Partial Differential Equations

On the existence of solutions for a strongly degenerate system

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Abstract

We establish the existence of a solution in a certain sense to a strongly degenerate problem consisting in a coupled nonlinear parabolic-elliptic system. The diffusion term in the parabolic equation is of the form $-\operatorname{div} a(x, t, u, \nabla u)$, where a is an operator of the Leray–Lions type. Moreover, the second equation is nonuniformly elliptic.

Sur l'existence de solutions pour un système fortement dégénéré. On montre l'existence d'une solution dans un certain sens d'un problème fortement dégénéré constitué par un système non-linéaire de deux équations aux dérivées partielles couplées du type parabolique-elliptique, le terme de diffusion de l'équation parabolique étant de la forme $-\operatorname{div} a(x, t, u, \nabla u)$, où a est un opérateur du type de Leray–Lions. En outre, la seconde équation de ce système est non-uniformement elliptique.

Version française abrégée

On étudie un problème aux limites d'évolution constitué par deux équations aux dérivées partielles non-linéaires du type parabolique-elliptique. Ce problème peut être considéré comme une généralisation du problème du thermisteur [1–5,8–10]. Le résultat énoncé dans cette Note est une version plus générale d'un résultat de Xu [8].

Soient $N \geqslant 2$, $T > 0$, et $\Omega \subset \mathbb{R}^N$ un ensemble ouvert, borné et suffisament régulier, on pose $\Omega_T = \Omega \times (0, T)$ et $\Gamma_T = \partial\Omega \times (0, T)$. Étant données les fonctions $u_0 : \Omega \mapsto \mathbb{R}$ et $\varphi_0 : \Gamma_T \mapsto \mathbb{R}$, on considère le problème

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div} a(x, t, u, \nabla u) = \sigma(u)|\nabla\varphi|^2 \quad \text{dans } \Omega_T, \\ \operatorname{div}(\sigma(u)\nabla\varphi) = 0 \quad \text{dans } \Omega_T, \\ u(\cdot, 0) = u_0 \quad \text{dans } \Omega, \quad u = 0, \ \varphi = \varphi_0 \quad \text{sur } \Gamma_T. \end{cases} \tag{1}$$

Dorénavant, on prend $p \geqslant 2$ et on pose $p' = \frac{p}{p-1}$. Les hypothèses sur les données sont (H.1)–(H.5) (voir plus bas).

On observe donc que l'équation elliptique peut être dégénérée sur l'ensemble $\{u = +\infty\}$. En effet, l'hypothèse (H.5) permet de considérer le cas $\sigma(s) \to 0$ quand $s \to +\infty$ (ce qui se passe dans les plupart des cas d'intérêt pratique). Par conséquent, on ne peut pas s'attendre à obtenir d'estimations à priori pour $\nabla\varphi$, et donc φ n'appartient pas à aucun espace de Sobolev. Ceci veut dire que l'hypothèse (H.5) empêche la recherche de solutions faibles de (1). Néanmoins, si l'on considère la expression $\Phi = \sigma(u)\nabla\varphi$ comme une seule fonction, alors on peut montrer que $\Phi \in L^2(\Omega_T)^N$, et ceci va nous permettre donner une nouvelle formulation du système (1) pour lequel on montre l'existence d'une solution : il s'agît de la solution de capacité (voir la Définition 2.1 ci-après), une notion introduite par Xu [8] dans l'étude d'une version modifiée du problème du thermisteur.

Le résultat principal de cette Note est le suivant :

Théorème 0.1. *Sous les hypothèses* (H.1)–(H.6)*, le problème* (1) *admet au moins une solution de capacité.*

1. Introduction and statement of the problem

We prove an existence result to a coupled nonlinear parabolic-elliptic system, which may be regarded as a generalized version of the well-known thermistor problem arising in electromagnetism [1–5,8–10]. Also, this Note extends a previous result due to Xu [8].

