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LONG TIME BEHAVIOUR OF FRACTIONAL IMPULSIVE STOCHASTIC DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. This paper is first devoted to the local and global existence of mild solutions for a class of fractional impulsive stochastic differential equations with infinite delay driven by both $\mathbb{K}$-valued Q-cylindrical Brownian motion and fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. A general framework which provides an effective way to prove the continuous dependence of mild solutions on initial value is established under some appropriate assumptions. Furthermore, it is also proved the exponential decay to zero of solutions to fractional stochastic impulsive differential equations with infinite delay.

Keywords: Impulsive differential equations, fractional derivative, fractional Brownian motion, infinite delay, exponential asymptotic behaviour.

1. **Introduction.** In the past decades, the theory of impulsive differential equations has become an active area due to its wide applications in several fields such as communications, mechanics, electrical engineering, medicine, biology, etc. For the basic theory of impulsive differential equations, we refer the readers to [2, 4, 5, 6, 10, 21, 27, 33] and the references therein.

It is well-known that the deterministic models arising in mathematical finance, climate and weather derivatives, often fluctuate due to the presence of some kind of noise. Hence, the stochastic models driven by a Brownian motion or fractional Brownian motion have attracted the researchers great interest (see, e.g., [5, 7, 9, 13, 20, 28, 37]). It is worth mentioning that in [9] the authors established a technical lemma (Lemma 2) which is crucial to the stochastic integral with respect to fractional Brownian motion when considering the Hurst parameter $H \in (1/2, 1)$. In our paper, we adopt this point of view to deal with the fractional Brownian motion.

On the other hand, the fractional differential equations which are presented in the modeling of many real problems (e.g. in physical phenomena) have been the object of extensive study in order to analyze not only non-random fractional phenomena in physics, but also stochastic processes driven

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by a fractional Brownian motion, see [1, 10, 22, 23, 24, 34] and references therein. Therefore, our main aim in this paper is to address the issue of existence, uniqueness and asymptotic behaviour of mild solutions for the following fractional stochastic impulsive differential equations with infinite delay,

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x_t) + g(t, x_t)\frac{dB(t)}{dt} + h(t)\frac{dB_Q^H(t)}{dt}, \quad t \geq 0, \\ \qquad t \neq t_k, \quad \frac{1}{2} < \alpha < 1, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, \\ x(t) = \phi(t), \quad t \in (-\infty, 0], \end{cases} \tag{1}$$

where D_t^α is the Caputo fractional derivative of order $\frac{1}{2} < \alpha < 1$, $A : D(A) \subseteq \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of an α-order fractional compact and analytic operator $T_\alpha(t)(t \geq 0)$ in a separable Hilbert space $\mathbb{H}$. As usual, $B(t)$ and $B_Q^H(t)$ denote, respectively, a $\mathbb{K}$-valued Q-cylindrical Brownian motion and fractional Brownian motion defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The fixed time t_k, where the impulses take place, satisfy $0 = t_0 < t_1 < \cdots < t_k \to +\infty$ as $k \to \infty$.

Let $\mathbb{K}$ be another separable Hilbert space, we consider the functions $x_t : (-\infty, 0] \to L^2(\Omega; \mathbb{H})$ defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in (-\infty, 0]$, which are continuous everywhere except for countable points $t_k(k \in \mathbb{N}^+)$, at which there exist $x(t_k^+)$ and $x(t_k^-)$, and $x(t_k) = x(t_k^-)$ (for each k, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right-hand and left-hand limits of $x(t)$ at $t = t_k$ respectively). The nonlinear maps $f : \mathbb{R}^+ \times \mathcal{PC} \to \mathbb{H}$, $g : \mathbb{R}^+ \times \mathcal{PC} \to L(\mathbb{K}, \mathbb{H})$ and $h : \mathbb{R}^+ \to L_Q^0(\mathbb{K}, \mathbb{H})$ satisfy some appropriate assumptions, where $L(\mathbb{K}, \mathbb{H})$, $L_Q^0(\mathbb{K}, \mathbb{H})$ and $\mathcal{PC}$ are different phase spaces which will be described precisely later. Moreover, the impulse function I_k belongs to $C(\mathbb{H}; \mathbb{H})$ and ϕ denotes the initial value to be chosen in an appropriate phase space.

The theory of local existence of solutions for impulsive non-stochastic or impulsive stochastic differential equations has experienced a good development up to date. In [24], the local existence of mild solutions for neutral impulsive stochastic integro-differential equations has been investigated. Subsequently, using a Banach fixed point theorem, the existence and uniqueness of local solutions for fractional impulsive differential equations with infinite delay are established in [12]. Henceforth, studies of existence of solutions for impulsive stochastic differential equations have been launched as can be seen, for instance, [33, 34, 35, 36]. Notice that, most of the previous research concerns the case of the local existence of solutions for such equations, there has been little regarding the case of the global existence of solutions except for [10, 39], in which the global existence of solutions for fractional impulsive differential equations was obtained. Motivated by the work in [39], we derive the global existence of solutions for problem (1.1).

We would like to emphasize that it is meaningful and necessary to discuss the qualitative properties for fractional stochastic differential equations with impulsive perturbations and infinite delay. However, to the best of our knowledge, no results have been reported on the exponential asymptotic behavior of solutions for fractional impulsive stochastic differential equations with infinite delay. Hence the aim of this paper is to prove some results in this field. When studying the exponential decay to zero of solutions to problem (1), without loss of generality, we assume that the α-order fractional solution operator $T_\alpha(t)(t > 0)$ and α-resolvent family $S_\alpha(t)(t > 0)$ are tempered.

We organize the rest of this paper as follows. Section 2 presents some basic notations, preliminaries and lemmas which will be used throughout this paper. Based on the properties of the α-order fractional solution operator $T_\alpha(t)(t > 0)$ and α-resolvent family $S_\alpha(t)(t > 0)$, as well as iterate technique, the existence and uniqueness of mild solutions to problem (1) are derived in Section 3. The last section is devoted to show the exponential asymptotic behavior of mild solutions for fractional impulsive stochastic differential equations with infinite delay under some appropriate conditions.

2. **Preliminaries.** In this section we introduce the basic definitions of $\mathbb{K}$-valued Q-cylindrical fractional Brownian motion as well as Brownian motion, also we recall some important definitions and lemmas which will be used in the remainder of this paper. Although the content of this section can be found in several published works, we prefer to include it in our paper to make it more readable and as much self-contained as possible.

Throughout this paper, let $\mathbb{H}$ and $\mathbb{K}$ be two separable Hilbert spaces and $L(\mathbb{K},\mathbb{H})$ be the space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathbb{H}$, $\mathbb{K}$ and $L(\mathbb{K},\mathbb{H})$, and use $(\cdot,\cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$ without any confusion. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$).

Let $B=(B(t))_{t\geq 0}$ and $B_Q^H=(B_Q^H(t))_{t\geq 0}$ be a $\mathbb{K}$-valued Q-cylindrical Brownian motion and a fractional Brownian motion respectively, defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ with $TrQ<\infty$, where Q is a symmetric nonnegative trace class operator from $\mathbb{K}$ into itself. We assume that there exists a complete orthonormal basis $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k=\lambda_k e_k$, $k=1,2,\cdots$. Then for arbitrary $t\in[0,T]$, $B(\cdot)$, $B_Q^H(\cdot)$ have the expansions

$$B(t)=\sum_{k=1}^{\infty}\sqrt{\lambda_k}\beta_k(t)e_k, \qquad B_Q^H(t)=\sum_{k=1}^{\infty}\sqrt{\lambda_k}\beta_k^H(t)e_k, \quad t\geq 0,$$

where $\{\beta_k\}_{k\geq 1}$ and $\{\beta_k^H\}_{k\geq 1}$ are, respectively, a sequence of two-sided one-dimensional real valued standard Brownian motions and a sequence of fractional Brownian motions mutually independent on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$.

For $\varphi,\psi\in L(\mathbb{K},\mathbb{H})$, we define $(\varphi,\psi)=Tr[\varphi Q\psi^*]$, where ψ^* is the adjoint operator of ψ. Then, for any bounded operator $\psi\in L(\mathbb{K},\mathbb{H})$,

$$\|\psi\|_Q^2=Tr[\psi Q\psi^*]=\sum_{k=1}^{\infty}\|\sqrt{\lambda_k}\psi e_k\|^2.$$

If $\|\psi\|_Q^2<\infty$, then ψ is called a Q-Hilbert-Schmidt operator, we denote by $L_Q^0(\mathbb{K},\mathbb{H})$ the space of all $\xi\in L(\mathbb{K},\mathbb{H})$ such that $\xi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. By Proposition 2.8 in [11], if ψ is an $L(\mathbb{K},\mathbb{H})$-valued stochastic process on $T\times\Omega$ such that $\psi(t)$ is measurable relative to $\mathcal{F}_t$ for all $t\in[0,T]$, and satisfies

$$\int_0^T E\|\psi(t)\|^2dt<\infty,$$

then we have the following property,

$$E\Big\|\int_0^T\psi(s)dB(s)\Big\|^2\leq Tr(Q)\int_0^T E\|\psi(s)\|^2ds. \tag{2}$$

Lemma 1. (*[9]) Let $\varphi:[0,T]\to Q$ such that, for any α, $\beta\in[0,T]$ with $\alpha>\beta$,*

$$\int_\beta^\alpha\|\varphi(s)\|_Q^2ds<\infty.$$

Then,

$$E\left\|\int_\beta^\alpha \varphi(s)dB_Q^H(s)\right\|^2 \le cH(2H-1)(\alpha-\beta)^{2H-1}\int_\beta^\alpha \|\varphi(s)\|_Q^2 ds. \tag{3}$$

Below we briefly state the definitions and properties of fractional calculus.

Definition 2. *([18,31]) The fractional integral of order $\alpha > 0$ with the lower limit 0 for a function f is defined as*

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \quad t>0,$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 3. *([18,32]) The Caputo fractional derivative of order $\alpha > 0$ with the lower limit 0 for a function f is defined as*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds = I_t^{n-\alpha}f^{(n)}(t), \text{ for } n-1<\alpha<n,\ n\in\mathbb{N},$$

where the function $f(t)$ has absolutely continuous derivatives up to order $n-1$. If $0<\alpha<1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f^{(1)}(s)ds.$$

Obviously, the Caputo fractional derivative of a constant is equal to zero. The Laplace transform of the Caputo fractional derivative of order $\alpha>0$ is given as

$$L\{D_t^\alpha f(t);\lambda\} = \lambda^\alpha F(\lambda) - \sum_{k=0}^{n-1}\lambda^{\alpha-k-1}f^{(k)}(0), \qquad n-1\le\alpha<n,$$

where $F(\lambda)=\int_0^\infty e^{-\lambda t}f(t)dt$ denotes the Laplace transform of a function $f(t)$.

If f is an abstract function with values in $\mathbb{H}$, then the integrals which appear in Definition 2 and Definition 3 are taken in the Bochner sense. A measurable function $u:[0,\infty)\to\mathbb{H}$ is Bochner integrable if $\|u\|$ is Lebesgue integrable.

A two–parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)} = \frac{1}{2\pi i}\int_C \frac{t^{\alpha-\beta}e^t}{t^\alpha - z}dt, \quad \alpha,\beta>0, \quad z\in\mathbb{C},$$

where the path C is a loop which starts and ends at $-\infty$ and encircles the disc $|t|\le|z|^{\frac{1}{2}}$ in the positive sense. For short, $E_\alpha(z)=E_{\alpha,1}(z)$. It is an entire function which provides a simply generalization of the exponential function $E_1(z)=e^z$, and plays a vital role in the theory of fractional differential equations. The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral,

$$\int_0^\infty e^{-\lambda t}t^{\beta-1}E_{\alpha,\beta}(\omega t^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha-\omega}, \quad Re\lambda > \omega^{\frac{1}{\alpha}},\ \omega>0,$$

and for more details we refer to [31,38].

