

# CONTINUOUS-TIME PORTFOLIO CHOICE UNDER MONOTONE MEAN-VARIANCE PREFERENCES—STOCHASTIC FACTOR CASE

JAKUB TRYBULA AND DARIUSZ ZAWISZA

First arXiv version: 13 March 2014

This version: Accepted manuscript<sup>1</sup>

Final version: published in *Math. Oper. Res.* 44 (2019), 966 – 987

<https://doi.org/10.1287/moor.2018.0952>

**ABSTRACT.** We consider an incomplete market with a non-tradable stochastic factor and a continuous time investment problem with an optimality criterion based on monotone mean-variance preferences. We formulate it as a stochastic differential game problem and use Hamilton-Jacobi-Bellman-Isaacs equations to find an optimal investment strategy and the value function. What is more, we show that our solution is also optimal for the classical Markowitz problem and every optimal solution for the classical Markowitz problem is optimal also for the monotone mean-variance preferences. These results are interesting because the original Markowitz functional is not monotone, and it was observed that in the case of a static one-period optimization problem the solutions for those two functionals are different. In addition, we determine explicit Markowitz strategies in the square root factor models.

## 1. INTRODUCTION.

Since Markowitz published his famous paper [26], the mean-variance criterion has become a very popular topic in the investment literature. First, the problem has been solved for a myopic investor in the static optimization framework. Then, it has been extended and has been solved in the multi-period framework, when intertemporal trading is allowed. The exact solution can be found in Li and Ng [22] (discrete time setting) and Zhou and Li [38] (continuous time framework).

On the other hand, it is commonly accepted that a proper decision functional should reflect the fact that the main motivation of a rational investor is to earn money and thus, when choosing between two prospects (investment returns)  $X$  and  $Y$ , such that  $X \leq Y$ , the investor will always choose  $Y$ . This type of behavior is often formulated in the form of a monotonicity condition. Namely, we say that the functional  $\rho$  satisfies a monotonicity condition if the relation  $X \leq Y$  implies  $\rho(X) \leq \rho(Y)$ . Note that in case of portfolio optimization problem, the use of a not monotone functional may lead to

---

*Date:* January 15, 2020.

*2010 Mathematics Subject Classification.* 91G10; 91A15; 91A23; 93E20.

*Key words and phrases.* continuous optimization; stochastic control; stochastic factor model, Heston model.

<sup>1</sup>History:Received–December 22, 2015; Accepted–May 20, 2018; Published Online–May 29, 2019

irrational decisions because there might be a better (greater) prospect which is excluded by the functional. Therefore, such axiom is usually incorporated in the majority of modern theories of a rational investor behavior (e.g. the expected utility theory - von Neumann and Morgenstern [32], the dual theory of choice - Yaari [33], the max-min theory - Gilboa and Schmeidler [15], dynamic variational preferences - Maccherroni et al. [25], coherent and convex risk measures - Artzner et al. [1], Föllmer and Schied [13]). Nevertheless, it is well known that the mean-variance functional is not monotone. For this reason Maccheroni et al. [25] created a new class of preferences that *coincide with mean-variance preferences on their domain of monotonicity but differ where the mean-variance preferences fail to be monotone*:

$$(1.1) \quad V_\theta(X) := \inf_{Q \in \mathcal{Q}} \left\{ \mathbb{E}^Q[X] + \frac{1}{2\theta} C(Q|P) \right\}, \quad X \in \mathcal{L}^2(P),$$

where  $\theta > 0$  is a risk aversion coefficient,  $P$  is a given probability measure,  $\mathcal{L}^2(P)$  is the set of square-integrable random variables with respect to the measure  $P$ ,  $\mathcal{Q}$  is the class of all probability measures, such that

$$C(Q|P) := \begin{cases} \mathbb{E}^P \left[ \left( \frac{dQ}{dP} \right)^2 \right] - 1, & \text{if } Q \ll P, \\ +\infty, & \text{otherwise.} \end{cases}$$

Moreover, they have shown that *the functional associated with this class of preferences is the best approximation of the mean-variance functional among those which are monotonic*. For more details about monotone mean-variance preferences and their other advantages over mean-variance preferences, we refer to Maccheroni et al. [25].

It is also worth mentioning that ((1.1)) can be rewritten as

$$(1.2) \quad V_\theta(X) = -\Lambda_\theta(X) - \frac{1}{2\theta},$$

where

$$\Lambda_\theta(X) := \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q \left[ -X - \frac{1}{2\theta} \frac{dQ}{dP} \right], \quad X \in \mathcal{L}^2(P),$$

so additional motivation to investigate further properties of such type performance criterion lies in the fact that the above functional satisfies the following axioms:

*Convexity:* If  $\alpha \in (0, 1)$ , then  $\Lambda_\theta(\alpha X + (1 - \alpha)Y) \leq \alpha \Lambda_\theta(X) + (1 - \alpha) \Lambda_\theta(Y)$ .

*Monotonicity:* If  $X \leq Y$ , then  $\Lambda_\theta(Y) \leq \Lambda_\theta(X)$ .

*Translation invariance:* If  $\beta \in \mathbb{R}$ , then  $\Lambda_\theta(X + \beta) = \Lambda_\theta(X) - \beta$ .

Namely,  $\Lambda_\theta(X)$  is a convex risk measure (see Föllmer and Schied [13] or Frittelli and Rosazza-Gianin [14]). Note that this type of measures is commonly accepted as a tool for assigning the risk to financial positions, so the minimization of  $\Lambda_\theta(X)$  can be interpreted as searching for the risk minimizing portfolio. In this case the function  $C(Q|P)$  is called a penalty function.

The portfolio optimization problem based on the functional  $V_\theta(X)$  has been solved by Maccheroni et al. [25] in the static setting. Since this functional has never been investigated

in intertemporal setting, it is important to describe an optimal financial strategy which the investor can follow in order to maximize  $V_\theta(X)$  and investigate whether the same differences between monotone and not monotone counterpart arise in the continuous time optimization framework.

In this paper we assume that an investor has access to the market, where he can freely buy and sell a risk-less bond and a risky asset whose price is a diffusion with dynamics affected by a correlated non-tradable (but observable) stochastic factor. Nowadays, stochastic factor models have become very popular in continuous time portfolio optimization theories. Such models can incorporate many empirical findings about stochastic market returns, for example stochastic nature of the volatility or factor dependence on the excess return. Many researchers have tried to investigate what is the impact of the factor on the risky asset for various optimization criteria, usually under the hyperbolic/constant absolute risk aversion utility framework. The topic was explored, among others, by Kim and Omberg [19], Campbell and Viceira [5], Fleming and Hernández [11], Liu [23], Taksar and Zeng [37] and Zariphopoulou [34]. Besides, stochastic factor models are fundamental examples of incomplete market models and it is important to check various properties of the Markowitz portfolio in the case of these models because they may lead to various paradoxes. For example, as documented in Bäuerle and Grether [3] and Cui et al. [6], they might be responsible for generating so called free cash flow streams.

The problem of looking for a risk minimizing portfolio, with various modifications of performance criterion ((1.1)), was considered by many authors. For example Mataramvura and Øksendal [27] studied this issue in a jump diffusion setting with a general penalty function of the form

$$C(Q|P) = h\left(\frac{dQ}{dP}\right).$$

The same kind of problem was examined by Elliott and Siu [7] in the context of an optimal reinsurance problem and by Elliott and Siu [9, 8] in a regime switching market.

Risk based portfolio problems are also useful for derivative pricing in incomplete markets. Namely, one possibility is to determine the value by considering the so called risk indifference price. For more information about an indifference price in a jump diffusion market see Øksendal and Sulem [29], while for the stochastic factor model it is worth to read Elliott and Siu [10]. It merits mentioning that in many papers the utility function is used to take into account the non-linear form of the investor's satisfaction in functional ((1.1)). This approach is called the robust utility portfolio optimization and was taken up in Hernández and Schied [17] (stochastic factor model), Øksendal and Sulem [28] (jump-diffusion risk), Bordigioni et al. [4] (more general semimartingale setting).

All of the aforementioned papers examine the problem without presenting any detailed solution for a specific choice of  $C(Q|P)$  or consider one specific example using the entropic penalty function of the form

$$C(Q|P) = \frac{dQ}{dP} \ln\left(\frac{dQ}{dP}\right).$$

Due to the best of our knowledge the quadratic penalty

$$C(Q|P) = \mathbb{E}^P \left[ \left( \frac{dQ}{dP} \right)^2 \right] - 1$$

has never been studied in detail in the dynamic optimization framework.

The problem of maximizing ((1.1)) is a max-min problem, hence it naturally forms a stochastic differential game. In the literature there are two main approaches for determining the solution to such games. First of them exploits maximum principle and Backward Stochastic Differential Equations (BSDE), while the second one is based on the dynamic programming principle and Hamilton-Jacobi-Bellman equations (Hamilton-Jacobi-Bellman-Isaacs for differential games).

The latter is more suitable for our approach. In our case the associated HJBI equation can be simplified to a linear form by applying certain transformations. To complete the reasoning it is sufficient to use a suitable version of a verification theorem. By this method we obtain a formula for the optimal strategy, which is characterized by the aforementioned linear equation. Further, the solution is proved to be optimal in the classical Markowitz framework, which is in opposition to the one-period setting. It seems that the continuous time trading (even in an incomplete market setting) ensures enough flexibility to react faster against changing conditions and allows the mean-variance investor to "behave like a monotone investor".

It should be noticed that our paper is not the first one where such a qualitative difference between discrete and continuous time models is presented. Cui et al. [6] have shown that a complete financial market is time-consistent in efficiency in the continuous time framework. It is also worth mentioning that the time inconsistency (see for instance Basak and Chabakauri [2]) and the lack of monotonicity are two major drawbacks of the Markowitz optimization, commonly discussed in the literature. Moreover, Kallsen et al. [18] reported that the expression for the hedging error in the variance optimal hedging problem is much more involved in the discrete than in the continuous time setting.

The paper is organized as follows. In Section 2 we describe the set up of the problem. We formulate the verification theorem and derive the HJBI equation. In Section 3 we transform our equation to a linear form and prove some useful properties of its solution. In Section 4 we solve an auxiliary portfolio optimization problem and then in Section 5 we compare the result with the solution to the classical mean-variance optimization problem. Finally, in Section 6 we formulate our main theorem. In the last section we use methods from previous chapters to determine explicit formulas for the mean-variance optimal portfolio in specific stochastic volatility models and we compare it further with the Black-Scholes model.

It is worth to note here that in this work in many places we write about PDEs and talk about solutions. In this article by a solution to PDE we always mean a classical smooth solution.

## 2. GENERAL MODEL DESCRIPTION.

Let  $(\Omega, \mathcal{F}, P)$  be a probability space with a filtration  $(\mathcal{F}_t, 0 \leq t \leq T)$  possibly enlarged to satisfy the usual assumptions and generated by two independent Brownian motions  $(W_t^1, 0 \leq t \leq T)$ ,  $(W_t^2, 0 \leq t \leq T)$  defined on  $(\Omega, \mathcal{F}, P)$ . Suppose that an investor has access to the market, where he can freely buy and sell two securities: a bond  $(B_t, s \leq t \leq T)$  and a share  $(S_t, s \leq t \leq T)$ . We assume that the dynamics of  $(S_t, s \leq t \leq T)$  depend on one non-tradable (but observable) external factor  $(Z_t, s \leq t \leq T)$ . This factor can be used to model stochastic volatility or other varying economic conditions. Processes mentioned above are described by the system

$$(2.1) \quad \begin{cases} dB_t = rB_t dt, \\ dS_t = \mu(Z_t)S_t dt + \sigma(Z_t)S_t dW_t^1, \\ dZ_t = a(Z_t)dt + b(Z_t)(\rho dW_t^1 + \bar{\rho} dW_t^2), \quad Z_s = z, \end{cases}$$

where the coefficients  $\sigma > 0$ ,  $\mu$ ,  $a$ ,  $b$  have all the required regularity conditions, in order to ensure that a unique strong solution to ((2.1)) exists. The interest rate  $r > 0$  is constant,  $\rho \in [-1, 1]$  is a correlation coefficient and  $\bar{\rho} := \sqrt{1 - \rho^2}$ . The coefficients are time-independent only for notational convenience.

