On certain doubly non-uniformly and singular non-uniformly elliptic systems

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Abstract

We consider the steady state of the thermistor problem consisting of a coupled set of nonlinear elliptic equations governing the temperature and the electric potential. We study the existence of weak solutions under two kind of assumptions. The first one considers the case in which the two diffusion coefficients are not bounded below far from zero, arising to a doubly non-uniformly elliptic system. In the second one, we assume in addition that the thermal conductivity blows up for a finite value of the temperature, arising to a singular and non-uniformly coupled system.

Keywords: Non-uniformly and singular elliptic systems; Nonlinear elliptic equations; Thermistor problem; Sobolev spaces

1. Introduction

The heat produced by an electrical current passing through a conductor device is governed by the so-called thermistor problem. This problem consists of a system of nonlinear parabolic–elliptic system describing the temperature, u, and the electric potential φ [2,8]. Here, we consider the steady-state case, resulting in a coupled non-linear elliptic system. Let $\mathscr{J}$ be the current density, $\mathscr{Q}$ the heat flux and $\mathscr{E} = -\nabla\varphi$ the electric field; then by Ohm's and Fourier's law we have

$$\mathscr{J} = \sigma(u)\mathscr{E},\ \mathscr{Q} = -a(u)\nabla u,$$

where $a(u)$ and $\sigma(u)$ are, respectively, the thermal and electric conductivities. Also, from the usual conservation laws $\nabla \cdot \mathscr{J} = 0$, $\nabla \cdot \mathscr{Q} = \mathscr{E} \cdot \mathscr{J}$ we obtain

$$\begin{cases} -\nabla \cdot (a(u)\nabla u) = \sigma(u)|\nabla\varphi|^2 & \text{in } \Omega, \\ \nabla \cdot (\sigma(u)\nabla\varphi) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \varphi = \varphi_0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open, bounded and smooth enough set in $\mathbb{R}^N$, $N \geqslant 1$. We note that, the right-hand side of the equation for the temperature may be written as $\nabla \cdot (\sigma(u)\varphi\nabla\varphi)$ thanks to the equation verified by φ; this is true, for instance, if $\varphi \in H^1(\Omega)$ (this aspect is also discussed in Remark 2).

The steady-state thermistor problem has been studied by several authors along the last two decades. Among them, we refer to Cimatti [4–7] and Cimatti and Prodi [9]. In these papers, the authors have obtained some existence results of weak solutions in both, two and three dimensions, using the so-called Diesselhorst transformation, and under the conditions $u = u_0$ on $\partial\Omega$, and u_0 being a constant value, or $u = u_0 \geqslant u_m > 0$ on $\partial\Omega$, together with the hypothesis $0 < a_m \leqslant a(u)$, or $a(u) = a_0$ constant, or even under the Wiedemann–Franz law (that is, $a(s) = Ls\sigma(s)$, $L > 0$ a constant value) with metallic conduction, and certain assumptions on $\sigma(u)$. We notice that in all these papers is assumed that $a(s) \geqslant a_0 > 0$, for all s.

In the present work we show two existence results of weak solution to the steady-state thermistor problem under the general assumption that both $a(s)$ and $\sigma(s)$ are not bounded below far from zero (Theorems 1 and 2). In this way, system (1) becomes doubly non-uniformly elliptic; in general, we cannot expect the regularity $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, or that u belongs to some Sobolev space. We point out that the technique used here in the analysis of the non-singular case is not based on the derivation of L^∞-estimates for the temperature. On the other hand, our setting for the singular and non-uniformly elliptic problem leads us to derive L^∞-estimates for the temperature, and this implies more regularity for both, the electric potential φ and the temperature u itself.

2. Setting of the non-uniformly elliptic problem

We consider the steady-state thermistor problem in divergence form, namely

$$\begin{cases} -\nabla \cdot (a(u)\nabla u) = \nabla \cdot (\sigma(u)\varphi\nabla\varphi) & \text{in } \Omega, \\ \nabla \cdot (\sigma(u)\nabla\varphi) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \varphi = \varphi_0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

together with the following hypotheses on data:

(H.1) $\sigma \in C(\mathbb{R})$ and $0 < \sigma(s) \leqslant \bar{\sigma}$, for all $s \in \mathbb{R}$.

(H.2) $a \in C(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$, $\int_0^{+\infty} a(s)\,\mathrm{d}s = +\infty$, and $A(r) = \int_0^r a(s)\,\mathrm{d}s$ is a strictly increasing function.