Next we recall some basic definitions concerning the sectorial operator A, α-order fractional solution operator $T_\alpha(t)$ and α-resolvent family $S_\alpha(t)$.

Definition 4. ([16]) *A linear closed densely defined operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$, such that the following two conditions are satisfied,*
(1) $\sigma(A) \subset \sum_{\omega,\theta} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$,

(2) $\|\mathbb{R}(\lambda, A)\| \leq \frac{M}{|\lambda-\omega|}, \ \lambda \in \sum_{\omega,\theta}$.

Definition 5. *Let A be a closed and linear operator with domain $D(A)$ defined on $\mathbb{H}$. Let $\rho(A)$ be the resolvent set of A. We say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(\mathbb{H})$, such that $\{\lambda^\alpha : Re(\lambda) > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha I - A)^{-1} y = \int_0^\infty e^{-\lambda t} S_\alpha(t) y dt, \quad Re(\lambda) > \omega, \ y \in \mathbb{H},$$

where $S_\alpha(t)$ is called the α-resolvent family generated by A.

Definition 6. ([30, 36]) *A solution operator $T_\alpha(t)$ of (1) is called analytic if $T_\alpha(t)$ admits an analytic extension to a sector $\sum_{\theta_0} := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| < \theta_0\}$ for some $\theta_0 \in (0, \frac{\pi}{2}]$. An analytic solution operator is said to be of analyticity type (ω_0, θ_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$, there is a positive constant $M = M(\omega, \theta)$ such that $\|T_\alpha(t)\| \leq M e^{\omega Re(t)}$, $t \in \sum_\theta := \{t \in \mathbb{C}\backslash\{0\} : |\arg t| < \theta\}$. Denote $\mathbb{A}^\alpha(\omega_0, \theta_0) := \{\mathbb{A}$ generates analytic solution operators $T_\alpha(t)$ of type $(\omega_0, \theta_0)\}$.*

Definition 7. ([3]) *A family $T_\alpha(t) : \mathbb{R}^+ \to L(\mathbb{H})$ is called an α-order fractional solution operator generated by A if the following conditions are satisfied:*
(1) *$T_\alpha(t)$ is strongly continuous for $t \geq 0$ and $T_\alpha(0) = I$;*
(2) *$T_\alpha(t)D(A) \subset D(A)$ and $AT_\alpha(t)x = T_\alpha(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;*
(3) *for all $x \in D(A)$, $T_\alpha(t)x$ is a solution of the following operator equation*

$$x(t) = x + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Ax(s) ds, \quad t \geq 0.$$

Definition 8. *An α-order fractional solution operator $T_\alpha(t)(t \geq 0)$ is called compact if for every $t > 0$, $T_\alpha(t)$ is a compact operator.*

Arguing as in the proof of Lemma 3.8 in [15], we obtain the continuity of the α-order fractional solution operator $T_\alpha(t)$ and α-resolvent family $S_\alpha(t)$ in the uniform operator topology for $t > 0$.

Lemma 9. *Assume $A \in \mathbb{A}^\alpha(\omega_0,\theta_0)$, and that the α-order fractional solution operator $T_\alpha(t)(t > 0)$ and the α-resolvent family $S_\alpha(t)(t > 0)$ are compact. Then the following properties are fulfilled:*

(1) $\lim\limits_{h\to 0} \|T_\alpha(t+h) - T_\alpha(t)\| = 0$ *and* $\lim\limits_{h\to 0} \|S_\alpha(t+h) - S_\alpha(t)\| = 0$ *for* $t > 0$;

(2) $\lim\limits_{h\to 0^+} \|T_\alpha(t+h) - T_\alpha(h)T_\alpha(t)\| = 0$ *and* $\lim\limits_{h\to 0^+} \|S_\alpha(t+h) - S_\alpha(h)S_\alpha(t)\| = 0$ *for* $t > 0$;

(3) $\lim\limits_{h\to 0^+} \|T_\alpha(t) - T_\alpha(h)T_\alpha(t-h)\| = 0$ *and* $\lim\limits_{h\to 0^+} \|S_\alpha(t) - S_\alpha(h)S_\alpha(t-h)\| = 0$ *for* $t > 0$.

Lemma 10. ([36]) *If $A \in \mathbb{A}^\alpha(\omega_0,\theta_0)$, then for every $\omega > \omega_0$ ($\omega_0 \in \mathbb{R}^+$), there exists a constant $M = M(\omega, \theta)$ such that, for all $t > 0$,*

$$\|T_\alpha(t)\| \leq M e^{\omega t} \quad and \quad \|S_\alpha(t)\| \leq M e^{\omega t}(1 + t^{\alpha-1}). \tag{4}$$

Furthermore, let $M_T := \sup_{0\le t\le T} \|T_\alpha(t)\|$, $N_T := \sup_{0\le t\le T} Me^{\omega t}(1+t^{1-\alpha})$. *Then we obtain that*

$$\|T_\alpha(t)\| \le M_T \quad \textit{and} \quad \|S_\alpha(t)\| \le N_T t^{\alpha-1}. \tag{5}$$

3. **Existence, uniqueness and continuous dependence of mild solutions.** In this section, we establish the existence of mild solutions to fractional impulsive stochastic differential equation (1). Before doing this, we first present the abstract phase space $\mathcal{PC}$.

Let $L^2(\Omega;\mathbb{H})$ denote the Banach space of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables equipped with the norm $\|u(\cdot)\|_{L^2}^2 = E\|u(\cdot)\|^2$, where the expectation E is defined by $Eu = \int_\Omega u(\cdot)d\mathbb{P}$. The abstract phase space $\mathcal{PC}$ is defined by

$$\begin{aligned}\mathcal{PC} = \Big\{ &\xi : (-\infty,0] \to L^2(\Omega;\mathbb{H}) \text{ is } \mathcal{F}_0\text{-adapted and continuous except in at most a} \\ &\text{countable number of points } \{\theta_k\}, \text{ at which there exist } \xi(\theta_k^+) \text{ and } \xi(\theta_k^-) \\ &\text{with } \xi(\theta_k) = \xi(\theta_k^-), \text{ and } \sup_{\theta\in(-\infty,0]} e^{\gamma\theta} E\|\xi(\theta)\|^2 < \infty \Big\},\end{aligned}$$

for some fixed parameter $\gamma > 0$. If $\mathcal{PC}$ is endowed with the norm

$$\|\xi\|_{\mathcal{PC}} = \left(\sup_{\theta\in(-\infty,0]} e^{\gamma\theta} E\|\xi(\theta)\|^2 \right)^{\frac{1}{2}}, \qquad \xi \in \mathcal{PC},$$

then, $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

In order to establish the main result, we impose the following conditions.

(H_1) $f : [0,\infty)\times\mathcal{PC} \to \mathbb{H}$, $g : [0,\infty)\times\mathcal{PC} \to L(\mathbb{K},\mathbb{H})$ are continuous and there exist two functions $l_1(t)$, $l_2(t) \in L^\infty(\mathbb{R}^+,\mathbb{R}^+)$ such that

$$E\|f(t,x) - f(t,y)\|^2 \le l_1(t)\|x-y\|_{\mathcal{PC}}^2, \quad E\|g(t,x) - g(t,y)\|^2 \le l_2(t)\|x-y\|_{\mathcal{PC}}^2,$$

for every x, $y \in \mathcal{PC}$, and almost every $t > 0$. Moreover, $g(t,\cdot)$ is measurable relative to $\mathcal{F}_t$ for all $t \in [0,\infty)$ satisfying $\int_0^\infty E\|g(t,x)\|^2 dt < \infty$.

(H_2) For $h : [0,\infty) \to Q$, there exists a constant $q > 1$ such that, for every $t \in [0,T]$,

$$\int_0^t \|h(s)\|_Q^{2q} ds < \infty.$$

(H_3) The functions $I_k : L^2(\Omega;\mathbb{H}) \to L^2(\Omega;\mathbb{H})$ are continuous for each $k \in \mathbb{N}^+$, and there exist two positive constants b_1, $b_2 > 0$ such that

$$E\|I_k(x)\|^2 \le b_1 E\|x\|^2 + b_2, \text{ for all } x \in L^2(\Omega;\mathbb{H}).$$

There exists a positive constant N such that, for all $k \in \mathbb{N}^+$,

$$E\|I_k(x) - I_k(y)\|^2 \le NE\|x-y\|^2, \text{ for all } x,y \in L^2(\Omega;\mathbb{H}).$$

(H_4) $\beta = \inf\limits_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\} > 0$, $\eta = \sup\limits_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\} < \infty$.

Now, we introduce the definition of mild solutions for problem (1).

Definition 11. *Set $\mathcal{F}_t = \mathcal{F}_0$ for all $t \in (-\infty, 0]$. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, T] \to \mathbb{H}$ is called a mild solution of the equation (1) if $x(t) = \phi(t)$ for $t \in (-\infty, 0]$ with $\phi \in \mathcal{PC}$, and for $t \in [0, T]$, $x(t)$ satisfies the integral equation*

$$x(t) = \begin{cases} T_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s)f(s, x_s)ds + \int_0^t S_\alpha(t-s)g(s, x_s)dB(s) \\ + \int_0^t S_\alpha(t-s)h(s)dB_Q^H(s), \quad t \in [0, t_1], \\ T_\alpha(t-t_1)(I_1(x(t_1^-)) + x(t_1^-)) + \int_{t_1}^t S_\alpha(t-s)f(s, x_s)ds \\ + \int_{t_1}^t S_\alpha(t-s)g(s, x_s)dB(s) + \int_{t_1}^t S_\alpha(t-s)h(s)dB_Q^H(s), \quad t \in (t_1, t_2], \\ \cdots, \\ T_\alpha(t-t_m)(I_m(x(t_m^-)) + x(t_m^-)) + \int_{t_m}^t S_\alpha(t-s)f(s, x_s)ds \\ + \int_{t_m}^t S_\alpha(t-s)g(s, x_s)dB(s) + \int_{t_m}^t S_\alpha(t-s)h(s)dB_Q^H(s), \quad t \in (t_m, T], \end{cases} \tag{6}$$

where $t_m = \max\{t_k,\ t_k < T,\ k = 0, 1, 2, \cdots\}$, and

$$T_\alpha(t) = E_{\alpha,1}(At^\alpha) = \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}\frac{\lambda^{\alpha-1}}{\lambda^\alpha - A}d\lambda, \quad S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha) = \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}\frac{1}{\lambda^\alpha - A}d\lambda,$$

here the integral contour Γ_θ is oriented counter-clockwise.

In the sequel C denotes a generic positive constant, which may be different from line to line and even in the same line. Next we are in a position to state the main theorem.

Theorem 12. *Assume (H_1)-(H_4) holds true, let $A \in \mathbb{A}^\alpha(\omega_0, \theta_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in \mathbb{R}^+$, and assume that the α-order fractional solution operator $T_\alpha(t)(t > 0)$ and the α-resolvent family $S_\alpha(t)(t > 0)$ are compact. Assume also that there exist two constants p and q which satisfy $\frac{1}{p} + \frac{1}{q} = 1$, where $1 < p < \frac{1}{2(1-\alpha)}$. Then, for every initial value $\phi \in \mathcal{PC}$ and every $T > 0$, the problem (1) has at least one mild solution defined on $(-\infty, T]$.*

Proof. We start the proof by defining an abstract phase space $\mathcal{PC}^T$ as follows, for a fixed $T > 0$,

$$\mathcal{PC}^T = \Big\{x(\cdot,\cdot) : (-\infty, T] \times \Omega \to \mathbb{H} \text{ such that } x(t,\cdot) \text{ is } \mathcal{F}_t\text{-adapted},\ x(t,\cdot) \in L^2(\Omega; \mathbb{H}) \\ \text{for all } t \le T,\ x|_{J_k} \in C(J_k; L^2(\Omega; \mathbb{H})) \text{ and } \sup_{t \in (-\infty, T]} e^{\gamma t} E\|x(t)\|^2 < \infty\Big\},$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k \in \mathbb{N}^+$. Then the abstract space $\mathcal{PC}^T$ endowed with the norm

$$\|x\|_{\mathcal{PC}^T} = \left(\sup_{t \in (-\infty, T]} e^{\gamma t} E\|x(t)\|^2\right)^{\frac{1}{2}}, \qquad x \in \mathcal{PC}^T,$$

is a Banach space. Notice that, when considering $T = 0$, we have $\mathcal{PC}^T = \mathcal{PC}^0 = \mathcal{PC}$.