For $N \geqslant 2$ and $T > 0$, let $\Omega \subset \mathbb{R}^N$ be an open, bounded and smooth enough set, and write $\Omega_T = \Omega \times (0, T)$ and $\Gamma_T = \partial\Omega \times (0, T)$. Given the functions $u_0 : \Omega \mapsto \mathbb{R}$ and $\varphi_0 : \Gamma_T \mapsto \mathbb{R}$, we consider the system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) = \sigma(u)|\nabla\varphi|^2 \quad \text{dans } \Omega_T, \\ \operatorname{div}(\sigma(u)\nabla\varphi) = 0 \quad \text{dans } \Omega_T, \\ u(\cdot,0) = u_0 \quad \text{dans } \Omega, \quad u = 0,\ \varphi = \varphi_0 \quad \text{sur } \Gamma_T. \end{cases} \tag{1}$$

For $p \geqslant 2$, let $p' = \frac{p}{p-1}$ and suppose the following hypotheses on data:

(H.1) $a : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.
(H.2) There exists $\alpha > 0$ such that, for all $s \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^N$, and almost everywhere in Ω_T, $[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) \geqslant \alpha|\xi - \eta|^p$.
(H.3) $a(x,t,s,0) = 0$.
(H.4) There exist a nonnegative function $b \in L^{p'}(\Omega_T)$ and a constant $\beta > 0$ such that, for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, and for almost every $(x,t) \in \Omega_T$, $|a(x,t,s,\xi)| \leqslant \beta[b(x,t) + |s|^{p-1} + |\xi|^{p-1}]$.
(H.5) $\sigma \in C(\mathbb{R})$ and $0 < \sigma(s) \leqslant \bar{\sigma}$, for all $s \in \mathbb{R}$.
(H.6) $\varphi_0 \in L^2(0,T; H^1(\Omega)) \cap L^\infty(\Omega_T)$ and $u_0 \in L^2(\Omega)$.

Our interest is focused on the search of solutions to system (1) bearing in mind, apart from the form of the diffusion term in the parabolic equation of (1), that the hypothesis on σ prevents from the search of weak solutions. Indeed, since $\sigma(s)$ may converge to zero as $|s| \to \infty$, the elliptic equation of (1) becomes degenerate when u is infinite. Thus, no a priori estimates for $\nabla\varphi$ will be available and so, φ might not belong to a Sobolev space. However, taking the function $\Phi = \sigma(u)\nabla\varphi$ instead of φ, it is possible to prove that $\Phi \in L^2(\Omega_T)^N$, which enable us to give a new formulation of (1). Its solution is called capacity solution. The notion of capacity solution was first introduced by Xu in the analysis of a modified version of the thermistor problem [7–10].

From this point on, we use the following notation: $L^p(X)$ stands for $L^p(0,T;X)$, X being a Banach space. We also introduce the following spaces together with their respective dual spaces: $H = H_0^1(\Omega)$, $H' = H^{-1}(\Omega)$, $V = W_0^{1,p}(\Omega)$, $V' = W^{-1,p'}(\Omega)$, $\mathcal{H} = L^2(H)$, $\mathcal{H}' = L^2(H')$, $\mathcal{V} = L^p(V)$ and $\mathcal{V}' = L^{p'}(V')$. Also, we write $a(u, \nabla u)$ instead of $a(x,t,u,\nabla u)$. Finally, when passing to the limit, the subsequences will be denoted in the same way as their respective original sequences, except where explicitly noted.

2. Notion of capacity solution and main result

Definition 2.1. A triplet (u, φ, Φ) is called a capacity solution to problem (1) if the following conditions are fulfilled:

(C.1) $u \in \mathbf{W} = \{v \in \mathcal{V} \mid \frac{\partial v}{\partial t} \in \mathcal{V}'\}$, $\varphi \in L^\infty(\Omega_T)$, $\Phi \in L^\infty(L^2(\Omega))^N$.
(C.2) $\frac{\mathrm{d}u}{\mathrm{d}t} - \operatorname{div} a(u, \nabla u) = \operatorname{div}(\varphi \Phi)$ in $\mathcal{V}'$, $\operatorname{div} \Phi = 0$ in $\mathcal{H}'$.
(C.3) For every $S \in C_0^1(\mathbb{R})$, $S(u)\varphi - S(0)\varphi_0 \in \mathcal{H}$ and $S(u)\Phi = \sigma(u)[\nabla(S(u)\varphi) - \varphi \nabla S(u)]$.
(C.4) $u(\cdot, 0) = u_0$.

