Topological method for couple systems of impulsive stochastic semilinear differential inclusions with fractional Brownian motion

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Abstract

In this paper we prove the existence of mild solutions for a first-order impulsive semilinear stochastic differential inclusion with an infinite-dimensional fractional Brownian motion. We consider the cases in which the right hand side can be either convex or nonconvex-valued. The results are obtained by using two different fixed point theorems for multivalued mappings, more precisely, the technique is based on a multivalued version of Perov's fixed point theorem and a new version of a nonlinear alternative of Leray–Schauder's fixed point theorem in generalized Banach spaces.

Key words and phrases: Mild solutions, fractional Brownian motion, impulses, matrix convergent to zero, generalized Banach space, fixed point, set-valued analysis, differential inclusions.
AMS (MOS) Subject Classifications: 34A37,60H15, 60H20.

1 Introduction

The theory of stochastic differential and partial differential inclusions has become an active area of investigation due to their applications in several fields from the applied sciences such as mechanics, electrical engineering, medical biology, ecology amongst others.

Recently, stochastic differential and partial differential inclusions have been extensively studied. For instance, in [1, 6] the authors investigated the existence of solutions of nonlinear stochastic differential inclusions by means of a Banach fixed point theorem and a semigroup approach. Balasubramaniam [5] obtained existence of solutions of functional stochastic differential inclusions by Kakutani's fixed point theorem, Balasubramaniam et al. [6] initiated the study of existence of solutions of semilinear stochastic delay evolution inclusions in a Hilbert space by using the nonlinear alternative of Leray-Schauder type [19], some existence results for impulsive neutral stochastic evolution inclusions in Hilbert Space, where a class of first-order evolution inclusions with a convex and nonconvex cases are considered, is studied in [32] by using a fixed point theorem due to Dhage and Covitz, as well as Nadler's theorem for contraction multivalued maps.

It is also worth emphasizing that impulsive differential systems and evolution differential systems are used to describe numerous models of real processes and phenomena appearing in the applied sciences, for instance, in physics, related to chemical technology, population dynamics, biotechnology and economics. That is why in recent years they have been the objectives of many investigations. We refer to the monographs by Bainov and Simeonov [3], Benchohra *et al.* [7], amongst others, to see several studies on the properties of their solutions. The reader can also find a detailed and extensive bibliography in the previously mentioned books (see also Da Prato and Zabczyk [16], Gard [20], Gikhman and Skorokhod [21], Sobzyk [41]). As a motivating example, let us refer to a stochastic model for drug distribution in a biological system which was described by Tsokos and Padgett [43] as a closed system with a simplified heart, one organ or capillary bed, and re-circulation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs of Mao [29], Øksendal, [34], Tsokos and Padgett [43], Sobczyk [41] and Da Prato and Zabczyk [16], and for some literature in the case of stochastic differential inclusions, see for instance [14, 15, 27]. More specifically, in this paper we are interested in proving the existence of solutions for a system of stochastic impulsive differential inclusions of the following type:

$$\begin{cases} dx(t) & \in & (Ax(t)+F^1(t,x(t),y(t)))dt+\sum_{l=1}^{\infty}\sigma_l^1(t,x(t)),y(t))dB_l^H(t), \ t\in J, t\neq t_k, \\ dy(t) & \in & (Ay(t)+F^2(t,x(t),y(t)))dt+\sum_{l=1}^{\infty}\sigma_l^2(t,x(t),y(t))dB_l^H(t), \ t\in J, t\neq t_k, \\ \Delta x(t) & = & I_k(x(t_k)), \quad t=t_k \quad k=1,2,\dots,m \\ \Delta y(t) & = & \overline{I}_k(y(t_k)), \\ x(0) & = & x(b), \\ y(0) & = & y(b), \end{cases} \tag{1.1}$$

where $J:=[0,b]$, X is a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ induced by norm $\|\cdot\|$, $A: D(A)\subset X\longrightarrow X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(S(t))_{t\geq 0}$ in X and $F^1,F^2:[0,b]\times X\times X\longrightarrow\mathcal{P}(X)$ are given set-valued functions, where $\mathcal{P}(X)$ denotes the family of nonempty subsets of X, $I_k, \overline{I}_k \in C(X,X)$ $(k = 1,2,\ldots,m)$, $\sigma_l^1, \sigma_l^2 : J \times X \times X \to L_Q^0(Y,X)$. Here, $L_Q^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from another separable Hilbert space Y into X, and B_l is a fractional Brownian motion which will be defined in the next section. Moreover, the fixed times t_k satisfy $t_0 = 0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = b$, $y(t_k^-)$ and $y(t_k^+)$ denotes the left and right limits of $y(t)$ at $t = t_k$.

$$\begin{cases} \sigma(.,x) = (\sigma_1(.,x), \sigma_2(.,x), \ldots), \\ \|\sigma(.,x)\|^2 = \displaystyle\sum_{l=1}^{\infty} \|\sigma_l(.,x)\|^2_{L_Q^0} < \infty \end{cases} \tag{1.2}$$

with $\sigma(.,x) \in \ell^2$ for all $x \in X$, where

$$\ell^2 = \{\phi = (\phi_l)_{l\geq 1} : X \times X \to L_Q^0(Y,X) \quad : \|\phi(x)\|^2 = \sum_{l=1}^{\infty} \|\phi_l(x)\|^2_{L_Q^0} < \infty\}.$$

We remark that when F^1 and F^2 are single-valued mappings, this problem has been analyzed in [8]. In our current analysis, in order to prove the existence of solutions for our problem, we need to use different and more sophisticate tools from the field of set-valued analysis. Consequently, our theory in this paper generalizes the one in [8].

In the deterministic and single valued framework, the above system was used to analyze initial value problems and boundary value problems for nonlinear competitive or cooperative differential systems from mathematical biology [30] and mathematical economics [26] which can be set in the operator form (1.1).

Some existence results of solutions for differential equations with infinite Brownian motion was obtained in [45]. Recently, Precup [38] proved the role of matrix convergence and vector metric in the study of semilinear operator systems.
When the space X is finite dimensional, some existence results of mild solutions for Problem (1.1) in the particular case $Ay = \lambda y$ $(\lambda \in \mathbb{R})$ have been obtained in [23]. The goal of this paper is to study the existence of mild solutions of systems of semilinear stochastic differential inclusions with infinite fractional Brownian motions.

The paper is organized as follows. In Section 2 we recall some definitions and facts which will be needed in our analysis. Section 3 is concerned with the case in which we assume that $1 \in \Lambda(S(b))$ (where $\Lambda(S(b))$ denotes its resolvent set) and we prove some existence results based on a nonlinear alternative of Leray-Schauder type theorem in generalized Banach spaces in the convex case. Finally we establish a multivalued version of Perov's fixed point theorem [35] and prove another result on the existence of solution in a non-convex case.

2 Preliminaries

In this section, we introduce some notations, and recall some definitions, and preliminary facts which are used throughout this paper. Actually we will borrow them from [8]. Although we could simply refer to this paper whenever we need it, we prefer to include this summary in order to make our paper as much self-contained as possible.

2.1 Some results on stochastic integrals with respect to fractional Brownian motions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).

For a stochastic process $x(\cdot,\cdot) : [0,T]\times\Omega \to X$ we will write $x(t)$ (or simply x when no confusion is possible) instead of $x(t,\omega)$.

Definition 2.1. *Given $H \in (0,1)$, a continuous centered Gaussian process B^H is said to be a two-sided one-dimensional fractional Brownian motion (fBm) with Hurst parameter H, if its covariance function $R_H(t,s) = E[B^H(t))B^H(s)]$ satisfies*

$$R_H(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \quad t,s \in [0,T].$$

It is known that $B^H(t)$ with $H > \frac{1}{2}$ admits the following Volterra representation

$$B^H(t) = \int_0^t K_H(t,s)dB(s) \tag{2.1}$$

where B is a standard Brownian motion given by

$$B(t) = B^H((K_H^*)^{-1}\xi_{[0,t]}),$$

and the Volterra kernel the kernel $K(t,s)$ is given by

$$K_H(t,s) = c_H s^{1/2-H}\int_s^t (u-s)^{H-\frac{3}{2}}\left(\frac{u}{s}\right)^{H-\frac{1}{2}} du, \quad t \geq s,$$

where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2H-2,H-\frac{1}{2})}}$ and $\beta(\cdot,\cdot)$ denotes the Beta function, $K(t,s) = 0$ if $t \leq s$, and it holds

$$\frac{\partial K_H}{\partial t}(t,s) = c_H\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}},$$

and the kernel K_H^* is defined as follows. Denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \chi_{[0,t]}, \chi_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s),$$

and consider the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by,

$$(K_H^*\phi)(t) = \int_s^T \phi(t)\frac{\partial K_H}{\partial t}(t,s)dt.$$

Notice that,

$$(K_H^*\chi_{[0,t]})(s) = K_H(t,s)\chi_{[0,t]}(s).$$

The operator K_H^* is an isometry between $\mathcal{E}$ and $L^2([0,T])$ which can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,T]$ we have

$$\langle K_H^*\chi_{[0,t]}, K_H^*\chi_{[0,t]}\rangle_{L^2([0,T])} = \langle \chi_{[0,t]}, \chi_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

In addition, for any $\phi \in \mathcal{H}$,

$$\int_0^T \phi(s)dB^H(s) = \int_0^T (K_H^*\phi)(s)dB(s),$$

if and only if $K_H^*\phi \in L^2([0,T])$.

Remark 2.1. *In the sequel, the notation c_H will be used to denote the value of a constant which depends on the Hurst parameter (not necessarily the one used above) and that can be different from line to line or even in the same line.*

Moreover, the following useful result holds

Lemma 2.1. *[33] There exists a positive constant c_H such that for any $\phi \in L^{1/H}([0,T])$ it holds*

$$H(2H-1)\int_0^T\int_0^T |\phi(y)||\phi(z)||y-z|^{2H-2}dydz \le c_H\|\phi\|^2_{L^{1/H}([0,T])}. \tag{2.2}$$

Next we are interested in considering an fBm with values in a Hilbert space and giving the definition of the corresponding stochastic integral.

Definition 2.2. *An $\mathcal{F}_t$-adapted process ϕ on $[0,T]\times\Omega \to X$ is an elementary or simple process if for a partition $\psi = \{\bar{t}_0 = 0 < \bar{t}_1 < \ldots < \bar{t}_n = T\}$ and $(\mathcal{F}_{\bar{t}_i})$-measurable X-valued random variables $(\phi_{\bar{t}_i})_{1\le i\le n}$, ϕ_t satisfies*

$$\phi_t(\omega) = \sum_{i=1}^n \phi_i(\omega)\chi_{(\bar{t}_{i-1},\bar{t}_i]}(t), \quad for \;\; 0 \le t \le T, \quad \omega \in \Omega.$$

The Itô integral of the simple process ϕ is defined as

$$I_H(\phi) = \int_0^T \phi_l(s)dB_l^H(s) = \sum_{i=1}^n \phi_l(\bar{t}_i)(B_l^H(\bar{t}_i) - B_l^H(\bar{t}_{i-1})), \tag{2.3}$$

whenever $\phi_{\bar{t}_i} \in L^2(\Omega, \mathcal{F}_{\bar{t}_i}\mathbb{P}, X)$ for all $i \le n$.

