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The Vanishing Square, The Fibonacci Sequence, and The Golden Ratio

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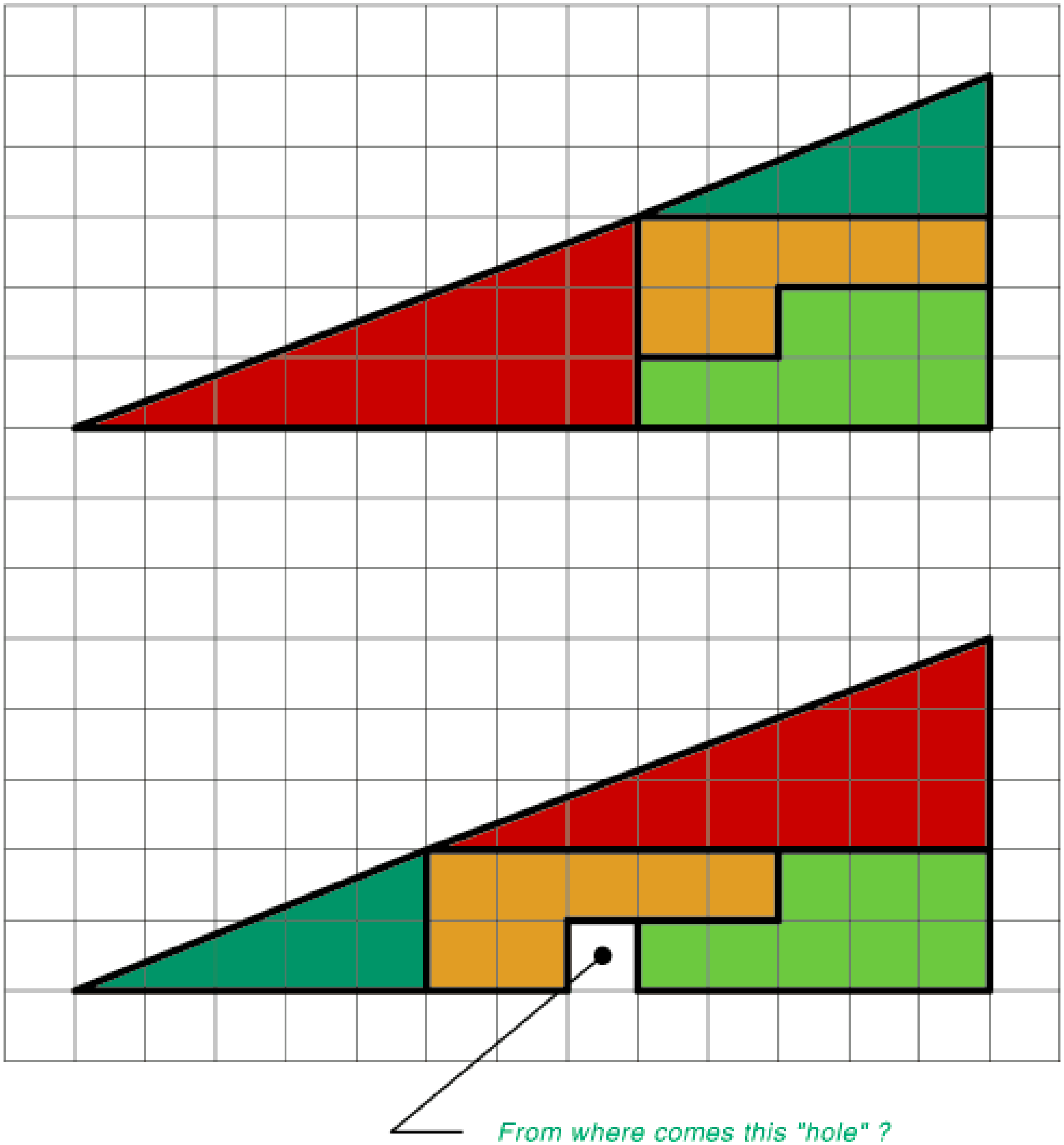
HOW CAN THIS BE TRUE ?



Below the four

parts are

moved around

The partitions

are exactly the

same, as those

used above

From where comes this "hole" ?

Hints for the Triangle Puzzle

- What is the slope of the red triangle?
- What is the slope of the blue-green triangle?
- Shouldn't these slopes be the same?
- Is the hypotenuse of the overall triangle really a straight line?
- Is the overall triangle really a "triangle" at all?

Solution to the Triangle Puzzle

- The overall shape is actually not a triangle at all, but a quadrilateral (4-sided figure) instead!
- The slopes of the red and blue-green segments are so close (2/5 = 0.400; 3/8 = 0.375) that the eye is fooled into believing that they are identical.

Heights and Bases in the Triangle Figure

- Note that the height and base of the blue-green triangle are 2 units and 5 units, respectively, and the height and base of the red triangle are 3 units and 8 units, respectively.
- Also the height and base of the overall quadrilateral are 5 and 13, respectively.
- These are all Fibonacci numbers!

The Fibonacci Sequence

- The Fibonacci Sequence is:

  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …
- Each number in the sequence (after the first two) is the sum of the two immediately previous numbers.
- These numbers form the solution to a famous problem posed by the mathematician Fibonacci in 1202 AD.

Fibonacci

- Fibonacci = Leonardo of Pisa (≈ 1175–1250) [“Fibonacci” = “Filius Bonaccio” = “son of Bonaccio”]
- Fibonacci was born in Pisa, Italy and educated in North Africa, where his father was in charge of a customs-house.
- Fibonacci traveled extensively throughout the Mediterranean, observing all the commercial arithmetic systems used by various countries.

Abacists vs. Algoriths

- Fibonacci wrote *Liber Abaci* (= "Book of Counting") in 1202 to introduce the Hindu-Arabic numbers (which we use today) to Europeans. (2nd edition: 1228)
- This led to an almost 250-year conflict in Europe between the abacists (who wanted to retain Roman numerals) and the algoriths (who wanted to use the Hindu-Arabic numbers instead). (Who won?)

The Rabbit Problem

- *Liber Abaci* is equally famous today for a problem that Fibonacci posed in Book XII:
  - *A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair, which from the second month on becomes productive?*

What's Up, Doc?

- In other words, we begin with 1 pair of adult rabbits (i.e., 1 male and 1 female).
- Each month (right on schedule!) each <u>adult</u> pair of rabbits produces 1 additional pair (always 1 male and 1 female).
- But each young pair cannot have offspring until they reach adulthood – that is, until after 1 month has passed.
- (Rabbit incest going on here – shocking!)

Rabbit Transit – Not!

- We also are assuming that none of the rabbits die, and all the rabbits are fertile.
- We are also assuming none of the rabbits are removed and no additional rabbits are introduced (because of the outside wall).

Rabbit Totals

Month	Adult Pairs	Young Pairs
1	1	1
2	2	1
3	3	2
4	5	3
5	8	5
6	13	8
7	21	13
8	34	21
9	55	34
10	89	55
11	144	89
12	233	144

Solution to the Rabbit Problem

- This gives the solution to the rabbit problem:

  The total number of pairs of rabbits produced from the first pair after 12 months =

  233 (the number of adult pairs)

  \+ 144 (the number of young pairs)

  – 1 (the original pair)

  = **376**

- The numbers in the "young pairs" column form the **Fibonacci sequence**.

The Fibonacci Sequence

- The usual notation for the Fibonacci numbers is:

$F_1 = 1$	$F_6 = 8$
$F_2 = 1$	$F_7 = 13$
$F_3 = 2$	$F_8 = 21$
$F_4 = 3$	$F_9 = 34$
$F_5 = 5$	$F_{10} = 55$, etc.

- Therefore, $F_{n+2} = F_n + F_{n+1}$, for $n \geq 1$.

The Fibonacci Sequence

Fibonacci couldn't sleep ---
Counted rabbits instead of sheep.
(Katherine O'Brien)

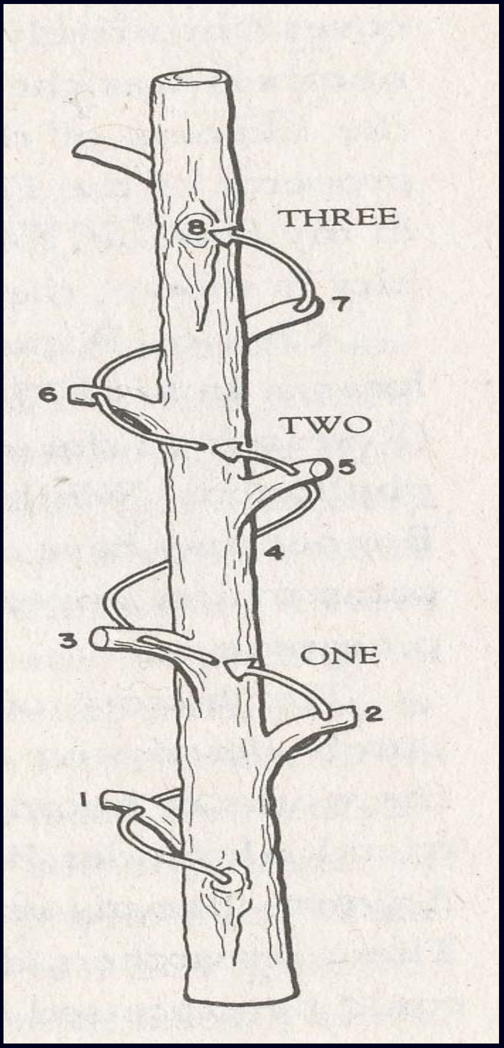
Each wife of Fibonacci,
Eating nothing that wasn't starchy,
Weighed as much as the two before her,
His fifth was some signora!
(J. A. Lindon)

Fibonacci Numbers in Nature: Phyllotaxis ("leaf arrangement")

- Leaves (or stems) on a branch do not often grow directly above/below each other since the lower ones would not get enough sunlight and moisture.
- On the elm and basswood, going from one leaf to the next involves ½ of a turn:
  ½ phyllotactic ratio
- On the hazel, blackberry, and beech, going from one leaf to the next involves ⅓ of a turn:
  ⅓ phyllotactic ratio

Phyllotaxis (continued)

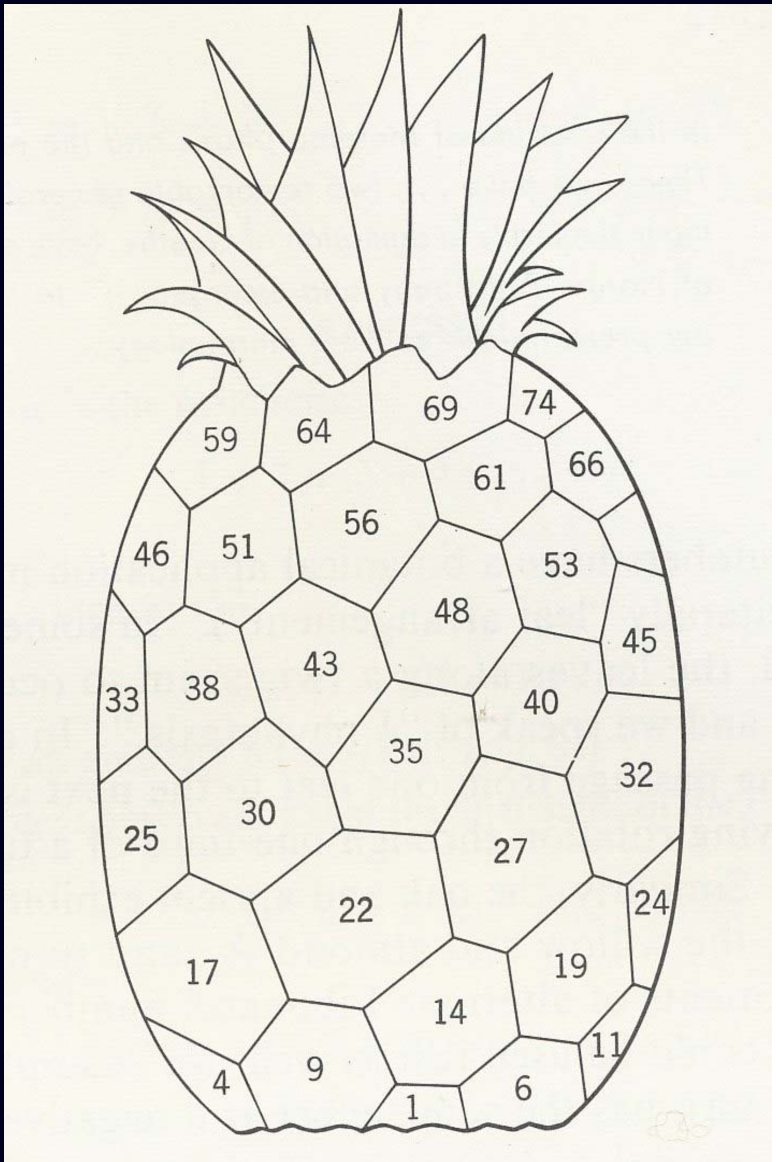
- On the apple, coast live oak, and apricot, going from one leaf to the next involves 2/5 of a turn: 2/5 phyllotactic ratio
- On the poplar, pear and weeping willow, going from one leaf to the next involves 3/8 of a turn: 3/8 phyllotactic ratio
- On the willow and almond: 5/13 phyllotactic ratio
- Each of these cases involves <u>alternate</u> Fibonacci numbers!

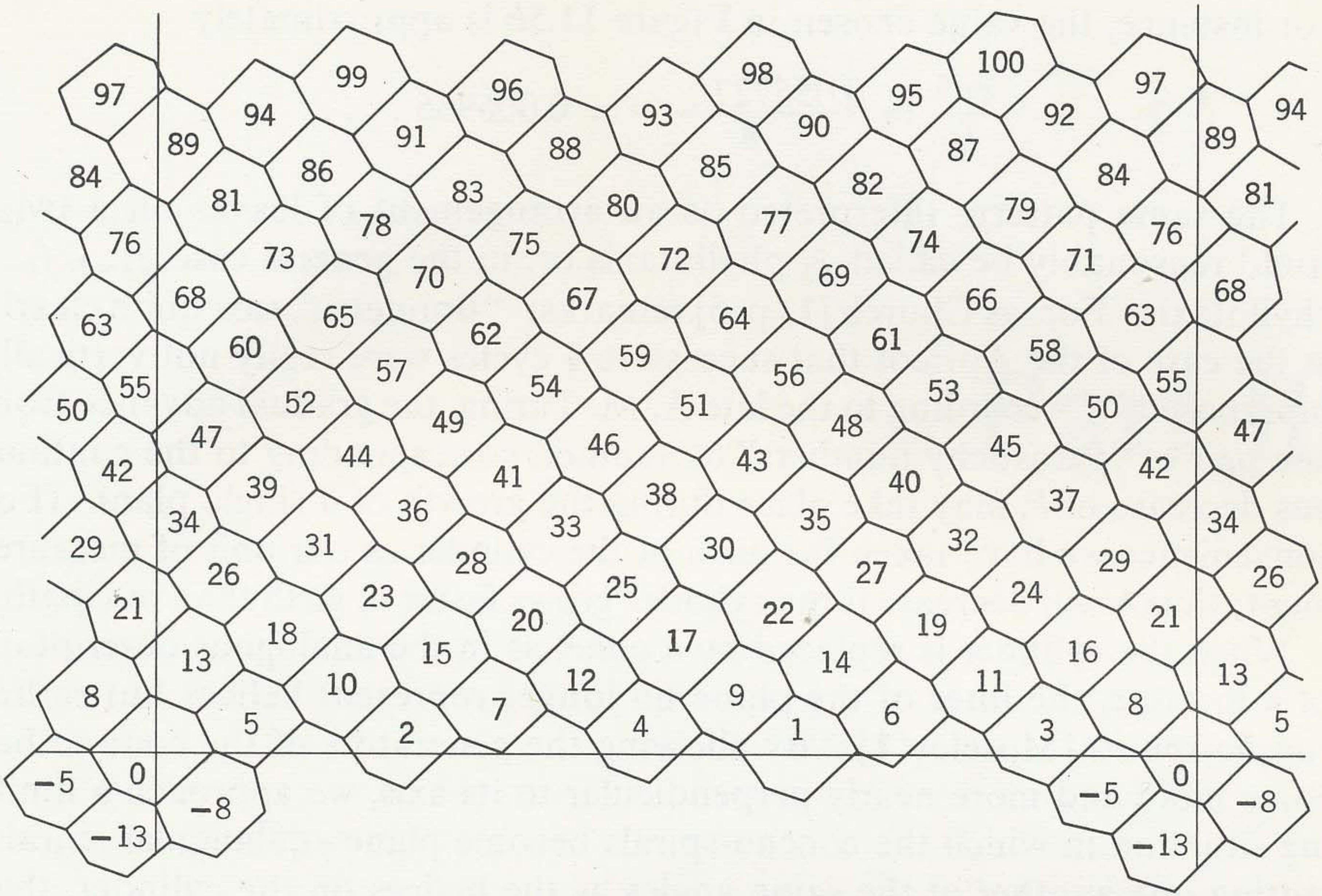


Pinecones and Pineapples

Pineapples and Fibonacci

- The scales of a pineapple are hexagons. There are 3 sets of parastitchies (parallel spirals): 1) sloping gently from lower left to upper right, 2) more steeply from lower right to upper left, and 3) very steeply from bottom to top.
- The number of spirals of each type is generally 5, 8, 13, or 21: Fibonacci numbers!

