RESEARCH PAPER

Optimality conditions for Linear Copositive Programming problems with isolated immobile indices

O. I. Kostyukova[a] and T. V. Tchemisova[b]

[a]Institute of Mathematics, National Academy of Sciences of Belarus, Surganov str. 11, 220072, Minsk, Belarus; [b]Mathematical Department, University of Aveiro, Campus Universitário Santiago, 3810-193, Aveiro, Portugal

ARTICLE HISTORY
Compiled October 15, 2019

Abstract

In the present paper, we apply our recent results on optimality for convex Semi-Infinite Programming to a problem of Linear Copositive Programming. We prove explicit optimality conditions that use concepts of immobile indices and their immobility orders, and do not require the Slater constraint qualification to be satisfied. The only assumption that we impose here is that the set of immobile indices consists of isolated points, and hence is finite. This assumption is weaker than the Slater condition; therefore, the optimality conditions obtained in the paper are more general when compared with those usually used in Linear Copositive Programming. We present an example of a problem in which the new optimality conditions allow one to test the optimality of a given feasible solution while the known optimality conditions fail to do this. Further, we use the immobile indices to construct a pair of *regularized* dual copositive problems and show that regardless of whether the Slater condition is satisfied or not, the duality gap between the optimal values of these problems is zero. An example of a problem is presented for which the standard strict duality fails, but the duality gap obtained by using the regularized dual problem vanishes.

KEYWORDS
Convex Programming, Semi-Infinite Programming, Copositive Programming, Constraint Qualifications, immobile index, optimality conditions, strong duality.

Abbreviations: SIP (Semi-Infinite Programming), CP (Copositive Programming), LCP (Linear Copositive Programming), QP (Quadratic Programming), CQ (Constraint Qualification)

AMS CLASSIFICATION
90C25, 90C30, 90C34

CONTACT T. V. Tchemisova. Email: tatiana@ua.pt

1. Introduction

Copositive Programming (CP) is a relatively new field of Conic Optimization, which is most actively developing in recent years. Despite the fact that the first works on CP have appeared in the last century [1–3], the term “Copositive Programming” was introduced in 2000 by Bomze et al. in [4].

Linear Copositive Programming (LCP) problems have the form

$$LCP: \qquad \sup_{x\in\mathbb{R}^n} c^T x \quad \text{s.t.} \sum_{i=1}^{n} A_i x_i + A_0 \in \mathcal{C},$$

where $A_i, i = 0, 1, \ldots, n$ are symmetric matrices and $\mathcal{C}$ denotes the cone of matrices which are positively semidefined on the non-negative orthant $\mathbb{R}^k_+$:

$$\mathcal{C} := \{D \in \mathcal{S}(k) : \; t^T D t \geq 0 \,\forall t \in \mathbb{R}^k_+\}. \tag{1}$$

LCP can be considered as a generalization of Semidefinite Programming (SDP), since its general problem consists in optimizing over the cone $\mathcal{C}$ of so-called *copositive matrices*. We refer the interested readers to a recent article [5] for a survey on copositive matrices, to the monograph [6] for their algebraic properties, and to the paper [7] for open problems in the theory of completely positive and copositive matrices.

Copositive models arise in Quadratic Programming (QP) with linear and binary constraints [8,9], fractional QP [3,10], Graph Theory and Combinatorics [2,11], among others. The diversity of copositive formulations in different domains of optimization (continuous and discrete, deterministic and stochastic, robust optimization with uncertain objective and others) is described in [9,12], et al. According to M. Dür [9], CP is "a powerful modeling tool which interlinks the quadratic and binary worlds". Being formally very similar to that of SDP, the copositive programs are NP-hard since testing copositivity of matrices is *co-NP-complete* [13]. Different algorithms for copositivity detection are described e.g. in [12,14–16]. A clustered bibliography on copositive optimization can be found in [17].

Optimality conditions is an important issue in the study of any optimization problem since they permit not only to test the optimality of a given feasible solution, but also to develop efficient numerical methods. Usually, optimality conditions are formulated for individual classes of optimization problems. This permits to exploit efficiently special properties of a problem, its objective and constraint functions, and the structure of the feasible set. Often, optimality conditions are based on the topological study of feasible sets and use certain assumptions, so-called *constraint qualifications* (CQs), for references see [18–20]. Therefore, to verify such optimality conditions, one should, first of all, check the corresponding CQs.

Testing CQs is not always an easy task, and moreover, in practice, the known CQs often fail, see [21,22] and the references therein. According to [22], in the absence of CQs, the standard duality theory does not guarantee the vanishing of the duality gap (the difference between the optimal values of the given problem and the corresponding dual one), and the property of so-called *strong duality* (the existence of an optimal solution of the dual problem in addition to zero duality gap) may not occur. The failure of CQs can lead to numerical difficulties such as an increase of the expected number of iterations and even to incorrect solutions.

All of the above makes it possible to conclude that an interesting and important challenge, both theoretically and practically, is to develop new optimality conditions that either do not use any CQ (*CQ-free optimality conditions*) or use assumptions that are weaker than the known CQs.

As a rule, the optimality conditions for CP are formulated under the Slater condition consisting in the strict feasibility. This CQ is also used to guarantee the strict duality in copositive optimization. The optimality conditions for CP problems are usually drawn on the base of the analogous conditions for equivalent problems of Semi-Infinite Programming (SIP) [cf. 27], and therefore, the wider the range of application of the optimal conditions for SIP, the more effective the conditions obtained forCP.

In our previous papers, see e.g. [19,23–26], we used a notion of immobile indices and their immobility orders for problems of convex SIP, and formulated new optimality conditions under assumptions that are weaker than the commonly used CQs. Our goal now is to apply our approach proposed for SIP to the problems of LCP and to obtain for the latter new optimality conditions and dual formulations that guarantee strong duality.

In this paper, given a problem in the form (LCP), we formulate for it an equivalent semi-infinite problem, and define immobile indices and their immobility orders. Based on the optimality conditions for SIP, obtained in our paper [26], we prove new optimality conditions for the problem (LCP). These conditions use the assumption about the isolation of the immobile indices which is equivalent to one about finiteness of the set of immobile indices. Both these assumptions are weaker than the Slater condition. Further, we reformulate the constraints of the LCP problem with the help of special cones and obtain a new pair of regularized primal and dual problems. These problems use the information about the immobile indices, the cones in their constraints are explicitly described, and we show that the duality gap for this dual pair is zero. To illustrate our approach, we present two examples. In the first example, we consider an LCP problem in which the new optimality conditions allow one to detect the optimality of the given feasible solution while the optimality conditions from [27] are not able to do this. The second example presents an LCP problem for which the standard strict duality fails, but the duality gap obtained by using the regularized dual problem vanishes.

The paper is organized as follows. Section 1 hosts the Introduction. In Section 2, we formulate an LCP problem, the equivalent SIP problem and define the immobile indices and their immobility orders. Section 3 is devoted to new optimality conditions for LCP and contains an illustrative example. Some duality issues are discussed in Section 4 and a new pair of regularized primal and dual problems in a conic form is formulated. The final Section 5 contains some conclusions and final remarks.

2. Equivalent formulations of the LCP problem. Immobile indices and their properties

Here and in what follows, we use the next notation: given an integer k, $\mathbb{R}^k_+$ denotes the set of all k-dimensional vectors with non-negative components and $\mathcal{S}(k)$ stays for the space of symmetric $k \times k$ matrices. The space $\mathcal{S}(k)$ is considered here as a vector space equipped with the trace inner product $A \bullet B := trace(AB)$, for $A, B \in \mathcal{S}(k)$. Given a set $\mathcal{D}$, we denote by $conv(\mathcal{D})$ its convex hull.

