the stability of a ring of vortices on a sphere. Love [38] also studied the stability of Kirchhoff's elliptic vortex. Marchioro and Pulvirenti [39] showed the relationship between the incompressible Euler equations (1.1.4) and the Kirchhoff–Helmholtz model.

Each of the above works considers the case of water in a fixed domain. But a natural question is: what happens in an actual water wave. A number of mathematicians have studied the problem of how a free surface responds to the motion of submerged point vortices. Some of the earliest works were from Ter-Krikorov and Filippov [40, 27] who gave the existence theory for steady waves solution without surface tension. Later Tyvand investigated the short time behavior of a pair of strong and weak vortices when they were suddenly placed near the fluid surface [41, 42]. Kuznetsov and Ruban [43] derived exact equations of motion of point vortices when they interact with surface waves using conformal maps and asymptotic methods. They also examined how the speed of these moving vortices can affect surface waves. Moreover, systems with one or more vortices have been considered by many authors. For instance, Shatah, Walsh, and Zeng [33] constructed a family of steady capillary-gravity waves in infinite depth with both a single point vortex and a vortex patch. Varholm [44] obtained similar results for waves in finite depth with one or more vortices. Other authors have done numerical and experimental studies of two point vortices such as Fish [45], Marcus and Berger [46], Telste [47], Willmarth, Tryggvason, and Hirsa [48]. Finally, many real world applications for point vortices have also been studying. For example, the finite dipole serves as a model for fish schoolings [49].

When a point vortex is placed in traveling waves, the vorticity transport equation (1.4.4) implies that it is transported with the fluid flow and remains as such a vortex for all t. The splitting

$$\psi = \psi_{\mathcal{H}} + \epsilon\Gamma,$$

where $\psi_{\mathcal{H}}$ is harmonic, represents the irrotational part of the fluid, and $\epsilon\Gamma$ being the vortical contribution from point vortices. From the relation (1.4.3), the Newtonian potentials for a point vortex is given by

$$\Gamma_{\text{point vortex}}(t, x) = \frac{1}{2\pi} \log |x - \bar{x}|.$$

Another question we usually ask is: what is the motion of the vortex when it is placed inside the traveling wave? By the Kirchhoff–Helmholtz model [50, 34], the point vortex moves with the velocity field obtained by removing the rotational part. That means

$$\partial_t \bar{x}_{\text{point vortex}} = \left(v - \epsilon \nabla^{\perp}\Gamma\right)\big|_{x=\bar{x}} = \left(v - \frac{\epsilon}{2\pi}\frac{(-x_2+\bar{x}_2, x_1-\bar{x}_1)}{|x-\bar{x}|^2}\right)\Bigg|_{x=\bar{x}}.$$

Similarly, the Newtonian potential for a finite dipole is the sum of two Newtonian potentials of point vortex, and the motion for the finite dipole is given by two separated motion equations for each vortex. For the point dipole, we have

$$\Gamma_{\text{point dipole}}(t,x) = \xi \cdot \nabla\Gamma_{\text{point vortex}}.$$

1.6 Plan of the thesis

After establishing the background knowledge, we give a brief discussion about the structure of the thesis. Each of the remaining sections is a journal article.

Section 2 has been accepted for publication [51]. In this section, we present results about the existence and uniqueness of solutions of elliptic equations with transmission and Wentzell boundary conditions. We provide Schauder estimates and existence results in Hölder spaces. As an application, we develop an existence theory for small-amplitude two-dimensional traveling waves in an air-water system with surface tension. The water region is assumed to be irrotational and of finite depth, and we permit a general distribution of vorticity in the atmosphere.

Section 3 has been submitted. This project considers the existence and stability properties of two-dimensional solitary waves traversing an infinitely deep body of water. We assume that above the water is vacuum, and that the waves are acted upon by gravity with surface tension effects on the air–water interface. In particular, we consider the case where there is a finite dipole in the bulk of the fluid, that is, the vorticity is a sum of two weighted δ-functions. Using an implicit function theorem argument, we construct a family of solitary waves solutions for this system that is exhaustive in a neighborhood of 0. Our main result is that this family is conditionally orbitally unstable. This is proved using a modification of the Grillakis–Shatah–Strauss method recently introduced by Varholm, Wahlén, and Walsh.

Chapter 2

Wind-waves

2.1 Introduction

2.1.1 Elliptic theory

Let $\Omega \subset \mathbb{R}^n$ be a connected bounded $C^{2,\beta}$ domain for $n > 1$ and $\beta \in (0,1)$. Suppose that there exists a $C^{2,\beta}$ hypersurface Γ that divides Ω into two connected regions such that

$$\Omega = \Omega_1 \cup \Gamma \cup \Omega_2, \qquad \Omega_1 \cap \Omega_2 = \emptyset, \qquad \partial\Omega_1 \cap \partial\Omega_2 = \Gamma,$$

and denote by $S := \partial\Omega$. Let $\nu = (\nu_1, \ldots, \nu_n)$ be the normal vector field on the interface Γ pointing outward from Ω_1. We define the co-normal derivative operator on Γ

$$\partial_N := \sum_{i,j=1}^{n} a^{ij} \nu_i \partial_{x_j},$$

and the tangential differential operator along Γ

$$\mathcal{D}_s := \sum_{t=1}^{n} w^{st} \partial_{x_t}, \quad 1 \leq s \leq n,$$

where $w := I_n - \nu \otimes \nu$, and I_n is the $n \times n$ identity matrix.

Our main object of study is the following transmission problem with a Wentzell boundary condition

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } S, \\ [\![u]\!] = 0 & \text{on } \Gamma, \\ Bu = g & \text{on } \Gamma, \end{cases} \tag{2.1.1}$$

where

$$Lu := -\sum_{i,j=1}^{n} \partial_{x_i}(a^{ij}(x)\partial_{x_j}u) + \sum_{i=1}^{n} b^i(x)\partial_{x_i}u + c(x)u, \tag{2.1.2}$$

$$Bu := -\sum_{s,t=1}^{n} \mathcal{D}_s\big(\mathfrak{a}^{st}(x)\mathcal{D}_t u\big) + \alpha\,[\![\partial_N u]\!] + \sum_{s=1}^{n} \mathfrak{b}^s(x)\mathcal{D}_s u + \mathfrak{c}(x)u, \quad \alpha = \pm 1. \tag{2.1.3}$$

Here, we are using $[\![\cdot]\!] := (\cdot)|_{\Omega_1} - (\cdot)|_{\Omega_2}$ to denote the jump operator across Γ. We think of $\alpha = +1$ as favorable and $\alpha = -1$ as unfavorable. We shall assume uniform ellipticity condition on the operators L and B; that is, there exist constants $\lambda, \mu > 0$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n, \tag{2.1.4}$$

and

$$\mu|\xi|^2 \leq \sum_{s,t=1}^{n} \mathfrak{a}^{st}(x)\xi_s\xi_t \quad \text{for all } x \in \Gamma \text{ and } \xi \in \mathbb{R}^n \text{ such that } \xi \cdot \nu(x) = 0. \tag{2.1.5}$$

The coefficients a^{ij} and $\mathfrak{a}^{st}$ satisfy $a^{ij} = a^{ji}, \mathfrak{a}^{st} = \mathfrak{a}^{ts}$ for all $i, j, s, t = 1, \ldots, n$. We also assume that a^{ij}, b^i, c are in $L^\infty(\Omega)$, and $\mathfrak{a}^{st}, \mathfrak{b}^s, \mathfrak{c}$ are in $L^\infty(\Gamma)$.

Note that B contains second-order tangential derivatives of u. This is characteristic of so-called *Wentzell-type* boundary conditions, whose study was initiated by Wentzell in [52]. They arise, for example, in stochastic equations [53] or as an asymptotic model for roughness of the boundary or other more complex geometrical effects [54]. They also appear in water waves and continuum mechanics, which is our principal interest here. For instance, the Young–Laplace Law states that at the interface between two immiscible fluids, the pressure experiences a jump proportional to the curvature. In a free boundary problem where the interface is given as the graph of an unknown function, this naturally leads to quasilinear versions of Wentzell-type conditions. More generally, the curvature of a hyperplane is the first variation of its surface area. Thus, these types of conditions are frequently encountered in free boundary problems where the shape of the interface contributes to the energy.

Transmission conditions refers to the jump operator in B. They are commonly found in multiphase problems, where physically their purpose is to enforce continuity of the normal stress across a material interface. Many researchers alternatively call these *diffraction problems* (see, for example, [55, 56]).

We first have an a priori estimate for classical solutions in Hölder spaces.

Theorem 2.1.1 (Schauder estimate)**.** *Assume that* $a^{ij} \in C^{1,\beta}(\overline{\Omega_1}) \cap C^{1,\beta}(\overline{\Omega_2})$*,* $b^i, c \in C^{0,\beta}(\overline{\Omega_1}) \cap C^{0,\beta}(\overline{\Omega_2})$*, and* $\mathfrak{a}^{st} \in C^{1,\beta}(\Gamma)$*,* $\mathfrak{b}^s, \mathfrak{c} \in C^{0,\beta}(\Gamma)$*; and suppose that*

$$\|a^{ij}\|_{C^{1,\beta}(\Omega_k)}, \|b^i\|_{C^{0,\beta}(\Omega_k)}, \|c\|_{C^{0,\beta}(\Omega_k)}, \|\mathfrak{a}^{st}\|_{C^{1,\beta}(\Gamma)}, \|\mathfrak{b}^s\|_{C^{0,\beta}(\Gamma)}, \|\mathfrak{c}\|_{C^{0,\beta}(\Gamma)} < \Lambda_2$$

for some constant $\Lambda_2 > 0$*, for all* $i, j, s, t = 1, \dots, n$*, and* $k = 1, 2$*. Suppose that* $u \in C^0(\overline{\Omega}) \cap C^{2,\beta}(\overline{\Omega_1}) \cap C^{2,\beta}(\overline{\Omega_2})$ *solves equation* (2.1.1) *with* $\alpha = \pm 1$*. Then for any* $\Omega' \subset\subset \overline{\Omega} \backslash (\Gamma \cap S)$*, if* $f \in C^{0,\beta}(\overline{\Omega_1}) \cap C^{0,\beta}(\overline{\Omega_2})$ *and* $g \in C^{0,\beta}(\Gamma)$*, the following estimate holds:*

$$\begin{aligned} \|u_1\|_{C^{2,\beta}(\Omega_1')} + &\|u_2\|_{C^{2,\beta}(\Omega_2')} \\ &\leq C\left(\|u\|_{C^0(\Omega)} + \|f\|_{C^{0,\beta}(\Omega_1)} + \|f\|_{C^{0,\beta}(\Omega_2)} + \|g\|_{C^{0,\beta}(\Gamma)}\right) \end{aligned} \tag{2.1.6}$$

for some constant $C = C(n, \beta, \Lambda_2, \lambda, \mu, \Omega_1', \Omega_2') > 0$*, where* $\Omega_k' := \Omega' \cap \Omega_k$ *and* $u_k := u|_{\Omega_k}$*.*

Note that the estimates in the theorem hold on subsets that are positively separated from $\Gamma \cap S$ where the boundary may not be smooth. The next theorems will assume that $\Omega \subset \mathbb{T}^{n-1} \times \mathbb{R}$, where $\mathbb{T}$ is a torus and $S \cap \Gamma = \emptyset$.

Theorem 2.1.2 (Existence and uniqueness of solutions in $C^{2,\beta}$)**.** *Suppose the coefficients* a^{ij}, b^i, c*,* $\mathfrak{a}^{st}, \mathfrak{b}^s, \mathfrak{c}$ *exhibit the same regularity as in Theorem 2.1.1 and assume in addition that* $c \geq 0$ *and* $\mathfrak{c} > 0$*. Then for all* $f \in C^{0,\beta}(\overline{\Omega_1}) \cap C^{0,\beta}(\overline{\Omega_2})$ *and* $g \in C^{0,\beta}(\Gamma)$*, the problem* (2.1.1) *with* $\alpha = 1$ *has a unique* $C^{0,\beta}(\overline{\Omega}) \cap C^{2,\beta}(\overline{\Omega_1}) \cap C^{2,\beta}(\overline{\Omega_2})$ *solution.*

We emphasize that this theorem only holds if we assume $c \geq 0$, $\mathfrak{c} > 0$, and α has a favorable sign. However, for a general c, $\mathfrak{c}$, and α, we are still able to assert the Fredholm solvability of the problem. Letting

$$X := C^{2,\beta}(\overline{\Omega_1}) \cap C^{2,\beta}(\overline{\Omega_2}) \cap \{u|_S = 0\} \cap C^{0,\beta}(\overline{\Omega}), \tag{2.1.7}$$

which is a Banach space with respect to the norm

$$\|u\|_X := \|u\|_{C^{2,\beta}(\Omega_1)} + \|u\|_{C^{2,\beta}(\Omega_2)} + \|u\|_{C^{0,\beta}(\Omega)},$$

we have the following theorem.

Theorem 2.1.3 (Fredholm solvability)**.** *Suppose the coefficients* $a^{ij}, b^i, c, \mathfrak{a}^{st}, \mathfrak{b}^s, \mathfrak{c}$ *exhibit the same regularity as in Theorem 2.1.1 and* $\alpha = \pm 1$*. Then either*

(i) the homogeneous problem (2.1.1) *with* $f = g = 0$ *has nontrivial solutions that form a finite dimensional subspace of* X*, or*

(ii) the homogeneous problem has only the trivial solution in which case the inhomogeneous problem has a unique solution in X *for all* $f \in C^{0,\beta}(\overline{\Omega_1}) \cap C^{0,\beta}(\overline{\Omega_2})$ *and* $g \in C^{0,\beta}(\Gamma)$*.*

A great deal of research has been devoted to studying elliptic problems with linear and nonlinear Wentzell boundary conditions, but they remain comparatively less well-understood. One of the earliest works to consider Wentzell conditions was Korman [57] who, like us, was interested in their connection to a problem in water waves. Specifically, he investigated a model describing three-dimensional periodic capillary-gravity waves where the gravity pointed upward. A Schauder theory was later provided by Luo and Trudinger [58] for the linear case. In the quasilinear setting, Luo [59] gave a priori estimates for uniformly elliptic Wentzell conditions, while later Luo and Trudinger [60] studied the degenerate case. More recently, Nazarov and Paletskikh [61] derived local Hölder estimates in the spirit of De Giorgi for divergence form elliptic equations with measurable coefficients and a Wentzell condition imposed on a portion of the boundary. See also the survey by Apushkinskaya and Nazarov [62] for a summary of the progress made on the nonlinear problem.

Transmission boundary conditions are of great importance to physics and other applied sciences. They are also of interest from a purely mathematical perspective as they arise naturally in the weak formulation of PDEs with discontinuous coefficients. The study of transmission problems dates back to the 1950s and 1960s. Schechter [63] and Šeftel' [64] investigated even-order elliptic equations on a smooth and bounded domain with smooth coefficients. Schechter obtained estimates and provided an existence for weak solutions. His strategy involved transforming the transmission problem into a mixed boundary value problem for a system of equations. On the other hand, Šeftel' found a priori L^p-estimates. Oleĭnik [65] also studied transmission problems for second-order elliptic equations with smooth coefficients; approximating equations were used to derive results for weak solutions. One of the most foundational work was done by Ladyzhenskaya and Ural'tseva [55], who considered second-order elliptic equations on a bounded domain and then obtained estimates for weak and classical solutions in Sobolev and Hölder spaces, respectively. In contrast to Schechter's approach, Ladyzehnskaya and Ural'tseva exploited cleverly chosen test functions to deduce their a

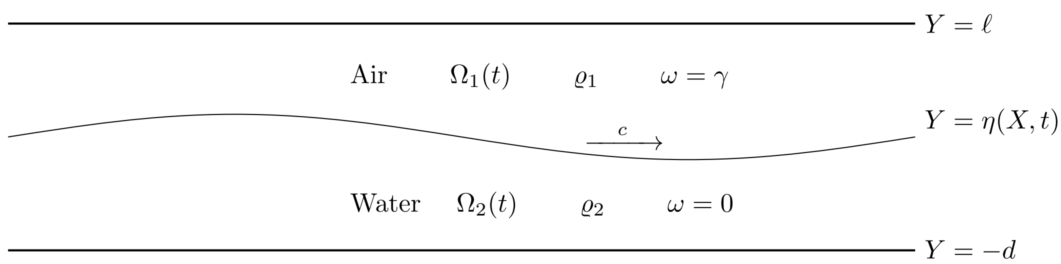

Figure 2.1: The air–water system

priori estimates. More recently, Borsuk [66, 67, 68] has treated linear and quasilinear transmission problems on non-smooth domains.

Apushkinskaya and Nazarov [69] considered Sobolev and Hölder solutions of linear elliptic and parabolic equations for two-phase systems. However, they only examined the problem with a favorable sign $\alpha = +1$ of the transmission term, and did not study Fredholm property. Note that in water wave applications, the sign is typically unfavorable. With that in mind, in this chapter we make the effort to also include Schauder estimates and Fredholm solvability for $\alpha = -1$ as well; see also Remarks 2.2.1, 2.2.5, and 2.2.7. Our approach is to view the Wentzell boundary condition as a non-local $(n-1)$-dimensional elliptic equation, treating the jump in the co-normal derivative term as forcing that can be controlled using techniques from the literature on transmission problems.

2.1.2 Steady wind-driven capillary-gravity water waves

Our second set of results considers an application of the above elliptic theory to a problem in water waves. In particular, we will prove the existence of small amplitude periodic wind-driven capillary-gravity waves in a two-phase air-water system. One of the main novelties here is that we also allow for a general distribution of vorticity in the air region. For simplicity we take the flow in the water to be irrotational. When discussing these results, we adopt notational conventions common in studies of steady water waves which occasionally conflict with our notations in the elliptic theory part.

Let us now formulate the problem more precisely. Fix a Cartesian coordinate system $(X, Y) \in \mathbb{R}^2$ so that the X-axis points in the direction of wave propagation and the Y-axis is vertical. The ocean bed is assumed to be flat and at the depth $Y = -d$, while the interface between the water and the atmosphere is a free surface given as the graph of $\eta = \eta(X, t)$. We then normalize η so that the free surface is oscillating around the line $Y = 0$. The atmospheric domain is assumed to be bounded in Y; that is, the air region lies below $Y = \ell$ for some fixed $\ell > 0$. At a given time t, the fluid domain is

$$\Omega(t) = \Omega_1(t) \cup \Omega_2(t),$$

where Ω_1 is the air region,

$$\Omega_1(t) := \{(X,Y) \in \mathbb{R}^2 : \eta(X,t) < Y < \ell\},$$

and Ω_2 is the water region,

$$\Omega_2(t) := \{(X,Y) \in \mathbb{R}^2 : -d < Y < \eta(X,t)\}.$$

We also denote $\mathcal{I}(t) := \partial\Omega_1(t) \cap \partial\Omega_2(t)$. Here we think of $\mathcal{I}(t)$ as playing the role of Γ in the notation of the previous subsection.

Let $u = u(X,Y,t)$ and $v = v(X,Y,t)$ be the horizontal and vertical fluid velocities, respectively, and denote by $P = P(X,Y,t)$ the pressure. We say that this is a *traveling wave* provided that there exists a wave speed $c > 0$ such that the change of variables

$$(X,Y) \mapsto (x,y) := (X - ct, Y)$$

eliminates time dependence. The velocity field is assumed to be incompressible and, in the moving frame, (u, v, η, P) are taken to be 2π-periodic in x.

For water waves, the governing equations are the incompressible steady Euler system:

$$\begin{cases} u_x + v_y & = 0, \\ \varrho(u-c)u_x + \varrho v u_y & = -P_x, \\ \varrho(u-c)v_x + \varrho v v_y & = -P_y - g\varrho, \end{cases} \qquad \text{in } \Omega \tag{2.1.8}$$

where $g > 0$ is the gravitational constant, and $\varrho = \varrho_1 \mathbb{1}_{\Omega_1} + \varrho_2 \mathbb{1}_{\Omega_2}$ with ϱ_1 and ϱ_2 assumed to be constant densities of Ω_1 and Ω_2, respectively. We assume $\varrho_1 < \varrho_2$. The symbol $\mathbb{1}_{\Omega_i}$ stands for the characteristic function on Ω_i. As above, $[\![\cdot]\!]$ denotes the jump over $\mathcal{I}$, that is, $[\![\cdot]\!] = (\cdot)|_{\Omega_1} - (\cdot)|_{\Omega_2}$.

The kinematic and dynamic boundary conditions for the lidded atmosphere problem with surface tension σ are

$$\begin{cases} v = 0 & \text{on } y = \ell, \\ v = 0 & \text{on } y = -d, \\ v = (u-c)\eta_x & \text{on } y = \eta(x), \\ [\![P]\!] = -\sigma \frac{\eta_{xx}}{(1+(\eta_x)^2)^{3/2}} & \text{on } y = \eta(x). \end{cases} \tag{2.1.9}$$

Note that the last condition will give rise to nonlinear Wentzell and transmission terms. In particular, the right hand side can be viewed as a second-order elliptic operator acting on η, while the jump in the pressure will relate to a jump in $(u-c)^2 + v^2$ via Bernoulli's theorem that we discuss below.

We consider waves without (horizontal) stagnation, that is, we will always assume

$$u - c < 0 \quad \text{in } \overline{\Omega}. \tag{2.1.10}$$

As (u, v) is divergence free according to (2.1.8), we can define the *pseudostream* function $\psi = \psi(x, y)$ for the flow by

$$\psi_x = \sqrt{\varrho} v, \qquad \psi_y = \sqrt{\varrho}(u - c) \qquad \text{in } \Omega. \tag{2.1.11}$$

The level sets of ψ are called *streamlines*. Without stagnation (2.1.10), we have $\psi_y < 0$, which implies that each streamline is given as the graph of a function of x via a simple Implicit Function Theorem argument. The boundary conditions in (2.1.9) show that the air-water interface, bed, and lid are each level sets of ψ. We will take $\psi = 0$ on the upper lid so that $\psi = -p_0$ on $y = -d$, where p_0 is defined by

$$p_0 := \int_{-d}^{\eta(x)} \sqrt{\varrho(x,y)}(u(x,y) - c)\, \mathrm{d}y.$$

It can be shown that p_0 does not depend on x (see, for example, [23]). Bernoulli's theorem states that

$$E := P + \frac{\varrho}{2}((u-c)^2 + v^2) + g\varrho y$$

is constant along streamlines. Evaluating the jump of E on the interface gives

$$\left[\!\left[|\nabla\psi|^2\right]\!\right] + 2g\,[\![\varrho]\!]\,(\eta + d) + \sigma\kappa = Q \quad \text{on } y = \eta(x),$$

where κ is the signed curvature of the air-water interface and $Q := 2\,[\![E + g\varrho d]\!]$.

Recall that in two dimensions, the *vorticity* ω is defined to be

$$\omega := v_x - u_y.$$

If there is no stagnation (2.1.10), there exists a function γ, called the *vorticity strength function*, such that

$$\omega(x,y) = \gamma(\psi(x,y)) \quad \text{for all } (x,y) \in \Omega.$$

The vorticity plays a key role in the wind generation of water waves as we will discuss below. Mathematically, it substantially complicates the analysis.

Finally, we will use the following notational conventions. For any integer $k \geq 0$, $\alpha \in (0,1)$, and an open region $R \subset \mathbb{R}^n$, we define the space $C^{k+\alpha}_{\mathrm{per}}(\overline{R})$ to be the set of $C^{k+\alpha}(\overline{R})$ functions that are 2π-periodic in their first argument.

Our main theorem is an existence result for traveling capillary-gravity water waves in the presence of wind.

Theorem 2.1.4 (Existence of small amplitude wind-driven water waves)**.** *Fix d, ℓ, $c > 0$, and $p_0 < p_1 < 0$. For any vorticity function $\gamma \in C^{0,\alpha}([p_1, 0])$ and $\sigma > 0$ sufficiently large, there exists a C^1 curve*

$$\mathcal{C}'_{\mathrm{loc}} := \{(u(s), v(s), \eta(s), Q(s)) : s \in (-\epsilon, \epsilon)\}$$

of traveling wave solutions to the capillary-gravity water wave problem (2.1.8)–(2.1.10) *such that*

1. *Each $(u, v, \eta, Q) \in \mathcal{C}'_{\mathrm{loc}}$ is of class*

$$(u(s), v(s), \eta(s), Q(s)) \in \left(C^{\alpha}_{\mathrm{per}}(\overline{\Omega}) \cap C^{1+\alpha}_{\mathrm{per}}(\overline{\Omega}(s)\backslash\mathcal{I}(s))\right)^2 \times C^{2+\alpha}_{\mathrm{per}}(\mathbb{R}) \times \mathbb{R} =: \mathscr{S},$$

   where $u(s)$ and $v(s)$ are even and odd in the first coordinate, respectively, $\eta(s)$ is even in x, and $\Omega(s)$ is the domain corresponding to $\eta(s)$;

2. *$(u(0), v(0), \eta(0), Q(0)) = (U_*(y), 0, 0, Q_*)$, where (U_*, Q_*) is laminar solution.*

We prove this theorem using a local bifurcation theoretic strategy that draws on the ideas of Constantin and Strauss [29], who studied rotational periodic gravity water waves in a single fluid. Indeed, following the publication of [29], traveling water waves with vorticity have been an extremely active area of research (see, for example, the surveys in [70, 71]).

Our most direct influence is the work of Bühler, Shatah, and Walsh [72] on the existence of steady gravity waves in the presence of wind. These authors studied exactly the system (2.1.8)–(2.1.10) taking $\sigma = 0$. One of the main objectives of that paper was to construct waves that were dynamically accessible from an initial state where the flow is laminar and the horizontal velocity experiences a jump over the interface. More specifically, this meant that the circulation along each streamline was prescribed in order to ensure that its values in the air and water regions were distinct (see Remark 2.3.3). We also adopt this approach in the present work, though the addition of surface tension necessitate many nontrivial adaptations.

2.1.3 History of the problem

Steady capillary and capillary-gravity waves have been the subject of extensive research. Because we are particularly interested in the role of vorticity, we will restrict our discussion to rotational waves. In this setting, progress is much more recent and begins with the work of Wahlén [73, 74], who proved the existence of small-amplitude periodic capillary and capillary-gravity waves in two-dimensions for a single fluid system. As in [29], this was done for a general vorticity function γ. Contrary to the gravity wave case, Wahlén showed that with surface tension there can be double bifurcation points; this is a rotational analogue of the famous Wilton ripples [75]. Later, Walsh considered two-dimensional periodic capillary-gravity waves with density stratification [76, 77].

Recently, Martin and B-V Matioc proved the existence of steady small-amplitude capillary-gravity water waves with piecewise constant vorticity [78]. While they consider a one-layer model, the analysis has a similar flavor to that in the present work. A-V Matioc and B-V Matioc also constructed weak solutions for steady capillary-gravity water waves in a single fluid [79].

The waves we construct can also be viewed as internal waves moving along the interface between two immiscible fluid layers confined in a channel. Versions of this problem have been investigated by many authors. For instance, Amick–Turner [80] and Sun [81, 82] considered the existence of solitary waves in a channel where the flow is irrotational at infinity. Amick–Turner built their solitary waves as limits of periodic waves with the period tending to infinity. Sun, on the other hand, exploited the fact that the leading-order form of the wave is given by the Benjamin–Ono equation, and then used singular integral operator estimates to control the remainder. The existence of continuously stratified channel flows has also been verified in a number of regimes. Note that these are rotational, since heterogeneity in the density produces vorticity. Specifically, Turner [83] and Kirchgässner [84] investigated small-amplitude continuously stratified waves using a variational scheme and a center manifold reduction method, respectively. A large-amplitude existence theory was also provided by

Bona, Bose, and Turner [85], Lankers and Friesecke [86], and Amick [87]. We remark that, in all of these works, the vorticity vanishes at infinity. Finally, internal waves with surface tension on the interface were recently considered by Nilsson [88]. In that paper, each fluid layer was assumed to be irrotational and constant density. Using spatial dynamics and a center manifold reduction, Nilsson proved the existence of both periodic and solitary wave solutions.

As mentioned above, steady water waves in the presence of wind was studied by Bühler, Shatah, and Walsh in [72]. Our main contribution relative to that work is to account for capillary effects on the air-water interface. It is known that surface tension is important in the formation of wind-driven waves. Indeed, high frequency and small-amplitude capillary-gravity waves are the first to form when wind blows over a quiescent body of water.

One of the most successful explanations for the mechanism behind the wind generation of water waves was given by Miles [1]. His main observation was that vorticity in the air region can create a certain resonance phenomenon that destabilizes the system. Importantly, this so-called *critical layer instability* can occur even when the horizontal velocity is continuous — or nearly continuous — over the interface, and therefore does not require exceedingly strong wind speeds like the Kelvin–Helmholtz model. The mathematical ideas underlying Miles's theory were recently reexamined and rigorously proved by Bühler, Shatah, Walsh, and Zeng [89]. In that work, the authors also allowed surface tension. This is somewhat important as the interface Euler problem itself is ill-posed when there is a jump in the tangential velocity and there is no surface tension (see, for example, [90]). In a forthcoming work, the author intends to study the stability of the family of waves constructed in Theorem 2.1.4. This will serve as a model for wind generation of water waves in the spirit of Miles, but with an initial state that is not purely laminar.

2.1.4 Plan of the article

We now briefly discuss the strategies we use to derive these results. The elliptic theory is proved in Section 2.2. Our approach is based on the work of Luo and Trudinger [58], who gave Schauder estimates for elliptic equations with Wentzell boundary conditions.

In Section 2.3, we construct capillary-gravity water waves where the air region is rotational. Following Bühler, Shatah, and Walsh [72], the first step in this procedure is to reformulate the interface Euler system (2.1.8)–(2.1.10) as a quasilinear elliptic equation on a fixed domain. Due to surface tension, there is now a nonlinear Wentzell condition on the image of the interface in these new coordinates. We construct the non-laminar waves using local bifurcation theory. This entails studying the spectrum of the linearized equation at a laminar flow, and here we make essential use of the elliptic theory developed in Section 2.2. One major difficulty that arises is that this linearized problem is of Sturm–Liouville type, but associated to an indefinite inner product. Consequently, to successfully determine the spectral behavior, we must work in Pontryagin spaces. A similar issue was encountered by Wahlén in [73, 74]. Finally, we apply the Crandall–Rabinowitz local bifurcation theorem to obtain Theorem 2.1.4.

2.2 Elliptic Theory

To simplify subsequent calculations, it is convenient to first change variables. Fix a point $x^0 \in \Gamma$. Then by the assumption on Ω, there is a neighborhood $\mathcal{U}$ of x^0 and a $C^{2,\beta}$ diffeomorphism that maps $\mathcal{U}$ to some ball $B \subset \mathbb{R}^n$ so that Γ maps to $\{x_n = 0\}$, Ω_1 to $B \cap \{x_n > 0\}$, and Ω_2 to $B \cap \{x_n < 0\}$ (see, for example, [58]). Then it suffices to assume that Γ is the hyper-plane $\{x_n = 0\}$, and consequently, Ω_1 and Ω_2 lie inside the upper-half and lower-half planes respectively.

In this case, the co-normal derivative operator simplifies to

$$\partial_N u = -\sum_{j=1}^{n} a^{nj} \partial_{x_j} u,$$

and the Wentzell and transmission condition on Γ becomes

$$Bu = -\sum_{s,t=1}^{n-1} \partial_{x_s} \left(\mathfrak{a}^{st} \partial_{x_t} u \right) + \alpha \left[\!\left[\partial_N u \right]\!\right] + \sum_{s=1}^{n-1} \mathfrak{b}^s \partial_{x_s} u + \mathfrak{c} u.$$

We also denote by ∇' the tangential gradient on Γ in this case.

