Towards a Geometric Theory for nD Behaviors: Conditioned Invariance and Detectability Subspaces ⋆

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Abstract: We introduce the definitions of observer, conditioned invariance and detectability subspaces for discrete multidimensional behavioral systems, based on our previous work for the continuous 1D case, as a step forward in the attempt to develop a geometric theory for nD behaviors.

Keywords: Multidimensional behavioral systems, invariance, observer, conditioned-invariance, detectability

1. INTRODUCTION

The properties of conditioned invariance and detectability subspaces have been introduced in [Pereira and Rocha 2017] for one-dimensional (1D) behavioral continuous time systems, based on the theory of 1D behavioral observers developed in [Valcher and Willems 1999] and [Trumpf et al. 2011]. The definitions proposed in that paper tried to incorporate the ones given for classical state space systems, [Basile and Marro 1969], [Trentelman et al. 2001] into the behavioral framework basically by replacing subspaces of the state space by sub-behaviors of relevant behaviors.

Important contributions to the development of a geometric theory for discrete multidimensional systems have been made based on multidimensional state space models such as the (2D) Fornasini-Marchesini model, [Conte and Perdon 1988].

Here we generalize the behavioral definitions and results of [Pereira and Rocha 2017] to discrete multidimensional (nD) systems. Whereas nD conditioned invariance can be defined and characterized similarly to the 1D case, requiring only some small technical adjustments, the same does not apply to nD detectability subspaces. This is due to the fact that an underlying stability notion is needed, and it is not clear in what extent the different notions of stability for multidimensional behavioral systems, see for instance [Valcher 2000], [Pillai and Shankar 1998] and [Rocha 2008], generalize the definition given for the one-dimensional case, [Polderman and Willems 1998].

In this paper we adopt the definition of nD behavioral stability with respect to a cone S introduced in [Rocha 2008], which is the discrete counterpart of the definition of [Pillai and Shankar 1998] for the continuous nD case, since the definition of [Valcher 2000] is only given for the discrete 2D case and the corresponding results seem difficult to generalize for $n > 2$.

The structure of our presentation is as follows. Section 2 contains the background material; nD behavioral observers are introduced in Section 3, while Sections 4 and 5 are respectively devoted to conditioned invariance and detectability subspaces; finally Section 6 contains some concluding remarks.

2. PRELIMINARIES

In this paper we consider discrete multidimensional systems over $\mathbb{Z}^n$ with behavior that can be described as the solution set of a system of linear partial difference equations with constant coefficients:

$$H(\underline{\sigma}, \underline{\sigma}^{-1})w \equiv 0, \tag{1}$$

where $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n)$, $\underline{\sigma}^{-1} = (\sigma_1^{-1}, \ldots, \sigma_n^{-1})$, the σ_i's are the elementary nD shift operators (defined by $\sigma_i w(\underline{k}) = w(\underline{k} + e_i)$, for $\underline{k} \in \mathbb{Z}^n$, where e_i is the i-th element of the canonical basis of $\mathbb{R}^n$), $H(\underline{s}, \underline{s}^{-1})$ is an nD Laurent-polynomial matrix in the indeterminates $\underline{s} = (s_1, \ldots, s_n)$, and w - the system variable - is a vector function with $\mathtt{w}$ components that correspond to the system signals. The behavior $\mathcal{B}_w$ described by (1) is the kernel of the shift operator $H(\underline{\sigma}, \underline{\sigma}^{-1})$ acting on the (vector) signal universe $\mathcal{U}_w$, which is here considered to be equal to $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^n}$, i.e.:

$$\mathcal{B} = \ker H(\underline{\sigma}, \underline{\sigma}^{-1}) := \{w \in \mathcal{U}_w \mid H(\underline{\sigma}, \underline{\sigma}^{-1})w \equiv 0\}. \tag{2}$$

For this reason $\mathcal{B}_w$ is often referred to as a *kernel behavior*. Throughout this paper, the term behavior will always mean kernel behavior. Moreover, for the sake of simplicity, whenever no confusion arises, we shall omit the indeterminates $(\underline{s}, \underline{s}^{-1})$ when writing an nD Laurent-polynomial matrix as well as the

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elementary shift operators $(\underline{\sigma}, \underline{\sigma}^{-1})$ when writing the corresponding nD shift operator.