Consider an LCP problem in the form

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{s.t. } t^T \mathcal{A}(x) t \geq 0 \;\; \forall t \in \mathbb{R}^k_+, \tag{2}$$

where $x = (x_1, ..., x_n)^T$ is a $n-$vector of variables, $t = (t_1, \dots, t_k)^T \in \mathbb{R}^k_+$ is a $k-$ vector of indices, the matrix function $\mathcal{A}(x)$ is given by

$$\mathcal{A}(x) := \sum_{i=1}^{n} A_i x_i + A_0, \tag{3}$$

and the data are the matrices $A_i \in \mathcal{S}(k), i = 0, 1, \dots, n$, and the vector $c \in \mathbb{R}^n$.

Problem (2) is a linear conic problem [18], since its constraints can be rewritten in the form $\mathcal{A}(x) \in \mathcal{C}$, where the cone $\mathcal{C}$ is defined in (1). Evidently, this problem is equivalent to the following SIP problem with a compact index set:

$$\min_{x \in \mathbb{R}^n} c^T x \;\; \text{s.t. } t^T \mathcal{A}(x) t \geq 0 \; \forall t \in T := \{l \in \mathbb{R}^k_+ : e^T l = 1\}. \tag{4}$$

Here and in what follows, $e := \sum_{i=1}^{k} e_i = (1, 1, ..., 1)^T \in \mathbb{R}^k$, where e_i is the i-th vector of the canonic basis of $\mathbb{R}^k$. Notice that the unit simplex T in (4) can be replaced by any base of the cone $\mathbb{R}^k_+$.

It should be mentioned that problem (4) is a particular case of convex SIP problems with k - dimensional index set T, which are considered in our paper [19].

Denote by X the set of feasible solutions of the equivalent problems (2) and (4):

$$X = \{x \in \mathbb{R}^n : t^T \mathcal{A}(x) t \geq 0 \;\; \forall t \in T\}.$$

Given a feasible solution $x \in X$ of the SIP problem (4), the *lower level problem* has the form

$$LLP(x): \quad \min\ t^T\mathcal{A}(x)t \quad \text{s.t. } t \in T.$$

The Slater condition is one of the most commonly used in convex optimization CQs. Let us recall this condition for problems (2) and (4).

The Slater condition for the LCP problem (2) has the form [18,28]

$$\exists\ \bar{x} \in \mathbb{R}^n \ \text{ such that } \ t^T\mathcal{A}(\bar{x})t > 0\ \forall t \in \mathbb{R}^k_+ \setminus \{\mathbf{0}\}, \tag{5}$$

or, equivalently, $\exists\ \bar{x} \in \mathbb{R}^n$ such that $\mathcal{A}(\bar{x}) \in int\,\mathcal{C}$. Here $\mathbf{0} := (0, 0, \ldots, 0)^T \in \mathbb{R}^k$.

The constraints of the SIP problem (4) satisfy the Slater condition if

$$\exists\ \bar{x} \in \mathbb{R}^n \ \text{ such that } \ t^T\mathcal{A}(\bar{x})t > 0 \quad \forall t \in T. \tag{6}$$

Evidently, the Slater conditions (5) and (6) are equivalent.

Following [19], we will say that an index $t \in T$ is immobile if the corresponding constraint of the SIP problem (4) is active for all feasible x. Denote by T^* the set of all immobile indices in this problem, i.e.

$$T^* := \{t \in T : t^T\mathcal{A}(x)t = 0\ \forall x \in X\}. \tag{7}$$

It is evident that the set of immobile indices $\tilde{T}^*$ for the LCP problem (2) is generated by the set T^* as follows: $\tilde{T}^* := \{\tau \in \mathbb{R}^k : \tau = \alpha t,\ \alpha \in \mathbb{R}_+,\ t \in T^*\}$. Hence, in what follows, we will refer to the set T^* as to the set of immobile indices for problem (2) as well.

The next proposition is a corollary of Proposition 2 in [25] and of Corollary 5.1.1 in [29].

Proposition 2.1. *Given the LCP problem (2), the Slater condition (5) is equivalent to the emptiness of the set* T^*.

From the proposition above, it follows that the emptiness of the set of immobile indices can be considered as a CQ. If the set T^* is empty, then, according to Proposition 2.1, the constraints of problem (2) satisfy the Slater condition, and in this case the optimality conditions for this problem are known from the literature. To formulate the conditions from [27], we need to define the dual cone to the cone of copositive matrices.

The dual cone of the cone $\mathcal{C}$ defined in (1) is a so-called *cone of completely positive matrices* [9]:

$$\mathcal{C}^* := conv\{tt^T : t \in \mathbb{R}^k_+\}. \tag{8}$$

Let us say that a feasible solution x^0 of problem (2) satisfies the *Karush-Kuhn-Tucker*

(KKT) condition if there exists a matrix $\Omega \in \mathcal{C}^*$ such that

$$-c_i + \Omega \bullet A_i = 0,\ i = 1, \dots, n,\ \ \Omega \bullet \mathcal{A}(x^0) = 0. \tag{9}$$

Following [27], the optimality conditions for problem (2) can be formulated in the form of the following theorem.

Theorem 2.2. *If a feasible solution $x^0 \in X$ satisfies the KKT condition (9), then x^0 is a minimizer of problem (2). On the other hand, under the Slater condition (5) a minimizer x^0 of problem (2) must satisfy the KKT condition.*

If $T^* \neq \emptyset$, then a minimizer x^0 of problem (2) may not satisfy the KKT condition, and hence Theorem 2.2 does not always allow to recognize the optimality of a minimizer.

In our study, we will prove new optimality conditions for LCP without special assumptions about the (non-)emptiness of the set T^* and hence without the Slater CQ. For our considerations, we will essentially use the equivalence of problems (2) and (4), and the results of [26].

Set $P := \{1, 2, ..., k\}$. Given a vector $t \in T^*$, define the following sets of its coordinates:

$$P_0(t) := \{p \in P :\ t_p = 0\}, \quad P_*(t) := P \setminus P_0(t),$$

and the polyhedral convex cone of feasible directions at t relative to T:

$$L(t) := \{l \in \mathbb{R}^k : e^T l = 0,\ l_p \geq 0,\ p \in P_0(t)\}. \tag{10}$$

The next proposition permits to understand better the structure of the set $L(t)$.

Proposition 2.3. *For $t \in T^*$, the set $L(t)$ defined in (10) admits the following representation:*

$$L(t) = \{l \in \mathbb{R}^k : l = \sum_{p \in P \setminus \{s(t)\}} (e_p - e_{s(t)})\alpha_p,\ \alpha_p \geq 0, p \in P_0(t)\}, \tag{11}$$

where $s(t)$ is a fixed coordinate from the set $P_*(t) \neq \emptyset$.

Notice that the vectors $e_p - e_{s(t)}, p \in P_0(t)$, are the extremal rays in $L(t)$.

For problem (4), let us reformulate the definition of the immobility orders of the immobile indices from [19,23].

Definition 2.4. Given a SIP problem in the form (4), let $t \in T^*$ and $l \in L(t),\ l \neq 0$. The immobility order $q(t, l)$ of the index t along the direction l is defined as follows:

- $q(t, l) = 0$ if $\exists\, \bar{x} = x(t, l) \in X$ such that $l^T \mathcal{A}(\bar{x}) t \neq 0$;
- $q(t, l) = 1$ if $l^T \mathcal{A}(x) t = 0\ \forall x \in X$ and $\exists\, \bar{x} = x(t, l) \in X$ such that $l^T \mathcal{A}(\bar{x}) l \neq 0$;
- $q(t, l) = \infty$ if $l^T \mathcal{A}(x) t = 0, l^T \mathcal{A}(x) l = 0\ \forall x \in X$.