Let $(X, \langle\cdot,\cdot\rangle, |\cdot|_X)$, $(Y, \langle\cdot,\cdot\rangle, |\cdot|_Y)$ be separable Hilbert spaces. Let $\mathcal{L}(Y,X)$ denote the space of all linear bounded operators from Y into X. Let $e_n, n = 1,2,\dots$ be a complete orthonormal basis in Y and $Q \in \mathcal{L}(Y,X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$ where λ_n, $n = 1,2,\dots$, are non-negative real numbers. Let $(\beta_n^H)_{n\in N}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. If we define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n^H(t)e_n, \tag{2.4}$$

then it is well defined as an Y-valued Q-cylindrical fractional Brownian motion (see [16]) and we have

$$E\langle\beta_l^H(t), x\rangle\langle\beta_k^H(s), y\rangle = R_{H_{lk}}(t,s)\langle Q(x), y\rangle, \quad x, y \in Y \quad and\, s,t \in [0,T]$$

such that

$$R_{H_{lk}} = \frac{1}{2}\{\mid t\mid^{2H} + \mid s\mid^{2H} + \mid t-s\mid^{2H}\}\delta_{lk} \quad t,s \in [0,T],$$

where

$$\delta_{lj} = \begin{cases} 1 & j = l, \\ 0, & j \neq l. \end{cases}$$

In order to define Wiener integrals with respect to a $Q - fBm$, we introduce the space $L_Q^0 := L_Q^0(Y,X)$ of all $Q-$Hilbert-Schmidt operators $\varphi : Y \longrightarrow X$. We recall that $\varphi \in L(Y,X)$ is called a $Q-$Hilbert-Schmidt operator, if

$$\|\varphi\|_{L_Q^0}^2 = \|\varphi Q^{1/2}\|_{HS}^2 = tr(\varphi Q\varphi^*) < \infty.$$

Definition 2.3. *Let $\phi(s), s \in [0,T]$, be a function with values in $L_Q^0(Y,X)$. The Wiener integral of ϕ with respect to fBm given by (2.4) is defined by*

$$\int_0^T \phi(s)dB^H(s) = \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda_n}\phi(s)e_n d\beta_n^H$$

$$= \sum_{n=1}^{\infty}\int_0^T \sqrt{\lambda_n}K_H^*(\phi e_n)(s)d\beta_n. \tag{2.5}$$

Notice that if

$$\sum_{n=1}^{\infty} \|\phi Q^{1/2}e_n\|_{L^{1/H}([0,T];X)} < \infty, \tag{2.6}$$

the next result ensures the convergence of the series in the previous definition.

Lemma 2.2. *[8] For any $\phi : [0,T] \to L_Q^0(Y,X)$ such that (2.6) holds, and for any $\alpha, \beta \in [0,T]$ with $\alpha > \beta$,*

$$E\left|\int_\alpha^\beta \phi(s)dB^H(s)\right|_X^2 \leq c_H H(2H-1)(\alpha-\beta)^{2H-1}\sum_{n=1}^{\infty}\int_\alpha^\beta \left|\phi(s)Q^{1/2}e_n\right|_X^2 ds. \quad (2.7)$$

If in addition

$$\sum_{n=1}^{\infty} |\phi Q^{1/2}e_n|_X \text{ is uniformly convergent for } t \in [0,T],$$

then,

$$E\left|\int_\alpha^\beta \phi(s)dB^H(s)\right|_X^2 \leq c_H H(2H-1)(\alpha-\beta)^{2H-1}\int_\alpha^\beta \|\phi(s)\|_{L_Q^0}^2 ds. \quad (2.8)$$

2.2 Some results on fixed point theorems and set-valued analysis

For $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$ and $|x| = (|x_1|, \dots, |x_n|)$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \dots, n$.

Definition 2.4. *Let X be a nonempty set. A vector-valued metric on X is a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*

(i) $d(u,v) \geq 0$ for all $u, v \in X$; if $d(u,v) = 0$ then $u = v$;

(ii) $d(u,v) = d(v,u)$ for all $u, v \in X$;

(iii) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in X$.

The pair (X,d) a said to be a generalized metric space.

For $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Definition 2.5. *A generalized metric space* (X, d)*, where* $d(x, y) := \begin{pmatrix} d_1(x,y) \\ \cdots \\ d_n(x,y) \end{pmatrix}$*,* *is complete if* (X, d_i) *is a complete metric space for every* $i = 1, \ldots, n$*.*

Definition 2.6. *A square matrix of real numbers* M *is said to be convergent to zero if its spectral radius* $\rho(M)$ *is strictly less than* 1*. In other words, this means that all the eigenvalues of* M *are in the open unit disc (i.e.* $|\lambda| < 1$*, for every* $\lambda \in \mathbb{C}$ *with* $det(M - \lambda I) = 0$*, where* I *denotes the unit matrix of* $\mathcal{M}_{n\times n}(\mathbb{R})$*).*

Definition 2.7. *A non-singular matrix* $A = (a_{ij})_{1\le i,j\le n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ *is said to have the absolute value property if*

$$A^{-1}|A| \le I,$$

where

$$|A| = (|a_{ij}|)_{1\le i,j\le n} \in \mathcal{M}_{n\times n}(\mathbb{R}_+).$$

Lemma 2.3. *[40] Let* M *be a square matrix of nonnegative numbers. The following assertions are equivalent:*

(i) M *is convergent towards zero;*

(ii) the matrix $I - M$ *is non-singular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iii) $|\lambda| < 1$ *for every* $\lambda \in \mathbb{C}$ *with* $det(M - \lambda I) = 0$

(iv) $(I - M)$ *is non-singular and* $(I - M)^{-1}$ *has nonnegative elements.*

Some examples of matrices convergent to zero can be seen in [8].
We will use the following notation:

$$\mathcal{P}_{cl}(X) = \{y \in \mathcal{P}(X) : y \text{ closed }\},$$
$$\mathcal{P}_{b}(X) = \{y \in \mathcal{P}(X) : y \text{ bounded }\},$$
$$\mathcal{P}_{c}(X) = \{y \in \mathcal{P}(X) : y \text{ convex }\},$$
$$\mathcal{P}_{cp}(X) = \{y \in \mathcal{P}(X) : y \text{ compact }\}.$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$ defined by

$$H_d(A, B) \ := \ \begin{pmatrix} H_{d_1}(A,B) \\ \cdots \\ H_{d_n}(A,B) \end{pmatrix}.$$

Let (X, d) be a generalized metric space with

$$d(x,y) \ := \ \begin{pmatrix} d_1(x,y) \\ \dots \\ d_n(x,y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if $d_i,\ i=1,\dots,n$ are metrics on X, $H_d(A,B) = \ max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}$,
where $d(A,b) = \inf_{a\in A} d(a,b), d(a,B) = \inf_{b\in B} d(a,b)$. Then, $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

A multivalued map $F : X \longrightarrow \mathcal{P}(X)$ is convex (closed) valued if $F(y)$ is convex (closed) for all $y \in X$, F is bounded on bounded sets if $F(B) = \bigcup_{y\in B} F(y)$ is bounded in X for all $B \in \mathcal{P}_b(X)$. F is called upper semi-continuous (u.s.c. for short) on X if for each $y_0 \in X$ the set $F(y_0)$ is a nonempty, subset of X, and for each open set $\mathcal{U}$ of X containing $F(y_0)$, there exists an open neighborhood $\mathcal{V}$ of y_0 such that $F(\mathcal{V}) \in \mathcal{U}$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$. F is quasicompact if, for each subset $A \subset X$, $F(A)$ is relatively compact.

If the multivalued map F is completely continuous and possesses nonempty compact values, then F is *u.s.c.* if and only if F has a closed graph, i.e., $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

A multivalued map $F : J = [0,T] \longrightarrow \mathcal{P}_{cp,c}(X)$ is said to be measurable if for each $y \in X$, the mean-square distance between y and $F(t)$ is measurable.

Definition 2.8. *The set-valued map $F : J \times X \times X \to \mathcal{P}(X \times X)$ is said to be L^2-Carathéodory if*

(i) $t \mapsto F(t,v)$ *is measurable for each $v \in X \times X$;*

(ii) $v \mapsto F(t,v)$ *is u.s.c. for almost all $t \in J$;*

(iii) *for each $q > 0$, there exists $h_q \in L^1(J,\mathbb{R}^+)$ such that*

$$\|F(t,v)\|^2 := \sup_{f\in F(t,v)} \|f\|^2 \le h_q(t), \quad \textit{for all } \|v\|^2 \le q \textit{ and for a.e. } t \in J.$$

Remark 2.2. **(a)** *For each $x \in C(J,X)$, the set $S_{F,x}$ is closed whenever F has closed values. It is convex if and only if $F(t,x(t))$ is convex for a.e. $t \in J$.*

(b) *From [44], Theorem 5.10 (see also [28] when X is finite dimensional), we know that $S_{F,x}$is nonempty if and only if the mapping $t \longmapsto \inf\{\|v\| : \ v \in F(t,x(t))\}$ belongs to $L^2(J)$.*

Lemma 2.4. *[28] Let I be a compact interval and X be a Hilbert space. Let F be an L^2-Carathéodory multivalued map with $S_{F,y} \neq \emptyset$, and let Γ be a linear continuous mapping from $L^2(I,X)$ to $C(I,X)$. Then, the operator*

$$\Gamma \circ S_F : C(I,X) \longrightarrow \mathcal{P}_{cp,c}(L^2([0,T],X)), \quad y \longmapsto (\Gamma \circ S_F)(y) = \Gamma(S_F,y),$$

is a closed graph operator in $C(I,X)\times C(I,X)$, *where* $S_{F,y}$ *is known as the selectors set from* F *and given by* $f\in S_{F,y}=\{f\in L^2([0,T],X): f(t)\in F(t,y) \quad for\ a.e.t\in[0,T]\}$.

We denote the graph of G to be the set $gr(G)=\{(x,y)\in X\times Y, \quad y\in G(x)\}$.

Lemma 2.5. *[17] If* $G: X\to\mathcal{P}_{cl}(Y)$ *is u.s.c., then* $gr(G)$ *is a closed subset of* $X\times Y$. *Conversely, if* G *is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Lemma 2.6. *[18] If* $G: X\to\mathcal{P}_{cp}(Y)$ *is quasicompact and has a closed graph, then* G *is u.s.c.*

The following two results are easily deduced from the limit properties:

Lemma 2.7. *(See e.g. [2, Theorem 1.4.13]) If* $G: X\to\mathcal{P}_{cp}(X)$ *is u.s.c., then for any* $x_0\in X$,

$$\limsup_{x\to x_0} G(x)=G(x_0)$$

Lemma 2.8. *(See e.g. [2, Lemma 1.1.9]) If Let* $(K_n)_{n\in N}\subset K\subset X$ *be a sequence of subsets where* K *is compact in the separable Banach space* X. *Then*

$$\overline{co}(\limsup_{n\to\infty} K_n)=\cap_{N>0}\overline{co}(\cup_{n\geq N}K_n)$$

where $\overline{co}A$ *refers to the closure of the convex hull of* A.

The second one is due to Mazur (1933):

Lemma 2.9. *(Mazur's Lemma, [25, Theorem 21.4]) Let* X *be a normed space and* $\{x_k\}_{k\in N}\subset X$ *be a sequence weakly converging to a limit* $x\in X$. *Then there exists a sequence of convex combinations* $y_m=\sum_{k=1}^{m}\alpha_{mk}x_k$ *with* $\alpha_{mk}>0$ *for* $k=1,2,\ldots,m$ *and* $\sum_{k=1}^{m}\alpha_{mk}=1$, *which converges strongly to* x.

Definition 2.9. *A sequence* $(\upsilon_n)_{n\in\mathbb{N}}$*is said to be semi-compact if*

(1) it is integrably bounded, i.e. there exists $q\in L^1(J,R)$ *such that*

$$|\upsilon_n|_X\leq q(t)$$

for a.e. $t\in J$ *and every* $n\in\mathbb{N}$,

(2) the image sequence $(\upsilon_n)_{n\in\mathbb{N}}$ *is relatively compact in* X *for a.e.* $t\in[0,T]$.

This result is of particular importance if X is reflexive in which case (1) implies (2) in Definition 2.9.

Lemma 2.10. *Every semi-compact sequence $L^1([0,b],X)$ is weakly compact in $L^1([0,b],X)$.*

Recall that a set-valued operator G possesses a fixed point if there exists $y \in X$ such that $y \in G(y)$.

Now we can establish the following nonlinear alternatives of Leray and Schauder which will be needed in the proofs of our results (see [9, 19, 36]).

Lemma 2.11. *Let $(X, \|\cdot\|)$ be a generalized Banach space and $G : X \longrightarrow \mathcal{P}_{cl,cv}(X)$ be an upper semicontinuous and compact map. Then either,*

(a) *F has at least one fixed point, or*

(b) *the set $\mathcal{M} = \{x \in X \text{ and } \lambda \in (0,1), \text{with} \quad x \in \lambda G(u)\}$ is unbounded.*

Let us recall now the definition of resolvent set and family for a linear operator $A : E \to E$.

Definition 2.10. *The resolvent set $\Lambda(A)$ of A consists of all complex numbers λ for which the linear operator $\lambda I - A$ is invertible, i.e. $(\lambda I - A)^{-1}$ is a bounded linear operator in E. The family $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda \in \Lambda(A)$ is called the resolvent of A. All complex numbers λ not in $\Lambda(A)$ form a set called the spectrum of A.*

Our next result describes a basic theorem of reflexive spaces.

Theorem 2.1. *[10] E is reflexive if and only if $B_E = \{x \in E; ||x|| \leq 1\}$ is compact in the weak topology.*

3 Existence results

In this section we prove the existence of mild solution of the problem (1.1). Our approach is based on multivalued versions of Schaefer's fixed point theorem.