(H.3) $\varphi_0 \in H^1(\Omega)$.

(H.4) There exist an integer $M > 1$ and a function $\alpha : [M, +\infty) \to \mathbb{R}$ such that $\alpha(s) > 0$, for all $s \geqslant M$, α is non-increasing and $\sigma(s) \geqslant \alpha(s) > 0$.

(H.5) Let $p \in (2N/(N+2), 2)$ if $N \geqslant 2$, $p \in (1,2)$ if $N = 1$ and $p' = 2 - p$, then

$$\int_M^{+\infty} \frac{\mathrm{d}s}{\alpha(s)^{p/p'} A(s-1)^{\bar{q}/2}} < +\infty \quad \text{with} \begin{cases} \bar{q} = 2^* & \text{if } N \geqslant 3, \\ \bar{q} \in [2, +\infty) & \text{if } N = 2, \\ \bar{q} \in [1, +\infty) & \text{if } N = 1. \end{cases} \tag{3}$$

Remark 1. If we consider that σ and φ_0 are smooth enough, then φ will be smooth too, for instance $|\nabla\varphi| \in L^{\infty}(\Omega)$, and then we could obtain that $u \in L^{\infty}(\Omega)$; in particular, this implies $\sigma(u) \geqslant \alpha(\|u\|_{L^{\infty}(\Omega)})$, that is, $\sigma(u)$ becomes uniformly elliptic; thus, we have again that σ and φ_0 are smooth, starting over the cycle. However, this does not correspond with our setting, since both a and σ are non-uniformly elliptic.

The main result of this section now follows.

Theorem 1. *Under assumptions* (H.1)–(H.5), *problem*

$$\begin{cases} -\Delta A(u) = \nabla \cdot (\sigma(u)\varphi\nabla\varphi) & \text{in } \mathscr{D}'(\Omega), \\ \nabla \cdot (\sigma(u)\nabla\varphi) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \varphi = \varphi_0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

has a weak solution (u, φ) *in the following sense*:

$$\forall q < \frac{N}{N-1} \ \textit{if} \ N \geqslant 2, \quad q = 2 \ \textit{if} \ N = 1, \ A(u) \in W_0^{1,q}(\Omega), \tag{5}$$

$$\varphi - \varphi_0 \in W_0^{1,p}(\Omega), \quad \sigma(u)^{1/2}\nabla\varphi \in L^2(\Omega)^N, \tag{6}$$

$$\int_\Omega \nabla A(u)\nabla\xi = -\int_\Omega \sigma(u)\varphi\nabla\varphi\nabla\xi \quad \textit{for all } \xi \in \mathscr{D}(\Omega), \tag{7}$$

$$\int_\Omega \sigma(u)\nabla\varphi\nabla\phi = 0 \quad \textit{for all } \phi \in H_0^1(\Omega). \tag{8}$$

Furthermore, the term $\nabla \cdot (\sigma(u)\varphi\nabla\varphi)$ *is a Radon measure and* $u \geqslant 0$ *almost everywhere in* Ω.

The proof of this theorem is developed along the next paragraphs.

2.1. Approximate problems

Let $n \in \mathbb{N}$ and introduce the functions $a_n(s) = a(s) + 1/n$, $\sigma_n(s) = \sigma(s) + 1/n$, then we set the approximate prob lem given as follows:

$$\begin{cases} -\nabla \cdot (a_n(u_n)\nabla u_n) = \sigma_n(u_n)|\nabla \varphi_n|^2 & \text{in } \Omega, \\ \nabla \cdot (\sigma_n(u_n)\nabla \varphi_n) = 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \\ \varphi_n = T_n(\varphi_0) & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $T_n(s)$ is the truncation function at height n, that is

$$T_n(s) = \min(|s|, n)\,\mathrm{sign}\, s. \tag{10}$$

By virtue of the classical existence results [2], problem (9) has a solution such that $u_n \in H_0^1(\Omega)$, $\varphi_n - T_n(\varphi_0) \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

2.2. Estimates and passing to the limit

Since

$$\int_\Omega \sigma_n(u_n)\nabla\varphi_n\nabla\phi = 0 \quad \text{for all } \phi \in H_0^1(\Omega), \tag{11}$$

taking $\phi = \varphi_n - T_n(\varphi_0)$ yields

$$\int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2 = \int_\Omega \sigma_n(u_n)\nabla\varphi_n\nabla T_n(\varphi_0)$$

$$\leqslant \left(\int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2\right)^{1/2} \left(\int_\Omega \sigma_n(u_n)|\nabla\varphi_0|^2\right)^{1/2},$$

hence

$$\int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2 \leqslant \tilde{\sigma} \int_\Omega |\nabla\varphi_0|^2 \leqslant \tilde{\sigma}\|\varphi_0\|_{H^1(\Omega)} = C(\tilde{\sigma}, \varphi_0) = C_1, \tag{12}$$

therefore, $(f_n) = (\sigma_n(u_n)|\nabla\varphi_n|^2)$ is bounded in $L^1(\Omega)$.

