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REALIZATION OF 2D (2,2)-PERIODIC ENCODERS BY MEANS OF 2D PERIODIC SEPARABLE ROESSER MODELS

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It is well-known that convolutional codes are linear systems when they are defined over a finite field. A fundamental issue in the implementation of convolutional codes is to obtain a minimal state representation of the code. In comparison to the literature on one-dimensional (1D) time-invariant convolutional codes, there exists only relatively few results on the realization problem for the time-varying 1D convolutional codes and even fewer if the convolutional codes are two-dimensional (2D). In this paper we consider 2D periodic convolutional codes and address the minimal state space realization problem for this class of codes. This is, in general, a highly nontrivial problem. Here, we focus on separable Roesser models and show that in this case it is possible to derive, under weak conditions, concrete formulas for obtaining a 2D Roesser state space representation. Moreover, we study minimility and present necessary conditions for these representations to be minimal. Our results immediately lead to constructive algorithms to build these representations.

Keywords: Periodic 2D systems, convolutional codes, realizations.

1. Introduction

Since the sixties it has been widely known that convolutional codes and linear systems which are defined over a finite field are mathematically identical (Rosenthal, 2001). In the last decades there has been a new and increased interest in this connection and many advances have been derived from using the system-theoretical framework when dealing with convolutional codes. This approach has led to broad advances in fundamental issues in the area (Gluesing-Luerssen and Schneider, 2007; Rosenthal and York, 1999; Rosenthal, 2001; Kuijper and Polderman, 2004).

Multi-dimensional convolutional codes (nD convolutional codes where n stands for the dimension) are a natural generalization of one-dimensional (1D) convolutional codes. Standard 1D convolutional codes deal with the transmission and storage of data that evolve over time. Instead, nD convolutional codes are suited for dealing with n dimensional data, e.g., pictures, storage media, etc. (see (Basu and Swamy, 2002)). However, while the 1D convolutional codes have been thoroughly understood, little research has been done in the area of nD convolutional codes and much more needs to be done to make it attractive for applications. The literature about nD convolutional codes is limited but some important fundamental results have been already obtained. The algebraic theory of 2D and nD convolutional codes has been laid out by Fornasini and Valcher in (Valcher and Fornasini, 1994; Fornasini and Valcher, 1998; Fornasini and Valcher, 1994), Gluesing-Luerssen et al. (Gluesing-Luersen *et al.*, 2000), Lobo et al. (Lobo *et al.*, 2012) and Weiner (Weiner, 1998), see also the references therein. They introduced the general theory for the study of nD convolutional codes constituted by sequences indexed on Z^n or N^n, and discussed issues such as the characterization of such codes in terms of their internal properties and input-output representations. A fundamental issue that arises in this context is the so-called minimal realization problem: how to derive a state-space representation of the code with the minimal dimension (properly defined below), see (Napp *et al.*, 2010; Fornasini and Pinto, 2004; Jangisarakul

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and Charoenlarpnopparut, 2011; Charoenlarpnopparut and Bose, 2001). This representation is essential as it represents a blueprint for an actual physical device, typically built from shift registers. These representations are also of paramount importance for deriving efficient decoding algorithms using trellis diagrams, *e.g.*, the Viterbi decoding algorithm.

The minimal state space realization problem play a particularly important role in the analysis and design of multi-dimensional convolutional codes because of the large amount of data involved. However, the general problem of minimal state space realization of multidimensional systems has not been solved even for 2-dimensional systems. Nevertheless, for some special cases minimal state space realization methods have been derived, see (Zerz, 2000; Pinho *et al*., 2014; Pinho, 2016; Napp *et al*., 2010; Galkowski, 1996; Galkowski, 2001) and the references therein.

The state space formulation of convolutional codes can easily be extended to the time-varying case. The system matrices describing the convolutional code are typically considered to be constant over time. However, one can also consider time-varying linear systems in which the matrices representing the code also depend on time. The idea of considering 1D time-varying, and in particular, periodically time-varying convolutional codes, has attracted considerable attention of many researchers. After Costello conjectured in (Costello, 1974) that time-varying convolutional codes can achieve better properties than the time-invariant ones, many researchers have investigated such codes. The research in this area has focused on finding concrete encoders that yield 1D periodic convolutional codes with good distance properties (Mooser, 1983; Palazzo, 1993; Felstrom and Zigangirov, 1999; Guardia, 2018) and on state space representations of time-varying systems (Napp *et al*., 2018; Climent *et al*., 2009; Aleixo *et al*., 2011; Kuijper and Willems, 1997).

In this paper we continue this thread of research by considering periodic 2D convolutional codes and the corresponding minimal state space realization problem. Although some results have been obtained in the context of time-invariant 2D convolutional codes (Pinho *et al*., 2014; Fornasini *et al*., 2015) and 1D periodic convolutional codes (Climent *et al*., 2009), this problem remains unexplored in the context of periodic 2D convolutional codes. Here we aim at deriving state Roesser 2D state space representations (Aleixo and Rocha, 2017; Kaczorek, 2001) from a (2,2)-periodic two-dimensional generator matrix. This is, in general, a highly nontrivial problem and one needs to assume additional conditions to be able to build minimal state space representations. In this work we study the case of separable (2,2)-periodic two-dimensional generator matrices, *i.e.*, the encoders $G(z_1, z_2)$ that can be decomposed as $G(z_1, z_2) = V(z_2)H(z_1)$, where $V(z_2)$ and $H(z_1)$ are polynomial matrices with periodically time-varying coefficients of period 2. More concretely, both $V(z_2)$ and $H(z_1)$ are constructed based on two alternating invariant encoders $V_0(z_2)$ and $V_1(z_2)$, and $H_0(z_1)$ and $H_1(z_1)$, respectively. We first show that one cannot expect to obtain a realization of the periodic 2D convolutional code by realizing independently the time-invariant encoders on which $V(z_2)$ and $H(z_1)$ are based. However, we provide certain conditions that allow to obtain a minimal state Roesser 2D state space representation. Moreover, our results are constructive in the sense that we provide explicit formulas for the realization and a concrete methodology for obtaining such representations.