Let us make an assumption about isolation of the immobile indices.

Isolation Assumption. Given the LCP problem (2), suppose that the set T^* defined in (7) consists of isolated points.

This assumption permits us to establish the following property of the set T^*.

Proposition 2.5. *Given the LCP problem (2), the following conditions are equivalent:*

(i) all elements in T^* are isolated;
(ii) the set T^* is finite: $|T^*| < \infty$, and
(iii) the following inequalities take place:

$$q(t,l) \leq 1 \textit{ for all } l \in L(t) \setminus \{\boldsymbol{0}\} \textit{ and all } t \in T^*. \tag{12}$$

Proof. $(i) \Rightarrow (ii)$. It follows from condition (i) and the analyticity of the constraint function that the set T^* consists of a finite number of elements.

$(ii) \Rightarrow (iii)$. Suppose the contrary: there exists $t^* \in T^*$ and $l^* \in L(t^*)$, $l^* \neq \mathbf{0}$, such that $q(t^*, l^*) \geq 2$. Then, according to the definition of the immobility orders, we conclude that for all $x \in X$, the equalities $l^{*T}\mathcal{A}(x)t^* = 0$ and $l^{*T}\mathcal{A}(x)l^* = 0$, take place. These equalities imply that there exists $\theta_0 > 0$ such that for all $x \in X$, it holds

$$(t^* + \theta l^*)^T \mathcal{A}(x)(t^* + \theta l^*) = 0, \;\; (t^* + \theta l^*) \in T \;\; \forall \theta \in [0, \theta_0]. \tag{13}$$

By definition, the above means that $(t^* + \theta l^*) \in T^*$ for all $\theta \in [0, \theta_0]$. But the last relations contradict condition (ii).

$(iii) \Rightarrow (i)$. Suppose the contrary: there exist $t^* \in T^*$ and $l^* \in L(t^*)$, $l^* \neq \mathbf{0}$, and $\theta_0 > 0$ such that $(t^* + \theta l^*) \in T^*$ $\forall \theta \in [0, \theta_0]$. From the definition of the immobile indices we conclude that relations (13) hold true. It follows from these relations that $q(t^*, l^*) \geq 2$, but this contradicts condition (iii). The proposition is proved. $\square$

From Proposition 2.5, it is easy to conclude that the Isolation Assumption is equivalent to the following one.

Finiteness Assumption. Given the LCP problem in the form (2), suppose that $|T^*| < \infty$.

Notice that the Isolation and the Finiteness Assumptions are, in turn, equivalent to the following condition that can be easily checked in practice: given the LCP problem in the form (2), there exists a feasible $\bar{x}$ such that the corresponding active index set $T_a(\bar{x})$ is finite:

$$|T_a(\bar{x})| < \infty, \;\; \text{where} \;\; T_a(\bar{x}) := \{t \in T : t^T \mathcal{A}(\bar{x}) t = 0\}.$$

In what follows, for the sake of simplicity, we will use the Isolation Assumption, or equivalently, the Finiteness Assumption.

3. New optimality conditions for LCP problems under the Finiteness Assumption

As far as we know, all optimality conditions for LCP problems [see e.g. 18,27], are formulated under the Slater condition. In this Section, we will prove new optimality conditions that do not use this condition or any other "regularity condition" [cf. 18]. The only assumption we do here is that the set of immobile indices is finite.

According to the Finiteness Assumption, the set T^* can be written in the form

$$T^* = \{t^*(j), j \in J\}, \ \ |J| < \infty. \tag{14}$$

Denote:

$$P_0(j) := P_0(t^*(j)), \ P_*(j) := P_*(t^*(j)), \ \ L(j) := L(t^*(j)), \ s(j) := s(t^*(j)), \ j \in J.$$

Given $j \in J$ and $s(j) \in P_*(j)$, define the sets

$$P_{00}(j) := \{p \in P_0(j) : \ q(t^*_j, e_p - e_{s(j)}) = 0\},$$
$$P_{0*}(j) := P_0(j) \setminus P_{00}(j) = \{p \in P_0(j) : \ q(t^*_j, e_p - e_{s(j)}) > 0\}.$$

Notice that under the Finiteness Assumption, the set of immobile indices (14) and the corresponding coordinate sets $P_*(j), \ P_{0*}(j), \ P_{00}(j), \ j \in J$, can be constructed by the algorithm described in [24].

It is evident that for any $j \in J$ and any $x \in X$, the immobile index $t^*(j)$ is an optimal solution of the lower level problem $LLP(x)$ and $(t^*(j))^T \mathcal{A}(x) t^*(j) = 0$. Hence, from the optimality conditions for $LLP(x)$, it follows that for any $x \in X$ and any $j \in J$, there exist a vector $y(x,j) \in \mathbb{R}^k_+$ and a number $\lambda(x,j)$ such that

$$\mathcal{A}(x)t^*(j) - y(x,j) + \lambda(x,j)e = 0, \ \ (y(x,j))^T t^*(j) = 0. \tag{15}$$

Multiplying both sides of the first equality in (15) by $(t^*(j))^T$ and taking into account the equalities $(y(x,j))^T t^*(j) = 0$ and $e^T t^*(j) = 1$, we get $\lambda(x,j) = 0$.

Hence, conditions (15) can be rewritten as

$$\mathcal{A}(x)t^*(j) - y(x,j) = 0, \ \ y_p(x,j) = 0, p \in P_*(j), \ y_p(x,j) \geq 0, \ p \in P_0(j).$$

The last relations imply that for any $x \in X$ and any $j \in J$, it holds

$$e_p^T \mathcal{A}(x) t^*(j) = 0, \ \ p \in P_*(j), \ \ e_p^T \mathcal{A}(x) t^*(j) \geq 0, \ p \in P_0(j). \tag{16}$$

Notice that $(e_p - e_{s(j)}) \in L(j)$, $p \in P$, $j \in J$. Then, according to Definition 2.4 and the definition of the sets $P_{0*}(j)$, $j \in J$, the equalities $(e_p - e_{s(j)})^T \mathcal{A}(x) t^*(j) = 0, \ \ p \in P_{0*}(j)$,

$j \in J$, should take place for all $x \in X$. These equalities and that from (16) imply

$$e_p^T \mathcal{A}(x) t^*(j) = 0,\ p \in P \setminus P_{00}(j). \tag{17}$$

It follows from (16) and (17) that problem (2) is equivalent to the following one:

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} c^T x \\ &\text{s.t.}\quad t^T \mathcal{A}(x) t \geq 0 \ \ \forall t \in \mathbb{R}^k_+, \\ e_p^T \mathcal{A}(x) t^*(j) = 0,\ p \in P \setminus P_{00}(j),\ \ &e_p^T \mathcal{A}(x) t^*(j) \geq 0,\ p \in P_{00}(j),\ j \in J. \end{aligned} \tag{18}$$

Using Proposition 2.3, it is easy to show that for $t^*(j),\ j \in J$, the set of all feasible directions $l \in L(j)$, for which $q(t^*(j), l) > 0$, can be explicitly described as follows:

$$\begin{aligned} L^*(j) :=& \{l \in \mathbb{R}^k :\ e^T l = 0,\ l_p \geq 0,\ p \in P_{0*}(j),\ l_p = 0,\ p \in P_{00}(j)\} \\ =& \{l = \sum_{p \in P \setminus (P_{00}(j) \cup \{s(j)\})} (e_p - e_{s(j)}) \alpha_p,\ \alpha_p \geq 0,\ p \in P_{0*}(j)\}. \end{aligned} \tag{19}$$

Taking into account the representation above and applying Theorem 4.2 from [26] to problem (4), we can formulate the following lemma.