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14
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3
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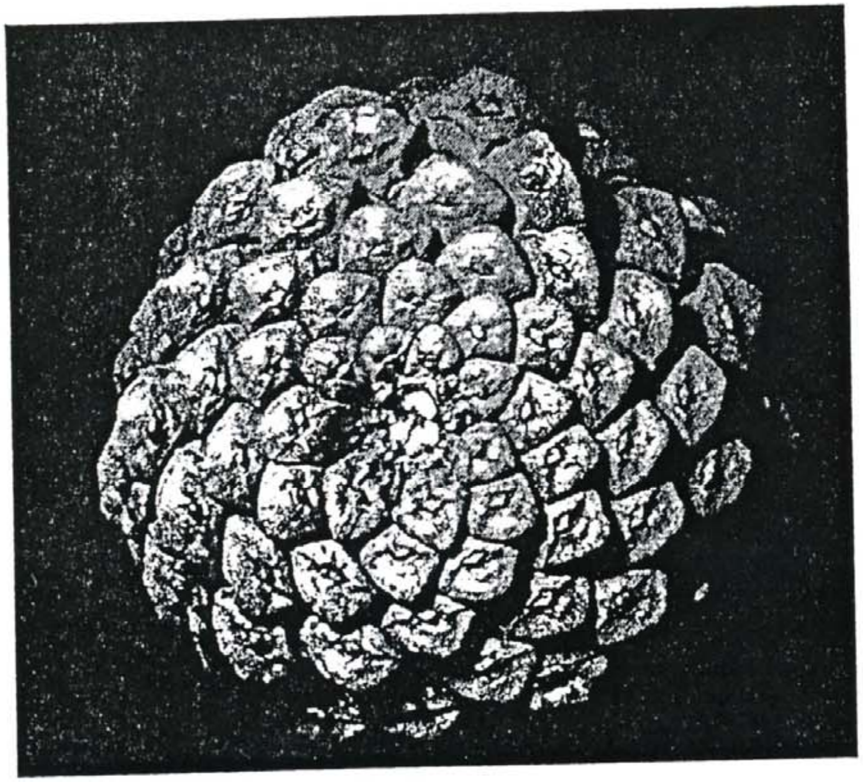
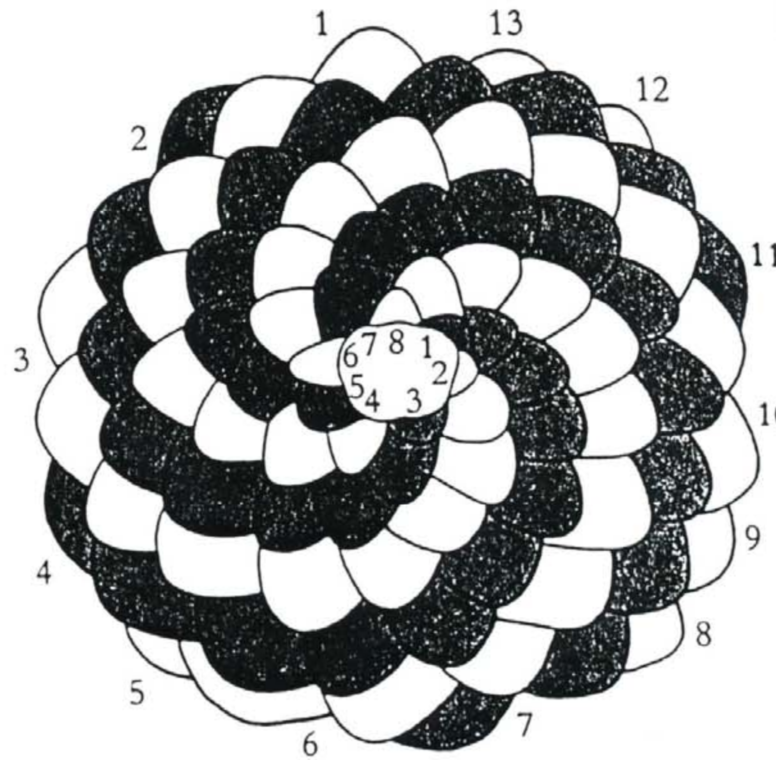
Pinecones

The pinecone also has spirals sloping from lower left to upper right, and from lower right to upper left.

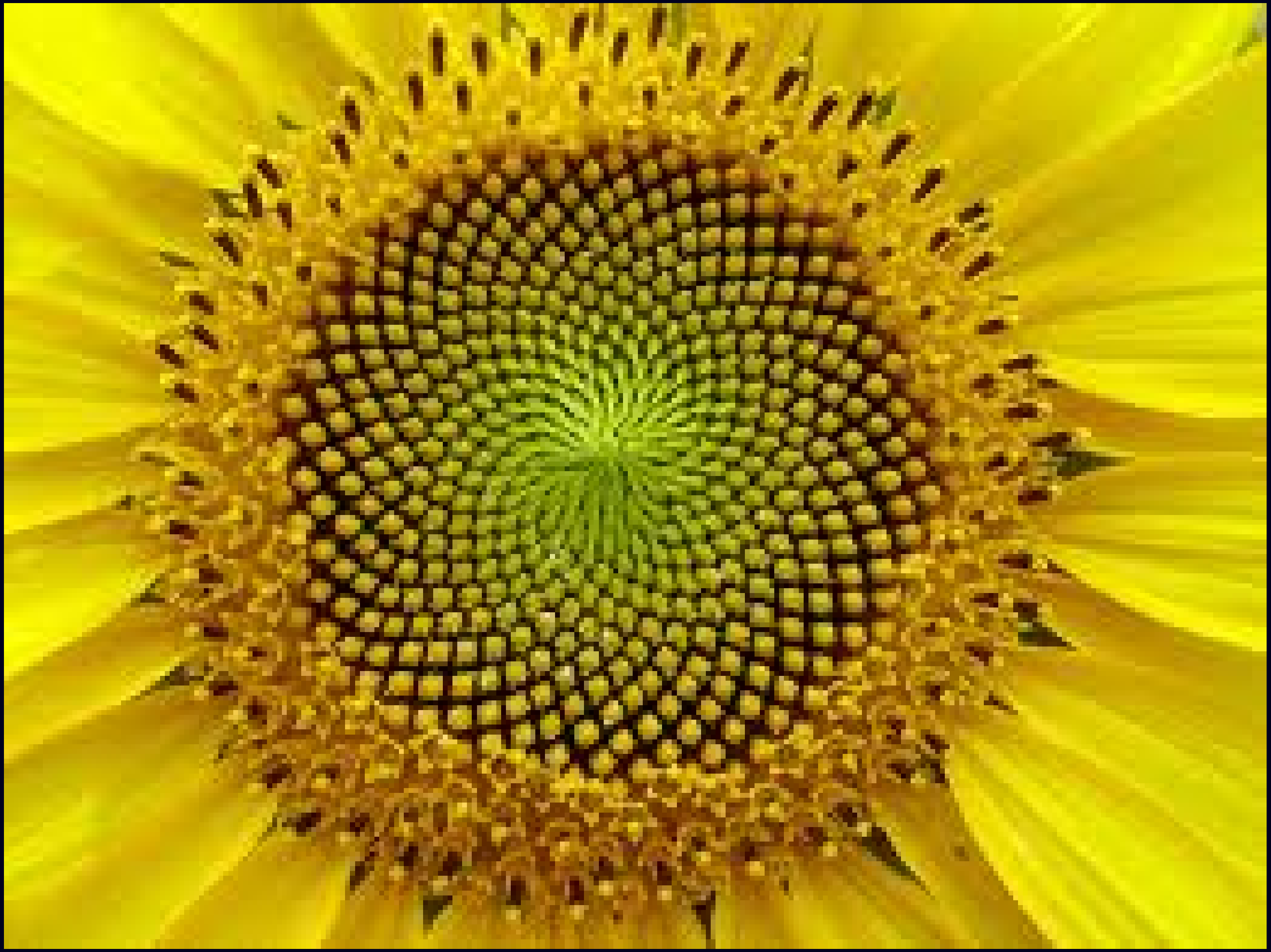
When viewed from one end, we see a series of clockwise and counterclockwise spirals.



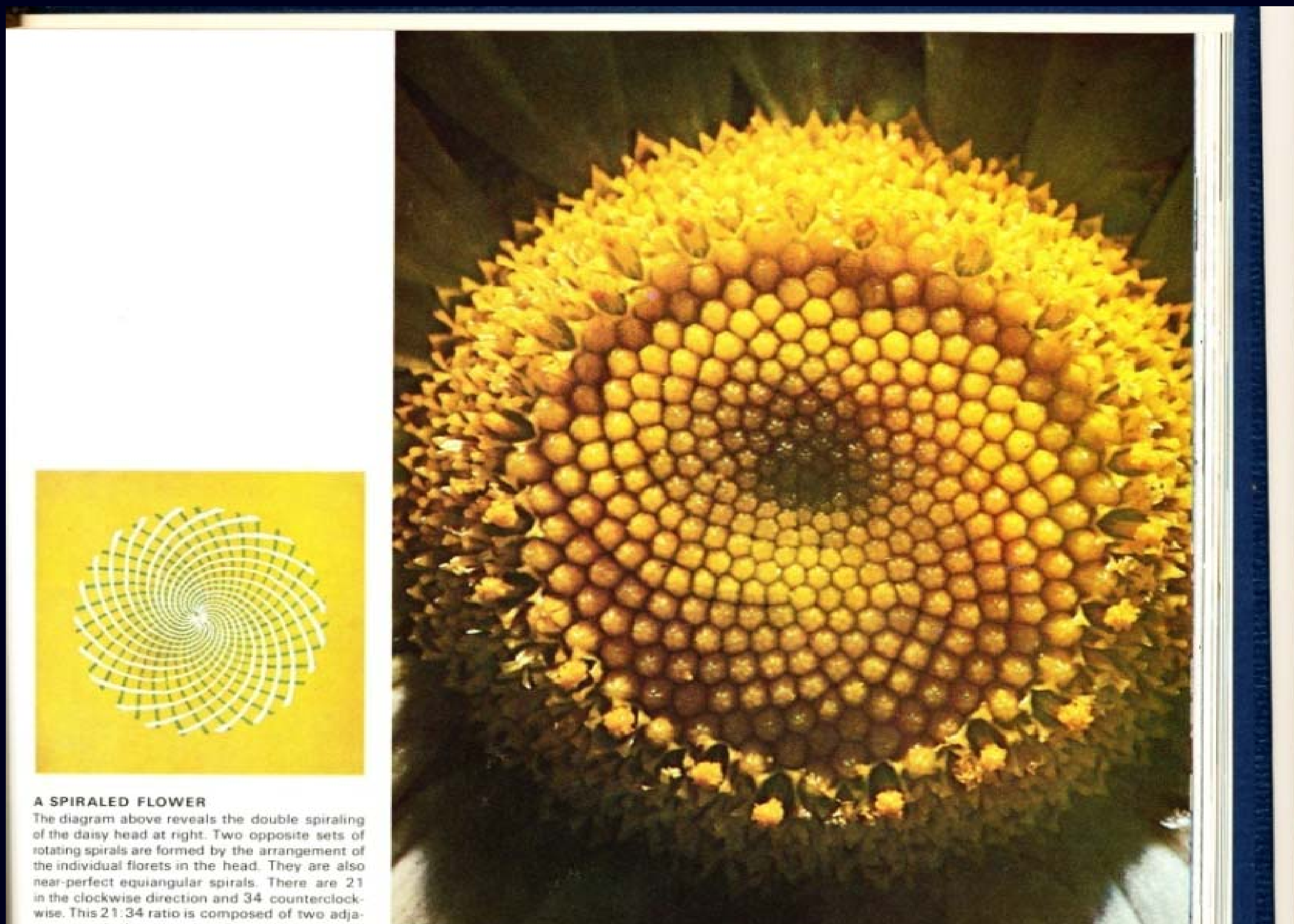


Here there are two sets of spirals:
8 numbered in a clockwise manner, and
13 numbered in a counterclockwise manner.

Sunflowers



Field Daisies



A SPIRALED FLOWER

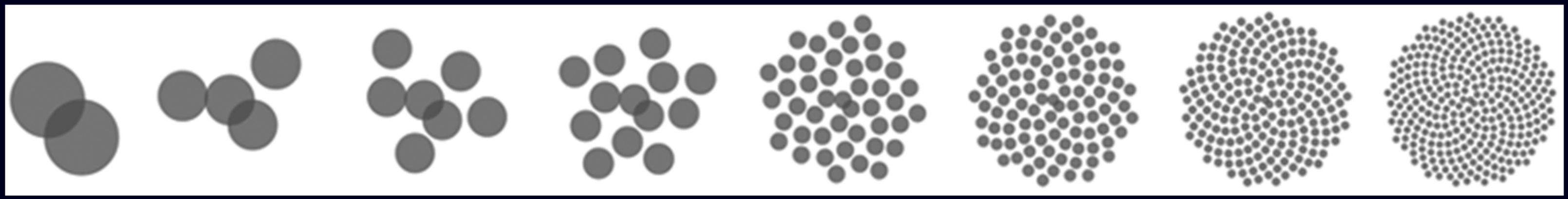
The diagram above reveals the double spiraling of the daisy head at right. Two opposite sets of rotating spirals are formed by the arrangement of the individual florets in the head. They are also near-perfect equiangular spirals. There are 21 in the clockwise direction and 34 counterclockwise. This 21:34 ratio is composed of two adjacent terms in the mysterious Fibonacci sequence

Sunflowers and Field Daisies

- The florets of sunflowers also contain both clockwise and counterclockwise spiral patterns. The number of spirals in each direction is usually 34 and 55, respectively.
- However, there are sunflowers containing spiral numbers of 55 and 89, 89 and 144, and even one (reported to *Scientific American* in 1951 by a Vermont couple) with 144 and 233 spirals!
- Most field daisies have 13, 21, or 34 petals ("She loves me; she loves me not…" – Valentines: Choose a daisy with an odd number of petals!)
- Again, all of these are Fibonacci numbers!

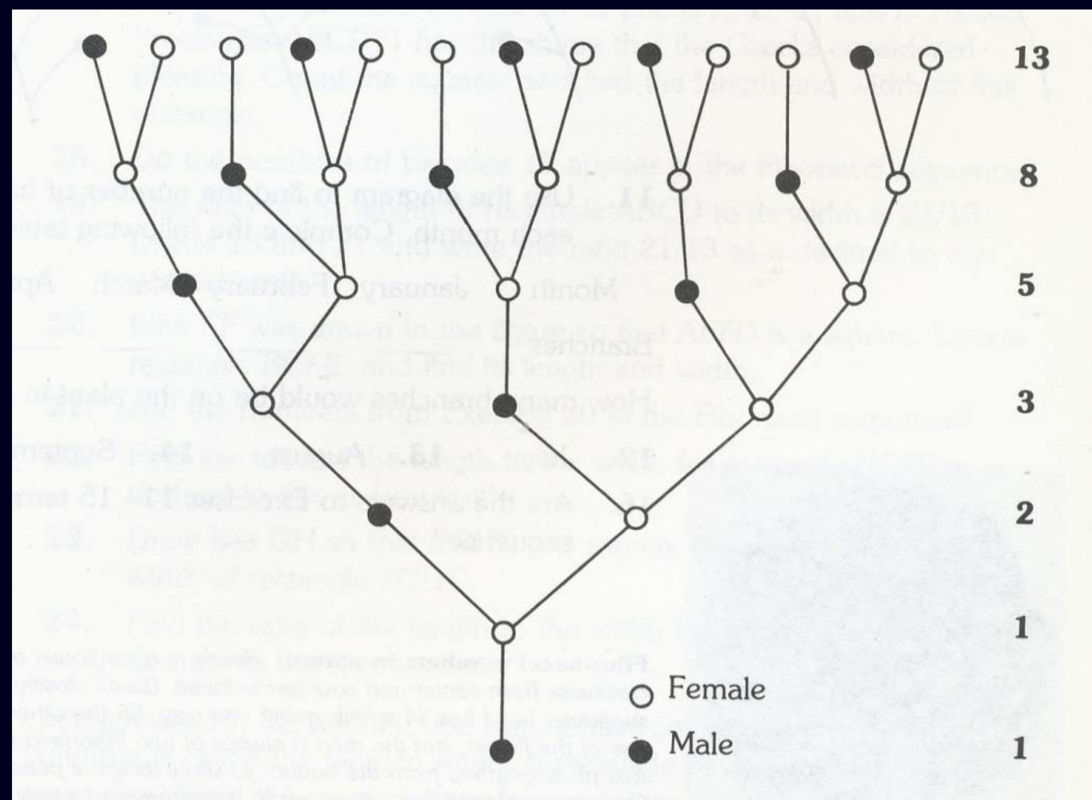
Why Fibonacci?

- Small primordia ("buds") grow around the center (tip). As more buds begin to develop, they repel each other, possibly because they are competing for essential nutrients.
- Also, the growing structure strives for <u>homogeneity</u> (structure is essentially the same everywhere). This apparently leads to an optimal solution using Fibonacci numbers.



Family Tree of a Male Bee

- While a female bee has two parents (father & mother), a male bee (from a non-fertilized egg) has only one parent (the mother).
- The number of ancestors of a male bee in each previous generation is a Fibonacci number!



The Fibonacci Quarterly

- The Fibonacci Association was formed in 1963 by Verner Hoggatt and Brother Alfred Brousseau, F.S.C., to "exchange ideas and stimulate research in Fibonacci numbers and related topics."
- Since 1963, the Fibonacci Association has published a mathematical journal entitled *The Fibonacci Quarterly* devoted entirely to the properties and application of Fibonacci numbers.

Brother Alfred Brousseau, F.S.C. (1907–1988)

An Interesting Property of the Fibonacci Sequence

- A property of the Fibonacci Sequence that is related to the Vanishing Square Puzzle is:

If *n* is odd, then

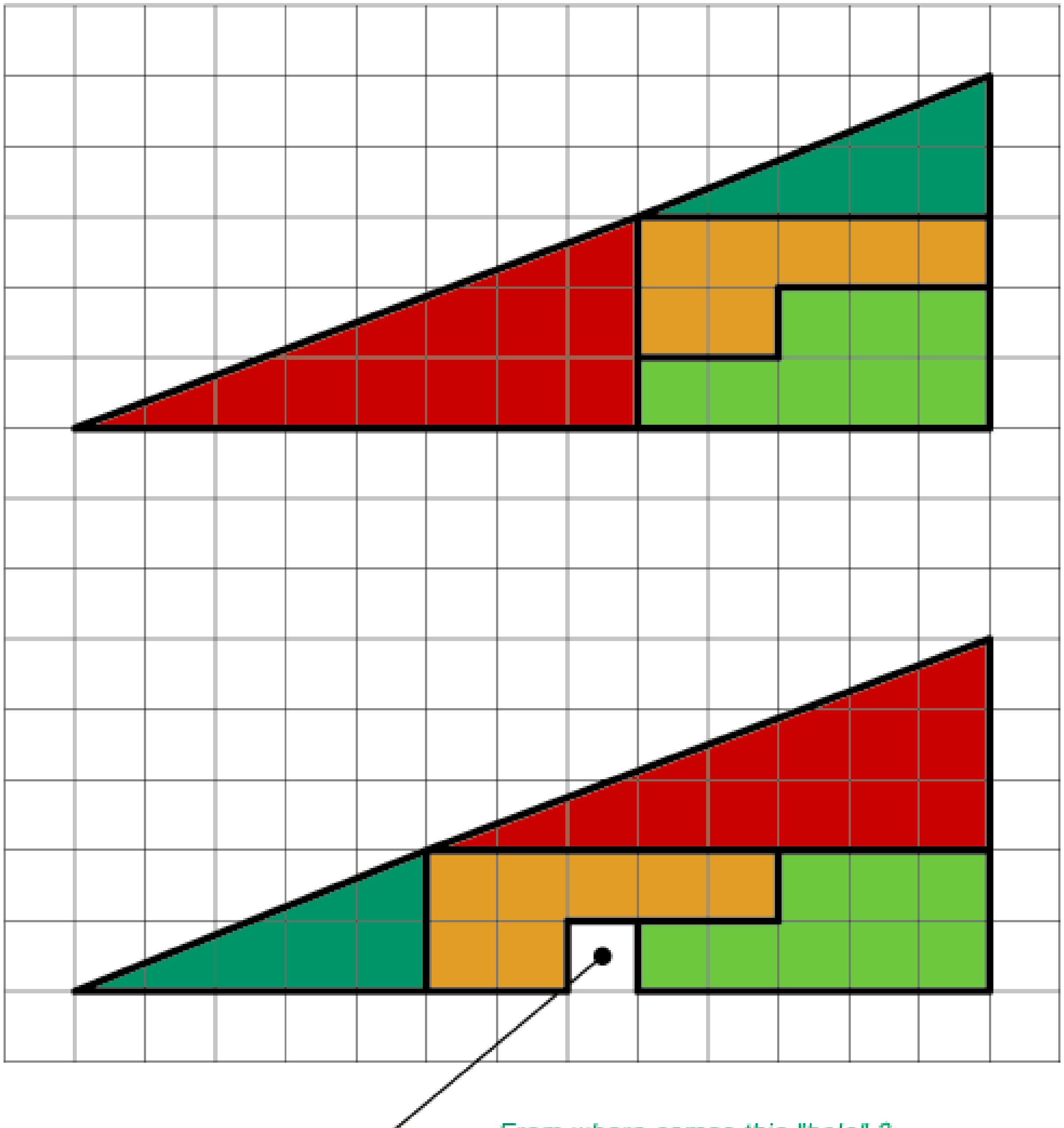
$$F_n \times F_{n+3} = (F_{n+1} \times F_{n+2}) + 1$$

- For example, when *n* = 3 (odd), we have:

$$F_3 \times F_6 = (F_4 \times F_5) + 1$$

$$2 \times 8 = (3 \times 5) + 1$$

HOW CAN THIS BE TRUE ?