**Performance functional.** In this paper we use the monotone mean-variance preferences of the form ((1.1)). Nevertheless, due to technical difficulties, our problem is solved first for the auxiliary functional given by

$$(2.2) \quad \bar{\Lambda}_\theta(X) := \sup_{Q \in \bar{\mathcal{Q}}} \mathbb{E}^Q \left[ -X - \frac{1}{2\theta} \frac{dQ}{dP} \right], \quad X \in \mathcal{L}^2(P),$$

where:

- $\bar{\mathcal{Q}}$  is of the form<sup>2</sup>

$$(2.3) \quad \bar{\mathcal{Q}} := \left\{ Q \sim P : \frac{dQ}{dP} = \mathcal{E} \left( \int \eta_{t,1} dW_t^1 + \eta_{t,2} dW_t^2 \right)_T, \quad (\eta_1, \eta_2) \in \mathcal{M} \right\};$$

- $\mathcal{E}(\cdot)_t$  denotes the Doleans-Dade exponential;
- $\mathcal{M}$  is the set of all progressively measurable processes  $\eta = (\eta_1, \eta_2)$  taking values in  $\mathbb{R}^2$ , such that

$$\mathbb{E}^P \left[ \left( \frac{dQ^\eta}{dP} \right)^2 \right] < +\infty \quad \text{and} \quad \mathbb{E}^P \left[ \frac{dQ^\eta}{dP} \right] = 1;$$

- $Q^\eta$  denotes the probability measure determined by  $\eta \in \mathcal{M}$ .

Note that the above set of assumptions allows us to change the probability measure by using the Girsanov Theorem and guarantees that our modification of the Maccheroni type objective function is well defined.

<sup>2</sup>For more information see Hernández and Schied [17].

In order to simplify formula ((2.2)), we define the additional family of stochastic processes  $(Y_t^\eta, s \leq t \leq T)$  which are given by the stochastic differential equations

$$dY_t^\eta = \eta_{t,1} Y_t^\eta dW_t^1 + \eta_{t,2} Y_t^\eta dW_t^2, \quad Y_s^\eta = y > 0, \quad \eta \in \mathcal{M}.$$

Then, it is easy to see that

$$Y_T^\eta = y \frac{dQ^\eta}{dP}, \quad \eta \in \mathcal{M}$$

and

$$\bar{\Lambda}_\theta(X) = \sup_{\eta \in \mathcal{M}} \mathbb{E}^\eta [-X - Y_T^\eta], \quad X \in \mathcal{L}^2(P),$$

where  $\mathbb{E}^\eta$  denotes the expectation with respect to the measure  $Q^\eta$  and

$$y = \frac{1}{2\theta}.$$

**Formulation of the problem.** We assume that at any time the investor can decide on the absolute value invested in the risky asset, and the value invested in the bank account. Let  $(\bar{X}_t^\pi, s \leq t \leq T)$  be the investor's wealth process with the following dynamics

$$d\bar{X}_t^\pi = (r\bar{X}_t^\pi + \bar{\pi}_t(\mu(Z_t) - r))dt + \bar{\pi}_t\sigma(Z_t)dW_t^1, \quad \bar{X}_s^\pi = \bar{x} > 0,$$

where  $\bar{x}$  denotes a current wealth of the investor, whereas the control  $\bar{\pi}_t$  can be interpreted as the absolute value invested in  $S_t$ . Note that  $\bar{\pi}_t$  as well as the portfolio wealth  $\bar{X}_T^\pi$  are allowed to be negative. In this work it is convenient to use forward values of  $\bar{\pi}_t$  and  $\bar{X}_t^\pi$ . Namely, let

$$\pi_t := e^{r(T-t)}\bar{\pi}_t \quad \text{and} \quad X_t^\pi := e^{r(T-t)}\bar{X}_t^\pi.$$

We have

$$(2.4) \quad dX_t^\pi = \pi_t(\mu(Z_t) - r)dt + \pi_t\sigma(Z_t)dW_t^1.$$

**Definition 2.1.** A control (or strategy)  $\pi = (\pi_s, t \leq s \leq T)$  is admissible on the time interval  $[t, T]$ , written  $\pi \in \mathcal{A}_{x,y,z,t}$ , if it satisfies the following assumptions:

- (i)  $\pi$  is progressively measurable;
- (ii) a unique solution to ((2.4)) exists and

$$\mathbb{E}_{x,y,z,t}^\eta \left[ \sup_{t \leq s \leq T} |X_s^\pi|^2 \right] < +\infty \quad \text{for all } \eta \in \mathcal{M}.$$

The investor's objective is to

$$(2.5) \quad \text{minimize} \quad \sup_{\eta \in \mathcal{M}} J^{\pi,\eta}(x, y, z, t)$$

over the class of admissible strategies  $\mathcal{A}_{x,y,z,t}$ , where

$$J^{\pi,\eta}(x, y, z, t) := \mathbb{E}_{x,y,z,t}^\eta [-X_T^\pi - Y_T^\eta].$$

The problem ((2.5)) is assumed to be a zero-sum stochastic differential game problem between the market and the investor with controls given by  $Q^\eta$  and  $\pi$  respectively. We

are looking for the value function  $V(x, y, z, t)$  and a saddle point i.e. a pair  $(\pi^*, \eta^*) \in \mathcal{A}_{x,y,z,t} \times \mathcal{M}$ , such that

$$J^{\pi^*, \eta}(x, y, z, t) \leq J^{\pi^*, \eta^*}(x, y, z, t) \leq J^{\pi, \eta^*}(x, y, z, t), \quad \forall \pi \in \mathcal{A}_{x,y,z,t}, \quad \forall \eta \in \mathcal{M}$$

and

$$V(x, y, z, t) := J^{\pi^*, \eta^*}(x, y, z, t).$$

**Remark 2.2.** If  $(\pi^*, \eta^*) \in \mathcal{A}_{x,y,z,t} \times \mathcal{M}$  is a saddle point of problem ((2.5)), then

$$\begin{aligned} \inf_{\pi \in \mathcal{A}_{x,y,z,t}} \sup_{\eta \in \mathcal{M}} J^{\pi, \eta}(x, y, z, t) &\leq \sup_{\eta \in \mathcal{M}} J^{\pi^*, \eta}(x, y, z, t) \leq J^{\pi^*, \eta^*}(x, y, z, t) \\ &\leq \inf_{\pi \in \mathcal{A}_{x,y,z,t}} J^{\pi, \eta^*}(x, y, z, t) \leq \sup_{\eta \in \mathcal{M}} \inf_{\pi \in \mathcal{A}_{x,y,z,t}} J^{\pi, \eta}(x, y, z, t). \end{aligned}$$

In addition, we always have

$$\sup_{\eta \in \mathcal{M}} \inf_{\pi \in \mathcal{A}_{x,y,z,t}} J^{\pi, \eta}(x, y, z, t) \leq \inf_{\pi \in \mathcal{A}_{x,y,z,t}} \sup_{\eta \in \mathcal{M}} J^{\pi, \eta}(x, y, z, t).$$

Summarizing

$$J^{\pi^*, \eta^*}(x, y, z, t) = \inf_{\pi \in \mathcal{A}_{x,y,z,t}} J^{\pi, \eta^*}(x, y, z, t),$$

so the control  $\pi^*$  is an optimal financial strategy.

For more information about differential games we refer to Fleming and Soner [12] and references therein.

**The verification theorem.** The investment problem stated in the previous section can be solved by applying the stochastic control theory. In this section we establish a link between the Hamilton-Jacobi-Bellman-Isaacs equation and the saddle point of our initial problem.

Let us recall, that

$$(2.6) \quad \begin{cases} dX_s^\pi = \pi_s(\mu(Z_s) - r)ds + \pi_s\sigma(Z_s)dW_s^1, \\ dY_s^\eta = \eta_{s,1}Y_s^\eta dW_s^1 + \eta_{s,2}Y_s^\eta dW_s^2, \\ dZ_s = a(Z_s)ds + b(Z_s)(\rho dW_s^1 + \bar{\rho}dW_s^2). \end{cases}$$

It is convenient to consider  $Q^\eta$ -dynamics of system ((2.6)). After applying the Girsanov transformation, we have

$$(2.7) \quad \begin{cases} dX_s^\pi = \pi_s(\mu(Z_s) - r + \sigma(Z_s)\eta_{s,1})ds + \pi_s\sigma(Z_s)dW_s^{\eta_1}, \\ dY_s^\eta = (\eta_{s,1}^2 + \eta_{s,2}^2)Y_s^\eta ds + \eta_{s,1}Y_s^\eta dW_s^{\eta_1} + \eta_{s,2}Y_s^\eta dW_s^{\eta_2}, \\ dZ_s = (a(Z_s) + b(Z_s)\rho\eta_{s,1} + b(Z_s)\bar{\rho}\eta_{s,2})dt + b(Z_s)(\rho dW_s^{\eta_1} + \bar{\rho}dW_s^{\eta_2}), \end{cases}$$

where  $(W_s^{\eta_1}, 0 \leq s \leq T)$  and  $(W_s^{\eta_2}, 0 \leq s \leq T)$  are  $Q^\eta$ -Brownian motions defined as

$$\begin{cases} dW_s^{\eta_1} = dW_s^1 - \eta_{s,1}ds, \\ dW_s^{\eta_2} = dW_s^2 - \eta_{s,2}ds. \end{cases}$$

Let  $\mathcal{L}^{\pi,\eta}$  be a differential operator given by

$$\begin{aligned} \mathcal{L}^{\pi,\eta}V(x, y, z, t) := & V_t + \pi(\mu(z) - r + \sigma(z)\eta_1)V_x + (\eta_1^2 + \eta_2^2)yV_y \\ & + (a(z) + b(z)\rho\eta_1 + b(z)\bar{\rho}\eta_2)V_z + \frac{1}{2}\pi^2\sigma^2(z)V_{xx} \\ & + \frac{1}{2}(\eta_1^2 + \eta_2^2)y^2V_{yy} + \frac{1}{2}b^2(z)V_{zz} + \pi\sigma(z)\eta_1yV_{xy} \\ & + \pi\sigma(z)b(z)\rho V_{xz} + b(z)(\rho\eta_1 + \bar{\rho}\eta_2)yV_{yz}. \end{aligned}$$

We can now formulate the Verification Theorem. The proof of this theorem is very similar to the proof of the analogous theorem from Mataramvura and Øksendal [27] (Theorem 3.2) or Zawisza [35, 36] (Theorem 3.1 and Theorem 6.1, respectively), so in this paper we omit it.