2. Prelimiaries

2.1. Time-invariant convolutional codes. Let $\mathbb{F}$ be a finite field and let $\mathbb{F}[z]$ be the polynomial ring. In a module theoretic point of view, we define a convolutional code as follows.

Definition 1. Let $\mathbb{F}$ be a finite field and n, k be positive integers with $k < n$. A *time-invariant convolutional code* $\mathcal{C}$ of rate k/n is a submodule $\mathbb{F}^n[z]$ described as

$$\mathcal{C} = \{w(z) \in \mathbb{F}^n[z] : w(z) = G(z)u(z),\ u(z) \in \mathbb{F}^k[z]\}$$

where $G(z) \in \mathbb{F}^{n\times k}[z]$ is a full column rank $n \times k$ polynomial matrix over $\mathbb{F}$, called the *encoder*, $u(z)$ taking values in $\mathbb{F}^k[z]$ is the *information vector* and $w(z)$ is the *codeword*.

The encoders of a code $\mathcal{C}$ are not unique; however they only differ by right multiplication by unimodular matrices over $\mathbb{F}[z]$. An encoder $G(z)$ is called *column reduced* if the sum of its column degrees attains the minimal possible value among all the encoders of the same code. If $G(z) \in \mathbb{F}^{n\times k}[z]$ has column degrees $\nu_1, \dots, \nu_k$, it can be written as

$$G(z) = G_{\text{hc}} \begin{bmatrix} z^{\nu_1} & & & \\ & z^{\nu_2} & & \\ & & \ddots & \\ & & & z^{\nu_k} \end{bmatrix} + G_{\text{rem}}(z)$$

where $G_{\text{rem}}(z)$ is a polynomial matrix such that the degree of column i is less than ν_i, $i = 1, \dots, k$, and $G_{\text{hc}} \in \mathbb{F}^{n\times k}$ is a matrix whose i-th column contains the coefficients of z^{ν_i} in the i-th column of $G(z)$. G_{hc} is called the *leading column coefficient matrix* and $G(z)$ is column reduced if and only if G_{hc} is full column rank.

We define the *degree* δ of a convolutional code as the sum of the column degrees of one, and hence any, column reduced encoder. Note that the list of column degrees (also known as Forney indices) of a column reduced encoder is unique up to a permutation. A code $\mathcal{C}$ of rate k/n and degree δ is said to be an (n, k, δ) code.

2.2. Periodically time-varying 1D convolutional codes.

In this section we consider 1D convolutional codes $\mathcal{C}$ with 2-periodic encoders. The definition of such encoders (or encoding maps) is introduced next together with the definition of the corresponding 2-periodic (time-varying) convolutional codes, see (Costello, 1974; Palazzo, 1993).

Definition 2. Given two polynomial matrices $G^0(z)$, $G^1(z) \in \mathbb{F}^{n\times k}[z]$, the *periodic encoding map* induced by G^0 and G^1 is defined as

$$\begin{array}{rccc} \Phi_{G^0,G^1}: & \mathbb{F}^k[z] & \longrightarrow & \mathbb{F}^n[z] \\ & u(z) & \longmapsto & w(z) \end{array}$$

where $w(z) = \sum_{i=0}^{+\infty} w_i z^i$ and $w_{2\ell+t} = \left(G^t(z)u(z)\right)_{2\ell+t}$, $t = 0, 1,\ \ell \in \mathbb{N}_0$, and, moreover, $\left(G^t(z)u(z)\right)_{2\ell+t}$ represents the $(2\ell+t)$-coefficient of the polynomial $G^t(z)u(z)$.

The corresponding *periodic convolutional code* $\mathcal{C}_p$ is

$$\begin{aligned} \mathcal{C}_p = \{w(z) \in \mathbb{F}^n[z] : w(z) = \Phi_{G^0,G^1}(u(z)),& \\ u(z) \in \mathbb{F}^k[z]\}.& \end{aligned} \tag{1}$$

Such codes will be called *2-periodic* convolutional codes.

2.3. State-space realizations.

In systems theory, input-state-output models are mainly used to describe the time evolution of the system signals, which, in the discrete-time case, are time sequences. Therefore, in the sequel, we sometimes identify an element $a(z) = \sum_{i=0}^{N} a_i z^i \in \mathbb{F}[z]$ with the finite support sequence $a_0 = (a(z))_0, a_1 = (a(z))_1, \ldots, a_N = (a(z))_N$ formed by its coefficients, and also use the notation $a(\ell)$ to denote $a_\ell = (a(z))_\ell$. The same applies for vectors with components in $\mathbb{F}[z]$.