Now, for $\phi \in \mathcal{PC}$, we define

$$\mathcal{PC}^T_\phi = \{x \in \mathcal{PC}^T :\ x(s) = \phi(s),\ s \le 0\}.$$

It is clear that $\mathcal{PC}_\phi^T$ is a closed subset of $\mathcal{PC}^T$, and consequently, it is a complete metric subspace of $\mathcal{PC}^T$.

In order to simplify our presentation, let us abbreviate $\|l_i\|_\infty$ by $l_i (i = 1, 2)$ according to the fact $l_i(t) \in L^\infty(\mathbb{R}^+, \mathbb{R}^+)$, and we write $\mathcal{PC}_\phi$ instead of $\mathcal{PC}_\phi^T$ for a fixed $T > 0$, when no confusion is possible.

Let us pick $\psi \in \mathcal{PC}_\phi$ and define

$$\begin{cases} x^0(t) = \psi(t), \\ x^n(t) = \chi_{(-\infty,0]}(t)\psi(t) + \sum_k \chi_{(t_k,t_{k+1}]}(t)\Big\{T_\alpha(t-t_k)(I_k(x^{n-1}(t_k^-)) + x^{n-1}(t_k^-)) \\ \quad + \int_{t_k}^t S_\alpha(t-s)f(s,x_s^{n-1})ds + \int_{t_k}^t S_\alpha(t-s)g(s,x_s^{n-1})dB(s) \\ \quad + \int_{t_k}^t S_\alpha(t-s)h(s)dB_Q^H(s)\Big\}, \qquad t \in (-\infty, T],\ k = 0, 1, 2\cdots, \end{cases} \tag{7}$$

where $I_0 = 0$ and $t_k < T$, χ is a characteristic function.

To ensure the existence of mild solutions, we split the proof into several steps.

Step 1. For all $n \in \mathbb{N}^+$, $x^n(\cdot) \in \mathcal{PC}_\phi$.

First, we claim that $x^n(\cdot)$ is $\mathcal{F}_t$-adapted for all $n \in \mathbb{N}^+$. Obviously, $x^0(t) = \psi(t) \in \mathcal{PC}_\phi$ implies that $x^0(t)$ is $\mathcal{F}_t$-adapted for all $t \in (-\infty, T]$. With the help of the continuity of f and I_k, as well as the fact that the limit of measurable functions is a measurable function, it is easy to see that $I_k(x^0(t_k^-))$ and $\int_{t_k}^t f(s, x_s^0)ds$ are $\mathcal{F}_t$-adapted. In addition, the stochastic integral $\int_{t_k}^t g(s, x_s^0)dB(s)$ is $\mathcal{F}_t$-adapted according to Definition 5.3 in [29]. For the last term $\int_{t_k}^t h(s)dB_Q^H(s)$, which is $\mathcal{F}_t$-adapted naturally. In conclusion, thanks to the Picard iterations technique, we obtain that $x^1(t)$ is also $\mathcal{F}_t$-adapted. By induction, $x^n(t)$ is $\mathcal{F}_t$-adapted for all $t \in (-\infty, T]$ and $n \in \mathbb{N}^+$.

Next, we want to prove $x^n(t) \in L^2(\Omega; \mathbb{H})$. Since $x^n(t) = \psi(t)$ on $(-\infty, 0]$, then $x^n(t) = \phi(t)$ for $t \in (-\infty, 0]$, and as $\phi \in \mathcal{PC}$, then we have $x^n(t) \in L^2(\Omega; \mathbb{H})$ for all $t \in (-\infty, 0]$. Furthermore, for every $t \in [0, t_1]$, by (2), (3), (5), (H_1)-(H_4) and Hölder's inequality,

$$\begin{aligned} E\|x^1(t)\|^2 &\le 4M_T^2\|\psi\|_{\mathcal{PC}_\phi}^2 + 8N_T^2(\eta l_1 + Tr(Q)l_2)\|\psi\|_{\mathcal{PC}_\phi}^2\frac{t^{2\alpha-1}}{2\alpha-1} \\ &\quad + 8N_T^2(\eta\sup_{s\in[0,t_1]}\|f(s,0)\|^2 + Tr(Q)\sup_{s\in[0,t_1]}\|g(s,0)\|^2)\frac{t^{2\alpha-1}}{2\alpha-1} \\ &\quad + 4N_T^2cH(2H-1)t^{2H-1}\int_0^t (t-s)^{2\alpha-2}\|h(s)\|_Q^2 ds \\ &:= 4M_T^2\|\psi\|_{\mathcal{PC}_\phi}^2 + C_1\|\psi\|_{\mathcal{PC}_\phi}^2\frac{t^{2\alpha-1}}{2\alpha-1} + C_2\frac{t^{2\alpha-1}}{2\alpha-1} + C_3t^{2H-1}, \end{aligned} \tag{8}$$

where we have used the notation

$$C_1 = 8N_T^2(\eta l_1 + Tr(Q)l_2), \qquad C_2 = 8N_T^2(\eta\sup_{s\in[0,T]}\|f(s,0)\|^2 + Tr(Q)\sup_{s\in[0,T]}\|g(s,0)\|^2),$$

and

$$C_3 \le 4N_T^2cH(2H-1)\left(\frac{\eta^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}\right)^{\frac{1}{p}}\left(\int_0^t \|h(s)\|_Q^{2q}ds\right)^{\frac{1}{q}} < \infty.$$

Notice that, if $t+\theta \in [0,t_1]$ (where $\theta \in (-\infty,0]$), in view of $\gamma > 0$, then it follows that

$$e^{\gamma\theta}E\|x^1(t+\theta)\|^2 \leq 4M_T^2\|\psi\|^2_{\mathcal{PC}_\phi} + C_1\|\psi\|^2_{\mathcal{PC}_\phi}\frac{t^{2\alpha-1}}{2\alpha-1} + C_2\frac{t^{2\alpha-1}}{2\alpha-1} + C_3t^{2H-1}, \tag{9}$$

if $t+\theta < 0$, we have

$$e^{\gamma\theta}E\|x^1(t+\theta)\|^2 \leq e^{-\gamma t}e^{\gamma(t+\theta)}E\|\psi(t+\theta)\|^2 \leq e^{-\gamma t}\|\psi\|^2_{\mathcal{PC}_\phi}. \tag{10}$$

Combining (9) and (10),

$$\begin{aligned}\|x_t^1\|^2_{\mathcal{PC}} &= \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}E\|x^1(t+\theta)\|^2 \\ &\leq \left(\|\psi\|^2_{\mathcal{PC}_\phi} + C_2\frac{\eta^{2\alpha-1}}{2\alpha-1} + C_3\eta^{2H-1}\right)\times\sum_{i=0}^{1}\left(4M_T^2 + C_1\frac{\eta^{2\alpha-1}}{2\alpha-1}\right)^i.\end{aligned}$$

By induction on n, for $t \in [0,t_1]$, we derive that

$$\|x_t^n\|^2_{\mathcal{PC}} \leq \left(\|\psi\|^2_{\mathcal{PC}_\phi} + C_2\frac{\eta^{2\alpha-1}}{2\alpha-1} + C_3\eta^{2H-1}\right)\times\sum_{i=0}^{n}\left(4M_T^2 + C_1\frac{\eta^{2\alpha-1}}{2\alpha-1}\right)^i. \tag{11}$$

For $t \in (t_1, t_2\wedge T]$, where $t_2 \wedge T = \min\{t_2, T\}$, similar to (8). By (2), (3), (5), (H_1)-(H_4) and Hölder's inequality,

$$\begin{aligned}E\|x^1(t)\|^2 &\leq 8M_T^2((b_1+1)\|\psi\|^2_{\mathcal{PC}_\phi} + b_2) + C_1\frac{(t-t_1)^{2\alpha-1}}{2\alpha-1}\|\psi\|^2_{\mathcal{PC}_\phi} \\ &\quad + C_2\frac{(t-t_1)^{2\alpha-1}}{2\alpha-1} + C_3(t-t_1)^{2H-1} \\ &\leq \left(\|\psi\|^2_{\mathcal{PC}_\phi} + 8M_T^2b_2 + C_2\frac{(t-t_1)^{2\alpha-1}}{2\alpha-1} + C_3(t-t_1)^{2H-1}\right) \\ &\quad \times\sum_{i=0}^{1}\left(8M_T^2(b_1+1) + C_1\frac{(t-t_1)^{2\alpha-1}}{2\alpha-1}\right)^i.\end{aligned} \tag{12}$$

Using the same argument as in (9) and (10), together with (11) and (12), for $t \in (t_1, t_2\wedge T]$, we have the following estimate,

$$\|x_t^1\|^2_{\mathcal{PC}} \leq \left(\|\psi\|^2_{\mathcal{PC}_\phi} + 8M_T^2b_2 + C_2\frac{\eta^{2\alpha-1}}{2\alpha-1} + C_3\eta^{2H-1}\right)\times\sum_{i=0}^{1}\left(8M_T^2(b_1+1) + C_1\frac{\eta^{2\alpha-1}}{2\alpha-1}\right)^i.$$

By induction on n, for $t \in (t_1, t_2\wedge T]$, we deduce

$$\|x_t^n\|^2_{\mathcal{PC}} \leq \left(\|\psi\|^2_{\mathcal{PC}_\phi} + 8M_T^2b_2 + C_2\frac{\eta^{2\alpha-1}}{2\alpha-1} + C_3\eta^{2H-1}\right)\times\sum_{i=0}^{n}\left(8M_T^2(b_1+1) + C_1\frac{\eta^{2\alpha-1}}{2\alpha-1}\right)^i. \tag{13}$$

In a similar way, combining (11) with (13), for each fixed $n \in \mathbb{N}^+$, for all $t \in [0,T]$, we find that

$$\|x_t^n\|^2_{\mathcal{PC}} \leq \left(\|\psi\|^2_{\mathcal{PC}_\phi} + 8M_T^2b_2 + C_2\frac{\eta^{2\alpha-1}}{2\alpha-1} + C_3\eta^{2H-1}\right)\times\sum_{i=0}^{n}\left(8M_T^2(b_1+1) + C_1\frac{\eta^{2\alpha-1}}{2\alpha-1}\right)^i < \infty. \tag{14}$$

Taking into account that $E\|x^n(t)\|^2 \leq \|x_t^n\|^2_{\mathcal{PC}}$, then (14) implies that $x^n(t) \in L^2(\Omega;\mathbb{H})$ for all $t \in [0,T]$.

Finally, since $x^0(\cdot) = \psi \in \mathcal{PC}_\phi$, it is easy to see that $x^n(t) = \phi(t)$ on $(-\infty, 0]$. Now we only need to prove that $x^n(\cdot) \in \mathcal{PC}^T$ for all $n \in \mathbb{N}^+$. Let us now check $x^1(\cdot) \in \mathcal{PC}^T$.

