

Convergence to equilibrium of global weak solutions for a Cahn-Hilliard-Navier-Stokes vesicle model

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Abstract

In this paper, we introduce a model describing the dynamic of vesicle membranes within an incompressible viscous fluid in $3D$ domains. The system consists of the Navier-Stokes equations, with an extra stress tensor depending on the membrane, coupled with a Cahn-Hilliard phase-field equation associated to a bending energy plus a penalization term related to the area conservation. This problem has a dissipative in time free-energy which leads, in particular, to prove the existence of global in time weak solutions. We analyze the large-time behavior of the weak solutions. By using a modified Łojasiewicz-Simon's result, we prove the convergence as time goes to infinity of each (whole) trajectory to a single equilibrium. Finally, the convergence of the trajectory of the phase is improved by imposing more regularity on the domain and initial phase.

Keywords: Vesicle membranes, Navier-Stokes equations, Cahn-Hilliard equation, energy dissipation, convergence to equilibrium, Łojasiewicz-Simon's inequalities.

1 Introduction

A type of differential equations modeling the dynamic of vesicle membranes within an incompressible viscous fluids was introduced by Helfrich [13]. The membranes are composed by lipid bilayers, which under appropriate conditions, withdraw into themselves forming a sort of bag, named vesicle. The static equilibrium configurations of the vesicle membranes can be obtained minimizing the Helfrich bending elastic energy, under the constraints of fixed area and volume.

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Numerous studies have been devoted to this type of models and a detailed description of them can be seen in [9] and references therein. A phase function is also used in subsequent papers to model vesicle membranes as diffuse interfaces. In [15] and [22], a coupled Allen-Cahn and Navier-Stokes problem is studied approaching both constraints, area and volume, via a penalization functional. On the other hand, without taking into account the vesicle-fluid interaction, a Cahn-Hilliard phase-field model for vesicle membranes is introduced in [2], and the well-posedness of a phase-field approximation satisfying area and volume constraints pointwisely is established in [7].

In this paper, a Cahn-Hilliard-Navier-Stokes model will be considered. Since the volume constraint is implicitly satisfied for the Cahn-Hilliard equation, only the surface area constraint will be approximated via penalization. Moreover, the resulting problem will be thermodynamically consistent because there exists a free-energy (kinetic plus bending plus penalized one) dissipative in time along the trajectories. This fact is used to prove the existence of global in time weak solutions.

On the other hand, the large-time behavior of the solutions will be analyzed following the way of [12], [16], [4]. Firstly, we prove that the ω -limit set for weak solutions is composed by critical points of the free-energy. After that, by using a modified Lojasiewicz-Simon's result we demonstrate the convergence of the whole trajectory to a single equilibrium. Finally, the convergence is improved for the phase assuming more regular data, but without using strong regularity for the velocity and pressure variables.

The main novelties in this paper are the following:

- The introduction of a new Navier-Stokes-Cahn-Hilliard problem modeling vesicle-fluid interactions.
- The proof of a modified Lojasiewicz-Simon's result associated to weak solutions of a fourth order elliptic problem, which is adapted to the exigences of this new model. In fact, it was not possible to apply any already known Lojasiewicz-Simon's result to this problem.
- Given a global weak solution, we choose a special regularized energy satisfying the energy's law inequality either in a integral version for all time interval and in a differential version a.e. in time (see (40) and (41) below).
- The convergence of each trajectory of weak solutions to a unique equilibrium point, imposing that the regularized energy coincides a.e. in time with the energy evaluated in the weak solution.

The current paper is organized as follows. We explain the model in Section 2 and give some preliminary results in Section 3. Section 4 is devoted to state and prove a new Lojasiewicz-Simon's result. In Section 5 we obtain, via a Galerkin method, the existence of global in time weak solutions satisfying an integral energy's law a.e. time interval. Moreover, we found a function defined for all time t equal to the free-energy a.e. that satisfies the integral energy's law for all time and also a differential version of the energy inequality a.e. time. Section 6 is devoted to the study of convergence at infinite time for global weak solutions. In fact, we prove that the ω -limit set consists of critical points. After that, by using the modified Lojasiewicz-Simon's result, we demonstrate that each trajectory converges to a single equilibrium. In Section 7, some global in time strong estimates are obtained for the phase variable, which allow to improve the norm where the phase trajectory converges.

Notations

- In general, the notation will be abridged. We set $L^p = L^p(\Omega)$, $p \geq 1$, $H_0^1 = H_0^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω , we denote by $L^p(0, T; X)$ the Banach space $L^p(0, T; X(\Omega))$. Also, boldface letters will be used for vectorial spaces, for instance $\mathbf{L}^2 = L^2(\Omega)^N$.
- The L^p -norm is denoted by $|\cdot|_p$, $1 \leq p \leq \infty$, the H^m -norm by $\|\cdot\|_m$ (in particular $|\cdot|_2 = \|\cdot\|_0$). The inner product of $L^2(\Omega)$ is denoted by (\cdot, \cdot) .
- We set \mathcal{V} the space formed by all fields $\mathbf{u} \in C_0^\infty(\Omega)^N$ satisfying $\nabla \cdot \mathbf{u} = 0$. We denote \mathbf{H} (respectively \mathbf{V}) the closure of \mathcal{V} in \mathbf{L}^2 (respectively \mathbf{H}^1). \mathbf{H} and \mathbf{V} are Hilbert spaces for the norms $|\cdot|_2$ and $\|\cdot\|_1$, respectively. Furthermore,

$$\begin{aligned} \mathbf{H} &= \{\mathbf{u} \in \mathbf{L}^2; \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ \mathbf{V} &= \{\mathbf{u} \in \mathbf{H}^1; \nabla \cdot \mathbf{u} = 0, \mathbf{u} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

- From now on, $C > 0$ will denote different constants, depending only on the data of the problem.
- We consider the following zero-mean spaces:

$$\begin{aligned} L_*^2 &= \left\{ \psi \in L^2; \int_{\Omega} \psi = 0 \right\}, \\ H_*^1 &= H^1 \cap L_*^2, \\ H_1^k &= \{\psi \in H^k \cap L_*^2; \partial_n \psi|_{\partial\Omega} = 0\}, \quad k = 2, 3, \end{aligned}$$

and the dual space $H_*^{-1} = (H_*^1)'$.

2 The model

Vesicle membranes within incompressible viscous fluids in a bounded 3D domain $\Omega \subset \mathbb{R}^3$ during a time interval $[0, +\infty)$, are modeled via a phase-field function $\phi : \overline{\Omega} \times [0, +\infty) \rightarrow \mathbb{R}$ such that two stable values $\phi = 1$ and $\phi = -1$ represent the exterior and interior of vesicle membranes, respectively, and $-1 < \phi < 1$ in the interface. Then, we will analyze the case where the so-called *bending energy* $\mathcal{E}_b(\phi)$ is given by a simplified elastic Willmore energy plus a penalization of the surface area constraint [10]:

$$\mathcal{E}_b(\phi) = \frac{1}{2\varepsilon} \int_{\Omega} (-\varepsilon \Delta \phi + \frac{1}{\varepsilon} F'(\phi))^2 d\mathbf{x} + \frac{M}{2} (\mathcal{A}(\phi) - \alpha)^2 \quad (1)$$

where $F'(\phi) = (\phi^2 - 1)\phi$ denotes the derivative of the Ginzburg-Landau potential

$$F(\phi) = \frac{1}{4}(\phi^2 - 1)^2,$$

$M > 0$ is a penalization constant, $\varepsilon > 0$ is related to the interface width, and

$$\mathcal{A}(\phi) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{\varepsilon} F(\phi) \right) d\mathbf{x},$$

is an approximation of the surface area.

Remark 1 *Other possible approximation of the surface area is to consider*

$$\mathcal{A}(\phi) = \int_{\Omega} \frac{\varepsilon}{2} |\nabla \phi|^2 d\mathbf{x}$$

(see [2]). *The same results of this paper may be extended to this case.*

To model the dynamic in time of the vesicle-fluid interactions, we will introduce the following Navier-Stokes-Cahn-Hilliard equations in $Q = \Omega \times (0, +\infty)$:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} - \lambda w \nabla \phi + \nabla q = 0, \quad (2)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (3)$$

$$\partial_t \phi + \mathbf{u} \cdot \nabla \phi - \gamma \Delta w = 0, \quad (4)$$

where

$$w := \frac{\delta \mathcal{E}_b(\phi)}{\delta \phi}$$

is the chemical potential, see (9) below. The coefficients $\nu > 0$, $\lambda > 0$ and $\gamma > 0$ depend on viscosity, elasticity and mobility, respectively. The system (2)-(4) is completed with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \partial_n \phi|_{\partial\Omega} = 0, \quad \partial_n (\Delta \phi)|_{\partial\Omega} = 0, \quad \partial_n w|_{\partial\Omega} = 0, \quad (5)$$

and the initial conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \phi|_{t=0} = \phi_0 \quad \text{in } \Omega. \quad (6)$$

For compatibility, we will assume $\mathbf{u}_0|_{\partial\Omega} = 0$ with $\nabla \cdot \mathbf{u}_0 = 0$ in Ω and $\partial_n \phi_0|_{\partial\Omega} = 0$.

By integrating the phase-equation (4), using the free-divergence $\nabla \cdot \mathbf{u} = 0$, the non-slip condition $\mathbf{u}|_{\partial\Omega} = 0$, and the last boundary condition of (5), $\partial_n w|_{\partial\Omega} = 0$, it is easy to deduce that the total volume of ϕ in Ω is conserved:

$$\frac{d}{dt} \int_{\Omega} \phi(x, t) \, d\mathbf{x} = 0. \quad (7)$$

In order to complete the problem (2)-(4), we compute the variational derivative $w = \frac{\delta \mathcal{E}_b(\phi)}{\delta \phi}$.

On the one hand, for all $\phi, \bar{\phi} \in H^1$,

$$\left\langle \frac{\delta \mathcal{A}(\phi)}{\delta \phi}, \bar{\phi} \right\rangle = \int_{\Omega} \varepsilon \nabla \phi \cdot \nabla \bar{\phi} + \frac{1}{\varepsilon} F'(\phi) \bar{\phi},$$

hence if $\phi \in H^2$ and $\partial_n \phi|_{\partial\Omega} = 0$, integrating by parts, we can identify

$$\mu(\phi) := \frac{\delta \mathcal{A}(\phi)}{\delta \phi} = -\varepsilon \Delta \phi + \frac{1}{\varepsilon} F'(\phi). \quad (8)$$

Note that, if $\phi \in H^3$ and $\partial_n \phi|_{\partial\Omega} = 0$, then $\partial_n \Delta \phi|_{\partial\Omega} = 0$ is equivalent to $\partial_n \mu(\phi)|_{\partial\Omega} = 0$.

On the other hand, by using (8), the bending energy (1) can be rewritten as

$$\mathcal{E}_b(\phi) = \frac{1}{2\varepsilon} \int_{\Omega} \mu(\phi)^2 \, d\mathbf{x} + \frac{1}{2} M(\mathcal{A}(\phi) - \alpha)^2.$$

Then, for all $\phi, \bar{\phi} \in H^2$,

$$\left\langle \frac{\delta \mathcal{E}_b(\phi)}{\delta \phi}, \bar{\phi} \right\rangle = \frac{1}{\varepsilon} \int_{\Omega} \mu(\phi) (-\varepsilon \Delta \bar{\phi} + \frac{1}{\varepsilon} F''(\phi) \bar{\phi}) + M(\mathcal{A}(\phi) - \alpha) \left\langle \frac{\delta \mathcal{A}(\phi)}{\delta \phi}, \bar{\phi} \right\rangle.$$

If $\phi \in H_2^4$ and $\bar{\phi} \in H_1^2$, after some integrations by parts, using $\nabla \mu(\phi) \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\nabla \bar{\phi} \cdot \mathbf{n}|_{\partial\Omega} = 0$, we can identify

$$w = \frac{\delta \mathcal{E}_b(\phi)}{\delta \phi} = -\Delta \mu(\phi) + \frac{1}{\varepsilon^2} F''(\phi) \mu(\phi) + M(\mathcal{A}(\phi) - \alpha) \mu(\phi). \quad (9)$$

Remark 2 The variational derivatives $w = \frac{\delta \mathcal{E}_b(\phi)}{\delta \phi}$ and $\mu(\phi) = \frac{\delta \mathcal{A}(\phi)}{\delta \phi}$ given in (8) and (9) have been identified as $L^2(\Omega)$ -functions via the $L^2(\Omega)$ scalar product.

From (8) and (9), we can decompose w as

$$w = \varepsilon \Delta^2 \phi + G(\phi) \quad (10)$$

where

$$\begin{aligned} G(\phi) &:= -\frac{1}{\varepsilon} \Delta(F'(\phi)) + \frac{1}{\varepsilon^2} F''(\phi) \mu(\phi) + M(\mathcal{A}(\phi) - \alpha) \mu(\phi) \\ &= -\frac{2}{\varepsilon} F''(\phi) \Delta \phi - \frac{1}{\varepsilon} F'''(\phi) |\nabla \phi|^2 + \frac{1}{\varepsilon^3} F''(\phi) F'(\phi) \\ &\quad + M(\mathcal{A}(\phi) - \alpha) (-\varepsilon \Delta \phi + \frac{1}{\varepsilon} F'(\phi)). \end{aligned} \quad (11)$$

2.1 A equivalent “zero-mean” problem

With the aim to use the conservative property given in (7), if we define

$$m_0 = \langle \phi_0 \rangle := \frac{1}{|\Omega|} \int_{\Omega} \phi_0(x) \, d\mathbf{x},$$

we can introduce the following zero-mean variables:

$$\psi(x, t) := \phi(x, t) - m_0 \quad \text{and} \quad z := w - \langle G(\phi) \rangle.$$

Observe that,

$$\int_{\Omega} \psi(t, \cdot) \, d\mathbf{x} = 0 \quad \text{and} \quad \int_{\Omega} z(t, \cdot) \, d\mathbf{x} = 0.$$

In fact, since $\partial_n \phi|_{\partial\Omega} = 0$, in particular $\partial_n(F'(\phi))|_{\partial\Omega} = 0$, hence $\int_{\Omega} -\Delta(F'(\phi)) \, d\mathbf{x} = 0$. Therefore, integrating (10) and (11),

$$\langle w \rangle = \langle G(\phi) \rangle = \frac{1}{\varepsilon^2} \langle F''(\phi) \mu(\phi) \rangle + M(\mathcal{A}(\phi) - \alpha) \langle \mu(\phi) \rangle. \quad (12)$$

Reciprocally, given (ψ, z) we can recover (ϕ, w) as $\phi = \psi + m_0$ and $w = z + \langle G(\phi) \rangle$.