Let $v_n = A_n(u_n)$, $A_n(r) = \int_0^r a_n(s)\,\mathrm{d}s$ and consider the elliptic problem

$$-\Delta v_n = f_n \quad \text{in } \Omega,$$

$$v_n = 0 \quad \text{on } \partial\Omega.$$

From Boccardo–Gallouët estimates [3,11], we deduce that

$$(v_n) \text{ is bounded in } W_0^{1,q}(\Omega), \text{ for all } q < \frac{N}{N-1}$$

$$\text{if } N \geqslant 2,\ q = 2 \text{ if } N = 1. \tag{13}$$

As a result, there exist a subsequence (still denoted in the same way) and $v \in W_0^{1,q}(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{in } W_0^{1,q}(\Omega)\text{-weakly.} \tag{14}$$

Since the embeddings $W_0^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$, for all $r < N/(N-2)$ if $N \geqslant 2$, or $W_0^{1,q}(\Omega) = H_0^1(\Omega) \hookrightarrow C(\bar{\Omega})$ if $N = 1$, are compacts, we may also assume that

$$v_n \to v \quad \text{in } L^r(\Omega)\text{-strongly} \quad \text{if } N \geqslant 2, \tag{15}$$

$$v_n \to v \quad \text{in } C(\bar{\Omega})\text{-strongly} \quad \text{if } N = 1, \tag{16}$$

$$v_n \to v \quad \text{a.e. in } \Omega. \tag{17}$$

Moreover, since $f_n \geqslant 0$ in Ω, then $v_n \geqslant 0$ in Ω. Since A_n is strictly increasing, we also have $u_n \geqslant 0$ in Ω. Now, we show that $(A(u_n)) \subset H_0^1(\Omega)$ is bounded in $W_0^{1,q}(\Omega)$. Indeed,

$$|\nabla A(u_n)| = |a(u_n)\nabla u_n| \leqslant |a_n(u_n)\nabla u_n| = |\nabla A_n(u_n)| = |\nabla v_n|$$

and by virtue of (13), $(A(u_n))$ is also bounded in $W_0^{1,q}(\Omega)$; then there exist a subsequence (denoted in the same way) and $z \in W_0^{1,q}(\Omega)$ such that

$$A(u_n) \rightharpoonup z \quad \text{in } W_0^{1,q}(\Omega)\text{-weakly,} \tag{18}$$

$$A(u_n) \to z \quad \text{in } L^r(\Omega)\text{-strongly} \quad \textit{for all } r < \frac{N}{N-2} \text{ if } N \geqslant 2, \tag{19}$$

$$A(u_n) \to z \quad \text{in } C(\bar{\Omega})\text{-strongly} \quad \text{if } N = 1, \tag{20}$$

$$A(u_n) \to z \quad \text{a.e.} \quad \text{in } \Omega \tag{21}$$

But, since A is one-to-one, from (21) we deduce

$$u_n \to A^{-1}(z) = u \text{ a.e. in } \Omega \tag{22}$$

with $u \geqslant 0$ a.e. in Ω.

Thanks to the definition of σ_n, together with (22), we obtain

$$\sigma_n(u_n) \to \sigma(u) \quad \text{a.e. in } \Omega. \tag{23}$$

Also, by virtue of (H.1), $(\sigma_n(u_n))$ is bounded in $L^\infty(\Omega)$, and taking into account (23), we have

$$\sigma_n(u_n) \to \sigma(u) \quad \text{in } L^\infty(\Omega)\text{-weakly-}*. \tag{24}$$