Because the proof of the case $k = 0$ is similar, here we assume that $k \geq 1$. To do that, let us consider $\sigma > 0$ small enough, such that for t, $t + \sigma \in (-\infty, T] \cap (t_k, t_{k+1}]$, then

$$\begin{aligned}
&E\|x^1(t+\sigma) - x^1(t)\|^2 \leq 7\|T_\alpha(t+\sigma-t_k) - T_\alpha(t-t_k)\|^2 E\|I_k(x^0(t_k^-)) + x^0(t_k^-)\|^2 \\
&+ 7E\left\|\int_{t_k}^{t} (S_\alpha(t+\sigma-s) - S_\alpha(t-s))f(s,x_s^0)ds\right\|^2 + 7E\left\|\int_{t}^{t+\sigma} S_\alpha(t+\sigma-s)f(s,x_s^0)ds\right\|^2 \\
&+ 7E\left\|\int_{t_k}^{t} (S_\alpha(t+\sigma-s) - S_\alpha(t-s))g(s,x_s^0)dB(s)\right\|^2 + 7E\left\|\int_{t}^{t+\sigma} S_\alpha(t+\sigma-s)g(s,x_s^0)dB(s)\right\|^2 \\
&+ 7E\left\|\int_{t_k}^{t} (S_\alpha(t+\sigma-s) - S_\alpha(t-s))h(s)dB_Q^H(s)\right\|^2 + 7E\left\|\int_{t}^{t+\sigma} S_\alpha(t+\sigma-s)h(s)dB_Q^H(s)\right\|^2 \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned} \tag{15}$$

Due to the fact $T_\alpha(t)$ is compact for $t > 0$, by Lemma 9(1) and condition (H_3), we obtain that

$$I_1 \leq 14\|T_\alpha(t+\sigma-t_k) - T_\alpha(t-t_k)\|^2((b_1+1)\|\psi\|^2_{\mathcal{PC}_\phi} + b_2) \to 0 \text{ as } \sigma \to 0.$$

By (5), (H_1), (H_4), Hölder's inequality, Lebesgue's Theorem, Lemma 9(1) and for a fixed but sufficiently small $\epsilon > 0$,

$$\begin{aligned}
I_2 &\leq 14\eta \int_{t_k}^{t} \|S_\alpha(t+\sigma-s) - S_\alpha(t-s)\|^2 (l_1(s)\|\psi\|^2_{\mathcal{PC}_\phi} + \sup_{s\in(t_k,t_{k+1}\wedge T]} \|f(s,0)\|^2)ds \\
&\leq 14\eta(l_1\|\psi\|^2_{\mathcal{PC}_\phi} + \sup_{s\in(t_k,t_{k+1}\wedge T]} \|f(s,0)\|^2)\bigg(\int_{t_k}^{t-\epsilon} \|S_\alpha(t+\sigma-s) - S_\alpha(t-s)\|^2 ds \\
&\quad + 2\int_{t-\epsilon}^{t} \left(\|S_\alpha(t+\sigma-s)\|^2 + \|S_\alpha(t-s)\|^2\right) ds\bigg) \\
&\leq 14\eta(l_1\|\psi\|^2_{\mathcal{PC}_\phi} + \sup_{s\in(t_k,t_{k+1}\wedge T]} \|f(s,0)\|^2)\bigg(\int_{t_k}^{t-\epsilon} \|S_\alpha(t+\sigma-s) - S_\alpha(t-s)\|^2 ds \\
&\quad + 2N_T^2 \frac{(\sigma+\epsilon)^{2\alpha-1}}{2\alpha-1} + 2N_T^2 \frac{\epsilon^{2\alpha-1}}{2\alpha-1}\bigg).
\end{aligned} \tag{16}$$

Taking now limits when σ goes to zero, we obtain

$$\lim_{\sigma\to 0} I_2 \leq 28N_T^2\eta(l_1\|\psi\|^2_{\mathcal{PC}_\phi} + \sup_{s\in(t_k,t_{k+1}\wedge T]} \|f(s,0)\|^2)\frac{\epsilon^{2\alpha-1}}{2\alpha-1},$$

and as ϵ is arbitrarily small, then $I_2 \to 0$ as $\sigma \to 0$.

It is worth noticing that property (i) of Lemma 9 is not valid for $t = 0$, therefore we split the integral of (16) into two parts to avoid the singularity. A similar argument will be used in I_4 and I_6.

According to (2), (5), (H_1), (H_4), Lebegue's Theorem, Lemma 9 (1) and, for a fixed but sufficiently small $\epsilon > 0$, we have

$$\begin{aligned}
I_4 &\leq 14Tr(Q)\int_{t_k}^{t} \|S_\alpha(t+\sigma-s)-S_\alpha(t-s)\|^2(l_2(s)\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|g(s,0)\|^2)ds\\
&\leq 14Tr(Q)(l_2\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|g(s,0)\|^2)\bigg(\int_{t_k}^{t-\epsilon}\|S_\alpha(t+\sigma-s)-S_\alpha(t-s)\|^2ds\\
&\quad+2\int_{t-\epsilon}^{t}\left(\|S_\alpha(t+\sigma-s)\|^2+\|S_\alpha(t-s)\|^2\right)ds\bigg) \qquad (17)\\
&\leq 14Tr(Q)(l_2\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|g(s,0)\|^2)\bigg(\int_{t_k}^{t-\epsilon}\|S_\alpha(t+\sigma-s)-S_\alpha(t-s)\|^2ds\\
&\quad+2N_T^2\frac{(\sigma+\epsilon)^{2\alpha-1}}{2\alpha-1}+2N_T^2\frac{\epsilon^{2\alpha-1}}{2\alpha-1}\bigg).
\end{aligned}$$

Arguing again as in (16), we deduce that $I_4 \to 0$ as $\sigma \to 0$.

For I_6, there exist two constants p and q which are given in the theorem, by (3), (5), (H_2), (H_4), Lemma 9(1), Lebesgue's Theorem, Hölder's inequality and, once more, for a fixed but sufficiently small $\epsilon > 0$,

$$\begin{aligned}
I_6 &\leq 7cH(2H-1)\eta^{2H-1}\int_{t_k}^{t-\epsilon}\|S_\alpha(t+\sigma-s)-S_\alpha(t-s)\|^2\|h(s)\|^2_Q ds\\
&\quad+14cH(2H-1)\epsilon^{2H-1}\int_{t-\epsilon}^{t}(\|S_\alpha(t+\sigma-s)\|^2+\|S_\alpha(t-s)\|^2)\|h(s)\|^2_Q ds\\
&\leq 7cH(2H-1)\eta^{2H-1}\int_{t_k}^{t-\epsilon}\|S_\alpha(t+\sigma-s)-S_\alpha(t-s)\|^2\|h(s)\|^2_Q ds\\
&\quad+14N_T^2cH(2H-1)\epsilon^{2H-1}\left(\left(\int_{t-\epsilon}^{t}(t+\sigma-s)^{2(\alpha-1)p}ds\right)^{\frac{1}{p}}\right.\\
&\quad\left.+\left(\int_{t-\epsilon}^{t}(t-s)^{2(\alpha-1)p}ds\right)^{\frac{1}{p}}\right)\times\left(\int_{t-\epsilon}^{t}\|h(s)\|^{2q}_Q ds\right)^{\frac{1}{q}}\\
&\leq 7cH(2H-1)\eta^{2H-1}\int_{t_k}^{t-\epsilon}\|S_\alpha(t+\sigma-s)-S_\alpha(t-s)\|^2\|h(s)\|^2_Q ds\\
&\quad+14N_T^2cH(2H-1)\epsilon^{2H-1}\left(\left(\frac{(\sigma+\epsilon)^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}\right)^{\frac{1}{p}}+\left(\frac{\epsilon^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}\right)^{\frac{1}{p}}\right)\times\left(\int_{t-\epsilon}^{t}\|h(s)\|^{2q}_Q ds\right)^{\frac{1}{q}},
\end{aligned} \qquad (18)$$

and, as in the previous cases, we deduce $I_6 \to 0$ as $\sigma \to 0$.

For I_3, by (5), (H_1) and Hölder's inequality, we find that

$$\begin{aligned}
I_3 &\leq 14\sigma N_T^2\int_{t}^{t+\sigma}(t+\sigma-s)^{2\alpha-2}(l_1(s)\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|f(s,0)\|^2)ds\\
&\leq 14\sigma N_T^2(l_1\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|f(s,0)\|^2)\frac{\sigma^{2\alpha-1}}{2\alpha-1}\to 0 \quad \text{as} \quad \sigma\to 0.
\end{aligned} \qquad (19)$$

As for I_5, from (2), (5) and (H_1), we deduce

$$\begin{aligned} I_5 &\leq 14Tr(Q)N_T^2\int_t^{t+\sigma}(t+\sigma-s)^{2\alpha-2}(l_2(s)\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|g(s,0)\|^2)ds \\ &\leq 14Tr(Q)N_T^2(l_2\|\psi\|^2_{\mathcal{PC}_\phi}+\sup_{s\in(t_k,t_{k+1}\wedge T]}\|g(s,0)\|^2)\frac{\sigma^{2\alpha-1}}{2\alpha-1}\to 0 \quad \text{as} \quad \sigma\to 0. \end{aligned} \tag{20}$$

Finally, for I_7, by (3), (5), (H_2) and Hölder's inequality, taking the same constants p, q as for I_6, we obtain that

$$\begin{aligned} I_7 &\leq 7N_T^2cH(2H-1)\sigma^{2H-1}\left(\int_t^{t+\sigma}(t+\sigma-s)^{2(\alpha-1)p}ds\right)^{\frac{1}{p}}\left(\int_t^{t+\sigma}\|h(s)\|_Q^{2q}ds\right)^{\frac{1}{q}} \\ &\leq 7N_T^2cH(2H-1)\sigma^{2H-1}\left(\frac{\sigma^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}\right)^{\frac{1}{p}}\left(\int_t^{t+\sigma}\|h(s)\|_Q^{2q}ds\right)^{\frac{1}{q}}\to 0 \quad \text{as} \quad \sigma\to 0. \end{aligned} \tag{21}$$

Therefore, $E\|x^1(t+\sigma)-x^1(t)\|^2$ tends to zero as $\sigma\to 0$, which implies that $x^1(\cdot)\in C((-\infty,T]\cap(t_k,t_{k+1}];L^2(\Omega;\mathbb{H}))$. An induction argument shows $x^n(\cdot)\in C((-\infty,T]\cap(t_k,t_{k+1}];L^2(\Omega;\mathbb{H}))$ for $n\in\mathbb{N}^+$.

In conclusion, for all $n\in\mathbb{N}^+$, the assertion $x^n(\cdot)\in\mathcal{PC}_\phi$ holds true.

Step 2. We now show that $\{x^n\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in $\mathcal{PC}_\phi$.