By rewriting the equations (2)-(4) and (10) in these new variables (ψ, z) we arrive at

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} - \lambda z \nabla \psi + \nabla \tilde{q} = 0, \quad (13)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (14)$$

$$\partial_t \psi + \mathbf{u} \cdot \nabla \psi - \gamma \Delta z = 0, \quad (15)$$

$$\varepsilon \Delta^2 \psi + \overline{G}(\psi) - z = 0, \quad (16)$$

where

$$\overline{G}(\psi) := G(\psi + m_0) - \langle G(\psi + m_0) \rangle, \quad \tilde{q} := q - \lambda \langle G(\psi + m_0) \rangle \psi.$$

By using (11) and (12),

$$\begin{aligned}
\overline{G}(\psi) &= -\frac{2}{\varepsilon}F''(\psi + m_0)\Delta\psi - \frac{1}{\varepsilon}F'''(\psi + m_0)|\nabla\psi|^2 + \frac{1}{\varepsilon^3}F''(\psi + m_0)F'(\psi + m_0) \\
&\quad + M(\mathcal{A}(\psi + m_0) - \alpha) \left(-\varepsilon\Delta\psi + \frac{1}{\varepsilon}F'(\psi + m_0) \right) \\
&\quad - \frac{1}{\varepsilon^2} \frac{1}{|\Omega|} \int_{\Omega} \left(-\varepsilon\Delta\psi + \frac{1}{\varepsilon}F'(\psi + m_0) \right) F''(\psi + m_0) \\
&\quad - M(\mathcal{A}(\psi + m_0) - \alpha) \frac{1}{|\Omega|} \int_{\Omega} \left(-\varepsilon\Delta\psi + \frac{1}{\varepsilon}F'(\psi + m_0) \right).
\end{aligned} \tag{17}$$

System (13)-(16) is completed with the boundary and initial conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \partial_n\psi|_{\partial\Omega} = 0, \quad \partial_n(\Delta\psi)|_{\partial\Omega} = 0, \quad \partial_n z|_{\partial\Omega} = 0, \tag{18}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \psi|_{t=0} = \psi_0 := \phi_0 - \langle \phi_0 \rangle \quad \text{in } \Omega. \tag{19}$$

Consequently, problem (2)-(6) is equivalent to the “zero-mean” problem (13)-(19).

Note that, by denoting the bending energy with respect to the zero-mean unknown ψ as

$$\overline{\mathcal{E}}_b(\psi) = \mathcal{E}_b(\psi + m_0), \tag{20}$$

then it is easy to check that

$$z = \frac{\delta\overline{\mathcal{E}}_b(\psi)}{\delta\psi} = \varepsilon\Delta^2\psi + \overline{G}(\psi) \quad \forall \psi \in H_2^4, \tag{21}$$

where the identification (21) has been computed via the $L_*^2(\Omega)$ -scalar product. Latter, we will give a weak sense to this identification in order to define the concept of weak solution.

3 Some preliminary results

We assume Ω sufficiently regular (for instance $\partial\Omega \in C^3$) in order to have the H^3 -regularity of the following second order elliptic Poisson-Neumann problem:

$$(P_0) \quad \begin{cases} -\Delta v = f & \text{in } \Omega \\ \partial_n v|_{\partial\Omega} = 0, & \int_{\Omega} v \, d\mathbf{x} = 0, \end{cases}$$

where $f \in L_*^2(\Omega)$. In fact, the H^2 and H^3 -regularity of (P₀) provide, respectively, the existence of some constants $C > 0$, such that:

$$\|v\|_2 \leq C\|\Delta v\|_2 \quad \forall v \in H_1^2, \quad \|v\|_3 \leq C\|\Delta v\|_1 \quad \forall v \in H_1^3. \tag{22}$$

In particular, since $\Delta v \in L_*^2$, by applying the Poincare inequality for zero-mean functions, we have that

$$\|v\|_3 \leq C\|\nabla\Delta v\|_2 \quad \forall v \in H_1^3. \tag{23}$$

In order to define the inner product in H_*^{-1} , observe that if $r \in H_*^{-1}$, then applying the Lax-Milgram's Lemma, there exists a unique $u_r \in H_*^1$ such that $\langle r, v \rangle_{H_*^{-1}, H_*^1} = (u_r, v)_{H_*^1} = (\nabla u_r, \nabla v)_{L^2}$ for all $v \in H_*^1$. Therefore, $\|r\|_{H_*^{-1}} = \|\nabla u_r\|_{L^2}$ and $(r, s)_{H_*^{-1}} = (\nabla u_r, \nabla u_s)_{L^2}$ for all $s \in H_*^{-1}$.

In order to give a weak sense to $\Delta^2 v$ together with the boundary condition $\partial_n(\Delta v)|_{\partial\Omega} = 0$ for any $v \in H_1^3$, we introduce the operator $A : D(A) = H_1^3 \subset H_*^{-1} \mapsto H_*^{-1}$ as follows:

$$\langle A\psi, \tilde{\psi} \rangle_{H_*^{-1}, H_*^1} = -(\nabla \Delta \psi, \nabla \tilde{\psi})_{L^2} \quad \forall \psi \in H_1^3, \forall \tilde{\psi} \in H_*^1. \quad (24)$$

In particular, denoting $r = A\psi$ and $s = A\tilde{\psi}$, then $u_r = -\Delta\psi$ and $u_s = -\Delta\tilde{\psi}$, hence the inner product in H_*^{-1} remains $(A\psi, A\tilde{\psi})_{H_*^{-1}} = (\nabla \Delta \psi, \nabla \Delta \tilde{\psi})_{L^2}$.

Observe that $A \in \mathcal{L}(H_1^3, H_*^{-1})$ and it is self-adjoint and positive definite. Indeed, A is continuous because

$$\|A\psi\|_{H_*^{-1}} = \sup_{\tilde{\psi}} \frac{\langle A\psi, \tilde{\psi} \rangle_{H_*^{-1}, H_*^1}}{\|\tilde{\psi}\|_{H_*^1}} = \sup_{\tilde{\psi}} \frac{-(\nabla \Delta \psi, \nabla \tilde{\psi})_{L^2}}{|\nabla \tilde{\psi}|_2}$$

In particular, taking $\tilde{\psi} = \Delta\psi \in H_*^1$, one has

$$\|A\psi\|_{H_*^{-1}} = \|\nabla \Delta \psi\|_{L^2} \leq C\|\psi\|_3. \quad (25)$$

On the other hand, for all $\psi, \tilde{\psi} \in H_1^3$, since $\partial_n \psi|_{\partial\Omega} = \partial_n \tilde{\psi}|_{\partial\Omega} = 0$, we have that

$$\langle A\psi, \tilde{\psi} \rangle_{H_*^{-1}, H_*^1} = -(\nabla \Delta \psi, \nabla \tilde{\psi})_{L^2} = (\Delta \psi, \Delta \tilde{\psi})_{L^2} = -(\nabla \psi, \nabla \Delta \tilde{\psi})_{L^2} = \langle A\tilde{\psi}, \psi \rangle_{H_*^{-1}, H_*^1}.$$

Therefore, A is symmetric. In particular, $\langle A\psi, \psi \rangle_{H_*^{-1}, H_*^1} = \|\Delta \psi\|_{L^2}^2 \geq C\|\psi\|_2^2$ for all $\psi \in H_1^3$, hence A is positive definite. Finally, we are going to prove that there exists $A^{-1} \in \mathcal{L}(H_*^{-1}, H_1^3)$ by applying the Banach-Necas-Babuska's Theorem (see for instance [11]). For this, we consider the following continuous bilinear form $a(\cdot, \cdot) : H_1^3 \times H_*^1 \mapsto \mathbb{R}$:

$$a(\psi, \tilde{\psi}) = \langle A\psi, \tilde{\psi} \rangle_{H_*^{-1}, H_*^1} = -(\nabla \Delta \psi, \nabla \tilde{\psi})_{L^2} \quad \forall \psi \in H_1^3, \forall \tilde{\psi} \in H_*^1.$$

Owing to (23) and (25), there exists a constant $\beta > 0$ such that

$$\sup_{\tilde{\psi} \in H_*^1} \frac{a(\psi, \tilde{\psi})}{|\nabla \tilde{\psi}|_2} = \|A\psi\|_{H_*^{-1}} \geq \beta\|\psi\|_3.$$

On the other hand, if we assume $a(\psi, \tilde{\psi}) = 0$ for all $\psi \in H_1^3$, then taking in particular $\psi \in H_1^3$ the solution of problem (P₀) for $f = \tilde{\psi}$, then $-\Delta\psi = \tilde{\psi}$ and

$$0 = a(\psi, \tilde{\psi}) = -(\nabla \Delta \psi, \nabla \tilde{\psi})_{L^2} = |\nabla \tilde{\psi}|_2^2$$

hence $\tilde{\psi} \equiv Cte$. But, since $\int_{\Omega} \tilde{\psi} = 0$, then $\tilde{\psi} \equiv 0$. Therefore, the Banach-Necas-Babuska's Theorem implies that there exists $A^{-1} \in \mathcal{L}(H_*^{-1}, H_1^3)$. Moreover, A is self-adjoint.

The following lemma shows two compactness results of Aubin-Lions type, see [20].

Lemma 3 *Let us consider $T > 0$ and three Banach spaces such that $X \subset B \subset Y$ with continuous imbedding $B \rightarrow Y$ and continuous and compact imbedding $X \hookrightarrow B$.*

- *If the set F is bounded in $L^p(0, T; X)$ where $p < \infty$ and $\partial_t F = \{\partial_t f : f \in F\}$ is bounded in $L^1(0, T; Y)$, then F is relatively compact in $L^p(0, T; B)$.*
- *If the set F is bounded in $L^\infty(0, T; X)$ and $\partial_t F$ is bounded in $L^q(0, T; X)$ with $q > 1$, then F is relatively compact in $C([0, T]; B)$.*

Along this paper, we will use repeatedly the following classical interpolation and Sobolev inequalities (for 3D domains):

$$|v|_6 \leq C\|v\|_1, \quad |v|_3 \leq C|v|_2^{1/2}\|v\|_1^{1/2} \quad \forall v \in H^1$$

and the Gagliardo-Nirenberg inequality

$$|v|_\infty \leq C\|v\|_1^{1/2}\|v\|_2^{1/2} \quad \forall v \in H^2.$$

For the last inequality, see for example ([17], p. 334).

The following Lemma gives some global Lipschitz properties of $F(\phi)$ its derivatives and $\mathcal{A}(\phi)$ into H^2 -bounded sets.

Lemma 4 *Let us consider $K > 0$ a constant and any functions $\phi_i \in H^2(\Omega)$, with $\|\phi_i\|_2 \leq K$, for $i = 1, 2$, then the following inequalities are fulfilled for any p with $1 \leq p \leq \infty$:*

$$\begin{aligned} |F(\phi_i)|_p &\leq C(K), \quad |F'(\phi_i)|_p \leq C(K), \quad |F''(\phi_i)|_p \leq C(K), \quad |F'''(\phi_i)|_p \leq C(K) \quad i = 1, 2, \\ |F(\phi_1) - F(\phi_2)|_p &\leq C(K)|\phi_1 - \phi_2|_p, \quad |F'(\phi_1) - F'(\phi_2)|_p \leq C(K)|\phi_1 - \phi_2|_p, \\ |F''(\phi_1) - F''(\phi_2)|_p &\leq C(K)|\phi_1 - \phi_2|_p, \quad |F'''(\phi_1) - F'''(\phi_2)|_p \leq C(K)|\phi_1 - \phi_2|_p, \\ |\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)| &\leq C(K)\|\phi_1 - \phi_2\|_1, \end{aligned}$$

where $C(K) > 0$ are different constants depending on K .

Proof. Let us remember that $F(\phi) = \frac{1}{4}(\phi^2 - 1)^2$, $F'(\phi) = (\phi^2 - 1)\phi$, $F''(\phi) = 3\phi^2 - 1$, $F'''(\phi) = 6\phi$ and $\mathcal{A}(\phi) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{\varepsilon} F(\phi) \right) d\mathbf{x}$. We prove the first inequality:

$$|F(\phi_i)|_p = \frac{1}{4}|\phi^4 + 2\phi^2 - 1|_p \leq C(|\phi|_\infty^4 + 2|\phi|_\infty^2 + 1) \leq C(\|\phi\|_2^4 + 2\|\phi\|_2^2 + 1) \leq C(K),$$

The second, third and fourth inequalities can be obtained in an analogous way. We prove now the fifth one:

$$|F(\phi_1) - F(\phi_2)|_p \leq |F'(\theta\phi_1 + (1-\theta)\phi_2)|_\infty |\phi_1 - \phi_2|_p \leq C(K) |\phi_1 - \phi_2|_p.$$

Sixth, seventh and eighth inequalities can be proved in an analogous way. Finally, let us see the ninth one:

$$\begin{aligned} |\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)| &\leq \frac{1}{2} \|\nabla\phi_1\|_2^2 - \|\nabla\phi_2\|_2^2 + \int_\Omega |F(\phi_1) - F(\phi_2)| \\ &\leq \frac{1}{2} \|\nabla(\phi_1 + \phi_2)\|_2 \|\nabla(\phi_1 - \phi_2)\|_2 + C(K) |\phi_1 - \phi_2|_1 \\ &\leq C(K) \|\phi_1 - \phi_2\|_1. \end{aligned}$$

■

The following Lemma gives the global Lipschitz property of $G(\phi)$ into H^2 -bounded sets.

Lemma 5 *The map $G = G(\phi)$ given in (11) is well-posed from $H^2(\Omega)$ into $L^2(\Omega)$. Moreover, for any functions $\phi_i \in H^2(\Omega)$, with $\|\phi_i\|_2 \leq K$, for $i = 1, 2$, there exists $C = C(K) > 0$ such that*

$$|G(\phi_1) - G(\phi_2)|_2 \leq C(K) \|\phi_1 - \phi_2\|_2.$$

Remark 6 *In particular, since $G(0) = 0$, Lemma 5 implies $|G(\phi)|_2 \leq C(K) \|\phi\|_2 \leq C(K)$.*

Proof. From (11)

$$\begin{aligned} |G(\phi_1) - G(\phi_2)|_2 &\leq \frac{2}{\varepsilon} |F''(\phi_1)\Delta\phi_1 - F''(\phi_2)\Delta\phi_2|_2 \\ &\quad + \frac{1}{\varepsilon} |F'''(\phi_1)|\nabla\phi_1|^2 - F'''(\phi_2)|\nabla\phi_2|^2|_2 + \frac{1}{\varepsilon^3} |F'(\phi_1)F''(\phi_1) - F'(\phi_2)F''(\phi_2)|_2 \\ &\quad + \varepsilon M |(\mathcal{A}(\phi_1) - \alpha)\Delta\phi_1 - (\mathcal{A}(\phi_2) - \alpha)\Delta\phi_2|_2 \\ &\quad + \frac{M}{\varepsilon} |(\mathcal{A}(\phi_1) - \alpha)F'(\phi_1) - (\mathcal{A}(\phi_2) - \alpha)F'(\phi_2)|_2 := \sum_{i=1}^5 I_i. \end{aligned} \tag{26}$$