A state-space system

$$\left\{ \begin{array}{rcl} x(\ell+1) & = & Ax(\ell) + Bu(\ell) \\ w(\ell) & = & Cx(\ell) + Du(\ell) \end{array} \right., \ \ell \in \mathbb{N}_0,$$

denoted by (A, B, C, D), where $A \in \mathbb{F}^{\delta\times\delta}, B \in \mathbb{F}^{\delta\times k}, C \in \mathbb{F}^{n\times\delta}$ and $D \in \mathbb{F}^{n\times k}$, is said to be a state-space realization of the time-invariant (n, k, δ) convolutional code $\mathcal{C}$ if $\mathcal{C}$ is the set of codewords $w(z) \in \mathbb{F}^n[z]$ identified with the finite support output sequences w corresponding to finite support input sequences u (i.e., to information sequences $u(z) \in \mathbb{F}^k[z]$) and zero initial conditions, i.e., $x(0) = 0$.

If $G(z) \in \mathbb{F}^{n\times k}[z]$ is an encoder of $\mathcal{C}$, (A, B, C, D) is a state-space realization of $G(z)$ if

$$G(z) = C(I - Az)^{-1}Bz + D.$$

If $G(z) = \sum_{i\in\mathbb{N}} G_i z^i$, with $G_i \in \mathbb{F}^{n\times k}$, then

$$G_0 = D,\ \ G_i = CA^{i-1}B,\ i \geq 1. \tag{2}$$

Note that $G(z)$ admits many realizations. It is well-known that a state-space realization (A, B, C, D) of $G(z)$ is minimal, i.e., has minimal dimension among all the realizations of $G(z)$, if (A, B) is controllable and (A, C) is observable, i.e., the polynomial matrices $\begin{bmatrix} z^{-1}I - A & | & B \end{bmatrix}$ and $\begin{bmatrix} z^{-1}I - A \\ C \end{bmatrix}$ have, respectively, right and left polynomial inverses (in z^{-1}). The minimal dimension of a state-space realization of $G(z)$ is called the McMillan degree (Kailath, 1980) of $G(z)$ and it is represented as $\mu(G)$.

The next proposition, adapted from (Fornasini and Pinto, 2004; Gluesing-Luerssen and Schneider, 2007), provides a state-space realization for a given (not necessarily column reduced) encoder. Moreover, it states that state-space realizations of a code can be obtained from minimal realizations of column reduced encoders.

Proposition 1. *Let $G(z) \in \mathbb{F}^{n\times k}[z]$ be a polynomial matrix with rank k and column degrees $\nu_1, \ldots, \nu_k$. Consider $\bar{\delta} = \sum_{i=1}^{k} \nu_i$. Let $G(z)$ have columns $g_i(z) = \sum_{\ell=0}^{\nu_i} g_{\ell,i} z^\ell$, $i = 1, \ldots, k$ where $g_{\ell,i} \in \mathbb{F}^n$. For $i = 1, \ldots, k$ define the matrices*

$$A_i = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} \in \mathbb{F}^{\nu_i\times\nu_i},\ B_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{F}^{\nu_i},$$

$$C_i = \begin{bmatrix} g_{1,i} & \cdots & g_{\nu_i,i} \end{bmatrix} \in \mathbb{F}^{n\times\nu_i}.$$

Then a state-space realization of G is given by the matrix quadruple $(A, B, C, D) \in \mathbb{F}^{\bar{\delta}\times\bar{\delta}} \times \mathbb{F}^{\bar{\delta}\times k} \times \mathbb{F}^{n\times\bar{\delta}} \times \mathbb{F}^{n\times k}$ where

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{bmatrix},\ B = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{bmatrix},$$

$$C = \begin{bmatrix} C_1 & \cdots & C_k \end{bmatrix},\ D = \begin{bmatrix} g_{0,1} & \cdots & g_{0,k} \end{bmatrix} = G(0).$$

In the case where $\nu_i = 0$, the ith block of A and C are void and in B a zero column occurs.

In this realization (A, B) is controllable and if $G(z)$ is a column reduced encoder, (A, C) is observable and therefore the realization of $G(z)$ is minimal. Thus, the McMillan degree of a column reduced encoder is equal to the sum of its column degrees.

2.4. State-space realizations of 1D periodic convolutional codes.

Definition 3. Let $\Sigma_i = (A_i, B_i, C_i, D_i),\ i = 0, 1$, be two state-space systems with the same dimension. We define a *periodic state-space system* Σ_p as

$$\left\{ \begin{array}{rcl} x(\ell+1) & = & A(\ell)x(\ell) + B(\ell)u(\ell) \\ w(\ell) & = & C(\ell)x(\ell) + D(\ell)u(\ell) \end{array} \right. ,\ \ell \in \mathbb{N}_0 \quad (3)$$

where $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are periodic functions with period 2, such that

$$\big(A(2j), B(2j), C(2j), D(2j)\big) = (A_0, B_0, C_0, D_0)$$

and

$$\big(A(2j+1), B(2j+1), C(2j+1), D(2j+1)\big)$$
$$= (A_1, B_1, C_1, D_1), j \in \mathbb{N}_0.$$

The dimension of Σ_p is defined as the dimension of the state vector x. In this case we say that Σ_p is obtained from Σ_0 and Σ_1.
Moreover, Σ_p is a realization of a periodic encoding map Φ_{G^0,G^1} if the output of Σ_p that corresponds to an input $u(z)$ is equal to $\Phi_{G^0,G^1}(u(z))$, for all $u(z) \in \mathbb{F}^k[z]$.