3.1 The convex case

In this section, we will show same results concerning the existence results of mild solutions for system (1.1) in the convex case. Recall that the fixed times t_k satisfy $t_0 = 0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = b$ and we denote $J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, and $J = [0, b]$. In order to define a solution for Problem (1.1), consider the following space of picewise continuous functions

$$\begin{aligned} PC \;=\; & \{x : \Omega \times [0,b] \longrightarrow X,\; x \in C(J_k, X)),\; k = 0, \ldots, m \text{ such that} \\ & x(t_k^+, .) \text{ and } x(t_k^-, .) \text{ exist with } x(t_k^-, .) = x(t_k, .) \quad \text{almost surely and} \\ & \sup_{t \in [0,b]} \mathbb{E}|x(t, .)|_X^2 < \infty\}. \end{aligned}$$

Endowed with the norm

$$\|x\|_{PC} = \left(\sup_{s\in[0,b]} \mathbb{E}|x(s,.)|_X^2\right)^{\frac{1}{2}},$$

it is not difficult to check that PC is a Banach space with norm $\|\cdot\|_{PC}$.
$AC^i(J,X)$ is the space of functions $y : J \to X$ i times differentiable whose ith derivative, $y^{(i)}$, is absolutely continuous.

Lemma 3.1. *Let A be the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ such that $1 \in \Lambda(S(b))$ and let $f^i : J \to X, i = 1,2,$ be continuous. Let $I_k, \overline{I}_k \in C(X,X)$ for each $k = 1,\dots,m$, assume that $\sum_{l=1}^{\infty}\int_0^b \|\sigma_l^i(s,x,y)\|^2_{L_Q^0} < \infty$, $i = 1,2$ and let $x, y \in PC \cap AC^1$ be a classical solution of the problem*

$$\begin{cases} dx(t) = (f^1(t) + Ax(t))dt + \sum_{l=1}^{\infty}\sigma_l^1(t,x(t),y(t))dB_l^H(t), \ t\in J, t\neq t_k \\ dy(t) = (f^2(t) + Ay(t))dt + \sum_{l=1}^{\infty}\sigma_l^2(t,x(t),y(t))dB_l^H(t), \ t\in J, t\neq t_k \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k = 1,2,\dots,m \\ y(t_k^+) - y(t_k) = \overline{I}_k(y(t_k)), \\ x(0) = x(b), \\ y(0) = y(b). \end{cases} \tag{3.1}$$

Then it fulfills

$$x(t) = S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k)) + \int_0^b S(b-s)f^1(s)ds$$
$$+\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big) + \int_0^t S(t-s)f^1(s)ds$$
$$+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), \ for\, t\in J, a.e.\ \omega\in\Omega.$$

and

$$y(t) = S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))\overline{I}_k(y(t_k)) + \int_0^b S(b-s)f^2(s)ds$$
$$+\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s)\Big) + \int_0^t S(t-s)f^2(s)ds$$

$$+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s)+\sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k)),\ for\, t\in J,\, a.e.\ \omega\in\Omega.$$

Proof. Let (x,y) be a solution of Problem (3.1) and $L_1(s)=S(t-s)x(s)$ and $L_2(s)=S(t-s)y(s)$ for fixed $t\in J$. We have

$$\begin{aligned} L_1^{'}(s) &= -S^{'}(t-s)x(s)+S(t-s)x^{'}(s)\\ &= -AS(t-s)x(s)+S(t-s)x^{'}(s)\\ &= S(t-s)(x^{'}(s)-Ax(s))\\ &= S(t-s)(f^1(s)ds+\sum_{l=1}^{\infty}\sigma_l^1(t,x(s),y(s))dB_l^H(s)\end{aligned}$$

Let $0<t<t_1$. Integrating the previous equation, we deduce for $k=1$

$$L_1(t)-L_1(0)=\int_0^t S(t-s)f^1(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s).$$

Hence

$$x(t)=S(t)x(0)+\int_0^t S(t-s)f^1(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s).$$

More generally, for $t_k<t<t_{k+1}$

$$\int_0^{t_1} L_1^{'}(s)+\int_{t_2}^{t_1} L_1^{'}(s)+....+\int_{t_k}^{t} L_1^{'}(s)=\int_0^t S(t-s)f^1(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)$$

$$=L_1(t_1^-)-L_1(0)+L_1(t_2^-)-L_1(t_1^+)+...+L_1(t)-L_1(t_k^+)$$

$$=\int_0^t S(t-s)f^1(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s).$$

Therefore

$$x(t)=S(t)x(0)+\sum_{0<t_k<t}(L_1(t_k^+-L_1(t_k^-))+\int_0^t S(t-s)f^1(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s).$$

Since $x(0)=x(b)$ and $1\in\rho(S(T))$, then $(I-S(b))$ is invertible. Hence we obtain after substitution

$$\begin{aligned} x(t) &= S(t)(I-S(b))^{-1}\Big(\textstyle\sum_{k=1}^m(S(b-t_k))I_k(x(t_k))+\displaystyle\int_0^b S(b-s)f^1(s)ds\\ &\quad+\textstyle\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\Big)+\displaystyle\int_0^t S(t-s)f^1(s)ds\\ &\quad+\textstyle\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\\ &\quad+\textstyle\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)),\ \text{for}\ \ t\in J.\end{aligned}$$

□

This lemma leads to the definition of a mild solution.

Definition 3.1. *An $X-$valued stochastic process $u=(x,y)\in PC\times PC$ is said to be a mild solution of* (1.1) *with respect to the probability space* $(\Omega,\mathcal{F},\mathbb{P})$*, if:*

1) $u(t)$ is $\mathcal{F}_t$-adapted for all $t\in J_k=(t_k,t_{k+1}]\quad k=1,2,\dots,m$

2) $u(t)$ is right continuous and has limit on the left, and there exists selections f^i, $i=1,2$, such that $f^i(t)\in F^i(t,u(t))$ a.e. $t\in J$.

4) $u(t)$ satisfies, for each $t\in J$, a.e. $\omega\in\Omega$,

$$\begin{cases} x(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k))+\int_0^b S(b-s)f^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big)+\int_0^t S(t-s)f^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)+\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k)), \\ y(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))\overline{I}_k(y(t_k))+\int_0^b S(b-s)f^2(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s)\Big)+\int_0^t S(t-s)f^2(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s)+\sum_{0<t_k<t}S(t-t_k)\overline{I}_k(y(t_k)). \end{cases}$$

In this section, we assume again that $1\in\Lambda(S(b)))$. We are now in a position to state and prove our existence result for problem (1.1). First we will list the following hypotheses which will be imposed in our main theorem.
Consider the following assumptions: In the remaining of our work, we assume that $S(t)$ is compact for $t>0$ and that there exists $M>0$ such that

$$\|S(t)\|\leq M,\quad \text{for every}\quad t\in[0,b].$$

(H_1) The functions $\sigma_l^i: J\times X\times X\to L_Q^0(Y,X)$ are continuous and there exist positive constants α_i and β_i and c_i for each $i=1,2$, such that

$$\|\sigma^1(t,x,y)\|^2\leq\alpha_1|x|_X^2+\beta_1|y|_X^2+c_1,\quad \|\sigma^2(t,x,y)\|^2\leq\alpha_2|x|_X^2+\beta_2|y|_X^2+c_2$$

and

$$\sum_{l=1}^{\infty}\int_0^b\|\sigma_l^i(t,x,y)\|^2_{L_Q^0}dt<\infty,\ i=1,2,$$

for all x, $y\in X$ and $t\in J$.

(H_2) $F^i : [0,b] \times X \times X \longrightarrow \mathcal{P}_{c,cp}(X)$ is an integrable bounded multivalued map, i.e., there exists $p_i \in L^2(J,X), i = 1,2$ such that

$$|F^i(t,x,y)|_X^2 = \sup_{f^i \in F^i(t,x,y)} |f^i(t)|_X^2 \leq p_i(t), \quad \forall t \in J, \quad \forall (x,y) \in X \times X.$$

(H_3) Consider the functions $I_k, \overline{I}_k \in C(X,X)$ for which there exist constants $d_k, \overline{d}_k \geq 0$ and $e_k, \overline{e}_k \geq 0$ for each $k = 1,\ldots,m$ such that

$$|I_k(x)|_X^2 \leq d_k|x|_X^2 + e_k, \quad |\overline{I}_k(y)|_X^2 \leq \overline{d}_k|y|_X^2 + \overline{e}_k, \text{ for all } x,y \in X.$$

Consider the following operator $N(x,y) = (N_1(x,y), N_2(x,y)),\ (x,y) \in PC \times PC$ defined by

$$N(x,y) = \big\{(h,\overline{h}) \in PC \times PC\big\}$$

given by

$$\begin{aligned}
h(t) &= S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k)) \\
&+ \int_0^b S(t-s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^b S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big) \\
&+ \int_0^t S(t-s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s) \\
&+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), \quad \text{for} \quad t \in J, \text{ a.e. } \omega \in \Omega, \\
\overline{h}(t) &= S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))\overline{I}_k(y(t_k)) \\
&+ \int_0^b S(t-s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^b S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s)\Big) \\
&+ \int_0^t S(t-s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s) \\
&+ \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k)), \quad \text{for} \quad t \in J, \text{ a.e. } \omega \in \Omega,
\end{aligned}$$

where

$$f^i \in S_{F^i,u} = \{f^i \in L^2(J,X) : f^i(t) \in F^i(t,x,y) \text{ for each } t \in J, \text{and each } x,y \in PC\},\ i = 1,2.$$

Lemma 3.2. *Assume that, for* $i = 1,2$, $F^i : J \times X \times X \longrightarrow \mathcal{P}_{c,cp}(X)$ *is a Carathèodory map such that* $(H_1) - (H_3)$ *also hold. Then, operator* N *is completely continuous and u.s.c.*

Proof. First we show that $N=(N_1,N_2)$ is completely continuous. We split the proof into several steps.

Step 1.- N maps bounded sets into bounded sets in $PC\times PC$.
Indeed, it is enough to show that for any $q>0$ there exists a positive constant $l=(l_1,l_2)$ such that for each $(x,y)\in B_q=\{(x,y)\in PC\times PC : \mathbb{E}|x|_X^2\leq q,\ \mathbb{E}|y|_X^2\leq q\}$ one has

$$|h|_X^2\leq l_1,\quad |\overline{h}|_X^2\leq l_2.$$

Let $(h,\overline{h})\in(N_1,N_2)$. Then, there exists $f^i(t)\in F^i(t,x,y)$ for each $t\in J$, and each $x,y\in PC$, such that

$$\begin{aligned}\mathbb{E}|h(t)|_X^2 &= \mathbb{E}\Big|S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k))+\int_0^b S(t-s)f^1(s)ds\\ &\quad+\sum_{l=1}^{\infty}\int_0^b S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big)+\int_0^t S(t-s)f^1(s)ds\\ &\quad+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)+\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k))\Big|_X^2,\end{aligned}$$

and also

$$\begin{aligned}\mathbb{E}|h(t)|_X^2 &\leq 4\mathbb{E}\Big|S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k))+\int_0^b S(t-s)f^1(s)ds\\ &\quad+\sum_{l=1}^{\infty}\int_0^b S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big)\Big|_X^2+4\mathbb{E}\Big|\int_0^t S(t-s)f^1(s)ds\Big|_X^2\\ &\quad+4\mathbb{E}\Big|\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big|_X^2+4\mathbb{E}\Big|\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k))\Big|_X^2.\end{aligned}$$