By taking into account Lemma 4, we can estimate each term I_i as follows:

$$\begin{aligned} I_1 &= \frac{2}{\varepsilon} |F''(\phi_1)\Delta\phi_1 - F''(\phi_2)\Delta\phi_2|_2 \\ &\leq C(|F''(\phi_1) - F''(\phi_2)|\Delta\phi_1|_2 + |F''(\phi_2)|(\Delta\phi_1 - \Delta\phi_2)|_2) \\ &\leq C(\|\Delta\phi_1\|_2 |F''(\phi_1) - F''(\phi_2)|_\infty + |F''(\phi_2)|_\infty \|\Delta\phi_1 - \Delta\phi_2\|_2) \\ &\leq C(\|\phi_1\|_2 C(K) \|\phi_1 - \phi_2\|_\infty + C(K) \|\phi_1 - \phi_2\|_2) \\ &\leq C(K) \|\phi_1 - \phi_2\|_2. \end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{1}{\varepsilon} |F'''(\phi_1)|\nabla\phi_1|^2 - F'''(\phi_2)|\nabla\phi_2|^2|_2 \\
&\leq C(|(F'''(\phi_1) - F'''(\phi_2))|\nabla\phi_1|^2|_2 + |F'''(\phi_2)(|\nabla\phi_1|^2 - |\nabla\phi_2|^2)|_2) \\
&\leq C(|\nabla\phi_1|_6^2 |F'''(\phi_1) - F'''(\phi_2)|_6 + |F'''(\phi_2)|_6 |\nabla\phi_1 + \nabla\phi_2|_6 |\nabla\phi_1 - \nabla\phi_2|_6) \\
&\leq C(\|\phi_1\|_2^2 C(K) \|\phi_1 - \phi_2\|_6 + C(K) \|\phi_1 + \phi_2\|_2 \|\phi_1 - \phi_2\|_2) \\
&\leq C(K) \|\phi_1 - \phi_2\|_2.
\end{aligned}$$

$$\begin{aligned}
I_3 &= \frac{1}{\varepsilon^3} |F'(\phi_1)F''(\phi_1) - F'(\phi_2)F''(\phi_2)|_2 \\
&\leq C(|(F'(\phi_1) - F'(\phi_2))F''(\phi_1)|_2 + |F'(\phi_2)(F''(\phi_1) - F''(\phi_2))|_2) \\
&\leq C(|F'(\phi_1) - F'(\phi_2)|_6 |F''(\phi_1)|_3 + |F'(\phi_2)|_3 |F''(\phi_1) - F''(\phi_2)|_6) \\
&\leq C(K) \|\phi_1 - \phi_2\|_1.
\end{aligned}$$

To estimate I_4 and I_5 observe that

$$|\mathcal{A}(\phi_2) - \alpha| \leq C(|\nabla\phi_2|_2^2 + |\phi_2^2 - 1|_2^2) \leq C(\|\phi_2\|_1^2 + \|\phi_2\|_2^4 + 1) \leq C(K),$$

therefore,

$$\begin{aligned}
I_4 &= \varepsilon M |(\mathcal{A}(\phi_1) - \alpha)\Delta\phi_1 - (\mathcal{A}(\phi_2) - \alpha)\Delta\phi_2|_2 \\
&\leq C(|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)| |\Delta\phi_1|_2 + |\mathcal{A}(\phi_2) - \alpha| |\Delta(\phi_1 - \phi_2)|_2) \\
&\leq C(K)(\|\phi_1 - \phi_2\|_1 + \|\phi_1 - \phi_2\|_2) \leq C(K) \|\phi_1 - \phi_2\|_2,
\end{aligned}$$

and finally,

$$\begin{aligned}
I_5 &= \frac{M}{\varepsilon} |(\mathcal{A}(\phi_1) - \alpha)F'(\phi_1) - (\mathcal{A}(\phi_2) - \alpha)F'(\phi_2)|_2 \\
&\leq C(|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)| |F'(\phi_1)|_2 + |\mathcal{A}(\phi_2) - \alpha| |F'(\phi_1) - F'(\phi_2)|_2) \\
&\leq C(K) \|\phi_1 - \phi_2\|_1.
\end{aligned}$$

Plugging all previous estimates in (26), the proof of Lemma is finished. ■

4 A new Lojasiewicz-Simon's result

The use of Lojasiewicz-Simon inequalities is a classical procedure to study the convergence of trajectories at infinite time in dissipative systems. It is not easy to find in the literature a rigorous demonstrations of these types of inequalities associated to various Euler-Lagrange equations. Here, a particular Lojasiewicz-Simon's inequality associated to the critical points of the bending energy $\bar{\mathcal{E}}_b(\psi)$ for zero-mean functions ψ is deduced, by using the Theorem 9 presented below, which is a simplified version of Theorem 4.2, p. 41 of [14]. We make an extension of the Lemma 4.4 of [19] (which is applied to a second order elliptic problem) to the fourth-order elliptic problem that determines the critical points ψ_* of $\bar{\mathcal{E}}_b(\psi)$ and then, it

will be possible to prove a modified Lojasiewicz-Simon's result by relaxing the hypothesis of small $\|\psi - \psi_*\|_3$ by small $\|\psi - \psi_*\|_1$ and $|\overline{\mathcal{E}}_b(\psi) - \overline{\mathcal{E}}_b(\psi_*)|$.

We begin by recalling two definitions:

Definition 7 ([14] p. 22) *Let U be an open subset in a real Banach space $(X, \|\cdot\|_X)$ and Y a subspace of the dual space X' . A continuous map $\mathcal{M} : U \mapsto Y$ is called a **gradient map** if there exists a C^1 functional $\mathcal{E} : U \mapsto \mathbb{R}$ such that $\mathcal{M}(u) = \mathcal{E}'(u)$ for all $u \in U$, i.e.*

$$\mathcal{E}'(u)h = \langle \mathcal{M}(u), h \rangle \quad \forall u \in U, h \in X$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $X' \times X$.

Definition 8 ([14] p. 34) *A bounded linear operator $L : X_1 \mapsto X_2$ between two Banach spaces X_1 and X_2 is called a **Fredholm operator of index zero** if L has a closed range $R(L)$, a finite dimensional kernel $N(L)$ and $\dim(N(L)) = \dim(X_2/R(L)) < \infty$. A C^1 -map $\mathcal{M} : U \subset X_1 \mapsto X_2$ is called a **Fredholm map of index zero** if its Fréchet differential at each point is a Fredholm operator of index zero.*

For instance, an invertible operator plus a compact operator is a Fredholm operator of index zero (see, for example [1] pp. 98, 99)

We now give a simplified version of Theorem 4.2 of [14].

Theorem 9 *Assume the following hypotheses:*

- *Let H be a Hilbert space and let $A : D(A) \subset H \mapsto H$ be a linear self-adjoint and positive definite operator. We denote $(D(A), \langle \cdot, \cdot \rangle_A)$ the Hilbert space endowed with the scalar product $\langle u, v \rangle_A \equiv (Au, Av)_H$ for all $u, v \in D(A)$. Assume that the embedding $D(A) \subset H$ is continuous.*
- *Let $\mathcal{E} : D(A) \mapsto \mathbb{R}$ be a Fréchet-differentiable functional, and let $\mathcal{M} : D(A) \mapsto H$ be an analytic gradient map associated to \mathcal{E} (i.e. $\mathcal{M} = \mathcal{E}'$) with the following properties:*
 - *\mathcal{M} is a Fredholm map of index zero; i.e., for each $u \in D(A)$ the bounded linear operator $\mathcal{M}'(u) \in \mathcal{L}(D(A), H)$ is a Fredholm operator of index zero.*
 - *The map $\mathcal{R} : u \in D(A) \mapsto \mathcal{M}'(u)A^{-1} \in \mathcal{L}(H)$ is continuous.*
- *Let $u_* \in D(A)$ be a critical point of $\mathcal{E}(u)$, i.e. $\mathcal{E}'(u_*) = 0$.*

Then, there exists positive constants C , β and $\theta \in (0, 1/2]$ such that for all $u \in D(A)$ with $\|u - u_\|_{D(A)} \leq \beta$, it holds*

$$|\mathcal{E}(u) - \mathcal{E}(u_*)|^{1-\theta} \leq C \|\mathcal{E}'(u)\|_H.$$

Lemma 10 (Lojasiewicz-Simon inequality) *Let \mathcal{S} be the following set of equilibrium points related to the bending energy $\bar{\mathcal{E}}_b(\psi)$ given in (20):*

$$\mathcal{S} = \{\psi \in H_1^3(\Omega) : -\varepsilon(\nabla\Delta\psi, \nabla\tilde{\psi}) + (\bar{G}(\psi), \tilde{\psi}) = 0 \quad \forall \tilde{\psi} \in H_*^1\}. \quad (27)$$

Let $\psi_ \in \mathcal{S}$ and $K > 0$ fixed. Then, there exists positive constants β_1, β_2 and C and $\theta \in (0, 1/2]$, such that for all $\psi \in H_1^3$ with $\|\psi\|_2 \leq K$, $\|\psi - \psi_*\|_1 \leq \beta_1$ and $|\bar{\mathcal{E}}_b(\psi) - \bar{\mathcal{E}}_b(\psi_*)| \leq \beta_2$, it holds*

$$|\bar{\mathcal{E}}_b(\psi) - \bar{\mathcal{E}}_b(\psi_*)|^{1-\theta} \leq C \|z(\psi)\|_{H_*^{-1}} \quad (28)$$

where $z(\psi) := \varepsilon A\psi + \bar{G}(\psi)$ in the sense

$$\langle z(\psi), \tilde{\psi} \rangle_{H_*^{-1}, H_*^1} = -\varepsilon(\nabla\Delta\psi, \nabla\tilde{\psi}) + (\bar{G}(\psi), \tilde{\psi}), \quad \forall \tilde{\psi} \in H_*^1.$$

Proof.

Step 1: *There exists $\beta > 0$, $C > 0$ and $\theta \in (0, 1/2]$ (depending on ψ_*) such that for all $\psi \in H_1^3$ with $\|\psi - \psi_*\|_3 \leq \beta$, then (28) holds.*

The proof of this step is based on Theorem 9, choosing the spaces $H \equiv H_*^{-1}(\Omega)$, $D(A) \equiv H_1^3(\Omega)$ and by taking the following operators:

- $A : \psi \in H_1^3 \mapsto A\psi \in H_*^{-1}$ defined in (24). In particular, we have

$$\langle \psi, \xi \rangle_A := (A\psi, A\xi)_{H_*^{-1}} = (\nabla\Delta\psi, \nabla\Delta\xi)_{L^2} \quad \forall \xi, \psi \in H_1^3.$$

Note that $\langle \cdot, \cdot \rangle_A$ is an inner product in H_1^3 owing to (23).

- $\mathcal{E} : \psi \in H_1^3 \mapsto \mathcal{E}(\psi) = \bar{\mathcal{E}}_b(\psi) \in \mathbb{R}$,
- $\mathcal{E}' : \psi \in H_1^3 \mapsto \mathcal{E}'(\psi) \in (H_1^3)'$ defined as:

$$\begin{aligned} \langle \mathcal{E}'(\psi), \tilde{\psi} \rangle_{(H_1^3)', H_1^3} &= \varepsilon(\Delta\psi, \Delta\tilde{\psi})_{L^2} + (\bar{G}(\psi), \tilde{\psi})_{L^2} \\ &= -\varepsilon(\nabla\Delta\psi, \nabla\tilde{\psi})_{L^2} + (\bar{G}(\psi), \tilde{\psi})_{L^2} \quad \forall \psi, \tilde{\psi} \in H_1^3, \end{aligned}$$

- $\mathcal{M} \equiv \mathcal{E}' : \psi \in H_1^3 \mapsto \mathcal{M}(\psi) \in H_*^{-1}$ is an extension of $\mathcal{E}'(\psi)$ by density, defined as:

$$\langle \mathcal{M}(\psi), \tilde{\psi} \rangle_{H_*^{-1}, H_*^1} = \langle \varepsilon A\psi, \tilde{\psi} \rangle_{H_*^{-1}, H_*^1} + (\bar{G}(\psi), \tilde{\psi})_{L^2} \quad \forall \psi \in H_1^3, \forall \tilde{\psi} \in H_*^1,$$

- $\mathcal{M}'(\psi) : \xi \in H_1^3 \mapsto \mathcal{M}'(\psi)(\xi) := \varepsilon A\xi + \bar{G}'(\psi)(\xi) \in H_*^{-1}$, with

$$\begin{aligned} \mathcal{M}'(\psi)(\xi) &= \\ &= -\frac{2}{\varepsilon} \left[F''(\psi + m_0)\Delta\xi + F'''(\psi + m_0)(\Delta\psi\xi + \nabla\psi \cdot \nabla\xi) + \frac{1}{2}F^{iv}(\psi + m_0)|\nabla\psi|^2\xi \right] \\ &+ \frac{1}{\varepsilon^3} [F''(\psi + m_0)^2 + F'''(\psi + m_0)F'(\psi + m_0)]\xi \\ &+ M \left(-\varepsilon\Delta\psi + \frac{1}{\varepsilon}F'(\psi + m_0) \right) \int_{\Omega} \left(\varepsilon\nabla\psi \cdot \nabla\xi + \frac{1}{\varepsilon}F'(\psi + m_0)\xi \right) dx \\ &+ M(\mathcal{A}(\psi + m_0) - \alpha) \left(-\varepsilon\Delta\xi + \frac{1}{\varepsilon}F''(\psi + m_0)\xi \right). \end{aligned} \quad (29)$$

Note that $\mathcal{M}'(\psi)$ is a Fredholm operator of index zero, because $\mathcal{M}'(\psi)$ is the sum of the invertible operator εA and the compact operator $\overline{\mathcal{G}}'(\psi) : H_1^3 \mapsto H_*^{-1}$. Indeed, by using Lemma 4, if $\|\xi\|_{H_1^3} \leq C_1$, then $|\overline{\mathcal{G}}'(\psi)(\xi)|_2 \leq C_2$.