**Theorem 2.3** (Verification Theorem). *Suppose there exists a function*

$$V \in \mathcal{C}^{2,2,2,1}(\mathbb{R} \times (0, +\infty) \times \mathbb{R} \times [0, T]) \cap \mathcal{C}(\mathbb{R} \times [0, +\infty) \times \mathbb{R} \times [0, T])$$

and a Markov control

$$(\pi^*, \eta^*) \in \mathcal{A}_{x,y,z,t} \times \mathcal{M},$$

such that

$$(2.8) \quad \mathcal{L}^{\pi^*(x,y,z,t), \eta^*}V(x, y, z, t) \leq 0,$$

$$(2.9) \quad \mathcal{L}^{\pi, \eta^*(x,y,z,t)}V(x, y, z, t) \geq 0,$$

$$(2.10) \quad \mathcal{L}^{\pi^*(x,y,z,t), \eta^*(x,y,z,t)}V(x, y, z, t) = 0,$$

$$(2.11) \quad V(x, y, z, T) = -x - y$$

for all  $\pi \in \mathbb{R}$ ,  $\eta \in \mathbb{R}^2$  and  $(x, y, z, t) \in \mathbb{R} \times (0, +\infty) \times \mathbb{R} \times [0, T)$ ,

and

$$(2.12) \quad \mathbb{E}_{x,y,z,t}^{\eta} \left[ \sup_{t \leq s \leq T} |V(X_s^\pi, Y_s^\eta, Z_s, s)| \right] < +\infty$$

for all  $\pi \in \mathcal{A}_{x,y,z,t}$ ,  $\eta \in \mathcal{M}$  and  $(x, y, z, t) \in \mathbb{R} \times [0, +\infty) \times \mathbb{R} \times [0, T]$ .

Then

$$J^{\pi^*, \eta^*}(x, y, z, t) \leq V(x, y, z, t) \leq J^{\pi, \eta^*}(x, y, z, t), \quad \forall \pi \in \mathcal{A}_{x,y,z,t}, \quad \forall \eta \in \mathcal{M}$$

and

$$V(x, y, z, t) = J^{\pi^*, \eta^*}(x, y, z, t).$$



**Solution to the minimax problem.** To find a saddle point we start with analyzing the Hamilton-Jacobi-Bellman-Isaacs equation

$$(2.13) \quad \min_{\pi \in \mathbb{R}} \max_{\eta \in \mathbb{R}^2} \mathcal{L}^{\pi, \eta} V(x, y, z, t) = 0,$$

i.e.

$$\begin{aligned} & V_t + a(z)V_z + \frac{1}{2}b^2(z)V_{zz} \\ & + \min_{\pi \in \mathbb{R}} \max_{(\eta_1, \eta_2) \in \mathbb{R}^2} \left\{ \pi(\mu(z) - r + \sigma(z)\eta_1)V_x + (\eta_1^2 + \eta_2^2)yV_y \right. \\ & + b(z)(\rho\eta_1 + \bar{\rho}\eta_2)V_z + \frac{1}{2}\pi^2\sigma^2(z)V_{xx} + \frac{1}{2}(\eta_1^2 + \eta_2^2)y^2V_{yy} \\ & \left. + \pi\sigma(z)\eta_1yV_{xy} + \pi\sigma(z)b(z)\rho V_{xz} + b(z)(\rho\eta_1 + \bar{\rho}\eta_2)yV_{yz} \right\} = 0. \end{aligned}$$

We expect  $V(x, y, z, t)$  to be of the form

$$(2.14) \quad V(x, y, z, t) = -x + G(z, t)y, \quad \text{where} \quad G(z, T) = -1.$$

Then we have

$$\begin{aligned} & yG_t + a(z)yG_z + \frac{1}{2}b^2(z)yG_{zz} + \min_{\pi \in \mathbb{R}} \max_{(\eta_1, \eta_2) \in \mathbb{R}^2} \left\{ -\pi(\mu(z) - r + \sigma(z)\eta_1) \right. \\ & \left. + (\eta_1^2 + \eta_2^2)yG + 2b(z)(\rho\eta_1 + \bar{\rho}\eta_2)yG_z \right\} = 0. \end{aligned}$$

The maximum over  $(\eta_1, \eta_2)$  is attained at  $(\eta_1^*(\pi), \eta_2^*)$ , where

$$\begin{aligned} \eta_1^*(\pi) &= \frac{\sigma(z)}{2yG(z, t)}\pi - \rho b(z)\frac{G_z(z, t)}{G(z, t)}, \\ \eta_2^* &= -\bar{\rho}b(z)\frac{G_z(z, t)}{G(z, t)}. \end{aligned}$$

For  $(\eta_1^*(\pi), \eta_2^*)$  our equation is of the form

$$(2.15) \quad \begin{aligned} & yG_t + a(z)yG_z + \frac{1}{2}b^2(z)yG_{zz} + \min_{\pi \in \mathbb{R}} \left\{ -\pi(\mu(z) - r + \sigma(z)\eta_1^*(\pi)) \right. \\ & \left. + ((\eta_1^*(\pi))^2 + (\eta_2^*)^2)yG + 2b(z)(\rho\eta_1^*(\pi) + \bar{\rho}\eta_2^*)yG_z \right\} = 0. \end{aligned}$$

The minimum over  $\pi$  is attained at

$$(2.16) \quad \pi^* = -2yG(z, t) \left[ \frac{\mu(z) - r}{\sigma^2(z)} - \frac{\rho b(z)}{\sigma(z)} \frac{G_z(z, t)}{G(z, t)} \right].$$

It is worth to notice here that

$$(2.17) \quad \eta_1^*(\pi^*) = -\frac{\mu(z) - r}{\sigma(z)},$$

so the saddle point candidate

$$(2.18) \quad (\pi^*, (\eta_1^*(\pi^*), \eta_2^*))$$

looks as follows

$$\begin{aligned} \pi^* &= -2yG(z, t) \left[ \frac{\lambda(z)}{\sigma(z)} - \frac{\rho b(z)}{\sigma(z)} \frac{G_z(z, t)}{G(z, t)} \right], \\ \eta_1^*(\pi^*) &= -\lambda(z), \\ \eta_2^* &= -\bar{\rho}b(z) \frac{G_z(z, t)}{G(z, t)}, \end{aligned}$$

where

$$\lambda(z) = \frac{\mu(z) - r}{\sigma(z)}.$$

Now we substitute ((2.16)) into ((2.15)) and, after dividing by  $y$ , we get the final equation of the form

$$(2.19) \quad G_t + (a(z) - 2\rho b(z)\lambda(z)) G_z + \frac{1}{2}b^2(z)G_{zz} - \bar{\rho}^2b^2(z)\frac{G_z^2}{G} + \lambda^2(z)G = 0,$$

with the boundary condition  $G(z, T) = -1$ .

**Remark 2.4.** In Section 3 we rewrite equation ((2.19)) with the boundary condition  $G(z, T) = -1$  in a linear form and we provide a set of assumptions which ensure the existence of a classical solution (class  $\mathcal{C}^{2,1}$ ).

**Auxiliary results.** The following lemmas will be helpful in Section 4 to prove the main theorem and to establish the similarities between our paper and mean-variance optimization methods. In these lemmas we assume that initial conditions  $(x, y, r, t)$  are fixed.

**Lemma 2.5.** *Suppose that function  $V \in \mathcal{C}^{2,2,2,1}(\mathbb{R} \times (0, +\infty) \times \mathbb{R} \times [0, T])$  given by ((2.14)) is a classical solution to ((2.13)). Moreover, let  $(\pi^*, (\eta_1^*(\pi^*), \eta_2^*)) \in \mathcal{A}_{x,y,z,t} \times \mathcal{M}$  be determined by using ((2.18)). Then conditions ((2.8)) - ((2.11)) of Theorem 2.3 are satisfied.*

*Proof.* We already know that

$$\max_{\eta \in \mathbb{R}} \mathcal{L}^{\pi^*, \eta} V(x, y, z, t) = 0, \quad \mathcal{L}^{\pi^*, \eta^*} V(x, y, z, t) = 0$$

and

$$V(x, y, z, T) = -x - y,$$

which confirms ((2.8)), ((2.10)) and ((2.11)).

To prove ((2.9)) it is sufficient to use ((2.15)) and ((2.17)) and verify that

$$\min_{\pi \in \mathbb{R}} \mathcal{L}^{\pi, \eta^*(\pi^*)} V(x, y, z, t) = 0.$$

□

**Lemma 2.6.** *Suppose that function  $G$  is a classical solution to equation ((2.19)) and  $(\pi^*, (\eta_1^*(\pi^*), \eta_2^*)) \in \mathcal{A}_{x,y,z,t} \times \mathcal{M}$  is given by ((2.18)). Then*

$$2Y_s^{\eta^*} G(Z_s, s) = X_s^{\pi^*} - x + 2yG(z, t), \quad s \in [t, T].$$

*Proof.* It is sufficient to prove only that

$$dX_s^{\pi^*} = d(2Y_s^{\eta^*} G(Z_s, s)).$$

First of all, note that for the saddle point given by ((2.18)) system of equations ((2.6)) is of the form

$$(2.20) \quad \begin{aligned} dX_s^{\pi^*} = & -2Y_s^{\eta^*} G(Z_s, s) \left[ \lambda^2(Z_s) - \rho b(Z_s) \lambda(Z_s) \frac{G_z(Z_s, s)}{G(Z_s, s)} \right] ds \\ & - 2Y_s^{\eta^*} G(Z_s, s) \left[ \lambda(Z_s) - \rho b(Z_s) \frac{G_z(Z_s, s)}{G(Z_s, s)} \right] dW_s^1 \end{aligned}$$

and

$$dY_s^{\eta^*} = -\lambda(Z_s) Y_s^{\eta^*} dW_s^1 - \bar{\rho} b(Z_s) \frac{G_z(Z_s, s)}{G(Z_s, s)} Y_s^{\eta^*} dW_s^2.$$

Using ((2.19)), we can verify that

$$\begin{aligned} dG(Z_s, s) = & \left[ 2\rho b(Z_s) \lambda(Z_s) G_z(Z_s, s) + \bar{\rho}^2 b^2(Z_s) \frac{G_z^2(Z_s, s)}{G(Z_s, s)} - \lambda^2(Z_s) G(Z_s, s) \right] ds \\ & + G_z(Z_s, s) b(Z_s) (\rho dW_s^1 + \bar{\rho} dW_s^2). \end{aligned}$$

Moreover, we have

$$d(2Y_s^{\eta^*} G(Z_s, s)) = 2G(Z_s, s) dY_s^{\eta^*} + 2Y_s^{\eta^*} dG(Z_s, s) + 2dG(Z_s, s) dY_s^{\eta^*},$$

so, by substituting the appropriate dynamics into the above equation, we get the right hand side of ((2.20)).  $\square$

**Remark 2.7.** Note that the process  $(Y_s^{\eta^*}, t \leq s \leq T)$  is not directly observable in the financial market, but fortunately the above lemma ensures that for the fixed initial conditions  $(x, y, z, t)$ , rather than the Markov strategy

$$\pi_s^* = -2Y_s^{\eta^*} G(Z_s, s) \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right], \quad s \in [t, T],$$

we can use

$$\pi_s^* = -\left(X_s^{\pi^*} - x + 2yG(z, t)\right) \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right], \quad s \in [t, T].$$

### 3. CLASSICAL SMOOTH SOLUTION TO THE RESULTING EQUATION.

To solve equation ((2.19)) with the boundary condition  $G(z, T) = -1$  we consider following cases separately:

Case I:  $\rho^2 \neq \frac{1}{2}$

In this case the following ansatz is made (see Zariphopoulou [34])

$$G(z, t) = -F^\alpha(z, t), \quad \text{where} \quad F(z, T) = 1 \quad \text{and} \quad \alpha \in \mathbb{R} \setminus \{0\},$$

to obtain (by dividing by  $-\alpha F^{\alpha-1}$ )

$$\begin{aligned} F_t + (a(z) - 2\rho b(z)\lambda(z)) F_z + \frac{1}{2}b^2(z)F_{zz} + \frac{1}{\alpha}\lambda^2(z)F \\ + \left(\frac{1}{2}(\alpha - 1) - \alpha\bar{\rho}^2\right) b^2(z)\frac{F_z^2}{F} = 0. \end{aligned}$$

Note that for

$$\alpha = \frac{1}{2\rho^2 - 1},$$

we have

$$(3.1) \quad F_t + (a(z) - 2\rho b(z)\lambda(z)) F_z + \frac{1}{2}b^2(z)F_{zz} + (2\rho^2 - 1)\lambda^2(z)F = 0.$$

Case II:  $\rho^2 = \frac{1}{2}$

In this case if we substitute

$$G(z, t) = -e^{F(z, t)}, \quad \text{where} \quad F(z, T) = 0,$$

we get

$$(3.2) \quad F_t + \left(a(z) - \sqrt{2}b(z)\lambda(z)\right) F_z + \frac{1}{2}b^2(z)F_{zz} + \lambda^2(z) = 0.$$

Now we give a set of assumptions to ensure the existence of a classical smooth solution to equation ((2.19)) with the boundary condition  $G(z, T) = -1$  for any  $\rho \in [-1, 1]$ .