Below the four

parts are

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The partitions

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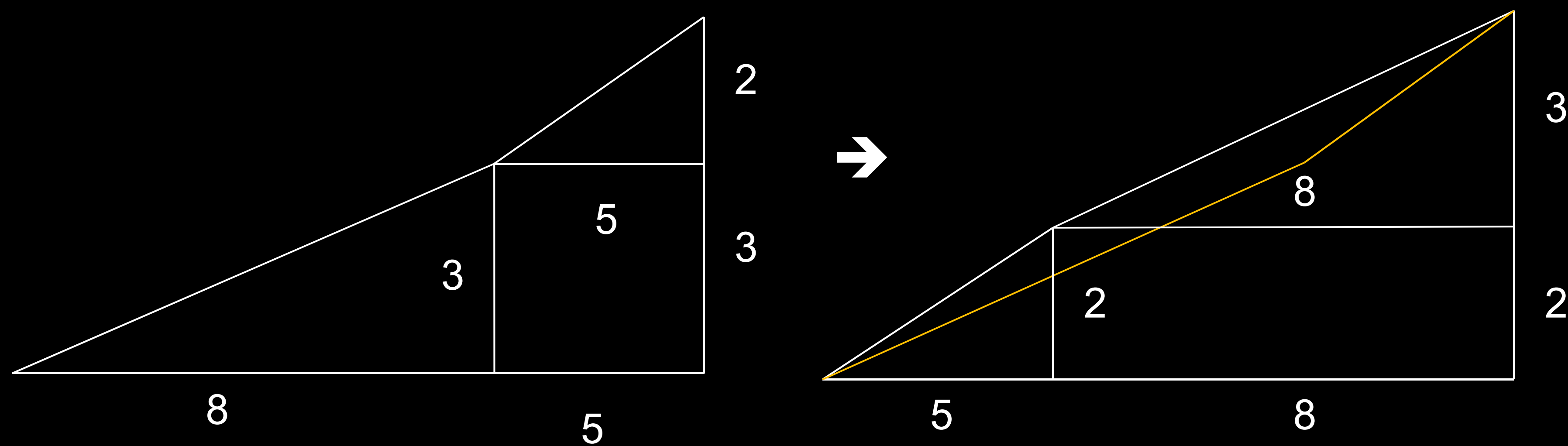
From where comes this "hole" ?

The Vanishing Square Revealed!

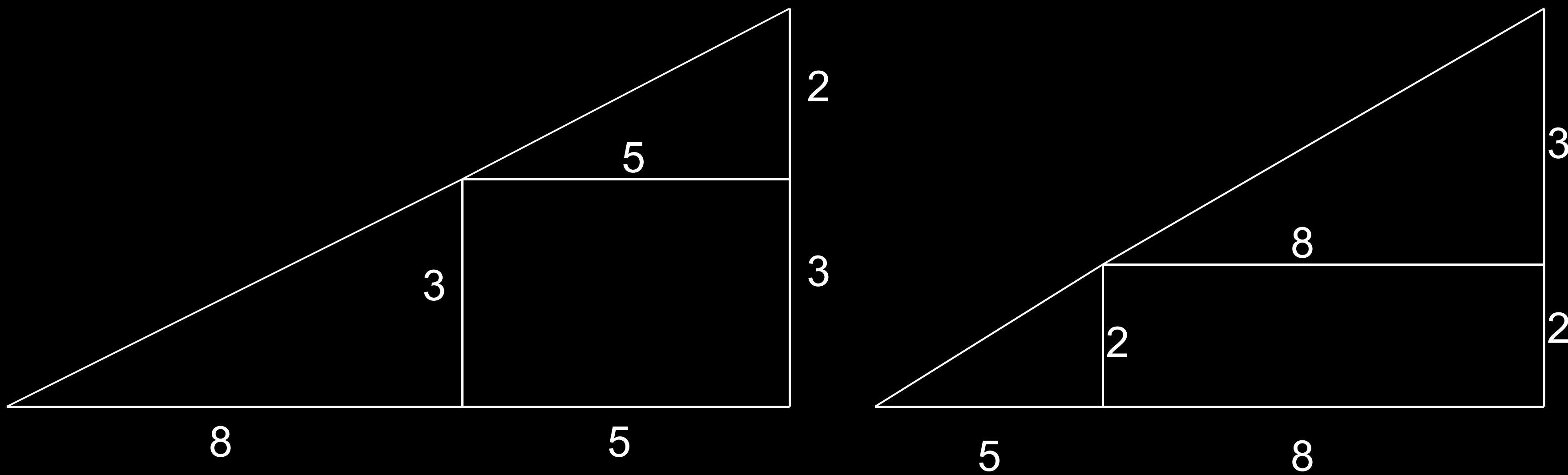
- On top of the Vanishing Square Puzzle: combine the gold and light green regions into a single rectangle, whose dimensions are Fibonacci numbers! (F_4 = 3 by F_5 = 5) Area = 3 x 5 = 15.
- Use the previous property to replace this rectangle with a new rectangle (F_3 = 2 by F_6 = 8). Area = 2 x 8 = 16.
- The new rectangle's dimensions are those of the remaining two triangular sides!!!! The area is 1 unit greater!!!!



Another Look at the Puzzle

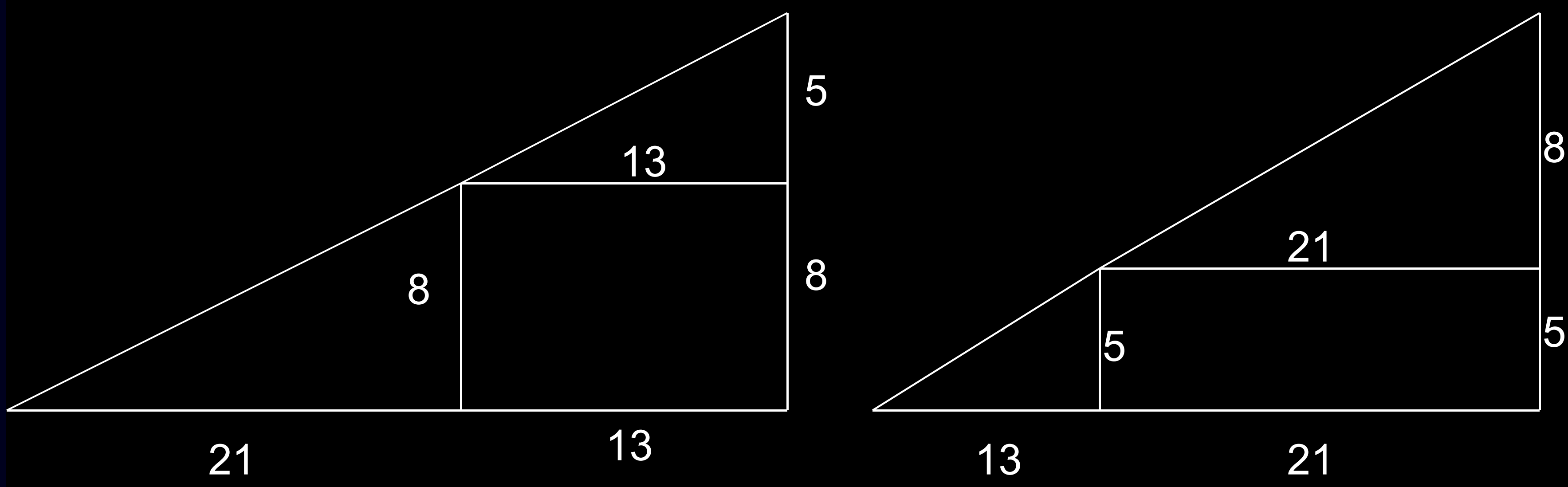
- **Total area = 32 square units** **Total area = 33 square units**
- **Slopes of the triangles in both figures are:**

  2/5 = 0.400 and 3/8 = 0.375 – close, as we saw earlier!



Puzzle for the Next Odd Case

- **Total area = 220.5 square units** **Total area = 221.5 square units**
- **Here the slopes of the triangles are:**
  5/13 = 0.3846 and 8/21 = 0.3809 – even closer than the previous case!



Successive Fibonacci Ratios

- We have looked at ratios of some Fibonacci numbers. What about the ratios of <u>consecutive</u> Fibonacci numbers?

1/1 = 1.000000	**34/21 = 1.619048**
2/1 = 2.000000	**55/34 = 1.617647**
3/2 = 1.500000	**89/55 = 1.618182**
5/3 = 1.666667	**144/89 = 1.617978**
8/5 = 1.600000	**233/144 = 1.618056**
13/8 = 1.625000	**377/233 = 1.618026**
21/13 = 1.615385	**610/377 = 1.618037**

- These ratios appear to approach a number close to 1.618…

The Golden Ratio

- In fact, the ratio of successive Fibonacci numbers gets closer and closer to a number ϕ (phi = “fee”) known as the “Golden Ratio”:

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

- This was first discovered by astronomer-mathematician Johannes Kepler.
- On the next slide are displayed the first 79 decimal digits of the Golden Ratio:


1.6180339
887498948
482045868
343656381
177203091
798057628
621354486
227052604
628189024

Definition of the Golden Ratio

- The Golden Ratio is one of the two solutions to the equation $x^2 - x - 1 = 0$, which are:

  1.6180339887… $= \dfrac{1+\sqrt{5}}{2}$ and

  0.6180339887… $= \dfrac{-1+\sqrt{5}}{2}$

  Strangely, the second solution is exactly 1 unit smaller than the first, as well as the reciprocal of the first solution!

Reciprocal of the Golden Ratio

The reason is that the original equation $x^2 - x - 1 = 0$, implies that $x^2 = x + 1$, so the Golden Ratio has the property that:

$$x = \frac{x}{1} = \frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x}$$

$$\Rightarrow\ x - 1 = \frac{1}{x}$$

so the reciprocal of the Golden Ratio is <u>exactly</u> 1 less than the Golden Ratio itself.

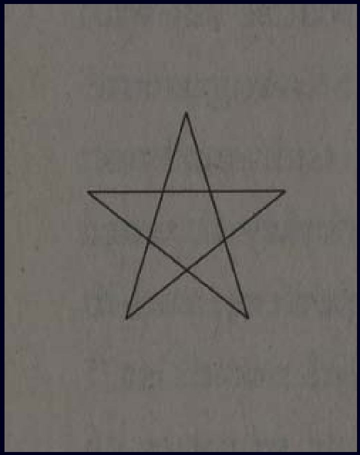
Reciprocal of the Golden Ratio

The golden mean is quite absurd;
It's not your ordinary surd.
If you invert it (this is fun!), ($1/x = x - 1$)
You'll get itself, reduced by one;
But if increased by unity, ($x + 1 = x^2$)
This yields its square, take it from me.

(Paul Bruckman, 1977)

Origin of the Golden Ratio

- The Golden Ratio was probably discovered by the Pythagoreans (≈ 6th or 5th century BC), a mathematical cult whose symbol was the pentagram.
- The Pythagoreans believed "number" was the cause of everything: 1 = reason, 2 = man, 3 = woman, 4 = justice, 5 = marriage, 10 = 1 + 2 + 3 + 4 = the four "elements" (fire, earth, air, water).



Rational vs. Irrational

- The discovery that some numbers are <u>not</u> rational (= a ratio of two integers – i.e., a fraction) was frightening to the Pythagoreans, who tried to suppress this information. (It leaked out!)
- In fact, it can be shown that the Golden Ratio is an <u>irrational</u> (non-rational) number.

Pythagoras

Did stagger us

And our reason encumber

With irrational number .

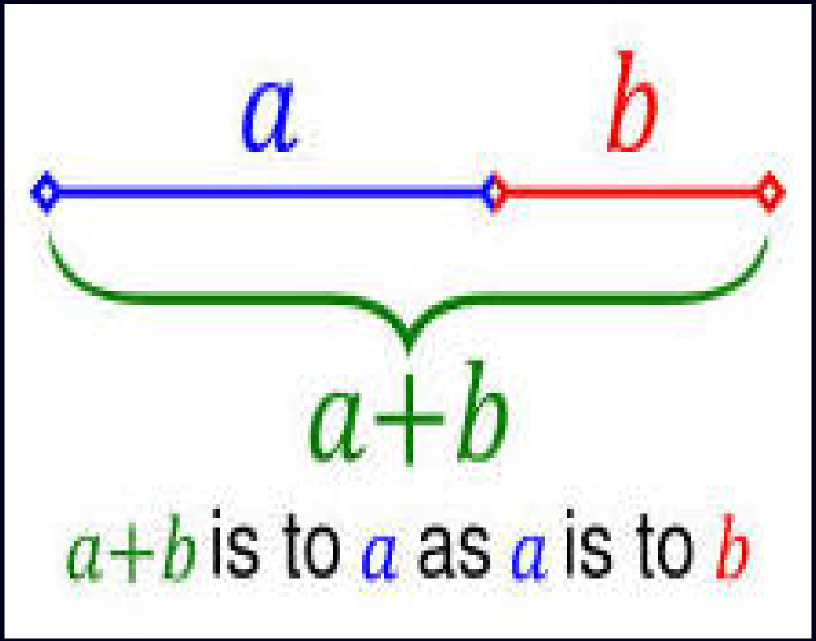
(Stephen Cushing, 1988)

Debut of the Golden Ratio

- The earliest known appearance of the golden ratio is in Euclid's *Elements* (≈ 300 BC).
- Euclid's *Elements* (13 books) is considered the greatest and most influential mathematics work ever written. (Only the Bible sold more copies than Euclid's *Elements* until the 20th century).

Euclid and the Golden Ratio

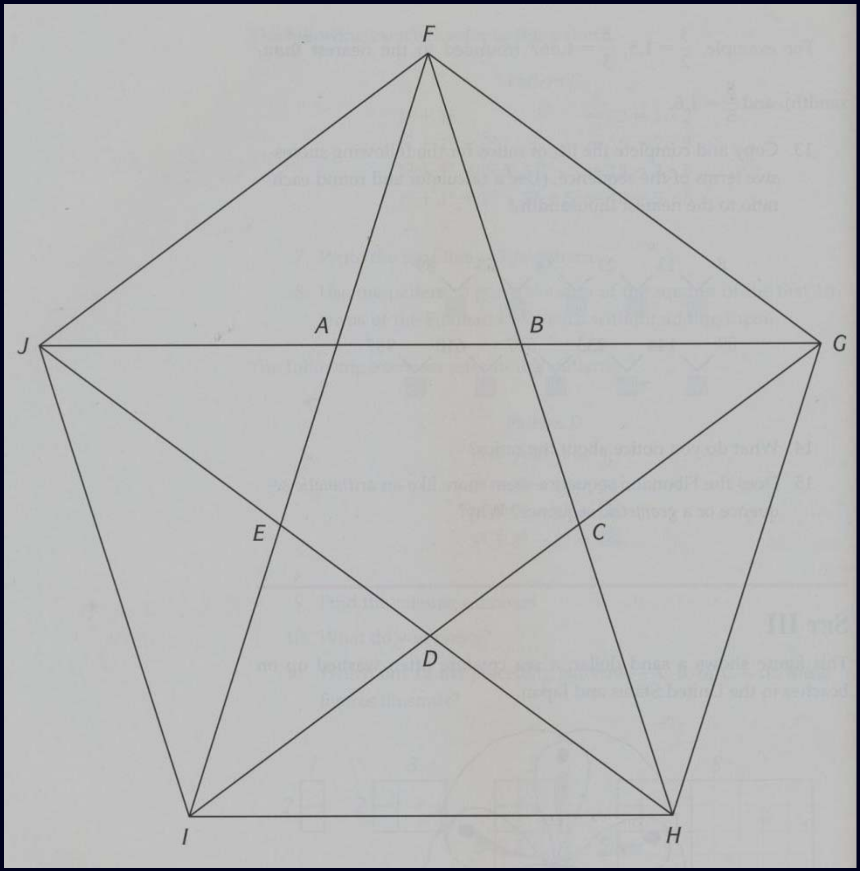
- In Books II and VI of Euclid's *Elements*, Euclid considers a segment broken at an internal point as shown:



- This is the Golden Ratio, but Euclid called it the "mean and extreme ratio." ("*a*" is the "mean" between the other two extremes.)

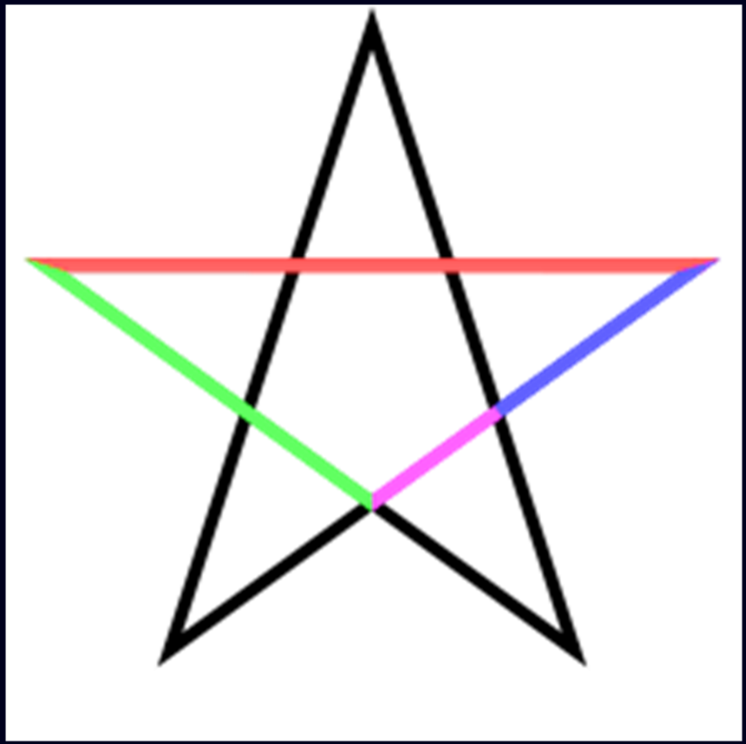
The Regular Pentagon

- In a regular pentagon, the ratio of a diagonal to a side is the Golden Ratio.
- Also, the diagonals of a regular pentagon partition each other in the Golden Ratio!
- By merely drawing the diagonals of a regular pentagon, we construct the Golden Ratio!