It was shown in [Oberst 1990, Zerz 2000] that given two behaviors $\mathcal{B}_w^1$ and $\mathcal{B}_w^2$ such that $\mathcal{B}_w^1 = \ker H_1 \subset \mathcal{B}_w^2 = \ker H_2$, there exists an nD Laurent-polynomial matrix E such that:

$$H_2 = EH_1. \tag{3}$$

Moreover, the quotient $\mathcal{B}_w^2/\mathcal{B}_w^1$ has the structure of a behavior and can be identified with $\ker M$ where:

$$M = \begin{bmatrix} E \\ L \end{bmatrix}, \tag{4}$$

and L is a minimal left annihilator (MLA) [1] of H_1, [Rocha and Wood 2001].

According to the behavioral approach, the system variable w is not *a priori* partitioned into inputs and outputs. However, we may be interested in splitting it into *sub-variables* (not necessarily inputs and outputs) depending on the different problems that we wish to study. In other situations, as for instance when a system is obtained by the interconnection of other systems, the overall system variable may be composed by joining the system variables of the elementary systems, and hence be naturally divided into sub-variables.

A behavior $\mathcal{B}_{(w_1,w_2)}$ with partioned variable (w_1, w_2) can be described by an equation of the type:

$$H_2(\underline{\sigma}, \underline{\sigma}^{-1})w_2 = H_1(\underline{\sigma}, \underline{\sigma}^{-1})w_1, \tag{5}$$

where, for $i = 1, 2$, $H_i(\underline{s}, \underline{s}^{-1})$ is an nD Laurent-polynomial matrix of size $g \times \mathtt{w}_i$, with $\mathtt{w}_i$ equal to the size of w_i and $g \in \mathbb{N}$.

According to the variable *elimination property* obtained in [Oberst 1990], the projection of $\mathcal{B}_{(w_1,w_2)}$ into (for instance) w_2, defined as:

$$\Pi_2(\mathcal{B}_{(w_1,w_2)}) = \{w_2 \in (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}^n} \mid \exists\, w_1 : (w_1, w_2) \in \mathcal{B}_{(w_1,w_2)}\} \tag{6}$$

is also a kernel behavior (with variable w_2), say $\mathcal{B}_{w_2}$. More concretely, a description of $\mathcal{B}_{w_2}$ may be obtained by applying to both sides of Equation (5) an operator $L(\underline{\sigma}, \underline{\sigma}^{-1})$ such that $L(\underline{s}, \underline{s}^{-1})$ is a MLA of $H_1(\underline{s}, \underline{s}^{-1})$, yielding as description for $\mathcal{B}_{w_2}$:

$$L(\underline{\sigma}, \underline{\sigma}^{-1})H_2(\underline{\sigma}, \underline{\sigma}^{-1})w_2 \equiv 0, \tag{7}$$

i.e., $\mathcal{B}_{w_2} = \ker LH_2$. Obviously, the same applies to w_1.

One of the behavioral properties that plays a crucial role in this paper is *autonomy*. We say that a behavior is *autonomous* if it has no free variables, more concretely, its projection on the i-th component is not free, i.e., if:

$$\Pi_i(\mathcal{B}_w) \neq \mathbb{R}^{\mathbb{Z}^n},\ i = 1, \ldots, \mathtt{w}.$$

$\mathcal{B}_w = \ker R(\underline{\sigma}, \underline{\sigma}^{-1})$ is autonomous if and only if $R(\underline{s}, \underline{s}^{-1})$ has full column rank (over $\mathbb{R}[\underline{s}, \underline{s}^{-1}]$), [Zerz 2000].

