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ON THE GEOMETRY OF THE COBLE-DOLGACHEV SEXTIC

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In this paper, we study the intersection of the Coble-Dolgachev sextic with special projective spaces. Let us recall that the Coble-Dolgachev sextic $\mathcal{C}_6$ is the branch divisor of the double cover map from $\mathcal{SU}_X(3)$ to $\mathbb{P}^8 = |3\Theta|$, where $\mathcal{SU}_X(3)$ is the moduli space of semi-stable vector bundles of rank 3 and trivial determinant on a fixed curve X of genus 2, and Θ is the Riemann theta divisor of $\mathrm{Pic}^1(X)$. The adjunction of divisors is an involution of $\mathrm{Pic}^1(X)$ that lifts to a non-trivial involution τ of $|3\Theta|$. The fixed locus $\mathrm{Fix}(\tau)$ is the disjoint union of two projective spaces $\mathbb{P}^4_+$ and $\mathbb{P}^3_-$. So we study the geometry of $\mathcal{C}_6 \cap \mathbb{P}^4_+$, which should be a degree 6 threefold in $\mathbb{P}^4_+$. It is in fact the union of the Igusa-Segre quartic and a tangent double hyperplane. As a result, we will also be able to determine the geometry of $\mathcal{C}_6 \cap \mathbb{P}^3_-$.

Introduction.

In the study of moduli spaces of vector bundles over algebraic (projective) curves, it is striking to see that we have a good geometric grasp of only very few examples. Although a lot has been done about explicit moduli spaces of vector bundles of rank 2, less is known about moduli spaces for rank 3, even on curves of genus 2.

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Let X be a curve of genus 2. Let Θ be the canonical Riemann theta divisor of $\mathrm{Pic}^1(X)$. We denote by $\mathcal{SU}_X(3)$ the moduli space of vector bundles on X of rank 3 with trivial determinant. This projective variety of dimension 8 is the double cover of $\mathbb{P}^8 = |3\Theta|$ branched along a hypersurface of degree 6 called the Coble-Dolgachev sextic and which we will denote $\mathcal{C}_6$. The adjunction map

$$L \mapsto \omega_X \otimes L^{-1}$$

is an involution of $\mathrm{Pic}^1(X)$ that induces an involution τ on $|3\Theta|$. Then its fixed locus $\mathrm{Fix}(\tau)$ is the disjoint union of a $\mathbb{P}^4$ and a $\mathbb{P}^3$. We call these spaces

$$\mathrm{Fix}(\tau) = \mathbb{P}^4_+ \sqcup \mathbb{P}^3_-.$$

The goal of this paper is to study the (not necessarily irreducible) scheme

$$V_{NR} = \mathcal{C}_6 \cap \mathbb{P}^4_+.$$

The main theorems surprisingly (or not) introduce (the compactification of) the moduli space of principally polarized Abelian surfaces with level-2 structure, the Igusa-Segre quartic, into the picture.

Theorem A. *The scheme V_{NR} is a degree 6 scheme in $\mathbb{P}^4_+$ which is the union of the Igusa-Segre quartic $\mathcal{I}_4$ and a double hyperplane $V_0 \cong \mathbb{P}^3$.*

It is somehow natural to see the Igusa-Segre quartic appear because it is the quotient of the moduli space $\mathcal{SU}_X(2)$ of vector bundles of rank 2 by the action of $\mathrm{Jac}(X)_2$ (subgroup of 2-torsion points of $\mathrm{Jac}(X)$), and the way it happens depends on the choice of a symplectic isomorphism ϕ from $\mathrm{Jac}(X)_2$ to $(\mathbb{F}_2)^4$. Which is exactly the data of a level-2 structure. Then a corollary of Theorem A follows easily from the well-known geometry of the Igusa-Segre quartic.

Theorem B. *The hyperplane V_0 is tangent to $\mathcal{I}_4$ at the point corresponding to the trivial vector bundle $\mathcal{O}_X^{\oplus 3}$ from the $\mathcal{SU}_X(3)$ perspective, but also to the point (Jac(X), ϕ) in the moduli space $\mathcal{I}_4 = \overline{\mathcal{A}_2(2)}$.*

Thanks to Theorem A, we can translate the beautiful geometry of $\mathcal{I}_4$ in terms of vector bundles through the analysis of the 2-torsion points of $\mathrm{Jac}(X)$. And that allows us to understand the other intersection.

Theorem C. The surface $S_{NR} = \mathcal{C}_6 \cap \mathbb{P}^3_-$ is a hexahedron, i.e. the union of 6 planes in $\mathbb{P}^3_-$.

This hexahedron realizes the $(15_4, 20_3)$-configuration. It contains 15 lines, each meeting some of the others at 4 intersection points. There are 20 intersection points, through each of which pass 3 lines.

In the following, we will denote by X a smooth projective curve of genus $g = g(X) \geq 2$ over the field of complex numbers $\mathbb{C}$.

1. The moduli space of vector bundles: generalities.

Before focusing our study to the case of rank-3 vector bundles, we will first expose a few useful facts about the case of general rank. Let E be a vector bundle of rank r on X. We define its *determinant* to be

$$\det(E) = \bigwedge^r E.$$

It is a line bundle whose degree will be called the *degree of the vector bundle E* and denoted $\deg(E)$. There is also a notion of *slope of* E, it is the number

$$\mu(E) = \frac{\deg(E)}{r}.$$

A vector bundle E is said to be *semi-stable* (respectively *stable*) if for any proper subbundle F the following inequality holds

$$\mu(F) \leq \mu(E) \text{ (resp. } \mu(F) < \mu(E)) .$$

In order to deal with moduli spaces, let us introduce an equivalence relation due to C.S. Seshadri [14]. Every semi-stable vector bundle E admits a strictly increasing Jordan-Hölder filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_{k-1} \subset E_k = E,$$

such that each successive quotient E_i/E_{i-1} is stable, for $i = 1, \ldots, k$. We call

$$\mathrm{gr}(E) = \bigoplus_{i=1}^{k} E_i/E_{i-1}$$

the *associated graded bundle* of E. Finally, two semi-stable vector bundles E and E' on X are said to be *S-equivalent* if $\mathrm{gr}(E) \cong \mathrm{gr}(E')$:

$$E \sim_S E' \overset{\text{def}}{\Longleftrightarrow} \mathrm{gr}(E) \cong \mathrm{gr}(E').$$

In particular, two stable bundles E and E' are S-equivalent if and only if they are isomorphic.