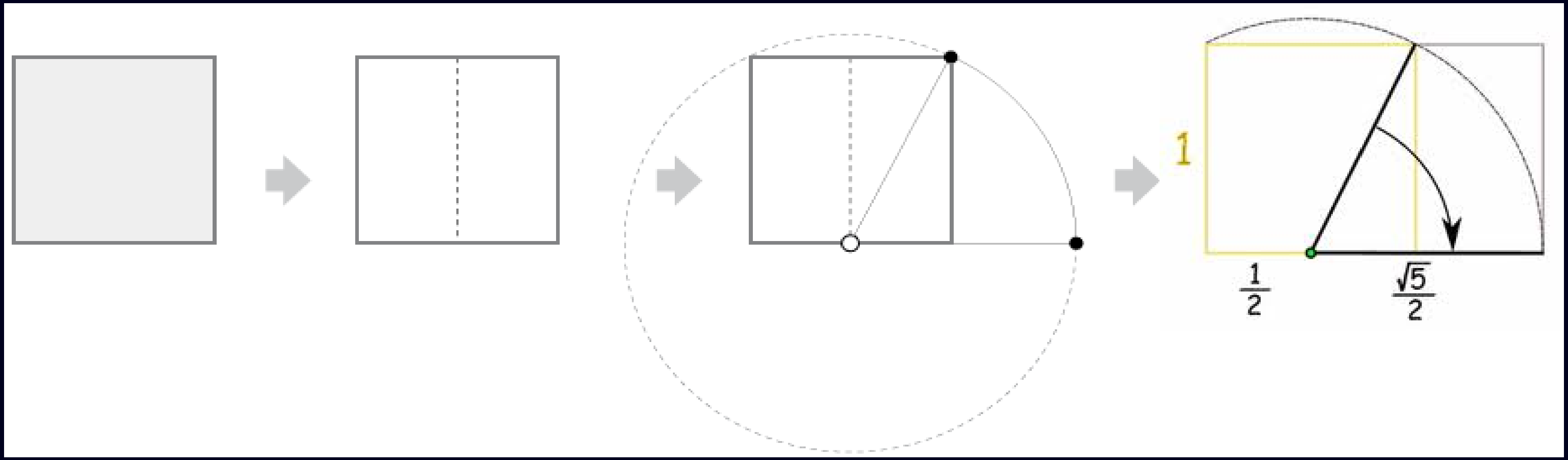
Return of the Pentagram!

- In fact, each of the 3 smallest lengths in color forms the Golden Ratio with its next largest length!
- The pentagram (as a filled in "star") has been carried over as a symbol in our American flag:



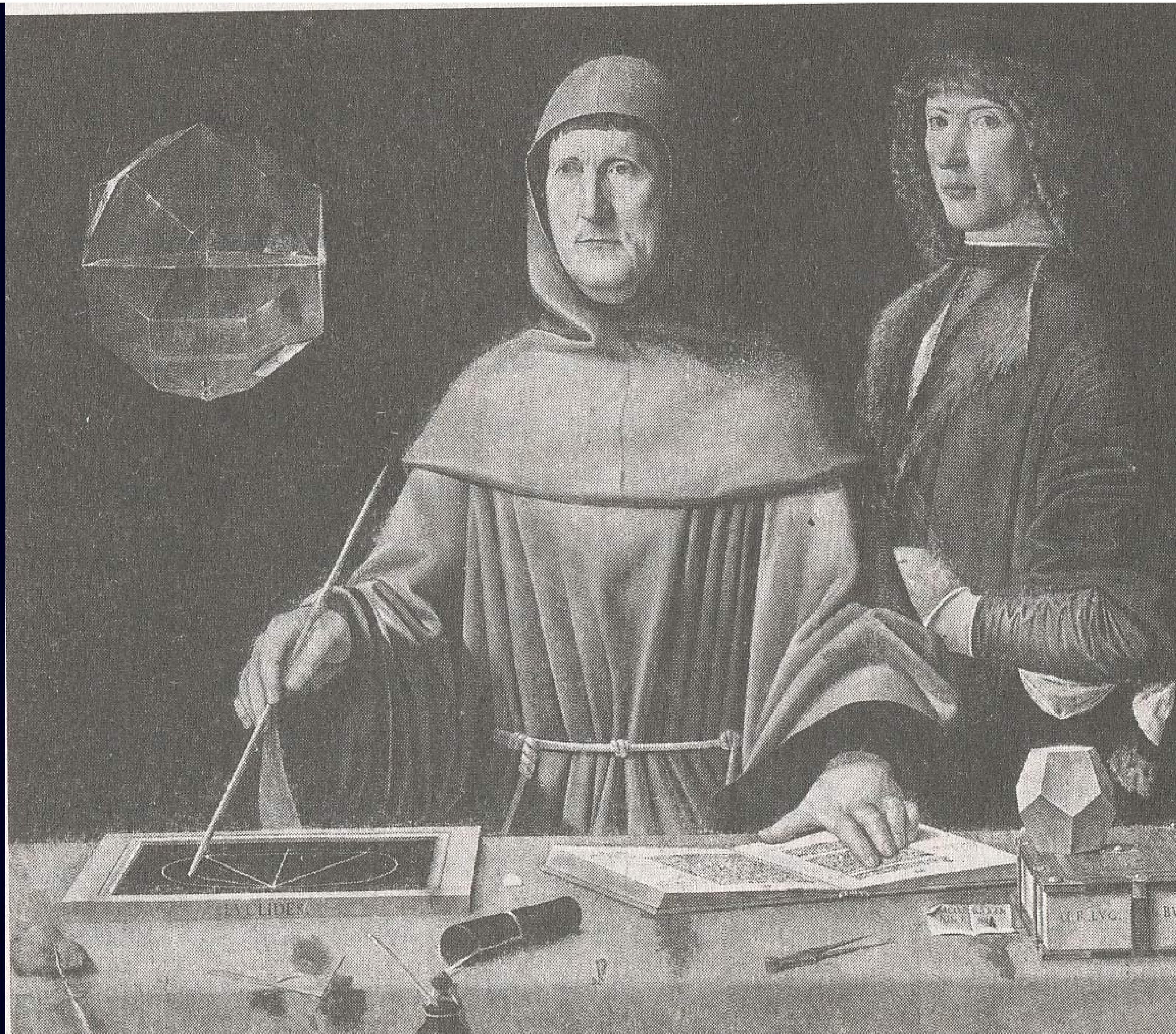
Another Construction of the Golden Ratio

With a square of side 1, form a diagonal from the midpoint of one side to an opposite corner. This diagonal has length $\frac{\sqrt{5}}{2}$. When added to half of one side, we get the Golden Ratio: $\frac{1}{2}+\frac{\sqrt{5}}{2}=\frac{1+\sqrt{5}}{2}$



Fra Luca Pacioli

- The Golden Ratio was popularized by Fra Luca Pacioli (a Franciscan friar) (1445–1517).
- Pacioli wrote *Divina Proportione* (1509) (3 vols., based on the work of Piero della Francesca) devoted to the Golden Ratio, which he called “The Divine Proportion.”
- The following portrait of Pacioli giving a lesson in mathematics is by Jacopo de’Barbari (1440–1515), and has been called the “best portrait of a mathematician ever produced”:



Humans and the Golden Ratio

- Fra Pacioli claimed the “ideal” human has all the following ratios equal to ϕ:
  - height of head : width of head
  - chin to eyes : chin to nose
  - eyes to mouth : eyes to nose
  - width of head : width of throat
  - width of forearm : width of wrist
  - navel to top of head : armpit to top of head
  - widest part of thigh : narrow part of thigh
  - width of calf : width of ankle

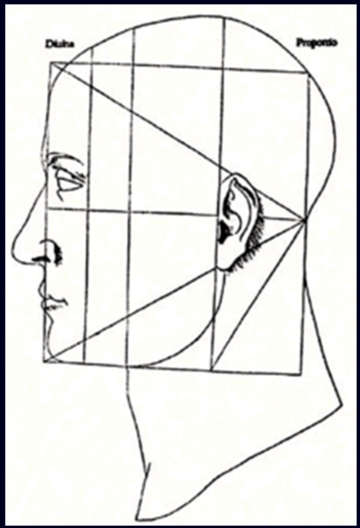
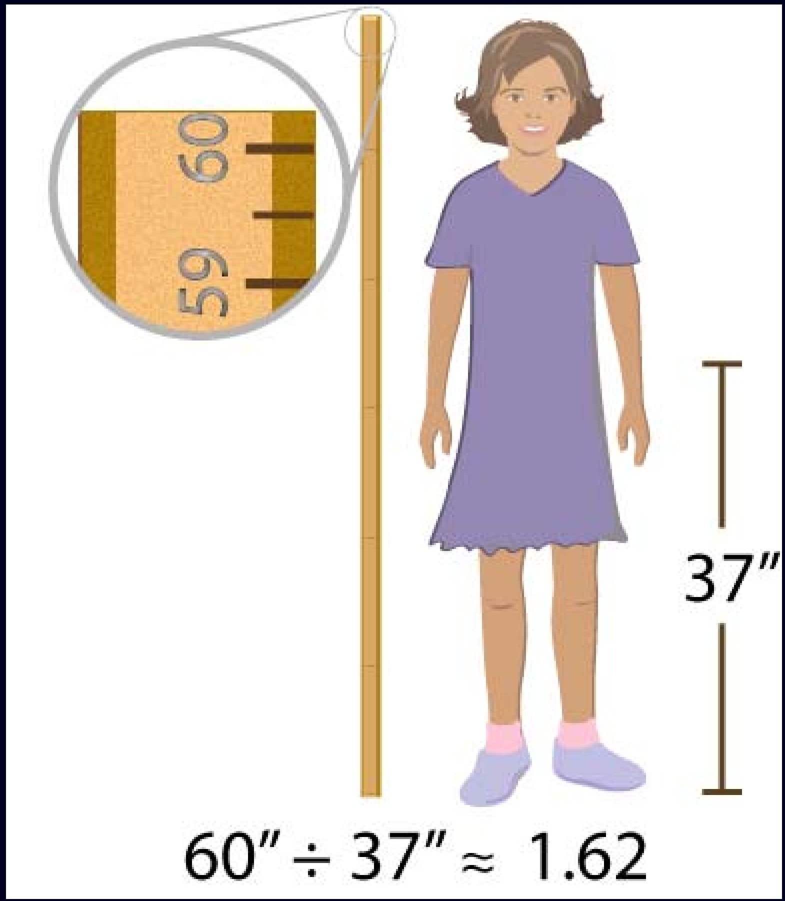


Illustration from Pacioli's *De Divina Proportione*

Women's Height and the Golden Ratio

- The average ratio of the height of a woman to the height of her navel is supposedly close to ϕ.

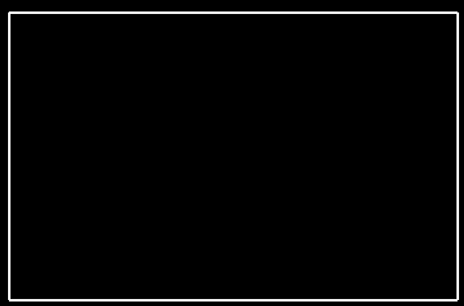


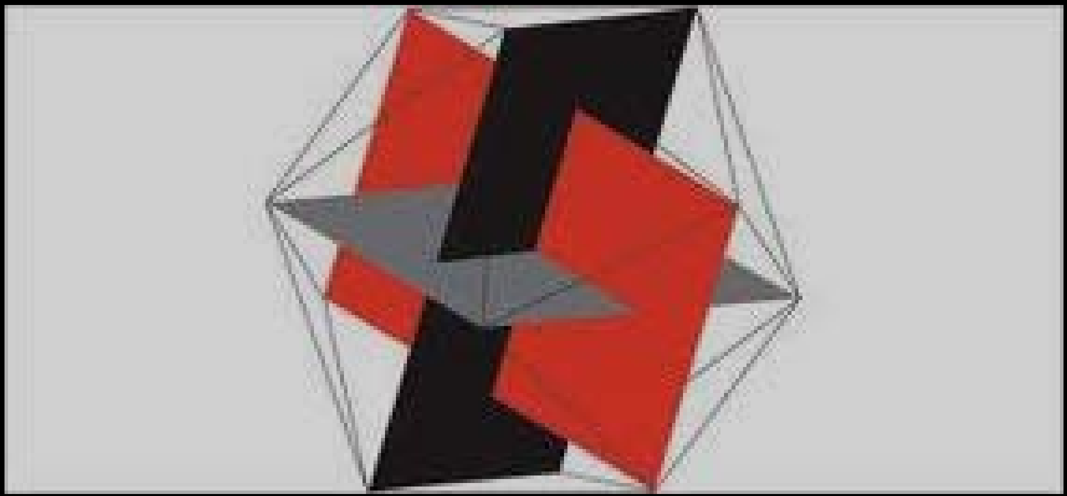
Fibonacci Is Very Hand-y!

- The ratios of the lengths of successive bones (2, 3, 5, 8 cm) of the middle finger in this illustration form ratios close to the Golden Ratio.

The Golden Rectangle

- A rectangle whose length is ϕ and whose width is 1 unit is called a Golden Rectangle.





- The 12 vertices of an icosahedron can be split into 3 groups of 4, with the vertices in each group of 4 forming a Golden Rectangle.

Common Golden Rectangles?

- Many familiar objects are in shapes "close to" the Golden Rectangle: index cards (3 x 5, 4 x 6), the page of a book, postcards, playing cards, posters, photographs, wall switch plates, electronic devices…
- Is this just "coincidence"? Or is there something about the dimensions of the Golden Rectangle that seems "right" to us?

Is the Golden Rectangle Most Aesthetically Pleasing?

Massage table

Coffee table

Brain tabled?

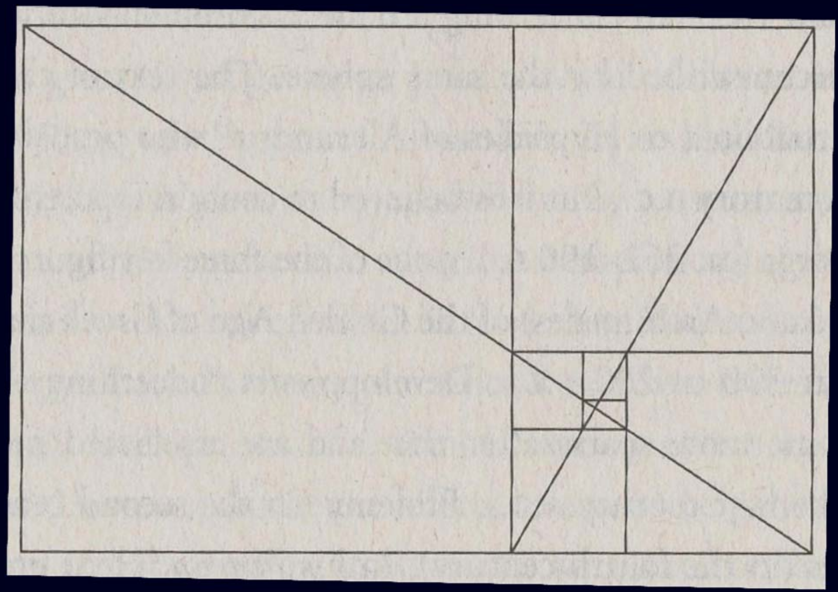
Holy Golden Rectangles, Batman?

- **Ark of the Covenant:**
- Exodus 25:10 – “Have them make a chest of acacia wood = two and a half cubits long, a cubit and a half wide, and a cubit and a half high…” (ratio 2.5:1.5 = 5:3 = 1.666; Φ = 1.618033)

- **Noah’s Ark:**
- Genesis 6:15 – “And this is the fashion that thou shalt make it of: The length of the ark shall be 300 cubits, the breadth of it 50 cubits, and the height of it 30 cubits…” (50:30 = 5:3 = 1.666; Φ = 1.618033)

The "Eye of God"

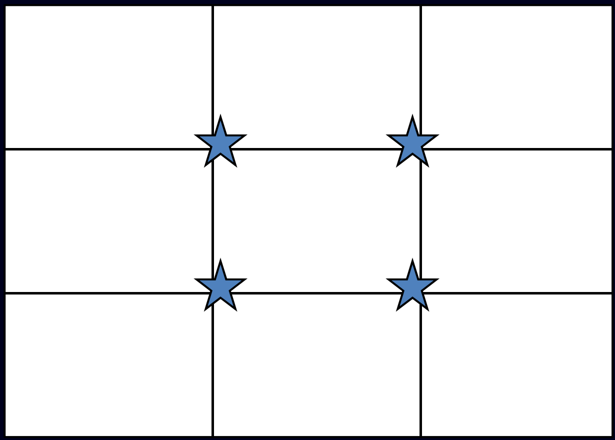
- Cut out a square from one side of a Golden Rectangle, and, amazingly, the remaining figure is a (smaller) Golden Rectangle!



- Repeating this process indefinitely leads to a point known as the "Eye of God."

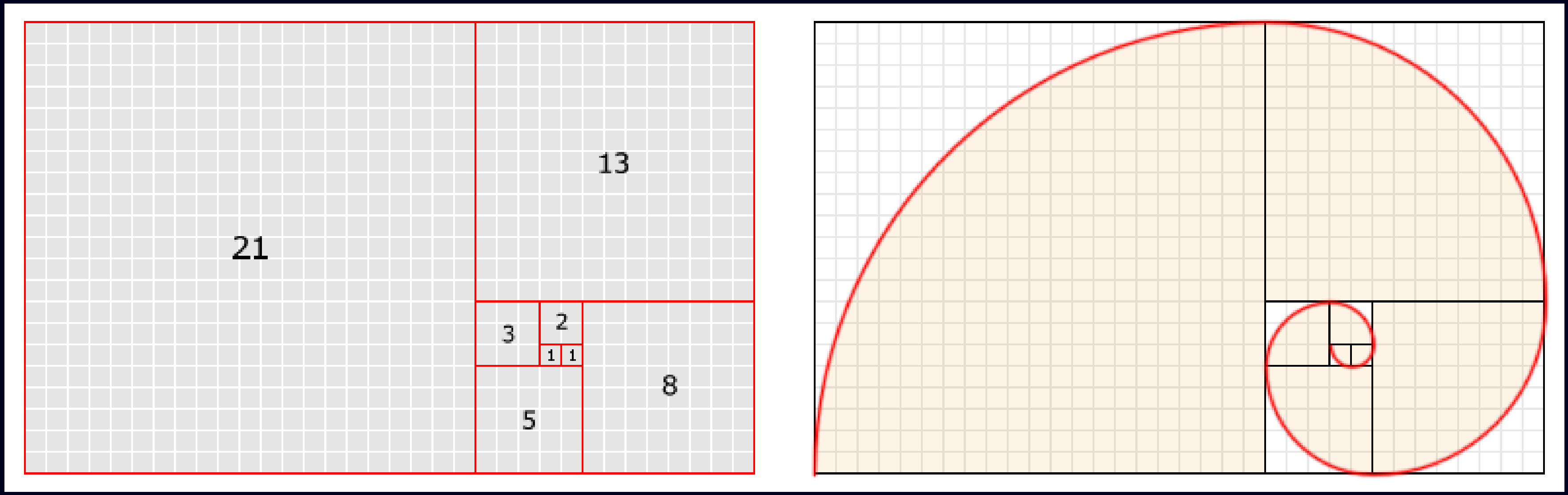
"Sweet Spots"

- In photography, an "Eye of God" point is often referred to as a "Sweet Spot". Many artists consciously or unconsciously place the focus of their compositions in a "Sweet Spot."
- The "Rule of Thirds": divide the frame into 9 compartments using two equally-spaced vertical lines, and two equally-spaced horizontal lines. Put the focus of the picture at one of the four intersections of these lines --- very close to the mathematical "Sweet Spots".