On the other hand, the map $\mathcal{R} : \psi \in H_1^3 \mapsto \mathcal{M}'(\psi)A^{-1} \in \mathcal{L}(H_*^{-1})$ is well-posed because $A^{-1} \in \mathcal{L}(H_*^{-1}; H_1^3)$ and $\mathcal{M}'(\psi) \in \mathcal{L}(H_1^3; H_*^{-1})$. It remains to prove that \mathcal{R} is (sequentially) continuous. Indeed, let $\psi_n \rightarrow \psi$ in H_1^3 as $n \rightarrow \infty$. Then,

$$\begin{aligned} \|\mathcal{R}(\psi_n) - \mathcal{R}(\psi)\|_{\mathcal{L}(H_*^{-1})} &= \|\mathcal{M}'(\psi_n)A^{-1} - \mathcal{M}'(\psi)A^{-1}\|_{\mathcal{L}(H_*^{-1})} \\ &\leq \|\mathcal{M}'(\psi_n) - \mathcal{M}'(\psi)\|_{\mathcal{L}(H_1^3; H_*^{-1})} \|A^{-1}\|_{\mathcal{L}(H_*^{-1}; H_1^3)} \end{aligned}$$

In order to bound $\|\mathcal{M}'(\psi_n) - \mathcal{M}'(\psi)\|_{\mathcal{L}(H_1^3; H_*^{-1})} = \|\overline{\mathcal{G}}'(\psi_n) - \overline{\mathcal{G}}'(\psi)\|_{\mathcal{L}(H_1^3; H_*^{-1})}$, by using (29) we observe that

$$\begin{aligned} &G'(\psi_n)(\xi) - G'(\psi)(\xi) \\ &= -\frac{2}{\varepsilon}(F''(\psi_n + m_0) - F''(\psi + m_0))\Delta\xi - \frac{1}{\varepsilon}F^{iv}(\psi_n + m_0)(\nabla\psi_n + \nabla\psi) \cdot (\nabla\psi_n - \nabla\psi)\xi \\ &+ \frac{1}{\varepsilon^3}(F''(\psi_n + m_0) + F''(\psi + m_0))(F''(\psi_n + m_0) - F''(\psi + m_0))\xi \\ &- \frac{2}{\varepsilon}\left[F'''(\psi_n + m_0)\Delta(\psi_n - \psi) + (F'''(\psi_n + m_0) - F'''(\psi + m_0))\Delta\psi\right]\xi \\ &- \frac{2}{\varepsilon}\left[F'''(\psi_n + m_0)\nabla(\psi_n - \psi) + (F'''(\psi_n + m_0) - F'''(\psi + m_0))\nabla\psi\right] \cdot \nabla\xi \\ &+ \frac{1}{\varepsilon^3}(F'''(\psi_n + m_0) - F'''(\psi + m_0))F'(\psi_n + m_0)\xi \\ &+ \frac{1}{\varepsilon^3}F'''(\psi + m_0)(F'(\psi_n + m_0) - F'(\psi + m_0))\xi \\ &+ M\left(-\varepsilon\Delta\psi_n + \frac{1}{\varepsilon}F'(\psi + m_0)\right) \int_{\Omega} \left(\nabla(\psi_n - \psi) \cdot \nabla\xi + \frac{1}{\varepsilon}(F'(\psi_n + m_0) - F'(\psi + m_0))\right) dx \\ &+ \left(-\varepsilon\Delta(\psi_n - \psi) + \frac{1}{\varepsilon}(F'(\psi_n + m_0) - F'(\psi + m_0))\right) \int_{\Omega} \left(\varepsilon\nabla\psi \cdot \nabla\xi + \frac{1}{\varepsilon}F'(\psi + m_0)\xi\right) dx \\ &+ M(\mathcal{A}(\psi_n + m_0) - \mathcal{A}(\psi + m_0))\left(-\varepsilon\Delta\xi + \frac{1}{\varepsilon}F''(\psi + m_0)\xi\right) \\ &+ M(\mathcal{A}(\psi + m_0) - \alpha)\frac{1}{\varepsilon}(F''(\psi_n + m_0) - F''(\psi + m_0))\xi. \end{aligned}$$

Then, by using Lemma 4, the H_*^{-1} norm of each term on the right side of the previous expression can be bounded by $C\|\psi_n - \psi\|_{L^2}$, $C\|\psi_n - \psi\|_{H^1}$ or $C\|\psi_n - \psi\|_{H^2}$ multiplied by $\|\xi\|_3$. In fact, let us precise the bound of some of these terms:

$$\begin{aligned} \|(F'''(\psi_n + m_0) - F'''(\psi + m_0))\Delta\psi \xi\|_{H_*^{-1}} &\leq C|F'''(\psi_n + m_0) - F'''(\psi + m_0)|_3 |\Delta\psi|_2 |\xi|_{\infty} \\ &\leq C(K)|\psi_n - \psi|_3 |\xi|_{\infty} \leq C(K)\|\psi_n - \psi\|_1 \|\xi\|_2, \end{aligned}$$

$$\begin{aligned} &\|F^{iv}(\psi_n + m_0)(\nabla\psi_n + \nabla\psi) \cdot (\nabla\psi_n - \nabla\psi) \xi\|_{H_*^{-1}} \\ &\leq C|F^{iv}(\psi_n + m_0)|_{\infty} |\nabla\psi_n + \nabla\psi|_3 |\nabla\psi_n - \nabla\psi|_2 |\xi|_{\infty} \leq C(K)\|\psi_n - \psi\|_1 \|\xi\|_2, \end{aligned}$$

$$\|F'''(\psi_n + m_0)\Delta(\psi_n - \psi)\xi\|_{H_*^{-1}} \leq |F'''(\psi_n + m_0)|_3 |\Delta(\psi_n - \psi)|_2 |\xi|_\infty \leq C(K) \|\psi_n - \psi\|_2 \|\xi\|_2,$$

$$\begin{aligned} \|(\mathcal{A}(\psi_n + m_0) - \mathcal{A}(\psi + m_0))\Delta\xi\|_{H_*^{-1}} &\leq C |\mathcal{A}(\psi_n + m_0) - \mathcal{A}(\psi + m_0)| |\Delta\xi|_2 \\ &\leq C(K) \|\psi_n - \psi\|_1 \|\xi\|_2, \end{aligned}$$

Thus,

$$\begin{aligned} \|\mathcal{M}'(\psi_n) - \mathcal{M}'(\psi)\|_{\mathcal{L}(H_1^3; H_*^{-1})} &= \sup_{\xi \in H_1^3 \setminus \{0\}} \frac{\|\mathcal{M}'(\psi_n)(\xi) - \mathcal{M}'(\psi)(\xi)\|_{H_*^{-1}}}{\|\xi\|_{H_1^3}} \\ &= \sup_{\xi \in H_1^3 \setminus \{0\}} \frac{\|\overline{G}'(\psi_n)(\xi) - \overline{G}'(\psi)(\xi)\|_{H_*^{-1}}}{\|\xi\|_3} \leq C \|\psi_n - \psi\|_{H^2}. \end{aligned}$$

Therefore, $\|\mathcal{M}'(\psi_n) - \mathcal{M}'(\psi)\|_{\mathcal{L}(H_1^3; H_*^{-1})} \rightarrow 0$ as $n \rightarrow \infty$ if $\psi_n \rightarrow \psi$ in H^3 (even if $\psi_n \rightarrow \psi$ in H^2), hence the continuity of the operator \mathcal{R} is proved.

Finally, by applying Theorem 9, there exists $\beta > 0$, $C > 0$ and $\theta \in (0, 1/2)$ such that for any $\psi \in H_1^3(\Omega)$ with $\|\psi - \psi_*\|_3 \leq \beta$, one has

$$|\overline{\mathcal{E}}_b(\psi) - \overline{\mathcal{E}}_b(\psi_*)|^{1-\theta} \leq C \|\mathcal{E}'(\psi)\|_{H_*^{-1}} = C \|z(\psi)\|_{H_*^{-1}}$$

and (28) holds.

Step 2: (Relaxing the local approximation $\|\psi - \psi_*\|_3 \leq \beta$ by $\|\psi\|_2 \leq K$, $\|\psi - \psi_*\|_1 \leq \beta_1$ and $|\overline{\mathcal{E}}_b(\psi) - \overline{\mathcal{E}}_b(\psi_*)| \leq \beta_2$). If $\psi \in H_1^3(\Omega)$ with $\|\psi\|_2 \leq K$, $\|\psi - \psi_*\|_1 \leq \beta_1$ and $|\overline{\mathcal{E}}_b(\psi) - \overline{\mathcal{E}}_b(\psi_*)| \leq \beta_2$, then there exists $C > 0$ and $\theta \in (0, 1/2]$ (depending on ψ_* , K , β_1 and β_2) such that (28) holds.

In this step, we follow Lemma 4.4 of [19] but imposing now the ‘‘proximity’’ condition between ψ and ψ_* only in the H^1 -norm instead of in the H^2 -norm as in [19].

Since $z(\psi_*) = 0$,

$$\|z(\psi)\|_{H_*^{-1}} = \|z(\psi) - z(\psi_*)\|_{H_*^{-1}} = \|\varepsilon A(\psi - \psi_*) + \overline{G}(\psi) - \overline{G}(\psi_*)\|_{H_*^{-1}}.$$

From (25), $\|\varepsilon A(\psi - \psi_*)\|_{H_*^{-1}} = |\varepsilon \nabla \Delta(\psi - \psi_*)|_2$ and from (23), there is a constant $M > 0$, such that $\|\psi - \psi_*\|_3 \leq M |\nabla \Delta(\psi - \psi_*)|_2$, hence,

$$\begin{aligned} \|z(\psi)\|_{H_*^{-1}} &\geq \varepsilon |\nabla \Delta(\psi - \psi_*)|_2 - \|\overline{G}(\psi) - \overline{G}(\psi_*)\|_{H_*^{-1}} \\ &\geq \frac{\varepsilon}{M} \|\psi - \psi_*\|_3 - C |\overline{G}(\psi) - \overline{G}(\psi_*)|_2. \end{aligned} \tag{30}$$

On the other hand, since $\|\psi\|_2 \leq K$, we can use the Lemma 5 to bound $|\overline{G}(\psi) - \overline{G}(\psi_*)|_2$ as

$$\begin{aligned} |\overline{G}(\psi) - \overline{G}(\psi_*)|_2 &\leq |G(\psi + m_0) - G(\psi_* + m_0)|_2 + |\langle G(\psi + m_0) \rangle - \langle G(\psi_* + m_0) \rangle|_2 \\ &\leq C(K) \|\psi - \psi_*\|_2. \end{aligned}$$

In particular, interpolating H_1^2 between H_*^1 and H_1^3 , we obtain that

$$C|\overline{G}(\psi) - \overline{G}(\psi_*)|_2 \leq C_1(K) \|\psi - \psi_*\|_1^{1/2} \|\psi - \psi_*\|_3^{1/2} \leq \frac{\varepsilon}{2M} \|\psi - \psi_*\|_3 + \frac{M}{2\varepsilon} C_1(K)^2 \|\psi - \psi_*\|_1.$$

Let $\beta > 0$, $\theta \in (0, 1/2)$ given in Step 1, by choosing $\beta_1 > 0$ and $\beta_2 > 0$, both sufficiently small, such that

$$\frac{M}{2\varepsilon} C_1(K)^2 \beta_1 \leq \frac{\varepsilon\beta}{4M} \quad \text{and} \quad \beta_2^{1-\theta} \leq \frac{\varepsilon\beta}{4M},$$

then, for any $\psi \in H_1^3(\Omega)$ satisfying $\|\psi\|_2 \leq K$, $\|\psi - \psi_*\|_1 \leq \beta_1$ and $|\overline{\mathcal{E}}_b(\psi) - \overline{\mathcal{E}}_b(\psi_*)| \leq \beta_2$, we have

$$C|\overline{G}(\psi) - \overline{G}(\psi_*)|_2 \leq \frac{\varepsilon}{2M} \|\psi - \psi_*\|_3 + \frac{\varepsilon\beta}{4M} \quad (31)$$

and

$$|\overline{\mathcal{E}}_b(\psi) - \overline{\mathcal{E}}_b(\psi_*)|^{1-\theta} \leq \frac{\varepsilon\beta}{4M}. \quad (32)$$

There are only two possibilities: either $\|\psi - \psi_*\|_3 \leq \beta$ and then (28) holds by using Step 1, or $\|\psi - \psi_*\|_3 > \beta$. In this latter case, from (30) and (31)

$$\|z(\psi)\|_{H_*^{-1}} \geq \frac{\varepsilon}{2M} \|\psi - \psi_*\|_3 - \frac{\varepsilon\beta}{4M} > \frac{\varepsilon\beta}{2M} - \frac{\varepsilon\beta}{4M} = \frac{\varepsilon\beta}{4M}.$$

On the other hand, from (32),

$$\frac{\varepsilon\beta}{4M} \geq |\overline{\mathcal{E}}_b(\psi) - \overline{\mathcal{E}}_b(\psi_*)|^{1-\theta}$$

hence (28) holds. ■

5 Weak Solutions

We define the *total free-energy* as $\overline{\mathcal{E}}(\mathbf{u}, \psi) = \mathcal{E}_k(\mathbf{u}) + \lambda \overline{\mathcal{E}}_b(\psi)$, where $\mathcal{E}_k(\mathbf{u}) = \frac{1}{2} \int_{\Omega} |\mathbf{u}|^2$ is the kinetic energy, and $\overline{\mathcal{E}}_b(\psi)$ is the bending energy defined in (20). That is,

$$\overline{\mathcal{E}}(\mathbf{u}, \psi) = \frac{1}{2} \int_{\Omega} |\mathbf{u}|^2 + \frac{\lambda}{2\varepsilon} \int_{\Omega} (-\varepsilon \Delta \psi + \frac{1}{\varepsilon} F'(\psi + m_0))^2 d\mathbf{x} + \frac{M}{2} (\mathcal{A}(\psi + m_0) - \alpha)^2. \quad (33)$$

Definition 11 Let $\mathbf{u}_0 \in \mathbf{H}$ and $\psi_0 = \phi_0 - m_0 \in H_1^2$, we say that (\mathbf{u}, ψ, z) is a *global weak solution* of (13)-(19) in $(0, +\infty)$, if

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, +\infty; \mathbf{H}) \cap L^2(0, +\infty; \mathbf{V}), \\ \psi &\in L^\infty(0, +\infty; H_1^2) \cap L_{\text{loc}}^2([0, +\infty); H_1^3), \quad z \in L^2(0, +\infty; H_*^1), \end{aligned} \quad (34)$$

satisfying the variational formulation a.e. $t \in (0, +\infty)$:

$$\langle \partial_t \mathbf{u}, \bar{\mathbf{u}} \rangle + ((\mathbf{u} \cdot \nabla) \mathbf{u}, \bar{\mathbf{u}}) + \nu(\nabla \mathbf{u}, \nabla \bar{\mathbf{u}}) - \lambda(z \nabla \psi, \bar{\mathbf{u}}) = 0, \quad \forall \bar{\mathbf{u}} \in \mathbf{V} \quad (35)$$

$$\langle \partial_t \psi, \bar{z} \rangle + (\mathbf{u} \cdot \nabla \psi, \bar{z}) + \gamma(\nabla z, \nabla \bar{z}) = 0, \quad \forall \bar{z} \in H_*^1 \quad (36)$$

$$-\varepsilon(\nabla \Delta \psi, \nabla \bar{\psi}) + (\bar{G}(\psi), \bar{\psi}) - (z, \bar{\psi}) = 0, \quad \forall \bar{\psi} \in H_*^1 \quad (37)$$

the initial conditions (19), and the energy inequality (in integral version)

$$\bar{\mathcal{E}}(\mathbf{u}(t_1), \psi(t_1)) - \bar{\mathcal{E}}(\mathbf{u}(t_0), \psi(t_0)) + \int_{t_0}^{t_1} (\nu |\nabla \mathbf{u}(s)|_2^2 + \lambda \gamma |\nabla z(s)|_2^2) ds \leq 0, \quad (38)$$

for a.e. $t_1, t_0 : t_1 \geq t_0 \geq 0$.

Energy inequality (38) shows the dissipative character of the model with respect to the total free-energy $\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t))$.

Observe that, using the operator A defined in (24), then (37) can be rewritten as

$$z(t) = \varepsilon A \psi(t) + \bar{G}(\psi(t)). \quad (39)$$

On the other hand, from (34), (35) and (36), one can deduce that

$$\partial_t \mathbf{u} \in L_{\text{loc}}^{4/3}([0, +\infty); \mathbf{V}') \quad \text{and} \quad \partial_t \psi \in L_{\text{loc}}^2([0, +\infty); H_*^{-1}),$$

hence, the following time-continuity can be deduced (see Lemma 1.4 of [21]),

$$\mathbf{u} \in C([0, +\infty); \mathbf{V}') \cap C_w([0, +\infty); \mathbf{H}),$$

$$\psi \in C([0, +\infty); H_*^{-1}) \cap C_w([0, +\infty); H_1^2).$$

Here, the weak continuity $\mathbf{u} \in C_w([0, +\infty); \mathbf{H})$ means that $\mathbf{u}(t) \in \mathbf{H}$ for all $t \in [0, +\infty)$ and $\mathbf{u}(s) \rightarrow \mathbf{u}(t)$ weak in \mathbf{H} as $s \rightarrow t$. In particular, the initial conditions (19) have a sense.