Another important property is *stability*. As it is well known, there exist different ways of defining stability for nD systems. Here we adopt the definition of stability with respect to a specified stability region introduced in [Rocha 2008], which is the discrete version of the definition given in [Pillai and Shankar 1998] for the continuous case.

[1] Given two nD Laurent-polynomial matrices L and H, L is said to be a minimal left annihilator of H if L is an annihilator of H, i.e., $LH = 0$ and, moreover, any other left annihilator Q of H is such that $Q = ML$ for some Laurent-polynomial matrix M [Bose 1995].

For this purpose we identify an *elementary direction* in $\mathbb{Z}^n$ with an element $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ whose components are coprime integers, and define a *direction* in $\mathbb{Z}^n$ as an integer linear combination of elementary directions. Moreover, we define a *stability cone* in $\mathbb{Z}^n$ as the set of all positive integer linear combinations of n linearly independent elementary directions [2]. A *half-line* associated with a direction $\underline{d} \in \mathbb{Z}^n$ is defined as the set of all points of the form $\alpha\underline{d}$ where α is a nonnegative integer; clearly, the half-lines in a stability cone S are the ones associated with the directions $\underline{d} \in S$. Now, stability with respect to a stability cone S is defined as follows: given a stability cone $S \subset \mathbb{Z}^n$, a (vector) signal $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^n}$ is said to be *S-stable* if it converges to zero along every half line in S. A behavior $\mathcal{B}_w$ is *S-stable* if all the signals in $\mathcal{B}_w$ are *S-stable*.

It turns out that every nD kernel behavior $\mathcal{B}_w \subset (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^n}$ which is stable with respect to some stability cone S is a finite dimensional linear subspace of the trajectory universe, $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^n}$, [Rocha 2008]. Finite dimensional nD behaviors are known as *strongly autonomous* [Pillai and Shankar 1998]. Thus, the notion of stability used here excludes the class of infinite dimensional autonomous behaviors. The definition of stability used in [Valcher 1998] does not leave out this class of systems, but is focussed on 2D behaviors and seems to be somewhat difficult to generalize to the higher dimensional case.

The S-stability of a behavior $\mathcal{B}_w = \ker R(\underline{\sigma}, \underline{\sigma}^{-1})$ can be characterized in terms of the zeros [3] of the matrix R and the elementary directions of the stability cone S. It was shown in [Rocha 2008] that $\mathcal{B}_w = \ker R(\underline{\sigma}, \underline{\sigma}^{-1})$ is S-stable if and only if $|\underline{\lambda}^{\underline{d}}| < 1$, for every zero $\underline{\lambda}$ of R and every elementary direction $\underline{d}$ in S, where $\underline{\lambda}^{\underline{d}} := \lambda_1^{k_1} \ldots \lambda_n^{k_n}$. If this condition is satisfied we also say that both the zero $\underline{\lambda}$ and the Laurent-polynomial matrix R are S-stable.

3. BEHAVIORAL ND OBSERVERS

Here we adopt the definitions given in [Valcher and Willems 1999] and [Trumpf et al. 2011] for the 1D case.

Let $\mathcal{B}_{(w_1,w_2)}$ be an nD behavior where the system variable is partitioned into two sub-variables: w_1, consisting of the measured components of the signal, and w_2, consisting of the signal components to be estimated. An observer for w_2 from w_1 is a behavior $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ that shares the measured variable w_1 with $\mathcal{B}_{(w_1,w_2)}$ and "produces" a variable $\widehat{w}_2$ which is to be seen as an estimate of w_2. The quality of this estimate depends on the properties of the error $e := \widehat{w}_2 - w_2$ or, more precisely, of the properties of its behavior $\mathcal{B}_e$ (the *error behavior*).