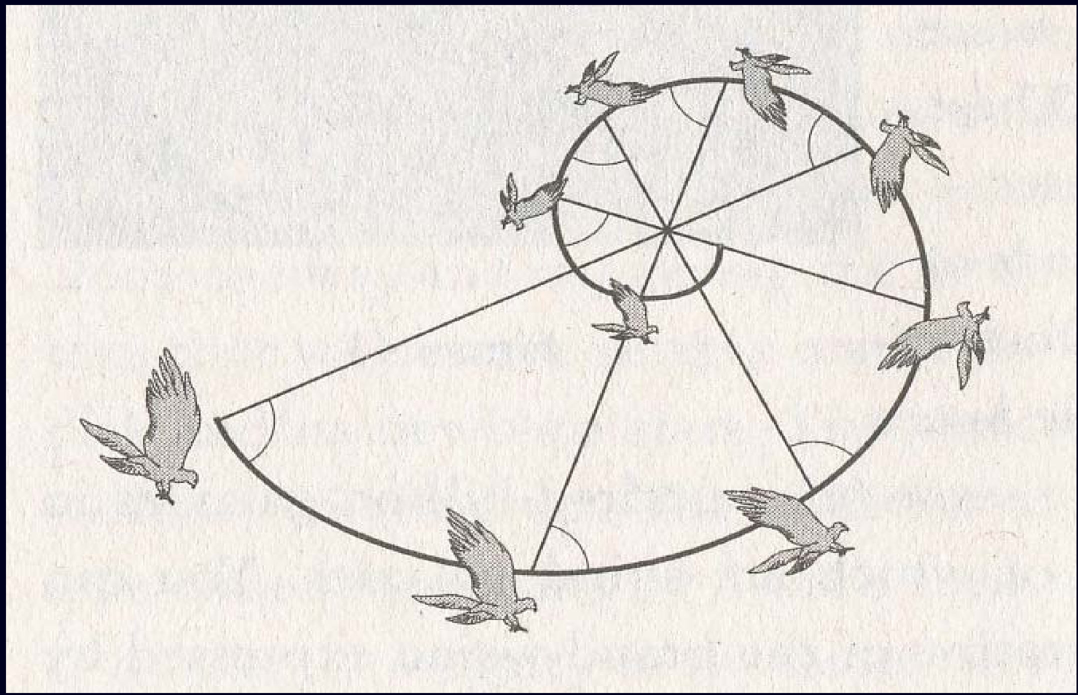
The Golden Spiral

- Connecting the vertices of these successive Golden Rectangles (as below) results in a spiral curve – often called the Golden Spiral (which also converges to an "Eye of God").



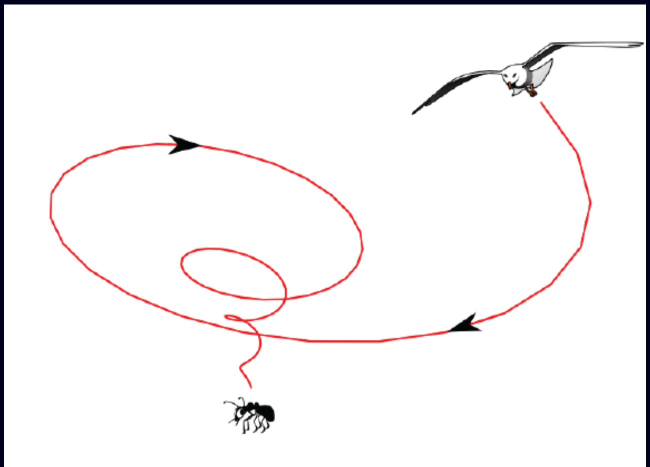
The Golden Spiral is Equiangular

- The Golden Spiral ($r = ae^{(0.0053468)\vartheta}$) is an example of an equiangular spiral, because a straight line from the “Eye” to any point on the spiral intersects the spiral in the same angle (≈ 89.7°).



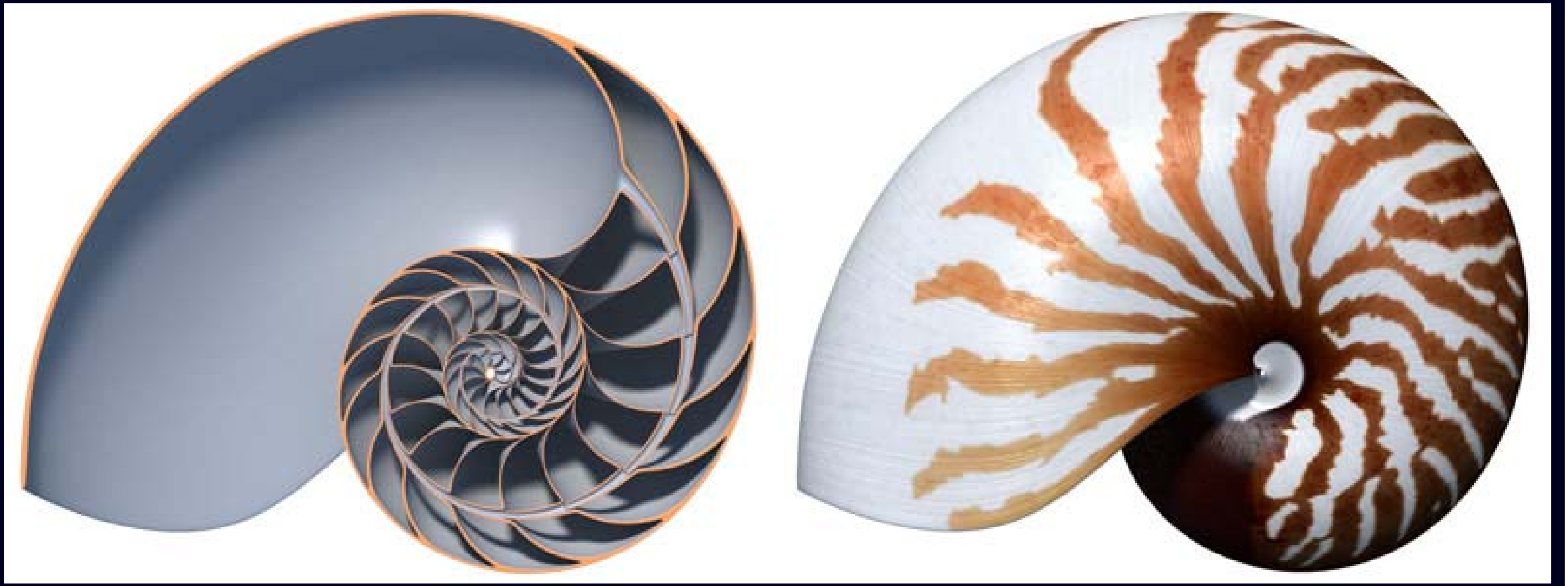
The Golden Spiral and Falcons

- Peregrine falcons (& hawks) follow an equiangular (Golden) Spiral when attacking their prey, approaching targets at speeds up to 200 mph.
- Falcons could fly faster directly to their victim, but then would have to turn their head (eyes are on either side of the head) 40 degrees to one side or the other.
- This would slow them down considerably (as shown in tunnel experiments).



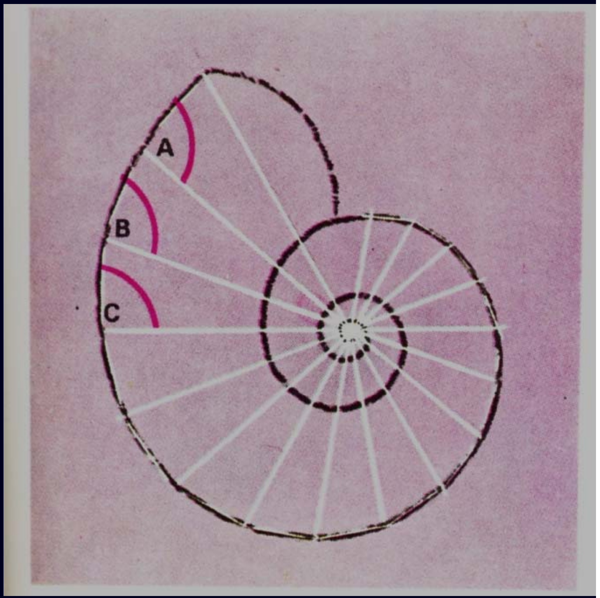
The Nautilus

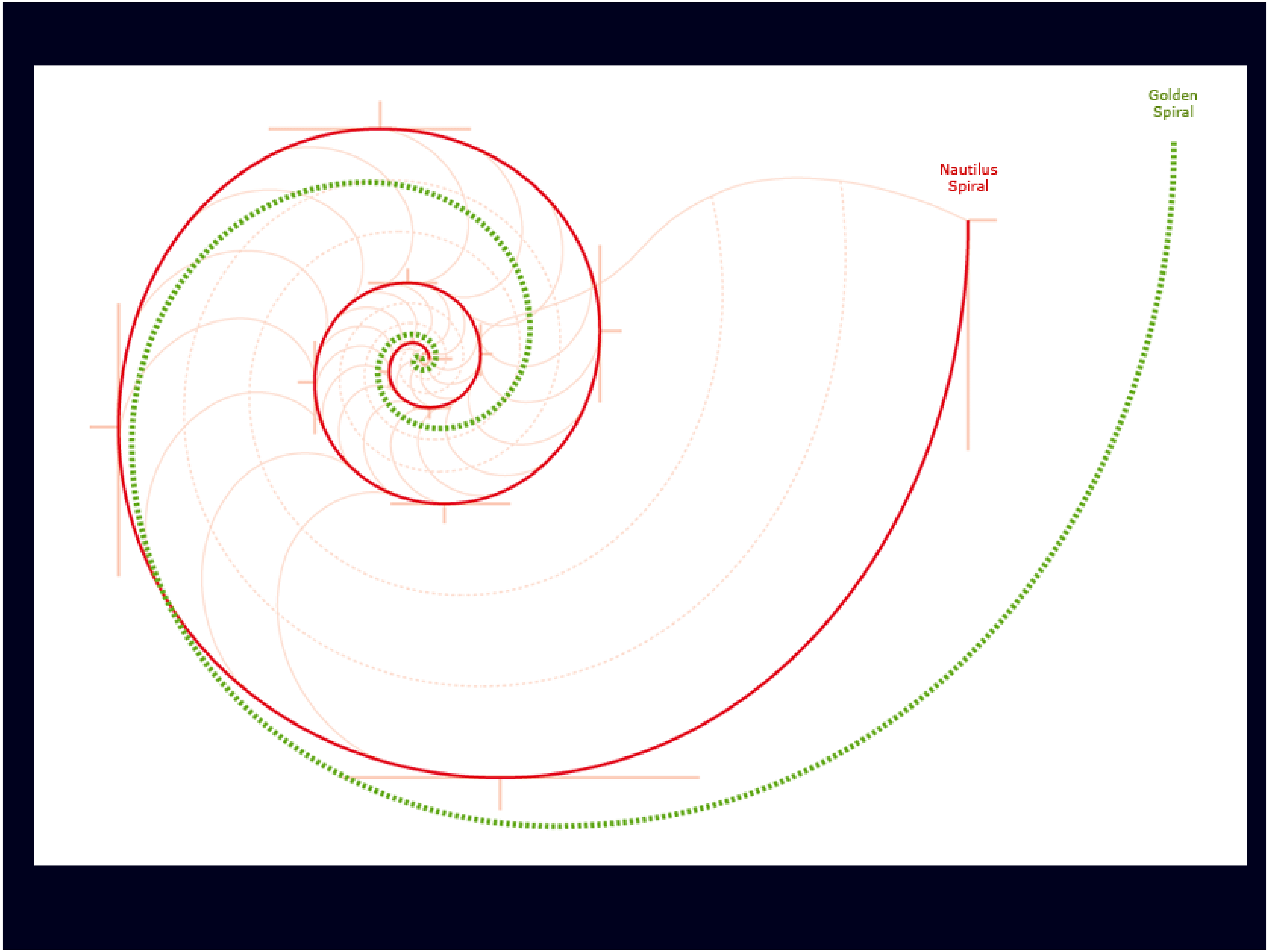
- The shell of a nautilus exhibits a shape that is similar to the Golden Spiral.



The Golden Spiral and the Nautilus

- As the mollusk inside the shell of the nautilus grows in size, it constructs larger chambers for itself, sealing off smaller ones as it progresses.
- For each increase in the length of the shell, there is almost a proportional increase in the radius, forming a spiral very close to the Golden Spiral, as shown on the next slide.




Golden
Spiral
Nautilus
Spiral

Other Golden Spirals in Nature

- The shells of snails, and the horns of rams and tusks of elephants (although not lying in a plane) also form spirals similar in shape to Golden Spirals.

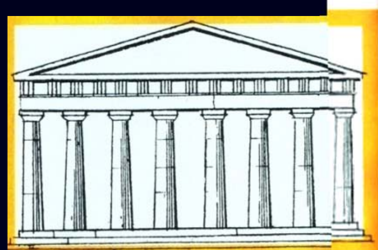
Horns, fangs and claws follow golden spirals, as do the shells of snails.

The Golden Ratio in Art

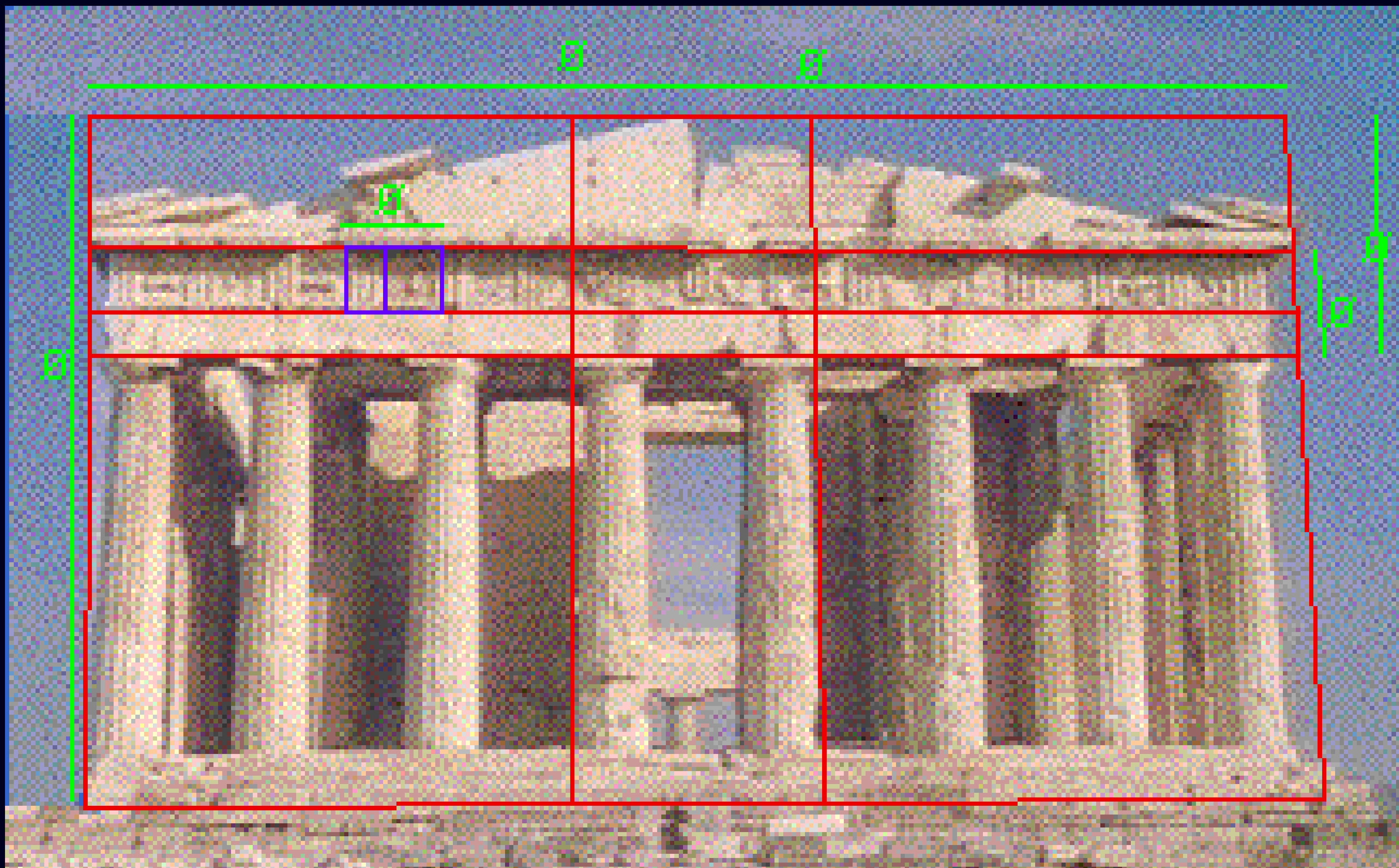
- Much of the controversy about the Golden Ratio in art began with two books by Matila Ghyka (1881–1965): *Esthétique des Proportions dans la Nature et dans les Arts* (1927) and *Le Nombre d'Or* (1931).
- These works contain much <u>inaccurate</u> and <u>anecdotal</u> information about the supposed influence of the Golden Ratio on art (from the time of the ancient Greeks -- and even earlier).

The Golden Ratio in Art

- In many mathematics and art history books, we find the claim that if the pediment of the Parthenon (built 447–432 BC) was restored, the overall shape would be inscribed in a Golden Rectangle.



The Parthenon

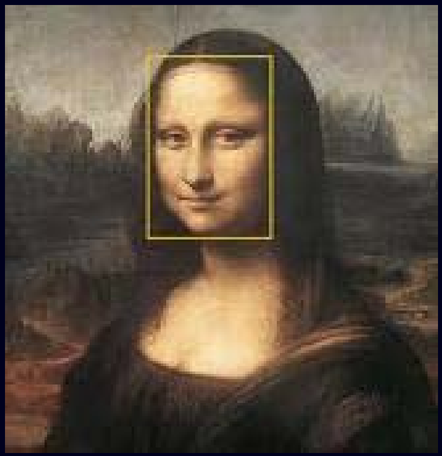


The Golden Ratio in Art

- However, while the Golden Ratio was probably discovered about the 5th century BC, it does not appear anywhere in mathematics or literature (to our knowledge) before Euclid (≈ 300 BC).
- Consequently, mathematicians are increasingly skeptical of claims that the Golden Ratio was employed in the construction of temples and sculpture before that time – including the Parthenon.