The following lemma gives two improved energy inequalities.

Lemma 12 *Given (\mathbf{u}, ψ, z) a global weak solution of (13)-(19) in $(0, +\infty)$, there exists a regularized energy function $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}(t) \in \mathbb{R}$ defined for all $t \geq 0$, which satisfies the following integral inequality for all t_1, t_0 with $t_1 \geq t_0 \geq 0$:*

$$\tilde{\mathcal{E}}(t_1) - \tilde{\mathcal{E}}(t_0) + \int_{t_0}^{t_1} (\nu |\nabla \mathbf{u}(s)|_2^2 + \lambda \gamma |\nabla z(s)|_2^2) ds \leq 0, \quad (40)$$

and the following differential inequality for a.e. $t \geq 0$:

$$\frac{d}{dt} \tilde{\mathcal{E}}(t) + \nu |\nabla \mathbf{u}(t)|_2^2 + \lambda \gamma |\nabla z(t)|_2^2 \leq 0, \quad \text{a.e. } t \geq 0. \quad (41)$$

Proof. From the weak continuity $\mathbf{u} \in C_w([0, +\infty); \mathbf{H})$ and $\psi \in C_w([0, +\infty); H_1^2)$, in particular the energy evaluated in the trajectory $\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t))$ exits for all t . Moreover, since the inequality (38) is satisfied for all $t_0, t_1 \in [0, +\infty) \setminus N$, where N is a set of null Lebesgue measure, then $\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t))$ is a real decreasing function in $[0, +\infty) \setminus N$, and $\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t)) \in L^\infty(0, +\infty)$. Therefore, we can define a regularized function $\tilde{\mathcal{E}}$ related to $\bar{\mathcal{E}}$ for all $t \geq 0$ as:

$$\tilde{\mathcal{E}}(0) := \bar{\mathcal{E}}(u_0, \psi_0), \quad \tilde{\mathcal{E}}(t) := \lim_{\substack{s \rightarrow t^- \\ s \in [0, +\infty) \setminus N}} \bar{\mathcal{E}}(\mathbf{u}(s), \psi(s)).$$

Then, this function $\tilde{\mathcal{E}}(t)$ is “continuous from the left” and is decreasing for all $t \geq 0$. Indeed, for any $t_1, t_2 \in [0, +\infty)$, for instance $t_1 < t_2$, we can choose sequences $\{s_n^1\}, \{s_n^2\} \subset [0, +\infty) \setminus N$ such that $s_n^1 \rightarrow t_1^-$, $s_n^2 \rightarrow t_2^-$ and, $s_n^1 \leq s_n^2$ for all $n \geq n_0$. Since s_n^1 and s_n^2 are not in N , we know that $\bar{\mathcal{E}}(\mathbf{u}(s_n^1), \psi(s_n^1)) \geq \bar{\mathcal{E}}(\mathbf{u}(s_n^2), \psi(s_n^2))$. By taking limit as $s_n^1 \rightarrow t_1^-$ and $s_n^2 \rightarrow t_2^-$, we obtain that $\tilde{\mathcal{E}}(t_1) \geq \tilde{\mathcal{E}}(t_2)$.

Since $\tilde{\mathcal{E}}(t)$ is decreasing for all $t \in [0, +\infty)$, it is derivable (and continuous) almost everywhere $t \in (0, +\infty)$ (see for example Theorem 3 of [18] p. 100).

On the other hand, since the inequality (38) is satisfied for all $t_0, t_1 \in [0, +\infty) \setminus N$, given any $t_0 \leq t_1$, we can take $\delta_n > 0$ and $\eta_n > 0$ such that $t_0 - \delta_n, t_1 - \eta_n \notin N$ and $\delta_n \rightarrow 0$ and $\eta_n \rightarrow 0$. Then

$$\bar{\mathcal{E}}(\mathbf{u}(t_1 - \delta_n), \psi(t_1 - \delta_n)) - \bar{\mathcal{E}}(\mathbf{u}(t_0 - \delta_n), \psi(t_0 - \delta_n)) + \int_{t_0 - \delta_n}^{t_1 - \eta_n} (\nu |\nabla \mathbf{u}(s)|_2^2 + \lambda \gamma |\nabla z(s)|_2^2) ds \leq 0.$$

By taking $\delta_n \rightarrow 0$ and $\eta_n \rightarrow 0$, and using the definition of $\tilde{\mathcal{E}}(t)$, we obtain (40) for any $t_0 \leq t_1$.

In particular, by choosing $t_0 = t$ and $t_1 = t + h$ in (40), we obtain

$$\frac{\tilde{\mathcal{E}}(t+h) - \tilde{\mathcal{E}}(t)}{h} + \frac{1}{h} \int_t^{t+h} (\nu |\nabla \mathbf{u}|_2^2 + \lambda \gamma |\nabla z|_2^2) ds \leq 0, \quad \forall t, h \geq 0. \quad (42)$$

Observe that, since the map $t \in [0, +\infty) \rightarrow \nu |\nabla \mathbf{u}(t)|_2^2 + \lambda \gamma |\nabla z(t)|_2^2 \in \mathbb{R}$ belongs to $L^1(0, +\infty)$, then

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} (\nu |\nabla \mathbf{u}|_2^2 + \lambda \gamma |\nabla z|_2^2) ds = \nu |\nabla \mathbf{u}(t)|_2^2 + \lambda \gamma |\nabla z(t)|_2^2 \quad \text{a.e. } t \geq 0.$$

By taking $h \rightarrow 0$ in (42), we obtain (41) a.e. $t \geq 0$. ■

Remark 13 *The fact that the energy is decreasing is essential in Lemma 12. For general problems with nonzero external forces, the thesis of this Lemma is not clear.*

5.1 Formal energy equality and large-time weak regularity

We will arrive at the energy equality related to (41) in a formal manner (assuming a sufficiently regular solution). A rigorous demonstration of existence of weak solutions (in particular, satisfying (38)) will be obtained in Section 5.2 via a Galerkin method, in the same way as was done in [8, pp. 71-73] for the Navier-Stokes problem. By assuming that (\mathbf{u}, ψ, z) is a sufficiently regular solution of (13)-(19), and taking $\bar{\mathbf{u}} = \mathbf{u}$, $\bar{z} = z$ and $\bar{\psi} = \partial_t \psi$ as test function in (35), (36) and (37) respectively, one arrives at the equalities:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\mathbf{u}|_2^2 + \nu |\nabla \mathbf{u}|_2^2 - \lambda (z \nabla \psi, \mathbf{u}) &= 0, \\ (\partial_t \psi, z) + (\mathbf{u} \cdot \nabla \psi, z) + \gamma |\nabla z|_2^2 &= 0, \\ \varepsilon \frac{d}{dt} \frac{1}{2} |\Delta \psi|_2^2 + (\bar{G}(\psi), \partial_t \psi) - (z, \partial_t \psi) &= 0. \end{aligned}$$

Adding the first equality plus the second and third ones multiplied by λ , the term $(z, \partial_t \psi)$ cancels, as well as the nonlinear convective term $(\mathbf{u} \cdot \nabla \psi, z)$ with the elastic term $-(z \nabla \psi, \mathbf{u})$, arriving at

$$\frac{1}{2} \frac{d}{dt} (|\mathbf{u}|_2^2 + \lambda \varepsilon |\Delta \psi|_2^2) + \lambda (\bar{G}(\psi), \partial_t \psi) + \nu |\nabla \mathbf{u}(t)|_2^2 + \lambda \gamma |\nabla z(t)|_2^2 = 0. \quad (43)$$

Since $\frac{\delta \bar{\mathcal{E}}_b(\psi)}{\delta \psi} = z = \varepsilon \Delta^2 \psi + \bar{G}(\psi)$, in particular

$$\frac{d}{dt} \bar{\mathcal{E}}_b(\psi(t)) = \left\langle \frac{\delta \bar{\mathcal{E}}_b(\psi)}{\delta \psi}, \partial_t \psi \right\rangle = (z, \partial_t \psi) = \varepsilon \frac{1}{2} \frac{d}{dt} |\Delta \psi|_2^2 + (\bar{G}(\psi), \partial_t \psi).$$

Then, equality (43) can be rewritten as the following *energy equality* (which is the equality version of (41)):

$$\frac{d}{dt} \bar{\mathcal{E}}(\mathbf{u}(t), \psi(t)) + \nu |\nabla \mathbf{u}(t)|_2^2 + \lambda \gamma |\nabla z(t)|_2^2 = 0, \quad (44)$$

From (44), assuming the initial regularity (\mathbf{u}_0, ψ_0) in $\mathbf{H} \times H_1^2$ hence $\bar{\mathcal{E}}(\mathbf{u}(0), \psi(0))$ is finite, we obtain that $\mathbf{u} \in L^\infty(0, +\infty; \mathbf{H}) \cap L^2(0, +\infty; \mathbf{V})$, $z \in L^2(0, +\infty; H_*^1)$, and $\bar{\mathcal{E}}_b(\psi(t)) \in L^\infty(0, +\infty)$. In particular,

$$\int_{\Omega} (-\varepsilon \Delta \psi + \frac{1}{\varepsilon} F'(\psi + m_0))^2 d\mathbf{x} \quad \text{and} \quad \mathcal{A}(\psi) = \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla \psi|^2 + \frac{1}{\varepsilon} F(\psi + m_0) \right) d\mathbf{x}$$

belong to $L^\infty(0, +\infty)$.

From $\mathcal{A}(\psi) \in L^\infty(0, +\infty)$ one has that $\nabla \psi \in L^\infty(0, +\infty; \mathbf{L}^2)$ and, since $\int_{\Omega} \psi = 0$, also $\psi \in L^\infty(0, +\infty; H_*^1)$. Secondly, from $\int_{\Omega} (-\varepsilon \Delta \psi + \frac{1}{\varepsilon} F'(\psi + m_0))^2 \in L^\infty(0, +\infty)$, we have that $\psi \in L^\infty(0, +\infty; H_1^2)$.

In summary, the following global in time “weak” regularity (in the time interval $(0, +\infty)$) hold:

$$\mathbf{u} \in L^\infty(0, +\infty; \mathbf{H}) \cap L^2(0, +\infty; \mathbf{V}), \quad \psi \in L^\infty(0, +\infty; H_1^2), \quad z \in L^2(0, +\infty; H_*^1). \quad (45)$$

By applying Remark 6 to $\overline{G}(\psi(t))$ defined in (17), since $\|\psi(t) + m_0\|_2 \leq K$, we obtain

$$|\overline{G}(\psi(t))|_2 \leq |G(\psi(t) + m_0)|_2 + |\langle G(\psi(t) + m_0) \rangle|_2 \leq C. \quad (46)$$

From (23), (25) and (39), we obtain

$$\|\psi\|_3 \leq C |\nabla \Delta \psi|_2 = C \|A\psi\|_{H_*^{-1}} \leq C(|z|_2 + |\overline{G}(\psi)|_2) \leq C(1 + |z|_2) \quad (47)$$

From this estimate and the regularity of z given in (45), we have

$$\psi \in L_{loc}^2([0, +\infty); H_1^3). \quad (48)$$

5.2 Existence of weak solutions satisfying energy inequality (38)

For instance, fixed the initial datum $(\mathbf{u}_0, \psi_0) \in \mathbf{H} \times H_1^2$, the existence of weak solutions of (13)-(19) in $(0, +\infty)$, furnished by a limit of an adequate Galerkin approximate solutions can be proved analogously to [3] or [4]. Moreover, these Galerkin solutions satisfy the corresponding energy equality (44) which suffices to prove rigorously the previous estimates (45) and (48). Finally, following similar arguments to those used in [8] for the Navier-Stokes equations, these weak solutions will satisfy the energy inequality (38).

Indeed, let $\{\mathbf{w}_i\}_n \geq 1$ and $\{\phi_i\}_n \geq 1$ “special” basis of \mathbf{V} and H_1^2 , respectively, formed by eigenfunctions of the Stokes problem

$$\mathbf{w}_i \in \mathbf{V} \quad \text{such that} \quad S\mathbf{w}_i = \lambda_i \mathbf{w}_i, \quad \text{with} \quad \|\mathbf{w}_i\|_{L^2} = 1, \quad \lambda_i \nearrow +\infty$$

and of the Poisson-Neumann problem

$$\phi_i \in H_1^2 \quad \text{such that} \quad -\Delta \phi_i = \mu_i \phi_i, \quad \text{with} \quad \|\phi_i\|_{L^2} = 1, \quad \mu_i \nearrow +\infty.$$

Here, we consider the Stokes operator S defined as $S\mathbf{w} \in \mathbf{V}$ such that $(\nabla(S\mathbf{w}), \nabla \mathbf{v}) = (\mathbf{w}, \mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}$.

Let \mathbf{V}^m and W^m be the finite-dimensional subspaces spanned by $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ and $\{\phi_1, \phi_2, \dots, \phi_n\}$ respectively. Note that if $\phi \in W^m$ then also $\Delta \phi \in W^m$ and $\Delta^2 \phi \in W^m$.

For each $m \geq 1$, we say that (\mathbf{u}_m, ψ_m) is a Galerkin solution, if $\mathbf{u}_m : [0, +\infty) \mapsto \mathbf{V}^m$ and $\psi_m : [0, +\infty) \mapsto W^m$, and satisfy

$$\left\{ \begin{array}{l} \langle \partial_t \mathbf{u}_m, \overline{\mathbf{u}}_m \rangle + ((\mathbf{u}_m \cdot \nabla) \mathbf{u}_m, \overline{\mathbf{u}}_m) + \nu (S\mathbf{u}_m, \overline{\mathbf{u}}_m) \\ \quad - \lambda (z_m \nabla \psi_m, \overline{\mathbf{u}}_m) = 0, \quad \forall \overline{\mathbf{u}}_m \in \mathbf{V}^m, \quad \text{a.e. } t > 0, \\ \langle \partial_t \psi_m, \overline{z}_m \rangle + (\mathbf{u}_m \cdot \nabla \psi_m, \overline{z}_m) + \gamma (\nabla z_m, \nabla \overline{z}_m) = 0, \quad \forall \overline{z}_m \in W^m, \quad \forall t > 0, \\ \mathbf{u}_m(0) = \mathbf{u}_{0m} := P_m(\mathbf{u}_0), \quad \psi_m(0) = \psi_{0m} = Q_m(\psi_0) \quad \text{in } \Omega, \end{array} \right. \quad (49)$$

where $z_m : [0, +\infty) \mapsto W^m$ is defined as $z_m := Q_m(\varepsilon \Delta^2 \psi_m + \overline{G}(\psi_m))$. Here, $P_m : L^2 \mapsto \mathbf{V}^m$ denotes the projection from L^2 onto \mathbf{V}^m and $Q_m : L^2 \mapsto W^m$ the projection from L^2 onto W^m . Since $\Delta^2 \psi_m \in W^m$ then

$$z_m = \varepsilon \Delta^2 \psi_m + Q_m(\overline{G}(\psi_m)) = \varepsilon A \psi_m + Q_m(\overline{G}(\psi_m)), \quad \text{in } \Omega \times (0, +\infty). \quad (50)$$