Definition 1. Given an nD behavior $\mathcal{B}_{(w_1,w_2)}$, let $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ be a behavior such that the universe $\mathcal{U}_{\widehat{w}_2}$ coincides with the universe $(\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}^n}$ of the variable w_2. $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ is said to be:

- a *tracking observer* for w_2 from w_1 if $\mathcal{B}_e$ is autonomous;
- an *S-asymptotic observer* for w_2 from w_1 if $\mathcal{B}_e$ is S-stable.

Moreover, w_2 is said to be *trackable* from w_1 in case a tracking observer $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ for w_2 from w_1 exists. If an S-asymptotic

[2] Note that, according to this definition, a stability cone is always a "full" cone, in the sense that it contains a basis of elementary directions.

[3] Recall that a *zero* of R is defined as $\underline{\lambda} \in (\mathbb{C} \setminus \{0\})^n$ such that $\operatorname{rank} R(\underline{\lambda}, \underline{\lambda}^{-1}) < \operatorname{rank} R(\underline{s}, \underline{s}^{-1})$, where, the first rank is taken over $\mathbb{C}$ and the second one over $\mathbb{R}[\underline{s}, \underline{s}^{-1}]$.

observer $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ for w_2 from w_1 exists, w_2 is said to be S-*detectable* from w_1.

An important role in the study of observers is played by the hidden behavior. If $\mathcal{B}_{(w_1,w_2)}$ is an nD behavior with measured variable w_1 and non-measured variable w_2, the *hidden behavior* of $\mathcal{B}_{(w_1,w_2)}$ - denoted by $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right)$ - consists of all the signals w_2 that are compatible with $w_1 \equiv 0$, i.e.,

$$\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \{w_2 \mid (0, w_2) \in \mathcal{B}_{(w_1,w_2)}\}.$$

If $\mathcal{B}_{(w_1,w_2)}$ is described by

$$R_2(\underline{\sigma}, \underline{\sigma}^{-1})w_2 = R_1(\underline{\sigma}, \underline{\sigma}^{-1})w_1, \tag{8}$$

the hidden behavior is obviously equal to $\ker R_2(\underline{\sigma}, \underline{\sigma}^{-1})$, i.e.,

$$\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2.$$

Note that $\ker R_2$ coincides with the error behavior associated to the trivial observer

$$R_2(\underline{\sigma}, \underline{\sigma}^{-1})\widehat{w}_2 = R_1(\underline{\sigma}, \underline{\sigma}^{-1})w_1. \tag{9}$$

Moreover, this hidden behavior, $\ker R_2$, is contained in the error behavior of any observer for w_2 from w_1. Indeed, let $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ be such an observer, and suppose that it is described by

$$\widetilde{R}_2(\underline{\sigma}, \underline{\sigma}^{-1})\widehat{w}_2 = \widetilde{R}_1(\underline{\sigma}, \underline{\sigma}^{-1})w_1.$$

Consider also the corresponding error

$$e = \widehat{w}_2 - w_2. \tag{10}$$

Equations (8), (9) and (10) can be written as

$$\underbrace{\begin{bmatrix} -R_1 & R_2 & 0 \\ -\widetilde{R}_1 & 0 & -\widetilde{R}_2 \\ 0 & -I & I \end{bmatrix}}_{R} \begin{bmatrix} w_1 \\ w_2 \\ \widehat{w}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} e \tag{11}$$

and a description for the error behavior $\mathcal{B}_e$ may be obtained by eliminating the variables w_1, w_2 and $\widehat{w}_2$. This is achieved by applying a MLA of R, say $L = [X\ Y\ Z]$, to both sides of equation (11), which yields

$$0 = Ze, \tag{12}$$

and consequently $\mathcal{B}_e = \ker Z$. But, it follows from the fact that $LR = 0$ that $XR_2 - Z = 0$, i.e., $Z = XR_2$. Thus

$$\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2 \subset \ker Z = \mathcal{B}_e.$$

Conversely, it is not difficult to prove that if the hidden behavior $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right)$ is contained in a behavior $\mathcal{E}$, then $\mathcal{E}$ is the error behavior of some observer for w_2 from w_1. For this reason $\mathcal{E}$ is called an *achievable error behavior*. These considerations can be summarized in the following proposition, cf [Trumpf et al. 2011].