By the construction of successive approximations, it is a straighforward consequence that $x^n(t)=x^{n-1}(t)$ on $(-\infty,0]$. On the other hand, for $t\in[0,t_1]$ and $n\geq 1$, by (2), (5), (H_1), (H_4) and Hölder's inequality, we have for some $0<p<1$,

$$\begin{aligned} E\|x^{n+1}(t)-x^n(t)\|^2 &\leq 2\eta\int_0^t\|S_\alpha(t-s)\|^2E\|f(s,x_s^n)-f(s,x_s^{n-1})\|^2ds \\ &+2Tr(Q)\int_0^t\|S_\alpha(t-s)\|^2E\|g(s,x_s^n)-g(s,x_s^{n-1})\|^2ds \\ &\leq 2N_T^2(\eta l_1+Tr(Q)l_2)\left(\int_0^t(t-s)^{\frac{2(\alpha-1)}{1-p}}ds\right)^{1-p}\left(\int_0^t\|x_s^n-x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}}ds\right)^p. \end{aligned} \tag{22}$$

Consequently, for all $t\in[0,t_1]$, we have

$$\sup_{t\in[0,t_1]}e^{\gamma t}E\|x^{n+1}(t)-x^n(t)\|^2\leq e^{\gamma T}C_4\left(\int_0^{t_1}\|x_s^n-x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}}ds\right)^p, \tag{23}$$

where we have used the notation

$$C_4=2N_T^2(\eta l_1+Tr(Q)l_2)\left(\frac{1-p}{2\alpha-1-p}\right)^{1-p}\eta^{2\alpha-1-p}.$$

Hence,

$$\left(\sup_{t\in[0,t_1]}e^{\gamma t}E\|x^{n+1}(t)-x^n(t)\|^2\right)^{\frac{1}{p}}\leq e^{\frac{\gamma T}{p}}C_4^{\frac{1}{p}}\int_0^{t_1}\|x_s^n-x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}}ds. \tag{24}$$

By repeating iterations of (24), for all $n\in\mathbb{N}^+$,

$$\left(\sup_{t\in[0,t_1]}e^{\gamma t}E\|x^{n+1}(t)-x^n(t)\|^2\right)^{\frac{1}{p}}\leq\frac{(e^{\frac{\gamma T}{p}}C_4^{\frac{1}{p}}\eta)^n}{n!}\times\|x_t^1-x_t^0\|_{\mathcal{PC}}^{\frac{2}{p}}. \tag{25}$$

Using a similar argument to the one we used with (8), for all $t \in [0, t_1]$, we have

$$\begin{aligned} \|x_t^1 - x_t^0\|_{\mathcal{PC}}^2 \leq \sup_{t\in[0,t_1]} E\|x^1(t) - x^0(t)\|^2 \leq 8(M_T^2+1)\|\psi\|_{\mathcal{PC}_\phi} \\ + C_1\|\psi\|_{\mathcal{PC}_\phi}^2 \frac{t_1^{2\alpha-1}}{2\alpha-1} + C_2\frac{t_1^{2\alpha-1}}{2\alpha-1} + C_3 t_1^{2H-1} := C_5. \end{aligned} \tag{26}$$

Replacing (26) into (25), for all $t \in [0, t_1]$,

$$\sup_{t\in[0,t_1]} e^{\gamma t} E\|x^{n+1}(t) - x^n(t)\|^2 \leq C_5 \frac{(e^{\gamma T} C_4 t_1^p)^n}{(n!)^p} \leq C\frac{(e^{\gamma T} C_4 t_1^p)^n}{(n!)^p}. \tag{27}$$

For $t \in (t_1, t_2 \wedge T]$, on the one hand, similar to (22), by (2), (5), (H_1)-(H_4), (27) and Hölder's inequality, we obtain

$$\begin{aligned} E\|x^{n+1}(t) - x^n(t)\|^2 &\leq 6M_T^2(N+1)E\|x^n(t_1^-) - x^{n-1}(t_1^-)\|^2 + C_4\left(\int_{t_1}^t \|x_s^n - x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}} ds\right)^p \\ &\leq 6CM_T^2(N+1)\frac{(e^{\gamma T}C_4 t_1^p)^{n-1}}{((n-1)!)^p} + C_4\left(\int_{t_1}^t \|x_s^n - x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}} ds\right)^p, \end{aligned}$$

which implies that

$$\left(\sup_{t\in(t_1,t_2\wedge T]} e^{\gamma t} E\|x^{n+1}(t) - x^n(t)\|^2\right)^{\frac{1}{p}} \leq e^{\frac{\gamma T}{p}} C_6^{\frac{1}{p}} \frac{(e^{\gamma T} C_4^{\frac{1}{p}} t_1)^{n-1}}{(n-1)!} + e^{\frac{\gamma T}{p}} C_7^{\frac{1}{p}} \int_{t_1}^{t_2\wedge T} \|x_s^n - x_s^{n-1}\|_{\mathcal{PC}}^{\frac{2}{p}} ds, \tag{28}$$

where we have used the notation

$$C_6 = 2^{1-p}(6CM_T^2(N+1)), \qquad C_7 = 2^{1-p}C_4.$$

On the other hand, for $t \in (t_1, t_2 \wedge T]$,

$$\begin{aligned} \|x_t^1 - x_t^0\|_{\mathcal{PC}}^2 \leq \sup_{t\in(t_1,t_2\wedge T]} E\|x^1(t) - x^0(t)\|^2 \leq (16M_T^2 b_1 + 16M_T^2 + 8)\|\phi\|_{\mathcal{PC}_\phi}^2 + 16M_T^2 b_2 \\ + C_1\|\psi\|_{\mathcal{PC}_\phi}^2 \frac{((t_2\wedge T) - t_1)^{2\alpha-1}}{2\alpha-1} + C_2\frac{((t_2\wedge T) - t_1)^{2\alpha-1}}{2\alpha-1} + C_3((t_2\wedge T) - t_1)^{2H-1}. \end{aligned}$$

Combining this with (28), by induction on n, we obtain

$$\begin{aligned} \sup_{t\in(t_1,t_2\wedge T]} e^{\gamma t} E\|x^{n+1}(t) - x^n(t)\|^2 &\leq C\frac{n^p(e^{\gamma T}C(t_2\wedge T)^p)^{n-1}}{((n-1)!)^p} + C\frac{(e^{\gamma T}C(t_2\wedge T)^p)^n}{(n!)^p} \\ &\leq C\frac{(e^{\gamma T}C(t_2\wedge T)^p)^n}{(n!)^p}. \end{aligned} \tag{29}$$

By repeating this procedure and induction, combining (29) and (27), for all $n \in \mathbb{N}^+$, $t \in [0, T]$, we deduce that

$$\|x^{n+1} - x^n\|_{\mathcal{PC}^T}^2 \leq C\frac{(e^{\gamma T}CT^p)^n}{(n!)^p}. \tag{30}$$

Therefore, for any $0 < n < m$, we deduce

$$\|x^m - x^n\|_{\mathcal{PC}^T} \leq \sum_{r=n}^{m-1}\left(C\frac{(e^{\gamma T}CT^p)^r}{(r!)^p}\right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n\to\infty, \tag{31}$$

which implies that $\{x^n(\cdot)\}_{n\in\mathbb{N}^+}$ is a Cauchy sequence in $\mathcal{PC}^T$. Therefore, the sequence $x^n(\cdot)$ possesses a limit function denoted by $\hat{x}(\cdot)$, such that, $x^n(\cdot) \to \hat{x}(\cdot)$ in $\mathcal{PC}^T$ as $n \to \infty$. Let us define

$$x(t,\omega) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \hat{x}(t,\omega), & t \in [0,T]. \end{cases}$$

It is easy to see that $x(t)$ is $\mathcal{F}_t$-adapted, $x^n(\cdot) \to x(\cdot)$ in $\mathcal{PC}_\phi$ as $n \to \infty$.

Step 3. We check the limit function x of the sequence $\{x^n\}_{n\in\mathbb{N}}$ is a solution of (1).

Taking into account condition (H_3), for every $k \geq 1$,

$$E\|I_k(x^{n-1}(t_k^-)) - I_k(x(t_k^-))\|^2 \leq NE\|x^{n-1}(t_k^-) - x(t_k^-)\|^2 \to 0 \text{ as } n \to \infty.$$

Moreover, in view of (2), (5), (H_1) and (H_4), when $n \to \infty$ we have for $t \in (t_k, t_{k+1}]$,

$$E\left\|\int_{t_k}^t S_\alpha(t-s)(f(s,x_s^{n-1}) - f(s,x_s))ds\right\|^2 \leq \eta l_1 N_T^2 \int_{t_k}^t (t-s)^{2\alpha-2}\|x_s^{n-1} - x_s\|^2_{\mathcal{PC}} ds \to 0,$$

and

$$E\left\|\int_{t_k}^t S_\alpha(t-s)(g(s,x_s^{n-1}) - g(s,x_s))dB(s)\right\|^2 \leq Tr(Q)l_2 N_T^2 \int_{t_k}^t (t-s)^{2\alpha-2}\|x_s^{n-1} - x_s\|^2_{\mathcal{PC}} ds \to 0.$$

Therefore, x is a solution of problem (1). This completes the proof. □

Remark 13. *Note that the main aim of this paper is to investigate a class of fractional impulsive stochastic differential equations, therefore, it is more reasonable to derive the real-value result of Theorem 12 with $T > t_1$, which ensures there is at least one impulse taking place on $[0,T]$.*

In what follows, a general result on the continuous dependence of mild solutions on initial value will be proved. In particular, we obtain the uniqueness of mild solutions to problem (1) by means of the conclusion below.

Theorem 14. *Under assumptions of Theorem 12, the mild solution of (1) is continuous with respect to the initial value $\phi \in \mathcal{PC}$. In particular, if $x(t)$, $y(t)$ are the corresponding mild solutions, on the interval $(-\infty, T]$, to the initial data ϕ and φ, then the following estimate holds,*

$$\|x_t - y_t\|^2_{\mathcal{PC}} \leq 2^{1-p}(3M_T^2 + 1)\|\phi - \varphi\|^2_{\mathcal{PC}} e^{(A_2 p + \frac{\ln A_3}{\beta})t}, \qquad \forall t \in [0,T], \tag{32}$$

where A_2 and A_3 are constants depending on η.

Proof. It follows from (4) that, for any $t \in [0, t_1]$, in view of (H_1), (2), (5) and Hölder's inequality, arguing as in the proof of (23),

$$\begin{aligned} E\|x(t) - y(t)\|^2 &\leq 3M_T^2 E\|\phi(0) - \varphi(0)\|^2 + 3N_T^2(l_1\eta + Tr(Q)l_2)\int_0^t (t-s)^{2\alpha-2}\|x_s - y_s\|^2_{\mathcal{PC}} ds \\ &\leq 3M_T^2\|\phi - \varphi\|^2_{\mathcal{PC}} + 3N_T^2(l_1\eta + Tr(Q)l_2)\left(\int_0^t (t-s)^{\frac{2(\alpha-1)}{1-p}} ds\right)^{1-p}\left(\int_0^t \|x_s - y_s\|^{\frac{2}{p}}_{\mathcal{PC}} ds\right)^p \\ &\leq 3M_T^2\|\phi - \varphi\|^2_{\mathcal{PC}} + A_1\left(\int_0^t \|x_s - y_s\|^{\frac{2}{p}}_{\mathcal{PC}} ds\right)^p, \end{aligned} \tag{33}$$

where we have used the notation

$$A_1 = 3N_T^2(l_1\eta + Tr(Q)l_2)\left(\frac{1-p}{2\alpha - p - 1}\right)^{1-p}\eta^{2\alpha-p-1}.$$

Observe that $\gamma > 0$, if $t+\theta \in [0,t_1]$ (where $\theta \in (-\infty,0]$), then

$$e^{\gamma\theta}E\|x(t+\theta)-y(t+\theta)\|^2 \le 3M_T^2\|\phi-\varphi\|_{\mathcal{PC}}^2 + A_1\bigg(\int_0^{t+\theta}\|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}}ds\bigg)^p,$$

on the other hand, if $t+\theta<0$, it follows

$$e^{\gamma\theta}E\|x(t+\theta)-y(t+\theta)\|^2 = e^{-\gamma t}e^{\gamma(t+\theta)}E\|\phi(t+\theta)-\varphi(t+\theta)\|^2 \le e^{-\gamma t}\|\phi-\varphi\|_{\mathcal{PC}}^2.$$

Therefore,

$$\begin{aligned}\|x_t-y_t\|_{\mathcal{PC}}^2 &= \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}E\|x(t+\theta)-y(t+\theta)\|^2\\ &\le (3M_T^2+1)\|\phi-\varphi\|_{\mathcal{PC}}^2 + A_1\bigg(\int_0^t\|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}}ds\bigg)^p,\end{aligned}$$

and

$$\|x_t-y_t\|_{\mathcal{PC}}^{\frac{2}{p}} \le 2^{\frac{1-p}{p}}(3M_T^2+1)^{\frac{1}{p}}\|\phi-\varphi\|_{\mathcal{PC}}^{\frac{2}{p}} + A_2\int_0^t\|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}}ds,$$

where

$$A_2 = 2^{\frac{1-p}{p}}A_1^{\frac{1}{p}}.$$

As a consequence of Gronwall's inequality, for $t\in[0,t_1]$,

$$\|x_t-y_t\|_{\mathcal{PC}}^2 \le 2^{1-p}(3M_T^2+1)\|\phi-\varphi\|_{\mathcal{PC}}^2 e^{A_2pt}, \tag{34}$$

and thus

$$E\|x(t_1)-y(t_1)\|^2 \le 2^{1-p}(3M_T^2+1)\|\phi-\varphi\|_{\mathcal{PC}}^2 e^{A_2pt_1} = B_1. \tag{35}$$