The nonuniform character of the elliptic equation in (1) is one of the main difficulties in the resolution of this system. Consider the new hypothesis (H.5′), namely,

$$\sigma \in C(\mathbb{R}) \quad \text{and} \quad 0 < \sigma_1 \leqslant \sigma(s) \leqslant \sigma_2, \quad \text{for all } s \in \mathbb{R}. \tag{H.5′}$$

We have the following result [6]:

Theorem 2.2. *Under hypotheses* (H.1)–(H.6)*, with* (H.5′) *instead of* (H.5)*, system* (1) *admits a weak solution* (u, φ)*, that is,* $u \in \mathcal{V}$*,* $\frac{\mathrm{d}u}{\mathrm{d}t} \in \mathcal{V}'$*,* $\varphi - \varphi_0 \in L^\infty(H) \cap L^\infty(\Omega_T)$*,* $u(\cdot, 0) = u_0$ *in* Ω *and*

$$\int_0^t \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \phi \right\rangle + \int_0^t \int_\Omega a(u, \nabla u) \nabla \phi = \int_0^t \int_\Omega \sigma(u) |\nabla \varphi|^2 \phi, \quad \forall \phi \in \mathcal{V} \cap L^\infty(\Omega_T),\ \forall t \in [0, T], \tag{2}$$

$$\int_\Omega \sigma(u) \nabla \varphi \nabla \xi = 0, \quad \forall \xi \in \mathcal{H},\ \text{a.e. } t \in (0, T). \tag{3}$$

Furthermore, the pair (u, φ) *satisfies the following properties*:

$$\|\varphi\|_{L^\infty(\Omega_T)} \leqslant \|\varphi_0\|_{L^\infty(\Omega_T)}, \quad \operatorname*{ess\,sup}_{t \in [0,T]} \int_\Omega \sigma(u) |\nabla \varphi|^2 \leqslant C(\sigma_1, \sigma_2, \varphi_0), \quad \|u\|_{\mathcal{H}} \leqslant C(u_0, \varphi_0, T, \sigma_1, \sigma_2, \alpha); \tag{4}$$

$$\left\langle \sigma(u) |\nabla \varphi|^2, \xi \right\rangle_{\mathcal{H}', \mathcal{H}} = -\int_{\Omega_T} \sigma(u) \varphi \nabla \varphi \nabla \xi, \quad \text{for all } \xi \in \mathcal{H}. \tag{5}$$

The main result of this Note now follows:

Theorem 2.3. *Under hypotheses* (H.1)–(H.6)*, there exists a capacity solution to problem* (1).

3. Proof of Theorem 2.3

Let $n \in \mathbb{N}$ and put $\sigma_n(s) = \sigma(s) + 1/n$. The approximate problem of (1) is given by

$$\frac{\partial u_n}{\partial t} - \operatorname{div} a(u_n, \nabla u_n) = \operatorname{div}\big(\sigma_n(u_n) \varphi_n \nabla \varphi_n\big) \quad \text{in } \Omega_T, \tag{6}$$

$$\operatorname{div}\big(\sigma_n(u_n) \nabla \varphi_n\big) = 0 \quad \text{in } \Omega_T, \tag{7}$$

$$u_n = 0, \quad \varphi_n = \varphi_0 \quad \text{on } \Gamma_T, \quad u_n(\cdot, 0) = u_0 \quad \text{in } \Omega. \tag{8}$$

By (H.5), $1/n \leqslant \sigma_n(s) \leqslant \bar{\sigma} + 1 = \hat{\sigma}$. According to Theorem 2.2, problem (6)–(8) admits a weak solution (u_n, φ_n) satisfying (2)–(5), where $(\operatorname{div}(\sigma_n(u_n)\varphi_n \nabla \varphi_n)) \subset \mathcal{H}' \hookrightarrow \mathcal{V}'$ is bounded. Also, the sequences (u_n), $(a(u_n, \nabla u_n))$ and $(\frac{\mathrm{d}u_n}{\mathrm{d}t})$ are bounded in $\mathcal{V}$, $L^{p'}(\Omega_T)^N$ and $\mathcal{V}'$, respectively.