Lemma 3.1. *Given the SIP problem (4), suppose that the Finiteness Assumption is fulfilled and the set T^* has the form (14). A vector $x^0 \in X$ is an optimal solution of problem (4) if and only if there exist numbers*

$$\lambda_j,\ \nu(j,p), p \in P \setminus \{s(j)\},\ \nu(j,p) \geq 0, p \in P_{00}(j),\ j \in J, \tag{20}$$

and vectors

$$\begin{aligned} &l(j,s) \in L^*(j),\ s \in S_j,\ j \in J;\ \ t(j) \in \mathbb{R}^k_+,\ j \in \bar{J}, \\ &\textit{with sets } S_j,\ j \in J, \textit{ and } \bar{J} \textit{ satisfying } \sum_{j \in J} |S_j| + |\bar{J}| \leq n, \end{aligned} \tag{21}$$

such that the following relations take place:

$$\begin{aligned} -c + \sum_{j \in J} \Bigg(\lambda_j \frac{\partial ((t^*(j))^T \mathcal{A}(x^0) t^*(j))}{\partial x} + \sum_{p \in P \setminus \{s(j)\}} \nu(j,p) \frac{\partial ((e_p - e_{s(j)})^T \mathcal{A}(x^0) t^*(j))}{\partial x} \\ + \sum_{s \in S_j} \frac{\partial ((l(j,s))^T \mathcal{A}(x^0) l(j,s))}{\partial x} \Bigg) + \sum_{j \in \bar{J}} \frac{\partial ((t(j))^T \mathcal{A}(x^0) t(j))}{\partial x} = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} \nu(j,p)(e_p - e_{s(j)})^T \mathcal{A}(x^0) t^*(j) = 0,\ p \in P_{00}(j); \\ (l(j,s))^T \mathcal{A}(x^0) l(j,s) = 0, s \in S_j, j \in J;\ \ (t(j))^T \mathcal{A}(x^0) t(j) = 0, j \in \bar{J}. \end{aligned} \tag{23}$$

Applying Lemma 3.1 and taking into account the equivalence of problems (2) and (4), we can prove the following optimality criterion for problem (2).

Theorem 3.2. *Let the Finiteness Assumption be fulfilled for the LCP problem (2). A vector $x^0 \in X$ is an optimal solution if and only if there exist vectors*

$$l(j) \in \tilde{L}(j), j \in J;\ \ \tau(j) \in \mathbb{R}^k_+, j \in I,\ \ |I| \le n, \tag{24}$$

such that

$$-c_i + \Omega \bullet A_i = 0, i = 1, 2, ..., n,\ \ \Omega \bullet \mathcal{A}(x^0) = 0, \tag{25}$$

$$\textit{with } \Omega = \sum_{j \in J} (l(j)(t^*(j))^T + t^*(j)(l(j))^T) + \sum_{j \in I} \tau(j)(\tau(j))^T. \tag{26}$$

Proof. Given $j \in J$, denote

$$\tilde{L}(j) := \{l \in \mathbb{R}^k : l_p \ge 0,\ p \in P_{00}(j)\}. \tag{27}$$

Let us show that conditions (22) and (23) with the scalars and vectors defined in (20) and (21), are equivalent to the following ones:

$$\begin{aligned} -c + \sum_{j \in J} \frac{\partial((l(j))^T \mathcal{A}(x^0) t^*(j))}{\partial x} + \sum_{j \in I} \frac{\partial((\tau(j))^T \mathcal{A}(x^0) \tau(j))}{\partial x} = 0, \\ (l(j))^T \mathcal{A}(x^0) t^*(j) = 0,\ j \in J;\ \ \ (\tau(j))^T \mathcal{A}(x^0) \tau(j) = 0, j \in I, \end{aligned} \tag{28}$$

with vectors (24).

First, let us show that conditions (22), (23) with numbers (20) and vectors (21) can be presented in the form (28) with some vectors (24).

Consider some index $t^*(j), j \in J$. It is evident that

$$\sum_{p \in P \setminus \{s(j)\}} \nu(j,p) \frac{\partial((e_p - e_{s(j)})^T \mathcal{A}(x^0) t^*(j))}{\partial x} = \frac{\partial((\tilde{l}(j))^T \mathcal{A}(x^0) t^*(j))}{\partial x}, \tag{29}$$

where $\tilde{l}(j) = (\tilde{l}_p(j), p \in P) \in \{l \in \mathbb{R}^k : e^T l = 0,\ l_p \ge 0, p \in P_{00}(j)\} \subset \tilde{L}(j)$,

$$\tilde{l}_p(j) := \nu(j,p),\ \ p \in P \setminus \{s(j)\},\ \ \tilde{l}_{s(j)}(j) := -\sum_{p \in P \setminus \{s(j)\}} \nu(j,p).$$

For any $\bar{l} \in L^*(j)$ and any $\beta(\bar{l}) \geq \max\limits_{p \in P_*(j)} \beta_p$, where

$$\beta_p(\bar{l}) = \infty, \ \text{ if } \bar{l}_p \geq 0; \ \beta_p(\bar{l}) = -\bar{l}_p / t^*_p(j), \ \text{ if } \bar{l}_p < 0, \ p \in P_*(j),$$

the vector $\tau := \bar{l} + \beta(\bar{l}) t^*(j)$ belongs to $\mathbb{R}^k_+$ and $\tau_p = 0$ for all $p \in P_{00}(j)$. Hence any vector $l(j,s) \in L^*(j)$ admits the following representation with $\beta(j,s) := \beta(l(j,s))$:

$$l(j,s) = t(j,s) - \beta(j,s) t^*(j), \quad \text{where } t(j,s) \in \mathbb{R}^k_+ \ \text{ and } t_p(j,s) = 0, \ p \in P_{00}(j). \tag{30}$$

Consequently,

$$\begin{aligned}\frac{\partial((l(j,s))^T \mathcal{A}(x^0) l(j,s))}{\partial x} =& \beta^2(j,s) \frac{\partial((t^*(j))^T \mathcal{A}(x^0) t^*(j))}{\partial x} \\ & - 2\beta(j,s) \frac{\partial((t(j,s))^T \mathcal{A}(x^0) t^*(j))}{\partial x} + \frac{\partial((t(j,s))^T \mathcal{A}(x^0) t(j,s))}{\partial x}.\end{aligned} \tag{31}$$

Notice that from the equalities $t^*_p(j) = 0$, $t_p(j,s) = 0, p \in P_{00}(j)$, it follows that

$$\beta t(j,s) \in \tilde{L}(j), \ \alpha t^*(j) \in \tilde{L}(j) \text{ for any } \beta \in \mathbb{R}, \ \alpha \in \mathbb{R}.$$

Then

$$l(j) := \tilde{l}(j) + \left(\sum_{s \in S_j} \beta^2(j,s) + \lambda_j \right) t^*(j) - 2 \sum_{s \in S_j} \beta(j,s) t(j,s) \in \tilde{L}(j), \tag{32}$$

and $l_p(j) = \nu(j,p) \geq 0, \ p \in P_{00}(j)$.