The Golden Ratio in Art

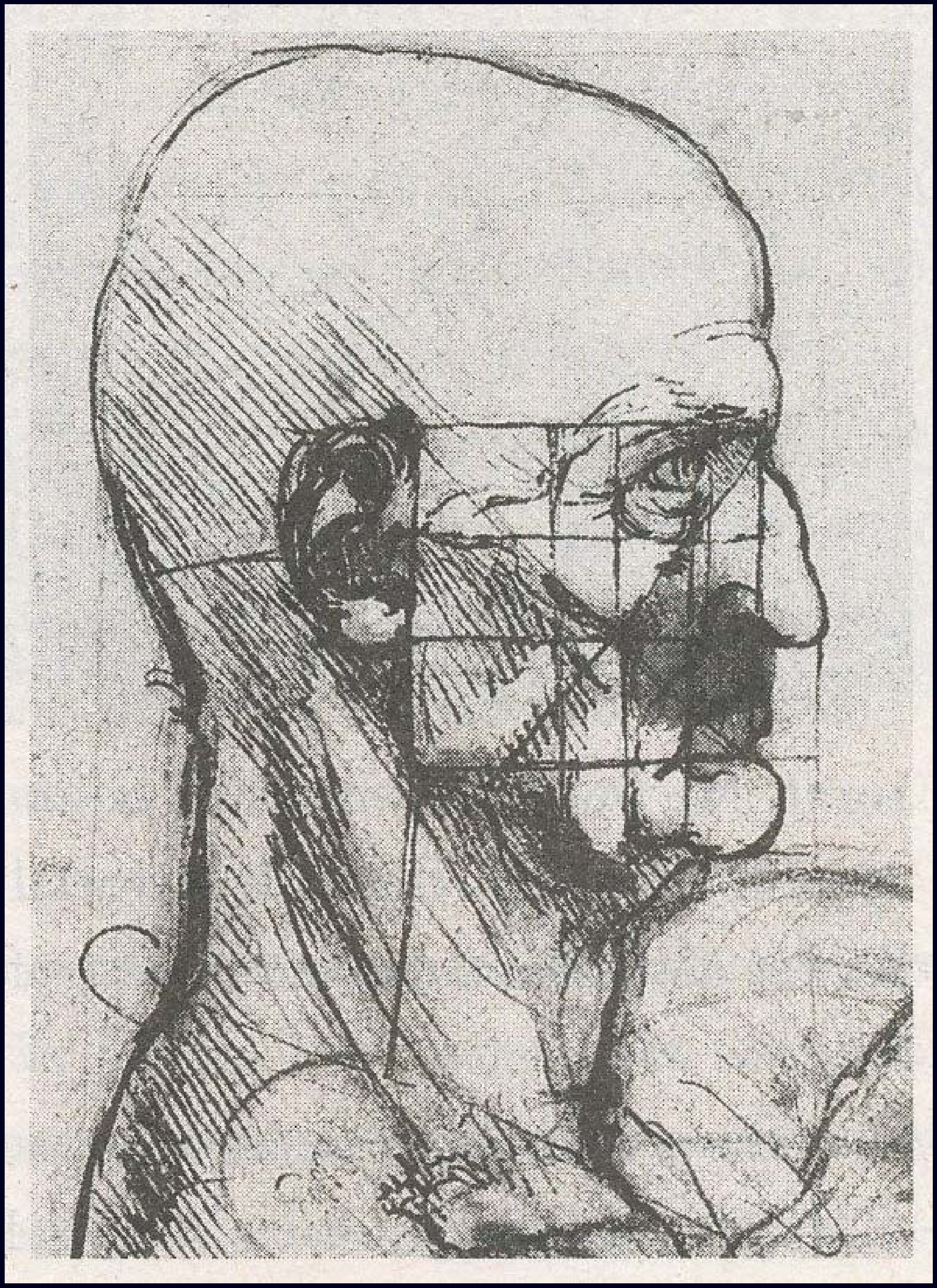
- Since Leonardo da Vinci (1452−1519) made 60 illustrations of solids (polyhedra) for Pacioli's book, there is much speculation that he may have used the Golden Ratio in his art work.
- Da Vinci wrote: "Let no one who is not a mathematician read my works."
- There are at least 5 works of da Vinci in which he may have used the Golden Ratio, including the "Mona Lisa". But there is no definitive evidence in each case.



The Golden Ratio in Art

- Other Leonardo da Vinci works that may have used the Golden Ratio: 2 very similar versions of “Madonna on the Rocks” (the first before 1486), a drawing of an old man’s head (≈ 1490: next slide), and the unfinished canvas of “St. Jerome” (1483) (later slide).
- However, da Vinci did not meet Pacioli until 1496, and some of these works pre-date that meeting! The first volume of Pacioli’s book was only completed in 1497.

Leonardo da Vinci: sketch for head of an old man, about 1490 (now in Galleria dell'Accademia in Venice)

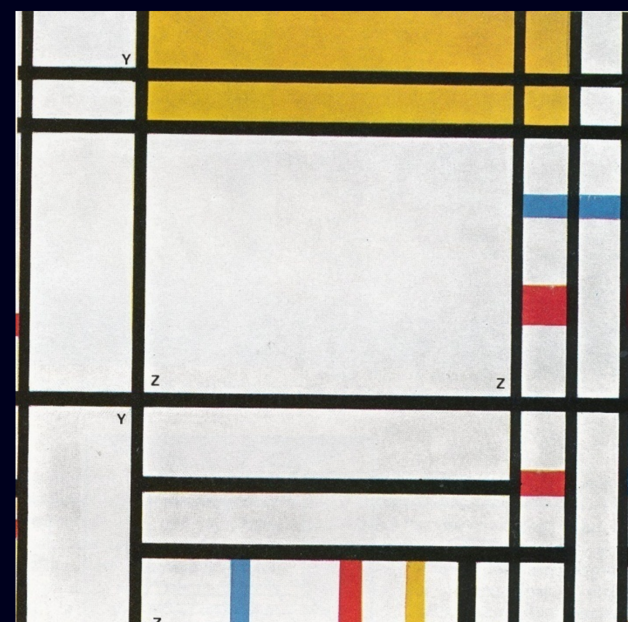


Leonardo da Vinci: *St. Jerome* (1483) (now in the Vatican Museum)

The Golden Ratio in Art

- Georges Seurat (1859−1891) and Piet Mondrian (1872−1944) probably did not use the Golden Ratio in their work, despite many claims to the contrary.

- Seurat's *Parade of a Circus*



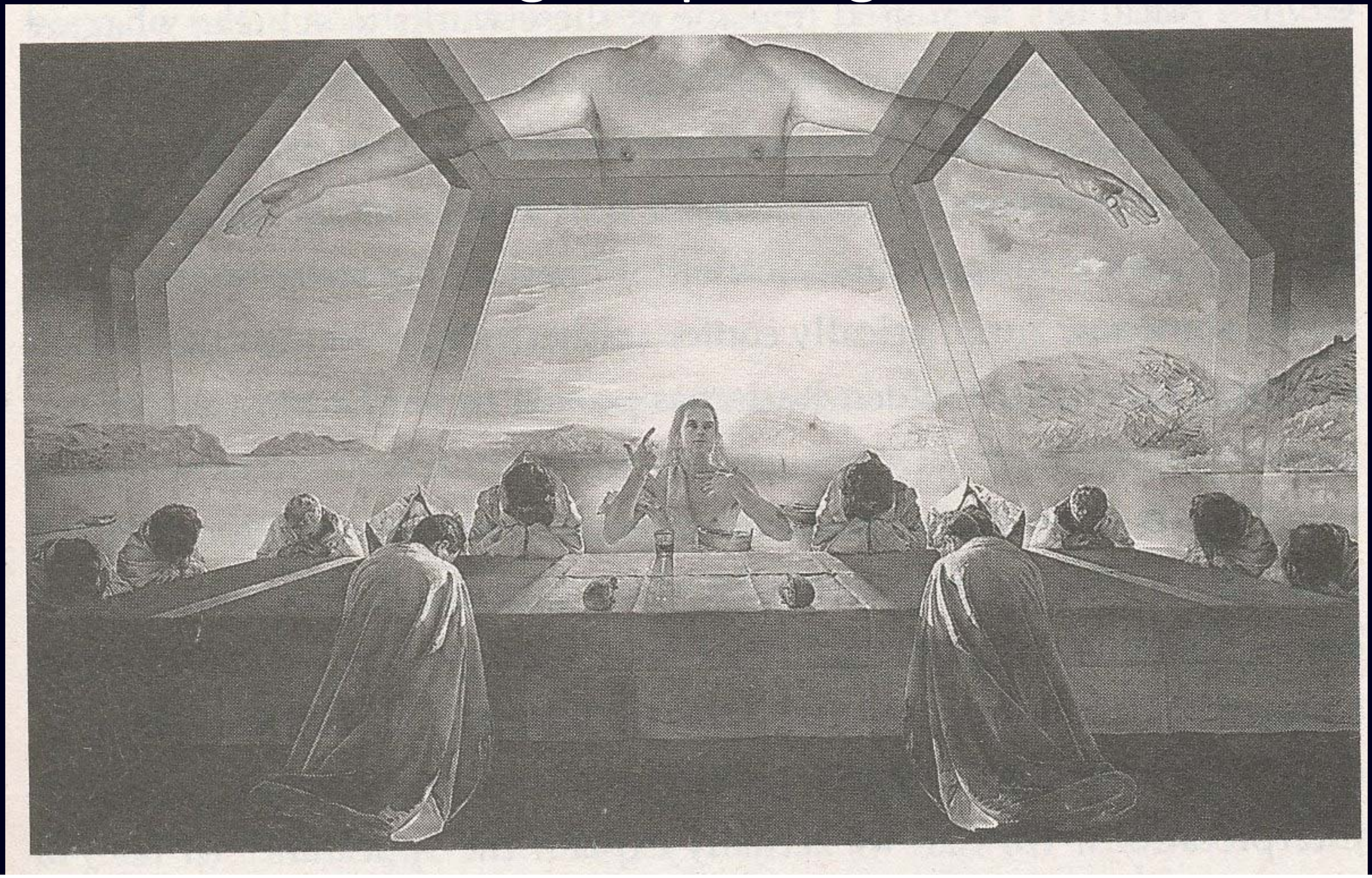
Mondrian's *Place de la Concorde*

The Golden Ratio in Art

- However, some artists who did use the Golden Ratio deliberately in their works are Paul Serusier (1864–1927), Juan Gris (1887–1927), Jacques Lipchitz (1891–1973), and Gino Severini (1883–1966).

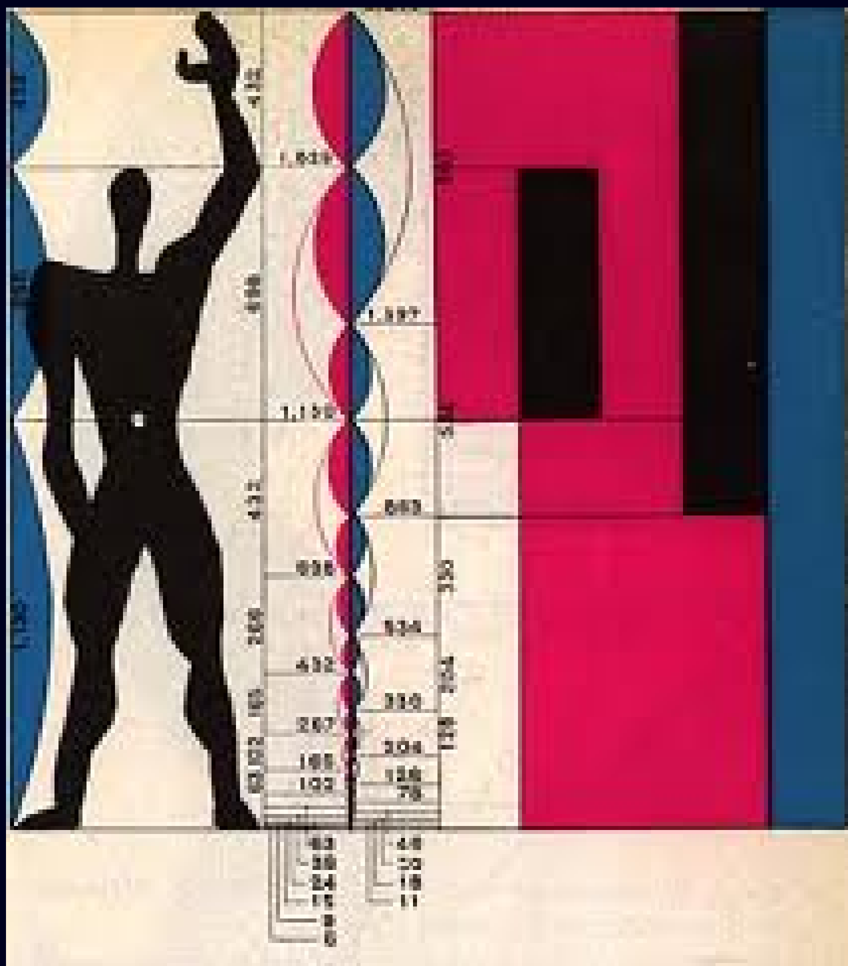
The Golden Ratio in Art

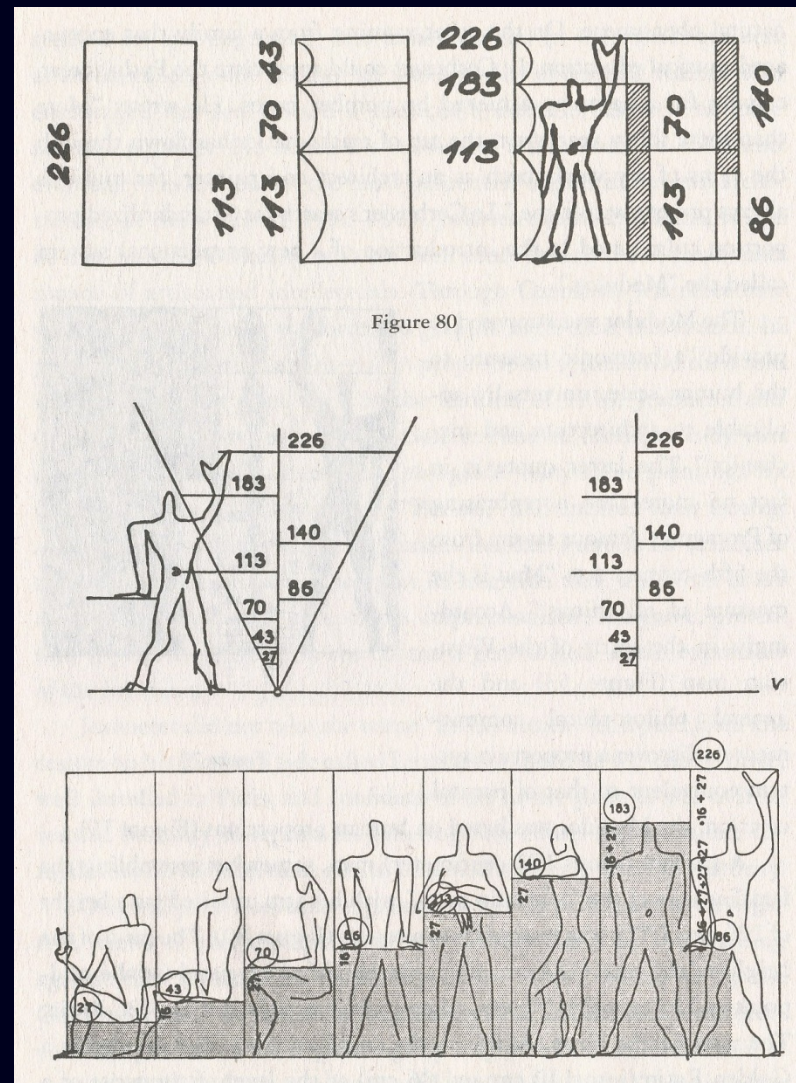
- Salvador Dali (*Sacrament of the Last Supper*) (1955). The border is a Golden Rectangle; the faces of the dodecahedron are regular pentagons.



Le Corbusier

- Le Corbusier (1887–1965): *Modulor*



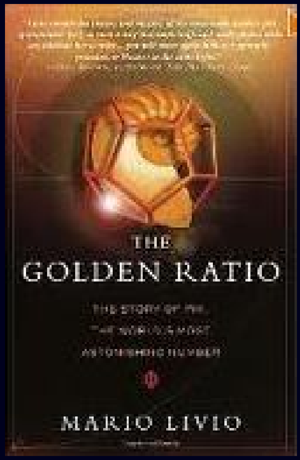


Just Remember….