**Remark 3.1.** From Theorem 1 of Heath and Schweizer [16] it follows that if  $a, b, b \cdot \lambda, \lambda^2$  are Lipschitz continuous,  $\lambda$  is continuous and bounded and  $b^2 > \epsilon > 0$ , then there exist the unique classical solutions (class  $\mathcal{C}^{2,1}(\mathbb{R} \times [0, T]) \cap \mathcal{C}(\mathbb{R} \times [0, T])$ )  $F_1$  and  $F_2$  to equations ((3.1)) and ((3.2)) respectively, that satisfy the Feynman-Kac representations:

$$\begin{aligned} F_1(z, t) &= \mathbb{E}_{z,t}^{\tilde{P}} \left[ \exp \left\{ (2\rho^2 - 1) \int_t^T \lambda^2(\tilde{Z}_s) ds \right\} \right], \\ F_2(z, t) &= \mathbb{E}_{z,t}^{\tilde{P}} \left[ \int_t^T \lambda^2(\tilde{Z}_s) ds \right], \end{aligned}$$

where

$$d\tilde{Z}_s = \left[ a(\tilde{Z}_s) - 2\rho b(\tilde{Z}_s)\lambda(\tilde{Z}_s) \right] ds + b(\tilde{Z}_s)d\tilde{W}_s, \quad \tilde{Z}_t = z$$

and  $(\tilde{W}_s, t \leq s \leq T)$  is a Brownian motion with respect to  $\tilde{P}$ .

Note that, if  $\lambda$  is a bounded function, then  $F_1$  and  $F_2$  are bounded and  $G$  is bounded and bounded away from 0 for any  $\rho \in [-1, 1]$ .

**Lemma 3.2.** *Suppose  $a, b, b \cdot \lambda, \lambda^2$  are Lipschitz continuous,  $\lambda$  is continuous and bounded,  $b^2 > \epsilon > 0$  and  $F \in \mathcal{C}^{2,1}(\mathbb{R} \times [0, T]) \cap \mathcal{C}(\mathbb{R} \times [0, T])$  is a bounded solution to equation ((3.1)) or ((3.2)). Then the first  $z$ -derivative of  $F$  is bounded.*

*Proof.* To get a bound for  $F_z$  it is sufficient to estimate the Lipschitz constant. First of all, note that for  $z_1, z_2 \in (-\infty, c]$  there exists  $L_c > 0$  such that

$$(3.3) \quad |e^{z_1} - e^{z_2}| \leq L_c |z_1 - z_2|.$$

Secondly, for solution to ((3.1)), using ((3.3)) and the fact that  $\lambda^2$  is Lipschitz continuous and bounded, we obtain the existence of  $L > 0$  such that

$$\begin{aligned} |F(z, t) - F(\bar{z}, t)| &\leq L \mathbb{E}^{\tilde{P}} \left[ \int_t^T \left| \tilde{Z}_s(z, t) - \tilde{Z}_s(\bar{z}, t) \right| ds \right] \\ &\leq LT \mathbb{E}^{\tilde{P}} \left[ \sup_{t \leq s \leq T} \left| \tilde{Z}_s(z, t) - \tilde{Z}_s(\bar{z}, t) \right| \right], \end{aligned}$$

where from notational convenience we wrote  $\mathbb{E}^{\tilde{P}} [f(\tilde{Z}_s(z, t))]$  instead of  $\mathbb{E}_{z,t}^{\tilde{P}} [f(\tilde{Z}_s)]$ . Now, using Jensen's inequality and Theorem 1.3.16 from Pham [30], we have

$$\exists C_T > 0: \quad \mathbb{E}^{\tilde{P}} \left[ \sup_{t \leq s \leq T} \left| \tilde{Z}_s(z, t) - \tilde{Z}_s(\bar{z}, t) \right| \right] \leq C_T |z - \bar{z}|,$$

which completes the proof in the first case. Naturally, we can get a similar estimate for solution to ((3.2)).  $\square$

#### 4. SOLUTION TO AUXILIARY OPTIMIZATION PROBLEM.

In this section we solve portfolio optimization problem ((2.5)).

**Theorem 4.1.** *Suppose that  $a, b, b \cdot \lambda, \lambda^2$  are Lipschitz continuous,  $\lambda$  is continuous and bounded,  $b$  is bounded and  $b^2 > \epsilon > 0$ . Then for each initial conditions  $(x, y, z, t)$  there exists a Markov saddle point*

$$(\pi^*, (\eta_1^*(\pi^*), \eta_2^*)) \in \mathcal{A}_{x,y,z,t} \times \mathcal{M}$$

for problem ((2.5)), such that for all  $s \in [t, T]$

$$\begin{aligned} \pi_s^* &= - (X_s^{\pi^*} - x + 2yG(z, t)) \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right], \\ \eta_{s,1}^*(\pi^*) &= -\lambda(Z_s), \\ \eta_{s,2}^* &= -\bar{\rho} b(Z_s) \frac{G_z(Z_s, s)}{G(Z_s, s)}, \end{aligned}$$

where  $G \in \mathcal{C}^{2,1}(\mathbb{R} \times [0, T]) \cap \mathcal{C}(\mathbb{R} \times [0, T])$  and is a bounded solution to

$$(4.1) \quad G_t + (a(z) - 2\rho b(z)\lambda(z)) G_z + \frac{1}{2} b^2(z) G_{zz} - \bar{\rho}^2 b^2(z) \frac{G_z^2}{G} + \lambda^2(z) G = 0,$$

with the terminal condition  $G(z, T) = -1$ .

*Proof.* It follows from Remark 3.1 and Lemma 3.2 that there exists a classical bounded solution to ((4.1)), which has bounded derivative  $G_z$ . If we set

$$V(x, y, z, t) := -x + G(z, t)y,$$

then it is sufficient to check whether function  $V$  and  $(\pi^*, (\eta_1^*(\pi^*), \eta_2^*))$  satisfy all conditions of the Verification Theorem. Due to calculations ((2.13)) - ((2.19)) and Lemma 2.5, conditions ((2.8)) - ((2.11)) are fulfilled. Now, we only have to prove that  $(\pi^*, (\eta_1^*(\pi^*), \eta_2^*))$  belongs to the set  $\mathcal{A}_{x,y,z,t} \times \mathcal{M}$  and condition ((2.12)) holds:

1. From Lemma 3.2 and Remark 3.1 we know that the function  $G_z/G$  is bounded, so  $(\eta_1^*(\pi^*), \eta_2^*) \in \mathcal{M}$ .

2. Since  $G$  is bounded and  $Y_s^{\eta^*}$  is a solution to the stochastic linear equation with bounded coefficients, using Hölder's inequality, we have

$$\mathbb{E}_{x,y,z,t}^{\eta} \left[ \sup_{t \leq s \leq T} |G(Z_s, s)Y_s^{\eta^*}| \right] \leq \sqrt{\mathbb{E}^P \left[ \left( \frac{dQ^{\eta}}{dP} \right)^2 \right]} \cdot \sqrt{\mathbb{E}_{x,y,z,t}^P \left[ \sup_{t \leq s \leq T} |G(Z_s, s)Y_s^{\eta^*}|^2 \right]} < +\infty,$$

for all  $\eta \in \mathcal{M}$ .

3. To prove the same with  $X_s^{\pi^*}$  we use the fact that from Lemma 2.6 and Remark 2.7 for the fixed initial conditions  $(x, y, z, t)$  the strategy  $\pi_s^*$  might be rewritten as

$$\pi_s^* = - \left( X_s^{\pi^*} - x + 2yG(z, t) \right) \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right], \quad s \in [t, T].$$

Now, let us define

$$\zeta(Z_s, s) := - \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right], \quad s \in [t, T].$$

Note that  $\zeta \cdot (\mu - r)$  and  $\zeta \cdot \sigma$  are bounded functions because  $\lambda$  and  $b$  are bounded. Therefore, with the assistance of equation ((2.20)), we get that the process

$$K_s := X_s^{\pi^*} - x + 2yG(z, t), \quad s \in [t, T],$$

is a solution to the following equation

$$dK_s = \zeta(Z_s, s)(\mu(Z_s) - r)K_s ds + \zeta(Z_s, s)\sigma(Z_s)K_s dW_s^1.$$

This is a linear stochastic equation with bounded coefficients, which implies that

$$\begin{aligned} \mathbb{E}_{x,y,z,t}^{\eta} \left[ \sup_{t \leq s \leq T} |X_s^{\pi^*}| \right] &\leq \sqrt{\mathbb{E}_{x,y,z,t}^{\eta} \left[ \sup_{t \leq s \leq T} |X_s^{\pi^*}|^2 \right]} \\ &\leq \sqrt[4]{\mathbb{E}^P \left[ \left( \frac{dQ^{\eta}}{dP} \right)^2 \right]} \cdot \sqrt[4]{\mathbb{E}_{x,y,z,t}^P \left[ \sup_{t \leq s \leq T} |X_s^{\pi^*}|^4 \right]} < +\infty, \quad \forall \eta \in \mathcal{M}. \end{aligned}$$

It means ((2.12)) is satisfied and confirms the admissibility of  $\pi_s^*$ .  $\square$

## 5. MEAN-VARIANCE OPTIMIZATION PROBLEM.

Since motivation of our objective function comes from the mean-variance optimization, it is worth to compare our results with the solution to the latter. To solve the mean-variance optimization problem we consider the following functional

$$U_\theta(X) := \mathbb{E}[X] - \frac{\theta}{2} \mathbb{E}[(X - \mathbb{E}[X])^2], \quad X \in \mathcal{L}^2(P),$$

where  $\theta > 0$  is a risk aversion coefficient and  $\mathbb{E}$  denotes the expectation with respect to the measure  $P$ .