Further, let us show that

$$(l(j))^T \mathcal{A}(x^0) t^*(j) = 0. \tag{33}$$

By construction, $e_p^T \mathcal{A}(x^0) t^*(j) = 0, p \in P \setminus P_{00}(j)$ (see (16)). Then from (23), we conclude that $\nu(j,p) e_p^T \mathcal{A}(x^0) t^*(j) = \nu(j,p) e_{s(j)}^T \mathcal{A}(x^0) t^*(j) = 0, \ p \in P_{00}(j)$. Hence

$$(l(j))^T \mathcal{A}(x^0) t^*(j) = \sum_{p \in P} l_p(j) e_p^T \mathcal{A}(x^0) t^*(j) = \sum_{p \in P_{00}(j)} l_p(j) e_p^T \mathcal{A}(x^0) t^*(j)$$

$$= \sum_{p \in P_{00}(j)} \nu(j,p) e_p^T \mathcal{A}(x^0) t^*(j) = 0.$$

Now, let us show that

$$(l(j))^T \mathcal{A}(x^0) t(j,s) = 0, \ \ s \in S_j, \tag{34}$$

where $t(j,s) = l(j,s) + \beta(j,s)t^*(j),\ s \in S_j$ (see (30)), and vectors $l(j,s), s \in S_j$, are defined in (21). Taking into account that by construction,

$$(t^*(j))^T \mathcal{A}(x^0)t^*(j) = 0,\ l^T \mathcal{A}(x^0)t^*(j) = 0\ \forall l \in L^*(j),$$

and that according to (23), it holds $(l(j,s))^T \mathcal{A}(x^0)l(j,s) = 0$, one has

$$\begin{aligned}(t(j,s))^T \mathcal{A}(x^0)t(j,s) =& \beta^2(j,s)(t^*(j))^T \mathcal{A}(x^0)t^*(j)\\ &+ 2\beta(j,s)(l(j,s))^T \mathcal{A}(x^0)t^*(j) + (l(j,s))^T \mathcal{A}(x^0)l(j,s) = 0.\end{aligned}$$

Hence, equalities (34) hold true.

It follows from (29)-(32) that equality (22) can be presented in the form

$$\begin{aligned}-c + \sum_{j\in J} \frac{\partial((l(j))^T \mathcal{A}(x^0)t^*(j))}{\partial x} + \sum_{j\in J}\sum_{s\in S_j} \frac{\partial((t(j,s))^T \mathcal{A}(x^0)t(j,s))}{\partial x}&\\ + \sum_{j\in \bar{J}} \frac{\partial((t(j))^T \mathcal{A}(x^0)t(j))}{\partial x} &= 0.\end{aligned} \tag{35}$$

Let $\{\tau(j), j \in I\} := \{t(j,s), s \in S_j, j \in J;\ \ t(j), j \in \bar{J}\}$. Then, evidently, equalities (33)- (35) can be rewritten as (28) with data (24).

Now we will show that relations (28) with vectors (24) can be presented as relations (22), (23) with numbers (20) and vectors (21). Fix $j \in J$. Then any vector $l \in \tilde{L}(j)$ admits representation $l = \beta t^*(j) + \tilde{l}$, where $\tilde{l} \in \{l \in \mathbb{R}^k : e^T l = 0,\ l_p \geq 0, p \in P_{00}(j)\}, \beta = e^T l$. Hence any vector $l(j) \in \tilde{L}(j)$ can be written in the form $l(j) = \beta(j)t^*(j) + \tilde{l}(j)$ with $\tilde{l}(j) \in \{l \in \mathbb{R}^k : e^T l = 0,\ l_p \geq 0, p \in P_{00}(j)\},\ \beta(j) = e^T l(j)$. Consequently,

$$\frac{\partial((l(j))^T \mathcal{A}(x^0)t^*(j))}{\partial x} = \beta(j)\frac{\partial((t^*(j))^T \mathcal{A}(x^0)t^*(j))}{\partial x} + \frac{\partial((\tilde{l}(j))^T \mathcal{A}(x^0)t^*(j))}{\partial x}. \tag{36}$$

Notice that since $e^T\tilde{l}(j) = 0$, we have

$$\begin{aligned}\frac{\partial((\tilde{l}(j))^T \mathcal{A}(x^0)t^*(j))}{\partial x} &= \sum_{p\in P} \frac{\partial(\tilde{l}_p(j)e_p^T \mathcal{A}(x^0)t^*(j))}{\partial x}\\ &= \sum_{p\in P\setminus\{s(j)\}} \frac{\partial(\tilde{l}_p(j)(e_p - e_{s(j)})^T \mathcal{A}(x^0)t^*(j))}{\partial x},\end{aligned} \tag{37}$$

where $s(j)$ is some coordinate from the set $P_*(j)$. Let us prove that

$$\tilde{l}_p(j)(e_p - e_{s(j)})^T \mathcal{A}(x^0)t^*(j) = 0,\ p \in P_{00}(j). \tag{38}$$

It was shown above that $(t^*(j))^T \mathcal{A}(x^0)t^*(j) = 0$ and, according to (28),

$(l(j))^T\mathcal{A}(x^0)t^*(j)=0$. Hence $\left(\tilde{l}(j)\right)^T\mathcal{A}(x^0)t^*(j)=0$ and therefore

$$0=\left(\tilde{l}(j)\right)^T\mathcal{A}(x^0)t^*(j)=\sum_{p\in P}\tilde{l}_p(j)e_p^T\mathcal{A}(x^0)t^*(j)=\sum_{p\in P_{00}(j)}\tilde{l}_p(j)e_p^T\mathcal{A}(x^0)t^*(j). \tag{39}$$

By construction, $\tilde{l}_p(j)\geq 0$, $e_p^T\mathcal{A}(x^0)t^*(j)\geq 0$, $p\in P_{00}(j)$. Then, it follows from (39) that $\tilde{l}_p(j)e_p^T\mathcal{A}(x^0)t^*(j)=0, p\in P_{00}(j)$. Taking into account that $s(j)\in P_*(j)$ and, hence $e_{s(j)}^T\mathcal{A}(x^0)t^*(j)=0$, we conclude that (38) holds true.

Finally, let us set

$$S_j:=\emptyset,\lambda_j:=\beta(j),\nu(j,p):=\tilde{l}_p(j),p\in P\setminus\{s(j)\},j\in J;\bar{J}:=I,t(j):=\tau(j),j\in I.$$

Then, it follows from (36)-(38) that relations (28) with vectors (24) can be represented in the form of relations (22), (23) with numbers (20) and vectors (21).

It is evident that the statements of the theorem follow from the proven above equivalence and the optimality criterion in the form of Lemma 3.1. □

The optimality conditions proved in Theorem 3.2 are formulated for any set of immobile indices (either empty or not). The only assumption that is done in the theorem is a not strong assumption that the set of immobile indices is finite. In our future work, we intend to show that this assumption can be omitted.

Notice that in the case $T^*=\emptyset$, i.e. $J=\emptyset$, Theorem 3.2 coincides with Theorem 2.2 and provides the same (KKT) conditions since the first term in (26) vanishes and the matrix (26) takes the form $\Omega=\sum\limits_{j\in I}\tau(j)(\tau(j))^T\in\mathcal{C}^*$.

In the case $T^*\neq\emptyset$, Theorem 3.2 gives more general optimality conditions than Theorem 2.2 since the fulfillment of conditions of Theorem 2.2 implies the fulfillment of conditions of Theorem 3.2, but not vice versa. Let us illustrate this with an example.