Proposition 2. Given a behavior $\mathcal{B}_{(w_1,w_2)}$, $\mathcal{E}$ is an achievable error behavior with respect to the estimation of w_2 from w_1 if and only if $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) \subset \mathcal{E}$.

This result states that the hidden behavior $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right)$ is the smallest achievable error behavior with respect to the estimation of w_2 from w_1.

Thus, clearly, w_2 is trackable from w_1 if and only if this hidden behavior is autonomous.

Corollary 3. Given a behavior $\mathcal{B}_{(w_1,w_2)}$, w_2 is trackable from w_1 if and only if $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right)$ is autonomous.

Similarly, it is not difficult to conclude that w_2 is S-detectable from w_1 if and only if the hidden behavior is S-stable.

Corollary 4. Given a behavior $\mathcal{B}_{(w_1,w_2)}$, w_2 is S-detectable from w_1 if and only if $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right)$ is S-stable.

4. CONDITIONED INVARIANCE

As it is well-known [Basile and Marro 1969, Trentelman et al. 2001], for state space systems, a subspace $\mathcal{V}_{\mathcal{X}}$ of the state space $\mathcal{X}$ is said to be conditioned invariant if there exists an observer for the state x modulo $\mathcal{V}_{\mathcal{X}}$, which means that $\mathcal{X}/\mathcal{V}_{\mathcal{X}}$ is an invariant subspace with respect to the corresponding error dynamics. In behavioral terms, according to the definition of invariance given in [Pereira and Rocha 2018], this is equivalent to saying that the quotient of the error behavior $\mathcal{E}$ by $\mathcal{V}_{\mathcal{X}}$, $\mathcal{E}/\mathcal{V}_{\mathcal{X}}$, is an autonomous behavior.

Inspired by this, together with the definition of conditioned invariance that we introduced in [Pereira and Rocha 2017], here we define conditioned invariance for nD behaviors as follows.

Definition 5. Let $\mathcal{B}_{(w_1,w_2)}$ be an nD behavior with measured variable w_1 and to-be-estimated variable w_2 in the universe $\mathcal{U}_{w_2} = (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}^n}$. A behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ is said to be *conditioned invariant* if there exists an observer behavior $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ for w_2 from w_1, with $\widehat{w}_2 \in \mathcal{U}_{w_2}$, with error behavior $\mathcal{B}_e$ such that $\mathcal{V} \subset \mathcal{B}_e$ and $\mathcal{B}_e/\mathcal{V}$ is autonomous.

In other words, roughly speaking, $\mathcal{V}$ is a conditioned invariant behavior if there exists a behavioral observer $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ for w_2 from w_1 which is a tracking observer modulo $\mathcal{V}$.

We next give a necessary and sufficient condition for conditioned invariance.

Proposition 6. Let $\mathcal{B}_{(w_1,w_2)}$ be an nD behavior with measured variable w_1 and to-be-estimated variable w_2 in the universe $\mathcal{U}_{w_2} = (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}^n}$. Assume that $\mathcal{B}_{(w_1,w_2)}$ is described by the matrix equation

$$R_2(\underline{\sigma}, \underline{\sigma}^{-1})w_2 = R_1(\underline{\sigma}, \underline{\sigma}^{-1})w_1.$$

Let further $\mathcal{V} = \ker V(\underline{\sigma}, \underline{\sigma}^{-1})$ be a sub-behavior of $\mathcal{U}_{w_2}$. Then $\mathcal{V}$ is conditioned invariant if and only if there exists an nD Laurent-polynomial $q(\underline{s}, \underline{s}^{-1}) \neq 0$ such that

$$\ker R_2(\underline{\sigma}, \underline{\sigma}^{-1}) \subset \ker q(\underline{\sigma}, \underline{\sigma}^{-1})V(\underline{\sigma}, \underline{\sigma}^{-1}).$$

Proof.