For $t\in(t_1,t_2\wedge T]$, similar to (33) and (34), by (H_1), (H_5), (2), (5) and Hölder's inequality, we find for $t+\theta>t_1$ ($\theta\in(-\infty,0]$) that

$$\begin{aligned}E\|x(t)-y(t)\|^2 &\le 3M_T^2E\|x(t_1^-)-y(t_1^-)+I_1(x(t_1^-))-I_1(y(t_1^-))\|^2\\ &\quad + 3N_T^2(l_1\eta+Tr(Q)l_2)\bigg(\int_{t_1}^t(t-s)^{\frac{2(\alpha-1)}{1-p}}ds\bigg)^{1-p}\bigg(\int_{t_1}^t\|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}}ds\bigg)^p\\ &\le 6M_T^2(N+1)E\|x(t_1^-)-y(t_1^-)\|^2 + A_1\bigg(\int_{t_1}^t\|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}}ds\bigg)^p.\end{aligned} \tag{36}$$

Replacing t by $t+\theta$ in (36), in view of $\gamma>0$, if $t+\theta\in(t_1,t_2\wedge T]$ ($\theta\in(-\infty,0]$), we have

$$e^{\gamma\theta}E\|x(t+\theta)-y(t+\theta)\|^2 \le 6M_T^2(N+1)E\|x(t_1^-)-y(t_1^-)\|^2 + A_1\bigg(\int_{t_1}^{t+\theta}\|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}}ds\bigg)^p. \tag{37}$$

It follows from (34) and (35) that for $t\in(t_1,t_2\wedge T]$ and $t+\theta<t_1$,

$$\begin{aligned}e^{\gamma\theta}E\|x(t+\theta)-y(t+\theta)\|^2 &\le 2^{1-p}(3M_T^2+1)\|\phi-\varphi\|_{\mathcal{PC}}^2e^{A_2p(t+\theta)}\\ &= 2^{1-p}(3M_T^2+1)\|\phi-\varphi\|_{\mathcal{PC}}^2e^{A_2p((t+\theta-t_1)+t_1)} \le B_1e^{A_2p\eta}.\end{aligned} \tag{38}$$

Combining (37) and (38),

$$\|x_t-y_t\|_{\mathcal{PC}}^{\frac{2}{p}} \le 2^{\frac{1-p}{p}}(6M_T^2(N+1)+e^{A_2p\eta})^{\frac{1}{p}}B_1^{\frac{1}{p}} + A_2\int_{t_1}^t\|x_s-y_s\|_{\mathcal{PC}}^{\frac{2}{p}}ds. \tag{39}$$

Applying Gronwall's inequality ([14, 17, 25, 26]), we derive for $t \in (t_1, t_2 \wedge T]$,

$$\|x_t - y_t\|^2_{\mathcal{PC}} \le 2^{1-p}(6M_T^2(N+1) + e^{A_2 p\eta})B_1 e^{A_2 p(t-t_1)}. \tag{40}$$

If $T < t_2$, this shows the assertion. We assume that $T > t_2$, and consequently,

$$E\|x(t_2) - y(t_2)\|^2 \le 2^{1-p}(6M_T^2(N+1) + e^{A_2 p\eta})B_1 e^{A_2 p(t_2-t_1)} = B_2. \tag{41}$$

Arguing similarly, we find that for $t \in (t_k, t_{k+1} \wedge T]$ with $k \ge 2$,

$$\|x_t - y_t\|^2_{\mathcal{PC}} \le 2^{1-p}(6M_T^2(N+1) + e^{A_2 p\eta})B_k e^{A_2 p(t-t_k)}, \tag{42}$$

we may as well consider the case $t_{k+1} < T$, because the estimate (32) holds true when $t_{k+1} > T$ with $k \ge 2$. Thus

$$E\|x(t_{k+1}) - y(t_{k+1})\|^2 \le 2^{1-p}(6M_T^2(N+1) + e^{A_2 p\eta})B_k e^{A_2 p(t_{k+1}-t_k)} = B_{k+1}. \tag{43}$$

For the sake of convenience, let $A_3 = 2^{1-p}(6M_T^2(N+1) + e^{A_2 p\eta})$, we deduce the following result by mathematical induction for $k \ge 2$,

$$B_k \le A_3 B_{k-1} e^{A_2 p(t_k - t_{k-1})} \le A_3^{k-1} B_1 e^{A_2 p(t_k - t_1)}. \tag{44}$$

Hence, it follows from (42) and (44), and for $t \in (t_k, t_{k+1} \wedge T]$,

$$\|x_t - y_t\|^2_{\mathcal{PC}} \le A_3 B_k e^{A_2 p(t-t_k)} \le A_3^k B_1 e^{A_2 p(t-t_1)}. \tag{45}$$

In view of the fact that condition (H_4) implies that $k\beta < t < (k+1)\eta$ for $t \in (t_k, t_{k+1}]$, taking into account (45), (34) and (40), we deduce for all $t \in (0, T]$,

$$\begin{aligned}\|x_t - y_t\|^2_{\mathcal{PC}} &\le 2^{1-p}(3M_T^2 + 1)\|\phi - \varphi\|^2_{\mathcal{PC}} e^{A_2 p t} e^{k \ln A_3} \\ &\le 2^{1-p}(3M_T^2 + 1)\|\phi - \varphi\|^2_{\mathcal{PC}} e^{(A_2 p + \frac{\ln A_3}{\beta})t}.\end{aligned} \tag{46}$$

The proof is finished. □

4. **Global existence and asymptotic behavior of mild solutions.** In this section, we first prove the global existence of mild solutions to stochastic impulsive differential equations before studying the exponential asymptotic behavior of mild solutions.

To start off we state some conditions which will be imposed later.

(C_1) The closed and linear sectorial operator $A \in \mathbb{A}^\alpha(\omega_0, \theta_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in \mathbb{R}$, generates an α-order fractional solution operator $T_\alpha(t)$ and an α-resolvent family $S_\alpha(t)$, on the separable Hilbert space $\mathbb{H}$ with

$$\|T_\alpha(t)\| \le Me^{-\mu t}, \quad \|S_\alpha(t)\| \le Me^{-\mu t}(1 + t^{\alpha-1}),\ \forall t > 0,\ \text{where } M \ge 1,\ Re(\mu) \in \mathbb{R}^+.$$

(C_2) There exist two positive constants C_f and C_g, such that, for any $x, y \in \mathcal{PC}$ and for all $t > 0$,

$$E\|f(t,x) - f(t,y)\|^2 \le C_f\|x - y\|^2_{\mathcal{PC}},\ E\|g(t,x) - g(t,y)\|^2 \le C_g\|x - y\|^2_{\mathcal{PC}},$$

and

$$\int_0^\infty e^{2\mu q s}\|f(s,0)\|^{2q} ds < \infty,\ \int_0^\infty e^{2\mu q s}\|g(s,0)\|^{2q} ds < \infty,$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(C_3) In addition to assumption (H_2), also suppose that

$$\int_0^\infty e^{2\mu q s}\|h(s)\|^{2q}_Q ds < \infty.$$

(C_4) Under condition (H_3), we impose an additional assumption on I_k, that is, for all $k \in \mathbb{N}^+$,

$$E\|I_k(x)\|^2 \leq b_1 E\|x\|^2, \text{ for all } x \in L^2(\Omega;\mathbb{H}).$$

4.1. **The global existence of mild solutions in $\mathcal{PC}$.** Now we state the global existence of mild solution to the problem (1).

Theorem 15. *Assume the conditions of Theorem 12 and (C_1). Then for every initial value $\phi \in \mathcal{PC}$, there exists a unique solution to problem (1) in the sense of Definition 11, defined on $[0,\infty)$.*

Proof. We derive the local existence and uniqueness of mild solution to (1) by means of the estimates (5) in Theorem 12, where M_T and N_T are constants depending on T which is finite. In order to extend the results to $[0,\infty)$, the constants of the estimates (5) must be independent of T. For this aim, we modify the estimates (5) slightly by condition (C_1), that is,

$$M^{'} = \sup_{t\in[0,\infty)} \|T_\alpha(t)\| < \infty \text{ and } N^{'} = \sup_{t\in[0,\infty)} Me^{-\mu t}(1+t^{1-\alpha}) < \infty.$$

Replacing M_T and N_T by $M^{'}$ and $N^{'}$, separately, in Theorem 12 and Theorem 14, the results still hold true. Now, we are ready to prove the global existence and uniqueness of mild solutions to (1).

By Theorem 12 and Theorem 14, we deduce that there exists a unique solution $x^{(1)}(t)$ to the initial value problem (1.1) such that

$$x^{(1)}(t) = \begin{cases} \phi(t), \qquad t \in (-\infty, 0], \\ T_\alpha(t)\phi(0) + \int_0^t S_\alpha(t-s)f(s,x_s^{(1)})ds + \int_0^t S_\alpha(t-s)g(s,x_s^{(1)})dB(s) \\ \quad + \int_0^t S_\alpha(t-s)h(s)dB_Q^H(s), \qquad t \in [0,t_1]. \end{cases}$$

Arguing as in the proof of Theorem 12, we derive the existence of $x^{(2)}(t)$ satisfying

$$x^{(2)}(t) = \begin{cases} x^{(1)}(t), \qquad t \in (-\infty, t_1], \\ T_\alpha(t-t_1)(I_1(x^{(1)}(t_1^-)) + x^{(1)}(t_1^-)) + \int_{t_1}^t S_\alpha(t-s)f(s,x_s^{(2)})ds \\ \quad + \int_{t_1}^t S_\alpha(t-s)g(s,x_s^{(2)})dB(s) + \int_{t_1}^t S_\alpha(t-s)h(s)dB_Q^H(s), \quad t \in (t_1,t_2]. \end{cases}$$

Continuing the procedure in this way, we obtain a unique global solution to problem (1) in the sense of Definition 11. This completes the proof. □

4.2. **Exponential decay of mild solutions in $\mathcal{PC}$.** Motivated by the work of Caraballo et al. in [9], we turn our attention in this subsection to prove the exponential asymptotic behavior of the mild solutions to problem (1) in $\mathcal{PC}$.

Theorem 16. *Let conditions (H_4) and $(C_1)-(C_4)$ hold, assume that*

$$\gamma > 2\mu,$$

and

$$2\mu q - L - \frac{\ln w_1}{\beta} > 0, \tag{47}$$

where the positive constants L and w_1 will be explicitly written in the proof. Then, every mild solution $x(\cdot)$ of system (1) with the initial value $x_0 = \phi \in \mathcal{PC}$ satisfies

$$\|x_t\|_{\mathcal{PC}}^{2q} \leq C(1 + \|\phi\|_{\mathcal{PC}}^{2q})e^{-(2\mu q - L - \frac{\ln w_1}{\beta})t}, \quad \forall t \geq 0. \tag{48}$$

Proof. We split the proof into three steps.