Let us consider an LCP problem (2) with the following data:

$$n=5,\ k=5,\ c=(2.12,\ \ 1.24,\ \ -1.12,\ \ -3.48,\ \ 0.12)^T,$$

$$A_0=\begin{pmatrix}1&-1&0&0&0\\-1&1&0&0&0\\0&0&2&0&0\\0&0&0&2&0\\0&0&0&0&0.5\end{pmatrix},\ A_1=\frac{1}{2}\begin{pmatrix}2&-1&1&0&1\\-1&0&0&1&0\\1&0&0&1&1\\0&1&1&0&0\\1&0&1&0&0\end{pmatrix},\ A_2=\frac{1}{2}\begin{pmatrix}0&1&0&1&0\\1&-2&0&-1&1\\0&0&0&1&0\\1&-1&1&0&0\\0&1&0&0&0\end{pmatrix},$$

$$A_3=\frac{1}{2}\begin{pmatrix}2&-1&0&1&0\\-1&0&0&0&0\\0&0&2&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix},A_4=\frac{1}{2}\begin{pmatrix}0&1&0&1&1\\1&-2&0&0&-2\\0&0&0&0&1\\1&0&0&-2&0\\1&-2&1&0&2\end{pmatrix},A_5=\frac{1}{2}\begin{pmatrix}2&-1&1&0&-1\\-1&0&0&0&1\\1&0&0&0&-2\\0&0&0&0&1\\-1&1&-2&1&0\end{pmatrix}.$$

It is easy to check that for $t^* = (\frac{1}{2},\ \frac{1}{2}, 0, 0, 0)^T$, we have $(t^*)^T A_j t^* = 0, j = 0, 1, ..., 5$. Hence $(t^*)^T \mathcal{A}(x) t^* = 0$ for any $x \in \mathbb{R}^5$. This implies that the index t^* is immobile in our problem. On the other hand, one can check that vector $\bar{x} = (2, 0, 0, -1, -1)^T$ is a feasible solution and $t^T \mathcal{A}(\bar{x}) t > 0\ \ \forall t \in T \setminus \{t^*\}$. Hence, in this example, the set of immobile indices consists of a unique index $t^* =: t^*(1)$. Since $e_p^T \mathcal{A}(\bar{x}) t^* > 0, p = 3, 4, 5$, then $P_*(1) = \{1, 2\}$, $P_{0*}(1) = \emptyset$, $P_{00}(1) = \{3, 4, 5\}$. Notice that $T^* = \{t^*(1)\} \neq \emptyset$, and the constraints of our problem do not satisfy the Slater condition.

It is possible to verify that vector $x^0 = (1,\ -1,\ 1,\ 0,\ -1)^T$ is an optimal solution and the corresponding active index set is as follows:

$$T_a(x^0) := \{t \in T : t^T \mathcal{A}(x^0) t = 0\} = \{t(\alpha) := \alpha \bar{t} + (1 - \alpha) t^*,\ \alpha \in [0, 1]\},$$

where $\bar{t} = (0, \frac{1}{3},\ 0,\ 0,\ \frac{2}{3})^T$ and $e_p^T \mathcal{A}(x^0) t^* = 0, p = 1, 2, 3, 5$; $e_4^T \mathcal{A}(x^0) t^* = 2 > 0$.

First, notice that for x^0, the optimality conditions formulated in Theorem 2.2 are not satisfied. In fact, from the condition

$$\Omega \bullet \mathcal{A}(x^0) = 0 \text{ with } \Omega \in conv\{tt^T : t \in \mathbb{R}^k_+\}, \tag{40}$$

it follows that $\Omega \in conv\{\beta tt^T : t \in T_a(x^0),\ \ \beta \geq 0\}$. It is easy to check that $(t(\alpha))^T A_1 t(\alpha) = -(t(\alpha))^T A_3 t(\alpha) = \alpha(1 - \alpha)/3, \alpha \in [0, 1]$. Hence, for all $\tau \in T_a(x^0)$, we have $\tau^T A_1 \tau = -\tau^T A_3 \tau$. Consequently, for any Ω satisfying (40), it holds $\Omega \bullet A_1 = -\Omega \bullet A_3$. But $c_1 = 2.12 \neq -c_3 = 1.12$. Hence conditions (9) of Theorem 2.2 can not be satisfied, in spite of the fact that the active index set $T_a(x^0)$ consists of an infinite number of elements.

Now, let us show that the optimality conditions of Theorem 3.2 hold true. In fact, one can check that conditions (25), (26) are satisfied with $J = \{1\}$, $I = \{1\}$,

$$l(1) = (-2,\ 0,\ 2,\ 0,\ 4)^T \in \tilde{L}(1) = \{l \in \mathbb{R}^5 : l_p \geq 0, p \in P_{00}(1)\},$$
$$\tau(1) = \sqrt{3}(0.2,\ 0.4,\ 0,\ 0,\ 0.4)^T \in \mathbb{R}^5_+;\ \ t^*(1) = (1/2,\ 1/2,\ 0,\ 0,\ 0)^T,$$

and $\Omega = l(1)(t^*(1))^T + t^*(1)(l(1))^T + \tau(1)(\tau(1))^T$.

This example illustrates a situation where the new optimality conditions of Theorem 3.2 permit to reveal the optimality of some given solution of the LCP problem, but the conditions of Theorem 2.2 do not allow to do this. Notice that since Theorem 3.2 is a criterion, it will always (under the Finiteness Assumption) detect the optimality / non-optimality of a given feasible solution.

4. Dual formulations: the standard Lagrangian dual and the regularized dual problems

In this section, we will discuss some dual formulations of the LCP problem (2).

The (standard) Lagrangian dual problem for (2) is as follows [27]:

$$\max_{W}(-W \bullet A_0), \quad \text{s.t.}\ -c_i + W \bullet A_i = 0, i = 1, 2, ..., n;\ W \in \mathcal{C}^*, \tag{41}$$

where, as above, the cone $\mathcal{C}^* = conv\ \{ll^T : l \in \mathbb{R}^k_+\}$ is dual to $\mathcal{C}$.

It is well known [see 27, Theorem 3.1.] that if the constraints of problem (2) satisfy the Slater condition, then there is no gap between the optimal values of problems (2) and (41).

If the constraints of problem (2) do not satisfy the Slater condition, then the positive gap is possible. Notice that it may happen even in the case when problem (2) has an optimal solution.

Suppose that the Finiteness (or the Isolation) Assumption is satisfied and the set T^* has the form (14). Given $j \in J$, consider the set $\tilde{L}(j)$ defined in (27), and the following closed cone:

$$\begin{aligned} K(j) :&= \{D \in \mathcal{S}(k) : e_p^T D t^*(j) = 0, p \in P_*(j) \cup P_{0*}(j),\ e_p^T D t^*(j) \geq 0, p \in P_{00}(j)\} \\ &= \{D \in \mathcal{S}(k) :\ l^T D t^*(j) \geq 0\ \ \forall l \in \tilde{L}(j)\}. \end{aligned}$$

Notice that all cones $K(j),\ j \in J$, as well as the cone of copositive matrices $\mathcal{C}$ defined in (1), are convex and closed. It is easy to show that for any $j \in J$, the dual cone of $K(j)$ has the form $K^*(j) = \{l(t^*(j))^T + t^*(j)l^T :\ l \in \tilde{L}(j)\}$.

Denote:

$$\mathcal{K} := (\bigcap_{j \in J} K(j)) \cap \mathcal{C}.$$

It should be noted here that the cone $\mathcal{K}$ is a face of $\mathcal{C}$ and $\mathcal{F} \subset \mathcal{K}$, where

$$\mathcal{F} := \{\mathcal{A}(x),\ x \in X\}. \tag{42}$$

It is known [see e.g. 4,28] that given a family of closed convex cones $\mathcal{E}_i$, $i = 1, ..., m$, it holds $\left(\bigcap\limits_{i=1}^{m} \mathcal{E}_i\right)^* = cl\left(\sum\limits_{i=1}^{m} \mathcal{E}_i^*\right)$, where $cl(\mathcal{D})$ stays for the closure of a set $\mathcal{D}$. Hence the dual cone of $\mathcal{K}$ has the form $\mathcal{K}^* = cl\left(\sum\limits_{j \in J} K^*(j) + \mathcal{C}^*\right)$.