Using $(H_1)-(H_3)$ and (2.8) we have

$$\begin{aligned}
\mathbb{E}|h(t)|_X^2 &\leq 12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^m \sup_{z\in B(0,q)}\mathbb{E}|I_k(z)|_X^2+\|p_1\|_{L^1}^2\\
&\quad +c_H H(2H-1)b^{2H}(\alpha_1\mathbb{E}|x(t)|_X^2+\beta_1\mathbb{E}|y(t)|_X^2+c_1)\Big)+4M^2\|p_1\|_{L^1}\\
&\quad +4M^2(c_H H(2H-1)b^{2H}(\alpha_1\mathbb{E}|x(t)|_X^2+\beta_1\mathbb{E}|y(t)|_X^2+c_1))\\
&\quad +4M^2 m\sum_{k=1}^m(d_kq+e_k)\\
&\leq 4M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(\sum_{k=1}^m(d_kq+e_k)+\|p_1\|_{L^1}\\
&\quad +c_H H(2H-1)b^{2H}(\alpha_1 q+\beta_1 q+c_1)\Big)+4M^2\|p_1\|_{L^1}\\
&\quad +4M^2(c_H H(2H-1)b^{2H}(\alpha_1 q+\beta_1 q+c_1))\\
&\quad +4M^2 m\sum_{k=1}^m(d_kq+e_k):=l_1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathbb{E}|\overline{h}(t)|_X^2 &\leq 4M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(\sum_{k=1}^m\overline{d}_kq+\overline{e}_k)+\|p_2\|_{L^1}\\
&\quad +c_H H(2H-1)b^{2H}(\alpha_2 q+\beta_2 q+c_2)\Big)+4M^2\|p_2\|_{L^1}\\
&\quad +4M^2(c_H H(2H-1)b^{2H}(\alpha_2 q+\beta_2 q+c_2))\\
&\quad +4M^2 m\sum_{k=1}^m(\overline{d}_kq+\overline{e}_k)):=l_2.
\end{aligned}$$

Therefore

$$\begin{pmatrix}\mathbb{E}|h(t)|_X^2\\ \mathbb{E}|\overline{h}(t)|_X^2\end{pmatrix}\leq\begin{pmatrix}l_1\\ l_2\end{pmatrix}.$$

Step 2.- N maps bounded sets into equicontinuous sets of $PC\times PC$.
Let B_q be a bounded set in $PC\times PC$ as in Step 1. Let $\tau_1,\tau_2\in J,\tau_1<\tau_2$ and

$(x,y)\in B_q$, then there exists $f^i(t)\in F^i(t,x,y),\ i=1,2$, such that

$$\begin{aligned}\mathbb{E}|h(\tau_2)-h(\tau_1)|_X^2 \ \leq\ & 12M^2\Big\|S(\tau_2)-S(\tau_1)\Big\|^2\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}\sup_{z\in B(0,q)}E|I_k(z)|_X^2\\ &+\|p_1\|_{L^1}+c_H H(2H-1)b^{2H}(\alpha_1 E|x(t)|_X^2\\ &+\beta_1\mathbb{E}|y(t)|_X^2+c_1)\Big)\\ &+2\mathbb{E}\Big|\int_0^{\tau_2}(S(\tau_2-s)-S(\tau_1-s))f^1(s)ds+\int_{\tau_1}^{\tau_2}(S(\tau_1-s)f^1(s)ds\\ &+\sum_{l=1}^{\infty}\int_0^{\tau_2}(S(\tau_2-s)-S(\tau_1-s))\sigma_l^1(s,x(s),y(s))dB_l^H(s)\\ &+\sum_{l=1}^{\infty}\int_{\tau_1}^{\tau_2}(S(\tau_2-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\\ &+\sum_{0<t_k<\tau_2}(S(\tau_2-t_k)-S(\tau_1-t_k))I_k(x(t_k))+\sum_{\tau_1<t_k<\tau_2}S(\tau_2-t_k)I_k(x(t_k))\Big|_X^2.\end{aligned}$$

From $(H_1)-(H_3)$ and (2.8), we obtain

$$\begin{aligned}&\sup_{t\in J}\mathbb{E}\Big|h(\tau_2)-h(\tau_1)\Big|_X^2\\ &\quad\leq 12M^2\Big\|(S(\tau_2)-S(\tau_1)\Big\|^2\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sup_{z\in B(0,q)}\sum_{k=1}^{m}\mathbb{E}|I_k(z)|_X^2\\ &\qquad+\|p_1\|_{L^1}+c_H H(2H-1)b^{2H}(\alpha_1\mathbb{E}|x(t)|_X^2+\beta_1\mathbb{E}|y(t)|_X^2+c_1)\Big)\\ &\qquad+12\int_0^{t_2}\Big\|S(\tau_2-s)-S(\tau_1-s)\Big\|^2p_1(s)ds+12\int_{\tau_1}^{\tau_2}\Big\|S(\tau_1-s)\Big\|^2p_1(s)ds\\ &\qquad+12c_H H(2H-1)t_2^{2H-1}\int_0^{\tau_1}\Big\|S(\tau_2-s)-S(\tau_1-s)\Big\|^2\mathbb{E}\Big\|\sigma^1(s,x(s),y(s))\Big\|^2ds\\ &\qquad+12c_H H(2H-1)(\tau_2-\tau_1)^{2H-1}\int_{\tau_1}^{\tau_2}\Big\|S(\tau_2-s)\Big\|^2\mathbb{E}\Big\|\sigma^1(s,x(s),y(s))\Big\|^2ds\\ &\qquad+12m\sum_{0<t_k<\tau_1}\Big\|S(\tau_1-t_k)-S(\tau_2-t_k)\Big\|^2\sup_{z\in B(0,q)}\mathbb{E}\Big|I_k(z)\Big|_X^2\\ &\quad+12m\sum_{\tau_1<t_k<\tau_2}\Big\|S(\tau_2-t_k)\Big\|^2\sup_{z\in B(0,q)}\mathbb{E}|I_k(z)|_X^2,\end{aligned}$$

which gives

$$
\begin{aligned}
&\mathbb{E}\Big|h(\tau_2)-h(\tau_1)\Big|^2_X\\
&\quad\leq 12M^2\Big\|(S(\tau_2)-S(\tau_1)\Big\|^2\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sup_{z\in B(0,q)}\sum_{k=1}^{m}\mathbb{E}|I_k(z)|^2_X+\|p_1\|_{L^1}\\
&\quad+c_H H(2H-1)b^{2H-1}(\alpha_1\mathbb{E}|x(t)|^2_X+\beta_1\mathbb{E}|y(t)|^2_X+c_1)\Big)\\
&\quad+12\int_0^{\tau_1}\Big\|S(\tau_2-s)-S(\tau_1-s)\Big\|^2p_1(s)ds+12\int_{\tau_1}^{\tau_2}\Big\|S(\tau_2-s)\Big\|^2p_1(s)ds\\
&\quad+12c_H H(2H-1)(\tau_2)^{2H-1}\int_0^{\tau_2}\Big\|S(\tau_2-s)-S(\tau_1-s)\Big\|^2\Big(\alpha_1\mathbb{E}|x(t)|^2_X\\
&\quad+\beta_1\mathbb{E}|y(t)|^2_X+c_1)ds\Big)\\
&\quad+12c_H H(2H-1)(t_2-t_1)^{2H-1}\int_{\tau_1}^{\tau_2}\Big\|S(\tau_1-s)\Big\|^2\Big(\alpha_1\mathbb{E}|x(t)|^2_X+\beta_1\mathbb{E}|y(t)|^2_X+c_1)ds\Big)\\
&\quad+12m\sum_{0<t_k<\tau_1}\Big\|S(\tau_2-t_k)-S(\tau_1-t_k)\Big\|^2\sup_{z\in B(0,q)}\mathbb{E}|I_k(z)|^2_X\\
&\quad+12m\sum_{\tau_1<t_k<\tau_2}\Big\|S(\tau_2-t_k)\Big\|^2\sup_{z\in B(0,q)}\mathbb{E}|I_k(z)|^2_X.
\end{aligned}
$$

Therefore, we arrive at

$$
\begin{aligned}
\mathbb{E}\Big|h(\tau_2)-h(\tau_1)\Big|^2_X &\leq 18M^2\Big\|(S(t_2)-S(t_1)\Big\|^2\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}\sup_{z\in B(0,q)}\mathbb{E}|I_k(z)|^2_X+\|p_1\|_{L^1}\\
&\quad+c_H H(2H-1)b^{2H}(\alpha_1 q+\beta_1 q+c_1)\Big)\\
&\quad+12\int_0^{\tau_1}\Big\|S(\tau_2-s)-S(\tau_1-s)\Big\|^2p_1(s)ds+12\int_{\tau_1}^{\tau_2}\Big\|S(\tau_1-s)\Big\|^2p_1(s)ds\\
&\quad+12c_H H(2H-1)(t_2)^{2H-1}\int_0^{t_2}\Big\|S(\tau_2-s)-S(\tau_1-s)\Big\|^2\Big(\alpha_1 q\\
&\quad+\beta_1 q+c_1\Big)ds\\
&\quad+12c_H H(2H-1)(\tau_2-\tau_1)^{2H-1}\int_{\tau_1}^{\tau_2}\Big\|S(t_2-s)\Big\|^2\Big(\alpha_1 q\\
&\quad+\beta_1 q+c_1\Big)ds\\
&\quad+12m\sum_{0<t_k<t_1}\Big\|S(\tau_2-t_k)-S(\tau_1-t_k)\Big\|^2\sup_{z\in B(0,q)}\mathbb{E}|I_k(z)|^2_X\\
&\quad+12m\sum_{\tau_1<t_k<\tau_2}\Big\|S(\tau_2-t_k)\Big\|^2\sup_{z\in B(0,q)}\mathbb{E}|I_k(z)|^2_X.
\end{aligned}
$$

Similarly, we obtain that

$$\begin{aligned}\mathbb{E}\Big|\overline{h}(t_2)-\overline{h}(t_1)\Big|_X^2 &\leq 12M^2\Big\|(S(\tau_2)-S(\tau_1)\Big\|^2\Big\|(I-S(b))^{-1}\Big\|^2\Big(\sup_{z\in B(0,q)}\sum_{k=1}^{m}\mathbb{E}|\overline{I}_k(z)|_X^2+\|p_2\|_{L^1}\\ &+12c_H H(2H-1)b^{2H}(\alpha_2 q+\beta_2 q+c_2)\Big)\\ &+12\int_0^{\tau_1}\Big\|S(t_2-s)-S(t_1-s)\Big\|^2 p_2(s)ds+12\int_{\tau_1}^{\tau_2}\Big\|S(t_2-s)\Big\|^2 p_2(s)ds\\ &+12c_H H(2H-1)(t_2)^{2H-1}\int_0^{\tau_1}\Big\|S(\tau_2-s)-S(\tau_1-s)\Big\|^2\Big(\alpha_2 q\\ &+\beta_2 q+c_2\Big)ds\\ &+12c_H H(2H-1)(\tau_2-\tau_1)^{2H-1}\int_{\tau_1}^{\tau_2}\Big\|S(\tau_2-s)\Big\|^2\Big(\alpha_2 q\\ &+\beta_2 q+c_2\Big)ds\\ &+12m\sum_{0<t_k<\tau_1}\Big\|S(\tau_2-t_k)-S(\tau_1-t_k)\Big\|^2\sup_{z\in B(0,q)}\mathbb{E}|\overline{I}_k(z)|_X^2\\ &+12m\sum_{\tau_1<t_k<\tau_2}\Big\|S(\tau_2-t_k)\Big\|^2\sup_{z\in B(0,q)}\mathbb{E}|\overline{I}_k(z)|_X^2.\end{aligned}$$

The right-hand term tends to zero as $|\tau_2-\tau_1|\to 0$ since $S(t)$ is a strongly continuous operator and the compactness of $S(t)$ for $t>0$ implies the continuity in the uniform operator topology [37]. This proves the equicontinuity.

Step 3.- $(N(B_q)(t)$ is precompact in $X\times X$.