"If part:" Assume that there exists $q(\underline{s}, \underline{s}^{-1}) \neq 0$ such that $\ker R_2 \subset \ker qV =: \mathcal{E}$. From what was said in Section 2,

$$\mathcal{E}/\mathcal{V} = \ker \begin{bmatrix} qI \\ L \end{bmatrix}$$

where L is a MLA of V. Since $\begin{bmatrix} qI \\ L \end{bmatrix}$ has full column rank, then $\mathcal{E}/\mathcal{V}$ is autonomous. Moreover, note that $\mathcal{E}$ is an achievable error behavior as it contains the hidden behavior $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2$ (cf Proposition 2). In this way we conclude that $\mathcal{V}$ is conditioned invariant.

"Only if part:" Assume that $\mathcal{V}$ is conditioned invariant. Then there exists an error behavior $\mathcal{E} \supset \mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2$ such that $\mathcal{V} \subset \mathcal{E}$ and $\mathcal{E}/\mathcal{V}$ is autonomous.

If $\mathcal{E} = \ker E$, this can be translated in terms of Laurent-polynomial matrices as the existence of $\bar{E}(\underline{s}, \underline{s}^{-1})$ and $F(\underline{s}, \underline{s}^{-1})$ such that $E = \bar{E}R_2$, $E = FV$, and $\begin{bmatrix} F \\ L \end{bmatrix}$ (where L is a MLA of V) has full column rank.

Let $U(\underline{s}, \underline{s}^{-1})$ be such that $Q = U \begin{bmatrix} F \\ L \end{bmatrix}$ is square and full rank. Then

$$QV = U \begin{bmatrix} F \\ L \end{bmatrix} V = U \begin{bmatrix} FV \\ LV \end{bmatrix} = U \begin{bmatrix} FV \\ 0 \end{bmatrix} =$$
$$= U \begin{bmatrix} E \\ 0 \end{bmatrix} = [U_1 \ U_2] \begin{bmatrix} E \\ 0 \end{bmatrix} = U_1 E,$$

where $[U_1 \ U_2]$ is obviously a partition of U. Pre-multiplying the equality $QV = U_1 E$ by the adjoint matrix $\widetilde{Q}$ of Q, one obtains

$$\widetilde{Q}QV = \widetilde{Q}U_1 E \Leftrightarrow qV = \widetilde{Q}U_1 \bar{E} R_2 \Leftrightarrow qV = NR_2,$$

where $q = \det Q$ and $N = \widetilde{Q}U_1\bar{E}$. This implies that there exists an nD Laurent polynomial $q(\underline{s}, \underline{s}^{-1})$ such that $\ker R_2 \subset \ker qV$. $\Box$

5. DETECTABILITY SUBSPACES

Similarly to what was done in [Pereira and Rocha 2017], we define S-detectability subspaces as behaviors modulo which the error dynamics of a suitable observer is S-stable.

Definition 7. Let $\mathcal{B}_{(w_1,w_2)}$ be an nD behavior with measured variable w_1 and to-be-estimated variable w_2 in a universe $\mathcal{U}_{w_2} = (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}^n}$. Let further S be a stability cone in $\mathbb{Z}^n$. A behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ is said to be an *S-detectability subspace* if there exists an observer behavior $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$, with $\widehat{w}_2 \in U_{w_2}$ with error behavior $\mathcal{B}_e$ such that $\mathcal{V} \subset \mathcal{B}_e$ and $\mathcal{B}_e/\mathcal{V}$ is S-stable.

The following result characterizes S-detectability subspaces.