Now, if we fix a line bundle L on the curve X, we will denote by $\mathcal{S}U_X(r, L)$ the (coarse in general) *moduli space of S-equivalence classes of semi-stable vector bundles on X of rank r and fixed determinant L*. We will also denote by $\mathcal{U}_X(r, d)$ the *moduli of S-equivalence classes of semi-stable vector bundles on X of rank r and degree d*. The construction of the latter moduli space is due to C. S. Seshadri [14], and he shows that it is an irreducible normal projective variety of dimension

$$\dim \mathcal{U}_X(r, d) = r^2(g-1) + 1, \tag{1.1}$$

where g is the genus of the smooth complete curve X. Moreover, $\mathcal{U}_X(r, d)$ is a fibration over $\mathrm{Pic}^d(X)$, the space of line bundles (or divisors) of degree d on X:

$$\det : \ \mathcal{U}_X(r, d) \to \mathrm{Pic}^d(X).$$

The fiber over a point L of $\mathrm{Pic}^d(X)$ is exactly $\mathcal{SU}_X(r, L)$. And we see that

$$\dim \mathcal{SU}_X(r, L) = (r^2 - 1)(g - 1). \tag{1.2}$$

It also follows that the spaces $\mathcal{SU}_X(r, L)$ are also normal irreducible and have universal properties similar to those of $\mathcal{U}_X(r, d)$. For instance, for both kinds of spaces, $\mathcal{U}_X(r, d)$ and $\mathcal{SU}_X(r, L)$, the singular points correspond exactly to classes of decomposable bundles, i.e. *strictly semi-stable bundles*, except when $g = r = 2$ and d is even. J.-M. Drezet and M. S. Narasimhan [3] also prove that these spaces are locally factorial, i.e. Weil divisors are Cartier divisors, and manage to describe the Picard groups. Let F be a fixed vector bundle on X of degree $(-d + r(g(X) - 1))/(r, d)$ and rank $r/(r, d)$ and L be a fixed line bundle of degree d. We define the sets

$$\Delta_F = \{E \in \mathcal{U}_X(r, d) \ : \ h^0(X, E \otimes F) > 0\},$$
$$\Delta_{F,L} = \{E \in \mathcal{SU}_X(r, L) \ : \ h^0(X, E \otimes F) > 0\}.$$

According to A. Hirschowitz [5], we can find some *convenient* vector bundles F so that there exists a stable bundle $E \in \mathcal{U}_X(r, d)$ making $h^0(X, E \otimes F) = 0$. In this case, $\Delta_F \subset \mathcal{U}_X(r, d)$ and $\Delta_{F,L} \subset \mathcal{S}U_X(r, L)$ are divisors.

Theorem 1.1. (J.-M. Drezet, M. S. Narasimhan [3]). *Let F be a fixed* convenient *vector bundle on X of degree $(-d + r(g(X) - 1))/(r, d)$ and rank $r/(r, d)$ and L be a fixed line bundle of degree d.*

(i) The line bundles

$$\Theta_{\mathcal{U}_X(r,d)} = \mathcal{O}_{\mathcal{U}_X(r,d)}(\Delta_F), \qquad (1.3)$$
$$\Theta_{\mathcal{S}U_X(r,L)} = \mathcal{O}_{\mathcal{S}U_X(r,L)}(\Delta_{F,L})$$

associated to the divisors $\Delta_F \subset \mathcal{U}_X(r,d)$ *and* $\Delta_{F,L} \subset \mathcal{S}U_X(r,L)$*) do not depend on the choice of* F*.*
(ii) The Picard group of $\mathcal{S}U_X(r,L)$ *is generated by* $\Theta_{\mathcal{S}U_X(r,d)}$*.*
(iii) The canonical sheaf of $\mathcal{S}U_X(r,L)$ *is given by*

$$\omega_{\mathcal{S}U_X(r,L)} = \Theta_{\mathcal{S}U_X(r,L)}{}^{-2(r,d)}. \qquad (1.4)$$

This generator of the Picard group is often called the *generalized theta divisor*, for it generalizes the traditional notion of theta divisor on Jacobians of curves. If X is a smooth projective algebraic of genus g, then the variety $\mathrm{Pic}^{g-1}(X)$ has a canonical *Riemann theta divisor* W_{g-1} or $\Theta_{\mathrm{Pic}^{g-1}(X)}$ defined as

$$\Theta_{\mathrm{Pic}^{g-1}(X)} = W_{g-1} = \{L \in \mathrm{Pic}^{g-1}(X) \;:\; h^0(X,L) > 0\},$$

which is a translate of the theta divisor on $\mathrm{Jac}(X)$. If no confusion arises, the generalized theta divisor will be denoted by Θ^{gen}, and the canonical Riemann theta divisor of $\mathrm{Pic}^{g-1}(X)$ just by Θ.

We will now focus our attention to moduli spaces with trivial determinant $\mathcal{S}U_X(r,\mathcal{O}_X) = \mathcal{S}U_X(r)$. For any $E \in \mathcal{S}U_X(r)$, we define

$$D_E = \{L \in \mathrm{Pic}^{g-1}(X) \;:\; h^0(X, E \otimes L) > 0\} \subset \mathrm{Pic}^{g-1}(X).$$

It is known that D_E is either the whole space $\mathrm{Pic}^{g-1}(X)$ or a divisor of the linear system $|r\Theta|$. The former case only happens for some special $E \in \mathcal{S}U_X(r)$, so we get a rational map Φ_r:

$$\begin{aligned} \mathcal{S}U_X(r) &\overset{\Phi_r}{- - - >} |r\Theta|, \\ E &\longmapsto D_E. \end{aligned} \qquad (1.5)$$

The map Φ_r is defined by the linear system $|\Theta^{gen}|$. This follows from a theorem of A. Beauville, M. S. Narasimhan and S. Ramanan [2] which states that there is a canonical isomorphism

$$H^0(\mathcal{S}U_X(r), \Theta^{gen})^* \cong H^0(\mathrm{Pic}^{g-1}(X), r\Theta). \qquad (1.6)$$