Now, we seek for estimates to the sequence (φ_n) in some Sobolev space $W^{1,p}(\Omega)$, with $1 < p < 2$. By virtue of (H.5), $2/p'$ is the conjugate exponent of $2/p$. Applying Young's inequality and taking into account (12), we obtain

$$\begin{aligned}\int_\Omega |\nabla\varphi_n|^p &\leqslant \left(\int_\Omega \sigma_n(u_n)^{-p/p'}\right)^{p'/2} \left(\int_\Omega \sigma_n(u_n)|\nabla\varphi_n|^2\right)^{p/2} \\ &\leqslant C_1^{p/2} \left(\int_\Omega \sigma_n(u_n)^{-p/p'}\right)^{p'/2}.\end{aligned}$$

Let us show the following estimate:

$$\int_\Omega \sigma_n(u_n)^{-p/p'} \leqslant C_2. \tag{25}$$

From $0 < \sigma(s) \leqslant \sigma_n(s) \leqslant \tilde{\sigma}$, for all $s \in \mathbb{R}$, it yields

$$\tilde{\sigma}^{-p/p'} \leqslant \sigma_n(s)^{-p/p'} \leqslant \sigma(s)^{-p/p'} \quad \text{for all } s \in \mathbb{R},$$

hence

$$\int_\Omega \sigma_n(u_n)^{-p/p'} \leqslant \int_\Omega \sigma(u_n)^{-p/p'} \leqslant \int_{\{|u_n| \leqslant M\}} \sigma(u_n)^{-p/p'} + \int_{\{u_n > M\}} \sigma(u_n)^{-p/p'}.$$

Thanks to (H.1), σ^{-1} is bounded on compact sets of $\mathbb{R}$, in particular, there exists a constant value $C_M > 0$ such that $\min_{|s| \leqslant M} \sigma(s) = C_M$, and this implies that $\sigma(u_n)^{-p/p'} \chi_{\{|u_n| \leqslant M\}} \leqslant C_M^{-p/p'}$, and

$$\int_{\{|u_n| \leqslant M\}} \sigma(u_n)^{-p/p'} \leqslant C_M^{-p/p'} |\Omega| = C(M, p, p', \Omega) = C_3.$$

On the other hand, by virtue of (H.4), we deduce

$$\begin{aligned}
\int_{\{u_n > M\}} \sigma(u_n)^{-p/p'} &\leqslant \int_{\{u_n > M\}} \alpha(u_n)^{-p/p'} \leqslant \sum_{i \geqslant M} \int_{\{i \leqslant u_n < i+1\}} \alpha(u_n)^{-p/p'} \\
&\leqslant \sum_{i \geqslant M} \int_{\{i \leqslant u_n < i+1\}} \alpha(i+1)^{-p/p'} \\
&\leqslant \sum_{i \geqslant M} \alpha(i+1)^{-p/p'} \, |\{u_n \geqslant i\}|
\end{aligned} \tag{26}$$

In order to derive some estimate to $|\{u_n \geqslant i\}|$, we first study $|\{v_n = A_n(u_n) \geqslant i\}|$. To do so, we take $T_i(v_n)$ as a test function in the equation of u_n; then

$$\int_\Omega \nabla v_n \nabla T_i(v_n) = \int_\Omega \sigma_n(u_n) |\nabla \varphi_n|^2 T_i(v_n) \leqslant C_1 i,$$

the left-hand side can be written as $\int_\Omega \nabla v_n \nabla T_i(v_n) = \int_\Omega |\nabla T_i(v_n)|^2 = I_{i,n}$. By Sobolev's inequality we have

$$\begin{aligned}
I_{i,n} &\geqslant C \left(\int_\Omega |T_i(v_n)|^{\bar{q}} \right)^{2/\bar{q}} \geqslant C \left(\int_{\{v_n \geqslant i\}} |T_i(v_n)|^{\bar{q}} \right)^{2/\bar{q}} \\
&= C \left(\int_{\{v_n \geqslant i\}} i^{\bar{q}} \right)^{2/\bar{q}} = C i^2 |\{v_n \geqslant i\}|^{2/\bar{q}},
\end{aligned}$$

where $\bar{q} = 2^* = 2N/(N-2)$ and $C = C(\Omega, N)$, if $N \geqslant 3$, $\bar{q} \in [2, +\infty)$ and $C = C(\Omega, \bar{q})$, if $N \leqslant 2$. Consequently,

$$|\{v_n \geqslant i\}|^{2/\bar{q}} \leqslant \frac{C_1 i}{i^2 C} = \frac{C_1}{iC},$$