Since $(\partial_t \mathbf{u}_m, \partial_t \psi_m) \in \mathbf{V}^m \times W^m$ and $(S \mathbf{u}_m, -\Delta z_m) \in \mathbf{V}^m \times W^m$, the variational formulation (49) yields to the point-wise equalities

$$\partial_t \mathbf{u}_m + \nu S \mathbf{u}_m + P_m((\mathbf{u}_m \cdot \nabla) \mathbf{u}_m - \lambda z_m \nabla \psi_m) = 0, \quad \text{in } \Omega \times (0, +\infty). \quad (51)$$

$$\partial_t \psi_m - \gamma \Delta z_m + Q_m(\mathbf{u}_m \cdot \nabla \psi_m) = 0, \quad \text{in } \Omega \times (0, +\infty). \quad (52)$$

The existence and uniqueness of local in time solution of (49)-(50) in $\Omega \times (0, T)$, for any $T > 0$ sufficiently small, can be proved using the Leray-Schauder's Theorem in the same way as the Theorem 5.2 of [3]. By following the argument given in Subsection 5.1, the energy equality

$$\overline{\mathcal{E}}(\mathbf{u}_m(t), \psi_m(t)) - \overline{\mathcal{E}}(\mathbf{u}_m(t_0), \psi_m(t_0)) + \int_{t_0}^t (\nu |\nabla \mathbf{u}_m|_2^2 + \lambda \gamma |\nabla z_m|_2^2) d\tau = 0 \quad (53)$$

holds for all $t, t_0 : t \geq t_0 \geq 0$. Therefore, the solution of (49)-(50) can be extended to the whole time interval $(0, +\infty)$, and the following uniform in time estimates (independent of m) can be obtained as in Subsection 5.1:

$$\mathbf{u}_m \text{ in } L^\infty(0, +\infty; \mathbf{H}) \cap L^2(0, +\infty; \mathbf{V}), \quad \psi_m \text{ in } L^\infty(0, +\infty; H_1^2), \quad z_m \text{ in } L^2(0, +\infty; H_*^1).$$

By applying these estimates in (51)-(52) one has the estimates for the time derivatives

$$\partial_t \mathbf{u}_m \text{ in } L_{loc}^{4/3}([0, +\infty); \mathbf{V}') \quad \text{and} \quad \partial_t \psi_m \text{ in } L_{loc}^2([0, +\infty); H_*^{-1}).$$

By following the argument given in Subsection 5.1 to prove (46) and (48), one has the estimates

$$\overline{G}(\psi_m) \text{ in } L^\infty((0, +\infty) \times \Omega) \quad \text{and} \quad \psi_m \text{ in } L_{loc}^2([0, +\infty); H_1^3).$$

By using compactness Lemma 3 for the spaces $\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}'$ and $H_1^2 \subset W_*^{1,p}(\Omega) \subset H_*^{-1}$ with $p < 6$, and considering the continuous embedding $W^{1,p}(\Omega) \subset C(\overline{\Omega})$ ($p > 3$), we have that \mathbf{u}_m is relatively compact in $L^2(0, T; \mathbf{H})$, ψ_m is relatively compact in $C([0, T] \times \overline{\Omega})$ and $\nabla \psi_m$ in $C([0, T]; L^p(\Omega))$, with $p < 6$ for any $T > 0$. In particular, from Lemma 4, $F(\psi_m)$ (and $F'(\psi_m)$, $F''(\psi_m)$) is relatively compact in $C([0, T] \times \overline{\Omega})$. Finally, by using again Lemma 3 for the spaces $H_1^3 \subset H_1^2 \subset H_*^{-1}$ then ψ_m is relatively compact in $L^2(0, T; H_1^2)$.

In order to prove the existence of a weak solution (\mathbf{u}, ψ, z) of the variational problem (35)-(37) a.e. $t \in (0, +\infty)$, obtained as the limit of a (subsequence) of $(\mathbf{u}_m, \psi_m, z_m)$, it suffices

to pass to the limit in (50)-(52), which will be possible due to the previous estimates and compactness. For this, given any $T > 0$ and $(\bar{\mathbf{u}}, \bar{w}) \in \mathbf{V} \times H_1^2$ we take $(\bar{\mathbf{u}}_m, \bar{w}_m) \in \mathbf{V}^m \times W^m$ with $(\bar{\mathbf{u}}_m, \bar{w}_m) \rightarrow (\bar{\mathbf{u}}, \bar{w})$ in $\mathbf{V} \times H_1^2$ strongly. We rewrite (50) as

$$(z_m, \bar{w}_m) = \varepsilon \langle A\psi_m, \bar{w}_m \rangle + (\bar{G}(\psi_m), \bar{w}_m), \quad \forall t \in [0, T]. \quad (54)$$

To prove the weak convergence

$$\bar{G}(\psi_m) \rightarrow \bar{G}(\psi) \quad \text{weakly in } L^2((0, T) \times \Omega), \quad (55)$$

we analyze for example, the term $F''(\psi_m + m_0)\Delta\psi_m$. Since $F''(\psi_m + m_0)$ converges to $F''(\psi + m_0)$ in $C([0, T] \times \bar{\Omega})$ and $\Delta\psi_m$ converges weakly to $\Delta\psi$ in $L^2((0, T) \times \Omega)$, hence $F''(\psi_m + m_0)\Delta\psi \rightarrow F''(\psi + m_0)\Delta\psi$ weakly in $L^2((0, T) \times \Omega)$. The proof for the rest of the terms of $\bar{G}(\psi_m)$ is easier, and then, one has the convergence of (54) towards the limit equation (37).

For example, the convergence of the term $(z_m \nabla \psi_m, \bar{\mathbf{u}}_m)$ of (49)₁ towards $(z \nabla \psi, \bar{\mathbf{u}})$ is deduced from the strong convergence of $\nabla \psi_m$ in $L^\infty(0, T; L^2(\Omega))$ and the weak convergence of z_m in $L^2(0, T; H_*^1)$. The convergence of the rest of terms of (49)₁ is rather standard in the Navier-Stokes setting, see [8], hence (35) holds.

Finally, the limit of (49)₂ is standard, arriving at (36).

In order to prove that the weak solutions furnished by previous Galerkin method satisfy the energy inequality (38), we follow similar arguments used in [8] for the Navier-Stokes equations.

Firstly, we are going to prove that

$$\bar{\mathcal{E}}(\mathbf{u}_m(t), \psi_m(t)) \rightarrow \bar{\mathcal{E}}(\mathbf{u}(t), \psi(t)) \quad \text{a.e. } t \geq 0. \quad (56)$$

In fact, to show the convergence of the most difficult term $\int_\Omega \mu(\psi_m + m_0)^2 d\mathbf{x}$ in (56), it suffices to prove that

$$\mu(\psi_m + m_0) \rightarrow \mu(\psi + m_0) \quad \text{strongly in } L^2(0, T; L^2), \quad \forall T > 0. \quad (57)$$

For this, we use that ψ_m is bounded in $L^2(0, T; H_1^3)$, and in particular $\Delta\psi_m$ is bounded in $L^2(0, T; H_*^1)$. Moreover, $\nabla F'(\psi_m + m_0) = F''(\psi_m + m_0)\nabla\psi_m$ is bounded in $L^\infty(0, T; L^2)$. Therefore,

$$\mu(\psi_m + m_0) \text{ is bounded in } L^2(0, T; H^1). \quad (58)$$

On the other hand, by using the parabolic equation (52),

$$\begin{aligned} \partial_t \mu(\psi_m + m_0) &= -\varepsilon \Delta \partial_t \psi_m + \frac{1}{\varepsilon} F''(\psi_m + m_0) \partial_t \psi_m \\ &= \varepsilon \Delta (Q_m(\mathbf{u}_m \cdot \nabla \psi_m)) - \gamma \varepsilon \Delta^2 z_m + \frac{1}{\varepsilon} F''(\psi_m + m_0) (\gamma \Delta z_m - Q_m(\mathbf{u}_m \cdot \nabla \psi_m)), \end{aligned}$$

Here, the second term at the right hand side has the following sense

$$-\langle \Delta^2 z_m, \tilde{\psi} \rangle_{(H_1^3)', H_1^3} = (\nabla z_m, \nabla \Delta \tilde{\psi})_{L^2}, \quad \forall \tilde{\psi} \in H_1^3.$$

Hence, the estimates already obtained for \mathbf{u}_m , ψ_m and z_m imply that

$$\partial_t \mu(\psi_m + m_0) \text{ is bounded in } L^2(0, T; (H_1^3)'), \quad (59)$$

where the dual space $(H_1^3)'$ appears due to the term $\Delta^2 z_m$.

Then, (58) and (59) and Lemma 3 (for the spaces $H_*^1 \subset L_*^2 \subset (H_1^3)'$) imply (57). On the other hand, since $\mathbf{u}_m \rightarrow \mathbf{u}$ strongly in $L^2(0, T; \mathbf{H})$ then (56) holds.

Secondly, since $\mathbf{u}_m \rightarrow \mathbf{u}$ and $z_m \rightarrow z$, both weakly in $L^2(0, T; H^1)$, we have that

$$\liminf_{m \rightarrow +\infty} \int_{t_0}^t (\nu |\nabla \mathbf{u}_m|_2^2 + \lambda \gamma |\nabla z_m|_2^2) d\tau \geq \int_{t_0}^t (\nu |\nabla \mathbf{u}|_2^2 + \lambda \gamma |\nabla z|_2^2) d\tau \quad \text{for all } t \geq t_0 \geq 0. \quad (60)$$

Thirdly, by taking \liminf in (53), we obtain that for all $t \geq t_0 \geq 0$,

$$\begin{aligned} \liminf_{m \rightarrow +\infty} \bar{\mathcal{E}}(\mathbf{u}_m(t), \psi_m(t)) + \liminf_{m \rightarrow +\infty} \int_{t_0}^t (\nu |\nabla \mathbf{u}_m|_2^2 + \lambda \gamma |\nabla z_m|_2^2) d\tau \\ \leq \limsup_{m \rightarrow +\infty} \bar{\mathcal{E}}(\mathbf{u}_m(t_0), \psi_m(t_0)). \end{aligned} \quad (61)$$

Finally, by using (56) and (60) in (61), we obtain the energy inequality (38) for a.e. $t, t_0 : t \geq t_0 \geq 0$.

6 Convergence at infinite time.

Let (\mathbf{u}, ψ, z) be a weak solution of (13)-(19) in $(0, +\infty)$ associated to an initial data $(\mathbf{u}_0, \psi_0) \in \mathbf{H} \times H_1^2$ (see Definition 11). From the energy inequality (38), there exists a real number $E_\infty \geq 0$ such that the total energy evaluated in the trajectory $(\mathbf{u}(t), \psi(t))$, satisfies

$$\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t)) \searrow E_\infty \text{ in } \mathbb{R} \quad \text{as } t \uparrow +\infty. \quad (62)$$

Let us define the ω -limit set of this global weak solution (\mathbf{u}, ψ) as follows:

$$\begin{aligned} \omega(\mathbf{u}, \psi) = \{(\mathbf{u}_\infty, \psi_\infty) \in \mathbf{H} \times H_1^2 : \exists \{t_n\} \uparrow +\infty \text{ s.t.} \\ (\mathbf{u}(t_n), \psi(t_n)) \rightarrow (\mathbf{u}_\infty, \psi_\infty) \text{ weakly in } \mathbf{L}^2 \times H_1^2\}. \end{aligned} \quad (63)$$

Let \mathcal{S} be the set of critical points of the bending energy defined in (27), that is

$$\mathcal{S} = \{\psi \in H_1^3 : -\varepsilon(\nabla \Delta \psi, \nabla \tilde{\psi}) + (\bar{\mathcal{G}}(\psi), \tilde{\psi}) = 0, \forall \tilde{\psi} \in H_*^1\}. \quad (64)$$

Theorem 14 *Assume that $(\mathbf{u}_0, \psi_0) \in \mathbf{H} \times H_1^2$. Fixed (\mathbf{u}, ψ, z) a weak solution of (13)-(19) in $(0, +\infty)$, then $\omega(\mathbf{u}, \psi)$ is nonempty and $\omega(\mathbf{u}, \psi) \subset \{0\} \times \mathcal{S}$. Moreover, for any $\psi_\infty \in \mathcal{S}$ such that $(0, \psi_\infty) \in \omega(\mathbf{u}, \psi)$, it holds $\bar{\mathcal{E}}_b(\psi_\infty) = E_\infty$. In particular, $\mathbf{u}(t) \rightarrow 0$ weakly in \mathbf{L}^2 and $\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t)) \rightarrow \bar{\mathcal{E}}_b(\psi_\infty)$ in \mathbb{R} as $t \uparrow +\infty$.*

Proof. Since $(\mathbf{u}, \psi) \in L^\infty(0, +\infty; \mathbf{H} \times H_1^2)$, there exists $\{t_n\} \uparrow +\infty$ and suitable limit functions $(\mathbf{u}_\infty, \psi_\infty) \in \mathbf{H} \times H_1^2$, such that

$$\mathbf{u}(t_n) \rightarrow \mathbf{u}_\infty \text{ weakly in } \mathbf{H}, \quad \psi(t_n) \rightarrow \psi_\infty \text{ weakly in } H_1^2. \quad (65)$$

We consider the initial and boundary-value problem associated to (13)-(19) restricted on the time interval $[t_n, t_n + 1]$ with initial values $\mathbf{u}(t_n)$ and $\psi(t_n)$. If we define

$$\mathbf{u}_n(s) := \mathbf{u}(s + t_n), \quad \psi_n(s) := \psi(s + t_n) \quad \text{and} \quad z_n(s) := z(s + t_n) \quad \text{for a.e. } s \in [0, 1],$$

then, $(\mathbf{u}_n, \psi_n, z_n)$ is a weak solution to the problem (13)-(19) in the time interval $[0, 1]$. From the energy inequality (38), we have that

$$\begin{aligned} \int_0^1 (\nu |\nabla \mathbf{u}_n(s)|_2^2 + \lambda \gamma |\nabla z_n(s)|_2^2) ds &= \int_{t_n}^{t_n+1} (\nu |\nabla \mathbf{u}(t)|_2^2 + \lambda \gamma |\nabla z(t)|_2^2) dt \\ &\leq \bar{\mathcal{E}}(\mathbf{u}(t_n), \psi(t_n)) - \bar{\mathcal{E}}(\mathbf{u}(t_n + 1), \psi(t_n + 1)) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

hence, $\nabla \mathbf{u}_n \rightarrow 0$ strongly in $L^2(0, 1; \mathbf{L}^2)$ and $\nabla z_n \rightarrow 0$ strongly in $L^2(0, 1; L^2)$. In particular, by using Poincaré inequality, one has

$$\mathbf{u}_n \rightarrow 0 \text{ strongly in } L^2(0, 1; \mathbf{V}) \quad \text{and} \quad z_n \rightarrow 0 \text{ strongly in } L^2(0, 1; H_*^1). \quad (66)$$

Moreover, since \mathbf{u}_n and $\partial_t \mathbf{u}_n$ are bounded in $L^\infty(0, 1; \mathbf{H})$ and $L^{4/3}(0, 1; \mathbf{V}')$ respectively, then using Lemma 3, $\mathbf{u}_n \rightarrow 0$ in $C([0, 1]; \mathbf{V}')$. In particular, $\mathbf{u}(t_n) = \mathbf{u}_n(0) \rightarrow 0$ in \mathbf{V}' , hence $\mathbf{u}_\infty = 0$ (owing to (65)). Consequently, the whole trajectory $\mathbf{u}(t) \rightarrow 0$ as $t \rightarrow +\infty$.