Proposition 8. Let $\mathcal{B}_{(w_1,w_2)}$ be an nD behavior with measured variable w_1 and to-be-estimated variable w_2 in the universe $\mathcal{U}_{w_2} = (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}^n}$. Assume that $\mathcal{B}_{(w_1,w_2)}$ is described by the matrix equation

$$R_2(\underline{\sigma}, \underline{\sigma}^{-1})w_2 = R_1(\underline{\sigma}, \underline{\sigma}^{-1})w_1.$$

Let further $\mathcal{V} = \ker V(\underline{\sigma}, \underline{\sigma}^{-1})$ be a sub-behavior of $\mathcal{U}_{w_2}$. Then $\mathcal{V}$ is an S-detectability subspace if and only if there exists an nD S-stable Laurent-polynomial matrix $Q(\underline{s}, \underline{s}^{-1})$ such that

$$\ker R_2(\underline{\sigma}, \underline{\sigma}^{-1}) \subset \ker Q(\underline{\sigma}, \underline{\sigma}^{-1})V(\underline{\sigma}, \underline{\sigma}^{-1}).$$

Proof.

"If part:" Assume that there exists $Q(\underline{s}, \underline{s}^{-1})$ S-stable Laurent-polynomial matrix such that $\ker R_2 \subset \ker QV =: \mathcal{E}$. Then

$$\mathcal{E}/\mathcal{V} = \ker \begin{bmatrix} Q \\ L \end{bmatrix}$$

where L is a MLA of V. Since Q is S-stable and $\ker \begin{bmatrix} Q \\ L \end{bmatrix} = \ker Q \cap \ker L$, we conclude that $\mathcal{E}/\mathcal{V}$ is S-stable. Moreover, since $\mathcal{E}$ contains the hidden behavior, $\ker R_2$, $\mathcal{E}$ is an achievable error behavior. Therefore $\mathcal{V}$ is an S-detectability subspace.

"Only if part:" Assume now that $\mathcal{V}$ is an S-detectability subspace. Then, there exists an achievable error behavior $\mathcal{E}$, which by Proposition 2 contains $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2$, such that $\mathcal{V} \subset \mathcal{E}$ and $\mathcal{E}/\mathcal{V}$ is S-stable. Let $E(\underline{s}, \underline{s}^{-1})$ be such that $\mathcal{E} = \ker E(\underline{\sigma}, \underline{\sigma}^{-1})$; then there exist nD Laurent-polynomial matrices $\bar{E}(\underline{s}, \underline{s}^{-1})$ and $F(\underline{s}, \underline{s}^{-1})$ such that $E = \bar{E}R_2$, $E = FV$, and $\begin{bmatrix} F \\ L \end{bmatrix}$ (with L a MLA of V) has kernel $\mathcal{E}/\mathcal{V}$. Thus $Q := \begin{bmatrix} F \\ L \end{bmatrix}$ must be an S-stable matrix. Now,

$$QV = \begin{bmatrix} F \\ L \end{bmatrix} V = \begin{bmatrix} FV \\ LV \end{bmatrix} = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

and hence $\ker QV = \ker E = \ker \bar{E}R_2$, implying that there exists an S-stable nD Laurent-polynomial matrix Q such that $\ker R_2 \subset \ker QV$. $\Box$

6. CONCLUSIONS

In this paper we have generalized the 1D definitions of tracking observer and asymptotic observer as well as the notions of conditioned invariance and detectability subspaces to the class of discrete nD behavioral systems.

In order to define asymptotic observers and, subsequently, detectability subspaces, we adopted the behavioral definition of S-stability introduced in [Rocha 2008]. According to this definition, a discrete nD behavior which is S-stable must be finite-dimensional (strongly autonomous). This condition may be too restrictive and limit the class of systems for which asymptotic observers exist. The use of other definitions of stability for nD behaviors is an important issue, to be investigated in future work.

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