We can then deduce the existence of functions $\varphi \in L^\infty(\Omega_T)$, $\Phi \in L^\infty(L^2(\Omega)^N)$ and $u \in \mathcal{V}$ such that $\varphi_n \rightharpoonup \varphi$ weakly–$\star$ in $L^\infty(\Omega_T)$, $\sigma_n(u_n)\nabla \varphi_n \rightharpoonup \Phi$ weakly–$\star$ in $L^\infty(L^2(\Omega_T)^N)$ and $u_n \rightharpoonup u$ weakly in $\mathcal{V}$. Furthermore, as (u_n) is relatively compact in $L^p(\Omega_T)$, we may assume that

$$u_n \longrightarrow u \quad \text{strongly in } L^p(\Omega_T) \text{ and a.e. in } \Omega_T, \tag{9}$$

and thus, $\frac{\mathrm{d}u_n}{\mathrm{d}t} \rightharpoonup \frac{\mathrm{d}u}{\mathrm{d}t}$ weakly in $\mathcal{V}'$. Therefore, the triplet (u, φ, Φ) satisfies condition (C.1) of Definition 2.1. Owing to (H.5) and (9), $\sigma_n(u_n) \rightharpoonup \sigma(u)$ weakly–$\star$ in $L^\infty(\Omega_T)$ and almost everywhere in Ω_T. Finally, there exists $\Upsilon \in L^{p'}(\Omega_T)^N$ such that $a(u_n, \nabla u_n) \rightharpoonup \Upsilon$ weakly in $L^{p'}(\Omega_T)^N$.

3.1. Strong convergence of (∇u_n)

Since (1) is nonlinear, (9) does not lead to $\Upsilon = a(u, \nabla u)$. To do so, it is enough to show that $\nabla u_n \to \nabla u$ almost everywhere in Ω_T. In fact, this is the goal of the next result [6]:

Lemma 3.1. *Assume* (H.1)–(H.6) *and let* (u_n, φ_n) *be a weak solution of system* (6)–(8). *Then* (u_n) *is relatively compact in* $L^q(W_0^{1,q}(\Omega))$ *for all* $q \in [1, p)$.

3.2. L^1*-convergence of* (φ_n)

In order to show the strong convergence of (φ_n) in $L^1(Q_T)$, up to a suitable subsequence, we need some useful results. First, we have the following lemma, which is a modified version of that due to Xu [8]:

Lemma 3.2. *Let* (u_n) *be a bounded sequence in* $\mathcal{V}$ *and relatively compact in* $L^p(\Omega_T)$. *Then there exists a subsequence* $(u_{n(k)}) \subset (u_n)$ *such that, for every* $\varepsilon > 0$, *there exist a constant value* $M = M(\varepsilon) > 0$ *and a function* $\psi \in L^1(W^{1,1}(\Omega))$ *such that* $0 \leqslant \psi \leqslant 1$, $\|\psi - 1\|_{L^1(\Omega_T)} + \|\nabla\psi\|_{L^1(\Omega_T)} \leqslant \varepsilon$ *and* $|u|, |u_{n(k)}| \leqslant M$ *in* $\{\psi > 0\}$ *for all* $k \geqslant 1$.

On the other hand, we can show the following intermediate results:

Lemma 3.3. *For every* $S \in C_0^1(\mathbb{R})$, $S(u_n)\varphi_n \rightharpoonup S(u)\varphi$ *weakly-*$\star$ *in* $L^\infty(\Omega_T)$.