2. The Heisenberg group and the Schrödinger representation.

Let us fix p a prime number and consider the group $(\mathbb{F}_p)^{2g} = (\mathbb{F}_p)^g \times (\mathbb{F}_p)^g$. We let $h(\ ,\ ) : (\mathbb{F}_p)^{2g} \times (\mathbb{F}_p)^{2g} \to \mathbb{F}_p$ be the standard symplectic form on $\mathbb{F}^{2g}$ defined by the matrix

$$\begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}.$$

Another way to express the standard symplectic form is

$$((\mathbb{F}_p)^g \times (\check{\mathbb{F}}_p)^g) \times ((\mathbb{F}_p)^g \times (\check{\mathbb{F}}_p)^g) \longrightarrow \mathbb{F}^p,$$
$$(u, \alpha) \times (v, \beta) \longmapsto \beta(u) - \alpha(v),$$

where $(\check{\mathbb{F}}_p)^g = \mathrm{Hom}((\mathbb{F}_p)^g, \mathbb{F}_p)$ denotes the group of linear forms on the $(\mathbb{F}_p)$-vector space $(\mathbb{F}_p)^g$. Let $\mathcal{H}_g(p)$ or $\mathcal{H}(p)$ be the central extension of the group $(\mathbb{F}_g)^{2p}$ with center isomorphic to the group $\boldsymbol{\mu}_p$ of p-th root of unity:

$$1 \to \boldsymbol{\mu}_p \to \mathcal{H}(p) \to (\mathbb{F}_p)^{2g} \to 0.$$

This group is called the *Heisenberg group*. It is set-theoretically equal to $(\mathbb{F}_p)^g \times (\check{\mathbb{F}}_p)^g \times \boldsymbol{\mu}_p$ and the group law is defined by:

$$(u, \alpha, x) \cdot (v, \beta, y) = (u + v, \alpha + \beta, e^{2\pi i(\beta(u) - \alpha(v))/p} x y).$$

We then define the *linear Schrödinger representation* of $\mathcal{H}(p)$ in $\mathbb{C}^{p^g} = \bigoplus_{v \in (\mathbb{F}_p)^g} \mathbb{C} f_v$ by extending linearly the action on the basis elements f_v:

$$(u, \alpha, x) \cdot f_v = x e^{2\pi i \alpha(v)/p} f_{u+v}.$$

If we look at the induced projective Schrödinger representation of $\mathcal{H}(p)$ in $\mathbb{P}(\mathbb{C}^{p^g})$, it is clear that the center $\boldsymbol{\mu}_p$ acts trivially on the projective space. In other words, we have a projective representation of $(\mathbb{F}_p)^{2g}$ in $\mathbb{P}^{p^g - 1}$, which lifts to the linear Schrödinger representation.

Let X be a curve of genus g. $(\mathrm{Jac}(X), \Theta_{\mathrm{Jac}(X)})$ is a principally polarized abelian variety and $\mathrm{Jac}(X)_p$ its subgroup of p-torsion points. $\mathrm{Jac}(X)_p$ acts on $\mathrm{Jac}(X)$ by translation:

$$t_\epsilon : x \mapsto x + \epsilon,$$

and also at the level of divisors:

$$t_\epsilon^*(D) = D - \epsilon.$$

By the theorem of the square, we see that the linear system $|p\Theta_{\mathrm{Jac}(X)}|$ is invariant under translations by p-torsion points. Similarly, if we identify $\mathrm{Pic}^{g-1}(X)$ with $\mathrm{Jac}(X)$, $\mathrm{Jac}(X)_p$ acts on $|p\Theta|$. This action leaves the map

$$\Phi_p : \mathcal{SU}_X(p) \to |p\Theta|$$

equivariant. We tie this with the Schrödinger representation thanks to the following well known theorem (see for instance [6]):

Theorem 2.1. *Let us fix a symplectic isomorphism* $\phi : \mathrm{Jac}(X)_p \to (\mathbb{F}_p)^{2g}$*, where the symplectic structure on* $Jac(X)_p$ *is the Weil pairing defined by the cup-product on* $H^1(X, \mathbb{F}_p) \cong Jac(X)_p$*. There exists a unique isomorphism*

$$\Phi : |p\Theta_{\mathrm{Jac}(X)}| \cong \mathbb{P}^{p^g-1}$$

which is ϕ*-equivariant with respect to the action of* $Jac(X)_p$ *on* $|p\Theta_{\mathrm{Jac}(X)}|$ *by translations and the action of* $(\mathbb{F}_p)^{2g}$ *on* $\mathbb{P}^{p^g-1}$ *by means of the Schrödinger representation.*

The choice of ϕ is called a *level-p structure*. Of course, we can restate the theorem for $|p\Theta|$, where $\Theta = \Theta_{\mathrm{Pic}^{g-1}(X)}$, when we identify $\mathrm{Pic}^{g-1}(X)$ with $\mathrm{Jac}(X)$.

3. The Coble-Dolgachev sextic.

From now on, X will denote a smooth projective curve of genus $g = 2$, therefore hyperelliptic. Let us restrict ourselves to the cases $r = 2$ and $r = 3$, i.e. vector bundles of rank 2 and rank 3. We keep the notation Θ for the canonical Riemann theta divisor of $\mathrm{Pic}^1(X)$ and Θ^{gen} for the generalized theta divisor of either $U_X(r, d)$ or $\mathcal{SU}_X(r, L)$. In our particular case, the maps Φ_r of (1.5) are well understood.

Theorem 3.1.

(i) The map $\Phi_2 : \mathcal{SU}_X(2) \to |2\Theta| \cong \mathbb{P}^3$ *is an isomorphism.*
(ii) The map $\Phi_3 : \mathcal{SU}_X(3) \to |3\Theta| \cong \mathbb{P}^8$ *is a finite map of degree* 2.

Proof. A proof of (i) can be found in [8]. For (ii), a first unpublished proof was given by D. Butler and I. Dolgachev using the Verlinde formula, but Y. Laszlo produced another beautiful proof in [7] by making a Hilbert polynomial computation.

It is easy to see that for all $L \in \mathrm{Jac}(X)$, we have $\mathcal{SU}_X(r) \cong \mathcal{SU}_X(r, L)$.

The double cover of $\mathbb{P}^8$ is the subject of study of this paper. If we apply (1.1), (1.2), (1.3) and (1.4) to the case of rank 2 and 3 and genus 2, we see that

$$\dim \mathcal{U}_X(2,0) = 5\,, \tag{3.1}$$

$$\dim \mathcal{SU}_X(3) = 8\,, \tag{3.2}$$

$$\omega_{\mathcal{SU}_X(3)} = \mathcal{O}_{\mathcal{SU}_X(3)}(-6\Theta^{gen})\,, \tag{3.3}$$

Let B be the branch divisor of Φ_3, of degree b in $\mathbb{P}^8$. So there is a divisor D of degree $b/2$ in $\mathbb{P}^8$ so that $B \sim_{\text{lin}} 2D$. The Hurwitz formula applied to the double cover gives:

$$\begin{aligned} \omega_{\mathcal{SU}_X(3)} &= \Phi_3^*(\omega_{\mathbb{P}^8} \otimes \mathcal{O}_{\mathbb{P}^8}(D)), \\ &= \Phi_3^*(\mathcal{O}_{\mathbb{P}^8}(-9 + \frac{b}{2})), \\ -6\Theta^{gen} &= (-9 + \frac{b}{2})\Theta^{gen} \text{ by (1.6) and (3.3).} \end{aligned}$$

So $b = 6$. This leads to the next definition, following Laszlo's denotation.