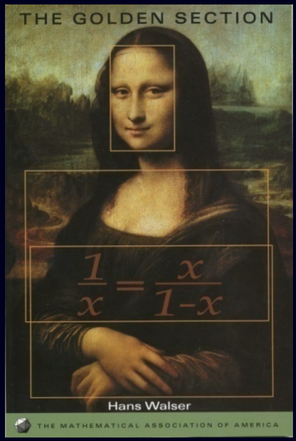
Sources

Gardner, Martin: *Mathematical Circus* (1979), published by Alfred Knopf (a gentle introduction)

Livio, Mario: *The Golden Ratio* (2002), published by Broadway Books (Random House)

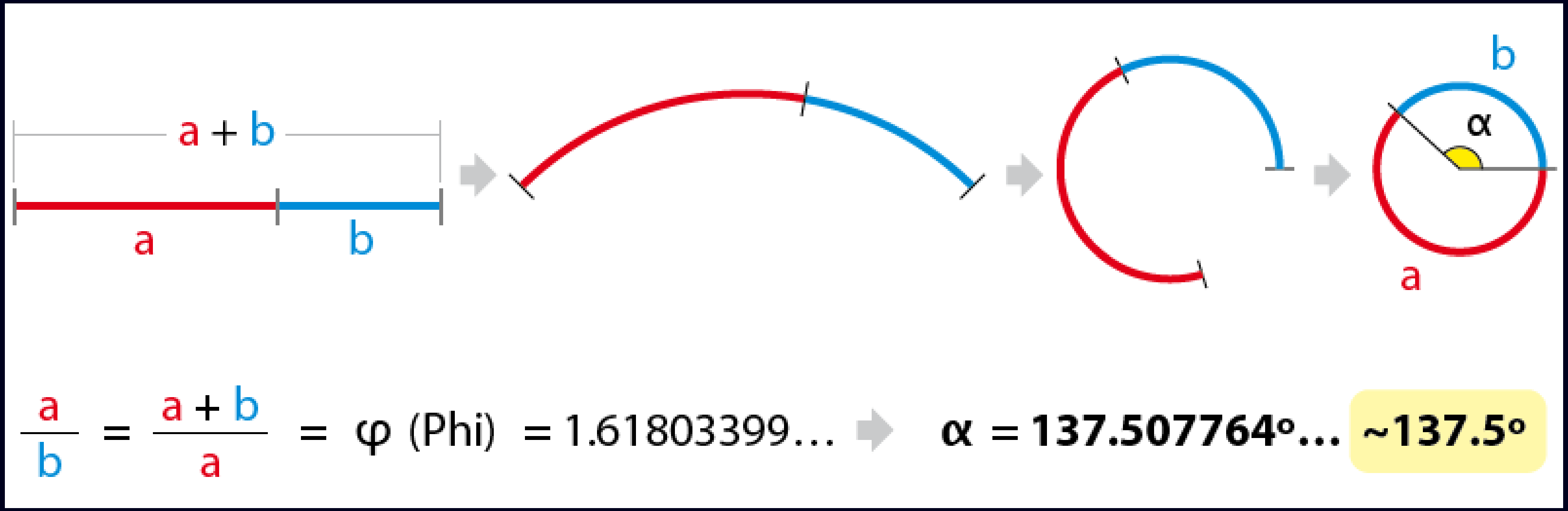


Walser, Hans: *The Golden Section* (2001), published by the Math Assoc. of America (MAA), translated by Peter Hilton and Jean Pedersen



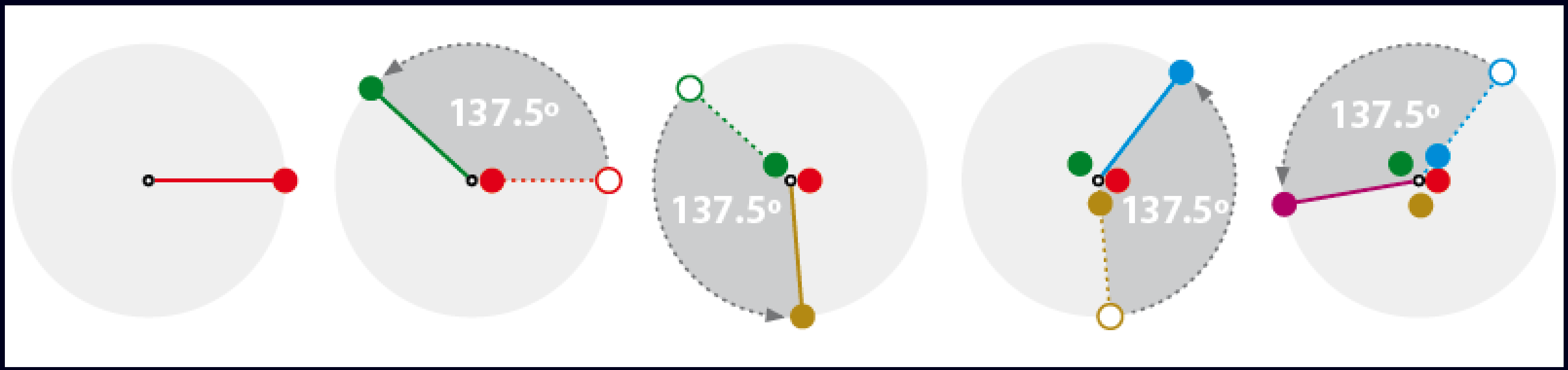
The Golden Angle

- A complete circle consists of 360°.
- If we divide 360° by ϕ, and subtract the result from 360º, we get ≈ 137.5°.
- This angle is known as the Golden Angle.



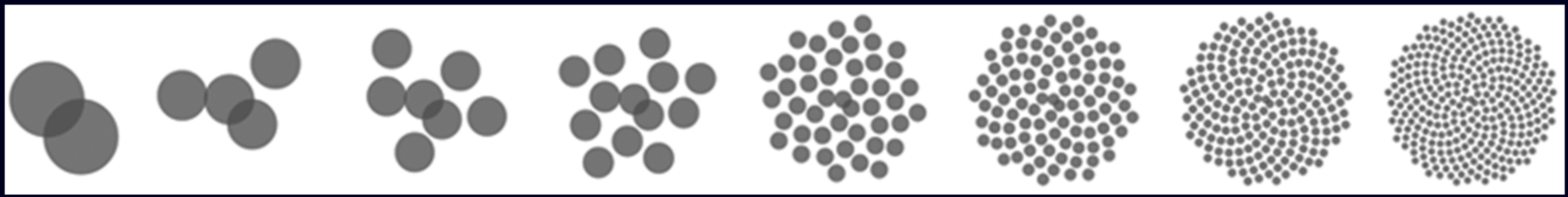
Return to Nature

- In 1907. German mathematician van Iterson discovered that if successive points are constructed using the Golden Angle on tightly wound spirals, the human eye picks out one family of spiral patterns winding clockwise and one counterclockwise – as we saw with sunflowers!



Douady & Couder Experiment

- Douady & Couder (1992–96) held a dish full of silicone oil in a magnetic field that was stronger near the dish's edge than at the center.
- Drops of a magnetic fluid (acting like tiny bar magnets) were dropped periodically into the center of the dish.
- The magnetic forces involved created a pattern that converged to a spiral in which the Golden Angle separated successive drops.

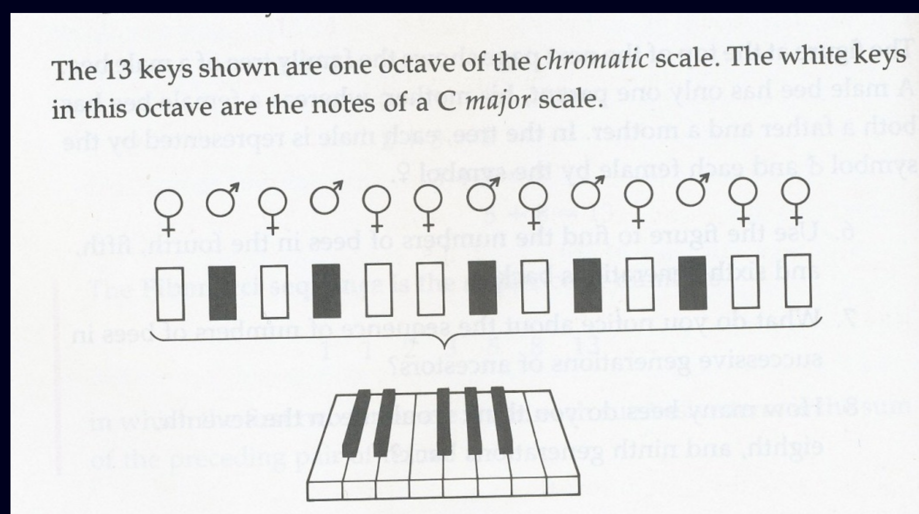


Nature's Decision

- Since physical systems usually settle into states that minimize the energy, the spirals that we have noted perhaps reflect states of minimal energy.
- Nature seems to prefer certain angles and ratios in order to "optimize" its creations.

The Fibonacci Sequence and Golden Ratio in Music

- Keyboard Octave: 8 white keys, 5 black keys (= 2 + 3) (all Fibonacci numbers), but actually just coincidence since the arrangement was made long before Pacioli’s book and an understanding of Fibonacci numbers.



Fibonacci and Music

- Traditionally, the Pythagoreans are also credited with the musical discovery that taut strings in the ratios 2:1, 3:2, 4:3 produce an "octave" (C–C), a "fifth" (C–G), and a "fourth" (C–F), respectively.
- The major sixth is often considered the most "beautiful" chord…. Tone used for tuning (A) is at 264 vibrations per second. Major sixth: C–A = 440/264 ≈ 5/3 (Fib #'s). Minor sixth: C–E = 528/330 ≈ 8/5 (Fib #'s).

The Golden Ratio in Music

- Construction of violins by Stradivari using the Golden Ratio
- Spurious claims of discovering the Golden Ratio in early Western music, or in Mozart’s piano sonatas – now largely debunked.
- Bela Bartok (e.g., *Music for Strings, Percussion and Celesta*, *Sonata for Two Pianos and Percussion*) and Claude Debussy (e.g., *La Mer*) may have used the Golden Ratio; assertions are controversial.

The Golden Ratio in Music

- A mathematical musician and teacher, Joseph Schillinger (1895-1943) deliberately used the Golden Ratio in his works. (Gershwin, Glenn Miller, and Benny Goodman were among his students.)

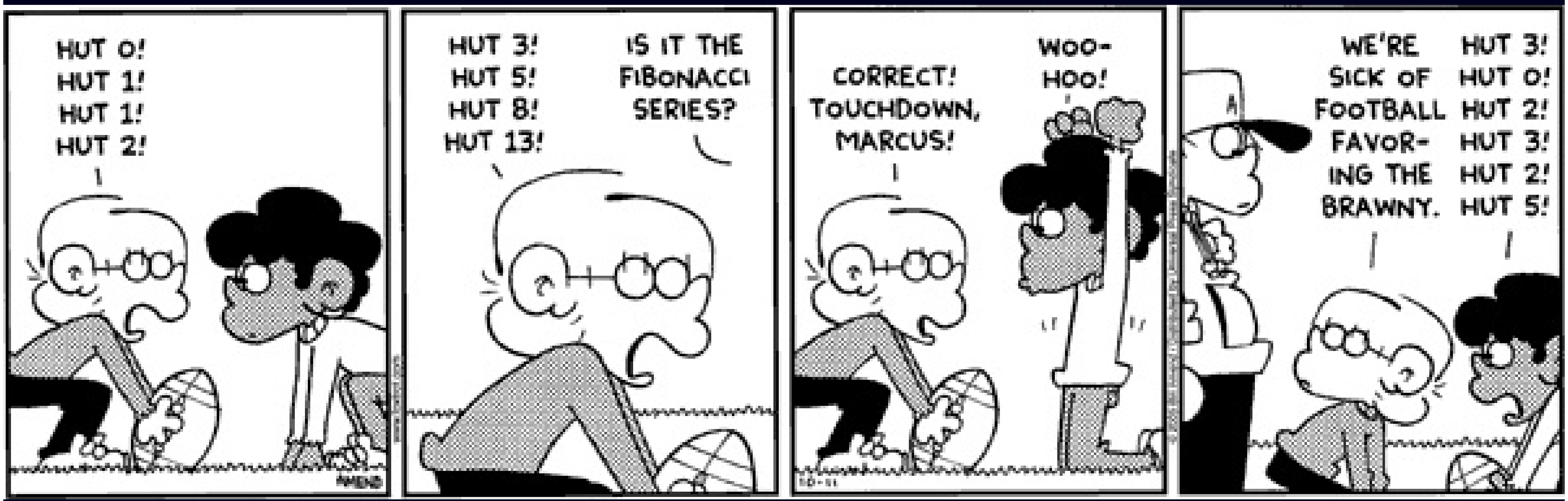
Other Ideas

- Golden Ratio expressed as a Continued Fraction using only 1's (and poem)
- Golden Ratio expressed as an infinite nested sum of square roots
- Lord Kelvin: "When you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind."

Other Ideas

- Golden Ratio in film: Russian director Sergei Eisenstein incorporated the Golden Ratio in his classic 1925 silent film *The Battleship Potemkin*. He separated the different portions of the film by using the Golden Ratio to begin significant scenes. To ensure the accuracy, he measured the length of each scene on the celluloid film!

Other Ideas



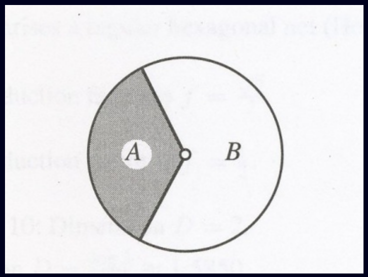
HUT 0! HUT 1! HUT 1! HUT 2!
HUT 3! HUT 5! HUT 8! HUT 13!
IS IT THE FIBONACCI SERIES?
CORRECT! TOUCHDOWN, MARCUS!
WOO-HOO!
WE'RE SICK OF FOOTBALL FAVOR-ING THE BRAWNY.
HUT 3! HUT 0! HUT 2! HUT 3! HUT 2! HUT 5!

Other Ideas

- Buckminster Fuller: "When I am working on a problem, I never think about beauty. I think only of how to solve the problem. But when I have finished, if the solution is not beautiful, I know it is wrong."
- The term "Golden Section" was probably first popularized by Martin Ohm in 1835.

Other Ideas

Fair game involving the
Golden Ratio



										Decimal place
1.61803	39887	49894	84820	45868	34365	63811	77203	09179	80576	50
28621	35448	62270	52604	62818	90244	97072	07204	18939	11374	100
84754	08807	53868	91752	12663	38622	23536	93179	31800	60766	
72635	44333	89086	59593	95829	05638	32266	13199	28290	26788	200
06752	08766	89250	17116	96207	03222	10432	16269	54862	62963	
13614	43814	97587	01220	34080	58879	54454	74924	61856	95364	300
86444	92410	44320	77134	49470	49565	84678	85098	74339	44221	
25448	77066	47809	15884	60749	98871	24007	65217	05751	79788	400
34166	25624	94075	89069	70400	02812	10427	62177	11177	78053	
15317	14101	17046	66599	14669	79873	17613	56006	70874	80710	500
13179	52368	94275	21948	43530	56783	00228	78569	97829	77834	
78458	78228	91109	76250	03026	96156	17002	50464	33824	37764	
86102	83831	26833	03724	29267	52631	16533	92473	16711	12115	
88186	38513	31620	38400	52221	65791	28667	52946	54906	81131	
71599	34323	59734	94985	09040	94762	13222	98101	72610	70596	
11645	62990	98162	90555	20852	47903	52406	02017	27997	47175	
34277	75927	78625	61943	20827	50513	12181	56285	51222	48093	
94712	34145	17022	37358	05772	78616	00868	83829	52304	59264	
78780	17889	92199	02707	76903	89532	19681	98615	14378	03149	
97411	06926	08867	42962	26757	56052	31727	77520	35361	39362	1000

A Useful Property of the Fibonacci Sequence

- An interesting property of the Fibonacci Sequence related to the Triangle Puzzle is:

  When *n* is odd:

$$F_n \times F_{n+2} = (F_{n+1})^2 + 1$$

- For example, when $n = 5$ (odd), we have:

$$F_5 \times F_7 = (F_6)^2 + 1$$

$$5 \times 13 = (8)^2 + 1$$

Another Property of the Fibonacci Sequence

- The sum of any ten consecutive Fibonacci numbers ($F_n + F_{n+1} + F_{n+2} + \ldots + F_{n+9}$), is equal to 11 times the seventh term in the sum (F_{n+6}).
- For example:

$$F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8 + F_9 + F_{10}$$
$$= 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55$$
$$= 143 = 11 \times 13 = 11 \times F_7.$$

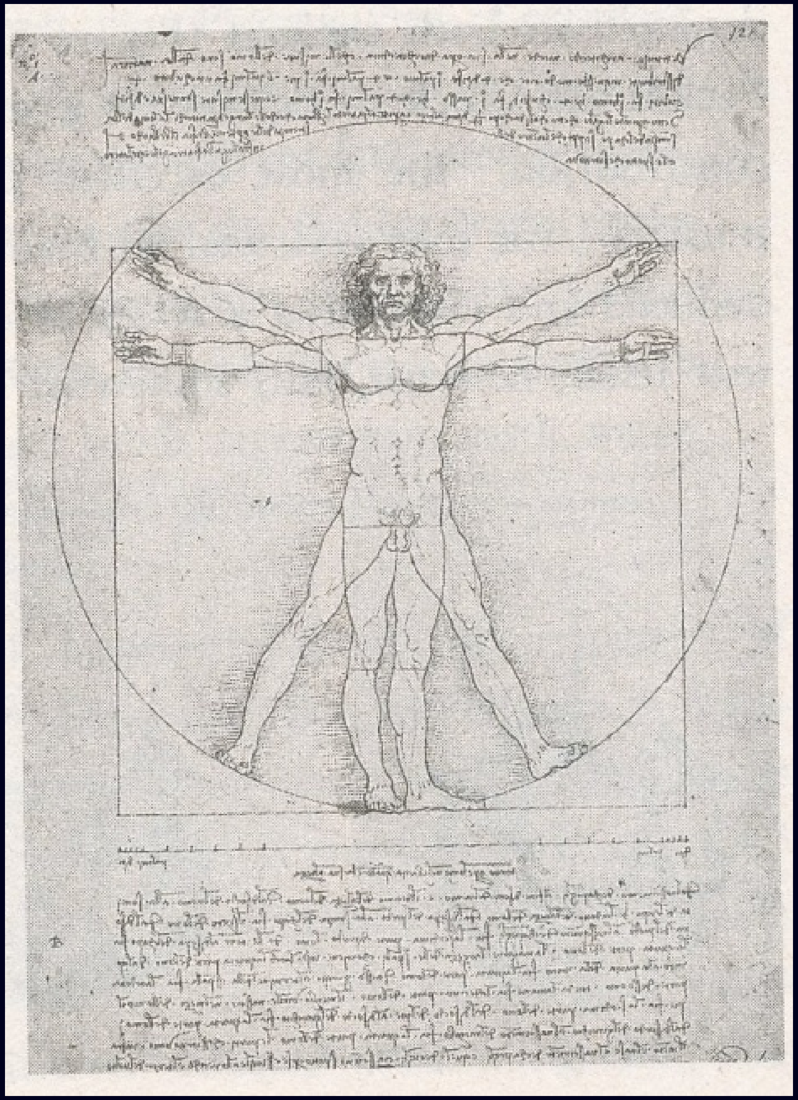
Another Property of the Fibonacci Sequence