On the other hand, from the large time regularity of (\mathbf{u}, ψ, z) given in (34), \mathbf{u}_n is bounded in $L^2(0, 1; \mathbf{H}^1)$, z_n in $L^2(0, 1; H_*^1)$ and ψ_n in $L^\infty(0, 1; H_1^2)$. Moreover, using (47), one also has that ψ_n is bounded in $L^2(0, 1; H_1^3)$ and from the ψ -equation (15) $\partial_t \psi_n$ is bounded in $L^2(0, 1; H_*^{-1})$. Therefore, owing to Lemma 3, there exists a subsequence of ψ_n (equally denoted) and a limit function $\bar{\psi}$ such that

$$\psi_n \rightarrow \bar{\psi} \text{ strongly in } C^0([0, 1] \times \bar{\Omega}) \cap L^2(0, 1; H_*^2) \text{ and weakly in } L^2(0, 1; H_1^3). \quad (67)$$

In particular, $\psi(t_n) = \psi_n(0) \rightarrow \bar{\psi}(0)$ in $C^0(\bar{\Omega})$, hence $\bar{\psi}(0) = \psi_\infty$ (owing to (65)). On the other hand, $\partial_t \psi_n$ converges weakly to $\partial_t \bar{\psi}$ in $L^2(0, 1; H_*^{-1})$, hence taking limits in the variational formulation $(\partial_t \psi_n, \tilde{z}) + (\mathbf{u}_n \cdot \nabla \psi_n, \tilde{z}) + \gamma (\nabla z_n, \nabla \tilde{z}) = 0$ for all $\tilde{z} \in H_*^1$ and using convergences (66)-(67), we have that $\partial_t \bar{\psi} \rightarrow 0$ in $L^2(0, 1; H_*^{-1})$. Therefore, $\partial_t \bar{\psi} = 0$ and $\bar{\psi}(t)$ is a constant function of H_1^3 for all $t \in [0, 1]$, hence since $\bar{\psi}(0) = \psi_\infty$, one has

$$\bar{\psi}(t) = \psi_\infty \in H_1^3 \quad \text{for all } t \in [0, 1]. \quad (68)$$

Finally, by using convergences (66)-(67) one also has arguing as in (55) that $\overline{G}(\psi_n) \rightarrow \overline{G}(\overline{\psi})$ weakly in $L^2(0, 1; L^2)$, hence taking limit as $n \rightarrow +\infty$ in the variational formulation $(z_n, \tilde{\psi}) = -\varepsilon(\nabla\Delta\psi_n, \nabla\tilde{\psi}) + (\overline{G}(\psi_n), \tilde{\psi})$ for all $\tilde{\psi} \in H_*^1$, we deduce

$$-\varepsilon(\nabla\Delta\overline{\psi}, \nabla\tilde{\psi}) + (\overline{G}(\overline{\psi}), \tilde{\psi}) = 0, \quad \forall \tilde{\psi} \in H_*^1, \text{ a.e. } t \in (0, 1).$$

Then, from (68), $\psi_\infty \in H_1^3$ and $-\varepsilon(\nabla\Delta\psi_\infty, \nabla\tilde{\psi}) + (\overline{G}(\psi_\infty), \tilde{\psi}) = 0$ for all $\tilde{\psi} \in H_*^1$ and the proof is finished. \blacksquare

Theorem 15 *Assume that $\tilde{\mathcal{E}}(t)$ belongs to the equivalence class of $\mathcal{E}(\mathbf{u}(t), \psi(t))$, that is, $\tilde{\mathcal{E}}(t) = \mathcal{E}(\mathbf{u}(t), \psi(t))$ a.e. $t \geq 0$. Under the hypotheses of Theorem 14, there exists a unique limit $\psi_\infty \in \mathcal{S}$ such that $\psi(t) \rightarrow \psi_\infty$ in H^2 -weakly as $t \uparrow +\infty$, i.e. $\omega(\mathbf{u}, \psi) = \{(0, \psi_\infty)\}$.*

Proof. Let $\psi_\infty \in \mathcal{S}$ such that $(0, \psi_\infty) \in \omega(\mathbf{u}, \psi)$, i.e. there exists $t_n \uparrow +\infty$ such that $\mathbf{u}(t_n) \rightarrow 0$ weakly in \mathbf{L}^2 and $\psi(t_n) \rightarrow \psi_\infty$ weakly in H_1^2 (and strongly in H_*^1).

Without loss of generality, it can be assumed that $\tilde{\mathcal{E}}(t) > E_\infty$ for all $t > 0$, because otherwise, if it exists some $\tilde{t} > 0$ such that $\tilde{\mathcal{E}}(\tilde{t}) = E_\infty$, then the energy inequality (40) implies

$$\tilde{\mathcal{E}}(t) = E_\infty, \quad |\nabla\mathbf{u}(t)|_2^2 = 0 \quad \text{and} \quad |\nabla z(t)|_2^2 = 0, \quad \forall t \geq \tilde{t}.$$

In particular, $\mathbf{u}(t) = 0$ and $z(t)$ is constant for all $t \geq \tilde{t}$, and by using the z -equation (36), $\partial_t \psi(t) = 0$, hence $\psi(t) = \psi_\infty$ for all $t \geq \tilde{t}$. In this setting the convergence of the whole ψ -trajectory towards ψ_∞ is trivial.

Therefore, we can assume that $\tilde{\mathcal{E}}(t) > E_\infty$ for all $t \geq 0$. In this case, the proof will be divided into three steps.

Step 1: *Assuming that there exists $t_1 > 0$ such that*

$$\|\psi(t) - \psi_\infty\|_1 \leq \beta_1 \quad \text{and} \quad |\overline{\mathcal{E}}_b(\psi(t)) - \overline{\mathcal{E}}_b(\psi_\infty)| \leq \beta_2$$

for a.e. $t \geq t_1 \geq 0$, where $\beta_1 > 0, \beta_2 > 0$ are the constants appearing in Lemma 10 (of Lojasiewicz-Simon's type), then the following inequalities hold:

$$\frac{d}{dt} \left((\overline{\mathcal{E}}(\mathbf{u}(t), \psi(t)) - E_\infty)^\theta \right) + C\theta (|\nabla\mathbf{u}(t)|_2 + |\nabla z(t)|_2) \leq 0, \quad \text{a.e. } t \in (t_1, \infty) \quad (69)$$

$$\int_{t_1}^{t_2} \|\partial_t \psi\|_{H_*^{-1}} \leq \frac{C}{\theta} (\overline{\mathcal{E}}(\mathbf{u}(t_1), \psi(t_1)) - E_\infty)^\theta, \quad \forall t_2 \in (t_1, \infty), \quad (70)$$

where $\theta \in (0, 1/2]$ is the constant appearing in Lemma 10.

Since E_∞ is constant, we can rewrite the energy inequality (41) as

$$\frac{d}{dt} (\tilde{\mathcal{E}}(t) - E_\infty) + C (|\nabla\mathbf{u}(t)|_2^2 + |\nabla z(t)|_2^2) \leq 0, \quad \text{a.e. } t \geq 0.$$

By taking into account that $|\nabla \mathbf{u}(t)|_2^2 + |\nabla z(t)|_2^2 \geq \frac{1}{2} (|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2)^2$ and the inequality $\frac{1}{2} (|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2) \geq C(|\mathbf{u}(t)|_2 + \|z(t)\|_{H_*^{-1}})$, we obtain

$$\frac{d}{dt}(\tilde{\mathcal{E}}(t) - E_\infty) + C(|\mathbf{u}(t)|_2 + \|z(t)\|_{H_*^{-1}}) (|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2) \leq 0, \quad \text{a.e. } t \geq 0.$$

By using this expression and the time derivative of the (strictly positive) function

$$H(t) := (\tilde{\mathcal{E}}(t) - E_\infty)^\theta > 0,$$

we obtain

$$\frac{dH(t)}{dt} + \theta(\tilde{\mathcal{E}}(t) - E_\infty)^{\theta-1} C(|\mathbf{u}(t)|_2 + \|z(t)\|_{H_*^{-1}}) (|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2) \leq 0, \quad \text{a.e. } t \geq 0. \quad (71)$$

On the other hand, by taking into account that $|\mathcal{E}_k(\mathbf{u}(t))| = \frac{1}{2} |\mathbf{u}(t)|_2^2$ and that $|\mathbf{u}(t)|_2 \leq K$, we have

$$|\mathcal{E}_k(\mathbf{u}(t))|^{1-\theta} = \frac{1}{2^{1-\theta}} |\mathbf{u}(t)|_2^{2(1-\theta)} = \frac{1}{2^{1-\theta}} |\mathbf{u}(t)|_2^{1-2\theta} |\mathbf{u}(t)| \leq C |\mathbf{u}(t)|_2 \quad \text{a.e. } t \geq 0. \quad (72)$$

Moreover, the Lojasiewicz-Simon inequality (28) holds a.e. $t \geq t_1$, that is

$$|\bar{\mathcal{E}}_b(\psi(t)) - E_\infty|^{1-\theta} \leq C \|z(t)\|_{H_*^{-1}}, \quad \text{a.e. } t \geq t_1. \quad (73)$$

Hence, (73) and (72) give

$$\begin{aligned} (\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t)) - E_\infty)^{1-\theta} &\leq |\mathcal{E}_k(\mathbf{u}(t))|^{1-\theta} + |\bar{\mathcal{E}}_b(\psi(t)) - E_\infty|^{1-\theta} \\ &\leq C(|\mathbf{u}(t)|_2 + \|z(t)\|_{H_*^{-1}}) \quad \text{a.e. } t \geq t_1. \end{aligned}$$

Therefore,

$$(\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t)) - E_\infty)^{\theta-1} (|\mathbf{u}(t)|_2 + \|z(t)\|_{H_*^{-1}}) \geq C \quad \text{a.e. } t \geq t_1. \quad (74)$$

By applying (74) in (71) and that $\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t)) = \tilde{\mathcal{E}}(t)$ a.e. t , one has

$$\frac{dH(t)}{dt} + \theta C (|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2) \leq 0, \quad \text{a.e. } t \geq t_1$$

hence (69) is proved. Notice that the hypothesis $\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t)) = \tilde{\mathcal{E}}(t)$ for almost every t is the key point to arrive at (69). In particular, this hypothesis implies that the integral and differential versions of the energy law (38) and (41) are satisfied by $\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t))$ a.e. in time. In fact, energy law (41), changing $\tilde{\mathcal{E}}(t)$ by $\bar{\mathcal{E}}(\mathbf{u}(t), \psi(t))$, is the crucial hypothesis imposed in Remark 2.4 of [16].

Fixed any $t_2 \in (t_1, \infty)$, integrating (69) into $[t_1, t_2]$, taking into account that $(\bar{\mathcal{E}}(\mathbf{u}(t_2), \psi(t_2)) - E_\infty)^\theta > 0$, we have

$$\theta C \int_{t_1}^{t_2} (|\nabla \mathbf{u}(t)|_2 + |\nabla z(t)|_2) dt \leq (\bar{\mathcal{E}}(\mathbf{u}(t_1), \psi(t_1)) - E_\infty)^\theta. \quad (75)$$

From the equation (36), by using the weak regularity $\psi \in L^\infty((0, +\infty) \times \Omega)$, then

$$\|\partial_t \psi(t)\|_{H_*^{-1}} \leq C(|\mathbf{u}(t)|_2 + |\nabla z(t)|_2) \quad a.e. \ t \geq 0.$$

By using this inequality in (75), then (70) is attained.

Step 2: *There exists a sufficiently large n_0 such that $\|\psi(t) - \psi_\infty\|_1 \leq \beta_1$ and $|\bar{\mathcal{E}}_b(\psi(t)) - \bar{\mathcal{E}}_b(\psi_\infty)| \leq \beta_2$ for all $t \geq t_{n_0}$ (β_1, β_2 given in Lemma 10).*

Since $\psi(t_n) \rightarrow \psi_\infty$ strongly in H_*^1 and $\bar{\mathcal{E}}(\mathbf{u}(t_n), \psi(t_n)) \searrow E_\infty = \bar{\mathcal{E}}_b(\psi_\infty)$ in \mathbb{R} (see (62)), then for any $\delta \in (0, \beta_1)$, there exists an integer $N(\delta)$ such that, for all $n \geq N(\delta)$,

$$\|\psi(t_n) - \psi_\infty\|_1 \leq \delta \quad \text{and} \quad \frac{1}{\theta} (\bar{\mathcal{E}}_b(\psi(t_n)) - E_\infty)^\theta \leq \delta. \quad (76)$$

For each $n \geq N(\delta)$, we define

$$\bar{t}_n := \sup\{t : t > t_n, \|\psi(s) - \psi_\infty\|_1 < \beta_1 \quad \forall s \in [t_n, t]\}.$$

It suffices to prove that $\bar{t}_{n_0} = +\infty$ for some n_0 . Assume by contradiction that $t_n < \bar{t}_n < +\infty$ for all n , hence $\|\psi(\bar{t}_n) - \psi_\infty\|_1 = \beta_1$ and $\|\psi(t) - \psi_\infty\|_1 < \beta_1$ for all $t \in [t_n, \bar{t}_n)$. By applying Step 1 for all $t \in [t_n, \bar{t}_n]$, from (70) and (76) we obtain,

$$\int_{t_n}^{\bar{t}_n} \|\partial_t \psi\|_{H_*^{-1}} \leq C\delta, \quad \forall n \geq N(\delta).$$

Therefore,

$$\|\psi(\bar{t}_n) - \psi_\infty\|_{H_*^{-1}} \leq \|\psi(t_n) - \psi_\infty\|_{H_*^{-1}} + \int_{t_n}^{\bar{t}_n} \|\partial_t \psi\|_{H_*^{-1}} \leq (1 + C)\delta,$$

which implies that $\lim_{n \rightarrow +\infty} \|\psi(\bar{t}_n) - \psi_\infty\|_{H_*^{-1}} = 0$.

On the other hand, we will prove that $\psi(\bar{t}_n)$ is bounded in H_1^2 . Indeed, from (62), $\bar{\mathcal{E}}(\mathbf{u}(\bar{t}_n), \psi(\bar{t}_n))$ is bounded in \mathbb{R} , therefore in particular

$$\mu(\psi(\bar{t}_n) + m_0) = -\varepsilon \Delta \psi(\bar{t}_n) + \frac{1}{\varepsilon} F'(\psi(\bar{t}_n) + m_0) \quad \text{is bounded in } L^2(\Omega). \quad (77)$$

But, since $F'(\psi(\bar{t}_n) + m_0)$ is bounded in $L^\infty(\Omega)$ (because $\psi \in C([0, +\infty) \times \bar{\Omega})$), then $\Delta \psi(\bar{t}_n)$ is bounded in $L^2(\Omega)$. Therefore using the H^2 -regularity of problem (P_0) (see (22)), one has that $\psi(\bar{t}_n)$ is bounded in H_1^2 .

Consequently, $\psi(\bar{t}_n)$ is relatively compact in H_*^1 , hence there exists a subsequence of $\psi(\bar{t}_n)$, which is still denoted as $\psi(\bar{t}_n)$, that converges to ψ_∞ in H_*^1 -strong. Hence $\|\psi(\bar{t}_n) - \psi_\infty\|_1 < \beta_1$ for a sufficiently large n , which contradicts the definition of \bar{t}_n .