2.2.1 Classical solutions

First, we prove our theorem on Schauder estimates for solutions in Hölder spaces. This relies on the observation that one can apply $(n-1)$-dimensional elliptic estimates for B on Γ with transmission boundary condition being lower ordered.

Proof of Theorem 2.1.1. Using the above change of variables, we rewrite the condition on Γ

$$-\sum_{s,t=1}^{n-1} \mathfrak{a}^{st}\partial_{x_s}\partial_{x_t}u - \sum_{s,t=1}^{n-1}(\partial_{x_s}\mathfrak{a}^{st})(\partial_{x_t}u) - \alpha\sum_{j=1}^{n} a^{nj}\partial_{x_j}u_1 \\ + \alpha\sum_{j=1}^{n} a^{nj}\partial_{x_j}u_2 + \sum_{s=1}^{n-1}\mathfrak{b}^s\partial_{x_s}u + \mathfrak{c}u = g.$$

We then cover Γ by a finite number of spheres in which the estimate in [91, Theorem 6.2] for B on Γ can be applied. This ensures the existence of a positive constant $C = C(n, \beta, L, B, \mu)$ such that

$$\|u_1\|_{C^{2,\beta}(\Gamma')} \leq C(\|u_1\|_{C^0(\Gamma)} + \|u_2\|_{C^{1,\beta}(\Gamma')} + \|g\|_{C^{0,\beta}(\Gamma)}). \tag{2.2.1}$$

Similarly, we have

$$\|u_2\|_{C^{2,\beta}(\Gamma')} \leq C(\|u_2\|_{C^0(\Gamma)} + \|u_1\|_{C^{1,\beta}(\Gamma')} + \|g\|_{C^{0,\beta}(\Gamma)}). \tag{2.2.2}$$

Next, we use a basic elliptic estimate for the Dirichlet problem in Ω_k' with boundary condition $u_k|_\Gamma$ (see, for example, [91, Theorem 6.6]), to obtain

$$\|u_k\|_{C^{2,\beta}(\Omega_k')} \leq C\left(\|u_k\|_{C^0(\Omega_k)} + \|u_k\|_{C^{2,\beta}(\Gamma')} + \|f\|_{C^{0,\beta}(\Omega_k)}\right). \tag{2.2.3}$$

Moreover, we have the following interpolation

$$\|u_k\|_{C^{1,\beta}(\Gamma')} \leq C_\epsilon\|u\|_{C^0(\Gamma)} + \epsilon\|u_k\|_{C^{2,\beta}(\Gamma')} \tag{2.2.4}$$

for some $\epsilon > 0$. Finally, evaluating (2.2.3) with $k = 1, 2$ and summing, using $[\![u]\!]|_\Gamma = 0$ and the estimates (2.2.1), (2.2.2), (2.2.4) and choosing appropriate ϵ give

$$\|u_1\|_{C^{2,\beta}(\Omega_1')} + \|u_2\|_{C^{2,\beta}(\Omega_2')} \\ \leq C\left(\|u\|_{C^0(\Omega)} + \|f\|_{C^{0,\beta}(\Omega_1)} + \|f\|_{C^{0,\beta}(\Omega_2)} + \|g\|_{C^{0,\beta}(\Gamma)}\right). \qquad \square$$

Remark 2.2.1. *A version of Theorem 2.1.1 was stated in [69, Theorem 2.3*] without proof under the assumption that* $\alpha = +1$ *in the boundary operator* (2.1.3)*. However, according to the above proof, this theorem holds regardless of the sign of* $[\![\partial_N u]\!]$*.*

Next, in preparation for proving the existence and uniqueness result, we first establish a maximum principle. Apushkinskaya and Nazarov state a similar result in [69, Theorem 3.1]. Using our notations, we have the lemma.

Lemma 2.2.2 (Maximum Principle)**.** *Suppose the coefficients $a^{ij}, b^i, c, \mathfrak{a}^{st}, \mathfrak{b}^s, \mathfrak{c}$ exhibit the same regularity as in Theorem 2.1.1 and assume in addition that $c \geq 0$ and $\mathfrak{c} > 0$. Let $\alpha = 1$ and suppose that $u \in C^0(\overline{\Omega}) \cap C^2(\overline{\Omega_1}) \cap C^2(\overline{\Omega_2})$ satisfies*

$$Lu \leq f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } S, \qquad Bu \leq g \quad \text{on } \Gamma.$$

Then we have the estimate

$$\sup_{\Omega} u \leq \sup_{\Gamma} \left| \frac{g}{\mathfrak{c}} \right| + C \sup_{\Omega} \left| \frac{f}{\lambda} \right| \tag{2.2.5}$$

for some positive constant $C = C(\operatorname{diam} \Omega, \lambda, \|\partial_{x_i} a^{ij}\|_{L^\infty}, \|b^i\|_{L^\infty})$, where the L^∞ norms are taken over Ω_1 and Ω_2.

Proof. We will follow very closely the classical arguments when proving this maximum principle in the interior. Rewriting L in non-divergence form gives

$$Lu = -\sum_{i,j=1}^{n} a^{ij} \partial_{x_i} \partial_{x_j} u + \sum_{i=1}^{n} \tilde{b}^i \partial_{x_i} u + cu,$$

where $\tilde{b} := b^i - \partial_{x_i} a^{ij}$. Setting

$$\tau := \frac{\|\tilde{b}^i\|_{L^\infty(\Omega_1)} + \|\tilde{b}^i\|_{L^\infty(\Omega_2)}}{\lambda},$$

choosing $\sigma \geq 1$ large enough so that $\sigma^2 - \tau\sigma \geq 1$, and without loss of generality, because of the boundedness of Ω, assuming Ω lies between $\{x_1 = 0\}$ and $\{x_1 = d\}$, let

$$v := \sup_{\partial\Omega_k} u^+ + \left(e^{\sigma d} - e^{\sigma x_1}\right) \sup_{\Omega_k} \frac{f^+}{\lambda},$$

where $u^+ := \max(u, 0)$ and $d := \operatorname{diam} \Omega$. Then

$$Le^{\sigma x_1} = (-a^{11}\sigma^2 + \tilde{b}^1\sigma + c)e^{\sigma x_1} \leq -\lambda(\sigma^2 - \tau\sigma)e^{\sigma x_1} + c\, e^{\sigma x_1} \leq -\lambda + c\, e^{\sigma x_1}.$$

Then since $c \geq 0$,

$$\begin{aligned} Lv &\geq c \sup_{\partial\Omega_k} u^+ + c\, e^{\sigma d} \sup_{\Omega_k} \frac{f^+}{\lambda} - (-\lambda + c\, e^{\sigma x_1}) \sup_{\Omega_k} \frac{f^+}{\lambda} \\ &\geq c(e^{\sigma d} - e^{\sigma x_1}) \sup_{\Omega_k} \frac{f^+}{\lambda} + \sup_{\Omega_k} f^+ \geq \sup_{\Omega_k} f^+, \end{aligned}$$

so we have $L(u - v) \leq 0$ in Ω_k. On the other hand, by construction $u - v \leq 0$ on $\partial\Omega_k$. Therefore, the maximum principle implies $u \leq v$ in Ω_k, and that there exists a positive constant $C = C(d, \lambda, \|\partial_{x_i} a^{ij}\|_{L^\infty}, \|b^i\|_{L^\infty})$ such that

$$\sup_{\Omega_k} u \leq \sup_{\partial\Omega_k} u^+ + C \sup_{\Omega_k} \frac{f^+}{\lambda} \quad \text{for } k = 1, 2.$$

Next, since $u|_S = 0$, if $u|_\Gamma \leq 0$ for all $x \in \Gamma$, then

$$\sup_{\partial\Omega_k} u^+ = 0 \leq \sup_\Gamma \left| \frac{g}{\mathfrak{c}} \right|.$$

If we suppose that u attains its local maximum at some point $x_0 \in \Gamma$ and $u(x_0) > 0$, then by the positive definiteness of the matrix $(\mathfrak{a}^{st})$,

$$\nabla' u(x_0) = 0 \quad \text{and} \quad \sum_{s,t=1}^{n-1} (\mathfrak{a}^{st} \partial_{x_s} \partial_{x_t} u)(x_0) \leq 0.$$

By the positive-definiteness of the matrix (a^{ij}), we have

$$\begin{aligned} \partial_N u_1(x_0) &= -(a^{nn} \partial_{x_n} u_1)(x_0) \geq 0 \\ \partial_N u_2(x_0) &= -(a^{nn} \partial_{x_n} u_2)(x_0) \leq 0, \end{aligned}$$

and hence $[\![\partial_N u(x_0)]\!] \geq 0$. Then the condition on Γ gives

$$(\mathfrak{c} u)(x_0) \leq g(x_0) + \sum_{s,t=1}^{n-1} (\mathfrak{a}^{st} \partial_{x_s} \partial_{x_t} u)(x_0) - [\![\partial_N u(x_0)]\!] \leq g(x_0),$$

so since $\mathfrak{c} > 0$ for all $x \in \Gamma$, we obtain

$$\sup_\Gamma u = u(x_0) \leq \frac{g(x_0)}{\mathfrak{c}(x_0)} \leq \sup_\Gamma \frac{g}{\mathfrak{c}}.$$

Therefore,

$$\sup_{\partial\Omega_k} u^+ = \sup_{\Gamma} u \le \sup_{\Gamma} \left| \frac{g}{\mathfrak{c}} \right|,$$

and hence we obtain the desired estimate (2.2.5) by using

$$\sup_{\Omega} u = \max(\sup_{\Omega_1} u, \sup_{\Omega_2} u). \qquad \square$$

Remark 2.2.3. *Note that if Ω is periodic in one variable, the lemma still holds by modifying the proof to assume that Ω lies between two hyperplanes parallel to the periodic direction.*

Using the notation of a Hölder seminorm, we have the following simple lemma whose proof will be omitted:

Lemma 2.2.4. *Suppose $u \in C^0(\overline{\Omega}) \cap C^{0,\beta}(\overline{\Omega_1}) \cap C^{0,\beta}(\overline{\Omega_2})$. Then $[u]_{0,\beta;\overline{\Omega}}$ is finite, and*

$$\|u\|_{C^{0,\beta}(\Omega)} \le C\big(\|u_1\|_{C^{0,\beta}(\Omega_1)} + \|u_2\|_{C^{0,\beta}(\Omega_2)}\big).$$

Now we can derive the existence and uniqueness of solution in Hölder spaces.

Proof of Theorem 2.1.2. Consider the family of problems indexed by $\theta \in [0,1]$:

$$\begin{cases} Lu &= & f & \text{in } \Omega, \\ u &= & 0 & \text{on } S, \\ [\![u]\!] &= & 0 & \text{on } \Gamma, \\ B_\theta u &= & g & \text{on } \Gamma, \end{cases} \tag{2.2.6}$$

where $B_\theta u = \theta B u + (1-\theta) B' u$ and

$$B'u := -\sum_{s=1}^{n-1} \partial_{x_s}^2 u + u.$$

We note that $B_1 = B$, $B_0 = B'$, and that

$$B_\theta u = -\sum_{s,t=1}^{n-1} \tilde{\mathfrak{a}}^{st} \partial_{x_s} \partial_{x_t} u + \sum_{s=1}^{n-1} \tilde{\mathfrak{b}}^s \, \partial_{x_s} u + \theta \, [\![\partial_N u]\!] + \tilde{\mathfrak{c}} u,$$

where all of the coefficients $\tilde{\mathfrak{a}}^{st}$, $\tilde{\mathfrak{b}}^s$, $\tilde{\mathfrak{c}}$ of B_θ are bounded in $C^{0,\beta}(\Gamma)$ independently of θ with $\tilde{\mathfrak{c}} > 0$

and

$$\min(1,\mu)|\xi|^2 =: \mu_\theta|\xi|^2 \leq \tilde{\mathfrak{a}}^{st}\xi_s\xi_t \quad \text{for all } x \in \Gamma,\ \xi \in \mathbb{R}^{n-1}.$$

Consider any solution $u \in C^0(\overline{\Omega}) \cap C^{2,\beta}(\overline{\Omega_1}) \cap C^{2,\beta}(\overline{\Omega_2})$ of (2.2.6). Then by estimates (2.1.6) and (2.2.5), the following inequality holds

$$\|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)} \leq C\left(\|f\|_{C^{0,\beta}(\Omega_1)} + \|f\|_{C^{0,\beta}(\Omega_2)} + \|g\|_{C^{0,\beta}(\Gamma)}\right), \tag{2.2.7}$$

where the constant C is independent of θ. Note that the above estimate is valid for Ω_k with $k = 1, 2$ since $S \cap \Gamma = \emptyset$.

Next, recalling the definition of X as in (2.1.7), let $Y = Y_1 \times Y_2$ where

$$Y_1 = C^{0,\beta}(\overline{\Omega_1}) \cap C^{0,\beta}(\overline{\Omega_2}), \quad \text{and} \quad Y_2 = C^{0,\beta}(\Gamma).$$

Then Y is a Banach space with respect to the norm

$$\|(f,g)\|_Y := \|f\|_{Y_1} + \|g\|_{Y_2} := \|f\|_{C^{0,\beta}(\Omega_1)} + \|f\|_{C^{0,\beta}(\Omega_2)} + \|g\|_{C^{0,\beta}(\Gamma)}.$$

Thus, problem (2.2.6) can be written as

$$\mathfrak{L}_\theta u := (Lu, B_\theta u) = (f, g),$$

where $\mathfrak{L}_\theta : X \to Y$, so the solvability of the problem (2.2.6) for arbitrary $f \in C^{0,\beta}(\overline{\Omega_1}) \cap C^{0,\beta}(\overline{\Omega_2})$ and $g \in C^{0,\beta}(\Gamma)$ is then equivalent to the invertibility of the mapping $\mathfrak{L}_\theta$. We note that $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are bounded operators.

On the other hand, by Lemma 2.2.4, Lemma 2.2.2, and estimate (2.2.7), we have

$$\begin{aligned}
\|u\|_{C^{0,\beta}(\Omega)} &\leq C\left(\|u\|_{C^{0,\beta}(\Omega_1)} + \|u\|_{C^{0,\beta}(\Omega_2)}\right) \\
&\leq C_\epsilon\|u\|_{C^0(\Omega)} + \epsilon\left(\|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)}\right) \\
&\leq C\left(\|g\|_{C^{0,\beta}(\Gamma)} + \|f\|_{C^{0,\beta}(\Omega_1)} + \|f\|_{C^{0,\beta}(\Omega_2)}\right)
\end{aligned}$$

for some $\epsilon > 0$, and hence

$$\begin{aligned}\|u\|_X &= \|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)} + \|u\|_{C^{0,\beta}(\Omega)} \\ &\leq C\left(\|f\|_{C^{0,\beta}(\Omega_1)} + \|f\|_{C^{0,\beta}(\Omega_2)} + \|g\|_{C^{0,\beta}(\Gamma)}\right) \\ &= C\left(\|f\|_{Y_1} + \|g\|_{Y_2}\right) = C\|\mathfrak{L}_\theta u\|_Y,\end{aligned}$$

where the constant C does not depend on θ. Thus, by the method of continuity (see, for example, [91, Theorem 5.2]), the surjectivity of $\mathfrak{L}_1$, which we are investigating, is equivalent to that of $\mathfrak{L}_0$ which is the problem

$$\begin{cases} Lu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } S, \\ [\![u]\!] &= 0 \quad \text{on } \Gamma, \\ B'u &= g \quad \text{on } \Gamma. \end{cases} \tag{2.2.8}$$

Finally, we recall that

$$B'u = -\sum_{s=1}^{n-1} \partial_{x_s}^2 u + u$$

is invertible on Γ. If $\varphi \in C^{2,\beta}(\Gamma)$ is the unique solution to $B'\varphi = g$ on Γ for a given $g \in C^{0,\beta}(\Gamma)$, then by [91, Lemma 6.38] we can make an extension to have $\varphi \in C^{2,\beta}(\overline{\Omega_1}) \cap C^{2,\beta}(\overline{\Omega_2})$. Now we have a Dirichlet problem

$$Lu_k = f \quad \text{in } \Omega_k, \qquad u_k = 0 \quad \text{on } S, \qquad u_k = \varphi \quad \text{on } \Gamma,$$

which has a unique solution $u_k \in C^{2,\beta}(\overline{\Omega_k})$ by [91, Theorem 6.14]. Therefore, by Lemma 2.2.4, we conclude that there is a unique solution in $C^{0,\beta}(\overline{\Omega}) \cap C^{2,\beta}(\overline{\Omega_1}) \cap C^{2,\beta}(\overline{\Omega_2})$ to the system (2.1.1). $\square$

Remark 2.2.5. *As a consequence of Theorem 2.1.2, we see that $\mathfrak{L}_1 = (L, B)$ is a Fredholm operator of index 0 despite the sign of the transmission term. Indeed, for $\theta \in [0, 1]$, consider the following linear operator*

$$\widetilde{\mathfrak{L}}_\theta := \left(Lu, (1-\theta)B + \theta\widetilde{B}\right),$$

where $\widetilde{\mathfrak{L}}_\theta : X \to Y$ *and*

$$\widetilde{B}u = -\sum_{s,t=1}^{n-1} \partial_{x_s}\left(\mathfrak{a}^{st}\partial_{x_t}u\right) - [\![\partial_N u]\!] + \sum_{s=1}^{n-1} \mathfrak{b}^s \partial_{x_s} u + \mathfrak{c}u$$

with coefficients $\mathfrak{a}^{st}$*,* $\mathfrak{b}^s$*, and* $\mathfrak{c}$ *satisfying the hypotheses of Theorem 2.1.2. Note that the sign of the transmission term is unfavorable. It is clear that the map* $\theta \mapsto \widetilde{\mathfrak{L}}_\theta \in \mathcal{L}(X,Y)$ *is continuous. Then Schauder estimate from Theorem 2.1.1 and Remark 2.2.1 give*

$$\begin{aligned}
&\|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)} + \|u\|_{C^{0,\beta}(\Omega)} \\
&\le C\|\widetilde{\mathfrak{L}}_\theta u\|_Y + C_\epsilon \|u\|_{C^0(\Omega)} + \epsilon\left(\|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)}\right)
\end{aligned}$$

for some small $\epsilon > 0$*, so*

$$(1-\epsilon)\left(\|u_1\|_{C^{2,\beta}(\Omega_1)} + \|u_2\|_{C^{2,\beta}(\Omega_2)}\right) + \|u\|_{C^{0,\beta}(\Omega)} \le C_\epsilon \|u\|_{C^0(\Omega)} + C\|\widetilde{\mathfrak{L}}_\theta u\|_Y.$$

Choosing $\epsilon > 0$ *small, we have*

$$\|u\|_X \le C\left(\|u\|_{C^0(\Omega)} + \|\widetilde{\mathfrak{L}}_\theta u\|_Y\right)$$

for some constant $C > 0$ *independent of* θ*, which implies that* $\widetilde{\mathfrak{L}}_\theta$ *has finite dimensional null space and closed range. Thus,* $\widetilde{\mathfrak{L}}_\theta$ *is semi-Fredholm. If* $\theta < \frac{1}{2}$*, the map* $\widetilde{\mathfrak{L}}_\theta$ *is invertible by Theorem 2.1.2 and hence has index* 0*. By the continuity of the index, it also holds for* $\theta \ge \frac{1}{2}$*, which means that we have Fredholm index* 0 *regardless of the sign of the transmission term.*

2.2.2 Fredholm property

In light of Remark 2.2.5, it suffices to take $\alpha = +1$. To simplify our notation, we write $\mathfrak{L}$ for $\mathfrak{L}_1$, which is the problem we are considering. If L and B do not satisfy the conditions $c \ge 0$ and $\mathfrak{c} > 0$, it is still possible to assert a Fredholm alternative, which we formulate as in Theorem 2.1.3.

Proof of Theorem 2.1.3. For all $\sigma, \tau \in \mathbb{R}$, notice that for $u \in X$, $(f,g) \in Y$,

$$\mathfrak{L}u = (f,g)$$

is equivalent to

$$\mathfrak{L}_{\sigma,\tau}u = (f + \sigma u, g + \tau u),$$

where $\mathfrak{L}_{\sigma,\tau}u := \big((L+\sigma)u, (B+\tau)u\big)$. From Theorem 2.1.2, the mapping $\mathfrak{L}_{\sigma,\tau}u : X \to Y$ is invertible for σ and τ sufficiently large. Now, applying $\mathfrak{L}_{\sigma,\tau}^{-1}$ to both sides, we obtain

$$u = \mathfrak{L}_{\sigma,\tau}^{-1}(f + \sigma u, g + \tau u|_\Gamma)$$

which can be written as

$$u - \mathfrak{L}_{\sigma,\tau}^{-1}(\sigma u, \tau u|_\Gamma) = \mathfrak{L}_{\sigma,\tau}^{-1}(f, g).$$

Letting $\mathcal{K}u : u \in X \subset Y_1 \mapsto \mathfrak{L}_{\sigma,\tau}^{-1}(\sigma u, \tau u|_\Gamma) \in Y_1$, and $h := \mathfrak{L}_{\sigma,\tau}^{-1}(f, g)$, the equation becomes

$$(I - \mathcal{K})u = h. \tag{2.2.9}$$

We claim that $\mathcal{K}$ is a compact operator. Let $\{(f_m, g_m)\} \subset Y$ be bounded, and define $u_m := \mathcal{K}(f_m, g_m) \in Y_1$. We want to show that $\{u_m\}$ has a convergent subsequence in Y_1. By definition of u_m and $\mathcal{K}$, we have

$$\begin{cases} Lu_m + \sigma u_m = f_m & \text{in } \Omega \\ Bu_m + \tau u_m = g_m & \text{on } \Gamma, \end{cases}$$

where $u_m \in X$, $f_m \in Y_1$, $g_m \in Y_2$. Thus, by Theorem 2.1.1, there exists a positive constant $C = C(n, \beta, L, B, \lambda, \mu)$ such that

$$\begin{aligned} \|u_m\|_{C^{2,\beta}(\Omega_1)} + \|u_m\|_{C^{2,\beta}(\Omega_2)} \le C\big(&\|u_m\|_{C^0(\Omega)} \\ &+\|f_m\|_{C^{0,\beta}(\Omega_1)} + \|f_m\|_{C^{0,\beta}(\Omega_2)} + \|g_m\|_{C^{0,\beta}(\Gamma)}\big). \end{aligned} \tag{2.2.10}$$

Note that the estimate holds for Ω_k since $S \cap \Gamma = \emptyset$. Since $C^{0,\beta}(\overline{\Omega}) \subset\subset C^0(\overline{\Omega})$ and $C^{2,\beta}(\overline{\Omega_k}) \subset\subset C^{0,\beta}(\overline{\Omega_k})$, $k = 1, 2$, using estimates as in the proof of Theorem 2.1.2, we find that

$$\|u_m\|_{C^0(\Omega)} \le C\|u_m\|_{C^{0,\beta}(\Omega)} \le C\big(\|f_m\|_{C^{0,\beta}(\Omega_1)} + \|f_m\|_{C^{0,\beta}(\Omega_2)} + \|g_m\|_{C^{0,\beta}(\Gamma)}\big).$$

Then the inequality (2.2.10) becomes

$$\|u_m\|_{C^{2,\beta}(\Omega_1)} + \|u_m\|_{C^{2,\beta}(\Omega_2)} \le C\big(\|f_m\|_{C^{0,\beta}(\Omega_1)} + \|f_m\|_{C^{0,\beta}(\Omega_2)} + \|g_m\|_{C^{0,\beta}(\Gamma)}\big),$$

or we can write this to be

$$\begin{aligned}\|u_m\|_{C^{2,\beta}(\Omega_1)} + \|u_m\|_{C^{2,\beta}(\Omega_2)} + \|u_m\|_{C^{0,\beta}(\Omega)} \\ \leq C\big(\|f_m\|_{C^{0,\beta}(\Omega_1)} + \|f_m\|_{C^{0,\beta}(\Omega_2)} + \|g_m\|_{C^{0,\beta}(\Gamma)}\big),\end{aligned}$$

which is equivalent to

$$\|u_m\|_X \leq C\|(f_m, g_m)\|_Y,$$

so $\|u_m\|_X$ is bounded in X. Since $X \subset\subset Y_1$, we conclude that $\{u_m\}$ contains a subsequence $\{u_{m_k}\}$ such that $u_{m_k} \to u$ in Y_1, which proves the claim that $\mathcal{K}$ is a compact operator.
Applying the Fredholm Alternative, equation (2.2.9) always has a solution $u \in X$ provided the homogeneous equation $(I - \mathcal{K})u = 0$ has only the trivial solution $u = 0$. When this condition is not satisfied, the kernel of $I - \mathcal{K}$ is a finite dimensional subspace of Y_1. Since the solutions of (2.2.9) are in one-to-one correspondence to the solutions of (2.1.1), we therefore can conclude the alternative stated in the theorem. □

Finally, the last result in this subsection gives Hölder continuity for a classical solution provided sufficient smoothness of the data and coefficients.

Proposition 2.2.6. *Suppose the coefficients* $a^{ij}, b^i, c, \mathfrak{a}^{st}, \mathfrak{b}^s, \mathfrak{c}$ *exhibit the same regularity as in Theorem 2.1.1. If* $u \in C^0(\overline{\Omega}) \cap C^2(\overline{\Omega_1}) \cap C^2(\overline{\Omega_2})$ *is a solution to equation* (2.1.1) *with* $\alpha = \pm 1$ *for* $f \in C^{0,\beta}(\overline{\Omega_1}) \cap C^{0,\beta}(\overline{\Omega_2})$ *and* $g \in C^{0,\beta}(\Gamma)$*, then* $u \in C^{0,\beta}(\overline{\Omega}) \cap C^{2,\beta}(\overline{\Omega_1}) \cap C^{2,\beta}(\overline{\Omega_2})$.

Proof. By the hypothesis, we have $u_k \in C^2(\Gamma)$, and hence, $\partial_N u_k \in C^1(\Gamma) \subset\subset C^{0,\beta}(\Gamma)$ for $k = 1, 2$. Thus, the boundary condition $Bu = g$ can be re-expressed as

$$-\sum_{s,t=1}^{n-1} \partial_{x_s}\left(\mathfrak{a}^{st}\partial_{x_t} u\right) + \sum_{s=1}^{n-1} \mathfrak{b}^s \partial_{x_s} u + \mathfrak{c}u = h \quad \text{on } \Gamma,$$

where $h := g - \alpha\, [\![\partial_N u]\!] \in C^{0,\beta}(\Gamma)$. By standard elliptic regularity theory, $u|_\Gamma \in C^{2,\beta}(\Gamma)$. Now the Dirichlet problem

$$Lu_k = f \quad \text{in } \Omega_k, \qquad u_k = 0 \quad \text{on } S, \qquad u_k = u|_\Gamma \quad \text{on } \Gamma$$

has a unique solution $u \in C^{2,\beta}(\overline{\Omega_1}) \cap C^{2,\beta}(\overline{\Omega_2})$. Using the fact that $C^{2,\beta}(\overline{\Omega_k}) \subset\subset C^{0,\beta}(\overline{\Omega_k})$ and Lemma 2.2.4, we conclude that $u \in C^{2,\beta}(\overline{\Omega_1}) \cap C^{2,\beta}(\overline{\Omega_2}) \cap C^{0,\beta}(\Omega)$. □

Remark 2.2.7. *If we change the boundary term B to*

$$\widehat{B}u := \sum_{s,t=1}^{n-1} \partial_{x_s}\left(\mathfrak{a}^{st}\partial_{x_t}u\right) + \alpha\,[\![\partial_N u]\!] + \sum_{s=1}^{n-1} \mathfrak{b}^s \partial_{x_s} u + \mathfrak{c}u,$$

where the signs of the second-order term is switched, we obtain the same results as in Theorems 2.1.1, 2.1.3, and Proposition 2.2.6. For Lemma 2.2.2 and Theorem 2.1.2 to be valid, we have to assume in addition that $\alpha = -1$ and $\mathfrak{c} < 0$, which means the signs of the second-order term and zeroth-order term must be opposite.

2.3 Steady capillary-gravity waves in the presence of wind

In this section, we will apply the results found above to investigate the existence of steady wind-driven water waves. There exists a well-known change of variables due to Dubreil-Jacotin that maps Ω to a strip (see [26]). We change variables $(x, y) \in \Omega \mapsto (x, -\psi) =: (q, p) \in D$. We recall that ψ is the (relative) pseudostream function for the flow defined by (2.1.11), along with the boundary conditions $\psi = 0$ on the upper lid, $\psi = -p_0$ at the bed, and $\psi \in C^{0,\alpha}(\overline{\Omega}) \cap C^{2,\alpha}(\overline{\Omega_1}) \cap C^{2,\alpha}(\overline{\Omega_2})$ for a fixed $\alpha \in (0, 1)$. Thus, the problem is now posed in a union of rectangles $D = D_1 \cup D_2 \subset \mathbb{R}^2$, where the air region is mapped to

$$D_1 := \{(q, p) \in D : 0 < q < 2\pi, p_1 < p < 0\},$$

and the water region is mapped to

$$D_2 := \{(q, p) \in D : 0 < q < 2\pi, p_0 < p < p_1\}.$$

With that in mind, we have definitions for the lid, the free surface, and the ocean bed respectively as follows

$$T := \{p = 0\}, \qquad I := \{p = p_1\}, \qquad B := \{p = p_0\}.$$

Under this change of coordinates, the Euler problem (2.1.8)−(2.1.10) becomes the following *height equation*

$$
\begin{cases}
(1+h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_ph_qh_{pq} = -\gamma(-p)h_p^3 & \text{in } D_1, \\
(1+h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_ph_qh_{pq} = 0 & \text{in } D_2, \\
\left[\!\left[\dfrac{1+h_q^2}{h_p^2} \right]\!\right] + 2g\,[\![\rho]\!]\,h - Q + \sigma \dfrac{h_{qq}}{(1+h_q^2)^{3/2}} = 0 & \text{on } p = p_1, \\
[\![h]\!] = 0 & \text{on } p = p_1, \\
h = 0 & \text{on } p = p_0, \\
h = \ell + d(h) & \text{on } p = 0,
\end{cases}
\tag{2.3.1}
$$

where $h(q,p)$ is the height above the bed of the point (x,y), where $x=q$ and (x,y) lies on $\{-\psi = p\}$, and the depth operator d is defined to be

$$
d(h) := \frac{1}{2\pi}\int_{-\pi}^{\pi} h(q,p_1)\,\mathrm{d}q.
$$

Note that ρ in the above equation is for (q,p)-coordinates after the transformation. The equivalence of (2.3.1) to the original system (2.1.8)–(2.1.10) can be proved following [92, Lemma A.2].

Our objective is to find solutions $(h,Q) \in \mathscr{S}'$, where

$$
\mathscr{S}' := \left(C^{2,\alpha}_{\text{per}}\left(\overline{D_1}\right) \cap C^{2,\alpha}_{\text{per}}\left(\overline{D_2}\right) \cap C^{0,\alpha}_{\text{per}}\left(\overline{D}\right)\right) \times \mathbb{R}
$$

and $h_p > 0$ in $\overline{D}$ because of no stagnation condition (2.1.10). Recall that the space $C^{k,\alpha}_{\text{per}}(\overline{R})$ is the set of $C^{k,\alpha}(\overline{R})$ functions that are 2π-periodic and even in their first coordinate. The presence of surface tension σ is manifested as the nonlinear second-order term in the boundary condition.

We will prove the following theorem stated in the Dubreil-Jacotin variables, which implies Theorem 2.1.4.