which yields, $|\{v_n \geqslant i\}| \leqslant (C_1/iC)^{\bar{q}/2} = C_4/i^{\bar{q}/2}$. Since $u_n \geqslant 0$ in Ω, $A_n(u_n) \geqslant A(u_n)$ in Ω, $\{A(u_n) \geqslant i\} \subset \{v_n = A_n(u_n) \geqslant i\}$ and

$$|\{A(u_n) \geqslant i\}| \leqslant |\{v_n \geqslant i\}| \leqslant \frac{C_4}{i^{\bar{q}/2}},$$

hence

$$|\{u_n \geqslant A^{-1}(i)\}| \leqslant \frac{C_4}{i^{\bar{q}/2}},$$

this can be expressed as

$$|\{u_n \geqslant l\}| \leqslant \frac{C_4}{A(l)^{\bar{q}/2}}.$$

Therefore, thanks to (3) in (H.5) and (26), we have

$$\int_{\{u_n > M\}} \sigma(u_n)^{-p/p'} \leqslant \sum_{i \geqslant M} \alpha(i+1)^{-p/p'} \frac{C_4}{A(i)^{\bar{q}/2}}$$
$$\leqslant C_4 \int_{M-1}^{+\infty} \frac{\mathrm{d}s}{\alpha(s+1)^{p/p'} A(s)^{\bar{q}/2}} = C_5.$$

This shows (25) and we deduce that

$$\int_\Omega |\nabla \varphi_n|^p \leqslant C_1^{p/2} C_2^{p'/2} = C_6, \tag{27}$$

which means that, $\varphi_n - T_n(\varphi_0)$ is bounded in $W_0^{1,p}(\Omega)$. We then take a subsequence (still denoted in the same way) and $\varphi \in W^{1,p}(\Omega)$ such that

$$\varphi_n \rightharpoonup \varphi \quad \text{in } W^{1,p}(\Omega)\text{-weakly}, \tag{28}$$

$$\varphi_n \to \varphi \quad \text{in } L^{\bar{r}}(\Omega)\text{-strongly} \quad \textit{for all } \bar{r} < p^* \text{ if } N \geqslant 2, \tag{29}$$

$$\varphi_n \to \varphi \quad \text{in } C(\bar{\Omega})\text{-strongly} \quad \text{if } N = 1, \tag{30}$$

$$\varphi_n \to \varphi \quad \text{a.e. in } \Omega. \tag{31}$$

From (H.5), $p > 2N/(N+2)$ which implies that $p^* = Np/(N-p) > 2$. In particular

$$\varphi_n \to \varphi \quad \text{in } L^2(\Omega)\text{-strongly}. \tag{32}$$

Thanks to (12) the sequence $(\sigma_n(u_n)^{1/2}\nabla\varphi_n)$ is bounded in $L^2(\Omega)^N$; thus there exist a subsequence (still denoted in the same way) and $\Phi \in L^2(\Omega)^N$ such that

$$\sigma_n(u_n)^{1/2}\nabla\varphi_n \rightharpoonup \Phi \quad \text{in } L^2(\Omega)^N\text{-weakly}. \tag{33}$$

From (24), (28) and (33) it is deduced that $\Phi = \sigma(u)^{1/2}\nabla\varphi \in L^2(\Omega)^N$. Moreover, taking into account (H.1), (24) and (33), we also have

$$\sigma_n(u_n)\nabla\varphi_n \rightharpoonup \sigma(u)\nabla\varphi \quad \text{in } L^2(\Omega)^N\text{-weakly}. \tag{34}$$

Going back to (11) and taking $\phi = \varphi_n \xi$, with $\xi \in \mathscr{D}(\Omega)$. Then

$$0 = \int_\Omega \sigma_n(u_n) \nabla \varphi_n \nabla(\varphi_n \xi) = \int_\Omega \sigma_n(u_n) |\nabla \varphi_n|^2 \xi + \int_\Omega \sigma_n(u_n) \nabla \varphi_n \varphi_n \nabla \xi$$

$$= \int_\Omega \sigma_n(u_n) |\nabla \varphi_n|^2 \xi - \int_\Omega \nabla \cdot (\sigma_n(u_n) \varphi_n \nabla \varphi_n)\, \xi$$

and so,

$$\sigma_n(u_n)|\nabla \varphi_n|^2 = \nabla \cdot (\sigma_n(u_n)\varphi_n \nabla \varphi_n) \text{ en } \mathscr{D}'(\Omega). \tag{35}$$