The investor's objective is to

$$(5.1) \quad \text{maximize} \quad \mathcal{I}^\pi(x, z, t)$$

over the class of admissible<sup>3</sup> strategies  $\mathcal{A}_{x,z,t}$ , where

$$\mathcal{I}^\pi(x, z, t) := \mathbb{E}_{x,z,t}[X_T^\pi] - \frac{\theta}{2} \mathbb{E}_{x,z,t}[(X_T^\pi - \mathbb{E}_{x,z,t}[X_T^\pi])^2].$$

We use again the stochastic control methods to obtain a solution. Namely, we can first use the standard Lagrange multipliers technique (see also Zhou and Li [38] for another method resulting in the same quadratic control problem). Note that

$$\begin{aligned} \sup_{\pi \in \mathcal{A}_{x,z,t}} \mathcal{I}^\pi(x, z, t) &= \sup_{\pi \in \mathcal{A}_{x,z,t}} \left\{ \mathbb{E}_{x,z,t}[X_T^\pi] - \frac{\theta}{2} \mathbb{E}_{x,z,t}[(X_T^\pi - \mathbb{E}_{x,z,t}[X_T^\pi])^2] \right\} \\ &= \sup_{A \in \mathbb{R}} \sup_{\pi \in \bar{\mathcal{A}}_{x,z,t}} \left\{ A - \frac{\theta}{2} \mathbb{E}_{x,z,t}[(X_T^\pi - A)^2] \right\} \\ (5.2) \quad &= \sup_{A \in \mathbb{R}} \left\{ A - \frac{\theta}{2} \inf_{\pi \in \bar{\mathcal{A}}_{x,z,t}} \mathbb{E}_{x,z,t}[(X_T^\pi - A)^2] \right\}, \end{aligned}$$

where

$$\bar{\mathcal{A}}_{x,z,t} = \{\pi \in \mathcal{A}_{x,z,t} : \mathbb{E}_{x,z,t}[X_T^\pi] = A\}, \quad A \in \mathbb{R}.$$

In that way we replace the unconstrained mean-variance optimization problem with the constrained maximization of the quadratic objective. Using Lagrange method it is sufficient to minimize the functional

$$\begin{aligned} I^{\pi(\gamma)}(x, z, t) &:= \mathbb{E}_{x,z,t} \left[ \left( X_T^{\pi(\gamma)} - A \right)^2 \right] - 2\gamma \mathbb{E}_{x,z,t} \left[ X_T^{\pi(\gamma)} \right] \\ (5.3) \quad &= \mathbb{E}_{x,z,t} \left[ \left( X_T^{\pi(\gamma)} - (A + \gamma) \right)^2 \right] - 2A\gamma - \gamma^2, \end{aligned}$$

over the class of admissible controls  $\pi \in \bar{\mathcal{A}}_{x,z,t}$ , determine the solution  $\pi^*(\gamma)$  and find  $\gamma^*$ , such that

$$\mathbb{E}_{x,z,t} \left[ X_T^{\pi^*(\gamma^*)} \right] = A.$$

---

<sup>3</sup>For more information see Definition 2.1.

We can use here results from Zawisza [36] where minimization of the robust quadratic functional

$$X_T^\pi \rightarrow \sup_{Q \in \mathcal{Q}} \mathbb{E}_{x,z,t}^Q [X_T^\pi - D]^2, \quad D \in \mathbb{R},$$

was considered (see also Laurent and Pham [21]). Namely, if we assume that  $\mathcal{Q} = \{P\}$ , from Theorem 4.1 in Zawisza [36], we have that the optimal strategy for functional ((5.3)) is given by

$$\pi_s^*(\gamma) = - (X_s^{\pi^*(\gamma)} - (A + \gamma)) \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} + \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{H_z(Z_s, s)}{H(Z_s, s)} \right], \quad s \in [t, T],$$

where  $H$  satisfies

$$H_t + (a(z) - 2\rho b(z)\lambda(z))H_z + \frac{1}{2}b^2(z)H_{zz} - \rho^2 b^2(z) \frac{H_z^2}{H} - \lambda^2(z)H = 0,$$

with the terminal condition  $H(z, T) = 1$ . It means that  $G = -\frac{1}{H}$  is a solution to

$$G_t + (a(z) - 2\rho b(z)\lambda(z))G_z + \frac{1}{2}b^2(z)G_{zz} - \bar{\rho}^2 b^2(z) \frac{G_z^2}{G} + \lambda^2(z)G = 0,$$

where  $G(z, T) = -1$ . In addition, we have

$$(5.4) \quad \pi_s^*(\gamma) = - (X_s^{\pi^*(\gamma)} - (A + \gamma)) \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right], \quad s \in [t, T],$$

which shows that the quadratic optimization is consistent with the monotone optimization with suitable chosen  $A$  and  $\gamma$  (e.g. Remark 2.7).

Now we find  $\gamma^*$ , such that

$$\mathbb{E}_{x,z,t} [X_T^{\pi^*(\gamma^*)}] = A.$$

Let us define

$$(5.5) \quad P_s := X_s^{\pi^*(\gamma^*)} - (A + \gamma^*), \quad s \in [t, T]$$

and recall that in the proof of Theorem 4.1 we set

$$\zeta(Z_s, s) = - \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right].$$

Since  $\zeta \cdot (\mu - r)$  and  $\zeta \cdot \sigma$  are bounded functions,  $P_s$  is a solution to the stochastic linear equation with bounded coefficients

$$dP_s = \zeta(Z_s, s)(\mu(Z_s) - r)P_s ds + \zeta(Z_s, s)\sigma(Z_s)P_s dW_s^1.$$

It means that

$$P_s = (x - (A + \gamma^*)) R_s,$$

where

$$R_s := \exp \left\{ \int_t^s \left[ \zeta(Z_u, u)(\mu(Z_u) - r) - \frac{1}{2}\zeta^2(Z_u, u)\sigma^2(Z_u) \right] du + \int_t^s \zeta(Z_u, u)\sigma(Z_u) dW_u^1 \right\}.$$



Using ((5.5)), we have

$$X_T^{\pi^*(\gamma^*)} = (x - A)R_T + A + \gamma^*(1 - R_T),$$

and then it is easy to see that

$$(5.6) \quad \gamma^*(A) = (A - x) \frac{\mathbb{E}_{x,z,t}[R_T]}{1 - \mathbb{E}_{x,z,t}[R_T]}, \quad \text{if } \mathbb{E}_{x,z,t}[R_T] \neq 1.$$

Finally, note that

$$A - \frac{\theta}{2} \mathbb{E}_{x,z,t} \left[ \left( X_T^{\pi^*(\gamma^*)} - A \right)^2 \right] = A - \frac{\theta}{2} \mathbb{E}_{x,z,t} \left[ \left( (x - A)R_T + \gamma^*(A)(1 - R_T) \right)^2 \right],$$

so the maximum over  $A$  is attained at

$$(5.7) \quad A^* = x + \frac{1}{\theta} \cdot \frac{1}{\mathbb{E}_{x,z,t}[\varphi_T^2]},$$

where

$$\varphi_T := \frac{R_T - \mathbb{E}_{x,z,t}[R_T]}{1 - \mathbb{E}_{x,z,t}[R_T]}.$$

Substituting  $\varphi_T$  into ((5.7)) and  $A^*$  into ((5.6)), we get

$$A^* = x + \frac{1}{\theta} \cdot \frac{(1 - \mathbb{E}_{x,z,t}[R_T])^2}{\text{Var}_{x,z,t}[R_T]} \quad \text{and} \quad \gamma^*(A^*) = \frac{1}{\theta} \cdot \frac{(1 - \mathbb{E}_{x,z,t}[R_T])\mathbb{E}_{x,z,t}[R_T]}{\text{Var}_{x,z,t}[R_T]},$$

where

$$\text{Var}_{x,z,t}[R_T] := \mathbb{E}_{x,z,t} \left[ (R_T - \mathbb{E}_{x,z,t}[R_T])^2 \right].$$

Taking into account ((5.4)), we conclude that the mean-variance optimal strategy is given by

$$\pi_s^* = - \left( X_s^{\pi^*} - x - \frac{1}{\theta} \cdot \frac{1 - \mathbb{E}_{x,z,t}[R_T]}{\text{Var}_{x,z,t}[R_T]} \right) \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right], \quad s \in [t, T],$$

whereas the monotone optimal strategy for initial conditions  $(x, y, z, t)$  looks as follows

$$\pi_s^* = - \left( X_s^{\pi^*} - x + 2yG(z, t) \right) \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right], \quad s \in [t, T].$$

In the next lemma we prove that both strategies are exactly the same if

$$\theta = \frac{1}{2y}.$$

**Lemma 5.1.** *Under conditions of Theorem 4.1 for each initial conditions  $(x, y, z, t)$  we have*

$$\frac{1 - \mathbb{E}_{x,z,t}[R_T]}{\text{Var}_{x,z,t}[R_T]} = -G(z, t).$$

*Proof.* We start with proving that

$$(5.8) \quad \mathbb{E}_{x,z,t} [R_T] = \mathbb{E}_{x,z,t} [R_T^2] = H(z, t).$$

First, let us recall that  $H$  is a classical solution to

$$(5.9) \quad H_t + (a(z) - 2\rho b(z)\lambda(z))H_z + \frac{1}{2}b^2(z)H_{zz} - \rho^2 b^2(z)\frac{H_z^2}{H} - \lambda^2(z)H = 0,$$

and

$$R_T = \exp \left\{ \int_t^T \left[ \zeta(Z_s, s)(\mu(Z_s) - r) - \frac{1}{2}\zeta^2(Z_s, s)\sigma^2(Z_s) \right] ds + \int_t^T \zeta(Z_s, s)\sigma(Z_s) dW_s^1 \right\},$$

where

$$\zeta(Z_s, s) = - \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right] = - \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} + \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{H_z(Z_s, s)}{H(Z_s, s)} \right]$$

and

$$dZ_s = a(Z_s)ds + b(Z_s)(\rho dW_s^1 + \bar{\rho} dW_s^2), \quad Z_t = z.$$

Equation ((5.9)) can be rewritten into

$$H_t + (a(z) + \rho b(z)\zeta(z, t)\sigma(z)) H_z + \frac{1}{2}b^2(z)H_{zz} + \zeta(z, t)(\mu(z) - r) H = 0,$$

so the solution has the Feynman-Kac representation (e.g. Theorem 1 of Heath and Schweizer [16]) of the following form

$$H(z, t) = \mathbb{E}_{x,z,t}^{\tilde{P}} \left[ \exp \left\{ \int_t^T \zeta(Z_s, s)(\mu(Z_s) - r) ds \right\} \right],$$

where

$$dZ_s = (a(Z_s) + \rho b(Z_s)\zeta(Z_s, s)\sigma(Z_s)) ds + b(Z_s)d\tilde{W}_s, \quad Z_t = z,$$

( $\tilde{W}_s, t \leq s \leq T$ ) is a Brownian motion with respect to  $\tilde{P}$ ,

$$\begin{aligned} d\tilde{W}_s &= \rho d\tilde{W}_s^1 + \bar{\rho} dW_s^2, \\ d\tilde{W}_s^1 &= dW_s^1 - \zeta(Z_s, s)\sigma(Z_s)ds, \end{aligned}$$

and

$$\frac{d\tilde{P}}{dP} = \exp \left\{ \int_t^T -\frac{1}{2}\zeta^2(Z_s, s)\sigma^2(Z_s) ds + \int_t^T \zeta(Z_s, s)\sigma(Z_s) dW_s^1 \right\}.$$

This allows to conclude that

$$H(z, t) = \mathbb{E}_{x,z,t} [R_T].$$

Now, note that

$$\begin{aligned} R_T^2 &= \exp \left\{ \int_t^T \left[ \left( \rho b(Z_s) \frac{H_z(Z_s, s)}{H(Z_s, s)} \right)^2 - \lambda^2(Z_s) \right] ds \right. \\ &\quad \left. - \int_t^T 2\zeta^2(Z_s, s)\sigma^2(Z_s) ds + \int_t^T 2\zeta(Z_s, s)\sigma(Z_s) dW_s^1 \right\}, \end{aligned}$$

and equation ((5.9)) can be rewritten as

$$H_t + (a(z) + 2\rho b(z)\zeta(z, t)\sigma(z)) H_z + \frac{1}{2}b^2(z)H_{zz} + \left( \rho^2 b^2(z) \frac{H_z^2}{H^2} - \lambda^2(z) \right) H = 0.$$

Again the Feynman-Kac representation guarantees the following equality

$$H(z, t) = \mathbb{E}_{x,z,t} [R_T^2].$$

Finally, let us recall that

$$G(z, t) = -\frac{1}{H(z, t)},$$

so using ((5.8)), we have

$$\frac{1 - \mathbb{E}_{x,z,t} [R_T]}{\text{Var}_{x,z,t} [R_T]} = \frac{1}{\mathbb{E}_{x,z,t} [R_T]} \cdot \frac{1 - \mathbb{E}_{x,z,t} [R_T]}{1 - \mathbb{E}_{x,z,t} [R_T]} = \frac{1}{H(z, t)} = -G(z, t).$$

□

## 6. GENERAL RESULT.

In this section we solve the portfolio optimization problem with performance functional given by ((1.1)).