Theorem 2.3.1 (existence)**.** *Let $p_1 < 0$, $\ell > 0$, and atmospheric vorticity function $\gamma \in C^{0,\alpha}((p_1,0))$ be given. Then there exists $\sigma_0 \geq 0$ such that for each $\sigma > \sigma_0$, there is a continuous curve $\mathcal{C}_{\text{loc}} \subset \mathscr{S}'$ of solution to* (2.3.1) *with the following properties:*

(i) $\mathcal{C}_{\text{loc}} := \{(h(\lambda), Q(\lambda)) : \lambda \in (\lambda^* + \epsilon, \lambda^* - \epsilon)\}$,
where $\lambda \in (\lambda^* + \epsilon, \lambda^* - \epsilon) \mapsto (h(\lambda), Q(\lambda)) \in \mathscr{S}'$ *is* C^1.

(ii) $(h(\lambda^*), Q(\lambda^*)) = (H(\lambda^*), Q(\lambda^*))$ *is a laminar solution.*

(iii) $h(\lambda)$ *is non-laminar for* $\lambda \neq \lambda^*$.

Remark 2.3.2. *In fact, there is a necessary and sufficient condition that we call* local bifurcation condition (LBC)*, which will be given explicit in Lemma 2.3.9. In particular,* (LBC) *always holds for σ sufficiently large. When σ is small, a local bifurcation argument can still be carried out, but the eigenvalue of the linearized problem may not be simple. In this case, a more sophisticated analysis is required (see, for example, [74, 73, 76]).*

2.3.1 Laminar solutions

We first consider *laminar flows* which are solutions of the height equation (2.3.1) that are independent of q. Physically, this entails a wave where all of the streamlines are parallel to the bed. These will serve as the trivial solution curve when we apply the Crandall-Rabinowitz theorem to obtain Theorem 2.3.1.

Let us define Γ_{rel} by

$$\partial_p(\Gamma_{\mathrm{rel}}(p)^2) = 2\gamma(-p), \qquad \ell = \int_{p_1}^{0} \frac{\mathrm{d}p}{\Gamma_{\mathrm{rel}}(p)}. \tag{2.3.2}$$

Remark 2.3.3. Γ_{rel} *is called the* (pseudo) relative circulation *and is given by*

$$\Gamma_{\mathrm{rel}}(p) = \frac{1}{2\pi} \int_{\{\psi=-p\}} |\nabla\psi| \, \mathrm{d}\mathcal{H}^1,$$

where $\mathcal{H}^1$ denotes one-dimensional Hausdorff measure. Note that circulation around a closed loop is conserved for the time-dependent problem by Kelvin's circulation law. For periodic domains, this includes the circulation along the streamlines $\{\psi = -p\}$. If the waves we construct are to be viewed as generated dynamically by the wind, the circulation along each streamline must agree with the initial configuration.

For laminar flows, since h does not depend on q, we can write $h = H(p)$, where H satisfies the following ODE:

$$\begin{cases} H_{pp} = -\gamma(-p) H_p^3 & \text{in } p_1 < p < 0, \\ H_{pp} = 0 & \text{in } p_0 < p < p_1, \\ \llbracket H_p^{-2} \rrbracket + 2g \llbracket \rho \rrbracket H - Q = 0 & \text{on } p = p_1, \\ H = 0 & \text{on } p = p_0, \\ H = \ell + d(H) & \text{on } p = 0. \end{cases} \tag{2.3.3}$$

Note that $d(H) = H(p_1)$. The above equation can be solved explicitly, but we still need some compatibility conditions to ensure continuity across the interface.

Lemma 2.3.4 (laminar flow)**.** *If the compatibility condition (2.3.2) is satisfied, then there exists a one-parameter family of solutions* $\{(H(\cdot;\lambda), Q(\lambda)) : \lambda > 0\}$ *to the laminar flow equation (2.3.3) with* $H_p > 0$*. Each member of the family has the explicit form*

$$H(p;\lambda) = \begin{cases} \displaystyle\int_{p_1}^{p} \frac{\mathrm{d}s}{\Gamma_{\mathrm{rel}}(s)} + \frac{p_1 - p_0}{\lambda}, & p_1 < p < 0, \\ \displaystyle\frac{p - p_0}{\lambda}, & p_0 < p < p_1, \end{cases} \tag{2.3.4}$$

and

$$Q(\lambda) = \frac{2g\,[\![\rho]\!]\,(p_1 - p_0)}{\lambda} + \Gamma_{\mathrm{rel}}(p_1)^2 - \lambda^2. \tag{2.3.5}$$

Moreover, the depth of the fluid at parameter value λ *is*

$$d(H(\cdot;\lambda)) = \frac{p_1 - p_0}{\lambda}. \tag{2.3.6}$$

Since the laminar flow is independent of the surface tension σ, the proof of Lemma 2.3.4 can be obtained by similar arguments as in [72, Lemma 4.2], which we will omit. Note that differentiating (2.3.5) with respect to λ gives

$$Q'(\lambda) = -\frac{2g\,[\![\rho]\!]\,(p_1 - p_0)}{\lambda^2} - 2\lambda$$

and

$$Q''(\lambda) = \frac{4g\,[\![\rho]\!]\,(p_1 - p_0)}{\lambda^3} - 2 < 0,$$

so $\lambda \mapsto Q(\lambda)$ is concave and has a unique maximum at λ_0 satisfying

$$\lambda_0^3 = -g\,[\![\rho]\!]\,(p_1 - p_0). \tag{2.3.7}$$

2.3.2 Linearized problem

Next, let us consider the linearization of the height equation (2.3.1) at one of the laminar solutions $(H(\cdot;\lambda), Q(\lambda))$ constructed in Lemma 2.3.4:

$$
\begin{cases}
(a^3 m_p)_p + (a m_q)_q = 0 & \text{in } D_1 \cup D_2, \\
-2\,[\![a^3 m_p]\!] + 2g\,[\![\rho]\!]\,m + \sigma m_{qq} = 0 & \text{on } p = p_1, \\
m = 0 & \text{on } p = p_0, \\
m - d(m) = 0 & \text{on } p = 0,
\end{cases}
\tag{2.3.8}
$$

where

$$
a := a(p;\lambda) = H_p(p;\lambda)^{-1} = \begin{cases} \Gamma_{\text{rel}}(p), & p_1 < p < 0, \\ \lambda, & p_0 < p < p_1. \end{cases}
$$

Since we seek solutions that are 2π-periodic and even in q, we first consider m of the form $m(q,p) = M(p)\cos(nq)$, for some $n \geq 0$. If $n = 0$, m does not depend on q and the linearized problem (2.3.8) becomes

$$
\begin{cases}
(a^3 M_p)_p = 0 & \text{in } p_1 < p < 0, \\
M_{pp} = 0 & \text{in } p_0 < p < p_1, \\
-\,[\![a^3 M_p]\!] + g\,[\![\rho]\!]\,M = 0 & \text{on } p = p_1, \\
M = 0 & \text{on } p = p_0, \\
M - d(M) = 0 & \text{on } p = 0.
\end{cases}
$$

This equation can be solved explicitly. Using the boundary condition at $p = p_0$ and the continuity of M across the interface, we find that in the water region

$$
M^{(2)}(p) = \frac{p - p_0}{p_1 - p_0} M(p_1), \quad \text{in } p_0 < p < p_1.
$$

For the air region, we first observe that

$$
M(0) = d(M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} M(p_1)\, \mathrm{d}q = M(p_1),
$$

and hence, M_p must vanish at least once inside $(p_1, 0)$ by Rolle's Theorem. We also have, from the above ODE, that $a^3 M_p$ is constant, so we conclude that $M_p \equiv 0$ in $(p_1, 0)$. Finally, the jump condition gives

$$
-\frac{\lambda^3}{p_1 - p_0} M(p_1) = g\,[\![\rho]\!]\,M(p_1),
$$

which implies that there can be a zero-mode solution if and only if $\lambda = \lambda_0$, where λ_0 is defined according to (2.3.7).

On the other hand, if $n > 0$, the linearized problem (2.3.8) becomes

$$\begin{cases} -(a^3 M_p)_p = -n^2 a M & \text{in } (p_0, p_1) \cup (p_1, 0), \\ 2\,[\![a^3 M_p]\!] - 2g\,[\![\rho]\!]\, M = -n^2 \sigma M & \text{on } p = p_1, \\ M = 0 & \text{on } p = p_0, \\ M = 0 & \text{on } p = 0. \end{cases} \tag{2.3.9}$$

To investigate the ODE (2.3.9), we consider the following general eigenvalue problem

$$\begin{cases} -\dfrac{1}{a}(a^3 u')' = \mu u & \text{in } (p_0, p_1) \cup (p_1, 0), \\ 2\,[\![a^3 u']\!] - 2g\,[\![\rho]\!]\, u = \mu \sigma u & \text{on } p = p_1, \\ u = 0 & \text{on } p = p_0, \\ u = 0 & \text{on } p = 0. \end{cases} \tag{P_μ}$$

In particular, we are interested in the case $\mu = -n^2$.

The eigenvalue problem (P_μ) closely resembles a Sturm–Liouville equation, but the eigenvalue occurs both in the interior and boundary conditions. Moreover, the associated inner product defining the relation between eigenfunctions is indefinite. For that reason, it is natural to reformulate it in a Pontryagin space. Here we follow the general approach of Wahlén [73, 74] and Walsh [76].

With that in mind, we introduce the complex Pontryagin space (see [93, 94])

$$\mathbb{H} := \left\{ \tilde{u} = (u, b) \in L^2([p_0, 0]) \times \mathbb{C} \right\}$$

with the indefinite inner product

$$[\tilde{u}_1, \tilde{u}_2] := \langle a u_1, u_2 \rangle_{L^2} - \frac{1}{2\sigma} b_1 \overline{b_2}.$$

We understand that the L^2-inner product is taken over $(p_0, 0)$. On $\mathbb{H}$, there is also an associated Hilbert space inner product, given by $\langle \tilde{u}, \tilde{v} \rangle_{\mathbb{H}} = [J\tilde{u}, \tilde{v}]$, where

$$J = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 2.3.5. $\mathbb{H}$ *is* π_1*-space, that is* $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$*, where*

$$\mathbb{H}_+ \subset \{x \in \mathbb{H} : [x,x] > 0, \ \textit{or } x = 0\},$$
$$\mathbb{H}_- \subset \{x \in \mathbb{H} : [x,x] < 0, \ \textit{or } x = 0\}$$

are complete subspaces with $\dim \mathbb{H}_+ = 1$ *or* $\dim \mathbb{H}_- = 1$.

We omit the proof of this proposition, as it is elementary. In fact, we have explicitly that $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$, where

$$\mathbb{H}_+ := L^2((p_0,0)) \times \{0\},$$
$$\mathbb{H}_- := \left\{0 \in L^2((p_0,0)\right\} \times \mathbb{C}.$$

Next, define the linear operator $K : D(K) \subset \mathbb{H} \to \mathbb{H}$ by

$$K\tilde{u} := \left(-\frac{1}{a}\left(a^3 u'\right)', \, 2\left[\!\left[a^3 u'\right]\!\right] - 2g\left[\!\left[\rho\right]\!\right] u(p_1)\right),$$

where

$$D(K) := \left\{\tilde{u} = (u,b) \in \left(H^2((p_0,p_1)) \cap H^2((p_1,0)) \cap C^0((p_0,0))\right) \times \mathbb{C} : \right.$$
$$\left. u(p_0) = u(0) = 0, \sigma u(p_1) = b\right\}.$$

Thus, there exists a nontrivial solution of (P_μ) if and only if μ is an eigenvalue of K. Moreover, it is clear that $D(K)$ is dense in $\mathbb{H}$, and the operator K is closed. Recalling the convention $[\![u]\!] = u|_{p_1^+} - u|_{p_1^-}$ and using integration by parts, we can show

$$[K\tilde{u}, \tilde{u}] = \left\langle a^3 u', u' \right\rangle_{L^2} + g\left[\!\left[\rho\right]\!\right] |u(p_1)|^2 = [\tilde{u}, K\tilde{u}] \in \mathbb{R}, \tag{2.3.10}$$

which implies that K is symmetric, and in fact, self-adjoint. The next proposition provides a condition under which the operator K is positive, that is, $[K\tilde{u}, \tilde{u}] > 0$ for all non-zero $\tilde{u} \in D(K)$.

Proposition 2.3.6. *K is self-adjoint with simple eigenvalues. Moreover, it has a maximal negative semidefinite subspace invariant under K that has dimension one*[1]*. For $\lambda > \lambda_0$, the operator K is*

[1]We recall that the subspace $\mathcal{M}$ of $\mathbb{H}$ is said to be negative semidefinite, provided that for all $u \in \mathcal{M}$, $[u,u] \leq 0$. Moreover, $\mathcal{M}$ is an invariant subspace under the operator K if $K(\mathcal{M}) \subset \mathcal{M}$. Also, $\mathcal{M}$ is said to be a maximal subspace in $\mathbb{H}$ if $\mathcal{M}$ is not contained in any other proper subspaces of $\mathbb{H}$.

positive with a unique negative eigenvalue.

Proof. It follows from the above discussion that K is self-adjoint. Since $\mathbb{H}$ is a π_1-space, [94, Theorem 12.1'] implies that K has a maximal negative semidefinite subspace invariant under K that has dimension one. By an argument similar to [73, Lemma 3.8] and [74, Lemma 2], we see that K has discrete spectrum and its eigenvalues are geometrically simple.

Next, since K has a maximal invariant negative semidefinite subspace which is of dimension one, it has at least one eigenvalue of negative-semidefinite type. By this, we mean the restriction of $[\cdot,\cdot]$ to the eigenspace corresponding to an eigenvalue is a negative semidefinite inner product. We caution that this does not say anything about the sign of the eigenvalue itself. Let μ be a general eigenvalue of K with corresponding non-zero eigenvector $\tilde{u}$. Taking the complex conjugate of the equation $K\tilde{u} = \mu\tilde{u}$, we see that $\bar{\mu}$ is also an eigenvalue of K (note that the coefficients of the operator K are real). Thus, either μ is real, or both μ and its complex conjugate are eigenvalues. For the latter case, the corresponding eigenvector $\tilde{u}$ must be neutral, that is, $[\tilde{u},\tilde{u}] = 0$. This follows from the observation that

$$\mu[\tilde{u},\tilde{u}] = [K\tilde{u},\tilde{u}] = [\tilde{u},K\tilde{u}] = \bar{\mu}[\tilde{u},\tilde{u}].$$

On the other hand, if $\mu \in \mathbb{R}$ is an eigenvalue with corresponding eigenvector $\tilde{u}$ such that $[\tilde{u},\tilde{u}] \neq 0$, then letting

$$\mathcal{N} := \operatorname{span}\,\tilde{u} = \ker(K - \mu I) \quad \text{and} \quad \mathcal{N}^{[\perp]} := \{\tilde{v} \in \mathbb{H} : [\tilde{v},\mathcal{N}] = 0\},$$

we have $\mathbb{H} = \mathcal{N}\,[\dot{+}]\,\mathcal{N}^{[\perp]}$ as an orthogonal direct sum. Note that we are using $[\perp]$ and $[\dot{+}]$ to emphasize that the orthogonality is with respect to the indefinite inner product $[\cdot,\cdot]$. If $\tilde{w}$ is in the range of $K - \mu I$, then $\tilde{w} = (K - \mu I)\tilde{v}$ for some $\tilde{v} \in \mathbb{H}$, and hence

$$[\tilde{w},\tilde{u}] = [K\tilde{v},\tilde{u}] - \mu[\tilde{v},\tilde{u}] = [\tilde{v},K\tilde{u}] - [\tilde{v},\mu\tilde{u}] = 0,$$

which implies that the range of $K - \mu I$ is in $\mathcal{N}^{[\perp]}$. Thus, μ is algebraically simple if it is real.

Finally, suppose $\lambda > \lambda_0$. By the Cauchy–Schwarz inequality,

$$|u(p_1)|^2 = \left|\int_{p_0}^{p_1} u'(p)\,\mathrm{d}p\right|^2 \leq \int_{p_0}^{p_1} a^3\,|u'|^2\,\mathrm{d}p \int_{p_0}^{p_1} a^{-3}\,\mathrm{d}p < -\frac{1}{g\,[\![\rho]\!]}\int_{p_0}^{p_1} a^3\,|u'|^2\,\mathrm{d}p,$$

which gives

$$g\,[\![\rho]\!]\,|u(p_1)|^2 + \int_{p_0}^{0} a^3\,|u'|^2\,\mathrm{d}p > 0.$$

Thus, $[K\tilde{u}, \tilde{u}] > 0$, that is, K is positive. Then $[\tilde{u}, \tilde{u}] \neq 0$, so $\tilde{u}$ is non-neutral. Hence all eigenvalues are real when $\lambda > \lambda_0$.

If μ is a negative semidefinite eigenvalue with the corresponding eigenvector $\tilde{u}$, then

$$\mu[\tilde{u}, \tilde{u}] = [K\tilde{u}, \tilde{u}] > 0,$$

and hence, it follows that $\mu < 0$. This means that any real negative semidefinite eigenvalue of K must be negative. In fact, there is only one such eigenvalue. Indeed, if ν is another eigenvalue with corresponding eigenvector $\tilde{v}$, then

$$[\tilde{u}, \tilde{v}] = \frac{1}{\mu}[\mu\tilde{u}, \tilde{v}] = \frac{1}{\mu}[K\tilde{u}, \tilde{v}] = \frac{1}{\mu}[\tilde{u}, K\tilde{v}] = \frac{\nu}{\mu}[\tilde{u}, \tilde{v}],$$

which implies $\mu = \nu$ or $[\tilde{u}, \tilde{v}] = 0$. Since any maximal invariant semidefinite subspace of K is one dimensional, we must have $\mu = \nu$. We have therefore shown K has a unique negative eigenvalue. $\square$

Define the Rayleigh quotient $\mathscr{R}$ corresponding to (P_μ) by

$$\mathscr{R}(\varphi; \lambda) := \frac{\int_{p_0}^0 a^3 \varphi_p^2 \,\mathrm{d}p + g \llbracket \rho \rrbracket \varphi(p_1)^2}{\int_{p_0}^0 a\varphi^2 \,\mathrm{d}p - \frac{\sigma}{2}\varphi(p_1)^2}, \quad \lambda > \lambda_0, \varphi \in \mathscr{A},$$

where the admissible set is defined by

$$\mathscr{A} := \Big\{ \varphi \in H^2((p_0, p_1)) \cap H^2((p_1, 0)) \cap C((p_0, 0)) : \\ \varphi(p_0) = \varphi(0) = 0 \text{ and } \int_{p_0}^0 a\varphi^2 \,\mathrm{d}p - \frac{\sigma}{2}\varphi(p_1)^2 < 0 \Big\}.$$

Note that we are considering φ only in the negative definite subspace of K because of the condition

$$\int_{p_0}^0 a\varphi^2 \,\mathrm{d}p - \frac{\sigma}{2}\varphi(p_1)^2 < 0. \tag{2.3.11}$$

Simple arguments can show that if for a fixed $\lambda > \lambda_0$, φ is a critical point of $\mathscr{R}(\cdot; \lambda)$, then φ solves (P_μ) for $\mu = \mathscr{R}(\varphi; \lambda)$.

Next, let us define

$$\nu(\lambda) := \sup_{\substack{\varphi \in \mathscr{A} \\ \varphi \not\equiv 0}} \mathscr{R}(\varphi; \lambda).$$

First, we want to show that -1 is in the range of ν. This is because we want our solutions to be

2π-periodic in q, and the null space of $\mathcal{F}_m(\lambda^*, 0)$ is spanned by $\varphi_1(p)\cos(q)$ (see Lemma 2.3.12), which is the case where $n = 1$ and hence $\mu = -n^2 = -1$ in (2.3.9).

Lemma 2.3.7. *Let $a_{\min} := \min_{[p_1,0]} a$ (which does not depend on λ). Then for each $n \geq 1$, $\nu(\lambda) < -n^2$ when λ satisfies*

$$\lambda^2 > -a_{\min}^2 - \frac{g\,[\![\rho]\!]}{n} + \frac{\sigma n}{2}.$$

Proof. Let $\varphi \in \mathscr{A}$ be given and fix any λ as in the hypothesis. Then

$$\begin{aligned}
\int_{p_1}^{0} \left(a^3\varphi_p^2 + n^2 a\varphi^2\right)\,\mathrm{d}p &\geq a_{\min}\int_{p_1}^{0}\left(a_{\min}^2\varphi_p^2 + n^2\varphi^2\right)\,\mathrm{d}p \\
&\geq -2na_{\min}^2\int_{p_1}^{0}\varphi_p\varphi\,\mathrm{d}p \\
&= -na_{\min}^2\int_{p_1}^{0}\left(\varphi^2\right)_p\,\mathrm{d}p = na_{\min}^2\varphi(p_1)^2.
\end{aligned}$$

On the other hand, since $a^{(2)} = \lambda$,

$$\begin{aligned}
\int_{p_0}^{p_1}\left(a^3\varphi_p^2 + n^2 a\varphi^2\right)\,\mathrm{d}p &= \lambda\int_{p_0}^{p_1}\left(a^2\varphi_p^2 + n^2\varphi^2\right)\,\mathrm{d}p \\
&\geq 2n\lambda^2\int_{p_0}^{p_1}\varphi_p\varphi\,\mathrm{d}p = n\lambda^2\varphi(p_1)^2.
\end{aligned}$$

Summing these together and using the hypothesis for λ, we find

$$\begin{aligned}
\int_{p_0}^{0}\left(a^3\varphi_p^2 + n^2 a\varphi^2\right)\,\mathrm{d}p &\geq \left(n\lambda^2 + na_{\min}^2\right)\varphi(p_1)^2 \\
&> \left(-g\,[\![\rho]\!] + \frac{\sigma n^2}{2}\right)\varphi(p_1)^2,
\end{aligned}$$

which implies

$$\int_{p_0}^{0} a^3\varphi_p^2\,\mathrm{d}p + g\,[\![\rho]\!]\,\varphi(p_1)^2 > -n^2\left(\int_{p_0}^{0} a\varphi^2\,\mathrm{d}p - \frac{\sigma}{2}\varphi(p_1)^2\right),$$

so $\mathscr{R}(\varphi;\lambda) < -n^2$. Thus, $\nu(\lambda) < -n^2$. □

Next, we need to verify that $\nu(\lambda) > -1$ for some $\lambda > \lambda_0$. Since this is not true in general, we will have it as one of our hypotheses.

Definition 2.3.8. We say that the *local bifurcation condition* is satisfied provided that

$$\sup_{\lambda > \lambda_0} \nu(\lambda) > -1. \tag{LBC}$$

This is necessary and sufficient for our main result Theorem 2.3.1 to hold. An explicit but not sharp condition is the following:

Lemma 2.3.9 (size condition). *For*

$$\sigma > \frac{2\lambda_0(p_1 - p_0)}{3} + \frac{2}{p_1^2}\int_{p_1}^{0} \left(\Gamma_{\text{rel}}^3 + p^2\Gamma_{\text{rel}}\right)\,\mathrm{d}p, \tag{2.3.12}$$

where λ_0 is defined as in (2.3.7), (LBC) holds.

Proof. Let

$$\varphi(p) := \begin{cases} \dfrac{p}{p_1}, & p_1 < p < 0, \\ \dfrac{p - p_0}{p_1 - p_0}, & p_0 < p < p_1. \end{cases}$$

We first check if φ is in the admissible set $\mathscr{A}$. We see that

$$\begin{aligned} \int_{p_0}^{0} a\varphi^2\,\mathrm{d}p - \frac{\sigma}{2}\varphi(p_1)^2 &= \int_{p_0}^{p_1} \lambda\left(\frac{p - p_0}{p_1 - p_0}\right)^2 \mathrm{d}p + \int_{p_1}^{0} \Gamma_{\text{rel}}\left(\frac{p}{p_1}\right)^2 \mathrm{d}p - \frac{\sigma}{2} \\ &= \frac{\lambda(p_1 - p_0)}{3} + \frac{1}{p_1^2}\int_{p_1}^{0} p^2\Gamma_{\text{rel}}\,\mathrm{d}p - \frac{\sigma}{2}, \end{aligned}$$

but from the hypothesis (2.3.12),

$$\frac{\lambda_0(p_1 - p_0)}{3} + \frac{1}{p_1^2}\int_{p_1}^{0} p^2\Gamma_{\text{rel}}\,\mathrm{d}p - \frac{\sigma}{2} < 0.$$

Thus, for $|\lambda - \lambda_0|$ small, $\varphi \in \mathscr{A}$. With this particular φ, we then compute

$$\begin{aligned} \mathscr{R}(\varphi;\lambda) &= \frac{\int_{p_0}^{p_1} \frac{\lambda^3}{(p_1 - p_0)^2}\,\mathrm{d}p + \int_{p_1}^{0} \frac{\Gamma_{\text{rel}}^3}{p_1^2}\,\mathrm{d}p + g\,[\![\rho]\!]}{\int_{p_0}^{p_1} \lambda\left(\frac{p - p_0}{p_1 - p_0}\right)^2 \mathrm{d}p + \int_{p_1}^{0} \Gamma_{\text{rel}}\left(\frac{p}{p_1}\right)^2 \mathrm{d}p - \frac{\sigma}{2}} \\ &= \frac{\frac{\lambda^3}{p_1 - p_0} + \frac{1}{p_1^2}\int_{p_1}^{0} \Gamma_{\text{rel}}^3\,\mathrm{d}p + g\,[\![\rho]\!]}{\frac{\lambda(p_1 - p_0)}{3} + \frac{1}{p_1^2}\int_{p_1}^{0} p^2\Gamma_{\text{rel}}\,\mathrm{d}p - \frac{\sigma}{2}}. \end{aligned}$$

Rewriting the hypothesis (2.3.12) gives

$$\frac{\sigma}{2} > \frac{\lambda_0^3}{p_1 - p_0} + \frac{\lambda_0(p_1 - p_0)}{3} + \frac{1}{p_1^2}\int_{p_1}^{0} \left(\Gamma_{\text{rel}}^3 + p^2\Gamma_{\text{rel}}\right)\,\mathrm{d}p + g\,[\![\rho]\!].$$

Then for $|\lambda - \lambda_0|$ small, we have

$$\frac{\sigma}{2} > \frac{\lambda^3}{p_1 - p_0} + \frac{\lambda(p_1 - p_0)}{3} + \frac{1}{p_1^2}\int_{p_1}^{0} \left(\Gamma_{\text{rel}}^3 + p^2\Gamma_{\text{rel}}\right)\,\mathrm{d}p + g\,[\![\rho]\!].$$

Recalling that φ satisfies inequality (2.3.11), the above estimate implies that $\mathscr{R}(\varphi;\lambda) > -1$, so (LBC) holds. $\square$

We note that the above proof can be further refined following arguments of [95, Theorem 4] to find a smaller lower bound on σ guaranteeing (LBC) than that in (2.3.12).

Lemma 2.3.10 (monotonicity of ν)**.** *If $\nu(\lambda) < 0$, then $\nu(\lambda)$ is decreasing in λ.*

Proof. Denoting derivatives with respect to λ by a dot, differentiating the eigenvalue problem (P_μ) with $u = \varphi \in \mathscr{A}$ gives

$$\begin{cases} -(3a^2\dot{a}\varphi_p)_p - (a^3\dot{\varphi}_p)_p = \dot{\nu}a\varphi + \nu\dot{a}\varphi + \nu a\dot{\varphi} & \text{in } (p_0,p_1)\cup(p_1,0), \\ 2\left[\!\left[3a^2\dot{a}\varphi_p + a^3\dot{\varphi}_p\right]\!\right] - 2g\left[\!\left[\rho\right]\!\right]\dot{\varphi} = \sigma\dot{\nu}\varphi + \sigma\nu\dot{\varphi} & \text{on } p = p_1, \\ \dot{\varphi} = 0 & \text{on } p = p_0, \\ \dot{\varphi} = 0 & \text{on } p = 0. \end{cases} \tag{$\dot{P}_\mu$}$$

Multiplying (P_μ) by $\dot{\varphi}$ and integrating yields

$$\int_{p_0}^0 a^3\varphi_p\dot{\varphi}_p\,\mathrm{d}p + g\left[\!\left[\rho\right]\!\right]\varphi(p_1)\dot{\varphi}(p_1) + \frac{\sigma}{2}\nu\varphi(p_1)\dot{\varphi}(p_1) = \int_{p_0}^0 \nu a\varphi\dot{\varphi}\,\mathrm{d}p. \tag{2.3.13}$$

On the other hand, multiplying $(\dot{P}_\mu)$ by φ and integrating gives

$$\begin{aligned} \int_{p_0}^0 3a^2\dot{a}\varphi_p^2\,\mathrm{d}p + g\left[\!\left[\rho\right]\!\right]\dot{\varphi}(p_1)\varphi(p_1) + \frac{\sigma}{2}\dot{\nu}\varphi(p_1)^2 + \frac{\sigma}{2}\nu\dot{\varphi}(p_1)\varphi(p_1) \\ + \int_{p_0}^0 a^3\dot{\varphi}_p\varphi_p\,\mathrm{d}p = \int_{p_0}^0 \left(\dot{\nu}a\varphi^2 + \nu\dot{a}\varphi^2 + \nu a\dot{\varphi}\varphi\right)\mathrm{d}p. \end{aligned} \tag{2.3.14}$$

Subtracting (2.3.13) from (2.3.14), we have the following Green's identity

$$\int_{p_0}^0 3a^2\dot{a}\varphi_p^2\,\mathrm{d}p + \frac{\sigma}{2}\dot{\nu}\varphi(p_1)^2 = \int_{p_0}^0 \left(\dot{\nu}a\varphi^2 + \nu\dot{a}\varphi^2\right)\mathrm{d}p.$$

Since $\dot{a} = \mathbb{1}_{(p_0,p_1)}$, we can simplify this to find

$$\int_{p_0}^{p_1} 3a^2\varphi_p^2\,\mathrm{d}p - \int_{p_0}^{p_1} \nu\varphi^2\,\mathrm{d}p = \left(\int_{p_0}^0 a\varphi^2\,\mathrm{d}p - \frac{\sigma}{2}\varphi(p_1)^2\right)\dot{\nu}.$$

Therefore, since $\nu < 0$ by assumption and the quantity in parenthesis is negative, we must have $\dot{\nu} < 0$. $\square$

Lemma 2.3.11. *Suppose that the* (LBC) *holds. Then there exists a unique value $\lambda^* > 0$ such that $\nu(\lambda^*) = -1$. Equivalently, there exists a unique value of λ for which there is a nontrivial solution to the linearized problem (2.3.8) with the ansatz $m(q,p) = M(p)\cos(q)$. Moreover, Q is an invertible function of λ in a neighborhood of λ^*.*

Proof. From Lemma 2.3.7, we have $\nu(\lambda) < -1$ for λ sufficiently large, and $\nu(\lambda) > -1$ for some λ by (LBC). By continuity, there exists λ^* such that $\nu(\lambda^*) = -1$. Moreover, Lemma 2.3.10 tells us that ν is a decreasing function when $\nu < 0$, so λ^* is unique.