Definition 3.2. The branch divisor of $\Phi_3 : \mathcal{SU}_X(3) \to \mathbb{P}^8$ is called the *Coble-Dolgachev sextic*, which we will denote by $\mathcal{C}_6$.

The name of the sextic came to be by analogy with the Coble quartic (see [9], [13]), and the fact that it was conjectured by I. Dolgachev that it is the dual of the Coble cubic, a result later proved by A. Ortega [11] (see also [10]).

Remark 3.3. Since $\mathbb{P}^8$ is smooth, we know what the singular loci of $\mathcal{SU}_X(3)$ and $\mathcal{C}_6$ the branch locus of the double cover are equal. We will denote this variety

$$\Sigma = \text{Sing}(\mathcal{SU}_X(3)) = \text{Sing}(\mathcal{C}_6). \tag{3.4}$$

We will now describe the involution of the double cover map Φ_3. Recall that X is a curve of genus 2, so it is hyperelliptic. We call h its hyperelliptic involution. Let τ be the adjunction involution on $\text{Pic}^1(X)$ given by

$$\begin{aligned} \text{Pic}^1(X) &\xrightarrow{\tau} \text{Pic}^1(X), \\ L &\longmapsto \omega_X \otimes L^{-1}. \end{aligned} \tag{3.5}$$

It induces, by pulling back, an involution on $|3\Theta|$, which we still denote by τ. Let also τ' be the involution of $\mathcal{SU}_X(3)$ given by

$$\begin{aligned} \mathcal{SU}_X(3) &\xrightarrow{\tau'} \mathcal{SU}_X(3), \\ E &\longmapsto E^*, \end{aligned} \tag{3.6}$$

where E^* denotes the dual vector bundle of E. Then the double cover involution σ is (see for instance [11])

$$\sigma = \tau' \circ h^* = h^* \circ \tau' : \ E \mapsto h^* E^*,$$

that is the ramification locus of $\mathcal{SU}_X(3)$ corresponds exactly to

$$\mathcal{C}_6 \cong \{E \in \mathcal{SU}_X(3) \ : \ \sigma(E) = h^* E^* \sim_S E\}$$

We end this section with a useful consequence of the theorem of Riemann-Roch:

$$\tau \circ \Phi_3 = \Phi_3 \circ \tau'. \tag{3.7}$$

This implies in particular,

lemma 3.4. *Let* $\mathrm{Fix}(\tau)$ *and* $\mathrm{Fix}(\tau')$ *be the fixed loci of* τ *and* τ' *respectively. If we identify the branch locus and the ramification locus* $\mathcal{C}_6$ *of the double cover* Φ_3 *of* $\mathbb{P}^8$*, then*

$$\mathrm{Fix}(\tau) \cap \mathcal{C}_6 \cong \mathrm{Fix}(\tau') \cap \mathcal{C}_6. \tag{3.8}$$

4. Involution-fixed spaces and intersection.

Now that we have introduced the notations and the algebraic varieties of study, we get to the heart of the problem. Recall - (3.5) - that the involution $\tau : |3\Theta| \to |3\Theta|$ is induced by the involution of $\mathrm{Pic}^1(X)$ defined by $L \mapsto \omega_X \otimes L^{-1}$. The involution τ is actually an involution on the vector space $H^0(\mathrm{Pic}^1(X), \mathcal{O}(3\Theta))$, therefore its fixed locus $\mathrm{Fix}(\tau)$ is the disjoint union of the projectivization of the invariant and anti-invariant spaces:

$$\begin{aligned} \mathrm{Fix}(\tau) &= \mathrm{Fix}(\tau)_+ \sqcup \mathrm{Fix}(\tau)_-, \\ &= \mathbb{P}^4_+ \sqcup \mathbb{P}^3_-. \end{aligned}$$

We want to study the geometry of the intersection

$$V_{NR} = \mathcal{C}_6 \cap \mathbb{P}^4_+.$$

First we have to show that V_{NR} is not $\mathbb{P}^4_+$, i.e. $\mathbb{P}^4_+ \not\subset \mathcal{C}_6$. This fact is known as discussed in the proof of Proposition 2.8.1 of [11] and in [10].

Recall that at the level of vector bundles, i.e. in $\mathcal{SU}_X(3)$, we also have an involution τ' (3.6), which, in virtue of (3.7), allows us to define

$$\begin{array}{rcccl} \mathrm{Fix}(\tau')_+ &=& \Phi_3^{-1}(\mathrm{Fix}(\tau)_+) &=& \Phi_3^{-1}(\mathbb{P}^4_+), \\ \mathrm{Fix}(\tau')_- &=& \Phi_3^{-1}(\mathrm{Fix}(\tau)_-) &=& \Phi_3^{-1}(\mathbb{P}^3_-). \end{array}$$

Then we deduce from (3.8) that

$$V_{NR} = \mathcal{C}_6 \cap \mathbb{P}^4_+ = \mathcal{C}_6 \cap \mathrm{Fix}(\tau')_+.$$

Our approach is now to construct vector bundles on X which belong to V_{NR}. Recall that we denoted by h the hyperelliptic involution of X.

Lemma 4.1. *Let* $F \in \mathcal{SU}_X(2)$ *be a vector bundle on* X *of rank* 2 *with trivial determinant. Then*

$$h^*F^* = F \text{ and } F^* = F.$$

Proof. Since the action of the hyperelliptic involution h is trivial on $|2\Theta|$, it follows that $h^*F = F$. Also $F = \det(F) \otimes F^* = F^*$, since we assumed that F had trivial determinant. And that proves the lemma. $\square$

Let

$$V_0 = \{\mathcal{O}_X\} \oplus \mathcal{SU}_X(2) = \{\mathcal{O}_X \oplus F \ : \ F \in \mathcal{SU}_X(2)\} \cong \mathbb{P}^3,$$

where the isomorphism with $\mathbb{P}^3$ comes from Theorem (1.3) (i).