Let Σ_0 and Σ_1 be two state-space realizations (of the same dimension) of two encoders $G^0(z)$ and $G^1(z)$. It is possible to show that the 2-periodic system Σ_p obtained from Σ_0 and Σ_1 is not always a state-space realization of Φ_{G^0,G^1}.

However, in the next theorem (Napp *et al.*, 2018) we provide a sufficient condition for a periodic state-space system to be a realization of a periodic encoding map.

Theorem 1. *Consider two encoders $G^0(z) \in \mathbb{F}^{n\times k}[z]$ and $G^1(z) \in \mathbb{F}^{n\times k}[z]$ with the same column degrees and let Σ_i be the realizations of $G^i(z), i = 0, 1$ obtained by Proposition 1. Then, the periodic state-space system Σ_p obtained from Σ_0 and Σ_1 is a realization of the periodic encoding map Φ_{G^0,G^1}.*
When the encoders given in the previous theorem are column reduced then the realization of the corresponding encoding map is minimal, as stated next (Napp *et al.*, 2018).

Theorem 2. *Let $G^0(z), G^1(z) \in \mathbb{F}^{n\times k}[z]$ be two column reduced encoders with the same column degrees and let Σ_i be the realizations of $G^i(z), i = 0, 1$ obtained by Proposition 1. Then, the 2-periodic state-space realization of the periodic encoding map Φ_{G^0,G^1} obtained from Σ_0 and Σ_1 is minimal.*

3. 2D (2,2)-periodic convolutional codes

In this paper we consider 2D convolutional codes $\mathcal{C}$ with (2,2)-periodic encoders. Analogously to the 1D case we introduce the definition of periodic encoding map.

Definition 4. Given four 2D polynomial matrices $G^{00}(z_1, z_2),\ G^{10}(z_1, z_2),\ G^{01}(z_1, z_2),\ G^{11}(z_1, z_2) \in \mathbb{F}^{n\times k}[z_1, z_2]$, the *(2,2)-periodic encoding map* induced by G^{00}, G^{10}, G^{01} and G^{11} is defined as

$$\begin{array}{rccc} \Phi_{G^{00},G^{10},G^{01},G^{11}}: & \mathbb{F}^k[z_1, z_2] & \longrightarrow & \mathbb{F}^n[z_1, z_2] \\ & u(z_1, z_2) & \longmapsto & w(z_1, z_2) \end{array}$$

where $w(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} w_{i,j} z_1^i z_2^j$ and

$$w_{2\ell+i,2m+j} = \big(G^{ij}(z_1, z_2)u(z_1, z_2)\big)_{2\ell+i,2m+j},$$
$$i, j = 0, 1,\ \ell, m \in \mathbb{N}_0,$$

and, moreover, $\big(G^{ij}(z_1, z_2)u(z_1, z_2)\big)_{2\ell+i,2m+j}$ represents the $(2\ell + i, 2m + j)$-coefficient of the polynomial $G^{ij}(z_1, z_2)u(z_1, z_2)$.

The corresponding *2D (2,2)-periodic convolutional code* $\mathcal{C}_p$ is

$$\mathcal{C}_p = \{w(z_1, z_2) \in \mathbb{F}^n[z_1, z_2] : \\ \exists\, u(z_1, z_2) \in \mathbb{F}^k[z_1, z_2] \text{ s.t. (4) holds}\}$$

$$w(z_1, z_2) = \Phi_{G^{00},G^{10},G^{01},G^{11}}(u(z_1, z_2)). \quad (4)$$

Such codes will be called *2D (2,2)-periodic* convolutional codes.
We consider a special class of 2D polynomial matrices $G^{ij}(z_1, z_2)$ that can be factored as:

$$G^{ij}(z_1, z_2) = V^j(z_2)H^i(z_1),$$

where $H^i(z_1) \in \mathbb{F}^{q\times k}[z_1]$ and $V^j(z_2) \in \mathbb{F}^{n\times q}[z_2]$ are 1D polynomial matrices, $i = 0, 1$.

Therefore the previous 2D (2,2)-periodic convolutional code $\mathcal{C}_p$ can be defined as

$$\mathcal{C}_p = \{w(z_1, z_2) \in \mathbb{F}^n[z_1, z_2] : \\ \exists\, u(z_1, z_2) \in \mathbb{F}^k[z_1, z_2] \text{ s.t. (5) holds}\}$$

$$w(z_1, z_2) = \Phi_{V^0H^0,V^0H^1,V^1H^0,V^1H^1}(u(z_1, z_2)). \quad (5)$$

4. 2D State space realizations

Here we focus on the state space realizations of the special class of 2D periodic convolutional codes introduced in the previous section by means of 2D periodic Roesser models. In general, this is a nontrivial matter, mainly due to the fact that a 2D periodic state space realization cannot be obtained by independently realizing each of the invariant polynomial operators $G^{ij} = V^j H^i$, (Aleixo and Rocha, 2017). However, in this paper we show that under certain conditions this problem does not arise, i.e., combining independent realizations of the invariant operators G^{ij} does yield a 2D periodic realization of the corresponding 2D periodic convolutional code. Before presenting our result, we first consider the invariant 2D case.