Next, as noted at the end of Lemma 2.3.4, Q is a concave function of λ according to (2.3.5), so we only need to show that $\lambda^* \neq \lambda_0$, where λ_0 is defined in (2.3.7) to be the critical point of Q. But $\lambda^* > \lambda$ by (LBC). □

2.3.3 Proof of local bifurcation

We are now prepared to prove Theorem 2.3.1. As stated above, our approach is based on the classical theory of Crandall–Rabinowitz on local bifurcation from simple (generalized) eigenvalues. Specifically, we will treat the family of laminar flows as our trivial solutions. Suppose the solution to the height equation (2.3.1) can be decomposed as $h(q,p) = H(p;\lambda) + m(q,p)$ and $Q = Q(\lambda)$. Then substituting it into the equation gives

$$\mathcal{F}(\lambda, m) = 0,$$

where $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4) : \Lambda \times \mathcal{O} \to Y$ with $\Lambda \subset \mathbb{R}$ to be a neighborhood of λ^*, and

$$\mathcal{O} := \left\{ m \in X : \inf(m_p + H_p) > 0 \text{ in } \overline{D} \text{ for all } \lambda \in \Lambda \right\},$$

$$\begin{aligned}
\mathcal{F}_1(\lambda, m) &:= (1 + (m_q^{(1)})^2)(m_{pp}^{(1)} + H_{pp}) + m_{qq}^{(1)}(m_p^{(1)} + H_p)^2 \\
&\qquad - 2m_q^{(1)}(H_p + m_p^{(1)})m_{pq}^{(1)} + \gamma(-p)(H_p + m_p^{(1)})^3, \\
\mathcal{F}_2(\lambda, m) &:= (1 + (m_q^{(2)})^2)(m_{pp}^{(2)} + H_{pp}) + m_{qq}^{(2)}(m_p^{(2)} + H_p)^2 \\
&\qquad - 2m_q^{(2)}(H_p + m_p^{(2)})m_{pq}^{(2)}, \\
\mathcal{F}_3(\lambda, m) &:= -\left[\!\!\left[\frac{1 + m_q^2}{(H_p + m_p)^2} \right]\!\!\right] - 2g\, [\![\rho]\!]\, (m + H) + Q - \frac{\sigma m_{qq}}{(1 + m_q^2)^{3/2}}, \\
\mathcal{F}_4(\lambda, m) &:= \Big(m + H - \ell - d(m) - d(H)\Big)\Big|_T.
\end{aligned} \tag{2.3.15}$$

The Banach spaces X and $Y = Y_1 \times Y_2 \times Y_3 \times Y_4$ are defined by

$$X := \left\{ h \in C^{2,\alpha}(\overline{D_1}) \cap C^{2,\alpha}(\overline{D_2}) \cap C^{0,\alpha}_{\text{per}}(\overline{D}) : h(p_0) = 0 \right\},$$

$$Y_1 := C^{0,\alpha}_{\text{per}}(\overline{D_1}), \quad Y_2 := C^{0,\alpha}_{\text{per}}(\overline{D_2}), \quad Y_3 := C^{\alpha}_{\text{per}}(I), \quad Y_4 := C^{2,\alpha}_{\text{per}}(T).$$

It is clear that $\mathcal{F}(\lambda, 0) = 0$ for all $\lambda > 0$. Let us record the Fréchet derivative of $\mathcal{F}$ with respect to m at $(\lambda^*, 0)$.

$$\begin{aligned}
\mathcal{F}_{1m}(\lambda^*, 0)\varphi &= \left(\partial_p^2 + H_p^2 \partial_q^2 + 3\gamma H_p^2 \partial_p\right) \varphi^{(1)}, \\
\mathcal{F}_{2m}(\lambda^*, 0)\varphi &= \left(\partial_p^2 + H_p^2 \partial_q^2\right) \varphi^{(2)}, \\
\mathcal{F}_{3m}(\lambda^*, 0)\varphi &= 2 \left[\!\left[H_p^{-3} \varphi_p \right]\!\right] - 2g \left[\!\left[\rho \right]\!\right] \varphi - \sigma \varphi_{qq}, \\
\mathcal{F}_{4m}(\lambda^*, 0)\varphi &= \Big(\varphi - d(\varphi)\Big)\Big|_T.
\end{aligned}$$

Note that in D_1, from (2.3.3), we have $\gamma = -H_{pp}/H_p^3$, so we can write

$$\begin{aligned}
\mathcal{F}_{1m}(\lambda^*, 0)\varphi &= \varphi_{pp} + a^{-2}\varphi_{qq} + 3H_p \partial_p \left(H_p^{-1}\right) \varphi_p \\
&= \varphi_{pp} + a^{-2}\varphi_{qq} + a^{-3} 3a^2 (\partial_p a)\varphi_p \\
&= a^{-3} \left(a^3 \varphi_p\right)_p + a^{-2}\varphi_{qq},
\end{aligned}$$

which is the same quantity as in D_2. Thus, the first expression can be written as

$$\mathcal{F}_{im}(\lambda^*, 0)\varphi = a^{-3}\partial_p \left(a^3 \partial_p \varphi^{(i)}\right) + a^{-2}\partial_q^2 \varphi^{(i)} \quad \text{for } i = 1, 2.$$

Lemma 2.3.12 (null space)**.** *The null space of $\mathcal{F}_m(\lambda^*, 0)$ is one-dimensional and spanned by $\varphi^*(q,p) := \varphi_1(p)\cos(q)$.*

Proof. Let φ be in the null space of $\mathcal{F}_m(\lambda^*, 0)$). Since φ is even and $C^{0,\alpha}$, we can express it via a cosine series

$$\varphi(q,p) = \sum_{n=0}^{\infty} \varphi_n(p)\cos(nq).$$

Clearly, $\mathcal{F}_m(\lambda^*, 0)\left(\varphi_n(p)\cos(nq)\right) = 0$ for every $n \geq 0$ meaning that φ_n must solve (2.3.9) for n. Since $\lambda^* > \lambda_0$, we can apply Proposition 2.3.6 to conclude that the null space of $\mathcal{F}_m(\lambda^*, 0)$ is one-dimensional. In particular, it is generated by $\varphi^*(q,p) := \varphi_1(p)\cos(q)$, where φ_1 is the unique solution to equation (2.3.9) for $n = 1$. ∎

Our next lemma characterizes the range of $\mathcal{F}_m(\lambda^*, 0)$.

Lemma 2.3.13 (range)**.** *$\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) \in Y$ is in the range of $\mathcal{F}_m(\lambda^*, 0)$ if and only if it satisfies the following orthogonality condition:*

$$\iint_{D_1} a^3 \mathcal{A}_1 \varphi^* \, \mathrm{d}q \, \mathrm{d}p + \iint_{D_2} a^3 \mathcal{A}_2 \varphi^* \, \mathrm{d}q \, \mathrm{d}p + \frac{1}{2} \int_I \mathcal{A}_3 \varphi^* \, \mathrm{d}q + \int_T a^3 \mathcal{A}_4 \varphi_p^* \mathrm{d}q = 0, \tag{2.3.16}$$

where φ^* generates the null space of $\mathcal{F}_m(\lambda^*, 0)$.

Proof. Denoting by $\mathcal{R}(\mathcal{F}_m(\lambda^*, 0))$ the range of $\mathcal{F}_m(\lambda^*, 0)$, we first suppose that $\mathcal{A} \in \mathcal{R}(\mathcal{F}_m(\lambda^*, 0))$. Let φ be given such that $\mathcal{F}_m(\lambda^*, 0)\varphi = \mathcal{A}$. Using integration by parts and the PDE for φ and φ^* (φ^* satisfies equation (2.3.9) with $n = 1$), we can compute

$$\begin{aligned}
&\left\langle a^3 \varphi^*, \mathcal{A}_1 \right\rangle_{L^2(D_1)} + \left\langle a^3 \varphi^*, \mathcal{A}_2 \right\rangle_{L^2(D_2)} \\
&= \iint_{D_1 \cup D_2} \left((a^3 \varphi_p)_p + a \varphi_{qq} \right) \varphi^* \, \mathrm{d}q \, \mathrm{d}p \\
&= - \iint_{D_1 \cup D_2} a^3 \varphi_p \varphi_p^* \, \mathrm{d}q \, \mathrm{d}p - \int_I \left[\!\left[a^3 \varphi_p \varphi^* \right]\!\right] \, \mathrm{d}q + \iint_{D_1 \cup D_2} a \varphi \varphi_{qq}^* \, \mathrm{d}q \, \mathrm{d}p \\
&= \iint_{D_1 \cup D_2} \left(a^3 \varphi_{pp}^* + a \varphi_{qq}^* \right) \varphi \, \mathrm{d}q \, \mathrm{d}p + \int_I \left(\left[\!\left[a^3 \varphi_p^* \varphi \right]\!\right] - \left[\!\left[a^3 \varphi_p \varphi^* \right]\!\right] \right) \, \mathrm{d}q - \int_T a^3 \varphi \varphi_p^* \, \mathrm{d}q.
\end{aligned}$$

Using the jump condition, the fact that φ and φ^* are continuous across the interface, and integration by parts, we obtain

$$\begin{aligned}
&\left\langle a^3 \varphi^*, \mathcal{A}_1 \right\rangle_{L^2(D_1)} + \left\langle a^3 \varphi^*, \mathcal{A}_2 \right\rangle_{L^2(D_2)} \\
&= \int_I \left(g \left[\!\left[\rho \right]\!\right] \varphi^* - \frac{\sigma}{2} \varphi^* \right) \varphi \, \mathrm{d}q - \int_I \left(g \left[\!\left[\rho \right]\!\right] \varphi + \frac{\sigma}{2} \varphi_{qq} + \frac{1}{2} \mathcal{A}_3 \right) \varphi^* \, \mathrm{d}q \\
&- \int_T a^3 \mathcal{A}_4 \varphi_p^* \, \mathrm{d}q - d(\varphi) \int_T a^3 \varphi_p^* \, \mathrm{d}q \\
&= \int_I \left(-\frac{\sigma}{2} \varphi^* \varphi - \frac{\sigma}{2} \varphi_{qq}^* \varphi - \frac{1}{2} \mathcal{A}_3 \varphi^* \right) \, \mathrm{d}q - \int_T a^3 \mathcal{A}_4 \varphi_p^* \, \mathrm{d}q.
\end{aligned}$$

Recalling that $\varphi^* = \varphi_1(p)\cos(q)$, the last equality comes from the observation that

$$\int_T a^3 \varphi_p^* \, \mathrm{d}q = a^3 \varphi_{1p}(0) \int_0^{2\pi} \cos(q) \, \mathrm{d}q = 0.$$

Finally, we have

$$\left\langle a^3 \varphi^*, \mathcal{A}_1 \right\rangle_{L^2(D_1)} + \left\langle a^3 \varphi^*, \mathcal{A}_2 \right\rangle_{L^2(D_2)} = -\frac{1}{2} \int_I \mathcal{A}_3 \varphi^* \, \mathrm{d}q - \int_T a^3 \mathcal{A}_4 \varphi_p^* \, \mathrm{d}q,$$

which can be rearranged to yield the identity (2.3.16).

Next, we prove the orthogonality condition (2.3.16) is sufficient. By similar arguments as in [73, Lemma 3.6], we define the following inner product for each λ:

$$\begin{aligned}
\langle(\mathcal{U}_1,&\mathcal{U}_2,\mathcal{U}_3,\mathcal{U}_4),(\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3,\mathcal{V}_4)\rangle_Y := \\
&\left\langle a^3\mathcal{U}_1,\mathcal{V}_1\right\rangle_{L^2(D_1)} + \left\langle a^3\mathcal{U}_2,\mathcal{V}_2\right\rangle_{L^2(D_2)} + \frac{1}{2}\left\langle \mathcal{U}_3,\mathcal{V}_3\right\rangle_{L^2(I)} + \left\langle a^3\mathcal{U}_4,\mathcal{V}_4\right\rangle_{L^2(T)}.
\end{aligned}$$

Note that the null space $\widehat{\mathcal{N}}$ of $\mathcal{F}_m(\lambda^*,0)$ can be identified with the subspace

$$\begin{aligned}
\widetilde{\mathcal{N}} :=&\{(\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3,\mathcal{V}_4)\in Y: \\
&\mathcal{V}_1=\mathcal{V}|_{D_1},\ \mathcal{V}_2=\mathcal{V}|_{D_2},\ \mathcal{V}_3=\mathcal{V}|_{I},\ \mathcal{V}_4=\mathcal{V}|_{T} \text{ for some } \mathcal{V}\in\widehat{\mathcal{N}}\}.
\end{aligned}$$

Then the necessary condition, which is shown above, implies that if $\mathcal{A}$ is in the range of $\mathcal{F}_m(\lambda^*,0)$, then

$$\langle(\varphi^*|_{D_1},\varphi^*|_{D_2},\varphi^*|_I,\varphi^*|_T),\mathcal{A}\rangle_Y=0,$$

so $\mathcal{R}(\mathcal{F}_m(\lambda^*,0))\subset\widetilde{\mathcal{N}}^{\perp}$. By Remark 2.2.5, $\mathcal{F}_m(\lambda^*,0)$ has Fredholm index 0, and hence

$$\operatorname{codim}\mathcal{R}(\mathcal{F}_m(\lambda^*,0))=\dim\widehat{\mathcal{N}}=\dim\widetilde{\mathcal{N}}=\operatorname{codim}\widetilde{\mathcal{N}}^{\perp}<\infty,$$

which means that $\mathcal{R}(\mathcal{F}_m(\lambda^*,0))=\widetilde{\mathcal{N}}^{\perp}$. This concludes the proof of the lemma. □

Lemma 2.3.14 (transversality)**.** *The following transversality condition holds*

$$\mathcal{F}_{\lambda m}(\lambda^*,0)\varphi^*\notin\mathcal{R}(\mathcal{F}_m(\lambda^*,0)), \tag{2.3.17}$$

where φ^* generates the null space of $\mathcal{F}_m(\lambda^*,0)$ solves equation (2.3.9) *for $n=1$.*

Proof. Using Lemma 2.3.13, it suffices to show that $\mathcal{A}:=\mathcal{F}_{\lambda m}(\lambda^*,0)\varphi^*$ does not satisfy the orthogonality condition (2.3.16). We must confirm that

$$\Xi:=\iint_{D_1}a^3\mathcal{A}_1\varphi^*\,\mathrm{d}q\,\mathrm{d}p+\iint_{D_2}a^3\mathcal{A}_2\varphi^*\,\mathrm{d}q\,\mathrm{d}p+\frac{1}{2}\int_I\mathcal{A}_3\varphi^*\,\mathrm{d}q+\int_T a^3\mathcal{A}_4\varphi_p^*\,\mathrm{d}q\neq0.$$

Computing the derivatives gives

$$\begin{aligned}
\mathcal{F}_{1\lambda m}(\lambda^*,0)\varphi^* &= 0,\\
\mathcal{F}_{2\lambda m}(\lambda^*,0)\varphi^* &= -\frac{2}{(\lambda^*)^3}\varphi^*_{qq},\\
\mathcal{F}_{3\lambda m}(\lambda^*,0)\varphi^* &= \left(-6(\lambda^*)^2\left(\varphi^*_p\right)^{(2)}\right)\Big|_I,\\
\mathcal{F}_{4\lambda m}(\lambda^*,0)\varphi^* &= 0,
\end{aligned}$$

so we have

$$\iint_{D_1} a^3\mathcal{A}_1\varphi^*\,\mathrm{d}q\,\mathrm{d}p = \int_T a^3\mathcal{A}_4\varphi^*_p\,\mathrm{d}q = 0.$$

Using (2.3.9) with $n=1$ to have $\varphi^* = (\lambda^*)^2\varphi^*_{pp}$, we can derive

$$(\lambda^*)^2\left(\varphi^*\varphi^*_p\right)_p = (\lambda^*)^2\left(\varphi^*_p\right)^2 + (\lambda^*)^2\varphi^*\varphi^*_{pp} = (\lambda^*)^2\left(\varphi^*_p\right)^2 + (\varphi^*)^2 \quad \text{in } D_2$$

so that using integration by parts and the fact that $\varphi^*_{qq} = -\varphi^*$ yields

$$\iint_{D_2} a^3\mathcal{A}_2\varphi^*\,\mathrm{d}q\,\mathrm{d}p = \iint_{D_2} 2(\varphi^*)^2\,\mathrm{d}q\,\mathrm{d}p$$

and

$$\begin{aligned}
\frac{1}{2}\int_I \mathcal{A}_3\varphi^*\,\mathrm{d}q &= -3\,(\lambda^*)^2\int_I \left(\varphi^*_p\right)^{(2)}\varphi^*\,\mathrm{d}q\\
&= -3\,(\lambda^*)^2\iint_{D_2}\left(\varphi^*_p\right)^2\,\mathrm{d}q\,\mathrm{d}p - 3\iint_{D_2}(\varphi^*)^2\,\mathrm{d}q\,\mathrm{d}p.
\end{aligned}$$

Finally, combining all terms gives

$$\Xi = -\iint_{D_2}(\varphi^*)^2\,\mathrm{d}q\,\mathrm{d}p - 3\,(\lambda^*)^2\iint_{D_2}\left(\varphi^*_p\right)^2\,\mathrm{d}q\,\mathrm{d}p < 0. \qquad \square$$

Now we are ready to prove our main theorem.

Proof of Theorem 2.3.1. Suppose conditions (2.3.2) and (LBC) are satisfied. Then $\mathcal{F}(\lambda,0)=0$ for all $\lambda>0$ and $\mathcal{F}_m$, $\mathcal{F}_\lambda$, $\mathcal{F}_{\lambda m}$ exist and are continuous, which means parts (*i*) and (*ii*) are confirmed. Moreover, Lemma 2.3.12 and Lemma 2.3.13 give dimension 1 for the null space of $\mathcal{F}_m(\lambda^*,0)$ and co-dimension 1 for the range of $\mathcal{F}_m(\lambda^*,0)$. Thus, $\mathcal{F}_m(\lambda^*,0)$ has Fredholm index 0, and hence part (*iii*) is justified. Lastly, the transversality condition in Lemma 2.3.14 fulfills part (*iv*). Therefore, the local bifurcation result follows directly from Theorem A.1. $\square$

Finally, back to our objective, we note that the existence of a solution of class $\mathscr{S}'$ to the height equation (2.3.1) is equivalent to the existence of a solution of class $\mathscr{S}$ to the Euler system (2.1.8)–(2.1.10) (see, for example, [29, Lemma 2.1] or [23, Lemma 2.1]).

Chapter 3

Water waves with a finite dipole

3.1 Introduction

This work is motivated by the following simple experiment. Imagine that a surface water wave passes over a thin submerged body. Boundary layer effects may then produce so-called *shed vortices* — highly localized vortical regions in the object's wake. A natural idealization for this phenomenon is a *finite dipole*, which is a weak solution of the Euler equations whose vorticity ω consists of a pair of Dirac δ-measures (called point vortices) of nearly opposite strength that are separated by a fixed distance.

Dipoles are used commonly in fluid dynamical models; see further discussion in subsection 3.1.3. It is well-known that, if the problem is posed in the plane, then there are exact (stable) solutions for which the pair of vortices translate in parallel at a fixed velocity. Here, we wish to study the far more complicated situation where the dipole lies inside a water wave. We prove that there exists traveling wave solutions to this system. However, our main result shows that they are conditionally orbitally unstable. Physically, this indicates that a pair of counter-rotating shed vortices moving with a wave will not persist over long periods of time. For instance, they may approach and then breach the surface.

3.1.1 Main equations

For each time $t \geq 0$, let $\Omega_t \subset \mathbb{R}^2$ be the fluid domain:

$$\Omega_t := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 < \eta(t, x_1)\right\},$$

where the a priori unknown function $\eta = \eta(t, x_1)$ describes the free surface between air and water. The water wave with a finite dipole problem is as follows. Let $v = v(t, \cdot) : \Omega_t \to \mathbb{R}^2$ be the fluid velocity. The vorticity $\omega = \omega(t, \cdot) : \Omega_t \to \mathbb{R}$ is the (scalar) curl of v. For a finite dipole, we must have

$$\omega := \partial_{x_2} v_1 - \partial_{x_1} v_2 = -\epsilon\gamma_1 \delta_{\bar{x}} + \epsilon\gamma_2 \delta_{\bar{y}}, \tag{3.1.1}$$

in the sense of distributions. Here $\bar{x} = \bar{x}(t)$ and $\bar{y} = \bar{y}(t)$ are the vortex centers, and $\epsilon\gamma_1$ and $-\epsilon\gamma_2$ are the strengths, respectively. We require that v is a weak solution of the incompressible Euler equations away from the two point vortices:

$$\begin{cases} \partial_t v + (v \cdot \nabla) v + \nabla p + g e_2 = 0 \quad \text{in } \Omega_t \backslash \{\bar{x}, \bar{y}\}, \\ \nabla \cdot v = 0 \quad \text{in } \Omega_t. \end{cases} \tag{3.1.2}$$

We require that there is finite excess kinetic energy, which corresponds to $v(t, \cdot) \in L^1_{\mathrm{loc}}(\Omega_t) \cap L^2(\Omega_t \backslash N_t)$ for any open set $N_t \subset \Omega_t \setminus \{\bar{x}, \bar{y}\}$.

On the free surface $S_t := \partial\Omega_t$, we have the kinematic and dynamic boundary condition:

$$\partial_t \eta = -\eta' v_1 + v_2, \qquad p = b\kappa \quad \text{on } S_t, \tag{3.1.3}$$

where primes indicate derivatives with respect to x_1, and $\kappa = \kappa(t, x_1)$ is the mean curvature of the surface

$$\kappa(t, x_1) = -\frac{\eta''(t, x_1)}{\langle \eta'(t, x_1) \rangle^3}.$$

Here we are using the Japanese bracket notation: $\langle \cdot \rangle := \left(1 + (\cdot)^2\right)^{\frac{1}{2}}$. Moreover, the constant $b > 0$ is the coefficient of surface tension.

Finally, the motion of the vortices is governed by the Kirchhoff–Helmholtz model [34, 50]:

$$\begin{cases} \partial_t \bar{x} = \left(v - \dfrac{\gamma_1}{2\pi} \epsilon \nabla^\perp \log |x - \bar{x}|\right)\Big|_{\bar{x}}, \\ \partial_t \bar{y} = \left(v + \dfrac{\gamma_2}{2\pi} \epsilon \nabla^\perp \log |x - \bar{y}|\right)\Big|_{\bar{y}}, \end{cases} \tag{3.1.4}$$

with $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. This system mandates that the point vortices are transported by the irrotational part of the fluid velocity field, and also attract each other due to the opposite vortex strengths.

3.1.2 Statement of main results

We are interested in both showing the existence of solitary waves solutions to (3.1.1)–(3.1.4) and determining their stability. As long as the two point vortices are separated from the surface, the fluid velocity v can be decomposed as

$$v = \nabla\Phi + \epsilon\nabla\Theta \tag{3.1.5}$$

in a neighborhood of S_t, where Φ is a harmonic function and Θ represents the influence of the dipole. Note that Θ can be written explicitly in terms of $\bar{x}$ and $\bar{y}$. To determine v, it is enough to know η and the restriction of Φ to the surface S_t:

$$\varphi = \varphi(t, x_1) := \Phi\left(t, x_1, \eta(t, x_1)\right). \tag{3.1.6}$$

For the steady problem, we look for solutions of the form

$$\eta = \eta^c(x_1 - ct), \quad \varphi = \varphi^c(x_1 - ct), \quad \bar{x} = cte_1 + (-a + \rho)e_2, \quad \bar{y} = cte_1 + (-a - \rho)e_2,$$

where (η^c, φ^c) are time-independent and spatially localized. Specifically, we work in the space

$$(\eta, \varphi, a, \rho) \in X = X_1 \times X_2 \times X_3 \times X_4 := H_e^k(\mathbb{R}) \times \left(\dot{H}_o^{k-1}(\mathbb{R}) \cap \dot{H}_o^{1/2}(\mathbb{R})\right) \times \mathbb{R} \times \mathbb{R} \tag{3.1.7}$$

with

$$H_e^k(\mathbb{R}) := \{f \in H^k(\mathbb{R}) : f \text{ is even in } x_1\}, \quad H_o^k(\mathbb{R}) := \{f \in H^k(\mathbb{R}) : f \text{ is odd in } x_1\},$$

and let $\dot{H}_o^k(\mathbb{R})$ be the corresponding homogeneous space. Then our first result is the existence of traveling capillary-gravity water waves with a finite dipole. This theorem is an analogue of the work of Varholm on the water wave problem with point vortices in finite depth [44].

Theorem 3.1.1 (Existence)**.** *Let*

$$\bar{x}(t) = cte_1 + (-a_0 + \rho_0)e_2, \qquad \bar{y}(t) = cte_1 + (-a_0 - \rho_0)e_2.$$

Then for every $k > \frac{3}{2}$, $a_0 \in (0, \infty)$, $\rho_0 \in (0, a_0)$, $\gamma_1^0 > 0$, *and* $\gamma_2^0 > 0$ *subject to the compatibility condition*

$$\gamma_2^0 = \frac{a_0^3 + \rho_0^3}{a_0^3 - \rho_0^3}\gamma_1^0, \tag{3.1.8}$$

there exists $\epsilon_1 > 0$, $c_1 > 0$, $\gamma_1^1 > 0$, $\gamma_2^1 > 0$, *and* C^1 *family of traveling water wave with a finite dipole:*

$$\begin{aligned}\mathscr{C}_{\text{loc}} = \{(\epsilon, c, \gamma_1, \gamma_2, \eta(\epsilon, c, \gamma_1, \gamma_2), \varphi(\epsilon, c, \gamma_1, \gamma_2), a(\epsilon, c, \gamma_1, \gamma_2), \rho(\epsilon, c, \gamma_1, \gamma_2)) :& \\ |\epsilon| < \epsilon_1, |c - c_0| < c_1, |\gamma_1 - \gamma_1^0| < \gamma_1^1, |\gamma_2 - \gamma_2^0| < \gamma_2^1\}& \\ \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times X.&\end{aligned}$$

Due to the variational structure of the problem, it is most natural to fix ϵ, γ_1, γ_2, and consider the curve in $\mathscr{C}_{\text{loc}}$ that results from varying the wave speed c. This gives rise to a one-parameter family of solitary waves indexed by the wave speed:

$$U_c := (\eta(c), \varphi(c), \bar{x}(c), \bar{y}(c)).$$

The compatibility condition (3.1.8) implies that the lower vortex at $\bar{y}$ must have a greater strength than the upper vortex at $\bar{x}$, that is, $\gamma_2 > \gamma_1$ for $0 < \rho_0 < a_0$. This is a consequence of the fact that $\bar{x}$ is closer to the free surface S_t and is therefore influenced by it more strongly. In contrast to finite dipoles in $\mathbb{R}^2$, which must have a total vorticity of 0, the interaction with the wave in fact forces the point vortices to have slightly unmatched strengths. It is worth mentioning that the compatibility condition (3.1.8) is not artificial. Indeed, as the family $\mathscr{C}_{\text{loc}}$ is exhaustive in a neighborhood of 0 in the space X, it must hold for any sufficiently small-amplitude, slow moving waves with even symmetry.

Returning to the time-dependent problem, we introduce two important spaces. Let

$$\mathbb{X} := \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \times \mathbb{X}_4 := H^1(\mathbb{R}) \times \dot{H}^{1/2}(\mathbb{R}) \times \mathbb{R}^2 \times \mathbb{R}^2, \tag{3.1.9}$$

and set

$$\mathbb{W} := \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3 \times \mathbb{W}_4 := H^{3+}(\mathbb{R}) \times \left(\dot{H}^{5/2+}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})\right) \times \mathbb{R}^2 \times \mathbb{R}^2, \tag{3.1.10}$$

where H^{k+} means H^{k+s} for some fixed $0 < s \ll 1$. We think of $\mathbb{W}$ as the *well-posedness* space for (3.1.1)–(3.1.4). A local well-posedness result for irrotational capillary-gravity water waves with this degree of regularity was proved by Alazard, Burq, and Zuily [96]. Very recently, Su [97] obtained local well-posedness for (3.1.1)–(3.1.4) in a somewhat smoother space than $\mathbb{W}$; our results will also hold in that setting with only minor modifications. On the other hand, $\mathbb{X}$ is the natural *energy space*. This is discussed in more detail in subsection 3.3.1. Finally, for the problem to be well-defined, the finite dipole must be away from the free surface, so we take

$$\mathcal{O} := \{u \in \mathbb{X} : \bar{x}_2 < \eta(\bar{x}_1) < -\bar{x}_2, \quad \bar{y}_2 < \eta(\bar{y}_1) < -\bar{y}_2, \quad \bar{x} \neq \bar{y}\}. \tag{3.1.11}$$

To state the main result, we introduce some terminology. First, observe that the entire system is invariant under the one-parameter affine symmetry group $T(s) : \mathbb{X} \to \mathbb{X}$ defined by

$$T(s)u := T(s)(\eta, \varphi, \bar{x}, \bar{y})^T = (\eta(\cdot - s), \varphi(\cdot - s), \bar{x} + se_1, \bar{y} + se_1)^T. \tag{3.1.12}$$

This suggests that stability or instability should be understood modulo $T(s)$. With that in mind, for each $\rho > 0$, we define the tubular neighborhood

$$\mathcal{U}_\rho := \left\{u \in \mathcal{O} : \inf_{s \in \mathbb{R}} \|T(s)U_c - u\|_{\mathbb{W}} < \rho\right\}.$$

Definition 3.1.2. We say U_c is *orbitally unstable* provided that there is a $\nu_0 > 0$ such that for every $0 < \nu < \nu_0$ there exists initial data in $\mathcal{U}_\nu$ whose corresponding solution exits $\mathcal{U}_{\nu_0}$ in finite time.

Our main theorem is as follows.

Theorem 3.1.3 (Instability)**.** *Assume that* (3.1.1)–(3.1.4) *is locally well-posed in* $\mathbb{W}$ *in the sense of Assumption B.1.7. For any $\epsilon \neq 0$ sufficiently small, the corresponding family of solitary capillary-gravity water waves with a finite dipole U_c furnished by Theorem 3.1.1 is conditionally orbitally unstable.*

One physical interpretation for this is that, while we can construct steady configurations of counter rotating vortices moving in parallel through a water wave, these will not tend to persist over long periods of time. Instead, we expect them to migrate to the surface of the water, fail to keep pace with the surface wave, or otherwise destabilize. Moreover, this result covers all sufficiently small amplitude, wave speed, and vortex strength waves with even symmetry because $\mathscr{C}_{\mathrm{loc}}$ comprises also such waves near 0 in X.

3.1.3 History of the problem

The study of point vortices was initiated by Helmholtz [34] and Kirchhoff [36], who independently developed the model (3.1.4). Since then, there has been extensive research on this subject. The majority of this work concerns vortices in fixed fluid domains. For instance, Love found a condition under which the motion of two pairs of vortices may be periodic [98] and investigated the stability of Kirchhoff's elliptic vortex [38]. Aref–Pomphrev [99] and Aref–Eckhardt [100] examined the chaotic behavior of the system of two pairs of vortices. Wan [101] proved the existence of steady concentrated vortex patches near the system of non-degenerate steady point vortices in the plane and on a bounded domain. Marchioro and Pulvirenti [39] later justified the connection between the incompressible Euler equation (3.1.2) and the Kirchhoff–Helmholtz model (3.1.4). Aref and Newton gave a thorough review of the results for N-vortex problem in the plane [102, 103] or on the surface of the sphere [103]. Recently, Smets–Van Schaftingen [104] and Cao–Liu–Wei [105] studied the existence of solutions to the point vortex problem in a bounded domain using either a variational or Lyapunov-Schmidt reduction approach. Kanso, Newton, and Tchieu also used the finite dipole as a model for fish schoolings [49]. They examined the formation of multi-pole systems, discussed their stability, and compared the model against the real world scenario. Point vortex models can also be used in studies of atmosphere and oceans [106].

When a dipole is placed inside a water wave, which is the case in this work, investigating existence and stability of solutions is much more involved mathematically as it requires developing an understanding of the interaction between the motion of the vortices and the free surface. Nonetheless, there have been a sizable number of studies in this regime. The first rigorous existence theory for steady solutions was given by Filippov [40] and Ter-Krikorov [27], who investigated the finite-depth regime neglecting surface tension. Later, Shatah, Walsh, and Zeng constructed a family of traveling capillary gravity waves in infinite depth water with a single point vortex [33]. Using a similar method, Varholm obtained analogues for capillary-gravity waves with one or more vortices in finite depth [44]. Our existence theory follows in large part from the techniques in these two papers.