Proposition 4.2. *Let S^2 be the second symmetric power map. Then,*

$$V_0 \in V_{NR} \text{ and } S^2\mathcal{SU}_X(2) \in V_{NR}.$$

Proof. Let us fix $F \in \mathcal{SU}_X(2)$. We first notice that the vector bundle S^2F is of rank 3 and has trivial determinant. Lemma (3) shows that $\mathcal{O}_X \oplus F$ and S^2F are in both $\mathcal{C}_6$ and $\mathrm{Fix}(\tau')$, so they belong to the intersection. We want then to prove that they actually belong to $\mathrm{Fix}(\tau')_+$. It follows directly from Proposition 5.2. of [12] that $S^2\mathcal{SU}_X(2)$ is a subset of $\mathrm{Fix}(\tau')_+$, and so does V_0, by the fact that $\mathcal{O}_X^{\oplus 3} \in V_0 \cap S^2\mathcal{SU}_X(2)$ and a connectedness and irreducibility argument. $\square$

As a direct consequence of the Proposition, V_{NR} is reducible. Moreover, $\mathrm{Jac}(X)_2$ acts on $\mathcal{SU}_X(2)$ by

$$\begin{array}{ccc} \mathrm{Jac}(X)_2 \times \mathcal{SU}_X(2) & \longrightarrow & \mathcal{SU}_X(2), \\ (\epsilon, F) \ |----> & & F \otimes \epsilon\,. \end{array}$$

This natural action becomes trivial on $S^2\mathcal{SU}_X(2) \subset \mathcal{SU}_X(3)$, so the map

$$\begin{array}{ccc} S^2 : \ \mathcal{SU}_X(2) & \longrightarrow & \mathcal{SU}_X(3)\,, \\ F \ |----> & & S^2F\,, \end{array}$$

factors through

$$S^2 : \ \mathcal{SU}_X(2)/\mathrm{Jac}(X)_2 \to V_{NR} \subset \mathcal{SU}_X(3)\,. \tag{4.1}$$

It turns out (see [12]) that this map (4.1) is an embedding.

Let us fix a level-2 structure on $\mathrm{Jac}(X)$, identified with $\mathrm{Pic}^1(X)$, that is a symplectic isomorphism

$$\phi : \ \mathrm{Jac}(X)_2 \to (\mathbb{F}_2)^4.$$

By Theorem 2.1, there is a unique isomorphism

$$\Phi : |2\Theta| \cong \mathbb{P}^3,$$

that is ϕ-equivariant. Again, Θ is the canonical Riemann theta divisor of $\mathrm{Pic}^1(X)$. We also know that the map of Theorem (1.1) (i)

$$\Phi_2 : \mathcal{SU}_X(2) \to |2\Theta|$$

is a ϕ-equivariant isomorphism. So the isomorphism

$$\Phi \circ \Phi_2 : \mathcal{SU}_X(2) \cong \mathbb{P}^3 \tag{4.2}$$

is compatible with the level-2 structure ϕ. Through the isomorphism (4.2), we identify

$$\mathcal{SU}_X(2)/\mathrm{Jac}(X)_2 \cong \mathbb{P}^3/(\mathbb{F}_2)^4.$$

It is known (see [1] Vol. IV, p. 210) that this variety is isomorphic to a quartic hypersurface in $\mathbb{P}^4$, which H. Baker calls the Segre quartic, the dual variety to the Segre cubic. But it is also commonly known as the Igusa quartic from its modular interpretation as (the Satake compactification of) the moduli space $\overline{\mathcal{A}_2(2)}$ of Abelian surfaces with level-2 structure [4]. We will hence call it the *Igusa-Segre quartic* $\mathcal{I}_4$. But the map (4.1) embeds $\mathbb{P}^3/(\mathbb{F}_2)^4$ into $V_{NR} \subset \mathbb{P}^4_+$, so it has to be the Igusa-Segre quartic (in $\mathbb{P}^4_+$.)

Theorem A. *The scheme V_{NR} is a degree 6 scheme in $\mathbb{P}^4_+$ which is the union of the Igusa-Segre quartic*

$$\mathcal{I}_4 = S^2(\mathcal{SU}_X(2)) = S^2(\mathcal{SU}_X(2)/\mathrm{Jac}(X)_2)$$

and a double hyperplane $V_0 \cong \mathbb{P}^3$.

Proof. We have already seen that as a consequence of Proposition 4.2 the hyperplane V_0 (4.1) lies in V_{NR}. But, as stated in Section 1, the singular locus Σ of $\mathcal{C}_6$ and $\mathcal{SU}_X(3)$ (3.4) consists of decomposable vector bundles. So $V_0 \subset \Sigma$, therefore the intersection multiplicity of $\mathbb{P}^4_+$ along V_0 is at least 2. Moreover, we have just shown that $\mathcal{I}_4 \subset V_{NR}$. By degree considerations, V_{NR} has to be

$$V_{NR} = \mathcal{I}_4 \cup V_0,$$

V_0 coming with multiplicity 2. $\square$

5. Vector bundles and the geometry of $\mathcal{I}_4$.

In this section, we will try to recover the geometry of $\mathcal{I}_4$. More precisely, we will identify the 15 lines and 15 nodes fitting in the symmetric (15_3)-configuration in terms of vector bundles.

We know that, scheme-theoretically,

$$\Sigma \cap \mathbb{P}^4_+ \subset \mathrm{Sing}(\mathcal{C}_6 \cap \mathbb{P}^4_+) = \mathrm{Sing}(V_{NR}).$$

Since $\mathcal{I}_4$ is a reduced component of V_{NR} and V_0 comes with multiplicity 2, we see that the support of $\mathrm{Sing}(V_{NR})$ is

$$\mathrm{support}(\mathrm{Sing}(V_{NR})) = \mathrm{Sing}(\mathcal{I}_4) \cup V_0.$$

Before we can actually investigate the intersection $\Sigma \cap \mathbb{P}^4_+$, we first intersect Σ with the whole fixed locus $\mathrm{Fix}(\tau)$, which, by Lemma 3.4, is the same thing as intersecting with $\mathrm{Fix}(\tau')$ in $\mathcal{SU}_X(3)$.

Proposition 5.1. *Let us consider $\mathcal{C}_6$ and its singular locus Σ from the ramification locus standpoint, i.e. in $\mathcal{SU}_X(3)$. Then*

$$\Sigma \cap \mathrm{Fix}(\tau') = \bigcup_{\epsilon \in \mathrm{Jac}(X)_2} V_\epsilon,$$

where V_0 still denotes $\{\mathcal{O}_X\} \oplus \mathcal{SU}_X(2)$ and for all $\epsilon \in \mathrm{Jac}(X)_2 - \{0\}$,

$$V_\epsilon = \{L_\epsilon \oplus F \ : \ F \in \mathcal{SU}_X(2, L_\epsilon) \ \text{ and } t_\epsilon(F) = F\},$$

where L_ϵ is the line bundle of $Pic^0(X)$ corresponding to $\epsilon \in \mathrm{Jac}(X)$, t_ϵ is the translation automorphism of $Pic^1(X)$ acting naturally on $SU_X(2, L_\epsilon)$ by tensor product.