Step 1. By Definition 11, (2), (3), (H_4) and (C_1)-(C_3), we obtain for $t \in [0, t_1]$,

$$\begin{aligned}
E\|x(t)\|^2 &\leq 4M^2e^{-2\mu t}\|\phi\|_{\mathcal{PC}}^2 + 16\eta M^2 e^{-2\mu t}\int_0^t (1 + (t-s)^{2\alpha-2})e^{2\mu s}(C_f\|x_s\|_{\mathcal{PC}}^2 + \|f(s,0)\|^2)ds \\
&+ 16Tr(Q)M^2e^{-2\mu t}\int_0^t (1 + (t-s)^{2\alpha-2})e^{2\mu s}(C_g\|x_s\|_{\mathcal{PC}}^2 + \|g(s,0)\|^2)ds \\
&+ 8cH(2H-1)\eta^{2H-1}M^2e^{-2\mu t}\int_0^t (1 + (t-s)^{2\alpha-2})e^{2\mu s}\|h(s)\|_Q^2 ds \\
&= 4M^2e^{-2\mu t}\|\phi\|_{\mathcal{PC}}^2 + 16M^2e^{-2\mu t}(\eta C_f + Tr(Q)C_g)\int_0^t (1 + (t-s)^{2\alpha-2})e^{2\mu s}\|x_s\|_{\mathcal{PC}}^2 ds \\
&+ 16M^2e^{-2\mu t}\int_0^t (1 + (t-s)^{2\alpha-2})e^{2\mu s}(\eta\|f(s,0)\|^2 + Tr(Q)\|g(s,0)\|^2)ds \\
&+ 8cH(2H-1)\eta^{2H-1}M^2e^{-2\mu t}\int_0^t (1 + (t-s)^{2\alpha-2})e^{2\mu s}\|h(s)\|_Q^2 ds.
\end{aligned} \tag{49}$$

In view of condition (C_3), which ensures the existence of a positive constant L_1, using Hölder's inequality, we derive

$$\begin{aligned}
&cH(2H-1)\eta^{2H-1}\int_0^t (1 + (t-s)^{2\alpha-2})e^{2\mu s}\|h(s)\|_Q^2 ds \\
&\leq cH(2H-1)\eta^{2H-1}\left(\int_0^t (1 + (t-s)^{2\alpha-2})^p ds\right)^{\frac{1}{p}}\left(\int_0^t e^{2\mu q s}\|h(s)\|_Q^{2q} ds\right)^{\frac{1}{q}} \leq L_1.
\end{aligned} \tag{50}$$

Observe that condition (C_2) ensures the existence of a positive constant L_2, by Hölder's inequality, one has

$$\begin{aligned}
&\int_0^t (1 + (t-s)^{2\alpha-2})e^{2\mu s}\|f(s,0)\|^2 ds \\
&\qquad \leq \left(\int_0^\eta (1 + (t-s)^{2\alpha-2})^p ds\right)^{\frac{1}{p}}\left(\int_0^\infty e^{2\mu q s}\|f(s,0)\|^{2q} ds\right)^{\frac{1}{q}} \leq L_2.
\end{aligned} \tag{51}$$

Analogously, one can prove that the following estimate holds true, thanks to condition (C_2), with a positive constant L_3,

$$\begin{aligned}
&\int_0^t (1 + (t-s)^{2\alpha-2})e^{2\mu s}\|g(s,0)\|^2 ds \\
&\qquad \leq \left(\int_0^\eta (1 + (t-s)^{2\alpha-2})^p ds\right)^{\frac{1}{p}}\left(\int_0^\infty e^{2\mu q s}\|g(s,0)\|^{2q} ds\right)^{\frac{1}{q}} \leq L_3.
\end{aligned} \tag{52}$$

Replacing (50)-(52) into (49), by Hölder's inequality,

$$\begin{aligned} E\|x(t)\|^2 &\le 4M^2e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}} + 16M^2e^{-2\mu t}(\eta L_2 + Tr(Q)L_3) + 8M^2e^{-2\mu t}L_1 \\ &\quad + 16M^2e^{-2\mu t}(\eta C_f + Tr(Q)C_g)\left(\int_0^t (1+(t-s)^{2\alpha-2})^p ds\right)^{\frac{1}{p}}\left(\int_0^t e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}}ds\right)^{\frac{1}{q}} \\ &\le 4M^2e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}} + 16M^2e^{-2\mu t}L_4 + 16M^2e^{-2\mu t}L_5\left(\int_0^t e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}}ds\right)^{\frac{1}{q}}, \end{aligned} \tag{53}$$

where we have used the notation

$$L_4 = \frac{L_1}{2} + \eta L_2 + Tr(Q)L_3,$$

and

$$L_5 \le 2^{\frac{p-1}{p}}(\eta C_f + Tr(Q)C_g)\left(\eta + \frac{\eta^{2(\alpha-1)p+1}}{2(\alpha-1)p+1}\right)^{\frac{1}{p}} < \infty.$$

Note that $e^{(\gamma-2\mu)\theta} < 1$ (for $\theta < 0$) since $\gamma > 2\mu$. For $t+\theta \in [0, t_1]$, multiplying by $e^{\gamma\theta}$ and replacing t by $t+\theta$ in (53), one has

$$\begin{aligned} e^{\gamma\theta}E\|x(t+\theta)\|^2 &\le 4M^2e^{-2\mu(t+\theta)}e^{\gamma\theta}\|\phi\|^2_{\mathcal{PC}} + 16M^2e^{-2\mu(t+\theta)}e^{\gamma\theta}L_4 \\ &\quad + 16M^2e^{-2\mu(t+\theta)}e^{\gamma\theta}L_5\left(\int_0^{t+\theta} e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}}ds\right)^{\frac{1}{q}} \\ &\le 4M^2e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}} + 16M^2e^{-2\mu t}L_4 + 16M^2e^{-2\mu t}L_5\left(\int_0^t e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}}ds\right)^{\frac{1}{q}}, \end{aligned} \tag{54}$$

and

$$e^{\gamma\theta}E\|x(t+\theta)\|^2 = e^{-\gamma t}e^{\gamma(t+\theta)}E\|\phi(t+\theta)\|^2 \le e^{-\gamma t}\|\phi\|^2_{\mathcal{PC}} \le e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}}, \text{ if } t+\theta < 0. \tag{55}$$

Therefore, for $t \in [0, t_1]$, (54) and (55) lead to

$$\|x_t\|^2_{\mathcal{PC}} \le 4M^2e^{-2\mu t}\|\phi\|^2_{\mathcal{PC}} + 16M^2e^{-2\mu t}L_4 + 16M^2e^{-2\mu t}L_5\left(\int_0^t e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}}ds\right)^{\frac{1}{q}}, \tag{56}$$

thus

$$\begin{aligned} \|x_t\|^{2q}_{\mathcal{PC}} &\le 3^{q-1}4^qM^{2q}e^{-2\mu q t}\|\phi\|^{2q}_{\mathcal{PC}} + 3^{q-1}4^{2q}M^{2q}e^{-2\mu q t}L_4^q \\ &\quad + 3^{q-1}4^{2q}M^{2q}e^{-2\mu q t}L_5^q\int_0^t e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}}ds. \end{aligned} \tag{57}$$

Multiplying both sides of (57) by $e^{2\mu q t}$,

$$e^{2\mu q t}\|x_t\|^{2q}_{\mathcal{PC}} \le 3^{q-1}4^qM^{2q}\|\phi\|^{2q}_{\mathcal{PC}} + 3^{q-1}4^{2q}M^{2q}L_4^q + 3^{q-1}4^{2q}M^{2q}L_5^q\int_0^t e^{2\mu q s}\|x_s\|^{2q}_{\mathcal{PC}}ds. \tag{58}$$

Gronwall's inequality implies

$$\|x_t\|^{2q}_{\mathcal{PC}} \le (3^{q-1}4^qM^{2q}\|\phi\|^{2q}_{\mathcal{PC}} + L_6)e^{-(2\mu q - L)t}, \tag{59}$$

where we have used the notation

$$L_6 = 3^{q-1}4^{2q}M^{2q}L_4^q,\ L = 3^{q-1}4^{2q}M^{2q}L_5^q,$$

and, consequently,

$$E\|x(t_1)\|^2 \leq (3^{q-1}4^q M^{2q}\|\phi\|_{\mathcal{PC}}^{2q} + L_6)^{\frac{1}{q}} e^{-\frac{2\mu q - L}{q}t_1} = (B_1^*)^{\frac{1}{q}}. \tag{60}$$

Step 2. By Definition 11, (2), (3), (H_4) and (C_1)-(C_4), similar to (49), we obtain for $t \in (t_1, t_2]$ that

$$\begin{aligned}
E\|x(t)\|^2 &\leq 8M^2 e^{-2\mu(t-t_1)}(1+b_1)E\|x(t_1^-)\|^2 + 16M^2(\eta C_f + Tr(Q)C_g)e^{-2\mu(t-t_1)} \\
&\quad \times \int_{t_1}^t (1+(t-s)^{2\alpha-2})e^{2\mu(s-t_1)}\|x_s\|_{\mathcal{PC}}^2 ds + 16M^2 e^{-2\mu(t-t_1)} \\
&\quad \times \int_{t_1}^t (1+(t-s)^{2\alpha-2})e^{2\mu(s-t_1)}(\eta\|f(s,0)\|^2 + Tr(Q)\|g(s,0)\|^2)ds \\
&\quad + 8cH(2H-1)\eta^{2H-1}M^2 e^{-2\mu(t-t_1)} \int_{t_1}^t (1+(t-s)^{2\alpha-2})e^{2\mu(s-t_1)}\|h(s)\|_Q^2 ds.
\end{aligned} \tag{61}$$

In order to prove the exponential decay to zero of solutions to problem (1), we need the results to the last two terms of (61) including $e^{-(2\mu-\frac{L}{q})t_1}$. For this purpose, we do the estimates separately as follows. On the one hand, using the same argument as in (50), condition (C_3) ensures the existence of a positive constant which is still denoted by L_1 for simplicity, such that

$$\begin{aligned}
& cH(2H-1)\eta^{2H-1}\int_{t_1}^t (1+(t-s)^{2\alpha-2})e^{(2\mu-\frac{L}{q})(s-t_1)+\frac{L}{q}(s-t_1)}\|h(s)\|_Q^2 ds \\
& \leq cH(2H-1)\eta^{2H-1}e^{-(2\mu-\frac{L}{q})t_1}\int_{t_1}^t (1+(t-s)^{2\alpha-2})e^{2\mu s}\|h(s)\|_Q^2 ds \\
& \leq cH(2H-1)\eta^{2H-1}e^{-(2\mu-\frac{L}{q})t_1}\left(\int_{t_1}^t (1+(t-s)^{2\alpha-2})^p ds\right)^{\frac{1}{p}}\left(\int_{t_1}^t e^{2\mu q s}\|h(s)\|_Q^{2q} ds\right)^{\frac{1}{q}} \\
& \leq e^{-(2\mu-\frac{L}{q})t_1}L_1.
\end{aligned} \tag{62}$$

On the other hand, similar to (51) and (52), using the same technique as in (62), by condition (C_2), there are two positive constants still denoted by L_2 and L_3 respectively, such that

$$\begin{aligned}
& \int_{t_1}^t (1+(t-s)^{2\alpha-2})e^{2\mu(s-t_1)}\|f(s,0)\|^2 ds \\
& \leq \left(\int_{t_1}^t (1+(t-s)^{2\alpha-2})^p ds\right)^{\frac{1}{p}} \times \left(\int_{t_1}^t e^{(2\mu q-L)(s-t_1)}e^{L(s-t_1)}\|f(s,0)\|^{2q} ds\right)^{\frac{1}{q}} \\
& \leq \left(\int_{t_1}^t (1+(t-s)^{2\alpha-2})^p ds\right)^{\frac{1}{p}} \times e^{-\frac{2\mu q-L}{q}t_1}\left(\int_{t_1}^t e^{2\mu q s}\|f(s,0)\|^{2q} ds\right)^{\frac{1}{q}} \\
& \leq L_2 e^{-\frac{2\mu q-L}{q}t_1},
\end{aligned} \tag{63}$$

and

$$
\begin{aligned}
&\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})e^{2\mu(s-t_1)}\|g(s,0)\|^2ds \\
&\leq \bigg(\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})^p ds\bigg)^{\frac{1}{p}} \times \bigg(\int_{t_1}^{t} e^{(2\mu q-L)(s-t_1)}e^{L(s-t_1)}\|g(s,0)\|^{2q}ds\bigg)^{\frac{1}{q}} \\
&\leq \bigg(\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})^p ds\bigg)^{\frac{1}{p}} \times e^{-\frac{2\mu q-L}{q}t_1}\bigg(\int_{t_1}^{t} e^{2\mu q s}\|g(s,0)\|^{2q}ds\bigg)^{\frac{1}{q}} \\
&\leq L_3 e^{-\frac{2\mu q-L}{q}t_1}.
\end{aligned}
\tag{64}
$$