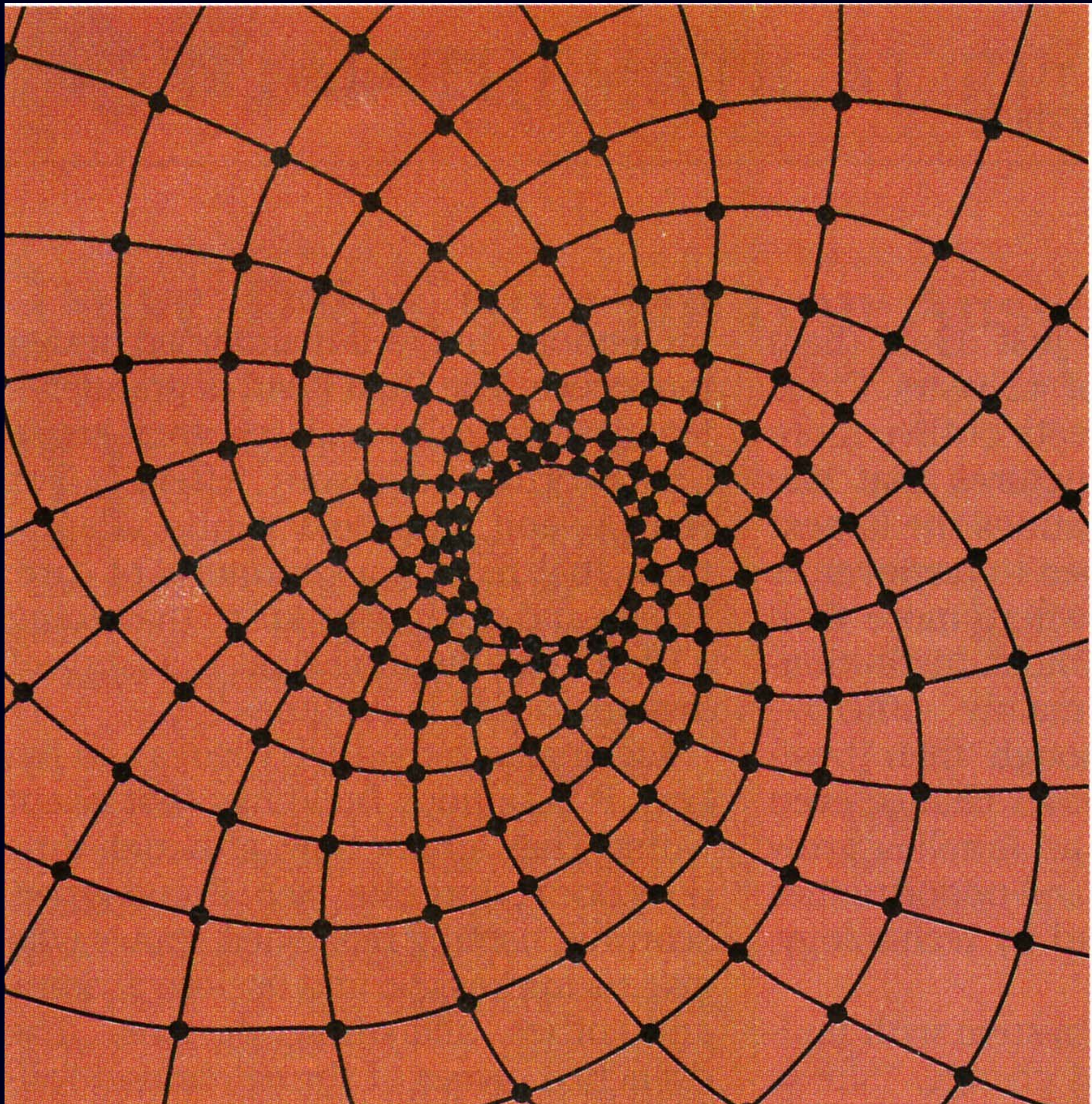
- The Fibonacci Sequence has many beautiful properties. Here is a typical one:
- The sum of the first n Fibonacci numbers, ($F_1 + F_2 + F_3 + \ldots + F_n$), is equal to one less than the second successor: $F_{n+2} - 1$.
- For example:

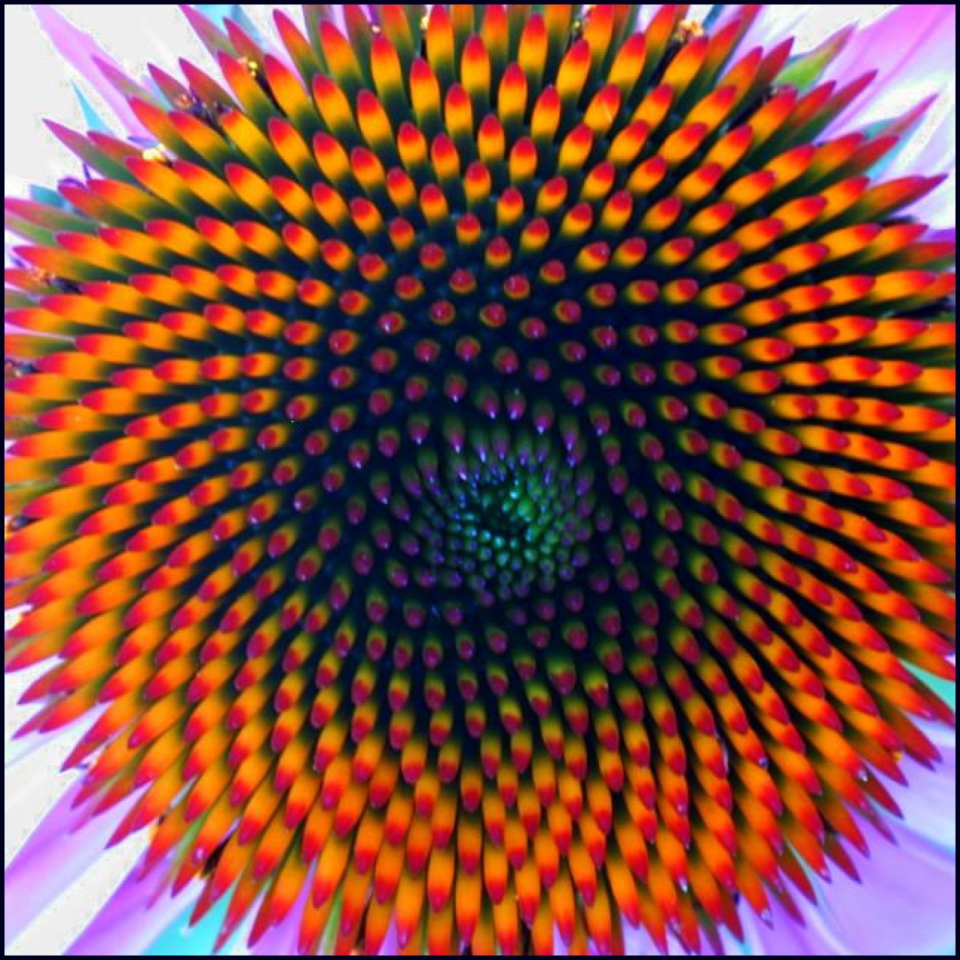
$$F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8$$
$$= 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 = 54$$
$$= 55 - 1 = F_{10} - 1$$

Da Vinci's *Vitruvian Man*

Vitruvian Man by da Vinci is based on the proportions stated by Marcus Vitruvius Pollio (70−25 BC):
(1) If a circle is drawn at the navel, the fingers and toes of the two hands and feet will touch the circle's circumference.
(2) In addition, the distance from the soles of the feet to the top of the head matches the width of the outstretched arms.



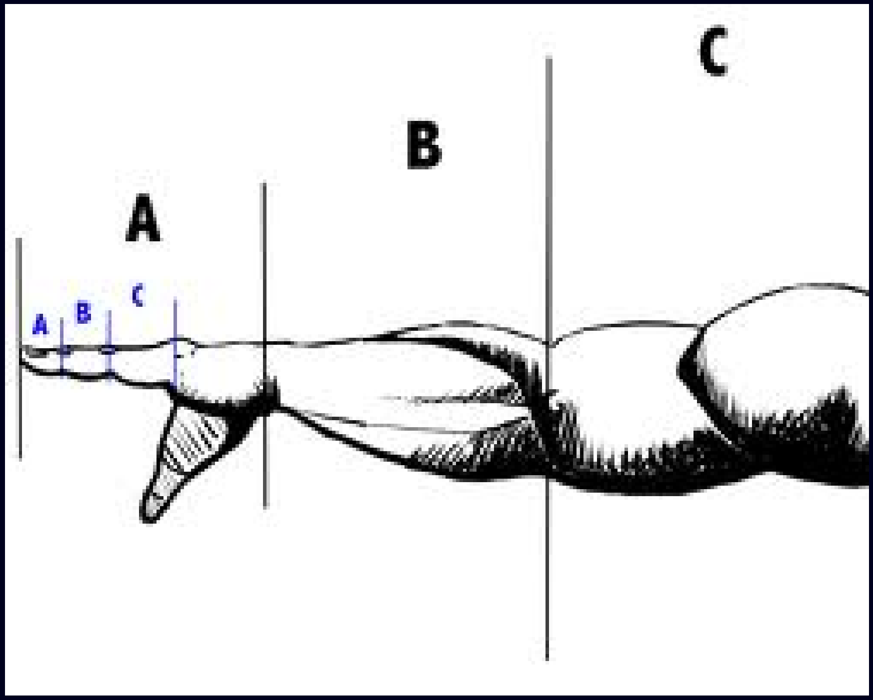


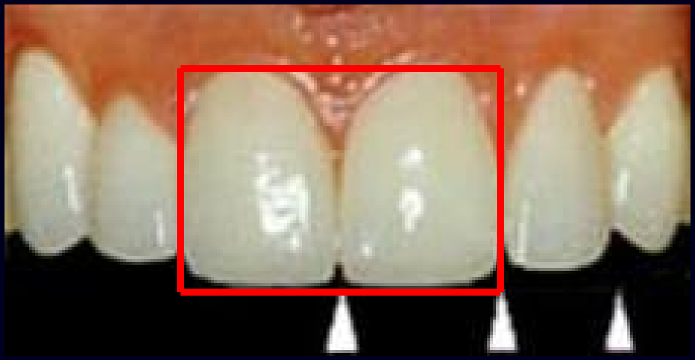


More Golden Ratios…

Length of arm

Top front teeth





Fibonacci Sequence in Terms of the Golden Ratio

- Each number in the Fibonacci Sequence can actually be expressed using the Golden Ratio using Binet's Formula:

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \left(\frac{1}{\phi}\right)^n\right)$$

- Amazingly, Binet's Formula always reduces to an integer value!

The Platonic Solids

- There are exactly 5 Platonic Solids (= regular polyhedrons): solids with all faces congruent to the same regular polygon, and with the same number of faces meeting at each vertex.
- Euclid concludes *The Elements* (in Book XIII) with the 5 Platonic Solids, and showing how they can be constructed (using just a straightedge and compass).

Platonic Solids (continued)

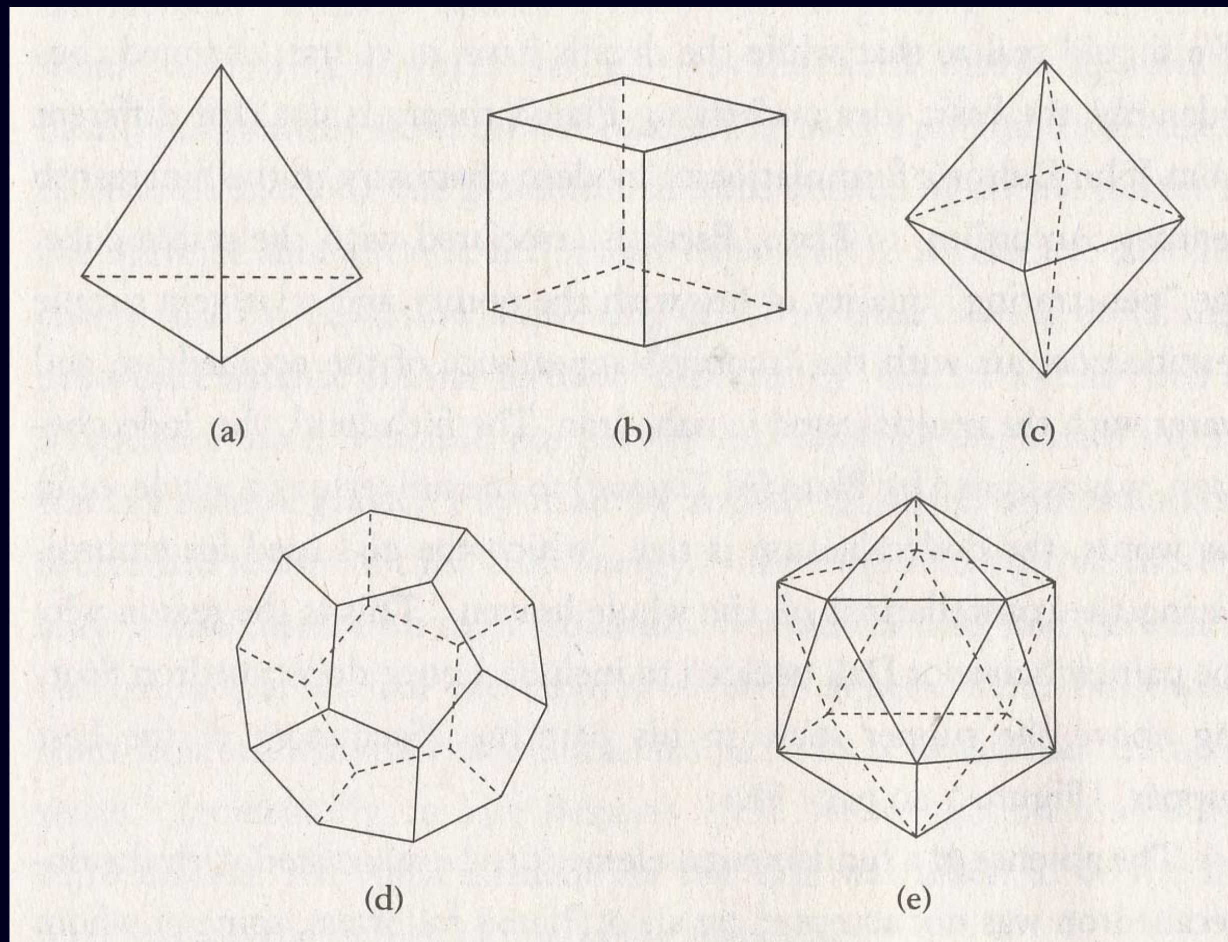
(a) Tetrahedron
(Fire)

(b) Hexahedron

(c) Octahedron
(Air)

(d) Dodecahedron
(Universe)

(e) Icosahedron
(Water)



The Golden Ratio and the Platonic Solids

- Euclid used properties of the Golden Ratio in his construction of the dodecahedron and icosahedron in Book XIII.
- Each face of a dodecahedron is a regular pentagon, whose diagonals partition each other in the Golden Ratio!

Fra Luca Pacioli

- Fra Luca Pacioli (Franciscan Friar, 1445−1517) brought the Golden Ratio to the attention of many in the Renaissance.
- Pacioli is actually more famous for his *Summa de arithmetica, geometria, proportioni et proportionalita* (1494), summarizing most of the mathematics known at that time (largely based on the work of Piero della Franesca).
- The Summa also contained first treatment of double-entry bookkeeping, earning Pacioli the nickname: "Father of Accounting."

Pacioli: "The Divine Proportion"

- In the portrait of Pacioli giving a lesson in mathematics by Jacopo de'Barbari (1440–1515), note the dodecahedron sitting on top of a copy of Pacioli's *Summa*.
- Also note the rhombicuboctahedron (!) (one of the 13 Archimedean Solids) suspended in mid-air, and half filled with water – symbolizing the purity and timelessness of mathematics.

Pacioli: “The Divine Proportion”

- Pacioli wrote a 3-volume treatise *Divina Proportione* (1509) (also based on the work of Piero della Francesca) devoted to the Golden Ratio
- Pacioli referred to the Golden Ratio as “The Divine Proportion.”
- In Chapter 5, Pacioli lists 5 reasons why he believed this name was appropriate:

“The Divine Proportion”

- 1. “That it is one only and not more.” (Unity “is the supreme epithet of God Himself”.)
- 2. The Golden Ratio involves three lengths (AC, CB, AB) just as there is a Holy Trinity.
- 3-4. God and the Golden Ratio are incomprehensible, and invariable (independent of the length of original segment to be divided).
- 5. We cannot construct the dodecahedron (symbolizing the universe), nor compare the other Platonic Solids to each other without the Golden Ratio.

The Golden Ratio in Art

- A 1992 paper in *The College Mathematics Journal* by George Markowsky entitled "Misconceptions about the Golden Ratio" seriously questions that the Parthenon incorporated the Golden Ratio in its design.
  - The dimensions of the Parthenon vary from source to source, giving width/height ratios in published calculations ranging from 1.72 to 2.25.
  - Parts of the structure (e.g., the edges of the pedestal) actually fall "outside" the presumed Golden Rectangle.

Notation for the Golden Ratio

- The notation ϕ (*phi*) for the golden ratio was given by Mark Barr, in honor of the sculptor Phidias, who worked on sculptures in the Parthenon and others such as the statue of Zeus at Olympia.
- Barr was presumably influenced by a belief that the Golden Ratio was used in sculpture and architecture since antiquity.

Fibonacci Numbers in Poetry

Recent claims that the poetry in Virgil's *Aeneid* involves the Golden Ratio or Fibonacci numbers is controversial.

However, Fibonacci numbers were used as meters in some classic Indian poetry.

Fibonacci Numbers in Poetry

A classic limerick:

A fly and a flea in a flue
Were imprisoned, so what could they do?
Said the fly, "Let us flee!"
"Let us fly!" said the flea,
So they fled through a flaw in the flue.

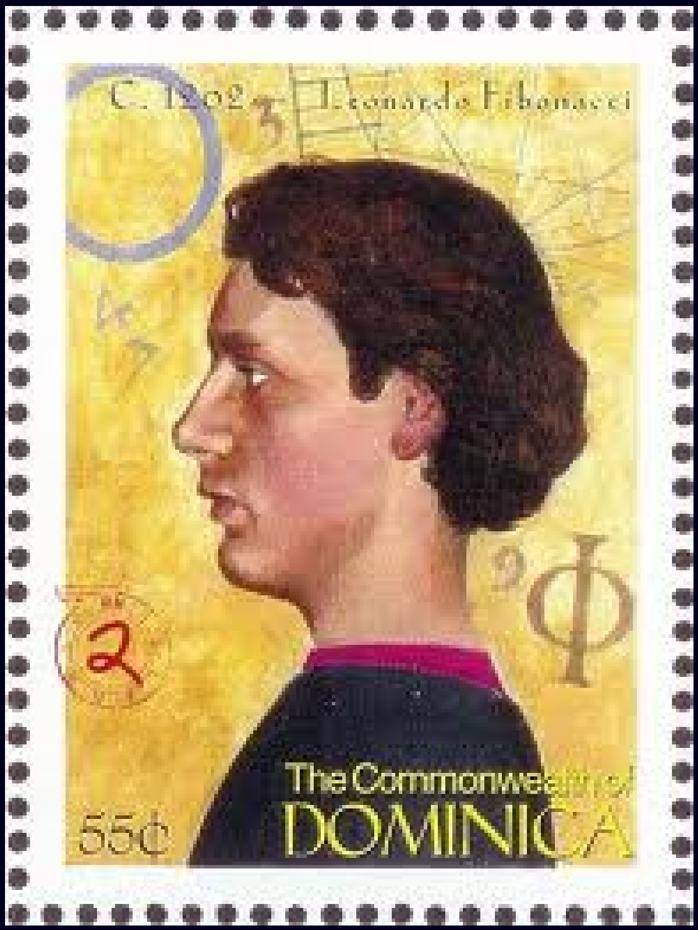
(Author unknown)

- Number of lines = 5
- Number of beats in each line = 2 or 3
- Total number of beats = 13

Golden Rectangles (??!?!)

Classic picture: the shooting of Lee Harvey Oswald by Jack Ruby

Fibonacci Stamps



Mathematics in Oz

- Was there a mathematical consultant on the set of "The Wizard of Oz?"
- Well, perhaps not, since the Scarecrow recites the Pythagorean Theorem wrongly after he gets his diploma"!!!

Great Pyramid of Giza

Of the Great Pyramid, Herodotus wrote in *The Histories* that “Its base is square, each side is 800 feet long, and its height the same.” This has been “translated” with much license to read: “the square of the Great Pyramid’s height is equal to the area of a triangular face.” This is equivalent to saying that the altitude of a triangular face and half the length of a base side are in the ratio ϕ to 1. (In fact, the true height is ≈ 481 feet, and the true base is only ≈ 756 feet!)