Our main source of inspiration is the recent paper by Varholm, Wahlén, and Walsh [107] that proves the orbital stability of traveling capillary gravity waves with a single point vortex. As we explain below, we will adopt a similar methodology. However, the dipole turns out to be significantly more difficult to analyze at a technical level. It is also of considerable importance to applications, as described above.

It is well-known in the physics literature that the governing equations for water waves with submerged point vortices have a Hamiltonian structure. Rouhi and Wright gave the formulation for the motion of vortices in the presence of a free surface in two and three dimensions [108]. A similar formulation was later given by Zakharov [109].

There have also been a number of numerical results about vortex pairs in a fluid. The closest to the current problem is the recent paper of Curtis, Carter, and Kalisch [110], who studied how constant vorticity shear profile affects the motion of the particles both at and beneath waves in infinitely deep water. Many authors have looked at the related scenario where a submerged dipole is sent moving towards the free surface rather than moving with the wave; see, for example, [47], [111], [41]. In all of these papers, the authors found cases where the vortices are able to breach the upper boundary. The exact opposite scenario was considered by Su [97], who proved that if a dipole initially moving away from the surface, the solution will persist over a long time scale. This is in stark contrast to the present work where we ask the dipole to move with the wave.

3.1.4 Plan of the article

This chapter contains two main sections. In Section 3.2, we show the existence of traveling capillary-gravity waves with a finite dipole. This follows from an implicit function theorem argument in the spirit of Varholm [44] and Shatah, Walsh, and Zeng [33]. Then, in Section 3.3, we prove that these waves are orbitally unstable.

We first establish that (3.1.1)–(3.1.4) can be formulated as an infinite-dimensional Hamiltonian system of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = J(u)DE(u)$$

with J being the Poisson map and E the energy functional. This is similar but distinct from the version due to Rouhi and Wright [108]. We offer a rigorous derivation in Theorem 3.3.2.

In two seminal papers [112, 113], Grillakis, Shatah, and Strauss provided a fairly simple method for determining the stability or instability of traveling wave solutions to systems of this form that are invariant under a continuous symmetry group. Among the hypotheses of this theory are that the Poisson map J is invertible, and that the initial value problem is globally well-posed in time.

Unfortunately, our J is state-dependent and not surjective. Moreover, we do not expect the problem to be well-posed in the energy space.

In this work, we will use a recent variant of the GSS method developed by Varholm, Wahlén, and Walsh [107]. Among other improvements, this machinery permits J to have merely a dense range, and also allow for a mismatch between the space where the problem is well-posed and the natural energy space. In the present context, the latter point relates to the fact that $\mathbb{W} \subsetneq \mathbb{X}$. As one of the hypotheses, we must compute the spectrum of the second variation of the augmented Hamiltonian defined in subsection 3.3.2. In particular, we show that it has a Morse index of 1.

For the convenience of the reader, several appendices have been included. Appendix B.1 contains a summary of the instability theory developed by Varholm–Wahlén–Walsh [107]. We also derive steady and unsteady equations for the velocity potential and stream functions in Appendix B.2. Finally, in Appendix B.3, we recorded the variations of the energy and momentum functional.

3.2 Existence theory

This section is devoted to proving the existence of traveling capillary-gravity water waves with a finite dipole. We will adopt a methodology introduced by Varholm [44] and Shatah, Walsh, and Zeng [33]. The first step is to reformulate (3.1.1)–(3.1.4) in the spirit of Zakharov [109], and Craig and Sulem [114]. This entails reducing the problem to a nonlocal system involving only surface variables.

Recalling the splitting for v in (3.1.5), we take $\Theta = \Theta_1 + \Theta_2 + \Theta_1^* + \Theta_2^*$ with

$$
\begin{aligned}
\Theta_1(x) &= -\frac{\gamma_1}{2\pi} \arctan\left(\frac{x_1 - \bar{x}_1}{|x - \bar{x}| + x_2 - \bar{x}_2}\right), \quad \Theta_1^*(x) = \frac{\gamma_1}{2\pi} \arctan\left(\frac{x_1 - \bar{x}_1^*}{|x - \bar{x}^*| + x_2 - \bar{x}_2^*}\right), \\
\Theta_2(x) &= \frac{\gamma_2}{2\pi} \arctan\left(\frac{x_1 - \bar{y}_1}{|x - \bar{y}| + x_2 - \bar{y}_2}\right), \quad \Theta_2^*(x) = -\frac{\gamma_2}{2\pi} \arctan\left(\frac{x_1 - \bar{y}_1^*}{|x - \bar{y}^*| + x_2 - \bar{y}_2^*}\right),
\end{aligned}
\tag{3.2.1}
$$

and $\bar{x}^* = (\bar{x}_1, -\bar{x}_2)$ and $\bar{y}^* = (\bar{y}_1, -\bar{y}_2)$ being the reflection of the two point vortices over the x_1-axis. This corresponds to making a branch cut straight down from the vortex centers. It is easy to see that $v \in L^2 + L^2$ in the complement of any neighborhood of $\bar{x}$ and $\bar{y}$.

It is often convenient to work with the harmonic conjugates of these functions. In particular, let

Γ be the harmonic conjugate of Θ, that is $\nabla\Theta = \nabla^\perp\Gamma$. Then we have $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_1^* + \Gamma_2^*$, where

$$\Gamma_1(x) = \frac{\gamma_1}{2\pi}\log|x-\bar{x}|, \qquad \Gamma_2(x) = -\frac{\gamma_2}{2\pi}\log|x-\bar{y}|,$$
$$\Gamma_1^*(x) = -\frac{\gamma_1}{2\pi}\log|x-\bar{x}^*|, \qquad \Gamma_2^*(x) = \frac{\gamma_2}{2\pi}\log|x-\bar{y}^*|.$$

We see that $-\Delta\Gamma_1 = -\gamma_1\delta_{\bar{x}}$, $-\Delta\Gamma_2 = \gamma_2\delta_{\bar{y}}$, and $-\Delta\Gamma = -\gamma_1\delta_{\bar{x}} + \gamma_2\delta_{\bar{y}}$, and hence

$$-\epsilon\Delta\Gamma = \omega.$$

Define

$$\Xi_1 := \Theta_1 - \Theta_1^*, \quad \Xi_2 := \Theta_2 - \Theta_2^*, \quad \Upsilon_1 := \Theta_1 + \Theta_1^*, \quad \Upsilon_2 := \Theta_2 + \Theta_2^*,$$

so that $\Theta = \Upsilon_1 + \Upsilon_2$. This will be convenient for computing $\partial_{\bar{x}}\Theta$. Also, let Ψ be the harmonic conjugate of Φ, so that

$$v = \nabla^\perp\Psi + \epsilon\nabla^\perp\Gamma,$$

and denote the restriction of Ψ to the surface S_t by

$$\psi = \psi(t, x_1) := \Psi(t, x_1, \eta(t, x_1)). \tag{3.2.2}$$

We represent the normal derivatives of these functions on the free surface using the Dirichlet–Neumann operator $\mathcal{G}(\eta) : \dot{H}^{1/2}(\mathbb{R}) \cap \dot{H}^k(\mathbb{R}) \to \dot{H}^{k-1}(\mathbb{R})$ defined by

$$\mathcal{G}(\eta)\phi := (-\eta'\partial_{x_1}\phi_{\mathcal{H}} + \partial_{x_2}\phi_{\mathcal{H}})|_{S_t}, \tag{3.2.3}$$

where $\phi_{\mathcal{H}} \in \dot{H}^{k+1/2}(\Omega)$ is the harmonic extension of ϕ to Ω_t determined uniquely by

$$\Delta\phi_{\mathcal{H}} = 0 \text{ in } \Omega, \quad \phi_{\mathcal{H}} = \phi \text{ on } S_t,$$

and $k \geq 0$. It is well known that for any $\eta \in H^{k_0}(\mathbb{R})$, $k_0 > 3/2$, $\mathcal{G}(\eta)$ is a bounded, invertible, and self-adjoint operator between these spaces when $k \in [1-k_0, k_0]$. Moreover, the mapping $\eta \mapsto \mathcal{G}(\eta)$ is C^∞ and $\mathcal{G}(0) = |\partial_{x_1}|$ (see, for example, the book by Lannes [115]).

Next, we rewrite the water wave problem as the following system for the unknowns $(\eta, \varphi, \bar{x}, \bar{y})$:

$$\begin{cases} \partial_t \eta &= \mathcal{G}(\eta)\varphi + \epsilon \nabla_\perp \Theta, \\ \partial_t \varphi &= -\dfrac{1}{2\langle \eta' \rangle^2}\left((\varphi')^2 - 2\eta'\varphi'\mathcal{G}(\eta)\varphi - (\mathcal{G}(\eta)\varphi)^2 \right) - \epsilon \partial_t \Theta - \epsilon \varphi' \partial_{x_1}\Theta - \dfrac{\epsilon^2}{2}|\nabla\Theta|^2 \\ & \quad -\eta + b\dfrac{\eta''}{\langle \eta' \rangle^3}, \\ \partial_t \bar{x} &= \nabla\Phi(\bar{x}) + \epsilon\nabla\Theta_1^*(\bar{x}) + \epsilon\nabla\Theta_2(\bar{x}) + \epsilon\nabla\Theta_2^*(\bar{x}), \\ \partial_t \bar{y} &= \nabla\Phi(\bar{y}) + \epsilon\nabla\Theta_1(\bar{y}) + \epsilon\nabla\Theta_1^*(\bar{y}) + \epsilon\nabla\Theta_2^*(\bar{y}). \end{cases} \tag{3.2.4}$$

Recall that $\varphi = \Phi|_{S_t}$ as in (3.1.6), and describes the irrotational part of the velocity field. Here we have made use of the differential operators

$$\nabla_\perp := \left(-\eta'\partial_{x_1} + \partial_{x_2}\right)\big|_{S_t}, \quad \nabla_\top := \left(\partial_{x_1} + \eta'\partial_{x_2}\right)\big|_{S_t}, \tag{3.2.5}$$

which come naturally as we take derivatives of functions restricted to the free surface.

Note that in (3.2.4), the equation for $\partial_t\eta$ can be derived from the kinematic boundary condition, but now Θ appears as a forcing term. We can see that the evolution of φ is determined by the unsteady Bernoulli equation (B.2.3). Finally, the equations for $\partial_t\bar{x}$ and $\partial_t\bar{y}$ come from the Kirchhoff–Helmholtz model (3.1.4).

Now we are prepared to prove the existence theorem. As this is done in the steady frame, we will simply write $S := S_t$ and $\Omega := \Omega_t$.

Proof of Theorem 3.1.1. For convenience, we prove this result using ψ, which immediately gives the stated theorem in terms of φ. For traveling waves solutions of (3.2.4), we have

$$\eta = \eta(x_1 - ct), \quad \psi = \psi(x_1 - ct), \quad \partial_t\bar{x} = ce_1, \quad \partial_t\bar{y} = ce_1.$$

First we rescale:

$$\eta =: \epsilon\tilde{\eta}, \quad \psi =: \epsilon\widetilde{\psi}, \quad \Psi =: \epsilon\widetilde{\Psi}, \quad c =: \epsilon\tilde{c},$$

so that the steady point vortex motion equations (3.1.4) now become

$$\begin{cases} -\widetilde{\Psi}_{x_2}(\bar{x}) - \Gamma_{2_{x_2}}(\bar{x}) - \Gamma^*_{1_{x_2}}(\bar{x}) - \Gamma^*_{2_{x_2}}(\bar{x}) = \tilde{c}, \\ -\widetilde{\Psi}_{x_2}(\bar{y}) - \Gamma_{1_{x_2}}(\bar{y}) - \Gamma^*_{1_{x_2}}(\bar{y}) - \Gamma^*_{2_{x_2}}(\bar{y}) = \tilde{c}. \end{cases}$$

Then the problem can be expressed as the abstract operator equation

$$\mathcal{F}(\epsilon, \tilde{c}, \gamma_1, \gamma_2; \tilde{\eta}, \widetilde{\psi}, a, \rho) = 0$$

with $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times X \to Y$ given by

$$\begin{aligned}
\mathcal{F}_1 &:= \frac{\epsilon \tilde{c}}{1 + (\epsilon \tilde{\eta}')^2} \Big(\tilde{\psi}' + \epsilon \tilde{\eta}' \mathcal{G}(\epsilon \tilde{\eta}) \tilde{\psi} \Big) + \epsilon \tilde{c} \Gamma_{x_2}|_S + \frac{\epsilon}{2(1 + (\epsilon \tilde{\eta}')^2)} \Big((\tilde{\psi}')^2 + (\mathcal{G}(\epsilon \tilde{\eta}) \tilde{\psi})^2 \Big) \\
&\quad + \frac{\epsilon}{1 + (\epsilon \tilde{\eta}')^2} \Big(\mathcal{G}(\epsilon \tilde{\eta}) \tilde{\psi} \nabla_\perp \Gamma + \tilde{\psi}' \nabla_\top \Gamma \Big) + \frac{\epsilon}{2} |(\nabla \Gamma)|_S|^2 + \tilde{\eta} + \frac{b}{\epsilon} \kappa(\epsilon \tilde{\eta}), \\
\mathcal{F}_2 &:= \epsilon \tilde{c} \tilde{\eta}' + \widetilde{\psi}' + (1, \epsilon \tilde{\eta}')^T \cdot \nabla \Gamma, \qquad (3.2.6) \\
\mathcal{F}_3 &:= \tilde{c} + \Big(\partial_{x_2} \tilde{\psi}_{\mathcal{H}} \Big) (\bar{x}) + \Gamma_{2_{x_2}}(\bar{x}) + \Gamma^*_{1_{x_2}}(\bar{x}) + \Gamma^*_{2_{x_2}}(\bar{x}), \\
\mathcal{F}_4 &:= \tilde{c} + \Big(\partial_{x_2} \tilde{\psi}_{\mathcal{H}} \Big) (\bar{y}) + \Gamma_{1_{x_2}}(\bar{y}) + \Gamma^*_{1_{x_2}}(\bar{y}) + \Gamma^*_{2_{x_2}}(\bar{y}),
\end{aligned}$$

where $\nabla\Gamma$ is evaluated at $x_2 = \epsilon\tilde{\eta}(x_1)$ and X is defined by (3.1.7). We take

$$Y = Y_1 \times Y_2 \times Y_3 \times Y_4 := H_e^{k-2}(\mathbb{R}) \times \Big(\dot{H}_o^{k-2}(\mathbb{R}) \cap \dot{H}_o^{-1/2}(\mathbb{R}) \Big) \times \mathbb{R} \times \mathbb{R}$$

for $k > \frac{3}{2}$ fixed.

It is clear that $\mathcal{F}(\epsilon_0, \tilde{c}_0, \gamma_1^0, \gamma_2^0; \tilde{\eta}_0, \widetilde{\psi}_0, a_0, \rho_0) = 0$ with

$$\begin{aligned}
\epsilon_0 &= 0, \\
\tilde{c}_0 &= -\Gamma_{2_{x_2}}(0, -a_0 + \rho_0) - \Gamma^*_{1_{x_2}}(0, -a_0 + \rho_0) - \Gamma^*_{2_{x_2}}(0, -a_0 + \rho_0) \qquad (3.2.7a) \\
&= -\frac{\gamma_1}{4\pi(a_0 - \rho_0)} + \frac{\gamma_2}{4\pi} \left(\frac{1}{a_0} + \frac{1}{\rho_0} \right),
\end{aligned}$$

and

$$\tilde{\eta}_0 = 0, \qquad \tilde{\psi}_0 = -\Gamma(x_1, 0) = 0, \qquad (3.2.7b)$$

and $\gamma_1^0, \gamma_2^0, a_0, \rho_0 \in \mathbb{R}$ if and only if the compatibility condition (3.1.8) holds. A simple computation shows that

$$\mathscr{L} := \left(D_{\tilde\eta}, D_{\tilde\psi}, \partial_a, \partial_\rho\right) \mathcal{F}\left(0, \tilde c_0, \gamma_1^0, \gamma_2^0; 0, \tilde\psi_0, a_0, \rho_0\right)$$
$$= \begin{pmatrix} g-\alpha^2\partial_{x_1}^2 & 0 & 0 & 0 \\ 0 & \partial_{x_1} & 0 & 0 \\ 0 & \left(\partial_{x_2}\langle\mathcal{H}(0),\cdot\rangle\right)|_{(0,-a_0+\rho_0)} & -\frac{\gamma_1^0}{4\pi(a_0-\rho_0)^2}+\frac{\gamma_2^0}{4\pi a_0^2} & \frac{\gamma_1^0}{4\pi(a_0-\rho_0)^2}+\frac{\gamma_2^0}{4\pi\rho_0^2} \\ 0 & \left(\partial_{x_2}\langle\mathcal{H}(0),\cdot\rangle\right)|_{(0,-a_0-\rho_0)} & -\frac{\gamma_1^0}{4\pi a_0^2}+\frac{\gamma_2^0}{4\pi(a_0+\rho_0)^2} & \frac{\gamma_2^0}{4\pi(a_0+\rho_0)^2}+\frac{\gamma_1^0}{4\pi\rho_0^2} \end{pmatrix} \tag{3.2.8}$$
$$\in \mathcal{L}(X,Y).$$

The invertibility of $\mathscr{L}$ is equivalent to the invertibility of the 2×2 real sub-matrix:

$$\mathscr{T} := \begin{pmatrix} -\dfrac{\gamma_1^0}{4\pi(a_0-\rho_0)^2}+\dfrac{\gamma_2^0}{4\pi a_0^2} & \dfrac{\gamma_1^0}{4\pi(a_0-\rho_0)^2}+\dfrac{\gamma_2^0}{4\pi\rho_0^2} \\ -\dfrac{\gamma_1^0}{4\pi a_0^2}+\dfrac{\gamma_2^0}{4\pi(a_0+\rho_0)^2} & \dfrac{\gamma_2^0}{4\pi(a_0+\rho_0)^2}+\dfrac{\gamma_1^0}{4\pi\rho_0^2} \end{pmatrix}. \tag{3.2.9}$$

By the compatibility condition (3.1.8), we have

$$\det\mathscr{T} = -\frac{\gamma_1^2}{16\pi^2}\frac{6(a_0^4-a_0^2\rho_0^2+\rho_0^4)}{(a_0+\rho_0)(a_0-\rho_0)^3(a_0^2+a_0\rho_0+\rho_0^2)^2} < 0.$$

Thus, $\mathscr{L}$ is an isomorphism. The Implicit Function Theorem then tells us that there exists a family $\mathcal{C}_{\mathrm{loc}}$ of solutions of the form

$$\mathcal{F}(\epsilon,\tilde c,\gamma_1,\gamma_2;\tilde\eta(\epsilon,\tilde c,\gamma_1,\gamma_2),\tilde\psi(\epsilon,\tilde c,\gamma_1,\gamma_2),a(\epsilon,\tilde c,\gamma_1,\gamma_2),\rho(\epsilon,\tilde c,\gamma_1,\gamma_2)) = 0$$

for all $|\epsilon|\ll 1$, $|\tilde c-\tilde c_0|\ll 1$, $|\gamma_1-\gamma_1^0|\ll 1$, and $|\gamma_2-\gamma_2^0|\ll 1$. Theorem 3.1.1 has, therefore, been proved. Again, the Implicit Function Theorem allows us to infer that $\mathscr{C}_{\mathrm{loc}}$ comprises all traveling wave solutions in a neighborhood of 0 in X. $\square$

For the stability analysis, we rely on asymptotic information about the traveling waves constructed above. Using implicit differentiation, one can readily compute that

$$\begin{aligned}\eta(\epsilon,\tilde c,\gamma_1,\gamma_2) &= -\epsilon^2(g-b\partial_{x_1}^2)^{-1}\left[\tilde c_0\Gamma_{x_2}(x_1,0)\right] \\ &\quad + O(|\epsilon|^3+|\epsilon||c-c_0|^2+|\epsilon||\gamma_1-\gamma_1^0|^2+|\epsilon||\gamma_2-\gamma_2^0|^2) \quad \text{in } C^1(U;X_1), \end{aligned} \tag{3.2.10a}$$
$$\psi(\epsilon,\tilde c,\gamma_1,\gamma_2) = O(|\epsilon|^3+|\epsilon||c-c_0|^2+|\epsilon||\gamma_1-\gamma_1^0|^2+|\epsilon||\gamma_2-\gamma_2^0|^2) \quad \text{in } C^1(U;X_2),$$

and

$$
\begin{aligned}
a(\epsilon, \tilde{c}, \gamma_1, \gamma_2) &= a_0 + \epsilon a_{0100}(c - c_0) + a_{0010}(\gamma_1 - \gamma_1^0) + a_{0001}(\gamma_2 - \gamma_2^0) \\
&\qquad + O(|\epsilon|^2 + |c - c_0|^2 + |\gamma_1 - \gamma_1^0|^2 + |\gamma_2 - \gamma_2^0|^2) \quad \text{in } C^1(U; \mathbb{R}), \\
\rho(\epsilon, \tilde{c}, \gamma_1, \gamma_2) &= \rho_0 + \epsilon \rho_{0100}(c - c_0) + \rho_{0010}(\gamma_1 - \gamma_1^0) + \rho_{0001}(\gamma_2 - \gamma_2^0) \\
&\qquad + O(|\epsilon|^2 + |c - c_0|^2 + |\gamma_1 - \gamma_1^0|^2 + |\gamma_2 - \gamma_2^0|^2) \quad \text{in } C^1(U; \mathbb{R}),
\end{aligned}
\tag{3.2.10b}
$$

where $\widetilde{c}_0$, $\mathscr{T}$ are given by (3.2.7)–(3.2.9), and $U = B_{\epsilon_1}(0) \times B_{c_1}(c_0) \times B_{\gamma_1^1}(\gamma_1^0) \times B_{\gamma_2^1}(\gamma_2^0)$. The indices 0100, 0010, and 0001 are variations at the point $(0, \tilde{c}_0, \gamma_1^0, \gamma_2^0)$ with respect to $\tilde{c}$, γ_1, and γ_2, respectively. In particular,

$$
\begin{aligned}
a_{0100} &= \frac{1}{\det \mathscr{T}} \left(-\frac{\gamma_2^0}{4\pi(a_0 + \rho_0)^2} - \frac{\gamma_1^0}{4\pi\rho_0^2} + \frac{\gamma_1^0}{4\pi(a_0 - \rho_0)^2} + \frac{\gamma_2^0}{4\pi\rho_0^2} \right), \\
\rho_{0100} &= \frac{1}{\det \mathscr{T}} \left(\frac{\gamma_2^0}{4\pi(a_0 + \rho_0)^2} - \frac{\gamma_1^0}{4\pi a_0^2} + \frac{\gamma_1^0}{4\pi(a_0 - \rho_0)^2} - \frac{\gamma_2^0}{4\pi a_0^2} \right).
\end{aligned}
\tag{3.2.11}
$$

3.3 Instability theory

In this section, we show that the traveling waves constructed in Theorem 3.1.1 are orbitally unstable. To do so, we follow the general strategy of Varholm–Wahlén–Walsh [107] which is an adaptation of the classical Grillakis–Shatah–Strauss method [112, 113]. In subsection 3.3.1, we rewrite the equations of capillary-gravity waves with a finite dipole (3.1.1)–(3.1.4) as a Hamiltonian system and give an explicit form for its energy and momentum. Next, in subsections 3.3.2 we prove that the spectrum of the second variation of the augmented Hamiltonian has the required configuration. This is done in the spirit of Mielke [116]. Finally, in subsection 3.3.3, we complete the proof of our main result by computing the second derivative of the moment of instability for small waves in this family.

3.3.1 Hamiltonian formulation

We first show that the system of equations (3.2.4) has a Hamiltonian structure in terms of the state variable $u = (\eta, \varphi, \bar{x}, \bar{y})^T$. Define the energy $E = E(u)$ to be

$$
E(u) := K(u) + V(u), \tag{3.3.1}
$$

where K is the (excess) kinetic energy and V is the (excess) potential energy. The submerged dipole does not affect the latter, and so it we may take

$$V(u) := \int_{\mathbb{R}} \left(\frac{1}{2} g \eta^2 + b(\langle \eta' \rangle - 1) \right) \mathrm{d}x_1. \tag{3.3.2}$$

However, some care is needed in the deriving the correct expression for K. Formally, we take the classical kinetic energy $\frac{1}{2}\int_\Omega |v|^2 \, \mathrm{d}x$, split v according to (3.1.5), and then integrate by parts. The Newtonian potentials in Γ will naturally lead to singular terms; these we discard. What results is the following:

$$\begin{aligned} K(u) &:= K_0(u) + \epsilon K_1(u) + \epsilon^2 K_2(u) \\ &= \frac{1}{2}\int_{\mathbb{R}} \varphi \mathcal{G}(\eta)\varphi \, \mathrm{d}x_1 + \epsilon \int_{\mathbb{R}} \varphi \nabla_\perp \Theta \, \mathrm{d}x_1 + \epsilon^2 \left(\frac{1}{2} \int_{\mathbb{R}} \Theta|_{S_t} \nabla_\perp \Theta \, \mathrm{d}x_1 + \Gamma^* \right), \\ \Gamma^* &:= \frac{\gamma_1}{2}\Big(\Gamma_1^*(\bar{x}) + \Gamma_2(\bar{x}) + \Gamma_2^*(\bar{x})\Big) - \frac{\gamma_2}{2}\Big(\Gamma_1(\bar{y}) + \Gamma_1^*(\bar{y}) + \Gamma_2^*(\bar{y})\Big). \end{aligned}$$

Note that $K_0 = \frac{1}{2}\int_\Omega |\nabla \Phi|^2 \, \mathrm{d}x$, and hence represents the kinetic energy contributed by the purely irrotational part of the velocity. K_1 is the interaction between the irrotational and rotational parts, and K_2 is the kinetic energy attributed to the rotational part.

Recall the energy space $\mathbb{X}$ was defined by (3.1.9) and the well-posedness space $\mathbb{W}$ was defined by (3.1.10). As $\mathbb{X}$ is a Hilbert space, it is isomorphic to its continuous dual $\mathbb{X}^*$, and the isomorphism $I : \mathbb{X} \to \mathbb{X}^*$ takes the form

$$I = (1 - \partial_{x_1}^2, |\partial_{x_1}|, \mathrm{Id}_{\mathbb{R}^2}, \mathrm{Id}_{\mathbb{R}^2}),$$

where $\mathrm{Id}_{\mathbb{R}^2}$ is the 2×2 identity matrix. For the Dirichlet-Neumann operator in E to be well-defined, we want a smoother space than $\mathbb{X}$. For that, we choose

$$\mathbb{V} := \mathbb{V}_1 \times \mathbb{V}_2 \times \mathbb{V}_3 \times \mathbb{V}_4 := H^{3/2+}(\mathbb{R}) \times \left(\dot{H}^{1+}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R}) \right) \times \mathbb{R}^2 \times \mathbb{R}^2. \tag{3.3.3}$$

From the definition of the energy in (3.3.1), we see that $E \in C^\infty(\mathcal{O} \cap \mathbb{V}; \mathbb{R})$. Using the Gagliardo–Nirenberg interpolation inequality, we can confirm that the spaces $\mathbb{X}$, $\mathbb{V}$, and $\mathbb{W}$ satisfy Assumption B.1.1. In particular, there are constants $C > 0$ and $\theta \in (0, 1/4)$ such that

$$\|u\|_{\mathbb{V}}^3 \leq C \|u\|_{\mathbb{X}}^{2+\theta} \|u\|_{\mathbb{W}}^{1-\theta}.$$

Next, consider the closed operator $\widehat{J} : \mathcal{D}(\widehat{J}) \subset \mathbb{X}^* \to \mathbb{X}$ defined by

$$\widehat{J} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & (\epsilon\gamma_1)^{-1}\mathcal{J} & 0 \\ 0 & 0 & 0 & -(\epsilon\gamma_2)^{-1}\mathcal{J} \end{pmatrix}, \tag{3.3.4}$$

where $\mathcal{J}$ is a 2×2 real matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\mathcal{D}(\widehat{J}) = (H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})) \times (H^1(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R})) \times \mathbb{R}^2 \times \mathbb{R}^2.$$

Let $B \in C^1(\mathcal{O}; \mathrm{Lin}(\mathbb{X})) \cap C^1(\mathcal{O} \cap \mathbb{W}; \mathrm{Lin}(\mathbb{W}))$ be defined by

$$B(u) := \mathrm{Id}_{\mathbb{X}} + Z(u), \tag{3.3.5}$$

where

$$Z(u)\dot{w} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\epsilon(\gamma_1)^{-1}(\mathcal{J}\xi|_{S_t})^T & \epsilon(\gamma_2)^{-1}(\mathcal{J}\zeta|_{S_t})^T & \epsilon\xi^T|_{S_t} & \epsilon\zeta^T|_{S_t} \\ \gamma_1^{-1}\mathcal{J} & 0 & 0 & 0 \\ 0 & -(\gamma_2)^{-1}\mathcal{J} & 0 & 0 \end{pmatrix} \begin{bmatrix} \langle \xi|_{S_t}, \dot{\eta} \rangle \\ \langle \zeta|_{S_t}, \dot{\eta} \rangle \\ \dot{\bar{x}} \\ \dot{\bar{y}} \end{bmatrix}$$

for all $\dot{w} = (\dot{\eta}, \dot{\varphi}, \dot{\bar{x}}, \dot{\bar{y}})^T \in \mathcal{O}$ with

$$\xi := -\nabla_{\bar{x}}\Theta = (\Upsilon_{1_{x_1}}, \Xi_{1_{x_2}})^T, \qquad \zeta := -\nabla_{\bar{y}}\Theta = (\Upsilon_{2_{x_1}}, \Xi_{2_{x_2}})^T.$$

The next lemma constructs the Poisson map J in the Hamiltonian formulation and verifies that it satisfies Assumption B.1.2.

Lemma 3.3.1 (Properties of J)**.** *For each $u \in \mathcal{O}$, let $J(u) : \mathcal{D}(\widehat{J}) \subset \mathbb{X}^* \to \mathbb{X}$ be defined by*

$$J(u) := B(u)\widehat{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & J_{22} & J_{23} & J_{24} \\ 0 & J_{32} & (\epsilon\gamma_1)^{-1}\mathcal{J} & 0 \\ 0 & J_{42} & 0 & -(\epsilon\gamma_2)^{-1}\mathcal{J} \end{pmatrix}, \tag{3.3.6}$$

where

$$J_{22} := \epsilon\Upsilon_{1_{x_1}}|_{S_t} \ \langle \cdot, -\gamma_1^{-1}\Xi_{1_{x_2}}|_{S_t}\rangle + \epsilon\Xi_{1_{x_2}}|_{S_t} \ \langle \cdot, \gamma_1^{-1}\Upsilon_{1_{x_1}}|_{S_t}\rangle$$
$$+ \epsilon\Upsilon_{2_{x_1}}|_{S_t} \ \langle \cdot, \gamma_2^{-1}\Xi_{2_{x_2}}|_{S_t}\rangle + \epsilon\Xi_{2_{x_2}}|_{S_t} \ \langle \cdot, -\gamma_2^{-1}\Upsilon_{2_{x_1}}|_{S_t}\rangle,$$

$$J_{23} := \left(\gamma_1^{-1}\Xi_{1_{x_2}}|_{S_t}, \quad -\gamma_1^{-1}\Upsilon_{1_{x_1}}|_{S_t}\right), \quad J_{24} := \left(-\gamma_2^{-1}\Xi_{2_{x_2}}|_{S_t}, \quad \gamma_2^{-1}\Upsilon_{2_{x_1}}|_{S_t}\right),$$

$$J_{32} := \left(\langle \cdot, -\gamma_1^{-1}\Xi_{1_{x_2}}|_{S_t}\rangle, \quad \langle \cdot, \gamma_1^{-1}\Upsilon_{1_{x_1}}|_{S_t}\rangle\right)^T, \quad J_{42} := \left(\langle \cdot, \gamma_2^{-1}\Xi_{2_{x_2}}|_{S_t}\rangle, \quad \langle \cdot, -\gamma_2^{-1}\Upsilon_{2_{x_1}}|_{S_t}\rangle\right)^T.$$

Then $J(u)$ satisfies Assumption B.1.2.