From the equality

$$\int_\Omega \sigma_n(u_n) \varphi_n \nabla \varphi_n \nabla \xi = \int_\Omega \sigma_n(u_n)^{1/2} \varphi_n \sigma_n(u_n)^{1/2} \nabla \varphi_n \nabla \xi$$

and by virtue of (23), (32) and (33), passing to the limit, it yields

$$\int_\Omega \sigma(u)^{1/2} \varphi \sigma(u)^{1/2} \nabla \varphi \nabla \xi = \int_\Omega \sigma(u) \varphi \nabla \varphi \nabla \xi \quad \text{for all } \xi \in \mathscr{D}(\Omega),$$

so, $\sigma_n(u_n)|\nabla \varphi_n|^2 = \nabla \cdot (\sigma_n(u_n)\varphi_n \nabla \varphi_n) \to \nabla \cdot (\sigma(u)\varphi \nabla \varphi)$ en $\mathscr{D}'(\Omega)$. Since $\sigma_n(u_n)|\nabla \varphi_n|^2 \geqslant 0$ is bounded in $L^1(\Omega)$, we conclude that $\nabla \cdot (\sigma(u)\varphi\nabla\varphi)$ is a positive Radon measure.

This ends up the proof of Theorem 1. □

Remark 2. It is interesting to know if the equality $\nabla \cdot (\sigma(u)\varphi\nabla\varphi) = \sigma(u)|\nabla\varphi|^2$ holds in our setting. There are cases where this holds true (for instance in $N = 1$). We also know some situations where we find that this equality is true:

- In the regular case, that is, if $\varphi \in H^1(\Omega)$.
- In the non-regular case the situation is more complicated. Indeed, according to Theorem 1, we have

$$\varphi \in W^{1,p}(\Omega), \quad \sigma(u)^{-1} \in L^{1/(r-1)}(\Omega), \ \ r = \frac{2}{p}.$$

  Then, we may show that the equality still holds true in the following cases:
  - If $\sigma(u)$ is a weight of the Muckenhoupt class [13]; this means that, for all $x \in \mathbb{R}^N$,

$$\left(\frac{1}{|B_R(x)|} \int_{B_R(x)} \sigma(u) \right) \left(\frac{1}{|B_R(x)|} \int_{B_R(x)} \sigma(u)^{-1/(r-1)} \right)^{r-1} \leqslant C.$$

  - Or if the linear problem

$$\begin{cases} \in W_0^{1,p}(\Omega), \quad \sigma(u)^{1/2}\psi \in L^2(\Omega)^N, \\ \displaystyle\int_\Omega \sigma(u) \nabla \psi \nabla \phi = 0 \quad \text{for all } \phi \in H_0^1(\Omega), \end{cases}$$

  has only the trivial solution $\psi = 0$ (note that in this linear problem, we cannot take $\phi = \psi$ in the variational formulation).

In the general case and with the regularity deduced here for u and φ, we do not know if this equality still holds [12]).

3. Analysis of a singular and non-uniformly elliptic problem

The case described in the previous section does not lead to L^∞-estimates on the temperature u. The situation that we are presenting now considers the case of a singular thermal conductivity, that is, $a(s)$ blows up for a finite value $s=\tau>0$. Under certain hypotheses on data, we show that the temperature remains bounded in Ω. Specifically, it will be shown that $0\leqslant u(x)<\tau$, almost everywhere in Ω.

We consider the thermistor problem (1). All along this section, we will assume the following hypotheses:

(H.6) $\sigma\in C(\mathbb{R})$ and $0<\sigma(s)$ for all $s\in\mathbb{R}$.
(H.7) $a\in C(-\infty,\tau)$, $\tau>0$, $a(0)=0$, $a(s)>0$ for all $s\in(0,\tau)$, $a(s)\geqslant 0$ for all $s<\tau$, and $\int_0^\tau a(s)\,\mathrm{d}s=+\infty$.
(H.8) There exists $n_0>1/\tau$ such that $a(s)$ is an increasing function in the interval $(\tau-1/n_0,\tau)$.
(H.9) $\varphi_0\in H^1(\Omega)$.

Remark 3. Hypothesis (H.6) is very general. Indeed, like in the previous section we are not assuming σ to be uniformly elliptic. Besides, we do not make any assumption on the asymptotic behavior of $\sigma(s)$ for great values of s.

Remark 4. Hypotheses (H.7) and (H.8) yield $\lim_{s\to\tau^-}a(s)=+\infty$; so $a(s)$ becomes singular for the finite value $s=\tau$.