**Theorem 6.1.** *Under conditions of Theorem 4.1 for each initial conditions  $(x, y, z, t)$  the stochastic process*

$$(6.1) \quad \pi_s^* = - \left( X_s^{\pi^*} - x + 2yG(z, t) \right) \left[ \frac{\lambda(Z_s)}{\sigma(Z_s)} - \frac{\rho b(Z_s)}{\sigma(Z_s)} \frac{G_z(Z_s, s)}{G(Z_s, s)} \right], \quad s \in [t, T],$$

*is an optimal financial strategy in the portfolio optimization problem with preference criterion given by ((1.1)). Moreover, any optimal solution for the classical mean-variance optimization problem is optimal for functional ((1.1)).*

*Proof.* Let us define

$$(6.2) \quad \bar{V}_\theta(X_T^\pi) := \inf_{Q \in \mathcal{Q}} \left\{ \mathbb{E}^Q [X_T^\pi] + \frac{1}{2\theta} C(Q|P) \right\}.$$

Then it is easy to check that

$$(6.3) \quad \bar{V}_\theta(X_T^\pi) = -\bar{\Lambda}_\theta(X_T^\pi) - \frac{1}{2\theta}.$$

Secondly, using Theorem 2.1 from Maccheroni et al. [25], we obtain

$$(6.4) \quad U_\theta(X_T^\pi) \leq V_\theta(X_T^\pi) \leq \bar{V}_\theta(X_T^\pi), \quad \forall \pi \in \mathcal{A}_{x,y,z,t},$$

so obviously we have

$$(6.5) \quad \sup_{\pi \in \mathcal{A}_{x,y,z,t}} U_\theta(X_T^\pi) \leq \sup_{\pi \in \mathcal{A}_{x,y,z,t}} V_\theta(X_T^\pi) \leq \sup_{\pi \in \mathcal{A}_{x,y,z,t}} \bar{V}_\theta(X_T^\pi).$$

Theorem (4.1) and ((6.3)) guarantee that for each initial conditions  $(x, y, z, t)$  there exists an optimal financial strategy of the form ((6.1)) in the portfolio optimization problem with

the performance functional given by ((6.2)) and from Section 5 we know that it coincides with a solution to classical mean-variance optimization problem. Moreover, it turns out that for  $\pi_s^*$  given by ((6.1)) we have

$$U_\theta(X_T^{\pi^*}) = \bar{V}_\theta(X_T^{\pi^*}),$$

so taking into account ((6.4)) and ((6.5)), we get

$$\sup_{\pi \in \mathcal{A}_{x,y,z,t}} V_\theta(X_T^\pi) = V_\theta(X_T^{\pi^*}).$$

To prove the last part of the assertion, let's take any optimal strategy  $\hat{\pi}$  for the classical mean-variance optimization problem. Taking into account ((6.4)), we have

$$U_\theta(X_T^{\pi^*}) = U_\theta(X_T^{\hat{\pi}}) \leq \bar{V}_\theta(X_T^{\hat{\pi}}) \leq \bar{V}_\theta(X_T^{\pi^*}) = U_\theta(X_T^{\pi^*}).$$

Therefore,

$$\bar{V}_\theta(X_T^{\hat{\pi}}) = \bar{V}_\theta(X_T^{\pi^*}).$$

□

## 7. EXPLICIT SOLUTION TO THE SQUARE ROOT FACTOR PROCESS.

In the previous sections we developed the theory well enough to get more insight in the Markowitz optimal portfolio in the stochastic factor models. In this part, we consider some particular factor market models which admit an explicit solution to the mean-variance problem. The fundamental example of such a family of models are Heston stochastic volatility models. This kind of models usually do not satisfy the boundedness condition which was assumed in the previous sections and therefore the verification result needs a separate justification. Solutions obtained in this section can be compared further to the standard Black-Scholes model. We should also mention here that solutions to models considered in this section were obtained earlier by Shen and Zeng [31], even in slightly more general jump-diffusion framework. However, they used BSDE methods and their formula is not as explicit as ours.

The model, which we consider in this section, has the following form

$$(7.1) \quad \begin{cases} dS_t = [r + \lambda Z_t \sigma(Z_t)] S_t dt + \sigma(Z_t) \sqrt{Z_t} S_t dW_t^1, \\ dZ_t = \kappa(\xi - Z_t) dt + b \sqrt{Z_t} (\rho dW_t^1 + \bar{\rho} dW_t^2), \quad 2\kappa\xi > b^2, \quad \kappa > 0. \end{cases}$$

We avoid here the dynamics for the bank account  $B_t$  because we work with  $T$ -forward values in our framework. Apart from the unboundedness of the coefficients, the second difference here is that the factor process  $Z_t$  takes only positive values. Therefore, when considering the HJB equation, we have to limit ourselves to the space  $\mathbb{R} \times \mathbb{R}^+ \times [0, T]$ . In spite of this fact, we can easily follow the path in Section 5 to find the optimal portfolio candidate

$$\pi_s^* = - \left( X_s^{\pi^*} - x - \frac{1}{\theta} \frac{1}{H(z, t)} \right) \left[ \frac{\lambda}{\sigma(Z_s)} - \frac{\rho b}{\sigma(Z_s)} \frac{H_z(Z_s, s)}{H(Z_s, s)} \right], \quad s \in [t, T],$$

where  $H(z, t)$  is a solution to the resulting equation of the form

$$(7.2) \quad H_t + [\kappa(\xi - z) - 2\rho b\lambda z] H_z + \frac{1}{2}b^2 z H_{zz} - \rho^2 b^2 z \frac{H_z^2}{H} - \lambda^2 z H = 0, \quad z \geq 0,$$

with the terminal condition  $H(z, T) = 1$ .

Following Zeng and Taksar [37], we assume that

$$H(z, t) = e^{A(t)z + B(t)}.$$

Substituting the above ansatz into equation ((7.2)), we get

$$\begin{aligned} A'(t) + \left(\frac{1}{2}b^2 - \rho^2 b^2\right) A^2(t) + (-\kappa - 2\rho b\lambda) A(t) - \lambda^2 &= 0, \\ B'(t) + \kappa \xi A(t) &= 0. \end{aligned}$$

In order to solve these equations we consider the quadratic equation of the form

$$(7.3) \quad \left(\frac{1}{2}b^2 - \rho^2 b^2\right) x^2 + (-\kappa - 2\rho b\lambda) x - \lambda^2 = 0,$$

with the following discriminant

$$\Delta = \kappa^2 + 4\kappa\rho\lambda b + 2\lambda^2 b^2.$$

To avoid singularities and exceptions, we always assume that  $\Delta > 0$  and  $\rho^2 \neq \frac{1}{2}$ . In the other cases we have to consider further specification with respect to the time horizon  $T > 0$ . The complete analysis is presented in Proposition 3.1, Zeng and Taksar [37]. The mentioned paper is dedicated to the HARA utility objective with the risk aversion coefficient  $\alpha$ . Surprisingly, if we take Zeng and Taksar's resulting equation, substitute  $\alpha = 2$  and apply their results, then we get tools perfectly suited to our case.

Under our conditions equation ((7.3)) has the following two solutions

$$y_1 = \frac{\kappa + 2\rho b\lambda + \sqrt{\Delta}}{b^2 - 2\rho^2 b^2} \quad \text{and} \quad y_2 = \frac{\kappa + 2\rho b\lambda - \sqrt{\Delta}}{b^2 - 2\rho^2 b^2},$$

so

$$(7.4) \quad A(t) = \frac{e^{p(y_1 - y_2)(T-t)} - 1}{e^{p(y_1 - y_2)(T-t)} - \frac{y_2}{y_1}} y_2, \quad \text{where} \quad p = \frac{1}{2}b^2 - \rho^2 b^2$$

and

$$B(t) = \kappa \xi \int_t^T A(s) ds.$$

For further applications we need to have the following lemma.

**Lemma 7.1.** *Function  $A(t)$  given by ((7.4)) is less than or equal to 0 for all  $t \in [0, T]$ .*

*Proof.* Note that  $A(t)$  can be rewritten as follows

$$A(t) = \frac{e^{p(y_1 - y_2)(T-t)} - 1}{y_1 e^{p(y_1 - y_2)(T-t)} - y_2} y_1 y_2.$$

Moreover, we always have  $e^{p(y_1-y_2)(T-t)} = e^{\sqrt{\Delta}(T-t)} > 1$  and from Vieta's formulas, we get

$$y_1 y_2 = -\frac{\lambda^2}{p}.$$

Case I:  $y_1 y_2 = 0$  implies  $A(t) = 0$ .

Case II:  $y_1 y_2 < 0$  ( $p > 0$ )

In this case  $y_1 > 0 > y_2$  and the sign of  $A(t)$  is inherited from  $y_1 y_2$ .

Case III:  $y_1 y_2 > 0$  ( $p < 0$ )

In this case  $2\rho^2 > 1$  and at the same time  $\kappa + 2\rho b\lambda - \sqrt{\Delta} > 0$ , which implies that  $y_1 < y_2 < 0$  and  $\frac{y_2}{y_1} < 1$ . Formula ((7.4)) completes the reasoning.  $\square$

To formally prove that the strategy  $\pi^*$  is optimal, we should extend the class of optimal strategies.