As in section 2.3, we sometimes identify an element $a(z_1,z_2) = \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} a_{i,j} z_1^i z_2^j \in \mathbb{F}[z_1,z_2]$ with the finite support sequence $a_{0,0} = (a(z_1,z_2))_{0,0},\ a_{1,0} = (a(z_1,z_2))_{1,0}, \ldots, a_{N_1,N_2} = (a(z_1,z_2))_{N_1,N_2}$ formed by its coefficients, and also use the notation $a(\ell,m)$ to denote $a_{\ell,m} = (a(z_1,z_2))_{\ell,m}$. The same applies for vectors with components in $\mathbb{F}[z_1,z_2]$.

4.1. The invariant 2D case.

As is well-known, in the 2D invariant case, a separable Roesser model realization for a code

$$\begin{aligned}\mathcal{C} = \{ & w(z_1,z_2) \in \mathbb{F}^n[z_1,z_2] : \\ & w(z_1,z_2) = G(z_1,z_2)u(z_1,z_2),\, u(z_1,z_2) \in \mathbb{F}^k[z_1,z_2]\}\end{aligned}$$

where

$$G(z_1,z_2) = V(z_2)H(z_1)$$

can be obtained as the series connection of the 1D state space realizations of H and V. Indeed, if $(A^h, B^h, \tilde{C}^h, D^h)$ and $(A^v, \tilde{B}^v, C^v, D^v)$ are respectively state space realizations of $H(z_1)$ and $V(z_2)$, then the separable Roesser model $\Sigma = (A^h, A^v, A^{vh}, B^h, B^v, C^h, C^v, D)$:

$$\left\{\begin{aligned} x^h(i+1,j) &= A^h x^h(i,j) + B^h u(i,j) \\ x^v(i,j+1) &= A^{vh} x^h(i,j) + A^v x^v(i,j) + B^v u(i,j) \\ w(i,j) &= C^h x^h(i,j) + C^v x^v(i,j) + Du(i,j) \end{aligned}\right. \tag{6}$$

with $A^{vh} = \tilde{B}^v \tilde{C}^h$, $B^v = \tilde{B}^v D^h$, $C^h = D^v \tilde{C}^h$, and $D = D^v D^h$, is a realization of $\mathcal{C}$ in the sense that the codewords w in $\mathcal{C}$ coincide with the outputs of (6) produced by the same input u with zero initial conditions, i.e., $x^h(0,j) = 0$ and $x^v(i,0) = 0$, $i,j \in \mathbb{N}_0$. Moreover, we consider that the dimension of the horizontal and vertical states, $x^h(i,j)$ and $x^v(i,j)$, are δ_h and δ_v, respectively.

In the sequel the minimality of separable Roesser models will be studied. We start with some preliminary definitions and results.

Definition 5. The horizontal and vertical controllability matrices of the separable Roesser model (6) are defined, respectively, as:

$$\mathscr{C}_h = \left[\, B^h \,\middle|\, A^h B^h \,\middle|\, \cdots \,\middle|\, (A^h)^{\delta_h - 1} B^h \,\right] \in \mathbb{F}^{\delta_h \times k\delta_h} \tag{7}$$

$$\mathscr{C}_v = \left[B_{\delta_h} \,\middle|\, A^v B_{\delta_h} \,\middle|\, \cdots \,\middle|\, (A^v)^{\delta_v - 1} B_{\delta_h} \right] \in \mathbb{F}^{\delta_v \times \delta_v k(\delta_h+1)} \tag{8}$$

with $B_{\delta_h} = \left[\ B^v \ \middle|\ A^{vh}\mathscr{C}_h \ \right] \in \mathbb{F}^{\delta_v \times k(\delta_h+1)}$.

Definition 6. The vertical and horizontal observability matrices of the separable Roesser model (6) are defined, respectively, as:

$$\mathscr{O}_v = \left[(C^v)^\top \,\middle|\, (C^v A^v)^\top \,\middle|\, \cdots \,\middle|\, \left(C^v (A^v)^{\delta_v - 1} \right)^\top \right]^\top \in \mathbb{F}^{n\delta_v \times \delta_v} \tag{9}$$

$$\mathscr{O}_h = \left[(C_{\delta_v})^\top \,\middle|\, (C_{\delta_v} A^h)^\top \,\middle|\, \cdots \,\middle|\, \left(C_{\delta_v} (A^h)^{\delta_h - 1} \right)^\top \right]^\top \in \mathbb{F}^{\delta_h n(\delta_v+1) \times \delta_h} \tag{10}$$

with $C_{\delta_v} = \begin{bmatrix} C^h \\ \mathscr{O}_v A^{vh} \end{bmatrix} \in \mathbb{F}^{n(\delta_v+1)\times\delta_h}$.

The following proposition is well-known[1].

Proposition 2.

1. *The pair (A^h, B^h) is controllable if and only if $\operatorname{rank}\mathscr{C}_h = \delta_h$.*

2. *The pair (A^v, B_{δ_h}) is controllable if and only if $\operatorname{rank}\mathscr{C}_v = \delta_v$.*

3. *The pair (A^v, C^v) is observable if and only if $\operatorname{rank}\mathscr{O}_v = \delta_v$.*

4. *The pair (A^h, C_{δ_v}) is observable if and only if $\operatorname{rank}\mathscr{O}_h = \delta_h$.*

For separable Roesser models, separable controllability and separable observability are defined as follows

Definition 7. The 2D separable Roesser model (6) is said to be:

1. Separately locally controllable if (A^h, B^h) and (A^v, B_{δ_h}) are controllable.

2. Separately locally observable if (A^v, C^v) and (A^h, C_{δ_v}) are observable.

In (Hinamoto, 1980), Hinamoto presented a necessary and suficient condition for the minimality of a separable Roesser model, that we state in the next result using the language of codes.