We have the following existence result:

Theorem 2. *Under assumptions* (H.6)–(H.9), *problem* (1) *has a weak solution* (u,φ) *in the following sense*:

$$u\in L^\infty(\Omega),\ 0\leqslant u<\tau\quad \textit{almost everywhere in } \Omega, \tag{36}$$

$$\begin{cases}\forall q<\dfrac{N}{N-1}\quad \textit{if } N\geqslant 2,\ q=2 \textit{ if } N=1,\\ u\in W^{1,q}_{\mathrm{loc}}(\Omega),\ A(u)\in W^{1,q}_0(\Omega),\ \nabla A(u)=a(u)\nabla u,\end{cases} \tag{37}$$

$$\varphi-\varphi_0\in H^1_0(\Omega), \tag{38}$$

$$\int_\Omega a(u)\nabla u\nabla\xi=\int_\Omega \sigma(u)|\nabla\varphi|^2\xi\quad \textit{for all }\ \xi\in\mathscr{D}(\Omega), \tag{39}$$

$$\int_\Omega \sigma(u)\nabla\varphi\nabla\phi=0\quad \textit{for all }\ \phi\in H^1_0(\Omega). \tag{40}$$

P roof. We b egin b y introducing the truncation function $T^n(s)$ given as follows:

$$T^n(s) = \begin{cases} s & \text{if } s < \tau - 1/n, \\ \tau - 1/n, & \text{if } s \geqslant \tau - 1/n. \end{cases}$$

Then we define the regularized diffusion coefficients $a_n(s)$ and $\sigma_\tau(s)$, namely

$$a_n(s) = a(T^n(s)) + \frac{1}{n}, \qquad \sigma_\tau(s) = \sigma(T_\tau(s)),$$

where T_τ is the truncation at height τ defined in (10). Finally, we set the approximate problems

$$\begin{cases} -\nabla \cdot (a_n(u_n)\nabla u_n) = \sigma_\tau(u_n)|\nabla \varphi_n|^2 & \text{in } \Omega, \\ \nabla \cdot (\sigma_\tau(u_n)\nabla \varphi_n) = 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \\ \varphi_n = T_n(\varphi_0) & \text{on } \partial\Omega. \end{cases} \tag{41}$$

A straightforward application of the classical existence results [2] yields that problem (41) has a solution $u_n \in H_0^1(\Omega)$, $\varphi_n - T_n(\varphi_0) \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

From hypothesis (H.6), we deduce that there exist some constants, $C_\tau, c_\tau > 0$, such that $c_\tau \leqslant \sigma_\tau(s) \leqslant C_\tau$ for all $s \in \mathbb{R}$. In particular, taking $\phi = \varphi_n - T_n(\varphi_0)$ in the equation for ϕ yields

$$c_\tau \int_\Omega |\nabla \varphi_n|^2 \leqslant \int_\Omega \sigma_\tau(u_n)|\nabla \varphi_n|^2 \leqslant C_\tau \int_\Omega |\nabla \varphi_0|^2.$$

As a result, (φ_n) is bounded in $H^1(\Omega)$.

On the other hand, putting $A_n(s) = \int_0^s a_n(t)\,\mathrm{d}t$, $v_n = A_n(u_n)$, then $v_n \geqslant 0$ almost everywhere in Ω and (14)–(17) are still valid. Now, let us show that $(A(T^n(u_n))) \subset H_0^1(\Omega)$ is bounded in $W_0^{1,q}(\Omega)$ for all $q < N/(N-1)$. Indeed, using (H.8) and taking $n \geqslant n_0$, we have

$$|\nabla A(T^n(u_n))| \leqslant |a(T^n(u_n))\nabla u_n| \leqslant |a_n(T^n(u_n))\nabla u_n| \leqslant |a_n(u_n)\nabla u_n| = |\nabla v_n|,$$

where we have used (H.8) in the last inequality. Since (v_n) is bounded in $W_0^{1,q}(\Omega)$, the same is true for $(A(T^n(u_n)))$. Consequently, there exist a subsequence (still denoted in the same way) and $w \in W_0^{1,q}(\Omega)$ such that

$$A(T^n(u_n)) \to w \quad \text{in } W_0^{1,q}(\Omega) \text{ and almost everywhere.}$$

In particular, the set $\{w = +\infty\}$ has zero measure. On the other hand, since A and A^{-1}:$[0, +\infty) \mapsto [0, \tau)$ are bijective, we have, for $x \in \Omega$ such that $w(x) < +\infty$,

$$T^n(u_n) = A^{-1}(A(T^n(u_n))) \to A^{-1}(w(x)).$$

We denote $u(x) = A^{-1}(w(x))$; then $w = A(u)$, $0 \leqslant u(x) < \tau$ and also $u_n \to u$ almost everywhere in Ω.