**Definition 7.2.** A control (or strategy)  $\pi = (\pi_s, t \leq s \leq T)$  is admissible on the time interval  $[t, T]$ , written  $\pi \in \bar{\mathcal{A}}_{x,z,t}$ , if it satisfies the following assumptions:

- (i)  $\pi$  is progressively measurable;
- (ii) there exists a unique solution to the following equation

$$dX_s^\pi = \pi_s \lambda Z_s \sigma(Z_s) ds + \pi_s \sigma(Z_s) \sqrt{Z_s} dW_s^1, \quad s \in [t, T];$$

- (iii) there exists a localizing sequence of stopping times  $(\tau_n, n \in \mathbb{N})$ , such that

$$H(Z_{T \wedge \tau_n}, T \wedge \tau_n) [X_{T \wedge \tau_n}^\pi]^2$$

is uniformly integrable and

$$\mathbb{E}_{x,z,t} \left[ \int_t^{T \wedge \tau_n} [(X_s^\pi)^4 + 1] [\sigma(Z_s) + 1] Z_s ds \right] < +\infty,$$

where

$$H(z, t) = e^{A(t)z + B(t)}.$$

Note that if

$$\mathbb{E}_{x,z,t} \left[ \sup_{t \leq s \leq T} [X_s^\pi]^2 \right] < +\infty,$$

then the above conditions are satisfied, because  $\tau_n$  can be set as the first exit time from a sequence of open and bounded subsets exhausting the set  $\mathbb{R} \times \mathbb{R}^+$ .

**Theorem 7.3.** In model ((7.1)), under conditions  $\Delta > 0$ ,  $\frac{1}{2}b^2 - \rho^2 b^2 \neq 0$  and for each fixed initial conditions  $(x, z, t)$ , the stochastic process

$$\pi_s^* = - \left( X_s^{\pi^*} - x - \frac{1}{\theta} e^{-A(t)z - B(t)} \right) \left[ \frac{\lambda}{\sigma(Z_s)} - \frac{\rho b A(s)}{\sigma(Z_s)} \right], \quad s \in [t, T],$$

is an optimal financial strategy for the classical mean-variance functional (5.1).

*Proof.* Keeping in mind the proof of Theorem 4.1 in Zawisza [36] and calculations done in Section 5, it is sufficient to prove that for all  $D \in \mathbb{R}$  the strategy

$$\pi_D(x, z, s) = -(x - D) \left[ \frac{\lambda}{\sigma(z)} - \frac{\rho b A(s)}{\sigma(z)} \right],$$

is optimal for the problem

$$\mathbb{E}_{x,z,t}(X_T^\pi - D)^2 \rightarrow \min_{\pi \in \tilde{\mathcal{A}}_{x,z,t}}.$$

To verify that, we can follow the standard proof of the verification theorem. Namely, first let's take any admissible strategy  $\pi$  and apply the Itô formula to the function

$$V(x, z, t) = (x - D)^2 H(z, t)$$

and the process  $X_s^\pi$  to obtain

$$\mathbb{E}_{x,z,t} V(X_{T \wedge \tau_n}^\pi, Z_{T \wedge \tau_n}, T \wedge \tau_n) \leq \mathbb{E}_{x,z,t} (X_{T \wedge \tau_n}^\pi - D)^2,$$

where  $(\tau_n, n \in \mathbb{N})$  is the localizing sequence of stopping times from Definition 7.2. Then, we can pass to the limit under the expected value by taking the advantage from the uniform integrability condition for the family

$$H(Z_{T \wedge \tau_n}, T \wedge \tau_n) [X_{T \wedge \tau_n}^\pi]^2$$

and the fact that  $A(t) \leq 0$  (see Lemma 7.1),  $z \geq 0$  and consequently  $H(z, t) \leq 1$ .

Now, we take into consideration the strategy  $\pi_D^*$ . Applying the Itô formula, we obtain

$$\mathbb{E}_{x,z,t} V(X_{T \wedge \tau_n}^{\pi_D^*}, Z_{T \wedge \tau_n}, T \wedge \tau_n) = V(x, z, t),$$

where

$$\tau_n := \inf \{s \geq t : Z_s \notin \mathcal{O}_n\}$$

and  $\{\mathcal{O}_n\}_{n \in \mathbb{N}}$  is an increasing family of open and bounded sets exhausting  $\mathbb{R}^+$ .

Note that

$$X_s^{\pi_D^*} = (x - D)R_s + D \quad \text{and} \quad V(X_{T \wedge \tau_n}^{\pi_D^*}, Z_{T \wedge \tau_n}, T \wedge \tau_n) = (x - D)^2 R_{T \wedge \tau_n}^2 H(Z_{T \wedge \tau_n}, T \wedge \tau_n),$$

where

$$R_s = \exp \left\{ \int_t^s \lambda \sigma(Z_u) \zeta(u) Z_u - \frac{1}{2} \zeta^2(u) \sigma^2(Z_u) Z_u du + \int_t^s \zeta(u) \sigma(Z_u) \sqrt{Z_u} dW_u^1 \right\},$$

$$R_s^2 = \exp \left\{ \int_t^s [\lambda \sigma(Z_u) \zeta(u) + \zeta^2(u) \sigma^2(Z_u)] Z_u du - 2 \int_t^s \zeta^2(u) \sigma^2(Z_u) Z_u du + \int_t^s 2 \zeta(u) \sigma(Z_u) \sqrt{Z_u} dW_u^1 \right\}$$

and

$$\zeta(u) = - \left[ \frac{\lambda}{\sigma(Z_u)} - \frac{\rho b A(u)}{\sigma(Z_u)} \right].$$

Now, it is sufficient to apply the proof of Proposition A1 from Zeng and Taksar [37] (with  $\alpha = 2$ ) to prove that the family  $R_{T \wedge \tau_n}^2 H(Z_{T \wedge \tau_n}, T \wedge \tau_n)$  is uniformly integrable. Then, we are able to pass to the limit ( $n \rightarrow +\infty$ ), to get

$$\mathbb{E}_{x,z,t} (X_T^{\pi_D^*} - D)^2 = \mathbb{E}_{x,z,t} V(X_T^{\pi_D^*}, Z_T, T) = V(x, z, t).$$

□

**Theorem 7.4.** *Under conditions of Theorem 7.3, every Markowitz optimal strategy (in the class  $\bar{\mathcal{A}}_{x,z,t}$ ) is also optimal for monotone functional ((1.1)).*

*Proof.* We can deduce from (6.5) that it is sufficient for us to verify that

$$\pi_s^* = - \left( X_s^{\pi^*} - x - \frac{1}{\theta} e^{-A(t)z - B(t)} \right) \left[ \frac{\lambda}{\sigma(Z_s)} - \frac{\rho b A(s)}{\sigma(Z_s)} \right], \quad s \in [t, T],$$

is optimal for functional ((6.2)), when the infimum is taken under only one measure  $Q^{\eta^*}$ . Therefore, we will prove that  $\pi_s^*$  is the minimum for the functional

$$J^\pi(x, y, z, t) := \mathbb{E}_{x,y,z,t}^{\eta^*} \left[ -X_T^\pi - Y_T^{\eta^*} \right],$$

where  $\eta^*$  is determined by

$$\begin{aligned} \eta_1^* &= -\lambda(z), \\ \eta_2^* &= -\bar{\rho}b(z) \frac{G_z(z, t)}{G(z, t)} \quad \text{and} \quad G(z, t) = -\frac{1}{H(z, t)}. \end{aligned}$$

By applying the Itô formula (like in the proof of Lemma 2.6), we can prove

$$2Y_s^{\eta^*} G(Z_s, s) = X_s^{\pi^*} - x + 2yG(z, t), \quad s \in [t, T].$$

Uniform integrability condition for  $H(Z_{T \wedge \tau_n}, T \wedge \tau_n)[X_{T \wedge \tau_n}^{\pi^*}]^2$  ensures that  $Y_s^{\eta^*}$  is a square integrable continuous martingale and consequently an uniformly integrable martingale.

Now, let us define  $V(x, y, z, t) = -x + G(z, t)y$ . We can easily verify, by direct differentiation, that the suitable HJB equation is satisfied.

We prove first the following inequality

$$(7.5) \quad V(x, y, z, t) \leq J^\pi(x, y, z, t), \quad \pi \in \bar{\mathcal{A}}_{x,z,t}.$$

We can take an admissible strategy  $\pi$  and apply the Itô rule, in order to get

$$\begin{aligned} V(x, y, z, t) &\leq \mathbb{E}_{x,y,z,t}^{\eta^*} V(X_{T \wedge \tau_n}^\pi, Z_{T \wedge \tau_n}, T \wedge \tau_n) = \mathbb{E}_{x,y,z,t}^{\eta^*} \left[ -X_{T \wedge \tau_n}^\pi + G(Z_{T \wedge \tau_n}, T \wedge \tau_n) Y_{T \wedge \tau_n}^{\eta^*} \right] \\ &= \mathbb{E}_{x,y,z,t} \left[ -\frac{1}{y} Y_{T \wedge \tau_n}^{\eta^*} X_{T \wedge \tau_n}^\pi + \frac{1}{y} G(Z_{T \wedge \tau_n}, T \wedge \tau_n) [Y_{T \wedge \tau_n}^{\eta^*}]^2 \right], \end{aligned}$$

where  $(\tau_n, n \in \mathbb{N})$  is a localizing sequence of stopping times from Definition 7.2. The last equality is implied by the relation  $y \frac{dQ^{\eta^*}}{dP} = Y_T^{\eta^*}$  and the martingality of the process  $Y_t^{\eta^*}$ .

The maximization of the quadratic function  $u_1 a + u_2 a^2$  with respect to  $a$  yields

$$u_1 a + u_2 a^2 \leq -\frac{u_1^2}{4u_2}, \quad u_1, a \in \mathbb{R} \text{ and } u_2 < 0.$$

Substituting  $u_1 = -\frac{1}{y} X_{T \wedge \tau_n}^\pi$ ,  $u_2 = \frac{1}{y} G(Z_{T \wedge \tau_n}, T \wedge \tau_n)$  and  $a = Y_{T \wedge \tau_n}^{\eta^*}$ , we get

$$-\frac{1}{y} Y_{T \wedge \tau_n}^{\eta^*} X_{T \wedge \tau_n}^\pi + \frac{1}{y} G(Z_{T \wedge \tau_n}, T \wedge \tau_n) [Y_{T \wedge \tau_n}^{\eta^*}]^2 \leq \frac{1}{4y} H(Z_{T \wedge \tau_n}, T \wedge \tau_n) [X_{T \wedge \tau_n}^\pi]^2.$$



The strategy  $\pi$  is admissible, so the family

$$\frac{1}{4y}H(Z_{T \wedge \tau_n}, T \wedge \tau_n)[X_{T \wedge \tau_n}^\pi]^2,$$

is uniformly integrable. To assert formula (7.5) it is sufficient to apply the Fatou Lemma to the following reverse inequality

$$-V(x, y, z, t) \geq \mathbb{E}_{x,y,z,t} \left[ \frac{1}{y}Y_{T \wedge \tau_n}^{\eta^*} X_{T \wedge \tau_n}^\pi - \frac{1}{y}G(Z_{T \wedge \tau_n}, T \wedge \tau_n)[Y_{T \wedge \tau_n}^{\eta^*}]^2 \right].$$

The Fatou Lemma is possible because the integrand on the right hand side is bounded below by the uniformly integrable sequence.