Proof. We see from its definition in (3.3.4) that $\widehat{J}$ is injective and its domain $\mathcal{D}(\widehat{J})$ is dense in $\mathbb{X}^* = I\mathbb{X}$, which proves that part (i) and (ii) of Assumption B.1.2 are satisfied. Moreover, since $B(u)$ defined by (3.3.5) is both Fredholm index 0 and injective, it is an isomorphism on $\mathbb{X}$ and $\mathbb{W}$ giving part (iii). Parts (iv) and (v) of Assumption B.1.2 follow directly from the definitions (3.3.5) and (3.3.6). □

In order to apply the instability theory in Appendix B.1, we must show that DE can be realized as an element of $\mathbb{X}^*$. With that in mind, we define the extension $\nabla E \in C^0(\mathcal{O} \cap \mathbb{V}; \mathbb{X}^*)$ by

$$\langle \nabla E(u), v\rangle_{\mathbb{X}^* \times \mathbb{X}} := \langle E'_\eta(u), v_1\rangle_{H^{-1} \times H^1} + \langle E'_\varphi(u), v_2\rangle_{\dot{H}^{-1/2} \times \dot{H}^{1/2}}$$
$$+ (E'_{\bar{x}}(u), v_3)_{\mathbb{R}^2} + (E'_{\bar{y}}(u), v_4)_{\mathbb{R}^2}, \tag{3.3.7}$$

where

$$E'_\varphi(u) := \mathcal{G}(\eta)\varphi + \epsilon\nabla_\perp\Theta,$$

$$E'_\eta(u) := \frac{1}{2}\int_{\mathbb{R}} \varphi\langle D_\eta \mathcal{G}(\eta)\cdot, \varphi\rangle \,\mathrm{d}x_1 + \epsilon\varphi'\Theta_{x_1}|_{S_t} + \frac{\epsilon^2}{2}|(\nabla\Theta)|_{S_t}|^2 + g\eta - b\left(\frac{\eta'}{\langle\eta'\rangle}\right)',$$

$$E'_{\bar{x}}(u) := -\epsilon\int_{\mathbb{R}} \varphi\nabla_\perp\xi \,\mathrm{d}x_1 - \frac{\epsilon^2}{2}\int_{\mathbb{R}}\left(\xi\nabla_\perp\Theta + \Theta|_{S_t}\nabla_\perp\xi\right)\mathrm{d}x_1 + \nabla_{\bar{x}}\Gamma^*,$$

$$E'_{\bar{y}}(u) := -\epsilon\int_{\mathbb{R}} \varphi\nabla_\perp\zeta \,\mathrm{d}x_1 - \frac{\epsilon^2}{2}\int_{\mathbb{R}}\left(\zeta\nabla_\perp\Theta + \Theta|_{S_t}\nabla_\perp\zeta\right)\mathrm{d}x_1 + \nabla_{\bar{y}}\Gamma^*.$$

Here we use subscripts x_1 and x_2 to denote partial derivatives. See Appendix B.3 for details.

Theorem 3.3.2 (Hamiltonian formulation)**.** *The capillary-gravity water wave with a finite dipole problem* (3.2.4) *has a solution* $u := (\eta, \varphi, \bar{x}, \bar{y})^T \in C^1([0, t_0); \mathcal{O}\cap\mathbb{W})$ *if and only if it is a solution to the abstract Hamiltonian system*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = J(u)DE(u), \tag{3.3.8}$$

where E is the energy functional defined in (3.3.1) *and J is the skew-symmetric operator defined by* (3.3.6)*.*

Proof. Throughout the proof, we make repeated use of the identities

$$\nabla_\perp\psi_{x_1} = \nabla_\top\psi_{x_2} = \left(\psi_{x_2}|_{S_t}\right)', \quad \nabla_\perp\psi_{x_2} = -\nabla_\top\psi_{x_1} = -\left(\psi_{x_1}|_{S_t}\right)', \tag{3.3.9}$$

where ψ is any function harmonic in a neighborhood of S_t, and recall $\nabla_\perp$ and $\nabla_\top$ are defined in (3.2.5).

Suppose we have a solution u of (3.3.8). From the expressions for J in (3.3.6) and the differential equation (3.3.8), we see that

$$\partial_t\eta = D_\varphi E(u) = \mathcal{G}(\eta)\varphi + \epsilon\nabla_\perp\Theta,$$

which is the kinematic condition (3.1.3).

Next, we verify that

$$\partial_t\bar{x} = J_{32}\varphi + (\epsilon\gamma_1)^{-1}\mathcal{J}\nabla_{\bar{x}}E(u)$$

is equivalent to the ODE for $\bar{x}$ in (3.1.4). Explicitly, the first component of the equation is

$$\partial_t\bar{x}_1 = -(\epsilon\gamma_1)^{-1}\partial_{\bar{x}_2}E(u) + \left\langle D_\varphi E(u), -\gamma_1^{-1}\Xi_{1_{x_2}}|_{S_t}\right\rangle. \tag{3.3.10}$$

Using the fact that

$$\nabla_\top f = \partial_{x_1}\left(f|_S\right), \quad \nabla_\perp\Phi = \nabla_\top\Psi, \quad \nabla_\top\Phi = -\nabla_\perp\Psi, \quad \nabla_\perp\Theta = \nabla_\top\Gamma, \quad \nabla_\top\Theta = -\nabla_\perp\Gamma,$$

and the identities (3.3.9), (3.3.10) becomes

$$\begin{aligned}
\partial_t \bar{x} &= \frac{1}{\gamma_1}\int_{\mathbb{R}} \left(-\Xi_{1_{x_1}}|_{S_t}\nabla_\perp \Psi + \Psi|_{S_t}\nabla_\perp \Xi_{1_{x_1}}\right)\,\mathrm{d}x_1 \\
&\quad + \frac{\epsilon}{2\gamma_1}\int_{\mathbb{R}} \left(-\Xi_{1_{x_1}}|_{S_t}\nabla_\perp \Gamma - \Xi_{1_{x_2}}|_{S_t}\nabla_\perp \Theta\right)\,\mathrm{d}x_1 - \epsilon\Gamma^*_{1_{x_2}}(\bar{x}) - \epsilon\Gamma_{2_{x_2}}(\bar{x}) - \epsilon\Gamma^*_{2_{x_2}}(\bar{x}) \\
&=: \frac{1}{\gamma_1}\mathscr{A} + \frac{\epsilon}{2\gamma_1}\mathscr{B} - \epsilon\Gamma^*_{1_{x_2}}(\bar{x}) - \epsilon\Gamma_{2_{x_2}}(\bar{x}) - \epsilon\Gamma^*_{2_{x_2}}(\bar{x}).
\end{aligned}$$

Since Ψ and Θ_1^* are harmonic in Ω_t, for any $0 < r \ll 1$ we have

$$\begin{aligned}
\mathscr{A} &= -\int_{\partial B_r(\bar{x})} \Big(-\Theta_{1_{x_1}} N\cdot\nabla\Psi + \Psi N\cdot\nabla\Theta_{1_{x_1}}\Big)\,\mathrm{d}S_t \\
&= -\int_{\partial B_r(\bar{x})} \Big(\frac{\gamma_1}{2\pi}\frac{x_2-\bar{x}_2}{|x-\bar{x}|^2} N\cdot\nabla\Psi + \Psi\frac{\gamma_1}{2\pi}\frac{x_2-\bar{x}_2}{|x-\bar{x}|^3}\Big)\,\mathrm{d}S_t \\
&= -\frac{\gamma_1}{2\pi}\int_0^{2\pi} \Big(\frac{r\sin\theta}{r^2}\partial_r(\Psi) + \Psi\frac{r\sin\theta}{r^3}\Big) r\,\mathrm{d}\theta.
\end{aligned}$$

Expanding Ψ around $r = 0$ gives

$$\begin{aligned}
\mathscr{A} = -\frac{\gamma_1}{2\pi}\int_0^{2\pi} \Big[&\sin\theta(\Psi_{x_1}\cos\theta + \Psi_{x_2}\sin\theta) + \frac{\sin\theta}{r}(\Psi(\bar{x}) + \Psi_{x_1}(\bar{x})r\cos\theta \\
&+ \Psi_{x_2}(\bar{x})r\sin\theta)\Big]\,\mathrm{d}\theta + o(r) = -\gamma_1\Psi_{x_2}(\bar{x}) + o(r)
\end{aligned}$$

as $r \to 0$. A direct computation along the same lines shows $\mathscr{B} = 0$. Thus, (3.3.10) is equivalent to

$$\partial_t \bar{x}_1 = -\Psi_{x_2}(\bar{x}) - \epsilon\Gamma^*_{1_{x_2}}(\bar{x}) - \epsilon\Gamma_{2_{x_2}}(\bar{x}) - \epsilon\Gamma^*_{2_{x_2}}(\bar{x}),$$

which agrees the Kirchhoff–Helmholtz model (3.1.4). By nearly identical arguments, we likewise confirm that the same holds for $\partial_t \bar{x}_2$ and then $\partial_t \bar{y}$.

Finally, we claim that

$$\begin{aligned}
\partial_t \varphi = -D_\eta E(u) &+ \xi|_{S_t}\cdot(\gamma_1^{-1}\mathcal{J})\nabla_{\bar{x}}E(u) + \zeta|_{S_t}\cdot(\gamma_2^{-1}\mathcal{J})\nabla_{\bar{y}}E(u) \\
&+ \epsilon\xi|_{S_t}\left\langle D_\varphi E(u), (-\gamma_1^{-1}\mathcal{J})\xi\right\rangle + \epsilon\zeta|_{S_t}\left\langle D_\varphi E(u), (-\gamma_2^{-1}\mathcal{J})\zeta\right\rangle
\end{aligned} \tag{3.3.11}$$

is equivalent to the unsteady Bernoulli condition in (3.2.4). By a well-known formula for the derivative $\mathcal{G}(\eta)$ (see, for example, [116, Proposition 2.1]), we know that

$$\int_{\mathbb{R}} \varphi\langle D_\eta \mathcal{G}(\eta)\dot{\eta}, \varphi\rangle\,\mathrm{d}x_1 = \int_{\mathbb{R}} \frac{1}{\langle\eta'\rangle^2}\Big((\varphi')^2 - (\mathcal{G}(\eta)\varphi)^2 - 2\eta'\varphi'\mathcal{G}(\eta)\varphi\Big)\dot{\eta}\,\mathrm{d}x_1.$$

Then

$$\partial_t\varphi = -\frac{1}{\langle\eta'\rangle^2}\Big((\varphi')^2 - (\mathcal{G}(\eta)\varphi)^2 - 2\eta'\varphi'\mathcal{G}(\eta)\varphi\Big) + \epsilon\varphi'\Gamma_{x_2}|_{S_t} - \frac{\epsilon^2}{2}|(\nabla\Theta)|_{S_t}|^2 - V'_\eta(u)$$
$$+ \epsilon\Theta_{1_{x_1}}|_{S_t}\,\partial_t\bar{x}_1 + \epsilon\Theta_{1_{x_2}}|_{S_t}\,\partial_t\bar{x}_2 + \epsilon\Theta_{2_{x_1}}|_{S_t}\,\partial_t\bar{y}_1 + \epsilon\Theta_{2_{x_2}}|_{S_t}\,\partial_t\bar{y}_2.$$

Here we have used the fact that for $\Theta = (\Theta_1 + \Theta_1^* + \Theta_2 + \Theta_2^*)(x_1, x_2, \bar{x}, \bar{y})$,

$$\begin{aligned}(\partial_t\Theta)|_S &= (-\Theta_{1_{x_1}} - \Theta^*_{1_{x_1}})|_{S_t}\,\partial_t\bar{x}_1 + (-\Theta_{1_{x_2}} + \Theta^*_{1_{x_2}})|_{S_t}\,\partial_t\bar{x}_2 + (-\Theta_{2_{x_1}} - \Theta^*_{2_{x_1}})|_{S_t}\,\partial_t\bar{y}_1\\ &\quad + (-\Theta_{2_{x_2}} + \Theta^*_{2_{x_2}})|_{S_t}\,\partial_t\bar{y}_2\\ &= -\Upsilon_{1_{x_1}}|_{S_t}\,\partial_t\bar{x}_1 - \Xi_{1_{x_2}}|_{S_t}\,\partial_t\bar{x}_2 - \Upsilon_{2_{x_1}}|_{S_t}\,\partial_t\bar{y}_1 - \Xi_{2_{x_2}}|_{S_t}\,\partial_t\bar{y}_2.\end{aligned}$$

Thus, comparing this to the equations for φ in (3.2.4), the claim has been proved. □

The momentum associated to a solution of the system (3.3.8) is given by

$$P = P(u) = -\epsilon\gamma_1\bar{x}_2 + \epsilon\gamma_2\bar{y}_2 - \int_{\mathbb{R}} \eta'(\varphi + \epsilon\Theta|_{S_t})\,\mathrm{d}x_1. \tag{3.3.12}$$

It is clear that $P \in C^\infty(\mathcal{O}\cap\mathbb{V};\mathbb{R})$. Similarly to the Fréchet derivatives of the energy, DP can be extended to $\nabla P \in C^0(\mathcal{O}\cap\mathbb{V};\mathbb{X}^*)$:

$$\begin{aligned}\langle\nabla P(u), v\rangle_{\mathbb{X}^*\times\mathbb{X}} &:= \langle P'_\eta(u), v_1\rangle_{H^{-1}\times H^1} + \langle P'_\varphi(u), v_2\rangle_{\dot{H}^{-1/2}\times\dot{H}^{1/2}}\\ &\quad + (P'_{\bar{x}}(u), v_3)_{\mathbb{R}^2} + (P'_{\bar{y}}(u), v_4)_{\mathbb{R}^2}\end{aligned} \tag{3.3.13}$$

with

$$P'_\eta(u) = \varphi' + \epsilon\Theta_{x_1}|_{S_t}, \qquad P'_\varphi(u) = -\eta',$$
$$P'_{\bar{x}}(u) := -\epsilon\gamma_1 e_2 + \epsilon\int_{\mathbb{R}}\eta'\xi|_{S_t}\,\mathrm{d}x_1, \qquad P'_{\bar{y}}(u) := \epsilon\gamma_2 e_2 + \epsilon\int_{\mathbb{R}}\eta'\zeta|_{S_t}\,\mathrm{d}x_1.$$

It is immediate from their definitions in (3.3.7) and (3.3.13) that ∇E and ∇P satisfy Assumption B.1.3. Observe also that ∇P is in $\mathcal{D}(\widehat{J})$ and

$$J(u)\nabla P(u) = (-\eta', -\varphi', 1, 0, 1, 0)^T. \tag{3.3.14}$$

The next lemma records the fact that the momentum and the energy are conserved.

Lemma 3.3.3 (Conservation)**.** *Suppose that* $u \in C^0([0,t_0); \mathcal{O} \cap \mathbb{W})$ *is a distributional solution to the Cauchy problem* (3.3.8) *with initial data* $u_0 \in \mathcal{O} \cap \mathbb{W}$*. Then*

$$E(u(t)) = E(u_0) \quad \textit{and} \quad P(u(t)) = P(u_0) \quad \textit{for all } t \in [0,t_0).$$

The proof follows directly from computation and the regularity of the well-posedness space $\mathbb{W}$; see, [107, Theorem 5.3].

Next, we verify that the symmetry group T defined by (3.1.12) indeed satisfies Assumption B.1.4. The linear part of T is

$$dT(s)u = (\eta(\cdot - s), \varphi(\cdot - s), \bar{x}, \bar{y})^T \quad \text{for all } s \in \mathbb{R}, u \in \mathbb{X}, \tag{3.3.15}$$

and the infinitesimal generator of T is the unbounded affine operator

$$T'(0)u := (-\eta', -\varphi', e_1, e_1)^T \quad \text{for all } u \in \mathcal{D}(T'(0)) \tag{3.3.16}$$

with $\mathcal{D}(T'(0)) = H^2(\mathbb{R}) \times \left(\dot{H}^{3/2}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})\right) \times \mathbb{R}^2 \times \mathbb{R}^2$.

Lemma 3.3.4. *The group* $T(s)$ *satisfies Assumption B.1.4.*

The proof of this lemma is done by nearly identical arguments to that of [107, Lemma 5.4], as the symmetry groups are essentially the same. We therefore omit the details.

Finally, recall that

$$\{U_c = (\eta(c), \varphi(c), \bar{x}(c), \bar{y}(c)) : c \in \mathcal{I}\} \tag{3.3.17}$$

is a one-parameter family of solitary capillary-gravity water waves with a finite dipole constructed in Theorem 3.1.1, where we fix γ_1, γ_2, ϵ, and vary the wave speed c. Here $\mathcal{I}$ is an open interval containing

$$c_0 = -\frac{\epsilon\gamma_1}{4\pi(a_0 - \rho_0)} + \frac{\epsilon\gamma_2}{4\pi}\left(\frac{1}{a_0} + \frac{1}{\rho_0}\right).$$

The next lemma confirms that this family satisfies Assumption B.1.5 of the general stability theory.

Lemma 3.3.5. *Fix any choice of* $0 < \rho_0 < a_0$*, and consider the corresponding surface of solutions* $\mathscr{C}_{\mathrm{loc}}$ *furnished by Theorem 3.1.1. Then the corresponding one-parameter family of bound states* $\{U_c\}_{c \in \mathcal{I}}$ *satisfies Assumption B.1.5.*

Proof. From the proof of Theorem 3.1.1 in Section 3.2, it is clear that $\mathscr{C}_{\mathrm{loc}}$ is in fact C^∞, thus $c \mapsto U_c$ is likewise smooth, and part (i) of the assumption holds. The asymptotic form of the solutions given in (3.2.10) immediately shows that part (ii) holds. Part (iii) follows from the fact that the existence theory can be carried out in H^k spaces for any $k > \frac{3}{2}$. Finally, as these are solitary waves, it is obvious that the second alternative of part (iv) is satisfied. □

3.3.2 Spectrum of the augmented potential

We define the augmented Hamiltonian to be

$$E_c(u) := E(u) - cP(u).$$

The *moment of instability* is the scalar-valued function that results from evaluating E_c along the family $\{U_c\}$:

$$d(c) := E_c(U_c) = E(U_c) - cP(U_c). \tag{3.3.18}$$

By (3.3.8) and (3.3.14), we have $JDE(U_c) - cJDP(U_c) = 0$, and hence

$$DE_c(U_c) = DE(U_c) - cDP(U_c) = 0. \tag{3.3.19}$$

Thus, each traveling wave U_c is a critical point of the augmented Hamiltonian. Then differentiating d gives the identity

$$d'(c) = \left\langle DE(U_c) - cDP(U_c), \frac{\mathrm{d}U_c}{\mathrm{d}c} \right\rangle - P(U_c) = -P(U_c).$$

If we differentiate (3.3.19) with respect to c, we also obtain

$$\left\langle D^2E_c(U_c)\frac{\mathrm{d}U_c}{\mathrm{d}c}, \,\cdot\, \right\rangle = \langle DP(U_c), \,\cdot\, \rangle .$$

As in the work of Grillakis, Shatah, and Strauss [112, 113], we must show that the linearized Hamiltonian has Morse index 1. That is, $D^2E_c(U_c)$ can be associated to a bounded self-adjoint operator on $\mathbb{X}$ whose spectrum takes the form $\{-\mu_c^2\} \cup \{0\} \cup \Sigma_c$, where $\Sigma_c \subset \mathbb{R}_+$ is uniformly bounded away from 0 and $-\mu_c^2 < 0$. This corresponds to Assumption B.1.6.

We first note that 0 is in the spectrum. Indeed, for all $s \in \mathbb{R}$, $T(s)U_c$ is also a traveling wave solution. Therefore,

$$DE_c(T(s)U_c) = 0$$

for all s. Differentiating with respect to s gives

$$\left\langle D^2 E_c(T(s)U_c), T'(0)U_c \right\rangle = 0,$$

and hence, $T'(0)U_c$ is an eigenfunction for eigenvalue 0.

Following Mielke's approach [116], we will determine the remaining spectrum by first considering the *augmented potential*

$$V_c = V_c(\eta, \bar{x}, \bar{y}) := \min_{\varphi \in \mathbb{V}_2} E_c(\eta, \varphi, \bar{x}, \bar{y}) =: E_c(\eta, \varphi_m, \bar{x}, \bar{y}) \tag{3.3.20}$$

for $(\eta, \bar{x}, \bar{y}) \in \mathbb{V}_1 \times \mathbb{V}_3 \times \mathbb{V}_4$, which corresponds to fixing $(\eta, \bar{x}, \bar{y})$ and minimizing E_c over φ. Thus,

$$D_\varphi E_c(\eta, \varphi_m, \bar{x}, \bar{y}) = 0. \tag{3.3.21}$$

Because φ occurs quadratically in the energy, it is easy to see that this minimum is attained exactly when

$$\varphi_m(\eta, \bar{x}, \bar{y}) = \mathcal{G}(\eta)^{-1}[-c\eta' - \epsilon \nabla_\perp \Theta]. \tag{3.3.22}$$

Since we will be doing many calculations where φ is fixed, we adopt the notational convention that for $u = (u_1, u_2, u_3, u_4)$,

$$v := (u_1, u_3, u_4)$$

and write a variation in the direction v as $\dot{v}$. We also use the short hand $\mathbb{V}_{1,3,4} := \mathbb{V}_1 \times \mathbb{V}_3 \times \mathbb{V}_4$, and for convenience, define

$$u_m(v) := (\eta, \varphi_m, \bar{x}, \bar{y}) \in \mathbb{V}.$$

Also, when we evaluate derivatives of the Dirichlet–Neumann operator, we will encounter the quantities

$$\mathfrak{a} := (\nabla(\mathcal{H}\varphi_m))|_S, \quad \mathfrak{b} := \mathfrak{a} + \epsilon(\nabla\Theta)|_S - ce_1.$$

See Appendix B.2 for an explicit formula giving $\mathfrak{a}$ in terms of φ and η. Physically, $\mathfrak{b}$ is the restriction of the full relative velocity to the interface. Therefore, by the kinematic condition, $\mathfrak{b}_2 = \eta' \mathfrak{b}_1$; this also follows directly from (3.3.22).

While it is not completely obvious, we will see that the spectral properties of $D^2E_c(U_c)$ can be inferred from those of $D^2V_c(v)$. With that in mind, the first step is to derive a formula for the second variation of the augmented potential.

Lemma 3.3.6. *For all $v \in \mathbb{V}_{1,3,4} \cap \mathcal{O}_{1,3,4}$ and all variations $\dot{v} \in \mathbb{V}_{1,3,4}$, we have*

$$\begin{aligned}\left\langle D^2V_c(v)\dot{v}, \dot{v}\right\rangle_{\mathbb{V}_{1,3,4}^* \times \mathbb{V}_{1,3,4}} = &- \left\langle \mathcal{L}(v)\dot{v}, \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v}\right\rangle_{\mathbb{X}_2^* \times \mathbb{X}_2} \\ &+ \left\langle D_v^2E_c(u_m(v))\dot{v}, \dot{v}\right\rangle_{\mathbb{V}_{1,3,4}^* \times \mathbb{V}_{1,3,4}}, \end{aligned} \tag{3.3.23}$$

where $\mathcal{L}(v) \in \mathrm{Lin}(\mathbb{X}_{1,3,4}; \mathbb{X}_2^*)$ defined by

$$\mathcal{L}(v)\dot{v} := \mathcal{G}(\eta)(\mathfrak{a}_2\dot{\eta}) + (\mathfrak{b}_1\dot{\eta})' + \epsilon\nabla_\perp\xi \cdot \dot{\bar{x}} + \epsilon\nabla_\perp\zeta \cdot \dot{\bar{y}}. \tag{3.3.24}$$

The proof follows by a straightforward adaptation of [107, Lemma 6.2], and we therefore omit it.

In the next lemma, we refine expression (3.3.23) to derive a quadratic form representation of D^2V_c.

Lemma 3.3.7 (Quadratic form)**.** *For all $v \in \mathbb{V}_{1,3,4} \cap \mathcal{O}_{1,3,4}$, there is a self-adjoint linear operator $A(v) \in \mathrm{Lin}(\mathbb{X}_{1,3,4}; \mathbb{X}_{1,3,4}^*)$ such that*

$$\left\langle D^2V_c(v)\dot{v}, \dot{w}\right\rangle_{\mathbb{V}_{1,3,4}^* \times \mathbb{V}_{1,3,4}} = \langle A\dot{v}, \dot{w}\rangle_{\mathbb{X}_{1,3,4}^* \times \mathbb{X}_{1,3,4}}$$

for all $\dot{v}, \dot{w} \in \mathbb{V}_{1,3,4}$. The form of A is given in (3.3.25).

Proof. From [116, Proposition 2.1], we have

$$\int_{\mathbb{R}} \hat{\varphi}\langle D_\eta\mathcal{G}(\eta)\dot{\eta}, \varphi\rangle \,\mathrm{d}x_1 = \int_{\mathbb{R}} \dot{\eta}(\mathfrak{a}_1\hat{\varphi}' - \mathfrak{a}_2\mathcal{G}(\eta)\hat{\varphi}) \,\mathrm{d}x_1$$

and

$$\int_{\mathbb{R}} \varphi\langle\langle D_\eta^2\mathcal{G}(\eta)\dot{\eta}, \dot{\eta}\rangle, \varphi\rangle \,\mathrm{d}x_1 = 2\int_{\mathbb{R}} \left(\dot{\eta}^2\mathfrak{a}_1'\mathfrak{a}_2 + \mathfrak{a}_2\dot{\eta}\mathcal{G}(\eta)(\mathfrak{a}_2\dot{\eta})\right) \mathrm{d}x_1.$$

Letting the self-adjoint operator $\mathcal{M}$ defined by

$$\mathcal{M}\dot{\eta} := -\mathfrak{b}_1(\mathcal{G}(\eta)^{-1}(\mathfrak{b}_1\dot{\eta})')',$$

and using the fact that $\mathcal{G}(\eta)^{-1}$ is a self-adjoint operator, we can compute

$$
\begin{aligned}
\int_{\mathbb{R}} \mathcal{L}(v)\dot{v}\mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v}\,\mathrm{d}x_1 &= \int_{\mathbb{R}} \mathfrak{a}_2\dot{\eta}\mathcal{G}(\eta)(\mathfrak{a}_2\eta)\,\mathrm{d}x_1 + \int_{\mathbb{R}} \dot{\eta}\mathcal{M}\dot{\eta}\,\mathrm{d}x_1 + \int_{\mathbb{R}} (\mathfrak{a}_2\mathfrak{b}_1' - \mathfrak{a}_2'\mathfrak{b}_1)\dot{\eta}^2\,\mathrm{d}x_1 \\
&+ 2\epsilon\dot{\bar{x}}\cdot\int_{\mathbb{R}} \left(\mathfrak{a}_2\nabla_\perp\xi - \mathfrak{b}_1\left(\mathcal{G}(\eta)^{-1}\nabla_\perp\xi\right)'\right)\dot{\eta}\,\mathrm{d}x_1 + \epsilon^2\dot{\bar{x}}^T\left(\int_{\mathbb{R}} \nabla_\perp\xi\odot\mathcal{G}(\eta)^{-1}\nabla_\perp\xi\,\mathrm{d}x_1\right)\dot{\bar{x}} \\
&+ 2\epsilon\dot{\bar{y}}\cdot\int_{\mathbb{R}} \left(\mathfrak{a}_2\nabla_\perp\zeta - \mathfrak{b}_1\left(\mathcal{G}(\eta)^{-1}\nabla_\perp\zeta\right)'\right)\dot{\eta}\,\mathrm{d}x_1 + \epsilon^2\dot{\bar{y}}^T\left(\int_{\mathbb{R}} \nabla_\perp\zeta\odot\mathcal{G}(\eta)^{-1}\nabla_\perp\zeta\,\mathrm{d}x_1\right)\dot{\bar{y}} \\
&+ 2\epsilon^2\dot{\bar{x}}^T\left(\int_{\mathbb{R}} \nabla_\perp\xi\odot\mathcal{G}(\eta)^{-1}\nabla_\perp\zeta\,\mathrm{d}x_1\right)\dot{\bar{y}},
\end{aligned}
$$

where $x\odot y = (x\otimes y + y\otimes x)/2$ is the symmetric outer product. Next, we have

$$
\left\langle D_\eta^2 E_c(u_m)\dot{\eta},\dot{\eta}\right\rangle = \int_{\mathbb{R}} \mathfrak{a}_2\dot{\eta}\mathcal{G}(\eta)(\mathfrak{a}_2\eta)\,\mathrm{d}x_1 + \int_{\mathbb{R}} \left(g + \epsilon\mathfrak{b}_1\nabla_\top\Theta_{x_2} + \mathfrak{a}_2\mathfrak{b}_1'\right)\dot{\eta}^2\,\mathrm{d}x_1 + \int_{\mathbb{R}} \frac{b}{\langle\eta'\rangle^3}(\dot{\eta}')^2\,\mathrm{d}x_1,
$$

$$
\nabla_{\bar{x}}\langle D_\eta E_c(u_m),\dot{\eta}\rangle = \epsilon\int_{\mathbb{R}} (\mathfrak{a}_2\nabla_\perp\xi - \mathfrak{b}_1\nabla_\top\xi)\dot{\eta}\,\mathrm{d}x_1,
$$

$$
\nabla_{\bar{y}}\langle D_\eta E_c(u_m),\dot{\eta}\rangle = \epsilon\int_{\mathbb{R}} (\mathfrak{a}_2\nabla_\perp\zeta - \mathfrak{b}_1\nabla_\top\zeta)\dot{\eta}\,\mathrm{d}x_1,
$$

$$
\begin{aligned}
D_{\bar{x}}^2 E_c(u_m) = 2\epsilon^2 D_{\bar{x}}^2\Gamma^* - \epsilon\int_{\mathbb{R}} (\mathcal{G}(\eta)\varphi_m D_{\bar{x}}^2\Theta + \varphi_m' D_{\bar{x}}^2\Gamma)|_S\,\mathrm{d}x_1 + \epsilon^2\int_{\mathbb{R}} \nabla_\perp\xi\odot\xi\,\mathrm{d}x_1 \\
- \frac{\epsilon^2}{2}\int_{\mathbb{R}} (\nabla_\perp\Theta D_{\bar{x}}^2\Theta + \nabla_\top\Theta D_{\bar{x}}^2\Gamma)|_S\,\mathrm{d}x_1,
\end{aligned}
$$

$$
\begin{aligned}
D_{\bar{y}}^2 E_c(u_m) = 2\epsilon^2 D_{\bar{y}}^2\Gamma^* - \epsilon\int_{\mathbb{R}} (\mathcal{G}(\eta)\varphi_m D_{\bar{y}}^2\Theta + \varphi_m' D_{\bar{y}}^2\Gamma)|_S\,\mathrm{d}x_1 + \epsilon^2\int_{\mathbb{R}} \nabla_\perp\zeta\odot\zeta\,\mathrm{d}x_1 \\
- \frac{\epsilon^2}{2}\int_{\mathbb{R}} (\nabla_\perp\Theta D_{\bar{y}}^2\Theta + \nabla_\top\Theta D_{\bar{y}}^2\Gamma)|_S\,\mathrm{d}x_1,
\end{aligned}
$$

$$
\nabla_{\bar{x}}\nabla_{\bar{y}} E_c(u_m) = \epsilon^2\nabla_{\bar{x}}\nabla_{\bar{y}}\Gamma^* + \frac{\epsilon^2}{2}\int_{\mathbb{R}} \nabla_\perp(\xi\odot\zeta)\,\mathrm{d}x_1.
$$