Step 3: *There exists a unique ψ_∞ such that $\psi(t) \rightarrow \psi_\infty$ weakly in H_1^2 as $t \uparrow +\infty$.*

By using Steps 1 and 2, (70) can be applied, for all $t_1, t_0 : t_1 > t_0 \geq t_{n_0}$, hence

$$\|\psi(t_1) - \psi(t_0)\|_{H_*^{-1}} \leq \int_{t_0}^{t_1} \|\partial_t \psi\|_{H_*^{-1}} \rightarrow 0, \quad \text{as } t_0, t_1 \rightarrow +\infty.$$

Therefore, $(\psi(t))_{t \geq t_{n_0}}$ is a Cauchy sequence in H_*^{-1} as $t \uparrow +\infty$, hence there exists a unique $\psi_\infty \in H_*^{-1}$ such that $\psi(t) \rightarrow \psi_\infty$ in H_*^{-1} as $t \uparrow +\infty$. Finally, the convergence in H_1^2 -weak by sequences of $\psi(t)$ proved in Theorem 14, yields to $\psi(t) \rightarrow \psi_\infty$ in H_1^2 -weak, and the proof is finished. \blacksquare

7 Higher regularity for the phase variable

We consider the following zero-mean regular space:

$$H_2^4 = \{w \in H^4 \cap L_*^2; \partial_n w|_{\partial\Omega} = 0, \partial_n \Delta w|_{\partial\Omega} = 0\}.$$

From now on, we assume Ω sufficiently regular (for instance $\partial\Omega \in C^4$) such that the H^4 -regularity of the Poisson-Neumann problem (P₀) holds.

For any $w \in H_2^4$, taking $v = \Delta w$ in (P₀), since $\partial_n(\Delta\psi)|_{\partial\Omega} = 0$ and $\int_\Omega \Delta\psi = 0$, then the H^2 -regularity of (P₀) implies

$$\|\Delta w\|_2 \leq C|\Delta^2 w|_2 \quad \forall w \in H_2^4 \quad (78)$$

In particular, from (78) and the H^4 -regularity of (P₀), one has

$$\|w\|_4 \leq C|\Delta^2 w|_2 \quad \forall w \in H_2^4 \quad (79)$$

7.1 Global in time strong regularity for ψ

We will see that $\psi \in L^\infty(0, +\infty; H_1^3)$ if the data $\psi_0 \in H_1^3$. For this, we argue in a formal manner, that could be rigorously justified via the Galerkin method.

We add the z -equation (36) tested by $\partial_t \psi$ ($\in H_*^1$), and the ψ -equation (37) by $\gamma \Delta \partial_t \psi$ ($\in H_1^2$ owing to $\nabla \partial_t \Delta \psi \cdot \mathbf{n}|_{\partial\Omega} = 0$). If we integrate twice by parts in the term $\gamma(\bar{G}(\psi) - z, \Delta \partial_t \psi)$ of (37), taking into account that $\nabla \partial_t \psi \cdot \mathbf{n}|_{\partial\Omega} = 0$, $\nabla \bar{G}(\psi) \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $\nabla z \cdot \mathbf{n}|_{\partial\Omega} = 0$, one has

$$\gamma(\bar{G}(\psi) - z, \Delta \partial_t \psi) = \gamma(\Delta \bar{G}(\psi) - \Delta z, \partial_t \psi).$$

Therefore, the term $\gamma(\Delta z, \partial_t \psi)$ cancels, remaining:

$$\frac{\varepsilon}{2} \frac{d}{dt} |\nabla \Delta \psi|_2^2 + |\partial_t \psi|_2^2 = -(\mathbf{u} \cdot \nabla \psi, \partial_t \psi) - \gamma(\Delta \bar{G}(\psi), \partial_t \psi),$$

hence applying Holder and Young inequalities

$$\varepsilon \frac{d}{dt} |\nabla \Delta \psi|_2^2 + |\partial_t \psi|_2^2 \leq C(|\mathbf{u} \cdot \nabla \psi|_2^2 + |\Delta \bar{G}(\psi)|_2^2). \quad (80)$$

From the ψ -equation (16), by using (46) and (79), we obtain

$$\|\psi\|_4 \leq C |\Delta^2 \psi|_2 \leq C(|\bar{G}(\psi)|_2 + |z|_2) \leq C(1 + |z|_2). \quad (81)$$

In particular, since $\psi \in L^\infty(0, +\infty; H^2)$,

$$|\Delta \bar{G}(\psi)|_2 \leq C(1 + \|\psi\|_4) \leq C(1 + |z|_2). \quad (82)$$

By using (81) and (82) in (80), and taking into account that $\|\psi\|_3$ is equivalent to $|\nabla \Delta \psi|_2$ (see (23)), one has

$$\frac{d}{dt} \|\psi\|_3^2 + C_0(\|\psi\|_4^2 + |\partial_t \psi|_2^2) \leq C(1 + \|\mathbf{u}\|_1^2 + |z|_2^2). \quad (83)$$

By denoting

$$\Phi(t) := \|\psi\|_3^2, \quad B(t) := \|\mathbf{u}\|_1^2 + |z|_2^2,$$

then (83) yields to

$$\Phi'(t) + C_0 \Phi(t) \leq C(1 + B(t)). \quad (84)$$

Multiplying (84) by $e^{C_0 t}$ and integrating in time, we obtain

$$\Phi(t) \leq \Phi(0)e^{-C_0 t} + C e^{-C_0 t} \int_0^t e^{C_0 s} (1 + B(s)) ds.$$

In particular,

$$\Phi(t) \leq \Phi(0) + C(1 - e^{-C_0 t}) + C \int_0^t B(s) ds.$$

Since $B(t) \in L^1(0, +\infty)$, we have that $\Phi \in L^\infty(0, +\infty)$, that means

$$\psi \in L^\infty(0, +\infty; H_1^3). \quad (85)$$

Moreover, integrating in time in (83), we obtain

$$\psi \in L_{loc}^2([0, +\infty); H_2^4) \quad \text{and} \quad \partial_t \psi \in L_{loc}^2(0, +\infty; L_*^2). \quad (86)$$

In particular, using this improved regularity, the phase equations (15)-(16) are satisfied pointwisely a.e. $t \in (0, +\infty)$, if the data $\psi_0 \in H_1^3$.

Remark 16 *It has been possible to obtain higher estimates for the phase variable without improving estimates for the velocity and pressure.*

7.2 Improving the convergence of the phase trajectory

The previous extra global in time regularity obtained for the phase variable ψ allows to obtain a convergence of the whole trajectory of $\psi(t)$ in a more regular space, replacing the convergence in H^2 by H^3 . Indeed, fixed an initial data $(\mathbf{u}_0, \psi_0) \in \mathbf{H} \times H_1^3$, we replace the ω -limit and the equilibrium point sets defined in (63) and (64) by the following more regular sets:

$$\begin{aligned} \omega_{reg}(\mathbf{u}, \psi) &= \{(\mathbf{u}_\infty, \psi_\infty) \in \mathbf{H} \times H_1^3 : \exists \{t_n\} \uparrow +\infty \text{ s.t.} \\ &\quad (\mathbf{u}(t_n), \psi(t_n)) \rightarrow (\mathbf{u}_\infty, \psi_\infty) \text{ weakly in } \mathbf{H} \times H_1^3\}, \\ \mathcal{S}_{reg} &= \{\psi \in H_2^4(\Omega) : \varepsilon \Delta^2 \psi + \overline{G}(\psi) = 0 \text{ a.e in } \Omega\}. \end{aligned}$$

In this setting, theorems 14 and 15 imply in particular, the following statements:

Theorem 17 *The set $\omega_{reg}(\mathbf{u}, \psi)$ is nonempty and $\omega_{reg}(\mathbf{u}, \psi) \subset \{0\} \times \mathcal{S}_{reg}$. Moreover, for any $\psi_\infty \in \mathcal{S}_{reg}$ with $(0, \psi_\infty) \in \omega_{reg}(\mathbf{u}, \psi)$, then $\overline{\mathcal{E}}_b(\psi_\infty) = E_\infty$.*

Theorem 18 *There exists a unique $\psi_\infty \in \mathcal{S}_{reg}$ such that $\psi(t) \rightarrow \psi_\infty$ in H_1^3 weakly as $t \uparrow +\infty$, i.e. $\omega_{reg}(\mathbf{u}, \psi) = \{(0, \psi_\infty)\}$.*

Remark 19 *In the previous work [5], we have proved the convergence of the whole trajectory $(\mathbf{u}(t), \psi(t)) \rightarrow (0, \psi_\infty)$ weakly in $\mathbf{V} \times H_1^3$, by using different arguments that now. In [5], by applying a more standard Lojasiewicz-Simon inequality (given in Lemma 5.2 of [22]) jointly to the extra regularity (85) and (86), one has the existence and uniqueness of strong solutions (in velocity-pressure and phase) for sufficiently large times and the continuous dependence of local in time strong solutions. Now, we have proved the same type of results but using only the global weak regularity of the problem.*

7.3 Additional H^6 -regularity for the phase

It will be possible to obtain the following higher regularity in space for the phase variable $\psi \in L_{loc}^2(0, +\infty; H_2^6)$, by imposing only more regularity of the domain. For this, we consider the following zero-mean regular space:

$$H_2^6 = \{w \in H^6 \cap L_*^2; \partial_n w|_{\partial\Omega} = 0, \partial_n \Delta w|_{\partial\Omega} = 0\}$$

and the second order elliptic (nonhomogeneous) Neumann-Poisson problem:

$$(P_g) \quad \begin{cases} -\Delta v = f & \text{in } \Omega \\ \partial_n v|_{\partial\Omega} = g, & \int_{\Omega} v \, d\mathbf{x} = 0, \end{cases}$$

where $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$, satisfying the compatibility condition

$$\int_{\Omega} f \, d\mathbf{x} + \int_{\Omega} g \, d\mathbf{x} = 0.$$

We will use problem (P_g) for $v = \Delta^2\psi$, therefore we need to obtain an expression of $\partial_n(\Delta^2\psi)|_{\partial\Omega}$.

From (16), $\Delta^2\psi = \frac{1}{\varepsilon}(z - \overline{G}(\psi))$. Since $0 = \partial_n\psi|_{\partial\Omega} = \partial_n\Delta\psi|_{\partial\Omega} = \partial_n\mu(\psi)|_{\partial\Omega} = \partial_n z|_{\partial\Omega}$, then

$$\begin{aligned} \partial_n(\Delta^2\psi)|_{\partial\Omega} &= -\frac{1}{\varepsilon}\partial_n(\overline{G}(\psi))|_{\partial\Omega} = \frac{1}{\varepsilon^2}\partial_n(\Delta F'(\psi + m_0))|_{\partial\Omega} \\ &= \frac{1}{\varepsilon^2}\left(F^{iv}(\psi + m_0)|\nabla\psi|^2\partial_n\psi + 2F'''(\psi + m_0)\nabla\psi \cdot \partial_n(\nabla\psi) \right. \\ &\quad \left. + F'''(\psi + m_0)\Delta\psi\partial_n\psi + F''(\psi + m_0)\partial_n(\Delta\psi)\right)|_{\partial\Omega} \\ &= \frac{2}{\varepsilon^2}F'''(\psi + m_0)\nabla\psi \cdot \partial_n(\nabla\psi)|_{\partial\Omega} \end{aligned}$$

From now on, we assume Ω sufficiently regular (for instance $\partial\Omega \in C^6$) such that the H^6 -regularity of (P_0) and the H^2 -regularity of (P_g) hold. From the H^6 and the H^4 -regularity of (P_0) ,

$$\|w\|_6 \leq C\|\Delta w\|_4 \leq C\|\Delta^2 w\|_2 \quad \forall w \in H_2^6, \quad (87)$$

and from the H^2 -regularity of (P_g) for $v = \Delta^2\psi$ and $g = \partial_n(\Delta^2\psi)|_{\partial\Omega}$,

$$\|\Delta^2\psi\|_2 \leq C(\|\Delta^3\psi\|_2 + \|g\|_{1/2,\partial\Omega}). \quad (88)$$

From (15) and (16), and using (82), we have that

$$|\Delta^3\psi|_2 \leq C(|\partial_t\psi|_2 + \|\mathbf{u} \cdot \nabla\psi\|_2 + |\Delta\overline{G}(\psi)|_2) \leq C(|\partial_t\psi|_2 + \|\mathbf{u}\|_1 + 1 + |z|_2). \quad (89)$$

On the other hand, by using $\|\psi\|_2 \leq C$, (87), and (88) for $g = \frac{2}{\varepsilon^2}F'''(\psi + m_0)\nabla\psi \cdot \partial_n(\nabla\psi)$, we obtain

$$\begin{aligned} \|\psi\|_6 &\leq C(|\Delta^3\psi|_2 + \|F'''(\psi + m_0)\partial_j\psi\partial_{i_j}^2\psi n_i\|_{1/2,\partial\Omega}) \\ &\leq C(|\Delta^3\psi|_2 + \|\psi\|_3^2) \leq C(|\Delta^3\psi|_2 + \|\psi\|_2\|\psi\|_4). \end{aligned}$$

Taking into account the interpolation inequality $\|\psi\|_4 \leq \|\psi\|_2^{1/2}\|\psi\|_6^{1/2}$, we obtain that

$$\|\psi\|_6 \leq C(|\Delta^3\psi|_2 + \|\psi\|_6^{1/2}),$$

hence

$$\|\psi\|_6 \leq C(1 + |\Delta^3\psi|_2).$$

Therefore, taking into account (89)

$$\|\psi\|_6^2 \leq C(1 + |\partial_t\psi|_2^2 + \|\mathbf{u}\|_1^2 + |z|_2^2). \quad (90)$$

By using (90) in (83), we obtain

$$\frac{d}{dt}\|\psi\|_3^2 + C_0(\|\psi\|_6^2 + |\partial_t\psi|_2^2) \leq C(1 + \|\mathbf{u}\|_1^2 + |z|_2^2). \quad (91)$$

By integrating in time in (91), since the initial phase $\psi_0 \in H_1^3$, we obtain

$$\psi \in L_{loc}^2(0, +\infty; H_2^6).$$

Finally, from (16),

$$z \in L_{loc}^2(0, +\infty; H_1^2).$$

In particular, using these improved estimates and the phase equations (15)-(16), we have that the following sixth order equation

$$\partial_t\psi + \mathbf{u} \cdot \nabla\psi - \gamma\varepsilon\Delta^3\psi - \gamma\Delta(\overline{G}(\psi)) = 0,$$

is satisfied point-wisely a.e. $t \in (0, +\infty)$.

8 Conclusions and Perspectives

For the Navier-Stokes-Cahn-Hilliard model introduced in this paper we have proved the convergence of the trajectory of any global weak solution to a single equilibrium point. Moreover, the regularity for the phase is improved without the need of more regularity for the velocity and pressure variables, and therefore it is not necessary to impose large time or large viscosity constraints.

Bearing in mind the results obtained in this paper, it seems achievable to obtain (rational) convergence rate estimates of the convergence of trajectories in a similar way to [12]. Finally, it would be interesting to study if local minimizers of the elastic bending energy are stable, as was proved in [22] for a Navier-Stokes-Allen-Cahn problem modeling vesicle membranes.

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