Substituting (62)-(64) into (61), and using Hölder's inequality,

$$
\begin{aligned}
E\|x(t)\|^2 \leq\, & 8M^2e^{-2\mu(t-t_1)}(1+b_1)E\|x(t_1^-)\|^2 + 16M^2(\eta C_f + Tr(Q)C_g)e^{-2\mu(t-t_1)} \\
& \times \bigg(\int_{t_1}^{t}(1+(t-s)^{2\alpha-2})^p ds\bigg)^{\frac{1}{p}}\bigg(\int_{t_1}^{t} e^{2\mu q(s-t_1)}\|x_s\|_{\mathcal{PC}}^{2q}ds\bigg)^{\frac{1}{q}} \\
& + 16M^2e^{-2\mu(t-t_1)}(\eta L_2 + Tr(Q)L_3)e^{-\frac{2\mu q-L}{q}t_1} + 8M^2e^{-2\mu(t-t_1)}L_1e^{-\frac{2\mu q-L}{q}t_1} \\
& \leq 8M^2e^{-2\mu(t-t_1)}(1+b_1)E\|x(t_1^-)\|^2 + 16M^2e^{-2\mu(t-t_1)}L_4e^{-\frac{2\mu q-L}{q}t_1} \\
& + 16M^2e^{-2\mu(t-t_1)}L_5\bigg(\int_{t_1}^{t} e^{2\mu q(s-t_1)}\|x_s\|_{\mathcal{PC}}^{2q}ds\bigg)^{\frac{1}{q}}.
\end{aligned}
\tag{65}
$$

Arguing as in the proof of Step 1, due to the fact that $e^{(\gamma-2\mu)\theta} < 1$, we obtain for $t+\theta \in (t_1, t_2]$ (where $\theta \in (-\infty, 0]$)

$$
\begin{aligned}
e^{\gamma\theta}E\|x(t+\theta)\|^2 \leq\, & 8M^2e^{-2\mu(t+\theta-t_1)}e^{\gamma\theta}(1+b_1)E\|x(t_1^-)\|^2 + 16M^2e^{-2\mu(t+\theta-t_1)}e^{\gamma\theta} \\
& \times L_4e^{-\frac{2\mu q-L}{q}t_1} + 16M^2e^{-2\mu(t+\theta-t_1)}e^{\gamma\theta}L_5\bigg(\int_{t_1}^{t+\theta} e^{2\mu q(s-t_1)}\|x_s\|_{\mathcal{PC}}^{2q}ds\bigg)^{\frac{1}{q}} \\
& \leq 8M^2e^{-2\mu(t-t_1)}(1+b_1)E\|x(t_1^-)\|^2 + 16M^2e^{-2\mu(t-t_1)}L_4e^{-\frac{2\mu q-L}{q}t_1} \\
& + 16M^2e^{-2\mu(t-t_1)}L_5\bigg(\int_{t_1}^{t} e^{2\mu q(s-t_1)}\|x_s\|_{\mathcal{PC}}^{2q}ds\bigg)^{\frac{1}{q}}.
\end{aligned}
\tag{66}
$$

It follows from (59) and (60) that, for $t \in (t_1, t_2]$ and $t+\theta \leq t_1$,

$$
\begin{aligned}
e^{\gamma\theta}E\|x(t+\theta)\|^2 &\leq (3^{q-1}4^qM^{2q}\|\phi\|_{\mathcal{PC}}^{2q} + L_6)^{\frac{1}{q}}e^{-\frac{2\mu q-L}{q}t} \\
&= (3^{q-1}4^qM^{2q}\|\phi\|_{\mathcal{PC}}^{2q} + L_6)^{\frac{1}{q}}e^{-\frac{2\mu q-L}{q}t_1}e^{-\frac{2\mu q-L}{q}(t-t_1)} \\
&\leq (B_1^*)^{\frac{1}{q}}e^{\frac{L}{q}\eta}e^{-2\mu(t-t_1)}.
\end{aligned}
\tag{67}
$$

Hence, (66) and (67) imply that for all $t \in (t_1, t_2]$,

$$\begin{aligned}\|x_t\|^2_{\mathcal{PC}} \leq \left(8M^2(1+b_1) + e^{\frac{L}{q}\eta}\right)(B_1^*)^{\frac{1}{q}} e^{-2\mu(t-t_1)} + 16M^2 e^{-2\mu(t-t_1)} L_4 e^{-\frac{2\mu q - L}{q} t_1} \\ + 16M^2 e^{-2\mu(t-t_1)} L_5 \left(\int_{t_1}^t e^{2\mu q(s-t_1)} \|x_s\|^{2q}_{\mathcal{PC}} ds\right)^{\frac{1}{q}}.\end{aligned} \tag{68}$$

Thus

$$\begin{aligned}\|x_t\|^{2q}_{\mathcal{PC}} \leq 3^{q-1}\left(8M^2(1+b_1) + e^{\frac{L}{q}\eta}\right)^q B_1^* e^{-2\mu q(t-t_1)} + 3^{q-1} 4^{2q} M^{2q} e^{-2\mu q(t-t_1)} \\ \times L_4^q e^{-(2\mu q - L)t_1} + 3^{q-1} 4^{2q} M^{2q} e^{-2\mu q(t-t_1)} L_5^q \int_{t_1}^t e^{2\mu q(s-t_1)} \|x_s\|^{2q}_{\mathcal{PC}} ds.\end{aligned} \tag{69}$$

Multiplying both sides of (69) by $e^{2\mu q(t-t_1)}$,

$$\begin{aligned}e^{2\mu q(t-t_1)}\|x_t\|^{2q}_{\mathcal{PC}} \leq 3^{q-1}\left(8M^2(1+b_1) + e^{\frac{L}{q}\eta}\right)^q B_1^* + 3^{q-1} 4^{2q} M^{2q} L_4^q e^{-(2\mu q - L)t_1} \\ + 3^{q-1} 4^{2q} M^{2q} L_5^q \int_{t_1}^t e^{2\mu q(s-t_1)} \|x_s\|^{2q}_{\mathcal{PC}} ds.\end{aligned} \tag{70}$$

Solving the above Gronwall's inequality yields

$$\|x_t\|^{2q}_{\mathcal{PC}} \leq \left(3^{q-1}(8M^2(1+b_1) + e^{\frac{L}{q}\eta})^q B_1^* + L_6 e^{-(2\mu q - L)t_1}\right) e^{-(2\mu q - L)(t-t_1)}, \tag{71}$$

and, consequently,

$$E\|x(t_2)\|^2 \leq \left(3^{q-1}(8M^2(1+b_1) + e^{\frac{L}{q}\eta})^q B_1^* + L_6 e^{-(2\mu q - L)t_1}\right)^{\frac{1}{q}} e^{-\frac{2\mu q - L}{q}(t_2 - t_1)} = (B_2^*)^{\frac{1}{q}}. \tag{72}$$

Step 3. The same reasoning as above implies, for $t \in (t_k, t_{k+1}]$ with $k \geq 2$,

$$\|x_t\|^{2q}_{\mathcal{PC}} \leq \left(3^{q-1}(8M^2(1+b_1) + e^{\frac{L}{q}\eta})^q B_k^* + L_6 e^{-(2\mu q - L)t_k}\right) e^{-(2\mu q - L)(t-t_k)}, \tag{73}$$

and

$$E\|x(t_{k+1})\|^2 \leq \left(3^{q-1}(8M^2(1+b_1) + e^{\frac{L}{q}\eta})^q B_k^* + L_6 e^{-(2\mu q - L)t_k}\right)^{\frac{1}{q}} e^{-\frac{2\mu q - L}{q}(t_{k+1} - t_k)} = (B_{k+1}^*)^{\frac{1}{q}}. \tag{74}$$

For convenience, let $w_1 = 3^{q-1}(8M^2(1+b_1) + e^{\frac{L}{q}\eta})^q$. It is obvious $w_1 > 2$, such that, $\sum_{j=0}^{k-2} w_1^j \leq \frac{w_1^{k-1}}{w_1 - 1} \leq \frac{w_1^{k-1}}{w_1 - \frac{w_1}{2}} \leq 2w_1^{k-2}$. In addition, condition (H_4) implies that $k - 1 \leq \frac{t_k - t_1}{\beta}$ and $k\beta < t_k$.

Then, for $k \geq 2$, the mathematical induction method furnishes that

$$\begin{aligned}
B_k^* &\leq w_1 B_{k-1}^* e^{-(2\mu q-L)(t_k-t_{k-1})} + L_6 e^{-(2\mu q-L)t_k} \\
&\leq w_1^{k-1} B_1^* e^{-(2\mu q-L)(t_k-t_1)} + L_6 e^{-(2\mu q-L)t_k} \sum_{j=0}^{k-2} w_1^j \\
&\leq B_1^* e^{(t_k-t_1)\frac{\ln w_1}{\beta}} e^{-(2\mu q-L)(t_k-t_1)} + 2L_6 e^{-(2\mu q-L)t_k} w_1^{k-2} \\
&\leq B_1^* e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t_k-t_1)} + 2L_6 e^{-(2\mu q-L)t_k} e^{(k-2)\ln w_1} \\
&\leq B_1^* e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t_k-t_1)} + 2L_6 e^{-(2\mu q-L)t_k} e^{\frac{t_k}{\beta}\ln w_1} \\
&\leq B_1^* e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t_k-t_1)} + 2L_6 e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t_k}.
\end{aligned} \tag{75}$$

Therefore, by (60), (73) and (75) we deduce that, for $t \in (t_k, t_{k+1}]$ with $k \geq 2$,

$$\begin{aligned}
\|x_t\|_{\mathcal{PC}}^{2q} &\leq w_1 B_k^* e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t-t_k)} + L_6 e^{-(2\mu q-L)t} \\
&\leq w_1 \left(B_1^* e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t_k-t_1)} + 2L_6 e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t_k} \right) \\
&\quad \times e^{-(2\mu q-L-\frac{\ln w_1}{\beta})(t-t_k)} + L_6 e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t} \\
&\leq w_1 (3^{q-1} 4^q M^{2q} \|\phi\|_{\mathcal{PC}}^{2q} + 3L_6) e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t} \\
&\leq C(1 + \|\phi\|_{\mathcal{PC}}^{2q}) e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t},
\end{aligned}$$

which, thanks to (71) and (59), implies that, for all $t \geq 0$,

$$\|x_t\|_{\mathcal{PC}}^{2q} \leq C(1 + \|\phi\|_{\mathcal{PC}}^{2q}) e^{-(2\mu q-L-\frac{\ln w_1}{\beta})t}. \tag{76}$$

This completes the proof. □

Conclusions. We have proved some results concerning the local and global existence of mild solutions for a class of fractional impulsive stochastic differential equations with infinite delay driven by both $\mathbb{K}$-valued Q-cylindrical Brownian motion and fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. We also have proved the continuous dependence of mild solutions on initial values and have analyzed the exponential decay to zero of solutions to our fractional stochastic impulsive differential equations with infinite delay. However there are also some interesting points to be analyzed such as the existence and structure of random attractors for this model, since the global asymptotic behavior can be described by such objects (see e.g. [8,19] and the references cited therein). We plan to investigate this topic in the future.

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