Since (φ_n) is bounded in $H^1(\Omega)$, there exist a subsequence, which we shall denote in the same way, and $\varphi \in H^1(\Omega)$, such that $\varphi = \varphi_0$ on $\partial\Omega$ and $\varphi_n \to \varphi$ in $H^1(\Omega)$-weakly. Taking into account that $\sigma_\tau(u) = \sigma(u)$, it is straightforward that the convergence $\varphi_n \to \varphi$ is in fact strongly in $H^1(\Omega)$. Consequently, we may pass to the limit in the approximate problems; we then obtain

$$\int_\Omega \nabla A(u) \nabla \xi = \int_\Omega \sigma(u) |\nabla \varphi|^2 \xi \quad \text{for all } \xi \in \mathscr{D}(\Omega),$$

$$\int_\Omega \sigma(u) \nabla \varphi \nabla \phi = 0 \quad \text{for all } \phi \in H_0^1(\Omega).$$

It remains to prove that $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and that $\nabla A(u) = a(u) \nabla u$. These properties are based in the following result.

Proposition 1. *For every compact subset $\mathscr{K} \subset \Omega$ there exists a constant $\beta_{\mathscr{K}} > 0$ such that*

$$\operatorname*{ess\,inf}_{\mathscr{K}} A(u) \geqslant \beta_{\mathscr{K}}. \tag{42}$$

Proof of Proposition 1. We have already established that $w = A(u) \in W_0^{1,q}(\Omega)$, $w \geqslant 0$ a.e. in Ω.

Define $f_1 = T_1\left(\sigma(u)|\nabla \varphi|^2\right)$ and let $w_1 \in H_0^1(\Omega)$ be the unique solution to the variational problem

$$\int_\Omega \nabla w_1 \nabla \phi = \int_\Omega f_1 \phi \quad \text{for all } \phi \in H_0^1(\Omega).$$

Then $w_1 \geqslant 0$ a.e. in Ω, and since $\sigma(u)|\nabla\varphi|^2 \geqslant f_1$, a.e. in Ω, we deduce that $w \geqslant w_1$, a.e. in Ω. Now, $f_1 \in L^\infty(\Omega)$, and taking into account the estimates given in [1], it can be shown that $w_1 \in W^{2,p}_{\mathrm{loc}}(\Omega)$, for all $p \geqslant 1$; in particular, $w_1 \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$. Therefore, the hypotheses of Theorem 9.6 of [10] are verified by w_1 and hence $w_1 > 0$ in all Ω.

Finally, if $\mathscr{K} \subset \Omega$ is a compact subset, we get

$$\operatorname*{ess\,inf}_{\mathscr{K}} A(u) \geqslant \min_{\mathscr{K}} w_1 = \beta_{\mathscr{K}} > 0. \qquad \square$$

Since A^{-1} is globally Lipschitz on sets of the form $[\varepsilon, +\infty)$, $\varepsilon > 0$, and $w \in W_0^{1,q}(\Omega)$ we have, using $u = A^{-1}(w)$ that for any subdomain $\Omega' \subset \Omega$ with compact closure in Ω, $u \in W^{1,q}(\Omega')$ and

$$\nabla u = \frac{1}{a(u)} \nabla w \quad \text{in } \Omega'.$$

Note that $1/a(u) \in L^\infty(\Omega')$ thanks to Proposition 1. In conclusion, we have deduced that $\nabla w = \nabla A(u) = a(u) \nabla u$ in Ω', for any arbitrary subdomain $\Omega' \subset \Omega$ with compact closure in Ω.

This ends the proof of Theorem 2. $\square$

Remark 5. From the regularity $\varphi \in H^1(\Omega)$, we know that $\sigma(u)|\nabla u|^2 = \nabla \cdot (\sigma(u)\varphi\nabla\varphi)$. Moreover, if $\varphi_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ then $\varphi \in L^\infty(\Omega)$ and then $\nabla \cdot (\sigma(u)\varphi\nabla\varphi) \in H^{-1}(\Omega)$. In this case, it can be shown that $A(u) \in H_0^1(\Omega)$ and $u \in H_{\text{loc}}^1(\Omega)$.

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