As a consequence of Steps 1 and 2, together with the Arzelá-Ascoli theorem, it suffices to show that N maps B_q into a precompact set in $X\times X$. Let $0<t<b$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$. For $(x,y)\in B_q$ we define

$$\begin{aligned}h_\epsilon(t) &= S(\epsilon)S(t-\epsilon)(I-S(b))^{-1}\Big(\sum_{k=0}^{m}S(b-t_k)I_k(x(t_k))+\int_0^b S(b-s)f^1(s)ds\\ &+\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,(s),y(s))dB_l^H(s)\Big)+S(\epsilon)\int_0^{t-\epsilon}S(t-s)f^1(s)ds\\ &+\sum_{l=1}^{\infty}S(\epsilon)\int_0^{t-\epsilon}S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\\ &+S(\epsilon)\sum_{0<t_k<t-\epsilon}S(t-\epsilon-t_k)I_k(x(t_k))\end{aligned}$$

Since $S(t)$ is a compact operator, the set

$$H_\epsilon=\{\widetilde{h}_\epsilon(t)=(h_\epsilon(t),\overline{h}_\epsilon(t)):\widetilde{h}_\epsilon\in N_\epsilon(x,y)\ \text{ for }\ \text{each } (x,y)\in B_q\}$$

is precompact. Now,

$$\begin{aligned}
\mathbb{E}\Big|h(t)-h_\epsilon(t)\Big|_X^2 &\leq 3\mathbb{E}\Big|\int_{t-\epsilon}^t S(t-s)f^1(s)ds)\Big|_X^2\\
&+3\mathbb{E}\Big|\sum_{l=1}^\infty\int_{t-\epsilon}^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\Big|_X^2\\
&+3\mathbb{E}\Big|\sum_{t-\epsilon<t_k<t} S(t-t_k)I_k(x(t_k))\Big|_X^2\\
&\leq 3M^2\int_{t-\epsilon}^t p_1(s)ds\\
&+3M^2(c_H H(2H-1)\epsilon^{2H-1}\int_{t-\epsilon}^t(\alpha_1 q+\beta_1 q+c_1))ds\\
&+3M^2 m\sum_{t-\epsilon<t_k<t}\sup_{z\in B(0,q)}\mathbb{E}\Big|I_k(z)\Big|_X^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\Big|\overline{h}(t)-\overline{h}_\epsilon(t)\Big|_X^2 &\leq 3M^2\int_{t-\epsilon}^t p_2(s)ds\\
&+3M^2(c_H H(2H-1)\epsilon^{2H-1}\int_{t-\epsilon}^t(\alpha_2 q+\beta_2 q+c_2))ds\\
&+3M^2(m\sum_{t-\epsilon<t_k<t}\sup_{\overline{z}\in B(0,q)}\mathbb{E}\Big|\overline{I}_k(\overline{z})\Big|_X^2)
\end{aligned}$$

The right-hand side tends to 0, as $\epsilon\to 0$. Therefore, there are precompact sets arbitrarily close to the set $H=\{\widetilde{h}(t)=(h(t),\overline{h}(t)):\widetilde{h}\in N(x,y)\ \text{for each}\ (x,y)\in B_q\}$. This set is then precompact in $X\times X$.

Step 4.- $N=(N_1,N_2)$ has a closed graph.

Let $u_n=(x_n,y_n)\longrightarrow z_*=(x_*,y_*),(h_n,\overline{h}_n)\in N(u_n)$ and $(h_n,\overline{h}_n)\longrightarrow(h_*,\overline{h}_*)$ as $n\longrightarrow\infty$, we shall prove that $h_*\in N_1(u_*)$. The fact that $h_n\in N_1(u_n)$ and $\overline{h}_n\in N_2(u_n)$ means that there exists $f_n^i\in S_{F^i,u_n}$ for each $i=1,2$ such that

$$\begin{aligned}
h_n(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{0<t_k<t}S(b-t_k)I_k(x_n(t_k))+\int_0^b S(b-s)f_n^1(s)ds\\
&+\sum_{l=1}^\infty\int_0^b S(b-s)\sigma_l^1(s,x_n(s),y_n(s))dB_l^H(s)\Big)+\int_0^t S(t-s)f_n^1(s)ds\\
&+\sum_{l=1}^\infty\int_0^t S(t-s)\sigma_l^1(s,x_n(s),y_n(s))dB_l^H(s)+\sum_{0<t_k<t}S(t-t_k)I_k(x_n(t_k))
\end{aligned}$$

First, notice that

$$\begin{aligned}\Big\| h_n - S(t)(I &- S(b))^{-1}\Big(\sum_{k=1}^{m} S(b-t_k)I_k(x_n(t_k))\\ &+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x_n(s),y_n(s))dB_l^H(s)\Big)\\ &\quad -\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x_n(s),y_n(s))dB_l^H(s) - \sum_{0<t_k<k} S(t-t_k)I_k(x_n(t_k))\\ &- h_* + S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m} S(b-t_k)I_k(x_*(t_k))\\ &\quad -\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\\ &+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x_*(s),y_*(s))dB_l^H(s)\Big)\\ &\quad + \sum_{0<t_k<k} S(t-t_k)I_k(x(t_k))\Big\|_{PC} \longrightarrow 0, \quad \text{as } n\longrightarrow +\infty.\end{aligned}$$

Now, consider the continuous linear operator $\Gamma : L^2(J,X) \longrightarrow PC$ defined for each $i=1,2$, by

$$\Gamma(v^i)(t) = S(t)(I-S(b))^{-1}\int_0^b S(b-s)v^i(s)ds + \int_0^t S(t-s)v^i(s)ds.$$

From the definition of Γ we know that

$$\begin{aligned}\Big(h_n(t) &- S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m} S(b-t_k)I_k(x_n(t_k)) + \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x_n(s),y_n(s))dB_l^H(s)\Big)\\ &-\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x_n(s),y_n(s))dB_l^H(s) - \sum_{0<t_k<t} S(t-t_k)I_k(x_n(t_k))\Big) \in \Gamma(S_{F^1,u_n})\end{aligned}$$

and

$$\begin{aligned}(\overline{h}_n(t) &- S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m} S(b-t_k)\overline{I}_k(y_n(t_k)) + \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(s,x_n(s),y_n(s))dB_l^H(s)\Big)\\ &-\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s,x_n(s),y_n(s))dB_l^H(s) - \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y_n(t_k))\Big) \in \Gamma(S_{F^2,u_n})\end{aligned}$$

Since $u_n=(x_n,y_n)\longrightarrow z_*=(x_*,y_*)$ and $\Gamma\circ S_{F^i}$ is a closed graph operator thanks to Lemma 2.4, then there exists $f_*^i\in S_{F^i,u_*}$ for each $i=1,2$, such that

$$\begin{aligned} h_*(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}S(b-t_k)I_k(x_*(t_k))+\int_0^b S(b-s)f_*^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x_*(s),y_*(s))dB_l^H(s)\Big)+\int_0^t S(t-s)f_*^1(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x_*(s),y_*(s))dB_l^H(s)+\sum_{0<t_k<t}S(t-t_k)I_k(x_*(t_k)). \end{aligned}$$

Similarly,

$$\begin{aligned} \overline{h}_*(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}S(b-t_k)\overline{I}_k(\overline{y}_*(t_k))+\int_0^b S(b-s)f_*^2(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(s,x_*(s),y_*(s))dB_l^H(s)\Big)+\int_0^t S(t-s)f_*^2(s)ds \\ &+ \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s,x_*(s),y_*(s))dB_l^H(s)+\sum_{0<t_k<t}S(t-t_k)\overline{I}_k(y_*(t_k)). \end{aligned}$$

Hence $(h_*,h_*)\in(N_1(u_*),N_2(u_*))$, proving our claim. Lemma 2.6 yields that N is upper semicontinuous.

□

Now, we present our first result on the existence and compactness of solution set of Problem (1.1).

Theorem 3.1. *Assume that $F^i:[0,b]\times X\times X\longrightarrow\mathcal{P}_{c,cp}(X)$ is a Carathèodory map and $(H_1)-(H_3)$ hold as well. Then, Problem* (1.1) *possesses at least one mild solution on J. If further X is a reflexive space, then the solution set is compact in $PC\times PC$.*

Proof. **Part 1.-** Existence of solutions.

We transform Problem (1.1) into a fixed point problem. Consider the multivalued operator $N:PC\times PC\to\mathcal{P}(PC\times PC)$ defined in Lemma 3.2. It is clear that all solutions of Problem (1.1) are fixed points of the multivalued operator N defined previously. We shall show that N satisfies assumptions of Lemma 2.11. Since for each $(x,y)\in PC\times PC$, the nonlinearity F^i takes convex values, the selection set $S_{F^i,u}$ is convex, and therefore N has convex values. From Lemma 3.2, N is completely continuous and u.s.c.

Let us now obtain some a priori bounds on solutions. Let $(x,y)\in PC\times PC$ be a solution of the abstract nonlinear equation $x\in N_1(x,y)$ and $y\in N_2(x,y)$. Then there exists $f^i \in S_{F^i}$ for $t \in [0,b]$ for each $i = \{1,2\}$, namely

$$\begin{aligned}
x(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k)) \\
&+ \int_0^b S(b-s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\Big) \\
&+ \int_0^t S(t-s)f^1(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s) \\
&+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k))
\end{aligned}$$

and

$$\begin{aligned}
y(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}(S(b-t_k))\overline{I}_k(y(t_k)) \\
&+ \int_0^b S(b-s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(s,x(s),y(s))dB_l^H(s)\Big) \\
&+ \int_0^t S(t-s)f^2(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s,x(s),y(s))dB_l^H(s) \\
&+ \sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k)).
\end{aligned}$$

We first obtain an estimation for the third part,

$$\begin{aligned}
\mathbb{E}|x(t)|_X^2 &= \mathbb{E}\Big|S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k)) + \int_0^b S(b-s)f^1(s)ds \\
&+ \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s)\Big) + \int_0^t S(t-s)f^1(s)ds \\
&+ \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k))\Big|_X^2 \\
&\le 12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}(d_k\mathbb{E}|x(t_k)|_X^2 + e_k) + \|p_1\|_{L^1} \\
&+ c_H H(2H-1)b^{2H}(\alpha_1\mathbb{E}|x(t)|_X^2 + \beta_1\mathbb{E}|y(t)|_X^2 + c_1)\Big) + 4M^2\|p_1\|_{L^1} \\
&+ 4M^2 c_H H(2H-1)b^{2H}(\alpha_1\mathbb{E}|x(t)|_X^2 + \beta_1\mathbb{E}|y(t)|_X^2 + c_1) \\
&+ 4M^2 m\sum_{k=1}^{m}(d_k\mathbb{E}|x(t_k)|_X^2 + e_k).
\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{E}|y(t)|_X^2 &\leq 12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}(\overline{d}_k\mathbb{E}|y(t_k)|_X^2+\overline{e}_k)+\|p_2\|_{L^1}\\ &+c_H H(2H-1)b^{2H}(\alpha_2\mathbb{E}|x(t)|_X^2+\beta_2\mathbb{E}|y(t)|_X^2+c_2)\Big)+4M^2\|p_1\|_{L^1}\\ &+4M^2c_H H(2H-1)b^{2H}(\alpha_2\mathbb{E}|x(t)|_X^2+\beta_2\mathbb{E}|y(t)|_X^2+c_2)\\ &+4M^2m\sum_{k=1}^{m}(\overline{d}_k\mathbb{E}|y(t_k)|_X^2+\overline{e}_k).\end{aligned}$$

Consider the function $\mu,\overline{\mu}$ defined on J by

$$\mu(t)=\sup\{\mathbb{E}|x(s)|_X^2\ :0\leq s\leq t\}\ \text{ and }\ \overline{\mu}(t)=\sup\{\mathbb{E}|y(s)|_X^2\ :0\leq s\leq t\}.$$

This implies, for each $t\in J$,

$$\begin{aligned}\mu(t) &\leq 12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}(d_k\mu(t)+e_k)+\|p_1\|_{L^1}\\ &+c_H H(2H-1)b^{2H}(\alpha_1\mu(t)+\beta_1\overline{\mu}(t)+c_1)\Big)+4M^2\|p_1\|_{L^1}\\ &+4M^2(c_H H(2H-1)b^{2H}(\alpha_1\mu(t)+\beta_1\overline{\mu}(t)+c_1)\\ &+4M^2m\sum_{k=1}^{m}(d_k\mu(t)+e_k)\\ &= 12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}e_k+\|p_1\|_{L^1}+c_H H(2H-1)b^{2H}c_1\Big)\\ &+\mu(t)\Big(12M^4m\Big\|(I-S(b))^{-1}\Big\|^2(\sum_{k=1}^{m}d_k+c_H H(2H-1)b^{2H}\alpha_1)\\ &+4M^2(c_H H(2H-1)b^{2H}\alpha_1+4M^2m\sum_{k=1}^{m}d_k\Big)\\ &+\overline{\mu}(t)\Big(12M^4\Big\|(I-S(b))^{-1}\Big\|^2c_H H(2H-1)b^{2H}\beta_1+4M^2c_H H(2H-1)b^{2H}\beta_1\Big)\\ &= K_1+K_2\mu(t)+K_3\overline{\mu}(t)\end{aligned}$$

There exist constants $K_j,\overline{K_j}$ for each j=1,2,3 defined as follows

$$K_1=12M^4\Big\|(I-S(b))^{-1}\Big\|^2\Big(m\sum_{k=1}^{m}e_k+\|p_1\|_{L^1}+c_H H(2H-1)b^{2H}c_1\Big)$$

and

$$K_2=4M^4\Big\|(I-S(b))^{-1}\Big\|^2(m\sum_{k=1}^{m}d_k+c_H H(2H-1)b^{2H}\alpha_1)$$

$$+4M^2(c_H H(2H-1)b^{2H}\alpha_1 + 4M^2 m\sum_{k=1}^{m} d_k$$

and

$$K_3 = 12mM^4\left\|(I-S(b))^{-1}\right\|^2 c_H H(2H-1)b^{2H}\beta_1 + 4M^2(c_H H(2H-1)b^{2H}\beta_1.$$