Recall that, as it was shown in section 2, the primal LCP problem in the form (2) is equivalent to problem (18). Taking into account the notation introduced above, problem (18) can be rewritten as

$$\min_{x}\ c^T x \quad \text{s.t. } \mathcal{A}(x) \in \mathcal{K}. \tag{43}$$

Let us designate problem (43) as a *regularized primal problem*. Its dual (*regularized dual* *problem*) has the form

$$\max_W(-W \bullet A_0), \quad \text{s.t.} \ -c_i + W \bullet A_i = 0, i = 1, 2, ..., n, \ W \in \mathcal{K}^*. \tag{44}$$

It can be shown that for any feasible solution x^* of problem (43) and for any feasible solution W^* of problem (44) the following inequality (weak duality) holds:

$$c^T x^* \geq -W^* \bullet A_0.$$

It is easy to verify that under the Finiteness (Isolation) Assumption, for any optimal solution x^0 of problem (43), there exists a feasible solution W^0 of the corresponding dual problem (44) such that the strong duality property holds:

$$c^T x^0 = -W^0 \bullet A_0.$$

Indeed, here we can set matrix W^0 to be equal to matrix Ω defined in (26) for which, according to Theorem 3.2, conditions (25) hold.

Thus, we have proved the following proposition.

Proposition 4.1. *Suppose that the Finiteness (Isolation) Assumption is satisfied and the primal LCP problem (2) has an optimal solution. Then*

- *an optimal solution of the dual regularized problem (44) exists, and*
- *there is no gap between the optimal values of problem (2) ((43)), and its dual problem (44).*

Hence, the strong duality takes place for the primal problem (2) and its regularized dual problem (44), while, as it was mention above, for the pair constituted by the primal problem (2) and its (standard) Lagrangian dual problem (41), the strong duality may fail.

Let us illustrate these conclusions by an example which is a slight modification of the Example 2.2 from [30].

Consider an LCP problem in the form (2) with the following data:

$$n = 2, \ k = 3, c = (0, \ -1)^T;$$
$$A_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{45}$$

where $a > 0$.

For the $\mathcal{A}(x)$ constructed by formula (3) and for $t^* = (0, 0, 1)^T$, we have $(t^*)^T \mathcal{A}(x) t^* = 0, \ e_1^T \mathcal{A}(x) t^* = 0$ for all $x \in \mathbb{R}^2$.

It is easy to check that vector $\bar{x} = (-1, -1)^T$ is feasible in problem (2) with data (45), $t^T \mathcal{A}(\bar{x}) t > 0$ for all $t \in \mathbb{R}^3_+ \setminus \{t^*\}$, and $e_2^T \mathcal{A}(\bar{x}) t^* > 0$. Therefore, in this problem there is only one immobile index $t^*(1) := t^*$ and the corresponding coordinate sets defined in Section 3 are as follows: $P_*(1) = \{3\},\ P_{0*} = \{1\},\ P_{00} = \{2\},\ \tilde{L}(1) = \{l \in \mathbb{R}^3 : l_2 \geq 0\}$.

Vector $x^0 = (-1, 0)^T$ is an optimal solution of this problem and the optimal value of the cost function is equal to $val(P) = c^T x^0 = 0$.

Let us consider the corresponding Lagrangian dual problem (41),

$$\max_W(-W \bullet A_0), \quad \text{s.t. } W \bullet A_1 = 0,\ W \bullet A_2 = -1,$$
$$\text{with } W := \sum_{s \in S} t(s)(t(s))^T,\ t(s) \in \mathbb{R}^3_+,\ s \in S,$$

for some finite index set $S : |S| < \infty$. For data (45) this problem takes the form

$$\max(-a \sum_{s\in S} t_1^2(s)), \tag{46}$$
$$\text{s.t. } \sum_{s\in S} t_2^2(s) = 0; \sum_{s\in S}(-t_1^2(s) - 2t_2(s)t_3(s)) = -1, t_i(s) \geq 0, s \in S;\ i = 1, 2, 3.$$

It follows from the constraints of the dual problem above that for any dual feasible solution it holds $t_2(s) = 0, s \in S$, and $\sum\limits_{s\in S} t_1^2(s) = 1$. Hence, the optimal value of the cost function in problem (46) is equal to $val(D) = -a < 0$. Consequently, the duality gap is positive: $val(P) - val(D) = a > 0$.

Now, let us consider the regularized dual problem (44) with data (45). It is easy to check that for the matrix in the form

$$W^0 = l(1)(t^*(1))^T + t^*(1)(l(1))^T + \tau(1)(\tau(1))^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 1 \end{pmatrix}$$

with $l(1) = (0, 1/2, 0)^T \in \tilde{L}(1)$ and $\tau(1) = t^*(1) \in \mathbb{R}^3_+$, we have

$$-W^0 \bullet A_0 = 0, W^0 \bullet A_1 = c_1 = 0, \text{ and } W^0 \bullet A_2 = c_2 = -1.$$

Hence, W^0 is an optimal solution of the regularized dual problem and the optimal value of the cost function in this problem is equal to 0. Consequently, there is no duality gap between the primal LCP problem with data (45) and the corresponding regularized dual problem in the form (44).

The main contribution of this section consists in the formulation of the new (regularized) dual problem (44) for the LCP problem (2). This dual problem is constructed using the information about the immobile indices of the constraints of the primal problem. Under the Finiteness (or equivalently, Isolation) Assumption, Proposition 4.1 guarantees zero duality gap. These duality results may be used for constructing efficient numerical methods for LCP.

It is worth to be mentioned that the dual formulations obtained here correlate with those from [22,31], where CQ- free optimality conditions for a more general conic optimization problem were obtained by using the so-called *minimal representation* of the cone. Being applied to the LCP problem (2), these results consist of the following. The original (primal) problem (2) is replaced by the equivalent regularized primal problem in the form

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{s.t. } \mathcal{A}(x) \in \mathcal{C}_{min}, \tag{47}$$

where $\mathcal{C}_{min}$ is the minimal face of the cone $\mathcal{C}$ generated by the set $\mathcal{F}$ defined in (42). Then the regularized dual problem has the form

$$\max_{W}(-W \bullet A_0), \quad \text{s.t. } -c_i + W \bullet A_i = 0, i = 1, 2, ..., n, \ W \in \mathcal{C}^*_{min}, \tag{48}$$

where $\mathcal{C}^*_{min}$ is the dual cone to $\mathcal{C}_{min}$. Under the assumption that the primal problem has a finite optimal value, it was proved in [31] that for the dual pair (47) and (48), the strong duality holds, i.e. there is no positive duality gap and the dual optimal value is attained.

Notice here that in order to be able to efficiently apply these results, one should know explicit descriptions of the cones $\mathcal{C}_{min}$ and $\mathcal{C}^*_{min}$ which are not provided in [22,31]. To the best of our knowledge, explicit descriptions of the minimal cone $\mathcal{C}_{min}$ and its dual one $\mathcal{C}^*_{min}$ are known for SDP problems (see [21] and [25]), but not for LCP.

Therefore, given an LCP problem, the main difference between the previous formulations and those obtained in this section, is as follows: in [22,31], the regularization of the dual problem is based on the minimal representation of the cone of constraints, but this representation is defined *implicitly*, while the regularization based on the concepts of the immobile indices allows to *explicitly* describe the cones $\mathcal{K}$ and $\mathcal{K}^*$ and to obtain optimality conditions which are applicable even when the classical ones fail.