Proof. Let $E \in \Sigma \cap \mathrm{Fix}(\tau')$. We will study two cases. First, suppose E is S-equivalent to $L \oplus F$, for a line bundle $L \in \mathrm{Pic}^0(X)$ and a stable vector bundle $F \in \mathcal{SU}_X(2, L^{-1})$, and

$$L \oplus F \cong L^{-1} \oplus F^*.$$

Since we are assuming here that F is stable, then F^* is also stable and $\mathrm{Hom}(L, F^*) = 0$. So

$$L \cong L^{-1} \text{and} F \cong F^*.$$

Consequently, $L = L_\epsilon \in \mathrm{Jac}(X)_2$ is a 2-torsion point. But

$$F = F^* \otimes \det(F) = F^* \otimes L_\epsilon,$$

and, since $F \cong F^*$, it follows that $F \cong F \otimes L_\epsilon$. Notice that this last condition is vacuous when $\epsilon = 0$. So $E \in \bigcup_{\epsilon \in \mathrm{Jac}(X)_2} V_\epsilon$. Next, suppose that E is completely decomposable, i.e. it is S-equivalent to $L_1 \oplus L_2 \oplus L_3$, with $L_i \in \mathrm{Pic}^0(X)$ and $L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_X$. Then $E \cong E^*$ if and only if $\{L_1, L_2, L_3\} = \{L_1^{-1}, L_2^{-1}, L_3^{-1}\}$, thus if and only if $\{L_1, L_2, L_3\} = \{L, L^{-1}, \mathcal{O}_x\}$. Therefore $E \in V_0$. Finally, the reverse inclusion is easy to check. $\square$

Remark 5.2. It is easy to see that

$$V_0 \cap V_\epsilon \;=\; \{L_\epsilon \oplus L_\epsilon \oplus \mathcal{O}_X\} \text{ for } \epsilon \neq 0, \tag{5.3}$$

$$V_\epsilon \cap V_\eta \;=\; \{L_\epsilon \oplus L_\eta \oplus L_{\epsilon+\eta}\} \text{ for } \epsilon \neq \eta \text{ and } \epsilon, \eta \neq 0. \tag{5.4}$$

Remark 5.3. In view of (5.1), (5.2) and the fact that $V_0 \subset V_{NR}$, it follows that only the V_ϵ, $\epsilon \neq 0$, contribute to the singular locus of $\mathcal{I}_4$.

Let us now recall that there are maps

$$\begin{array}{ccc} \mathcal{U}_X(2,0) & \xrightarrow{\nu} & \Sigma \\ \downarrow^{\pi} & & \\ \mathrm{Jac}(X) & & \end{array}$$

given by

$$\nu:\; F \mapsto F \oplus \det(F)^{-1}, \tag{5.5}$$

$$\pi = \det:\; F \mapsto \det(F). \tag{5.6}$$

The map ν is clearly surjective and it is also injective on the open set of stable bundles, so ν is a birational map. It is well known that π is a projective bundle, the quotient of the trivial projective bundle $\mathrm{Jac}(X) \times \mathcal{SU}_X(2)$ under the (proper and discontinuous) diagonal action of $\mathrm{Jac}(X)_2$. The fiber of π over $L \in \mathrm{Jac}(X)$ is just the projective space $\mathcal{SU}_X(2, L)$. Since $\mathrm{Jac}(X)$ is smooth, $\mathcal{U}_X(2,0)$ is also smooth. Then, the fiber over a singular point of Σ, i.e. a point corresponding to a vector bundle of the form $L_1 \oplus L_2 \oplus L_3$, such that $L_1 \otimes L_2 \otimes L_3 \cong \mathcal{O}_X$, consists of three points:

$$\nu^{-1}(L_1 \oplus L_2 \oplus L_3) = \{L_1 \oplus L_2,\; L_1 \oplus L_2,\; L_2 \oplus L_3\},$$

neither two of which lie in the same $\mathbb{P}^3$ fiber of the map $\det:\; \mathcal{U}_X(2,0) \to J$. So we just proved the follwing:

Lemma 5.4. *$\mathcal{U}_X(2,0)$ is smooth and $\nu:\; \mathcal{U}_X(2,0) \to \Sigma$ is a resolution of singularities of Σ. And for $L \in$ $Pic^0(X)$, the restriction*

$$\nu_L = \nu|_{\mathcal{SU}_X(2,L)}:\; \mathcal{SU}_X(2,L) \to \Sigma$$

is an embedding.

Let us recall that we had fixed a level-2 structure on $\mathrm{Jac}(X)$ in Section 4, (4.4):

$$\phi : \mathrm{Jac}(X)_2 \to (\mathbb{F}_2)^4.$$

Again, from Theorem 2.1 and the way we identify $\mathcal{SU}_X(2, L)$ with $\mathbb{P}^3$ for any $L \in \mathrm{Pic}^0(X)$, the tensor product action of $\mathrm{Jac}(X)_2$ on $\mathcal{SU}_X(2, L)$ is just the action of $(\mathbb{F}_2)^4$ on $\mathbb{P}^3$ of the Shrödinger representation. It is then easy to see that for $\epsilon \in \mathrm{Jac}(X)_2 - \{0\}$, V_ϵ is the disjoint union of 2 lines in $\mathcal{SU}_X(2, L_\epsilon) \cong \mathbb{P}^3$:

$$V_\epsilon = \nu_\epsilon(\mathrm{Fix}(t_\epsilon)) \cong \mathrm{Fix}(t_\epsilon) \cong \mathbb{P}^1 \sqcup \mathbb{P}^1.$$

Looking back at $\mathbb{P}^4_+$, any line in $\mathbb{P}^4_+$ should intersect $V_0 \cong \mathbb{P}^3$ if it is not contained in the hyperplane. But we know from Remark 5.2 that $V_0 \cap V_\epsilon$ consists of only one point (set-theoretically), and since the lines of V_ϵ are disjoint and not contained in V_0, it follows that only one of them intersects V_0.