Substituting the above results into the expression (3.3.23), we arrive at

$$
\begin{aligned}
\langle D^2 V_c(v)\dot{v}, \dot{v}\rangle = \int_{\mathbb{R}} (g + \mathfrak{b}_2'\mathfrak{b}_1)\dot{\eta}^2 \,\mathrm{d}x_1 - \int_{\mathbb{R}} \left(\frac{b}{\langle \eta' \rangle^3}\dot{\eta}'\right)' \dot{\eta}\,\mathrm{d}x_1 - \int_{\mathbb{R}} \dot{\eta}\mathcal{M}\dot{\eta}\,\mathrm{d}x_1 \\
+ 2\epsilon\dot{\bar{x}} \cdot \int_{\mathbb{R}} \dot{\eta}\mathfrak{b}_1 \nabla_\top (\mathcal{G}(\eta)^{-1}\nabla_\perp \xi - \xi)\,\mathrm{d}x_1 + 2\epsilon\dot{\bar{y}} \cdot \int_{\mathbb{R}} \dot{\eta}\mathfrak{b}_1 \nabla_\top (\mathcal{G}(\eta)^{-1}\nabla_\perp \zeta - \zeta)\,\mathrm{d}x_1 \\
+ \dot{\bar{x}}^T \left(D^2_{\bar{x}} E_c(u_m) - \epsilon^2 \int_{\mathbb{R}} \nabla_\perp \xi \odot \mathcal{G}(\eta)^{-1}\nabla_\perp \xi\,\mathrm{d}x_1 \right) \dot{\bar{x}} \\
+ \dot{\bar{y}}^T \left(D^2_{\bar{y}} E_c(u_m) - \epsilon^2 \int_{\mathbb{R}} \nabla_\perp \zeta \odot \mathcal{G}(\eta)^{-1}\nabla_\perp \zeta\,\mathrm{d}x_1 \right) \dot{\bar{y}} \\
+ \dot{\bar{x}}^T \left(\nabla_{\bar{x}}\nabla_{\bar{y}} E_c(u_m) - 2\epsilon^2 \int_{\mathbb{R}} \nabla_\perp \xi \odot \mathcal{G}(\eta)^{-1}\nabla_\perp \zeta\,\mathrm{d}x_1 \right) \dot{\bar{y}}.
\end{aligned}
$$

Thus, inspecting the above formula, we see that the claimed quadratic form representation holds with the operator A defined as follows:

$$
A_{11}\dot{\eta} := (g + \mathfrak{b}_2'\mathfrak{b}_1)\dot{\eta} - \left(\frac{b}{\langle \eta' \rangle^3}\dot{\eta}'\right)' - \mathcal{M}\dot{\eta}, \tag{3.3.25a}
$$

$$
A_{13}\dot{\bar{x}} := \epsilon\mathfrak{b}_1 \nabla_\top (\mathcal{G}(\eta)^{-1}\nabla_\perp \xi - \xi) \cdot \dot{\bar{x}}, \tag{3.3.25b}
$$

$$
A_{13}^*\dot{\eta} := \epsilon \int_{\mathbb{R}} \dot{\eta}\mathfrak{b}_1 \nabla_\top (\mathcal{G}(\eta)^{-1}\nabla_\perp \xi - \xi)\,\mathrm{d}x_1, \tag{3.3.25c}
$$

$$
A_{14}\dot{\bar{y}} := \epsilon\mathfrak{b}_1 \nabla_\top (\mathcal{G}(\eta)^{-1}\nabla_\perp \zeta - \zeta) \cdot \dot{\bar{y}}, \tag{3.3.25d}
$$

$$
A_{14}^*\dot{\eta} := \epsilon \int_{\mathbb{R}} \dot{\eta}\mathfrak{b}_1 \nabla_\top (\mathcal{G}(\eta)^{-1}\nabla_\perp \zeta - \zeta)\,\mathrm{d}x_1, \tag{3.3.25e}
$$

$$
A_{33} := D^2_{\bar{x}} E_c(u_m) - \epsilon^2 \int_{\mathbb{R}} \nabla_\perp \xi \odot \mathcal{G}(\eta)^{-1}\nabla_\perp \xi\,\mathrm{d}x_1, \tag{3.3.25f}
$$

$$
A_{44} := D^2_{\bar{y}} E_c(u_m) - \epsilon^2 \int_{\mathbb{R}} \nabla_\perp \zeta \odot \mathcal{G}(\eta)^{-1}\nabla_\perp \zeta\,\mathrm{d}x_1, \tag{3.3.25g}
$$

$$A_{34} = A_{43} := \nabla_{\bar{x}} \nabla_{\bar{y}} E_c(u_m) - \epsilon^2 \int_{\mathbb{R}} \nabla_\perp \xi \odot \mathcal{G}(\eta)^{-1} \nabla_\perp \zeta \, \mathrm{d}x_1. \tag{3.3.25h}$$

This finishes the proof of Lemma 3.3.7. $\square$

The next lemma verifies that the second variation of the augmented Hamiltonian E_c has an extension to the energy space $\mathbb{X}$.

Lemma 3.3.8 (Extension of $D^2 E_c$)**.** *For all $v \in \mathbb{V}_{1,3,4} \cap \mathcal{O}_{1,3,4}$, there exists a self-adjoint operator $H_c(v) \in \mathrm{Lin}(\mathbb{X}, \mathbb{X}^*)$ such that*

$$\langle D^2 E_c(u_m(v)) \dot{u}, \dot{w} \rangle_{\mathbb{V}^* \times \mathbb{V}} = \langle H_c(v) \dot{u}, \dot{w} \rangle_{\mathbb{X}^* \times \mathbb{X}} \tag{3.3.26}$$

for all $\dot{u}, \dot{w} \in \mathbb{V}$ with

$$H_c(v)\dot{u} = \begin{pmatrix} \mathrm{Id}_{\mathbb{X}_1^*} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{\mathbb{X}_2^*} \\ 0 & \mathrm{Id}_{\mathbb{R}^2} & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{\mathbb{R}^2} & 0 \end{pmatrix} \begin{pmatrix} A(v) + \mathcal{L}(v)^* \mathcal{G}(\eta)^{-1} \mathcal{L}(v) & -\mathcal{L}(v)^* \\ -\mathcal{L}(v) & \mathcal{G}(\eta) \end{pmatrix} \begin{bmatrix} \dot{v} \\ \dot{\varphi} \end{bmatrix},$$

where $\mathcal{L}(v)$ and $A(v)$ are defined in Lemmas 3.3.6 and 3.3.7, respectively. The adjoint $\mathcal{L}(v)^* \in \mathrm{Lin}(\mathbb{X}_2; \mathbb{X}_{1,3,4}^*)$ is given by

$$\mathcal{L}(v)^* \dot{\varphi} = (\mathfrak{a}_2 \mathcal{G}(\eta)\dot{\varphi} - \mathfrak{b}_1 \dot{\varphi}', \epsilon \langle \nabla_\perp (\xi + \zeta), \dot{\varphi} \rangle),$$

and we have

$$\langle H_c(v)\dot{u}, \dot{u} \rangle_{\mathbb{X}^* \times \mathbb{X}} = \langle A(v)\dot{v}, \dot{v} \rangle_{\mathbb{X}_{1,3,4}^* \times \mathbb{X}_{1,3,4}} + \left\langle \mathcal{G}(\eta)(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}\dot{v}), \dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}\dot{v} \right\rangle_{\mathbb{X}_2^* \times \mathbb{X}_2} \tag{3.3.27}$$

for all $\dot{u} \in \mathbb{X}$.

Proof. It is straightforward to see that

$$\langle D_\varphi D_v E_c(u_m(v))\dot{v}, \dot{\varphi} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} = -\int_{\mathbb{R}} \dot{v} \mathcal{L}(v) \dot{v} \, \mathrm{d}x_1$$

holds for all $\dot{v} \in \mathbb{V}_{1,3,4}$ and $\dot{\varphi} \in \mathbb{V}_2$. Because of symmetry, it suffices to consider only the diagonal entries. For all $\dot{u} \in \mathbb{V}$, Lemmas 3.3.6 and 3.3.7 give

$$\begin{aligned}\langle D^2 E_c(u_m(v))\dot{u}, \dot{u}\rangle_{\mathbb{V}^* \times \mathbb{V}} &= \langle D_v^2 E_c(u_m(v))\dot{v}, \dot{v}\rangle + 2\langle D_\varphi D_v E_c(u_m(v))\dot{v}, \dot{\varphi}\rangle \\ &\quad + \langle D_\varphi^2 E_c(u_m(v))\dot{\varphi}, \dot{\varphi}\rangle \\ &= \langle A(v)\dot{v}, \dot{v}\rangle_{\mathbb{X}_{1,3,4}^* \times \mathbb{X}_{1,3,4}} \\ &\quad + \int_{\mathbb{R}} \Big[(\mathcal{L}(v)\dot{v})\mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v} - 2\dot{\varphi}\mathcal{L}(v)\dot{v} + \dot{\varphi}\mathcal{G}(\eta)\dot{\varphi}\Big]\,\mathrm{d}x_1 \\ &= \langle A(v)\dot{v}, \dot{v}\rangle_{\mathbb{X}_{1,3,4}^* \times \mathbb{X}_{1,3,4}} - \int_{\mathbb{R}} \mathcal{L}(v)\dot{v}(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v})\,\mathrm{d}x_1 \\ &\quad + \int_{\mathbb{R}} \dot{\varphi}\mathcal{G}(\eta)(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v})\,\mathrm{d}x_1.\end{aligned} \tag{3.3.28}$$

Using the fact that $\mathcal{G}(\eta)$ and $\mathcal{G}(\eta)^{-1}$ are self-adjoint operators, the integral is equal to

$$\begin{aligned}&\int_{\mathbb{R}} \Big[- \mathcal{L}(v)\dot{v}(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v}) + (\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v})\mathcal{G}(\eta)\dot{\varphi}\Big]\,\mathrm{d}x_1 \\ &\qquad = \int_{\mathbb{R}} \mathcal{G}(\eta)(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v})(\dot{\varphi} - \mathcal{G}(\eta)^{-1}\mathcal{L}(v)\dot{v})\,\mathrm{d}x_1.\end{aligned}$$

Substituting this into the equation (3.3.28) yields our desired result. □

We finish this subsection by showing that Assumption B.1.6 is satisfied.

Theorem 3.3.9 (Spectrum)**.** *Fix any choice of $0 < \rho_0 < a_0$ subject to the compatibility condition* (3.2.4)*, and consider the family of traveling wave solutions $\{U_c\}_{c \in \mathcal{I}}$ as in* (3.3.17)*. Then for all $c \in \mathcal{I}$, $I^{-1}H_c$ has one negative eigenvalue, 0 is in the spectrum, and the rest of the spectrum $\Sigma_c \subset (0, \infty)$ is bounded away from 0.*

Proof. From the asymptotic information furnished by the existence theorem (3.2.10), we infer that

$$\mathfrak{a}_1 = O(\epsilon^3), \quad \mathfrak{a}_2 = O(\epsilon^3), \quad \mathfrak{b}_1 = \mathfrak{a}_1 - c + o(\epsilon^2) = O(\epsilon), \quad \mathfrak{b}_2 = \mathfrak{a}_2 + o(\epsilon^2) = O(\epsilon).$$

Then from Lemmas 3.3.7 and 3.3.8, we can write

$$H_c = \begin{pmatrix} g - b\partial_{x_1}^2 & 0 & 0 \\ 0 & |\partial_{x_1}| & 0 \\ 0 & 0 & \mathcal{A} \end{pmatrix} + O(\epsilon^3) \in \mathrm{Lin}(\mathbb{X}, \mathbb{X}^*),$$

where

$$\mathcal{A} := \begin{pmatrix} \mathcal{A}_{33} & \mathcal{A}_{34} \\ \mathcal{A}_{43} & \mathcal{A}_{44} \end{pmatrix} = \frac{\epsilon^2}{4\pi} \begin{pmatrix} -\alpha & 0 & \alpha & 0 \\ 0 & \delta_1 + \alpha & 0 & -\beta \\ \alpha & 0 & -\alpha & 0 \\ 0 & -\beta & 0 & \delta_2 + \alpha \end{pmatrix},$$

and

$$\alpha := \frac{\gamma_1\gamma_2}{2}\left(\frac{1}{\rho^2} - \frac{1}{a^2}\right), \quad \beta := \frac{\gamma_1\gamma_2}{2}\left(\frac{1}{\rho^2} + \frac{1}{a^2}\right), \quad \delta_1 := \frac{\gamma_1^2}{(a-\rho)^2}, \quad \delta_2 := \frac{\gamma_2^2}{(a+\rho)^2}.$$

Setting $\epsilon = 0$, it follows that H_c has a zero eigenvalue of multiplicity 4, and the remainder of the spectrum is strictly positive. Thus, when $0 < |\epsilon| \ll 1$, $I^{-1}H_c$ will have positive spectrum $\Sigma_c \subset (0, \infty)$ along with four eigenvalues bifurcating from 0.

To determine these, we look more closely at the matrix $\mathcal{A}$. In particular, direct computation confirms that it has the eigenvalues:

$$0, \quad -2\alpha, \quad \frac{2\alpha + \delta_1 + \delta_2 + \sqrt{(\delta_1 - \delta_2)^2 + 4\beta^2}}{2}, \quad \frac{2\alpha + \delta_1 + \delta_2 - \sqrt{(\delta_1 - \delta_2)^2 + 4\beta^2}}{2}.$$

We know that 0 is in the spectrum of $I^{-1}H_c$ due translation invariance. Clearly, $-2\alpha < 0$, and the third eigenvalue above is positive. We claim that the last eigenvalue is also positive. Indeed,

$$2\alpha + \delta_1 + \delta_2 - \sqrt{(\delta_1 - \delta_2)^2 + 4\beta^2} > 0$$

is equivalent to

$$\alpha^2 + \alpha\delta_1 + \alpha\delta_2 + \delta_1\delta_2 - \beta^2 > 0.$$

Using the compatibility condition (3.1.8), we compute

$$\begin{aligned} \alpha^2 + \alpha\delta_1 + \alpha\delta_2 + \delta_1\delta_2 - \beta^2 &= \frac{\gamma_1\gamma_2}{2}\left(\frac{1}{\rho^2} - \frac{1}{a^2}\right)\left(\frac{\gamma_1^2}{(a-\rho)^2} + \frac{\gamma_2^2}{(a+\rho)^2}\right) - \frac{\gamma_1^2\gamma_2^2}{a^2\rho^2} \\ &= \frac{2(a+\rho)}{(a-\rho)(a^2 + a\rho + \rho^2)^2}\gamma_1^4 > 0. \end{aligned}$$

We then conclude that, for $|\epsilon| > 0$ sufficiently small, the spectrum of $I^{-1}H_c$ consists of precisely one negative eigenvalue, one zero eigenvalue, and the rest is positive. □

We have verified all of the assumptions B.1.1–B.1.6 of Varholm–Wahlén–Walsh instability theory. The next subsection shows that $d''(c) < 0$, which implies orbital instability.

3.3.3 Proof of Theorem 3.1.3

Using the expressions for the momentum P, $\bar{x}_2$, and $\bar{y}_2$, we can compute:

$$d'(c) = \epsilon\gamma_1(-a+\rho) - \epsilon\gamma_2(-a-\rho) - \int_{\mathbb{R}} \eta(\varphi' + \epsilon\nabla_\top\Theta)\, \mathrm{d}x_1.$$

Differentiating once more yields

$$d''(c) = \epsilon\gamma_1\partial_c(-a+\rho) + \epsilon\gamma_2\partial_c(a+\rho) - \int_{\mathbb{R}} \Big((\partial_c\eta)(\varphi' + \epsilon\nabla_\top\Theta) + \eta\, \partial_c(\varphi' + \epsilon\nabla_\top\Theta)\Big)\, \mathrm{d}x_1.$$

Recalling the definition of $\mathscr{T}$ in (3.2.9), using the compatibility (3.1.8) and variations for a and ρ in (3.2.11), we obtain

$$\begin{aligned}
d''(c) &= -\gamma_1(a_{\tilde{c}} - \rho_{\tilde{c}}) + \gamma_2(a_{\tilde{c}} + \rho_{\tilde{c}}) + O(\epsilon^3) \\
&= -\frac{\gamma_1}{\det\mathscr{T}}\left(-\frac{\gamma_2^0}{2\pi(a_0+\rho_0)^2} + \frac{-\gamma_1^0+\gamma_2^0}{4\pi\rho_0^2} + \frac{\gamma_1^0+\gamma_2^0}{4\pi a_0^2}\right) \\
&\quad + \frac{\gamma_2}{\det\mathscr{T}}\left(\frac{\gamma_1^0}{2\pi(a_0-\rho_0)^2} + \frac{-\gamma_1^0+\gamma_2^0}{4\pi\rho_0^2} - \frac{\gamma_1^0+\gamma_2^0}{4\pi a_0^2}\right) + O(\epsilon^3) \\
&= \frac{\gamma_1^2}{2\pi\det\mathscr{T}} \frac{6a_0\rho_0^2}{(a_0+\rho_0)(a_0-\rho_0)^2(a_0^2+a_0\rho_0+\rho_0^2)} + O(\epsilon^3).
\end{aligned}$$

Thus, since $\det\mathscr{T} < 0$, we conclude that $d''(c) < 0$ for $|\epsilon| \ll 1$ and $c = O(\epsilon)$. Hence, Theorem B.1.8 tells us that the corresponding water waves $\{U_c\}$ constructed in Theorem 3.1.1 are orbitally unstable.

Appendix A

Local bifurcation theory

Theorem A.1 (Crandall and Rabinowitz [117])**.** Let X and Y be Banach spaces and $I \subset \mathbb{R}$ be an open interval with $\lambda^* \in I$. Suppose that $\mathcal{F} : I \times X \to Y$ is a continuous map with the following properties:

(i) $\mathcal{F}(\lambda, 0) = 0$ for all $\lambda \in I$;

(ii) $D_1\mathcal{F}$, $D_2\mathcal{F}$, and $D_1D_2\mathcal{F}$ exist and are continuous, where D_i denotes the Fréchet derivative with respect to the i-th coordinate;

(iii) $D_2\mathcal{F}(\lambda^*, 0)$ is a Fredholm operator of index 0. In particular, the null space is one-dimensional and spanned by some element w^*.

(iv) $D_1D_2\mathcal{F}(\lambda^*, 0)w^* \notin \mathcal{R}(D_2\mathcal{F}(\lambda^*, 0))$.

There there exists a continuous local bifurcation curve $\{(\lambda(s), w(s)) \in \mathbb{R} \times X : |s| < \epsilon\}$ with $\epsilon > 0$ sufficiently small such that $(\lambda(0), w(0)) = (\lambda^*, w^*)$, and

$$\{(\lambda, w) \in \mathcal{U} : w \neq 0, \mathcal{F}(\lambda, w) = 0\} = \{(\lambda(s), w(s)) \in \mathbb{R} \times Y : |s| < \epsilon\}$$

for some neighborhood $\mathcal{U}$ of $(\lambda^*, 0)$ in $\mathbb{R} \times X$. Moreover, we have

$$w(s) = sw^* + o(s) \qquad \text{in } X,\ |s| < \epsilon.$$

If D_2^2 exists and is continuous, then the curve is of class C^1.

Appendix B

Water waves with a finite dipole

B.1 Abstract instability theory

This section summarizes the instability theory developed by Varholm, Wahlén, and Walsh in [107, Sections 2, 4]. We are considering the stability property of an abstract Hamiltonian

$$\frac{\mathrm{d}u}{\mathrm{d}t} = J(u)DE(u), \quad u|_{t=0} = u_0, \tag{B.1.1}$$

where J is the Poisson map and E is the energy. Let $\mathbb{X}$ be a Hilbert space, and $\mathbb{V}$ and $\mathbb{W}$ be reflexive Banach spaces such that

$$\mathbb{W} \hookrightarrow \mathbb{V} \hookrightarrow \mathbb{X}.$$

Let $\mathbb{X}^*$ be the (continuous) dual space of $\mathbb{X}$, which is naturally isomorphic to $\mathbb{X}$ via the mapping $I : \mathbb{X} \to \mathbb{X}^*$.

Assumption B.1.1 (Spaces)**.** There exist constants $\theta \in (0, 1]$ and $C > 0$ such that

$$\|u\|_{\mathbb{V}}^3 \leq C\|u\|_{\mathbb{X}}^{2+\theta}\|u\|_{\mathbb{W}}^{1-\theta} \tag{B.1.2}$$

for all $u \in \mathbb{W}$.

Let $\mathcal{O} \subset \mathbb{X}$ be an open set. Suppose that

$$J(u) := B(u)\widehat{J},$$

where for each $u \in \mathcal{O} \cap \mathbb{V}$, $B(u)$ is a bounded linear operator in $\mathbb{X}$, that is, $B(u) \in \mathrm{Lin}(\mathbb{X})$, and $\widehat{J} : \mathcal{D}(J) \subset \mathbb{X}^* \to \mathbb{X}$ is a closed linear operator.

Assumption B.1.2 (Poisson map)**.**

(i) The domain $\mathcal{D}(\widehat{J})$ is dense in $\mathbb{X}^*$.

(ii) $\widehat{J}$ is injective.

(iii) For each $u \in \mathcal{O} \cap \mathbb{V}$, the operator $B(u)$ is bijective.

(iv) The map $u \mapsto B(u)$ is of class $C^1(\mathcal{O} \cap \mathbb{V}; \mathrm{Lin}(\mathbb{X})) \cap C^1(\mathcal{O} \cap \mathbb{W}; \mathrm{Lin}(\mathbb{W}))$.

(v) For each $u \in \mathcal{O} \cap \mathbb{V}$, $J(u)$ is skew-adjoint in the sense that

$$\langle J(u)v, w \rangle = -\langle v, J(u)w \rangle$$

for all $v, w \in \mathcal{D}(\widehat{J})$.

We suppose that $\mathbb{V}$ is chosen so that $E \in C^3(\mathcal{O} \cap \mathbb{V}; \mathbb{R})$. In addition, assume that there exists a momentum functional $P \in C^3(\mathcal{O} \cap \mathbb{V}; \mathbb{X})$, and that both it and the energy are conserved by solutions of (B.1.1).

Assumption B.1.3 (Derivative extension)**.** There exist mappings ∇E, $\nabla P \in C^0(\mathcal{O} \cap \mathbb{V}; \mathbb{X}^*)$ such that $\nabla E(u)$ and $\nabla P(u)$ are extensions of $DE(u)$ and $DP(u)$, respectively, for every $u \in \mathcal{O} \cap \mathbb{V}$.

Suppose that there is a one-parameter family of affine maps $T(s) : \mathbb{X} \to \mathbb{X}$, with the linear part $dT(s) := T(s)u - T(s)0$ having the properties:

Assumption B.1.4 (Symmetry group)**.** The symmetry group $T(\cdot)$ satisfies the following.

(i) *(Invariance)* The neighborhood $\mathcal{O}$, and the subspaces $\mathbb{V}$ and $\mathbb{W}$, are all invariant under the symmetry group. Moreover, $I^{-1}\mathcal{D}(\widehat{J})$ is invariant under the *linear* symmetry group, or equivalently, $\mathcal{D}(\widehat{J})$ is invariant under the adjoint $dT^*(s) : \mathbb{X}^* \to \mathbb{X}^*$.

(ii) *(Flow property)* We have $T(0) = dT(0) = \mathrm{Id}_{\mathbb{X}}$, and for all $s, r \in \mathbb{R}$,

$$T(s + r) = T(s)T(r), \quad \text{and hence} \quad dT(s + r) = dT(s)dT(r).$$

(iii) *(Unitary)* The linear part $dT(s)$ is a unitary operator on $\mathbb{X}$ for each $s \in \mathbb{R}$, or equivalently,

$$dT^*(s)I = I\, dT(-s), \quad \text{for all } s \in \mathbb{R}. \tag{B.1.3}$$

Moreover, the linear part is an isometry on the spaces $\mathbb{V}$ and $\mathbb{W}$.

(iv) *(Strong continuity)* The symmetry group is strongly continuous on both $\mathbb{X}$, $\mathbb{V}$, and $\mathbb{W}$.

(v) *(Affine part)* The function $T(\cdot)0$ belongs to $C^3(\mathbb{R};\mathbb{W})$ and there exists an increasing function $w : [0,\infty) \to [0,\infty)$ such that

$$\|T(s)0\|_{\mathbb{W}} \neq w(\|T(s)0\|_{\mathbb{X}}), \quad \text{for all } s \in \mathbb{R}.$$

(vi) *(Commutativity with J)* For all $s \in \mathbb{R}$,

$$\begin{aligned} \widehat{J}I\, dT(s) &= dT(s)\widehat{J}I, \\ dT(s)B(u) &= B(T(s)u)dT(s), \quad \text{for all } u \in \mathcal{O} \cap \mathbb{V}. \end{aligned} \tag{B.1.4}$$

(vii) *(Infinitesimal generator)* The infinitesimal generator of T is the affine mapping

$$T'(0)u = \lim_{s\to 0}\big(s^{-1}(T(s)u - u)\big) = dT'(0) + T'(0)0,$$

with dense domain $\mathcal{D}(T'(0)) \subset \mathbb{X}$ consisting of all $u \in \mathbb{X}$ such that the limit exists in $\mathbb{X}$. Similarly, we may speak of the dense subspaces $\mathcal{D}(T'(0)|_{\mathbb{V}}) \subset \mathbb{V}$ and $\mathcal{D}(T'(0)|_{\mathbb{W}}) \subset \mathbb{W}$ on which the limit exists in $\mathbb{V}$ and $\mathbb{W}$, respectively. We assume that $\nabla P(u) \in \mathcal{D}(\widehat{J})$ for every $u \in \mathcal{D}(T'(0)|_{\mathbb{V}}) \cap \mathcal{O}$, and that

$$T'(0)u = J(u)\nabla P(u) \tag{B.1.5}$$

for all such u. Moreover, we assume that

$$\widehat{J}I\, dT'(0) = dT'(0)\widehat{J}I. \tag{B.1.6}$$

(viii) *(Density)* The subspace

$$\mathcal{D}(T'(0)|_{\mathbb{W}}) \cap \operatorname{Rng}\widehat{J}$$

is dense in $\mathbb{X}$.

(ix) *(Conservation)* For all $u \in \mathcal{O} \cap \mathbb{V}$, the energy is conserved by flow of the symmetry group:

$$E(u) = E(T(s)u), \quad \text{for all } s \in \mathbb{R}. \tag{B.1.7}$$

We say $u \in C^1(\mathbb{R}; \mathcal{O} \cap \mathbb{W})$ is a *bound state* of the Hamiltonian system (B.1.1) if u is a solution of the form

$$u(t) = T(ct)U,$$

for some $c \in \mathbb{R}$ and $U \in \mathcal{O} \cap \mathbb{W}$.

Assumption B.1.5 (Bound states)**.** There exists a one-parameter family of bound state solutions $\{U_c : c \in \mathcal{I}\}$ to the Hamiltonian system (B.1.1).

(i) The mapping $c \in \mathcal{I} \mapsto U_c \in \mathcal{O} \cap \mathbb{W}$ is C^1.

(ii) The non-degeneracy condition $T'(0)U_c \neq 0$ holds for every $c \in \mathcal{I}$. Equivalently, U_c is never a critical point of the momentum.

(iii) For all $c \in \mathcal{I}$,

$$U_c \in \mathcal{D}(T'''(0)) \cap \mathcal{D}(\widehat{J}IT'(0)), \tag{B.1.8}$$

and

$$\widehat{J}IT'(0)U_c \in \mathcal{D}(T'(0)|_{\mathbb{W}}). \tag{B.1.9}$$

(iv) Either $s \mapsto T(s)U_c$ is periodic, or $\liminf_{|s|\to\infty} \|T(s)U_c - U_c\|_{\mathbb{X}} > 0$.

Define $E_c(u) := E(u) - cP(u)$ to be the *augmented Hamiltonian*. Then we have the following assumption:

Assumption B.1.6 (Spectrum)**.** The mapping

$$u \in \mathbb{V} \mapsto \left\langle D^2 E_c(U_c)u, \cdot \right\rangle_{\mathbb{V}^* \times \mathbb{V}} \in \mathbb{V}^*$$

extends uniquely to a bounded linear operator $H_c : \mathbb{X} \to \mathbb{X}^*$ with the following properties

(i) $I^{-1}H_c$ is self-adjoint on $\mathbb{X}$.

(ii) The eigenvalues of $I^{-1}H_c$ satisfy

$$\operatorname{spec}(I^{-1}H_c) = \{-\mu_c^2\} \cup \{0\} \cup \Sigma,$$

where $-\mu_c^2 < 0$ is a simple eigenvalue corresponding to a unit eigenvector χ_c, 0 is a simple eigenvalue generated by T, and Σ is a subset of the positive real axis bounded away from 0.

Assumption B.1.7 (Local existence)**.** There exists $\nu_0 > 0$ and $t_0 > 0$ such that for all initial data $u_0 \in U_{\nu_0}$, there exists a unique solution to the ODE (B.1.1) on the interval $[0, t_0)$.

Let $d(c) := E_c(U_c) = E(U_c) - cP(U_c)$ be the *moment of instability*, where U_c is a traveling wave. Moreover, for each $\rho > 0$, the tubular neighborhood of radius ρ in $\mathbb{X}$ for the U_c-orbit generated by T is

$$\mathcal{U}_\rho := \left\{ u \in \mathcal{O} : \inf_{s\in\mathbb{R}} \|T(s)U_c - u\|_{\mathbb{W}} < \rho \right\}, \tag{B.1.10}$$

We have the following instability theorem; see [107, Theorem 2.6].

Theorem B.1.8 (Instability)**.** *Suppose that all assumptions B.1.1–B.1.7 are satisfied, and that there exists a family of traveling water waves U_c. Then if $d''(c) < 0$, the traveling wave U_c is orbitally unstable. That is, there exists a $\nu_0 > 0$ such that for every $0 < \nu < \nu_0$ there exists initial data in $\mathcal{U}_\nu$ whose corresponding solution exits $\mathcal{U}_{\nu_0}$ in finite time.*

B.2 Steady and unsteady equations

For the convenience of the reader, in this appendix we derive the nonlocal formulations for the water wave with a finite dipole problem (3.2.4).