Notice that in [27], some other duality relations for the pair of problems (2) and (41) are considered under the assumption that either the Slater condition (5) or the following one:

$$c \in int\ M \quad \text{with } M := cone\{a(t), t \in T\}, \ \ a(t) := (t^T A_1 t, ..., t^T A_n t)^T \in \mathbb{R}^n, \tag{49}$$

is satisfied. In Proposition 4.1 we do not require the fulfillment of any of these conditions. For instance, in the above example, the Slater condition and condition (49) are not fulfilled since $T^* \neq \emptyset$ and $c \notin int\ M$.

5. Conclusions

The main contribution of the work consists in the successful application of the new approach to optimality conditions, first developed for convex SIP problems, to the problems of LCP. This approach, based on the concept of the immobile indices, has permitted us to prove for the LCP problem (2) the first order optimality criterion without the commonly used Slater

CQ. The only assumption that we have done here is that the set of immobile indices (in the original LCP problem) consists of isolated points and, hence, is finite.

The results of the paper permit us to conclude that the idea of using the immobile indices for the derivation of new optimality conditions, effectively works not only in SIP, but also in LCP. Moreover, this approach to optimality conditions may be productive for wider classes of optimization problems.

The concept of immobile indices allowed us to formulate a new regularized dual problem for the primal LCP problem (2). Under a condition that this problem has an optimal solution, the duality gap between the optimal values of the cost functions in the primal problem and the regularized dual problem (44) vanishes and the dual optimal value is attained. This permits one to judge about the benefits of using the immobile indices in dual formulations.

In the future, we plan to generalize the results of the paper and obtain new optimality conditions for LCP without the Finiteness Assumptions and/or the equivalent Isolation Assumption as well as without other special conditions for the constraints of the problem. Namely, we intend to prove the conjecture:

Let T^* be the set of immobile indices in the LCP problem (2) and $t^*(j)$, $j \in J$, be the set of vertices of the bounded polyhedron $\mathrm{conv}T^*$. *Then Theorem 3.2 is true without Finiteness Assumption.*

For LPC problems, we intend to develop the *no- gap* duality theory as it is done in [21]. We plan also to extend our approach to new classes of CP problems, as well as to other optimization problems that admit copositive and conic reformulations.

Acknowledgements

This work was partially supported by the state research program "Convergence"(Republic Belarus), Task 1.3.01 and by Portuguese funds through CIDMA - Center for Research and Development in Mathematics and Applications, and FCT - Portuguese Foundation for Science and Technology, within the project UID/MAT/04106/2013.

We thank the anonymous referees for their very helpful comments and suggestions which aided us in considerably improving the presentation of this paper.

References

[1] Motzkin TS. Copositive quadratic forms. National Bureau of Standartds report. 1952; 1818:11–22.

[2] Motzkin TS, Straus EG. Maxima for graphs and a new proof of a theorem of Turán. Canadian Journal of Mathematics 1965; 17:533–540.

[3] Preisig JC. Copositivity and the minimization of quadratic functions with nonnegativity and quadratic equality constraints. SIAM Journal on Control and Optimization 1996; 34:1135–1150.

[4] Bomze IM, Dür M, de Klerk E, Roos C, Quist AJ, Terlaky T. On copositive programming and standard quadratic optimization problems. Journal Global Optim. 2000; 18:301–320.

[5] Hiriart-Urruty J.B., Seeder A. A variational approach to copositive matrices. SIAM Review. 2010; 52:593-629.

[6] Berman A., Shaked-Monderer N. Completely positive matrices. World Scientific Publishing Co.Inc. 2003.River Edge, NJ.

[7] Berman A., Dür M., Shaked-Monderer N. Open problems in the theory of completely positive and copositive matrices. Electronic Journal of Linear Algebra. 2015; 29:46-58.

[8] Burer S. On the copositive representation of binary and continuous nonconvex quadratic programs. Math. Program. 2009; 120:479-495.

[9] Dür M. Copositive Programming – a Survey. In: Diehl M, Glineur F, Jarlebring E, Michielis W., editors. Recent advances in Optimization and its applications in Engineering; 2010; Springer-Verlag Berlin Heidelberg X1: 535 p.

[10] Amaral P., Bomze I., Júdice J. Copositivity and constrained fractional quadratic problems. Math. Program. 2014; 146(1-2):325-350.

[11] de Klerk E, Pasechnik DV. Approximation of the stability number number of a graph via copositive programming. SIAM J. Optim. 2002; 12:875–892.

[12] Bomze IM. Copositive optimization - Recent developments and applications. EJOR 2012; 216(3):509–520.

[13] Murtu KG, Kabadi SN. Some NP-complete problems in quadratic and nonlinear programming. Math. Program. 1987; 39:117–129.

[14] Bunfuss S, Dür M. Algorithmic copositivity detection by simplicial partition. Linear Algebra Appl. 2008; 428:1511–1523.

[15] Dickinson P.J.C., Hildebrand R. Considering copositivity locally. J. Math. Anal. Appl. 2016; 437:1184-1195.

[16] Yang S.-j., Li X.-x. Algorithms for determining the copositivity of a given symmetric matrix. Linear Algebra Appl. 2009; 430:609-618.

[17] Bomze I., Schachinger W., Uchida G. Think co(mpletely)positive !-matrix properties, examples and a clustered bibliography on copositive optimization. J. Global Optim. 2012; 52:423-445.

[18] Bonnans JF., Shapiro A. Perturbation Analysis of Optimization Problems. New-York (NY): Springer-Verlag; 2000.

[19] Kostyukova OI, Tchemisova TV. Optimality conditions for convex Semi-Infinite Programming problems with finitely representable compact index sets. J. Optim. Theory Appl. 2017; 175(1):76-103.

[20] Ye JJ. Constraint Qualifications and necessary optimality conditions for optimization problems with variational inequality constraints. SIAM J. Optim. 2000; 10(4):943-962.

[21] Ramana MV. An exact duality theory for Semidefinite Programming and its complexity implications. DIMACS Technical report 95-02R 1995; RUTCOR, Rutgers University, New Brunswick (NJ).

[22] Tunçel L, Wolkowicz H. Strong duality and minimal representations for cone optimization. Comput. Optim. Appl. 2013; 53:619–648.

[23] Kostyukova OI, Tchemisova TV. Implicit optimality criterion for convex SIP problem with box constrained index set. TOP 2012; 20(2):475-502.

[24] Kostyukova OI, Tchemisova TV. A constructive algorithm for determination of immobile indices in convex SIP problems with polyhedral index sets. Working paper, University of Aveiro 2012. Available from: http://hdl.handle.net/10773/8888.

[25] Kostyukova OI, Tchemisova TV. Optimality Criteria without Constraint Qualification for Linear Semidefinite Problems. Special Issue "Algebraic Techniques in Graph Theory and Optimisation", JMS- Journal of Mathematical Sciences 2012; 182(2):126-143.

[26] Kostyukova OI, Tchemisova TV. On a constructive approach to optimality conditions for convex SIP problems with polyhedral index sets. Optimization 2014; 63(1):67–91.

[27] Ahmed F, Dür M, Still G. Copositive Programming via Semi-Infinite Optimization. J. Optim. Theory Appl. 2013; 159:322–340.

[28] Berman A, Shaked-Monderer N. Completely Positive Matrices. World Scientific; 2003.

[29] Goberna MA, López MA. Linear semi-infinite optimization. Wiley, Chichester; 1998.

[30] Shapiro A. Duality, optimality conditions and perturbation analysis. In: Saigal R, Vandenberghe L, Wolkowicz H., editors. Semidefinite Programming and Applications Handbook; 2000. Kluwer Academic Publishers, Boston:67–92.

[31] Borwein JM, Wolkowicz H. Characterization of optimality for the abstract convex program with finite-dimensional range. J. Aust.Math.Soc., Ser.A 1980/1981; 30(4):390–411.