Theorem 3. *Let $G(z_1,z_2) \in \mathbb{F}^{n\times k}[z_1,z_2]$ be an encoder of a convolutional code $\mathcal{C}$. Then the separable Roesser model $\Sigma = (A^h, A^v, A^{vh}, B^h, B^v, C^h, C^v, D)$ given by* (6) *is a minimal realization of the encoder $G(z_1,z_2)$ if and only if is separately locally controllable and separately locally observable.*

In the next theorem we provide a simpler characterization for the minimality of a separable Roesser model.

[1] Note that previously (before stating Proposition 1) we have given an alternative definition of controllable and observable pair.

Theorem 4. *Let $G(z_1, z_2) \in \mathbb{F}^{n\times k}[z_1, z_2]$ be an encoder of a convolutional code $\mathcal{C}$. Then the separable Roesser model $\Sigma = (A^h, A^v, A^{vh}, B^h, B^v, C^h, C^v, D)$ given by* (6) *is a minimal realization of the encoder $G(z_1, z_2)$ if and only if the following conditions hold:*

1. (A^h, B^h) *and* $\left(A^v, \left[\begin{array}{c|c} B^v & A^{vh}\end{array}\right]\right)$ *are controllable.*
2. (A^v, C^v) *and* $\left(A^h, \begin{bmatrix} C^h \\ A^{vh}\end{bmatrix}\right)$ *are observable.*

The next two auxiliary lemmas immediately prove the previous theorem.

Lemma 1. *The 2D separable Roesser model* (6) *is separately locally controllable if and only if (A^h, B^h) and $\left(A^v, \left[\begin{array}{c|c} B^v & A^{vh}\end{array}\right]\right)$ are controllable.*

Proof. "Only if" part. By definition of separately locally controllable we have that the matrices $\mathscr{C}_h$ and $\mathscr{C}_v$ have full row rank. Defining the matrices

$$M = \left[\begin{array}{c|c} B^v & A^{vh}\end{array}\right] \quad , \quad \overline{\mathscr{C}_h} = \left[\begin{array}{c|c} I_{\delta_v} & 0 \\ \hline 0 & \mathscr{C}_h\end{array}\right] \quad \text{and}$$

$$\overline{\overline{\mathscr{C}_h}} = \begin{bmatrix} \overline{\mathscr{C}_h} & & 0 \\ & \ddots & \\ 0 & & \overline{\mathscr{C}_h}\end{bmatrix}$$

we have that

$$B_{\delta_h} = \left[\begin{array}{c|c} B^v & A^{vh}\mathscr{C}_h\end{array}\right] = M\overline{\mathscr{C}_h},$$

and, in turn,

$$\mathscr{C}_v = \left[\begin{array}{c|c|c|c} B_{\delta_h} & A^v B_{\delta_h} & \cdots & (A^v)^{\delta_v - 1} B_{\delta_h}\end{array}\right] = \overline{\mathscr{C}_v}\,\overline{\overline{\mathscr{C}_h}}$$

with

$$\overline{\mathscr{C}_v} = \left[\begin{array}{c|c|c|c} M & A^v M & \cdots & (A^v)^{\delta_v - 1} M\end{array}\right].$$

Since $\mathscr{C}_h$ has full row rank, clearly both matrices $\overline{\mathscr{C}_h}$ and $\overline{\overline{\mathscr{C}_h}}$ also have full row rank. Moreover, by hypothesis, $\mathscr{C}_v$ has full row rank which implies that $\overline{\mathscr{C}_v}$ must also have full row rank. This means that $(A^v, M) = \left(A^v, \left[\begin{array}{c|c} B^v & A^{vh}\end{array}\right]\right)$ is controllable.

"If" part. Assuming the hypothesis, it suffices to prove that the pair (A^v, B_{δ_h}) is controllable, i.e., that the matrix $\mathscr{C}_v$ has full row rank. Adopting the notations of the "Only if" part, we have that

$$\mathscr{C}_v = \overline{\mathscr{C}_v}\,\overline{\overline{\mathscr{C}_h}}.$$

Since $\left(A^v, \left[\begin{array}{c|c} B^v & A^{vh}\end{array}\right]\right)$ is controllable then $\overline{\mathscr{C}_v}$ has full row rank. Furthermore, $\overline{\overline{\mathscr{C}_h}}$ has full row rank because $\mathscr{C}_h$ also has by the hypothesis of controllability of (A^h, B^h), and the result follows. ■

Lemma 2. *The 2D separable Roesser model* (6) *is separately locally observable if and only if (A^v, C^v) and $\left(A^h, \begin{bmatrix} C^h \\ A^{vh}\end{bmatrix}\right)$ are observable.*