Similarly,

$$\begin{array}{rcl}
\overline{\mu}(t) & \leq & 12M^4\left\|(I-S(b))^{-1}\right\|^2\Big(m\sum_{k=1}^{m}\overline{e}_k + \|p_2\|_{L^1} + c_H H(2H-1)b^{2H}c_2\Big) \\
& + & \overline{\mu}(t)\Big(12M^4\left\|(I-S(b))^{-1}\right\|^2(m\sum_{k=1}^{m}\overline{d}_k + c_H H(2H-1)b^{2H}\beta_2) \\
& + & 4M^2(c_H H(2H-1)b^{2H}\beta_2 + 4M^2\sum_{0<t_k<t}\overline{d}_k\Big) \\
& + & \mu(t)\Big(4M^4\left\|(I-S(b))^{-1}\right\|^2 c_H H(2H-1)b^{2H}\alpha_2 + 4M^4 c_H H(2H-1)b^{2H}\alpha_2\Big) \\
& = & \overline{K}_1 + \overline{K}_2\mu(t) + \overline{K}_3\overline{\mu}(t)
\end{array}$$

where

$$\overline{K}_1 = 12mM^4\left\|(I-S(b))^{-1}\right\|^2\Big(\sum_{0<t_k<t}\overline{e}_k + \|p_2\|_{L^1} + c_H H(2H-1)b^{2H}c_2\Big)$$

and

$$\overline{K}_2 = 12M^4\left\|(I-S(b))^{-1}\right\|^2 c_H H(2H-1)b^{2H}\alpha_2 + 4M^4 c_H H(2H-1)b^{2H}\alpha_2$$

and

$$\overline{K}_3 = \Big(12M^4\left\|(I-S(b))^{-1}\right\|^2(m\sum_{k=1}^{m}\overline{d}_k + c_H H(2H-1)b^{2H}\beta_2)\Big)$$

$$+4M^2\Big(c_H H(2H-1)b^{2H}\beta_2 + 4M^2 m\sum_{k=1}^{m}\overline{d}_k\Big).$$

On the other hand,

$$\mu(t) + \overline{\mu}(t) \leq \widetilde{K}_1 + \widetilde{K}_2(\mu(t) + \overline{\mu}(t))$$

Thus, we have

$$\mu(t) + \overline{\mu}(t) \leq \frac{\widetilde{K}_1}{1-\widetilde{K}_2} = M_*$$

the maximum being taken componentwise, and $\widetilde{K}_2$ is a suitable value lower than 1

$$\widetilde{K}_1 = \overline{K}_1 + K_1 \quad \widetilde{K}_2 = \max\{K_2 + \overline{K}_2, K_3 + \overline{K}_3\} < 1.$$

Thus

$$\mathbb{E}|x(t)|_X^2 + \mathbb{E}|y(t)|_X^2 \leq M_*,$$

and, consequently,

$$\|x\|_{PC}^2 \leq M_* \text{ and } \|y\|_{PC}^2 \leq M_*.$$

Let

$$U = \{(x, y) \in PC \times PC \quad : \|x\|_{PC}^2 < M_* + 1 \quad \text{and} \quad \|y\|_{PC}^2 < M_* + 1\}$$

and consider the operator $N : \overline{U} \to \mathcal{P}_{c,cp}(PC \times PC)$. From the choice of U, there is no $(x, y) \in \partial U$ such that $x \in \lambda N_1(x, y)$ and $y \in \lambda N_2(x, y)$ for some $\lambda \in (0, 1)$. As a consequence of the Leray and Schauder nonlinear alternative (Lemma 2.11), we deduce that N has a fixed point (x, y) in U, solution of Problem (1.1).

Part 2.- Compactness of the solution set. Let

$$S_F = \{(x, y) \in PC \times PC : \quad (x, y) \quad \text{is a solution of Problem (1.1)}\}$$

From Part 1, $S_F \neq \emptyset$ and there exists M_* such that for every $(x, y) \in S_F$,$\|x\|_{PC}^2 \leq M_*$ and $\|y\|_{PC}^2 \leq M_*$. Since N is completely continuous, then $N(S_F) = (N_1(S_{F^1}), N_2(S_{F^2}))$ is relatively compact in $PC \times PC$. Let $(x, y) \in S_F$ then $(x, y) \in N(x, y)$ and $S_F \subset \overline{N(S_F)}$. It remains to prove that S_F is a closed set in $PC \times PC$. Let $(x_n, y_n) \in S_F$ such that (x_n, y_n) converges to (x, y). For every $n \in N$, there exists $v_n^i(t) \in F^i(t, x_n, y_n)$ a.e. $t \in J$ for each $i \in \{1, 2\}$ such that

$$\begin{aligned} x_n(t) \quad = \quad & S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b - t_k))I_k(x_n(t_k)) \\ & + \int_0^b S(b - s)v_n^1(s)ds + \sum_{l=1}^{\infty}\int_0^b S(b - s)\sigma_l^1(t, x_n(s), y_n(s))dB_l^H(s)\Big) \\ & + \int_0^t S(t - s)v_n^1(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t - s)\sigma_l^1(s, x_n(s), y_n(s))dB_l^H(s) \\ & + \sum_{0<t_k<t} S(t - t_k)I_k(x_n(t_k)), \end{aligned}$$

and

$$
\begin{aligned}
y_n(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))\overline{I}_k(y_n(t_k)) \\
&+\int_0^b S(-s)\upsilon_n^2(s)ds+\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^2(t,x_n(s),y_n(s))dB_l^H(s)\Big) \\
&+\int_0^t S(t-s)\upsilon_n^2(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s,x_n(s),y_n(s))dB_l^H(s) \\
&+\sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y_n(t_k)).
\end{aligned}
$$

(H_2) implies that for a.e. $t\in J$, $|\upsilon_n^i|_X\le p_i(t)$, $\quad i=1,2$, hence $(\upsilon_n^i)_{n\in N}$ is integrably bounded. Note that this still remains true when S_F is a bounded set. Since X is reflexive, by Theorem 2.1, there exists a subsequence, still denoted by $(\upsilon_n^i)_{n\in N}$, which converges weakly to some limit $\upsilon^i\in L^2(J,X)$. Moreover, the mapping $\Gamma : L^2(J,X)\longrightarrow PC$ defined by

$$
\Gamma(g^i)(t)=\int_0^t S(t-s)g^i(s)ds
$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies [10]. Therefore for a.e. $t\in J$, the sequence $(x_n(t),y_n(t))$ converges strongly to $(x(t),y(t))$ and by the continuity of $(I_k,\overline{I}_k)$ (which we assumed in (H_3)) it follows that

$$
\begin{aligned}
x(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{0<t_k<t}(S(b-t_k))I_k(x(t_k)) \\
&+\int_0^b S(t-s)\upsilon^1(s)ds+\sum_{l=1}^{\infty}\int_0^b S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big) \\
&+\int_0^t S(t-s)\upsilon^1(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(s,x(s),y(s))dB_l^H(s) \\
&+\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)),
\end{aligned}
$$

and

$$\begin{aligned} y(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{0<t_k<t}(S(b-t_k))\overline{I}_k(y(t_k)) \\ &+\int_0^b S(t-s)v^2(s)ds+\sum_{l=1}^{\infty}\int_0^b S(t-s)\sigma_l^2(t,x(s),y(s))dB_l^H(s)\Big) \\ &+\int_0^t S(t-s)v^2(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^2(s,x(s),y(s))dB_l^H(s) \\ &+\sum_{0<t_k<t} S(t-t_k)\overline{I}_k(y(t_k)). \end{aligned}$$

Now we need to prove that $v^i(t)\in F^i(t,x(t),y(t))$, for a.e. $t\in J$. Lemma 2.9 yields the existence of constants $\alpha_i^n\geq 0$, $j=1,2\ldots,k(n)$ and $i=1,2$ such that $\sum_{j=1}^{k(n)}\alpha_i^n=1$ and the sequence of convex combinations $g_n^i(.)=\sum_{j=1}^{k(n)}\alpha_j^n v_j^i(.)$ converges strongly to some limit $v^i\in L^2(J,X)$. Since F^i takes convex values, using Lemma 2.8, we obtain that

$$\begin{aligned} v^i(t) &\in \bigcap_{n\geq 1}\overline{\{g_k^i(t):k\geq n\}},\quad a.e\quad t\in J \\ &\subseteq \bigcap_{n\geq 1}\overline{co}\{v_k^i(t),\quad k\geq n\} \\ &\subseteq \bigcap_{n\geq 1}\overline{co}\{\bigcup_{k\geq n}F^i(t,x_k(t),y_k(t))\} \end{aligned}$$

$$\subseteq \overline{co}\{\limsup_{k\to\infty}F^i(t,x_k(t),y_k(t))\}. \tag{3.2}$$

Since F^i is u.s.c. and has compact values, then by Lemma 2.7, we have

$$\limsup_{n\to\infty}F^i(t,x_n(t),y_n(t))\subseteq F^i(t,x(t),y(t))\quad \text{for a.e}\quad t\in J.$$

This and (3.2) imply that $v^i(t)\in\overline{co}(F^i(t,x(t),y(t)))$. Since, for each $i=1,2$, $F^i(.,.)$ has closed, convex values, we deduce that $v^i(t)\in F^i(t,x(t),y(t))$ for a.e. $t\in J$, for each $i=1,2$ as claimed. Hence $(x,y)\in S_{F^i}$ which proves that S_{F^i}, for each $i=1,2$, is closed, hence compact in $PC\times PC$.

□

For the next result we can prove the a priori estimates of solution for problem (1.1) by similar arguments to those used to prove Theorem 3.1 in Djebali *et al.* [18], so we omit the proof.

Theorem 3.2. *Assume hypotheses in Lemma 3.2 hold, but replacing $(H_1), (H_3)$ by the next ones:*

$(\bar{H}_1)$ *There exist positive constants α_i and $\beta_i$and c_i for each $i = 1, 2$ and $\gamma_1, \gamma_2 \in [0, 1)$ such that*

$$E\|\sigma^1(t, x, y)\|_X^2 \leq \alpha_1(E|x|_X)^{\gamma_1} + \beta_1(E|y|_X)^{\gamma_2} + c_1,$$

$$E\|\sigma^2(t, x, y)\|^2 \leq \alpha_2(E|x|_X)^{\bar{\gamma}_1} + \beta_2(E|y|_X)^{\bar{\gamma}_2} + c_2,$$

and

$$\sum_{l=1}^{\infty} \int_0^b \|\sigma_l^1(t, x, y)\|_{L_Q^0}^2 dt < \infty$$

for all $X-$valued stochastic processes x, $y \in X$ and $t \in J$.

$(\bar{H}_3)$ *There exist constants $d_k, \overline{d}_k \geq 0$ and $e_k, \overline{e}_k \geq 0, \nu_k, \bar{\nu}_k \in [0, 1)$ for each $k = 1, \ldots, m$ such that*

$$E|I_k(x)|_X^2 \leq d_k E|x|_X^{\nu_k} + e_k, \quad E|\overline{I}_k(y)|_X^2 \leq \overline{d}_k E|y|_X^{\bar{\nu}_k} + \overline{e}_k,$$

for all X-valued stochastic process $x, y \in X$.

Then, Problem (1.1) has at least one mild solution.

3.2 The nonconvex case

Now we present a second result for the problem (1.1) with a nonconvex valued right-hand side. Our considerations are based on a multivalued version of Perov's fixed point theorem proved by Petre and Petruşel [39] (see also Ouahab [35]).

Definition 3.2. *Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a matrix M convergent to zero such that*

$$d(N(x), N(y)) \leq Md(x, y) \text{ for all } x, y \in X.$$

Theorem 3.3. *Let (X, d) be a generalized complete metric space, and let $F : X \to \mathcal{P}_{cl}(X)$ be a multivalued map. Assume that there exist $A, B, C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$H_d(F(x), F(y)) \leq Ad(x, y) + Bd(y, F(x)) + Cd(x, F(x)) \tag{3.3}$$

where $A + C$ converge to zero. Then there exists $x \in X$ such that $x \in F(x)$.

Let us introduce the following hypotheses:

(H4) $F^i : J \times X \times X \longrightarrow \mathcal{P}_{cp}(X); (t, y) \longrightarrow F^i(t, x, y)$ is measurable for each $(x, y) \in X \times X$.