Lemma 5.5. *Let $\epsilon \in \mathrm{Jac}(X)_2 - \{0\}$. V_ϵ consists of two lines, one is in $\mathbb{P}^4_+$, more precisely in the singular locus of $\mathcal{I}_4$, and the other is in $\mathbb{P}^3_-$. The 15 lines in $\mathbb{P}^4_+$ are exactly the 15 lines forming the singular locus of $\mathcal{I}_4$.*

Proof. We know that the 15 lines in $\mathbb{P}^4_+$ are actually in $\Sigma \cap \mathbb{P}^4_+$ by Proposition 5.1. Then by (5.1), (5.2) and Remark 5.3, these 15 lines are in $\mathrm{Sing}(\mathcal{I}_4)$, but the singular locus of the Igusa-Segre quartic consists of exactly 15 lines already. □

Lemma 5.6. *There are in total 35 "nodes", i.e. the 15 pairs of lines V_ϵ intersect in 35 points: 15 of the nodes lie in $\mathbb{P}^4_+$ and 20 in $\mathbb{P}^3_-$. And through each of the nodes pass exactly 3 lines.*

Proof. From Remark 5.2, a choice of two non-zero 2-torsion points, say ϵ and η, determines a node $L_\epsilon \oplus L_\eta \oplus L_{\epsilon+\eta}$, but there are 3 ways to get the same node. So there are $\binom{15}{2}/3 = 35$ nodes. Since all these nodes lie in the singular locus $\Sigma \cap \mathrm{Fix}(\tau')$, we know from the configuration of the singular locus of the Igusa-Segre quartic that 15 nodes lie in $\mathbb{P}^4_+$ and 20 in $\mathbb{P}^3_-$. Finally, since

$$\{L_\epsilon \oplus L_\eta \oplus L_{\epsilon+\eta}\} = V_\epsilon \cap V_\eta \cap V_{\epsilon+\eta},$$

it is easy to see that each node is the intersection of three lines. □

The simple combinatorial fact that 15 points lie in $\mathbb{P}^4_+$ and 20 in $\mathbb{P}^3_-$ motivates the following theorem.

Theorem 5.7. *Let $w(\ ,\ )$ be the Weil pairing on Jac$(X)_2$, which correspond to the standard symplectic pairing on $(\mathbb{F}_2)^4$ through our choice of a level-2 structure (4.4).*

$$\text{The node } V_\epsilon \cap V_\eta \text{ is in } \begin{cases} \mathbb{P}^4_+ & \text{if } w(\epsilon,\eta) = 0 \in \mathbb{F}_2, \\ \mathbb{P}^3_- & \text{if } w(\epsilon,\eta) = 1 \in \mathbb{F}_2. \end{cases}$$

Proof. Given $\epsilon, \eta \in \mathrm{Jac}(X)_2 - \{0\}$, let

$$E_{\epsilon,\eta} = L_\epsilon \oplus L_\eta \oplus L_{\epsilon+\eta} \in \mathcal{SU}_X(3).$$

Then $\Phi_3(E_{\epsilon,\eta})$ is the divisor from the linear system $|3\Theta|$ supported on

$$D_{E_{\epsilon,\eta}} = \{L \in \mathrm{Pic}^1(X)\ :\ h^0(X, L\otimes L_\epsilon)+h^0(X, L\otimes L_\eta)+h^0(X, L\otimes L_{\epsilon+\eta}) > 0\}.$$

To determine whether $E_{\epsilon,\eta}$ (more rigorously $\Phi_3(E_{\epsilon,\eta})$) is in $\mathbb{P}^4_+$ or $\mathbb{P}^3_-$, we will investigate what theta characteristics lie in $D_{E_{\epsilon,\eta}}$.

Suppose $w(\epsilon,\eta) = 0 \in \mathbb{F}_2$. The Riemann-Mumford relation states that for any theta characteristic ϑ of $\mathrm{Pic}^1(X)$,

$$h^0(X,\vartheta) + h^0(X, L_\epsilon \otimes \vartheta) + h^0(X, L_\eta \otimes \vartheta) +$$

$$+\ h^0(X, L_{\epsilon+\eta} \otimes \vartheta) \equiv w(\epsilon,\eta)\,. \quad (\mathrm{mod} 2)$$

It follows that any odd theta characteristic ϑ (i.e. $h^0(X,\vartheta)$ odd) lies in $D_{E_{\epsilon,\eta}}$. By Lemma 2.2 of [12], we conclude that $\Phi_3(E_{\epsilon,\eta}) = \mathcal{O}(D_{E_{\epsilon,\eta}})$ is in $\mathbb{P}^4_+$.

Conversely, if $w(\epsilon,\eta) = 1 \in \mathbb{F}_2$, then we see that any even theta characteristic lies in $D_{E_{\epsilon,\eta}}$, so $\Phi_3(E_{\epsilon,\eta})$ is in $\mathbb{P}^3_-$. □

The Igusa-Segre quartic has another interesting property. The tangent spaces intersect the quartic along Kummer surfaces. In $\mathcal{SU}_X(2) \cong \mathbb{P}^3$, there is a natural Kummer surface $\mathcal{K}$:

$$\mathcal{K} = \{L \oplus L^{-1}\ :\ L \in \mathrm{Jac}(X)\} \cong \mathrm{Jac}(X)/\langle -1 \rangle.$$

Theorem B. *The hyperplane V_0 is tangent to $\mathcal{I}_4$ at the point corresponding to the trivial vector bundle $\mathcal{O}_X^{\oplus 3}$ from the $\mathcal{SU}_X(3)$ perspective, but also to the point (Jac$(X), \phi$) in the moduli space $\mathcal{I}_4 = \overline{\mathcal{A}_2(2)}$.*

Proof. The action (4.2) of $\mathrm{Jac}(X)_2$ on $\mathcal{SU}_X(2)$ restricts to $\mathcal{K}$ so that

$$\mathcal{K}/\mathrm{Jac}(X)_2 = \mathcal{K}.$$

So the second symmetric power map S^2 (4.3) embeds the Kummer surface $\mathcal{K}$ in $\mathcal{I}_4$ as the set

$$\mathcal{K}' = \{\mathcal{O}_X \oplus L \oplus L^{-1} \, : \, L \in \mathrm{Jac}(X)\}.$$

We notice that $\mathcal{K}'$ is also in V_0. So

$$\mathcal{K}' \subseteq \mathcal{I}_4 \cap V_0.$$

Clearly, 15 of the 16 nodes of $\mathcal{K}'$ are the points p_ϵ corresponding to $\mathcal{O}_X \oplus L_\epsilon \oplus L_\epsilon$, for $\epsilon \in \mathrm{Jac}(X)_2$. For $\epsilon \neq 0$, we see (Remark 5.2) that the node p_ϵ is

$$p_\epsilon = V_\epsilon \cap V_0,$$

i.e. the intersection of V_0 with the line of V_ϵ in $\mathbb{P}^4_+$. It is known that the last node $p_0 = \mathcal{O}_X^{\oplus 3}$ is given by the point of tangency [4], so V_0 is tangent to $\mathcal{I}_4$ at p_0. Finally, if A is the Abelian surface corresponding to p_0 in the moduli space $\mathcal{I}_4$, then

$$\mathcal{K}' = \mathcal{I}_4 \cap T_{p_0}\mathcal{I}_4 = A/\langle -1 \rangle.$$