Proof. The proof is analogous to the one of the previous lemma. ■

4.2. The periodic 2D case.

Analogously to the invariant 2D case, under certain conditions, in the 2D periodic case, a periodic separable Roesser model realization can be obtained as a series connection of two 1D periodic state space realizations of periodic operators $H^{2k+i} \equiv H^i$ and $V^{2\ell+j} \equiv V^j$, $i, j = 0, 1$, $k, l \in \mathbb{Z}$. Consider the (2,2)-periodic encoding map

$$\Phi_{V^0H^0, V^0H^1, V^1H^0, V^1H^1},$$

and let further $\Sigma_i^h = (A_i^h, B_i^h, \tilde{C}_i^h, D_i^h)$ and $\Sigma_j^v = (A_j^v, \tilde{B}_j^v, C_j^v, D_j^v)$ be state space realizations of the invariant operators H^i and V^j, $i, j = 0, 1$, respectively. Assume that Σ_0^h and Σ_1^h have the same state dimensions and that the same happens for Σ_0^v and Σ_1^v. Combining these realizations yields the following (2,2)-periodic 2D separable Roesser state space system Σ_p^{2D}:

$$\begin{aligned}
\begin{bmatrix} x^h(2\ell+i+1, 2m+j) \\ x^v(2\ell+i, 2m+j+1)\end{bmatrix} &= \begin{bmatrix} A_i^h & 0 \\ A_{ij}^{vh} & A_j^v\end{bmatrix}\begin{bmatrix} x^h(2\ell+i, 2m+j) \\ x^v(2\ell+i, 2m+j)\end{bmatrix} \\
&\quad + \begin{bmatrix} B_i^h \\ B_{ij}^v\end{bmatrix} u(2\ell+i, 2m+j) \\
w(2\ell+i, 2m+j) &= \begin{bmatrix} C_{ij}^h & C_j^v\end{bmatrix}\begin{bmatrix} x^h(2\ell+i, 2m+j) \\ x^v(2\ell+i, 2m+j)\end{bmatrix} \\
&\quad + D_{ij} u(2\ell+i, 2m+j)
\end{aligned} \tag{11}$$

with $A_{ij}^{vh} = \tilde{B}_j^v \tilde{C}_i^h$, $B_{ij}^v = \tilde{B}_j^v D_i^h$, $C_{ij}^h = D_j^v \tilde{C}_i^h$, and $D_{ij} = D_j^v D_i^h$.

Note that for each pair of *fixed* values of i and j this periodic 2D system is an invariant separable 2D state space system

$$\Sigma_{(i,j)} = \left(A_i^h, A_j^v, A_{ij}^{vh}, B_i^h, B_{ij}^V, C_{ij}^h, C_j^v, D_{ij}\right).$$

Similar to what happens in the 1D case, we say that Σ_p^{2D} is obtained from $\Sigma_{(0,0)}$, $\Sigma_{(1,0)}$, $\Sigma_{(0,1)}$ and $\Sigma_{(1,1)}$ and write $\Sigma_p^{2D} = \left(\Sigma_{(0,0)}, \Sigma_{(1,0)}, \Sigma_{(0,1)}, \Sigma_{(1,1)}\right)$.

As shown in the following example the 2D (2,2)-periodic Roesser state space system $\Sigma_p^{2D} = \left(\Sigma_{(0,0)}, \Sigma_{(1,0)}, \Sigma_{(0,1)}, \Sigma_{(1,1)}\right)$ is not necessarily a realization of the (2,2)-periodic encoding map

$$\Phi_{V^0H^0, V^0H^1, V^1H^0, V^1H^1}.$$

Example 1. Consider the (2,2)-periodic encoding map

$$\Phi_{V^0H^0, V^0H^1, V^1H^0, V^1H^1}$$

with

$$\begin{aligned} H^0(z_1) &= H_0^0 + H_1^0 z_1 + H_2^0 z_1^2 \\ &= \begin{bmatrix} 1+z_1^2 & 1 & 0 \\ z_1^2 & 1+z_1 & 1 \\ 1+z_1 & 1 & 1 \\ 1 & 1 & 1+z_1 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} H^1(z_1) &= H_0^1 + H_1^1 z_1 + H_2^1 z_1^2 \\ &= \begin{bmatrix} 1+z_1 & 1 & 0 \\ 1+z_1^2 & 1+z_1 & 1 \\ 1 & 1+z_1^2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \end{aligned}$$

$$V^0(z_2) = (1+z_2) I_4 \text{ and } V^1(z_2) = (1+2z_2) I_4.$$

Realizing $H^0(z_1)$ as in Proposition 1 we obtain the state-space realization $\Sigma_0^h = (A_0^h, B_0^h, \tilde{C}_0^h, D_0^h)$ with

$$A_0^h = \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \quad B_0^h = \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$$

$$\tilde{C}_0^h = \begin{bmatrix} 0&1&0&0 \\ 0&1&1&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{bmatrix} \quad D_0^h = \begin{bmatrix} 1&1&0 \\ 0&1&1 \\ 1&1&1 \\ 1&1&1 \end{bmatrix}.$$

Proceeding in the same way, we obtain a state-space realization $\Sigma_1^h = (A_1^h, B_1^h, \tilde{C}_1^h, D_1^h)$ for $H^1(z_1)$ with

$$A_1^h = \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{bmatrix} \quad B_1^h = \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix}$$

$$\tilde{C}_1^h = \begin{bmatrix} 1&0&0&0 \\ 0&1&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix} \quad D_1^h = \begin{bmatrix} 1&1&0 \\ 1&1&1 \\ 1&1&1 \\ 0&1&1 \end{bmatrix}.$$

As for $V^0(z_2)$ and $V^1(z_2)$, it is easily seen that they can be realized by $\Sigma_0^v = (A_0^v, \tilde{B}_0^v, C_0^v, D_0^v)$ and $\Sigma_1^v = (A_1^v, \tilde{B}_1^v, C_1^v, D_1^v)$ with

$$A_0^v = 0_4,\ \tilde{B}_0^v = C_0^v = D_0^v = I_4$$

and

$$A_1^v = 0_4,\ \tilde{B}_0^v = D_0^v = I_4,\ C_0^v = 2I_4,$$

where 0_4 denotes the 4×4 zero matrix.