Using the definitions of φ in (3.1.6) and $\mathcal{G}(\eta)$ in (3.2.3), we obtain

$$\nabla\Phi = \frac{1}{\langle\eta'\rangle^2}\begin{pmatrix} 1 & -\eta' \\ \eta' & 1 \end{pmatrix}\begin{pmatrix} \varphi' \\ \mathcal{G}(\eta)\varphi \end{pmatrix} = \frac{1}{\langle\eta'\rangle^2}\begin{pmatrix} \varphi' - \eta'\mathcal{G}(\eta)\varphi \\ \eta'\varphi' + \mathcal{G}(\eta)\varphi \end{pmatrix}. \tag{B.2.1}$$

Combining with the definitions of ψ in (3.2.2) gives

$$\begin{pmatrix} \mathcal{G}(\eta)\varphi \\ \varphi' \end{pmatrix} = \begin{pmatrix} \psi' \\ -\mathcal{G}(\eta)\psi \end{pmatrix}. \tag{B.2.2}$$

Then from the incompressible Euler equation (3.1.2), we can derive the *unsteady equation for velocity potential* on S

$$\begin{aligned}\partial_t\varphi = -\frac{1}{2\langle\eta'\rangle^2}\left((\varphi')^2 - 2\eta'\varphi'\mathcal{G}(\eta)\varphi - (\mathcal{G}(\eta)\varphi)^2\right) - \epsilon\partial_t\Theta + \epsilon\varphi'\partial_{x_2}\Gamma - \frac{\epsilon^2}{2}|\nabla\Gamma|^2 \\ - \eta + b\frac{\eta''}{\langle\eta'\rangle^3}.\end{aligned} \tag{B.2.3}$$

Using the relation (B.2.2), we also have the *unsteady equation for stream function* on S:

$$\partial_t\varphi = -\frac{1}{2\langle\eta'\rangle^2}\left((\mathcal{G}(\eta)\psi)^2 + 2\eta'\psi'\mathcal{G}(\eta)\psi - (\psi')^2\right) - \epsilon\partial_t\Theta - \epsilon\mathcal{G}(\eta)\psi\,\partial_{x_2}\Gamma - \frac{\epsilon^2}{2}|\nabla\Gamma|^2 - \eta + b\frac{\eta''}{\langle\eta'\rangle^3}. \quad \text{(B.2.4)}$$

For the traveling water waves, the *steady equation for velocity potential* on S is

$$-\frac{c}{\langle\eta'\rangle^2}(\varphi' - \eta'\mathcal{G}(\eta)\varphi) + c\epsilon\partial_{x_2}\Gamma + \frac{1}{2\langle\eta'\rangle^2}\left[(\varphi')^2 + (\mathcal{G}(\eta)\varphi)^2\right] + \frac{\epsilon}{\langle\eta'\rangle^2}\left[-\varphi'\nabla_\perp\Gamma + \mathcal{G}(\eta)\varphi\ \nabla_\top\Gamma\right] + \frac{\epsilon^2}{2}|\nabla\Gamma|^2 + \eta - b\frac{\eta''}{\langle\eta'\rangle^3} = 0, \quad \text{(B.2.5)}$$

and the *steady equation for stream function* on S is:

$$\frac{c}{\langle\eta'\rangle^2}(\psi' + \eta'\mathcal{G}(\eta)\psi) + c\epsilon\partial_{x_2}\Gamma + \frac{1}{2\langle\eta'\rangle^2}\left[(\psi')^2 + (\mathcal{G}(\eta)\psi)^2\right] + \frac{\epsilon}{\langle\eta'\rangle^2}\left[\mathcal{G}(\eta)\psi\nabla_\perp\Gamma + \psi'\ \nabla_\top\Gamma\right] + \frac{\epsilon^2}{2}|\nabla\Gamma|^2 + \eta - b\frac{\eta''}{\langle\eta'\rangle^3} = 0. \quad \text{(B.2.6)}$$

B.3 Variations of the energy and momentum

Finally, in this appendix we record the first and second Fréchet derivatives of the energy and momentum.

Recall that

$$\mathfrak{a} = (\nabla(\mathcal{H}\varphi))|_{S_t}, \quad \xi = (\Upsilon_{1_{x_1}}, \Xi_{1_{x_2}})^T, \quad \text{and} \quad \zeta = (\Upsilon_{2_{x_1}}, \Xi_{2_{x_2}})^T.$$

Let $\nabla\xi := (\Upsilon_{1_{x_1x_1}}, \Xi_{1_{x_2x_2}})^T$, $\nabla\zeta := (\Upsilon_{2_{x_1x_1}}, \Xi_{2_{x_2x_2}})^T$, and

$$D^2_{\bar{x}}\Theta := \begin{pmatrix} \Upsilon_{1_{x_1x_1}} & \Xi_{1_{x_1x_2}} \\ \Xi_{1_{x_1x_2}} & \Upsilon_{1_{x_2x_2}} \end{pmatrix}, \quad \text{and} \quad D^2_{\bar{y}}\Theta := \begin{pmatrix} \Upsilon_{2_{x_1x_1}} & \Xi_{2_{x_1x_2}} \\ \Xi_{2_{x_1x_2}} & \Upsilon_{2_{x_2x_2}} \end{pmatrix}.$$

Variations of $K_0(u)$

We compute that

$$D_\varphi K_0(u)\dot\varphi = \int_{\mathbb{R}} \dot\varphi \mathcal{G}(\eta)\varphi \, \mathrm{d}x_1, \quad D_\eta K_0(u)\dot\eta = \frac{1}{2}\int_{\mathbb{R}} \varphi \langle D_\eta \mathcal{G}(\eta)\dot\eta, \varphi\rangle \, \mathrm{d}x_1,$$

and

$$\langle D_\varphi^2 K_0(u)\dot\varphi, \dot\varphi\rangle = \int_{\mathbb{R}} \dot\varphi \mathcal{G}(\eta)\dot\varphi \, \mathrm{d}x_1,$$

$$\langle D_\varphi D_\eta K_0(u)\dot\varphi, \dot\eta\rangle = \int_{\mathbb{R}} \dot\varphi \langle D_\eta \mathcal{G}(\eta)\dot\eta, \varphi\rangle \, \mathrm{d}x_1 = \int_{\mathbb{R}} \dot\eta(\mathfrak{a}_1 \dot\varphi' - \mathfrak{a}_2 \mathcal{G}(\eta)\dot\varphi) \, \mathrm{d}x_1$$

$$\langle D_\eta^2 K_0(u)\dot\eta, \dot\eta\rangle = \frac{1}{2}\int_{\mathbb{R}} \varphi \langle\langle D_\eta^2 \mathcal{G}(\eta)\dot\eta, \dot\eta\rangle, \varphi\rangle \, \mathrm{d}x_1 = \int_{\mathbb{R}} (\mathfrak{a}_1' \mathfrak{a}_2 \dot\eta^2 + \mathfrak{a}_2 \dot\eta \mathcal{G}(\eta)(\mathfrak{a}_2\dot\eta)) \, \mathrm{d}x_1.$$

Variations of $K_1(u)$

Likewise, the first variations of K_1 are

$$D_\varphi K_1(u)\dot\varphi = \int_{\mathbb{R}} \dot\varphi \nabla_\perp \Theta \, \mathrm{d}x_1, \qquad D_\eta K_1(u)\dot\eta = \int_{\mathbb{R}} \dot\eta \varphi' \Theta_{x_1}|_S \, \mathrm{d}x_1,$$

$$\nabla_{\bar x} K_1(u) = -\int_{\mathbb{R}} \varphi \nabla_\perp \xi \, \mathrm{d}x_1, \qquad \nabla_{\bar y} K_1(u) = -\int_{\mathbb{R}} \varphi \nabla_\perp \zeta \, \mathrm{d}x_1,$$

and the second are given by

$$\langle D_\varphi D_\eta K_1(u)\dot\eta, \dot\varphi\rangle = \int_{\mathbb{R}} \dot\eta \dot\varphi' \Theta_{x_1}|_S \, \mathrm{d}x_1, \qquad \langle D_\eta^2 K_1(u)\dot\eta, \dot\eta\rangle = \int_{\mathbb{R}} \dot\eta^2 \varphi' \Theta_{x_1 x_2}|_S \, \mathrm{d}x_1,$$

$$D_{\bar x}^2 K_1(u) = \int_{\mathbb{R}} \varphi \nabla_\perp D_{\bar x}^2 \Theta \, \mathrm{d}x_1, \qquad D_{\bar y}^2 K_1(u) = \int_{\mathbb{R}} \varphi \nabla_\perp D_{\bar y}^2 \Theta \, \mathrm{d}x_1,$$

$$\nabla_{\bar x} D_\eta K_1(u)\dot\eta = -\int_{\mathbb{R}} \dot\eta \varphi' (\nabla \xi)|_S \, \mathrm{d}x_1, \qquad \nabla_{\bar y} D_\eta K_1(u)\dot\eta = -\int_{\mathbb{R}} \dot\eta \varphi' (\nabla \zeta)|_S \, \mathrm{d}x_1,$$

$$\nabla_{\bar x} D_\varphi K_1(u)\dot\varphi = -\int_{\mathbb{R}} \dot\varphi \nabla_\perp \xi \, \mathrm{d}x_1, \qquad \nabla_{\bar y} D_\varphi K_1(u)\dot\varphi = -\int_{\mathbb{R}} \dot\varphi \nabla_\perp \zeta \, \mathrm{d}x_1.$$

Variations of $K_2(u)$

It is straightforward to compute that

$$D_\eta K_2(u)\dot{\eta} = \frac{1}{2}\int_{\mathbb{R}} \dot{\eta}|(\nabla\Theta)|_S|^2 \,\mathrm{d}x_1,$$

$$\nabla_{\bar{x}} K_2(u) = \nabla_{\bar{x}}\Gamma^* - \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\xi\Theta)\,\mathrm{d}x_1, \quad \nabla_{\bar{y}} K_2(u) = \nabla_{\bar{y}}\Gamma^* - \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\zeta\Theta)\,\mathrm{d}x_1,$$

and

$$\langle D_\eta^2 K_2(u)\dot{\eta},\dot{\eta}\rangle = \int_{\mathbb{R}} \dot{\eta}^2\Big(\Theta_{x_1}\Theta_{x_1x_2} + \Theta_{x_2}\Theta_{x_2x_2}\Big)\Big|_S \,\mathrm{d}x_1,$$

$$D_{\bar{x}}^2 K_2(u) = 2D_{\bar{x}}^2\Gamma^* + \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\Theta D_{\bar{x}}^2\Theta + \xi\xi^T)\,\mathrm{d}x_1,$$

$$D_{\bar{y}}^2 K_2(u) = 2D_{\bar{y}}^2\Gamma^* + \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\Theta D_{\bar{y}}^2\Theta + \zeta\zeta^T)\,\mathrm{d}x_1,$$

$$\nabla_{\bar{x}}\nabla_{\bar{y}} K_2(u) = \nabla_{\bar{x}}\nabla_{\bar{y}}\Gamma^* + \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\xi\odot\zeta)\,\mathrm{d}x_1,$$

$$\nabla_{\bar{x}} D_\eta K_2(u)\dot{\eta} = -\int_{\mathbb{R}} \dot{\eta}((D_x\xi)\nabla\Theta)|_S\,\mathrm{d}x_1, \quad \nabla_{\bar{y}} D_\eta K_2(u)\dot{\eta} = -\int_{\mathbb{R}} \dot{\eta}((D_x\zeta)\nabla\Theta)|_S\,\mathrm{d}x_1.$$

Variations of $V(u)$

Similarly, we find that

$$D_\eta V(u)\dot{\eta} = \int_{\mathbb{R}} \dot{\eta}\left(g\eta - b\frac{\eta''}{\langle\eta'\rangle^3}\right)\,\mathrm{d}x_1,$$
$$\langle D_\eta^2 V(u)\dot{\eta},\hat{\eta}\rangle = \int_{\mathbb{R}} \left(g\hat{\eta}\dot{\eta} + \frac{b}{\langle\eta'\rangle^3}\hat{\eta}'\dot{\eta}'\right)\,\mathrm{d}x_1.$$

Variations of $P(u)$

Finally, the first variations of momentum $P(u)$ are given in Section 3.3.1.

The second derivatives are as follows:

$$\langle D_\eta D_\varphi P(u)\dot{\varphi},\dot{\eta}\rangle = -\int_{\mathbb{R}} \dot{\eta}'\dot{\varphi}\,\mathrm{d}x_1, \qquad \langle D_\eta^2 P(u)\dot{\eta},\dot{\eta}\rangle = \epsilon\int_{\mathbb{R}} \dot{\eta}^2\Theta_{x_1x_2}|_S\,\mathrm{d}x_1,$$

$$D_{\bar{x}}^2 P(u) = -\epsilon\int_{\mathbb{R}} \eta'(D_{\bar{x}}^2\Theta)|_S\,\mathrm{d}x_1, \qquad D_{\bar{y}}^2 P(u) = -\epsilon\int_{\mathbb{R}} \eta'(D_{\bar{y}}^2\Theta)|_S\,\mathrm{d}x_1,$$

$$\nabla_{\bar{x}} D_\eta P(u)\dot{\eta} = -\epsilon\int_{\mathbb{R}} \dot{\eta}(\nabla\xi)|_S\,\mathrm{d}x_1, \qquad \nabla_{\bar{y}} D_\eta P(u)\dot{\eta} = -\epsilon\int_{\mathbb{R}} \dot{\eta}(\nabla\zeta)|_S\,\mathrm{d}x_1.$$

Bibliography

[1] John W. Miles. On the generation of surface waves by shear flows. *J. Fluid Mech.*, 3:185–204, 1957.

[2] John W. Miles. On the generation of surfaces waves by shear flows. II. *J. Fluid Mech.*, 6:568–582, 1959.

[3] John W. Miles. On the generation of surface waves by shear flows. III. Kelvin-Helmholtz instability. *J. Fluid Mech.*, 6:583–598. (1 plate), 1959.

[4] John W. Miles. On the generation of surface waves by shear flows. IV. *J. Fluid Mech.*, 13:433–448, 1962.

[5] Leonhard Euler. Principes généraux de létat déquilibre dun fluide.masb, 11 [printed in 1757], 217–273. also in. *Opera omnia*, pages 2–53, 1755.

[6] Vladimir I. Arnold and Boris A. Khesin. *Topological methods in hydrodynamics*, volume 125 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[7] Claude Bardos and Edriss Titi. Euler equations for incompressible ideal fluids. *Russian Mathematical Surveys*, 62(3):409, 2007.

[8] Jean-Yves Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.

[9] Peter Constantin. Euler equations, navier-stokes equations and turbulence. In *Mathematical foundation of turbulent viscous flows*, pages 1–43. Springer, 2006.

[10] RW Ogden. Nonlinear elastic deformations dover publications inc. 1997.

[11] D. J. Acheson. *Elementary fluid dynamics.* Oxford Applied Mathematics and Computing Science Series. The Clarendon Press, Oxford University Press, New York, 1990.

[12] G. B. Whitham. *Linear and nonlinear waves.* Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974. Pure and Applied Mathematics.

[13] Jean-Marc Vanden-Broeck. *Gravity-capillary free-surface flows.* Cambridge Monographs on Mechanics. Cambridge University Press, Cambridge, 2010.

[14] A. Merzhanov, A. Filonenko, and I. Borovinskaya. New phenomena during combustion of condensed systems, doklady akademii nauk sssr. *Seriya Khimiya*, 208:892–894, 1973.

[15] Haruo Sato and Michael C. Fehler. *Seismic wave propagation and scattering in the heterogeneous earth.* AIP Series in Modern Acoustics and Signal Processing. American Institute of Physics, New York; Springer-Verlag, New York, 1998.

[16] A. Constantin and R. S. Johnson. Modelling tsunamis. *J. Phys. A*, 39(14):L215–L217, 2006.

[17] J. S. Russell. Report on waves. 311:390, 1844. In 14th meeting of the British Association for the Advancement of Science.

[18] Augustin Louis Cauchy. *Œuvres complètes. Series 1. Volume 1.* Cambridge Library Collection. Cambridge University Press, Cambridge, 2009. Reprint of the 1882 original.

[19] Leonhardus Eulerus. *Opera omnia. Series secunda. Opera mechanica et astronomica. Vol. XII. Commentationes mechanica ad theoriam corporum fluidorum pertinentes. vol. prius.* Societatis Scientiarum Naturalium Helveticae, Lausanne, 1954. Edidit Clifford Ambrose Truesdell.

[20] Laplace P s Marquis. Suite des recherches sur plusieurs points du syst'eme du monde (xxv–xxvii). *Mem. Pr esent esPr esent es Divers Savans Acad. R. Sci Inst. France*, pages 525–552, 1776.

[21] George Gabriel Stokes. *On the Theory of Oscillatory Waves*, page 197229. 1847.

[22] Cx K Batchelor and GK Batchelor. *An introduction to fluid dynamics.* Cambridge university press, 1967.

[23] Samuel Walsh. Stratified steady periodic water waves. *SIAM J. Math. Anal.*, 41(3):1054–1105, 2009.

[24] Joseph Pedlosky. *Geophysical fluid dynamics.* Springer Science & Business Media, 2013.

[25] Franz Gerstner. Theorie der wellen. *Annalen der Physik*, 32(8):412–445, 1809.

[26] M.-L. Dubreil-Jacotin. *Sur la détermination rigoureuse des ondes permanentes périodiques d'ampleur finie*. 1934.

[27] A. M. Ter-Krikorov. Exact solution of the problem of the motion of a vortex under the surface of a liquid. *Izv. Akad. Nauk SSSR Ser. Mat.*, 22:177–200, 1958.

[28] A. M. Ter-Krikorov. Théorie exacte des ondes longues stationnaires dans un liquide hétérogène. *J. Mécanique*, 2:351–376, 1963.

[29] Adrian Constantin and Walter Strauss. Exact steady periodic water waves with vorticity. *Comm. Pure Appl. Math.*, 57(4):481–527, 2004.

[30] Miles H. Wheeler. Large-amplitude solitary water waves with vorticity. *SIAM J. Math. Anal.*, 45(5):2937–2994, 2013.

[31] Miles H. Wheeler. The Froude number for solitary water waves with vorticity. *J. Fluid Mech.*, 768:91–112, 2015.

[32] Miles H. Wheeler. Solitary water waves of large amplitude generated by surface pressure. *Arch. Ration. Mech. Anal.*, 218(2):1131–1187, 2015.

[33] Jalal Shatah, Samuel Walsh, and Chongchun Zeng. Travelling water waves with compactly supported vorticity. *Nonlinearity*, 26(6):1529–1564, 2013.

[34] H. Helmholtz. über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. *J. Reine Angew. Math.*, 55:25–55, 1858.

[35] Lord Kelvin. On vortex motion. *Trans. Roy. Soc. Edinb.*, 25:217–260, 1869.

[36] Gustav Robert Kirchhoff. *Vorlesungen ber mathematische Physik*. Leipzig Teubner, 1876.

[37] J. J. Thomson. On Electrical Oscillations and the effects produced by the motion of an Electrified Sphere. *Proc. Lond. Math. Soc.*, 15:197–218, 1883/84.

[38] A. E. H. Love. On the Stability of certain Vortex Motions. *Proc. Lond. Math. Soc.*, 25:18–42, 1893/94.

[39] Carlo Marchioro and Mario Pulvirenti. Vortices and localization in Euler flows. *Comm. Math. Phys.*, 154(1):49–61, 1993.

[40] IG Filippov. Motion of vortex beneath the surface of a fluid. *Prikl. Mat. Mekh*, 25:242, 1961.

[41] Peder Tyvand. On the interaction between a strong vortex pair and a free surface. *Physics of Fluids A: Fluid Dynamics*, 2, 09 1990.

[42] Peder A. Tyvand. Motion of a vortex near a free surface. *J. Fluid Mech.*, 225:673–686, 1991. With an appendix by R. P. Tong.

[43] E. A. Kuznetsov and V. P. Ruban. Cherenkov interaction of vortices with a free surface. *Journal of Experimental and Theoretical Physics*, 88(3):492–505, Mar 1999.

[44] Kristoffer Varholm. Solitary gravity-capillary water waves with point vortices. *Discrete Contin. Dyn. Syst.*, 36(7):3927–3959, 2016.

[45] S. Fish. Vortex dynamics in the presence of free surface waves. *Physics of Fluids A: Fluid Dynamics*, 3(4):504–506, 1991.

[46] Daniel L. Marcus and Stanley A. Berger. The interaction between a counterrotating vortex pair in vertical ascent and a free surface. *Physics of Fluids A: Fluid Dynamics*, 1(12):1988–2000, 1989.

[47] John G Telste. Potential flow about two counter-rotating vortices approaching a free surface. *Journal of Fluid Mechanics*, 201:259–278, 1989.

[48] William W Willmarth, Grétar Tryggvason, Amir Hirsa, and D Yu. Vortex pair generation and interaction with a free surface. *Physics of Fluids A: Fluid Dynamics*, 1(2):170–172, 1989.

[49] Andrew A. Tchieu, Eva Kanso, and Paul K. Newton. The finite-dipole dynamical system. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 468(2146):3006–3026, 2012.

[50] G. Kirchhoff. über das Gleichgewicht und die Bewegung einer elastischen Scheibe. *J. Reine Angew. Math.*, 40:51–88, 1850.

[51] Hung Le. Elliptic equations with transmission and Wentzell boundary conditions and an application to steady water waves in the presence of wind. *Discrete Contin. Dyn. Syst.*, 38(7):3357–3385, 2018.

[52] A. D. Ventcel′. On boundary conditions for multi-dimensional diffusion processes. *Theor. Probability Appl.*, 4:164–177, 1959.

[53] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic Differential Equations and Diffusion Processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, second edition, 1989.

[54] V. Bonnaillie-Noël, M. Dambrine, F. Hérau, and G. Vial. On generalized Ventcel's type boundary conditions for Laplace operator in a bounded domain. *SIAM J. Math. Anal.*, 42(2):931–945, 2010.

[55] Olga A. Ladyzhenskaya and Nina N. Ural'tseva. *Linear and Quasilinear Elliptic Equations*. Academic Press, New York-London, 1968.

[56] A. A. Arkhipova and Osman Erlhamahmy. Regularity of solutions to a diffraction-type problem for nondiagonal linear elliptic systems in the Campanato space. *J. Math. Sci. (New York)*, 112(1):3944–3966, 2002. Function theory and applications.

[57] Philip Korman. Existence of solutions for a class of semilinear noncoercive problems. *Nonlinear Anal.*, 10(12):1471–1476, 1986.

[58] Yousong Luo and Neil S. Trudinger. Linear second order elliptic equations with Venttsel boundary conditions. *Proc. Roy. Soc. Edinburgh Sect. A*, 118(3-4):193–207, 1991.

[59] Yousong Luo. On the quasilinear elliptic Venttsel′ boundary value problem. *Nonlinear Anal.*, 16(9):761–769, 1991.

[60] Yousong Luo and Neil S. Trudinger. Quasilinear second order elliptic equations with Venttsel boundary conditions. *Potential Anal.*, 3(2):219–243, 1994.

[61] A. I. Nazarov and A. A. Paletskikh. On the Hölder property of the solutions of the elliptic Venttsel problem. *Dokl. Akad. Nauk*, 465(no. 5):532–536, 2015.

[62] D. E. Apushkinskaya and A. I. Nazarov. A survey of results on nonlinear Venttsel problems. *Appl. Math.*, 45(1):69–80, 2000.

[63] Martin Schechter. A generalization of the problem of transmission. *Ann. Scuola Norm. Sup. Pisa (3)*, 14:207–236, 1960.

[64] Z. G. Šeftel'. Estimates in L_p of solutions of elliptic equations with discontinuous coefficients and satisfying general boundary conditions and conjugacy conditions. *Soviet Math. Dokl.*, 4:321–324, 1963.

[65] O. A. Oleĭnik. Boundary-value problems for linear equations of elliptic parabolic type with discontinuous coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 25:3–20, 1961.

[66] Mikhail Borsuk. The transmission problem for elliptic second order equations in a conical domain. *Ann. Acad. Pedagog. Crac. Stud. Math.*, 7:61–89, 2008.

[67] Mikhail Borsuk. The transmission problem for quasi-linear elliptic second order equations in a conical domain. I, II. *Nonlinear Anal.*, 71(10):5032–5083, 2009.

[68] Mikhail Borsuk. *Transmission problems for elliptic second-order equations in non-smooth domains*. Frontiers in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2010.

[69] Darya E. Apushkinskaya and Aleksandr I. Nazarov. Linear two-phase Venttsel problems. *Ark. Mat.*, 39(2):201–222, 2001.

[70] Walter A. Strauss. Steady water waves. *Bull. Amer. Math. Soc. (N.S.)*, 47(4):671–694, 2010.

[71] Adrian Constantin. *Nonlinear water waves with applications to wave-current interactions and tsunamis*, volume 81 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

[72] Oliver Bühler, Jalal Shatah, and Samuel Walsh. Steady water waves in the presence of wind. *SIAM J. Math. Anal.*, 45(4):2182–2227, 2013.

[73] Erik Wahlén. Steady periodic capillary-gravity waves with vorticity. *SIAM J. Math. Anal.*, 38(3):921–943, 2006.

[74] Erik Wahlén. Steady periodic capillary waves with vorticity. *Ark. Mat.*, 44(2):367–387, 2006.

[75] J. R. Wilton. On ripples. *Phil.Mag.*, 29, 1915.

[76] Samuel Walsh. Steady stratified periodic gravity waves with surface tension I: Local bifurcation. *Discrete Contin. Dyn. Syst.*, 34(8):3241–3285, 2014.

[77] Samuel Walsh. Steady stratified periodic gravity waves with surface tension II: global bifurcation. *Discrete Contin. Dyn. Syst.*, 34(8):3287–3315, 2014.

[78] Calin Iulian Martin and Bogdan-Vasile Matioc. Existence of capillary-gravity water waves with piecewise constant vorticity. *J. Differential Equations*, 256(8):3086–3114, 2014.

[79] Anca-Voichita Matioc and Bogdan-Vasile Matioc. Capillary-gravity water waves with discontinuous vorticity: existence and regularity results. *Comm. Math. Phys.*, 330(2):859–886, 2014.

[80] C. J. Amick and R. E. L. Turner. A global theory of internal solitary waves in two-fluid systems. *Trans. Amer. Math. Soc.*, 298(2):431–484, 1986.

[81] S. M. Sun. Existence of solitary internal waves in a two-layer fluid of infinite depth. In *Proceedings of the Second World Congress of Nonlinear Analysts, Part 8 (Athens, 1996)*, volume 30, pages 5481–5490, 1997.

[82] S. M. Sun. Solitary internal waves in continuously stratified fluids of great depth. *Phys. D*, 166(1-2):76–103, 2002.

[83] R. E. L. Turner. Internal waves in fluids with rapidly varying density. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 8(4):513–573, 1981.

[84] Klaus Kirchgässner. Wave-solutions of reversible systems and applications. *J. Differential Equations*, 45(1):113–127, 1982.

[85] J. L. Bona, D. K. Bose, and R. E. L. Turner. Finite-amplitude steady waves in stratified fluids. *J. Math. Pures Appl. (9)*, 62(4):389–439 (1984), 1983.

[86] Katharina Lankers and Gero Friesecke. Fast, large-amplitude solitary waves in the 2D Euler equations for stratified fluids. *Nonlinear Anal.*, 29(9):1061–1078, 1997.

[87] Charles J. Amick. Semilinear elliptic eigenvalue problems on an infinite strip with an application to stratified fluids. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 11(3):441–499, 1984.

[88] Dag Viktor Nilsson. Internal gravity-capillary solitary waves in finite depth. *Math. Methods Appl. Sci.*, 40(4):1053–1080, 2017.

[89] Oliver Bühler, Jalal Shatah, Samuel Walsh, and Chongchun Zeng. On the wind generation of water waves. *Arch. Ration. Mech. Anal.*, 222(2):827–878, 2016.

[90] J. Thomas Beale, Thomas Y. Hou, and John S. Lowengrub. Growth rates for the linearized motion of fluid interfaces away from equilibrium. *Comm. Pure Appl. Math.*, 46(9):1269–1301, 1993.

[91] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[92] Robin Ming Chen and Samuel Walsh. Continuous dependence on the density for stratified steady water waves. *Arch. Ration. Mech. Anal.*, 219(2):741–792, 2016.

[93] János Bognár. *Indefinite Inner Product Spaces.* Springer-Verlag, New York-Heidelberg, 1974. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 78.

[94] I. S. Iohvidov, M. G. Krein, and H. Langer. *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric*, volume 9 of *Mathematical Research.* Akademie-Verlag, Berlin, 1982.

[95] Adrian Constantin and Walter Strauss. Periodic traveling gravity water waves with discontinuous vorticity. *Arch. Ration. Mech. Anal.*, 202(1):133–175, 2011.

[96] T. Alazard, N. Burq, and C. Zuily. On the water-wave equations with surface tension. *Duke Math. J.*, 158(3):413–499, 2011.

[97] Qingtang Su. Long time behavior of 2d water waves with point vortices. *arXiv preprint arXiv:1812.00540*, 2018.

[98] A. E. H. Love. On the Motion of Paired Vortices with a Common Axis. *Proc. Lond. Math. Soc.*, 25:185–194, 1893/94.

[99] H. Aref and N. Pomphrey. Integrable and chaotic motions of four vortices. I. The case of identical vortices. *Proc. Roy. Soc. London Ser. A*, 380(1779):359–387, 1982.

[100] B. Eckhardt and H. Aref. Integrable and chaotic motions of four vortices. II. Collision dynamics of vortex pairs. *Philos. Trans. Roy. Soc. London Ser. A*, 326(1593):655–696, 1988.

[101] Yieh Hei Wan. Desingularizations of systems of point vortices. *Phys. D*, 32(2):277–295, 1988.

[102] Hassan Aref. Point vortex dynamics: a classical mathematics playground. *J. Math. Phys.*, 48(6):065401, 23, 2007.

[103] Paul K. Newton. *The N-vortex problem*, volume 145 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 2001. Analytical techniques.

[104] Didier Smets and Jean Van Schaftingen. Desingularization of vortices for the Euler equation. *Arch. Ration. Mech. Anal.*, 198(3):869–925, 2010.

[105] Daomin Cao, Zhongyuan Liu, and Juncheng Wei. Regularization of point vortices pairs for the Euler equation in dimension two. *Arch. Ration. Mech. Anal.*, 212(1):179–217, 2014.

[106] Hassan Aref and Mark A. Stremler. Point vortex models and the dynamics of strong vortices in the atmosphere and oceans. In *Fluid mechanics and the environment: dynamical approaches (Ithaca, NY, 1999)*, volume 566 of *Lecture Notes in Phys.*, pages 1–17. Springer, Berlin, 2001.

[107] Varholm, Wahlen, and Walsh. On the stability of solitary water waves with a point vortex. 2018.

[108] Ali Rouhi and Jon Wright. Hamiltonian formulation for the motion of vortices in the presence of a free surface for ideal flow. *Phys. Rev. E (3)*, 48(3):1850–1865, 1993.

[109] V. E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9(2):190–194, Mar 1968.

[110] C. W. Curtis, J. D. Carter, and H. Kalisch. Particle paths in nonlinear Schrödinger models in the presence of linear shear currents. *J. Fluid Mech.*, 855:322–350, 2018.

[111] W. W. Willmarth, G. Tryggvason, A. Hirsa, and D. Yu. Vortex pair generation and interaction with a free surface. *Physics of Fluids A: Fluid Dynamics*, 1(2):170–172, 1989.

[112] Manoussos Grillakis, Jalal Shatah, and Walter Strauss. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.*, 74(1):160–197, 1987.

[113] Manoussos Grillakis, Jalal Shatah, and Walter Strauss. Stability theory of solitary waves in the presence of symmetry. II. *J. Funct. Anal.*, 94(2):308–348, 1990.

[114] W. Craig and C. Sulem. Numerical simulation of gravity waves. *J. Comput. Phys.*, 108(1):73–83, 1993.

[115] David Lannes. *The water waves problem*, volume 188 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013. Mathematical analysis and asymptotics.

[116] Alexander Mielke. On the energetic stability of solitary water waves. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, 360(1799):2337–2358, 2002. Recent developments in the mathematical theory of water waves (Oberwolfach, 2001).

[117] Michael G. Crandall and Paul H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Functional Analysis*, 8:321–340, 1971.

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