(H5) There exist functions $a_i,\ b_i \in L^1([0,T],\mathbb{R}^+)$ such that

$$\begin{cases} H^2_{d_1}(F^1(t,x,y),F^1(t,\overline{x},\overline{y})) \le a_1(t)|x-\overline{x}|^2_X + b_1(t)|y-\overline{y}|^2_X \\ H^2_{d_2}(F^2(t,x,y),F^2(t,\overline{x},\overline{y})) \le a_2(t)|x-\overline{x}|^2_X + b_2(t)|y-\overline{y}|^2_X \end{cases}$$

with

$$d_i(0,F^i(t,0,0) \le a_i(t)$$

for all $x,\ y,\ \overline{x},\ \overline{y} \in X$ for each $i=1,2$

(H6) There exist functions $\alpha_i,\ \beta_i \in L^1([0,T],\mathbb{R}^+)$ for each $i=1,2$ such that

$$\begin{cases} \|\sigma^1(t,x,y)-\sigma^1(t,\overline{x},\overline{y})\|^2 \le \alpha_1(t)\|x-\overline{x}\|^2 + \beta_1(t)\|y-\overline{y}\|^2 \\ \|\sigma^2(t,x,y)-\sigma^2(t,\overline{x},\overline{y})\|^2 \le \alpha_2(t)\|x-\overline{x}\|^2 + \beta_2(t)\|y-\overline{y}\|^2 \end{cases}$$

for all $x,\ y,\ \overline{x},\ \overline{y} \in X$ and $t \in J$.

(H7) there exist constants $d_k \ge 0$ and $\overline{d}_k \ge 0,\ k=1,\dots,m$ such that

$$|I_k(x)-I_k(\overline{x})|^2_X \le d_k|x-\overline{x}|^2_X$$

and

$$|\overline{I}_k(y)-\overline{I}_k(\overline{y})|^2 \le \overline{d}_k|y-\overline{y}|^2_X$$

for all $x,\ y,\ \overline{x},\ \overline{y} \in X$.

Theorem 3.4. *Assume that hypotheses (H4)-(H7) are fulfilled, and let A_1, A_2, B_1, B_2 be defined by*

$$A_1 = 2M\sqrt{\begin{array}{l} 3m\|(I-S(b))^{-1}\|^2\sum_{k=1}^m d_k + 3M^2\|(I-S(b))^{-1}\|^2\|a_1\|_{L^1} \\ +3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\alpha_1\|_{L^1} \\ +\|a_1\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\alpha_1\|_{L^1} + m\sum_{k=1}^m d_k \end{array}}$$

$$A_2 = 2M\sqrt{\begin{array}{l} \|b_1\|_{L^1} + 3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\beta_1\|_{L^1} \\ +\|b_1\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\beta_1\|_{L^1} \end{array}}$$

$$B_1 = 2M\sqrt{\begin{array}{l} 3m\|(I-S(b))^{-1}\|^2\sum_{k=1}^m \bar{d}_k + 3M^2\|(I-S(b))^{-1}\|^2\|a_2\|_{L^1} \\ +3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\alpha_2\|_{L^1} \\ +\|a_2\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\alpha_2\|_{L^1} + m\sum_{k=1}^m \bar{d}_k \end{array}}$$

and

$$B_2 = 2M\sqrt{\begin{matrix}\|b_2\|_{L^1} + 3M^2\|(I - S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\beta_2\|_{L^1} \\ +\|b_2\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\beta_1\|_{L^1}\end{matrix}}.$$

If the matrix

$$M_{\alpha,\beta} = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_1 \end{pmatrix}$$

converges to zero, then problem (1.1) *has at least one mild solution.*

Proof. In order to transform the problem (1.1) into a fixed point problem, let the multivalued operator $N : PC \times PC \to \mathcal{P}(PC \times PC)$ be defined as in Lemma 3.2. We shall show that N satisfies the assumptions of Theorem 3.3. Note that $(H4)$ implies that F^i, for each $i = 1, 2$ has at most linear growth, i.e.

$$\mathbb{E}|F^i(t, x, y)|_X^2 \leq a_i(t)\mathbb{E}|x|_X^2 + b_i(t)\mathbb{E}|y|_X^2$$

for a.e. $t \in J$ and all $x, y \in X$

(a) $N(x, y) \in \mathcal{P}_{cl}(PC \times PC)$ for each $(x, y) \in PC \times PC$. The proof is similar to that in Theorem 3.1, Part 1, and is omitted.

(b) There exists $M_{\alpha,\beta} \in \mathcal{M}_{2\times 2}(\mathbb{R}_+)$ convergent matrix to zero, such that

$$H_d(N(x, y), N(\overline{x}, \overline{y})) \leq M_{\alpha,\beta}\begin{pmatrix}\|x - \overline{x}\|_{PC} \\ \|y - \overline{y}\|_{PC}\end{pmatrix}, \quad \text{for all} \quad x, y, \overline{x}, \overline{y} \in PC.$$

Let $x, y, \overline{x}, \overline{y} \in PC$ and $h_i \in N_i(x, y)$, $i = 1, 2$. Then there exists $f^i(\cdot) \in S_{F^i,x,y}$ such that for each $t \in J$, we have

$$\begin{aligned} h_i(t) &= S(t)(I - S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b - t_k))\widetilde{I}_k(\overline{z^i}(t_k)) \\ &+ \int_0^b S(b - s)f^i(s)ds + \sum_{l=1}^{\infty}\int_0^b S(b - s)\sigma_l^i(t, x(s), y(s))dB_l^H(s)\Big) \\ &+ \int_0^t S(t - s)f^i(s)ds + \sum_{l=1}^{\infty}\int_0^t S(t - s)\sigma_l^i(t, x(s), y(s))dB_l^H(s) \\ &+ \sum_{0<t_k<t} S(t - t_k)\widetilde{I}_k(\overline{z^i}(t_k)) \end{aligned}$$

where

$$\widetilde{I}_k(\overline{z^1}(t_k)) = I_k(x(t_k)), \quad \text{and } \widetilde{I}_k(\overline{z^2}(t_k)) = \overline{I}_k(y(t_k)), \ k = 1, \ldots, m.$$

From $(H5)$,

$$\begin{cases} \mathbb{E}H_{d_1}^2(F(t, x, y), F(t, \overline{x}, \overline{y})) \leq a_1(t)\mathbb{E}|x - \overline{x}|_X^2 + b_1(t)\mathbb{E}|y - \overline{y}|_X^2, & \text{a.e. } t \in J, \\ \mathbb{E}H_{d_2}^2(F(t, x, y), F(t, \overline{x}, \overline{y})) \leq a_2(t)\mathbb{E}|x - \overline{x}|_X^2 + b_2(t)\mathbb{E}|y - \overline{y}|_X^2, & \text{a.e. } t \in J. \end{cases}$$

Hence there is $(w,\overline{w}) \in F^1(t,\overline{x}(t),\overline{y}(t)) \times F^2(t,\overline{x}(t),\overline{y}(t))$ such that

$$\mathbb{E}|f^1(t)-w|_X^2 \leq a_1(t)\mathbb{E}|x-\overline{x}|_X^2 + b_1(t)\mathbb{E}|y-\overline{y}|_X^2, \quad t \in J,$$

and

$$\mathbb{E}|f^2(t)-\overline{w}|^2 \leq a_2(t)\mathbb{E}|x-\overline{x}|_X^2 + b_2(t)\mathbb{E}|y-\overline{y}|_X^2, \quad t \in J.$$

Consider the multivalued maps $U_i : J \to \mathcal{P}(X)$, $i=1,2$ defined by

$$U_1(t) = \{w \in F^1(t,\overline{x}(t),\overline{y}(t)) : \mathbb{E}|f^1(t)-w|^2 \leq a_1(t)\mathbb{E}|x-\overline{x}|_X^2 + b_1(t)\mathbb{E}|y-\overline{y}|_X^2 \; a.e\, t \in J\}$$

and

$$U_2(t) = \{\overline{w} \in F^2(t,\overline{x}(t),\overline{y}(t)) : E|f^2(t)-\overline{w}|^2 \leq a_2(t)E|x-\overline{x}|_X^2 + b_2(t)E|y-\overline{y}|_X^2, \; a.e\, t \in [0,b]\}$$

that is $U_1 = \overline{B}(f^1(t), a_1(t)\mathbb{E}|x-\overline{x}|_X^2 + b_1(t)\mathbb{E}|y-\overline{y}|_X^2)$ and $U_2 = \overline{B}(f^2(t), a_2(t)\mathbb{E}|x-\overline{x}|_X^2 + b_2(t)\mathbb{E}|y-\overline{y}|_X^2)$. Since $f^i, a_i, b_i, x, y, \overline{x}, \overline{y}$ are measurable for each $i=1,2$, Theorem $III.4.1$ in [12], ensures that the closed ball U_i is measurable. In addition $(H4)$ and $(H5)$ imply that for each $(x,y) \in PC \times PC$ and $F^i(t,x(t),y(t))$ is measurable. Finally, the set $V_i(.) = U_i(.) \cap F^i(.,\overline{x}(.),\overline{y}(.))$ is nonempty. Therefore the intersection multivalued operator V_i is measurable with nonempty, closed values (see [25]), there exists a function $\overline{f}^i(t)$ which is a measurable selection for $V_i(.)$. Thus

$$\overline{f}^i(t) \in F^i(t,\overline{x}(t),\overline{y}(t)) \quad \text{for a.e. } t \in J.$$

Hence

$$\mathbb{E}|f^1(t)-\overline{f}^1(t)|_X^2 \leq a_1(t)\mathbb{E}|x-\overline{x}|_X^2 + b_1(t)\mathbb{E}|y-\overline{y}|_X^2, \quad \text{for a.e. } t \in J.$$

and

$$\mathbb{E}|f^2(t)-\overline{f}^2(t)|_X^2 \leq a_2(t)\mathbb{E}|x-\overline{x}|_X^2 + b_2(t)\mathbb{E}|y-\overline{y}|_X^2, \quad \text{for a.e. } t \in J.$$

Therefore

$$\begin{aligned}
\overline{h}_i(t) &= S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))\widetilde{I}_k(\overline{z^i}(t_k)) + \int_0^b S(b-s)\overline{f}^i(s)ds \\
&\quad + \sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^i(t,\overline{x}(s),\overline{y}(s))dB_l^H(s)\Big) + \int_0^t S(t-s)\overline{f}^i(s)ds \\
&\quad + \sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^i(t,\overline{x}(s),\overline{y}(s))dB_l^H(s) + \sum_{0<t_k<t} S(t-t_k)\widetilde{I}_k(\overline{z^i}(t_k)).
\end{aligned}$$

This implies that

$$
\begin{aligned}
&\mathbb{E}|h_1(t)-\overline{h}_1(t)|_X^2\\
&=\mathbb{E}\Big|S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(x(t_k))\\
&\quad+\int_0^b S(b-s)f^1(s)ds+\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\Big)\\
&\quad+\int_0^t S(t-s)f^1(s)ds+\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,x(s),y(s))dB_l^H(s)\\
&\quad+\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k))-S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))I_k(\overline{x}(t_k))\\
&\quad+\int_0^b S(b-s)\overline{f}^1(s)ds+\sum_{l=1}^{\infty}\int_0^b S(b-s)\sigma_l^1(t,\overline{x}(s),\overline{y}(s))dB_l^H(s)\Big)\\
&\quad-\int_0^t S(t-s)\overline{f}^1(s)ds-\sum_{l=1}^{\infty}\int_0^t S(t-s)\sigma_l^1(t,\overline{x}(s),\overline{y}(s))dB_l^H(s)\\
&\quad-\sum_{0<t_k<t}S(t-t_k)I_k(\overline{x}(t_k))\Big|_X^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathbb{E}|h_1(t)-\overline{h}_1(t)|_X^2\\
&\leq 4\mathbb{E}\Big|S(t)(I-S(b))^{-1}\Big(\sum_{k=1}^{m}(S(b-t_k))(I_k(x(t_k))-I_k(\overline{x}(t_k))\\
&\qquad+\int_0^b S(b-s)(f^1(s)-\overline{f}^1(s))ds\\
&+\sum_{l=1}^{\infty}\int_0^b S(b-s)(\sigma_l^1(t,x(s),y(s))-\sigma_l^1(t,\overline{x}(s),\overline{y}(s)))dB_l^H(s)\Big)\Big|^2\\
&\qquad+4\mathbb{E}\Big|\int_0^t S(t-s)(f^1(s)-\overline{f}^1(s))ds\Big|^2\\
&+4\mathbb{E}\Big|\sum_{l=1}^{\infty}\int_0^t S(t-s)(\sigma_l^1(t,x(s),y(s))-\sigma_l^1(t,\overline{x}(s),\overline{y}(s)))dB_l^H(s)\Big|^2\\
&\qquad+4\mathbb{E}\Big|\sum_{0<t_k<t}S(t-t_k)(I_k(x(t_k))-I_k(\overline{x}(t_k))\Big|^2.
\end{aligned}
$$