Lemma 3.4. *For every* $S \in C_0^1(\mathbb{R})$ *such that* $0 \leqslant S \leqslant 1$,

$$\limsup_{n\to\infty} \int_{\Omega_T} \sigma_n(u_n)\big|\nabla\big(S(u_n)\varphi_n - S(u)\varphi\big)\big|^2 \leqslant C\|S'\|_\infty\big(1 + \|S'\|_\infty\big).$$

Now we are ready to state the strong convergence of some subsequence $(\varphi_{n(k)})$ in $L^1(\Omega_T)$.

Lemma 3.5. *There exists a subsequence* $(\varphi_{n(k)}) \subset (\varphi_n)$ *such that* $\varphi_{n(k)} \to \varphi$ *strongly in* $L^1(\Omega_T)$.

Proof of Lemma 3.5. The sequence (u_n) satisfies the assumptions of Lemma 3.2, so that, for every $\varepsilon > 0$, there exist $M > 0$ and $\psi \in L^1(W^{1,1}(\Omega))$ as in Lemma 3.2. Furthermore, $\sigma_{n(k)}(u_{n(k)}) \geqslant C_M$ in $\{\psi > 0\}$ for all $k \geqslant 1$.

Choose $(S_m) \subset C_0^1(\mathbb{R})$ so that $0 \leqslant S_m \leqslant 1$, $S_m = 1$ in $[-M, M]$ and $\|S_m'\|_\infty \leqslant C/m$ for all $m \geqslant 1$ and some $C > 0$. Then we can write

$$\int_{\Omega_T} |\varphi_{n(k)} - \varphi| = \int_{\Omega_T} \big|S_m(u_{n(k)})\varphi_{n(k)} - S_m(u)\varphi\big|\psi + \int_{\Omega_T} |\varphi_{n(k)} - \varphi|(1 - \psi).$$

Taking into account Lemma 3.4 and properties of S_m, it can be shown that

$$\limsup_{k\to\infty} \int_{\Omega_T} |\varphi_{n(k)} - \varphi| \leqslant C\varepsilon + K_M\big[C\|S_m'\|_\infty\big(1 + \|S_m'\|_\infty\big)\big]^{1/2} \leqslant C\varepsilon + K_M C^{1/2}\left[\frac{C}{m}\left(1 + \frac{C}{m}\right)\right]^{1/2},$$

where $K_M = CC_M^{-1/2}(\operatorname{meas}\Omega)^{1/2}T^{1/2}$. Making $\varepsilon \to 0$ and $m \to \infty$ yields the result. □

The convergences deduced in the previous sections lead to (C.2) of Definition 2.1. Finally, in order to obtain (C.3), it is enough to make $k \to \infty$ in the equality

$$S(u_{n(k)})\sigma_{n(k)}(u_{n(k)})\nabla\varphi_{n(k)} = \sigma_{n(k)}(u_{n(k)})\big[\nabla\big(S(u_{n(k)})\varphi_{n(k)}\big) - \varphi_{n(k)}\nabla S(u_{n(k)})\big].$$

This ends the proof of Theorem 2.3.

Remark 1. The uniqueness of capacity solutions to problem (1) has not been analyzed yet and it is a very complex task. All known uniqueness results for the thermistor problem rely on restrictive hypotheses on data, for instance $\varphi_0 \in L^\infty(0, T; W^{1,\infty}(\Omega))$. However, in that setting there is no need to search for capacity solutions, because the regularity of the solutions leads us to the context of weak solutions.

Remark 2. The hypothesis $p \geqslant 2$ is used in two points: firstly, in the inclusion $L^2(0, T; H^{-1}(\Omega)) \hookrightarrow L^{p'}(0, T; W^{-1,p'}(\Omega))$, which yields $\operatorname{div}(\sigma(w)\varphi\nabla\varphi) \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, for every $w \in L^p(Q)$. Secondly, along the proof of the strong convergence in L^1 of a subsequence $(\varphi_{n(k)})$ (see [6] for details).

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