Since $\mathcal{K}' \cong \mathrm{Jac}(X)/\langle -1 \rangle$, we see that $A = \mathrm{Jac}(X)$. And to make everything correspond with the Heisenberg group action, there was an underlying choice of a level-2 structure, which was ϕ (4.4). So $p_0 = (\mathrm{Jac}(X), \phi)$. □

6. The hexahedron of $\mathbb{P}^3_-$.

From Lemma 5.5, we know that there are 15 lines in the intersection

$$S_{NR} = \mathcal{C}_6 \cap \mathbb{P}^3_- \cong \mathcal{C}_6 \cap \mathrm{Fix}(\tau')_-,$$

and more precisely in its singular locus, according to Proposition 5.1. This will allow us to examine and understand the reducible sextic surface S_{NR}. We also know by Lemma 5.6 that these 15 lines should intersect in 20 points, or nodes.

Proposition 6.1. *The 15 lines of S_{NR} realize the $(15_4, 20_3)$-configuration: 15 lines meet in 20 points. There are 4 intersection points on each line, and there are 3 lines through each intersection point.*

Proof. We already know that there are 15 lines, 20 intersection points, and 3 lines through each intersection point. We are left to show that on each line lie 4 intersection points. First, we notice that on each pair of lines V_ϵ, there are 7 intersection points. Indeed, the choice of $\epsilon \in \mathrm{Jac}(X)_2 - \{0\}$ leaves us with 14 other non-zero 2-torsion points, which obviously go in pairs:

$$V_\epsilon \cap V_\eta = V_\epsilon \cap V_{\epsilon+\eta} = \{L_\epsilon \oplus L_\eta \oplus L_{\epsilon+\eta}\}.$$

Which proves that there are 7 intersection points on the pair of lines V_ϵ: 3 on the line in $\mathbb{P}^4_+$, therefore 4 on the line in $\mathbb{P}^3_-$. □

Similarly the 15 lines of S_{NR} are part of its singular locus $\mathrm{Sing}(S_{NR})$. So when we intersect S_{NR} with a general hyperplane of $\mathbb{P}^3_-$, we obtain a sextic plane curve C with at least 15 singular points. Since the arithmetic genus of a sextic is 10, we see that the C cannot be irreducible.

Proposition 6.2. *The plane sextic curve C is a hexagon, i.e. it is the union of 6 lines.*

Proof. Say $C = C_1 \cup \ldots \cup C_k$, where C_i is an irreducible plane curve of degree d_i. An irreducible curve of genus g has at most g singular points (this is not a sharp bound but it will be enough for us.) So the maximal number of nodes $N_{\max}$ of C is the maximal number of nodes of the irreducible components plus the number of intersection points of the components. Let us compute the number of nodes of each partition of sum 6.

- $(d_1, d_2) = (5, 1) : N_{\max} \leq \binom{4}{2} + 5 = 11.$
- $(d_1, d_2) = (4, 2)$: $N_{\max} \leq 3 + 4 \times 2 = 11.$
- $(d_1, d_2) = (3, 3)$: $N_{\max} \leq 1 + 1 + 3 \times 3 = 11.$
- $(d_1, d_2, d_3) = (4, 1, 1)$: $N_{\max} \leq 3 + 2 \times 4 + 1 = 12.$
- $(d_1, d_2, d_3) = (3, 2, 1)$: $N_{\max} \leq 1 + 6 + 3 + 2 = 12.$
- $(d_1, d_2, d_3) = (2, 2, 2)$: $N_{\max} \leq 3 \times 4 = 12.$
- $(d_1, d_2, d_3, d_4) = (3, 1, 1, 1)$: $N_{\max} \leq 1 + 3 \times 3 + \binom{3}{2} = 13.$
- $(d_1, d_2, d_3, d_4) = (2, 2, 1, 1)$: $N_{\max} \leq 2 \times 2 + 4 \times 2 + 1 = 13.$
- $(d_1, d_2, d_3, d_4, d_5) = (2, 1, 1, 1, 1)$: $N_{\max} \leq 2 \times 4 + \binom{4}{2} = 14.$
- $(d_1, d_2, d_3, d_4, d_5, d_6) = (1, 1, 1, 1, 1, 1)$: $N_{\max} \leq \binom{5}{2} = 15.$

So we see that the only possible curve is a union of 6 lines, which indeed has 15 nodes. □

Theorem C. *The surface $S_{NR} = \mathcal{C}_6 \cap \mathbb{P}^3_-$ is a hexahedron, i.e. the union of 6 planes in $\mathbb{P}^3_-$.*

Proof. To produce the curve $C = S_{NR} \cap \mathbb{P}^2$ of Proposition 6.2, we chose a general plane. Therefore the sextic surface S_{NR} is completely reducible and is a hexahedron. And it is easy to check that the hexahedron realizes the $(15_4, 20_3)$-configuration. $\square$

Remark 6.3. It is worth noting that $\mathbb{P}^4_+$ and $\mathbb{P}^3_-$ are not in general position at all. A good way to see it is by looking at the expected dimension of the intersection with the singular locus Σ. Recall that $\mathrm{codim}_{\mathbb{P}^8}(\Sigma) = 3$. But $\Sigma \cap \mathbb{P}^4_+$ is of codimension 1 in $\mathbb{P}^4$. And $\Sigma \cap \mathbb{P}^3_-$ is of codimension 2 in $\mathbb{P}^3_-$.

Since the intersection with $\mathbb{P}^4_+$ allowed us to naturally recover the original genus-2 curve X via the Kummer surface or the Jacobian variety, it is logical to ask whether the hexahedron, i.e. the intersection with $\mathbb{P}^3_-$, determines X as well. The answer, which we state here and which is proved in [10], is positive.

Theorem 6.4. *The 6 planes of Theorem C correspond to the 6 Weierstraß points of the given curve X. That is, the six planes correspond to 6 points in the dual projective space $\check{\mathbb{P}}^3_-$ of $\mathbb{P}^3_-$, then on the unique rational curve passing through the 6 points, these 6 points are projectively equivalent to the 6 Weierstrass points of our given curve X.*

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