From (2.8) and $(H5)-(H8)$,

$$\begin{aligned}
\mathbb{E}|h_1(t)-\overline{h}_1(t)|_X^2 \leq\ & 12M^4m\|(I-S(b))^{-1}\|^2\sum_{k=1}^m d_k\mathbb{E}|x(t_k)-\overline{x}(t_k)|_X^2 \\
& +12M^4\|(I-S(b))^{-1}\|^2\Big(\int_0^b a_1(s)\mathbb{E}|x(s)-\overline{x}(s)|_X^2 \\
& +b_1(s)\mathbb{E}|y(s)-\overline{y}(s)|_X^2ds\Big) \\
& +12M^4\|(I-S(b))^{-1}\|^2c_HH(2H-1)b^{2H-1}\Big(\int_0^b \alpha_1(s)\mathbb{E}|x(s)-\overline{x}(s)|_X^2 \\
& +\beta_1(s)\mathbb{E}|y(s)-\overline{y}(s)|_X^2ds\Big) \\
& +4M^2\Big(\int_0^t a_1(s)\mathbb{E}|x(s)-\overline{x}(s)|_X^2+b_1(s)\mathbb{E}|y(s)-\overline{y}(s)|_X^2ds\Big) \\
& +4M^2c_HH(2H-1)b^{2H-1}\int_0^t \alpha_1(s)\mathbb{E}|x(s)-\overline{x}(s)|_X^2+\beta_1(s)\mathbb{E}|y(s)-\overline{y}(s)|_X^2ds \\
& +4mM^2\sum_{k=1}^m d_k\mathbb{E}|x(t_k)-\overline{x}(t_k)|_X^2.
\end{aligned}$$

Taking the supremum, we have

$$\begin{aligned}
\sup_{t\in J}\mathbb{E}|h_1(t)-\overline{h}_1(t)|_X^2 \leq\ & 4M^2\Big(3m\|(I-S(b))^{-1}\|^2\sum_{k=1}^m d_k+3M^2\|(I-S(b))^{-1}\|^2\|a_1\|_{L^1} \\
+\ & 3M^2\|(I-S(b))^{-1}\|^2c_HH(2H-1)b^{2H-1}\|\alpha_1\|_{L^1} \\
+\ & \|a_1\|_{L^1}+c_HH(2H-1)b^{2H-1}\|\alpha_1\|_{L^1}+m\sum_{k=1}^m d_k\Big)\sup_{t\in J}\mathbb{E}|x(t)-\overline{x}(t)|_X^2 \\
& +4M^2\Big(\|b_1\|_{L^1}+3M^2\|(I-S(b))^{-1}\|^2c_HH(2H-1)b^{2H-1}\|\beta_1\|_{L^1} \\
& +\|b_1\|_{L^1}+c_HH(2H-1)b^{2H-1}\|\beta_1\|_{L^1}\Big)\sup_{t\in J}\mathbb{E}|y(t)-\overline{y}(t)|_X^2.
\end{aligned}$$

Hence

$$\|h_1-\overline{h}_1\|_{PC} \leq A_1\|x-\overline{x}\|_{PC}+A_2\|y-\overline{y}\|_{PC},$$

and similarly

$$\begin{aligned}\sup_{t\in J}\mathbb{E}|h_2(t)-\overline{h}_2(t)|^2_X &\leq 4M^2\Big(3m\|(I-S(b))^{-1}\|^2\sum_{k=1}^m \bar{d}_k + 3M^2\|(I-S(b))^{-1}\|^2\|a_2\|_{L^1}\\ &+ 3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\alpha_2\|_{L^1}\\ &+ \|a_2\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\alpha_2\|_{L^1} + m\sum_{k=1}^m \bar{d}_k\Big)\sup_{t\in J}\mathbb{E}|y(t)-\overline{y}(t)|^2_X\\ &+4M^2\Big(\|b_2\|_{L^1} + 3M^2\|(I-S(b))^{-1}\|^2 c_H H(2H-1)b^{2H-1}\|\beta_2\|_{L^1}\\ &+\|b_2\|_{L^1} + c_H H(2H-1)b^{2H-1}\|\beta_1\|_{L^1}\Big)\sup_{t\in J}\mathbb{E}|x(t)-\overline{x}(t)|^2_X.\end{aligned}$$

Therefore

$$\|h_2-\overline{h}_2\|_{PC} \leq B_1\|x-\overline{x}\|_{PC} + B_2\|y-\overline{y}\|_{PC}.$$

By an analogous relation, obtained by exchanging the roles of x, y and $\overline{x}, \overline{y}$, we finally arrive at

$$H_d(N(x,y),N(\overline{x},\overline{y})) \leq M_{\alpha,\beta}\begin{pmatrix}\|x-\overline{x}\|_{PC}\\ \|y-\overline{y}\|_{PC}\end{pmatrix},$$

where

$$M_{\alpha,\beta} = \begin{pmatrix}A_1 & A_2\\ B_1 & B_1\end{pmatrix}.$$

Since $M_{\alpha,\beta}$ converges to zero, thanks to Theorem 3.3, we can ensure that N has a fixed point (x, y), which is a mild solution to (1.1). □

4 An example

In this section we use the abstract results proved in the above section to study the existence of mild solution for an impulsive Stokes differential inclusion.

Let $D \subset \mathbb{R}^3$ be a bounded open domain with the smooth boundary ∂D and and let $n(x)$ be the outward normal to D at the point $x \in \partial D$. Let

$$X = \{u \in (C_c^\infty(D))^3 : \nabla u = 0 \text{ in } \Omega \text{ and } n\cdot u = 0 \text{ on } \partial D\},$$

and let $E = \overline{Y}^{(L^2(D))^3}$ be the closure of Y in $(L^2(D))^3$. It is clear that, endowed with the standard inner product of the space $(L^2(D))^3$, defined by

$$\langle u, v\rangle = \sum_{i=1}^3 \langle u_i, v_i\rangle_{L^2(D)},$$

E is a Hilbert space. Let $P : (L^2(D))^3 \to X$ denote the orthogonal projection of $(L^2(D))^3$ onto X.

Consider the following system of impulsive stochastic Stokes type partial differential inclusions:

$$\begin{cases} u_t - P(\Delta u) \in F(t,u(t,x),v(t,x)) + \sigma_1(t)\frac{dB_Q^H}{dt}, & \text{a.e. } t \in [0,b],\ x \in D, \\ v_t - P(\Delta v) \in G(t,u(t,x),v(t,x)) + \sigma_2(t)\frac{dB_Q^H}{dt}, & \text{a.e. } t \in [0,b],\ x \in D, \\ u(t_k^+,x) - u(t_k^-,x) = I_k(u(t_k,x)), & \\ v(t_k^+,x) - v(t_k,x) = \overline{I}_k(v(t_k,x)), & k = 1,\dots,m \\ \nabla u = \nabla v = 0, & (t,x) \in [0,b] \times \partial D \\ u = v = 0, & (t,x) \in [0,b] \times \partial D \\ u(0,x) = u(b,x), \quad v(0,x) = v(b,x) & x \in D, \end{cases} \tag{4.1}$$

where $P(\Delta)$ is the Stokes operator. Let $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = (H^2(D) \cap H_0^1(D))^3 \cap X \\ Au = -P(\Delta u),\ u \in D(A). \end{cases}$$

Lemma 4.1. *(Fujita-Kato)(Theorem 7.3.4, [36]) The operator A, defined as above, is the generator of a compact and analytic C_0-semigroup of contractions in X.*

Let us assume that

($\mathcal{K}_1$) Let $f_i, g_i : [0,b] \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ i = 1,2$ are functions such that

$$f_1(t,x,u,v) \le f_2(t,x,u,v),\ g_1(t,x,u,v) \le g_2(t,x,u,v), \text{ for all } (t,x,u,v) \in [0,b]\times D\times\mathbb{R}\times\mathbb{R}.$$

($\mathcal{K}_2$) there exist $\phi_i, \psi_i \in L^1([0,b],\mathbb{R}_+) \cap L^\infty([0,b],\mathbb{R}_+)$ such that

$$|f_i(t,x,u,v)| \le \phi_i(t) \text{ and } |g_i(t,x,u,v)| \le \psi_i(t),\ i = 1,2$$

for each $(t,x,u,v) \in [0,b] \times D \times \mathbb{R} \times \mathbb{R}$.

($\mathcal{K}_3$) f_1, g_1 are l.s.c and f_2, g_2 are u.s.c.

($\mathcal{K}_4$) The function $\sigma_, \sigma_1 : [0,b] \longrightarrow L_Q^2(\mathcal{K},\mathcal{H})$ is bounded, that is, there exists a positive constant L such that

$$\int_0^b \|\sigma_i(s)\|^2_{L_Q^2} ds < L_i, \quad i = 1,2.$$

Lemma 4.2. *[13] Let $f_i, g_i : [0,b] \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ i = 1,2$ be functions satisfying $(\mathcal{K}_2) - (\mathcal{K}_3)$. Let $F : G : [0,b] \times L^2(D) \times L^2(D) \to \mathcal{P}(L^2(D))$ be multivalued maps defined by*

$$F(t,u,v) = \{f \in L^2(D) : f(x) \in [f_1(t,x,u,v), f_2(t,x,u,v]\}$$

and

$$G(t,u,v) = \{g \in L^2(D) : g(x) \in [g_1(t,x,u,v), g_2(t,x,u,v)]\}.$$

Then F and G are nonempty, u.s.c. with weakly compact and convex values. Moreover $F(.,.,.), G(.,.,.) \in \mathcal{P}_{cl,b,cv}(L^2(D))$.

Let

$$x(t)(\xi) = u(t,\xi) \quad t \in J, \quad \xi \in D,$$

$$I_k(x(t_k)) = K_k \frac{u(t_k^-,\xi)}{1 + |u(t_k^-,.)|_X}, \quad \xi \in D, \quad k = 1, \cdots, m,$$

$$I_k(y(t_k)) = \bar{K}_k \frac{v(t_k^-,\xi)}{1 + |v(t_k^-,.)|_X}, \quad \xi \in \Omega, \quad k = 1, \cdots, m,$$

$$x(0)(\xi) = u(0,\xi) = u(b,\xi) = x(b)(\xi), \quad y(0)(\xi) = v(0,\xi) = v(b,\xi) = y(b)(\xi) \quad \xi \in D,$$

where $K_k, \bar{K}_k \in \mathbb{R},\ k = 1, \ldots, m$. Assume that $(\mathcal{K}_1) - (\mathcal{K}_4)$ are satisfied. Thus problem (4.1) can be written in the abstract form

$$\begin{cases} x'(t) - A_1 x(t) & \in & F_1(t,x(t),y(t)) + \sigma_1(t)\frac{dB_Q^H}{dt}, \quad t \in [0,b] \\ y'(t) - A_2 y(t) & \in & F_2(t,x(t),y(t)) + \sigma_2(t)\frac{dB_Q^H}{dt}, \ t \in [0,b], \\ x(t_k^+) - x(t_k^-) & \in & I_k(x(t_k)), \\ y(t_k^+) - y(t_k^-) & \in & \overline{I}_k(y(t_k)), \quad k = 1, \ldots, m \\ x(0) = x_0, & & y(0) = y_0. \end{cases} \tag{4.2}$$

where $A_1 = A_2 = A$. Since for each $k = 1, \ldots, m$ we have

$$|I_k(x)| = \left| K_k \frac{x}{1 + |x|_X} \right|_X \leq |K_k|, \quad |\bar{I}_k(x)| = \left| \bar{K}_k \frac{x}{1 + |x|_X} \right|_X \leq |\bar{K}_k|, \quad \text{for all } x \in X.$$

Then, Theorem 3.1 ensures that problem (4.1) possesses at least on solution.

Acknowledgements. The research of T. Caraballo has been partially supported by FEDER and the Spanish Ministerio de Economía y Competitividad project MTM2015-63723-P and the Consejería de Innovación, Ciencia y Empresa (Junta de Andalucía) under grant 2010/FQM314 and Proyecto de Excelencia P12-FQM-1492.

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