To finish the proof, we should only verify that

$$V(x, y, z, t) = J^{\pi^*}(x, y, z, t) = \mathbb{E}_{x,y,z,t}^{\eta^*} \left[ -X_T^{\pi^*} + Y_T^{\eta^*} \right].$$

Here, we just need to employ the Itô formula

$$\begin{aligned} V(x, y, z, t) &= \mathbb{E}_{x,y,z,t}^{\eta^*} V(X_{T \wedge \tau_n}^{\pi^*}, Z_{T \wedge \tau_n}, T \wedge \tau_n) = \mathbb{E}_{x,y,z,t}^{\eta^*} \left[ -X_{T \wedge \tau_n}^{\pi^*} + G(Z_{T \wedge \tau_n}, T \wedge \tau_n)Y_{T \wedge \tau_n}^{\eta^*} \right] \\ &= \mathbb{E}_{x,y,z,t} \left[ -\frac{1}{y}Y_{T \wedge \tau_n}^{\eta^*} X_{T \wedge \tau_n}^{\pi^*} + \frac{1}{y}G(Z_{T \wedge \tau_n}, T \wedge \tau_n)[Y_{T \wedge \tau_n}^{\eta^*}]^2 \right], \end{aligned}$$

where  $(\tau_n, n \in \mathbb{N})$  is a suitable localizing sequence of stopping times. Now, we use the following duality

$$Y_s^{\eta^*} = \frac{1}{2} \frac{1}{G(Z_s, s)} X_s^{\pi^*} + \frac{1}{2} \frac{1}{G(Z_s, s)} [-x + 2yG(z, t)] = -\frac{1}{2}H(Z_s, s)X_s^{\pi^*} - \frac{1}{2}H(Z_s, s)[-x + 2yG(z, t)]$$

and pass to the limit ( $n \rightarrow +\infty$ ), using the uniform integrability condition for three families

$$H(Z_{T \wedge \tau_n}, T \wedge \tau_n)[X_{T \wedge \tau_n}^{\pi^*}]^2, \quad H^2(Z_{T \wedge \tau_n}, T \wedge \tau_n)[X_{T \wedge \tau_n}^{\pi^*}]^2 \quad \text{and} \quad H(Z_{T \wedge \tau_n}, T \wedge \tau_n)X_{T \wedge \tau_n}^{\pi^*}.$$

The first one has already been commented in the proof of Theorem 7.3, the rest are its implications. Namely, the second one is implied by the condition  $H(z, t) \leq 1$  and the last one can be proved by the application of the Cauchy-Schwartz inequality.  $\square$

Now, we would like to consider the solution with the emphasis on the following two special cases:

Case I: The volatility coefficient  $\sigma(z) = \bar{\sigma}$  is constant. Then

$$\pi_s^* = - \left( X_s^{\pi^*} - x - \frac{1}{\theta} e^{-A(t)z - B(t)} \right) \left[ \frac{\lambda}{\bar{\sigma}} - \frac{\rho b A(s)}{\bar{\sigma}} \right], \quad s \in [t, T].$$

Case II: The stochastic factor affects only the excess return but not the volatility i.e.  $\sigma(z) = \frac{\bar{\sigma}}{\sqrt{z}}$ . Then

$$\pi_s^* = - \left( X_s^{\pi^*} - x - \frac{1}{\theta} e^{-A(t)z - B(t)} \right) \left[ \frac{\lambda}{\bar{\sigma}} - \frac{\rho b A(s)}{\bar{\sigma}} \right] \sqrt{Z_s}, \quad s \in [t, T].$$

It is also worth to compare the above cases to the standard Black-Scholes market

$$dS_t = [r + \lambda\bar{\sigma}]S_t dt + \bar{\sigma}S_t dW_t^1,$$

where

$$\pi_s^* = - \left( X_s^{\pi^*} - x - \frac{1}{\theta} e^{\lambda^2(T-t)} \right) \frac{\lambda}{\bar{\sigma}}, \quad s \in [t, T].$$

In all three cases the absolute value invested in  $S_t$  is determined in the following way

$$\pi_s^* = (X_s^{\pi^*} - D(x, z, t))P(Z_s, s),$$

so is proportional to the excess of the current wealth  $X_s^{\pi^*}$  over some target level determined by  $D(x, z, t)$ . The term  $P(z, s)$  can be interpreted as the proportion value. We can see that the presence of the stochastic factor modulates the target level  $D(x, z, t)$ , but important are only the coefficients of the factor dynamics and the value of the stochastic factor at the beginning of the investing period. The proportion  $P(z, s)$  depends on the coefficients of the factor dynamics and when considering the model presented in Case II, on the current level of the stochastic factor as well. The direction and magnitude of the factor impact depends on many different configurations of the factor current value and model coefficients.

## 8. CONCLUSION.

We examined the continuous time portfolio optimization problem the stochastic factor model assuming that the preference criterion is the monotone mean-variance functional introduced by Maccheroni et al. [25]. We have solved the problem under general conditions and have proved that the the optimal mean-variance strategy is optimal also for the monotone criterion. This shows that continuous time mean-variance investor behaves like a monotone investor, even in economically important incomplete market models. It would be also interesting to check this property in other class of models, for example in jump-diffusion models.

## ACKNOWLEDGMENTS.

We would like to express our sincere gratitude to the Referees for a list of helpful remarks and a careful reading of the manuscript.

## REFERENCES

- [1] P. Artzner, F. Delbaen, JM. Eber, D. Heath *Coherent measures of risk*, Math. Finance 9 (1999), 203 – 228.
- [2] S. Basak, G. Chabakauri, *Dynamic mean-variance asset allocation*, Review of Financial Studies, 23 (2010), 2970 – 3016.
- [3] N. Bauerle, S. Grether, *Complete markets do not allow free cash ow streams*, Math. Meth. Oper. Res. 81 (2015) 137 – 146.

- [4] G. Bordigoni, A. Matoussi, M. Schweizer, *A stochastic control approach to a robust utility maximization problem*, Stochastic Analysis and Applications. The Abel Symposium 2005. Springer 2007, 125 – 151.
- [5] J.Y. Campbell, L.M. Viceira *Consumption and portfolio decisions when expected returns are time varying*, Q. J. Econ. 114 (1999) 433 – 495.
- [6] X. Cui, D. Li, S. Wang, S. Zhu, *Better than dynamic mean variance: Time inconsistency and free cash flow stream*, Math. Finance 22 (2012), 345 – 378.
- [7] R. J. Elliott, T. K. Siu, *A BSDE approach to a risk-based optimal investment of an insurer*, Automatica J. IFAC, 47 (2011), 253 – 261.
- [8] R. J. Elliott, T. K. Siu, *On risk minimizing portfolios under a Markovian regime-switching Black-Scholes economy*, Ann. Oper. Res. 176 (2010), 271 – 291.
- [9] R. J. Elliott, T. K. Siu, *Portfolio risk minimization and differential games*, Nonlinear Anal. 71 (2009), 2127 – 2135.
- [10] R. J. Elliott, T. K. Siu, *Risk-based indifference pricing under a stochastic volatility model*, Commun. Stoch. Anal. 4 (2010), 51 – 73.
- [11] WH. Fleming, D. Hernández-Hernández, *An optimal consumption model with stochastic volatility*, Finance Stoch. 7 (2003), 245 – 262.
- [12] W. Fleming, H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, 2nd Edition, Springer, New York, 2006.
- [13] H. Föllmer, A. Schied, *Convex measures of risk and trading constraints*, Finance and Stochastics, 6 (2012), 429 – 447.
- [14] M. Frittelli, E. Rosazza-Gianin, *Putting order in risk measures*, Journal of Banking and Finance, 26 (2002), 1473 – 1486.
- [15] I. Gilboa, D. Schmeidler *Maxmin expected utility with non-unique prior*, J. Math. Econ. 18 (1989) 141–153.
- [16] D. Heath, M. Schweizer, *Martingales versus PDEs in finance: an equivalence result with examples*, J. Appl. Probab. 37 (2000), 947 – 957.
- [17] D. Hernández, A. Schied, *A control approach to robust utility maximization with logarithmic utility and time-consistent penalties*, Stochastic Process. Appl. 117 (2007), 980 – 1000.
- [18] J. Kallsen, J. Muhle-Karbe, N. Shenkman, R. Vierthauer *Discrete time variance-optimal hedging in affine stochastic volatility models*, R. Kiesel, M Scherer, eds. Alternative Investments and Strategies (World Scientific, Singapore), 375–393.
- [19] TS. Kim, E. Omberg *Dynamic nonmyopic portfolio behavior*, Rev. Financ. Stud. 9 (1996), 141 – 161.

- [20] H. Kraft, T. Seiferling, FT. Seifried, *Optimal consumption and investment with epstein - zin recursive utility*, Finance Stoch. 21 (2017), 187 – 226.
- [21] J. Laurent, H. Pham, *Dynamic programming and mean-variance hedging*, Finance Stoch. 3 (1999), 83 – 110.
- [22] D. Li, W. Ng, *Optimal dynamic portfolio selection: multiperiod mean-variance formulation*, Math. Finance, 10 (2000), 387 – 406.
- [23] J. Liu, *Portfolio selection in stochastic environments*, Rev. Financ. Stud. 20 (2007), 1 – 39.
- [24] F. Maccheroni, M. Marinacci, A. Rustichini, *Ambiguity aversion, robustness, and the variational representation of preferences*, Econometrica 74 (2006), 1447 – 1498.
- [25] F. Maccheroni, M. Marinacci, A. Rustichini, M. Taboga, *Portfolio selection with monotone mean-variance preferences*, Math. Finance, 19 (2009), 487 – 521.
- [26] H. Markowitz, *Portfolio selection*, J. Finance, 7 (1952), 77 – 91.
- [27] S. Mataramvura, B. Øksendal, *Risk minimizing portfolios and HJBI equations for stochastic differential games*, Stochastics, 80 (2008), 317 – 337.
- [28] B. Øksendal, A. Sulem, *Forward-backward stochastic differential games and stochastic control under model uncertainty*, J. Optim. Theory Appl. 161 22–55.
- [29] B. Øksendal, A. Sulem, *Risk indifference pricing in jump diffusion markets*, Math. Finance, 19 (2009), 619 – 637.
- [30] H. Pham, *Continuous-time Stochastic Control and Optimization with Financial Applications*, Stochastic Modelling and Applied Probability, Springer-Verlag, Berlin, 2009.
- [31] Y. Shen, X. Zeng, *Optimal investment strategy for mean variance insurers with square root factor process*, Insur. Math. Econ. 62 (2015), 118 – 137.
- [32] J. von Neumann , O. Morgenstern, *Theory of games and economic behavior*, (Princeton: Princeton university press), 1944.
- [33] M. Yaari, *The dual theory of choice under risk*, Econometrica 55 (1987), 95–115.
- [34] T. Zariphopoulou, *A solution approach to valuation with unhegeable risks*, Finance Stoch. 5 (2001), 61 – 82.
- [35] D. Zawisza, *Robust portfolio selection under exponential preferences*, Appl. Math. 37 (2010) (Warsaw), 215 – 230.
- [36] D. Zawisza, *Target achieving portfolio under model misspecification: quadratic optimization framework*, Appl. Math. (Warsaw) 39 (2012), 425 – 443.
- [37] X. Zeng, M. Taksar *A stochastic volatility model and optimal portfolio selection*, Quant. Finance 13, 1547 – 1558.

- [38] X. Y. Zhou, D. Li, *Continuous-time mean-variance portfolio selection: a stochastic LQ framework*, Appl. Math. Optim. 42 (2000), 19 – 33.

JAKUB TRYBUŁA,  
DEPARTMENT OF MATHEMATICS  
UNIWEERSITY OF ECONOMICS  
RAKOWICKA 27,  
31-510 KRAKÓW, POLAND  
*E-mail address:* jakubtrybula@gmail.com

DARIUSZ ZAWISZA,  
INSTITUTE OF MATHEMATICS  
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE  
JAGIELLONIAN UNIVERSITY  
ŁOJASIEWICZA 6  
30-348 KRAKÓW, POLAND  
*E-mail address:* dariusz.zawisza@im.uj.edu.pl