Let us consider, for every $t_2 \in \mathbb{N}_0$,

$$u(0,t_2) = \begin{bmatrix} 0\\0\\1 \end{bmatrix},\ u(1,t_2) = \begin{bmatrix} 0\\0\\0 \end{bmatrix},\ u(t_1,t_2) = 0, t_1 \geq 2.$$

From (5) it follows that, for $m \in \mathbb{N}_0$, $j = 0, 1$,

$$\begin{aligned} w(1,2m+j) &= \big(V^j(z_2) H^1(z_1) u(z_1,z_2)\big)(1,2m+j) \\ &= \big(V^j(z_2) \bar{u}(z_1,z_2)\big)(1,2m+j) \end{aligned}$$

where

$$\bar{u}(1,2m+j) = H_0^1 u(1,2m+j) + H_1^1 u(0,2m+j)$$
$$= \begin{bmatrix} 1&1&0 \\ 1&1&1 \\ 1&1&1 \\ 0&1&1 \end{bmatrix} \begin{bmatrix} 0\\0\\0 \end{bmatrix} + \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix} \begin{bmatrix} 0\\0\\1 \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}$$

thus

$$w(1,2m+j) = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}, m \in \mathbb{N}_0, j = 0,1$$

or simply

$$w(1,t_2) = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix},\ \text{for } t_2 \in \mathbb{N}_0.$$

On the other hand, using (11), we have

$$w(1,0) = \begin{bmatrix} C_{10}^h & C_0^v \end{bmatrix} \begin{bmatrix} x^h(1,0) \\ x^v(1,0) \end{bmatrix} + D_{10} u(1,0)$$

Note that, due to the fact that the initial conditions must be zero (according to our definition of realization), $x^v(1,0) = 0$ and $x^h(0,0) = 0$. Moreover,

$$\begin{aligned} x^h(1,0) &= A_0^h x^h(0,0) + B_0^h u(0,0) \\ &= \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \begin{bmatrix} 0\\0\\1 \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} w(1,0) &= C_{10}^h x^h(1,0) = D_0^v \tilde{C}_1^h x^h(1,0) \\ &= I_4 \begin{bmatrix} 1&0&0&0 \\ 0&1&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix} \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix} = \begin{bmatrix} 0\\0\\1\\0 \end{bmatrix}, \end{aligned}$$

i.e., the output w of the 2D (2,2)-periodic Roesser state space system Σ_p^{2D} corresponding to v is different from the trajectory w corresponding to u according to (5). ♦

However, in the next theorem is shown that it is possible to obtain periodic 2D separable Roesser model realizations for 2D periodic encoding maps (5), by independently realizing the operators V^0, V^1, and H^0, H^1, provided that V^0 and V^1 have the same column degrees and the same happens for H^0 and H^1.

Theorem 5. *Consider the polynomial matrices $H^0(z_1), H^1(z_1) \in \mathbb{F}^{q \times k}[z_1]$, and assume that they have the same column degrees. Let Σ_i^h be the realizations of $H^i(z_1), i = 0, 1$, obtained by Proposition 1. Consider further the polynomial matrices $V^0(z_2), V^1(z_2) \in$*

$\mathbb{F}^{q\times n}[z_2]$, *and assume that they have the same column degrees. Let* Σ_j^v *be the realizations of* $V^j(z_2)$, $j=0,1$, *obtained by Proposition 1. Define the 2D periodic Roesser separable model* Σ_p^{2D} *obtained from* Σ_i^h *and* Σ_j^v *as in* (11). *Then* Σ_p^{2D} *is a state space realization of the 2D periodic encoding map given by* (5).

Proof. Note that since $H^0(z_1)$ and $H^1(z_1)$ have the same column degrees it follows that the corresponding realizations $\Sigma_i^h = (A_i^h, B_i^h, \tilde{C}_i^h, D_i^h)$, $i=0,1$, are such that $A_0^h = A_1^h$ and $B_0^h = B_1^h$. Let us consider $A^h := A_0^h = A_1^h$ and $B^h := B_0^h = B_1^h$. By the same reason the realizations $\Sigma_j^v = (A_j^v, \tilde{B}_j^v, C_j^v, D_j^v)$, $j=0,1$, are such that $A_0^v = A_1^v$ and $\tilde{B}_0^v = \tilde{B}_1^v$. Let $A^v := A_0^v = A_1^v$ and $\tilde{B}^v := \tilde{B}_0^v = \tilde{B}_1^v$.

Then, after simple, but cumbersome computations, one concludes that the output w of $\Sigma_p^{2D} = (\Sigma_{(0,0)}, \Sigma_{(1,0)}, \Sigma_{(0,1)}, \Sigma_{(1,1)})$, with

$$\Sigma_{(i,j)} = \Big(A^h, A^v, A_{ij}^{vh} = \tilde{B}^v\tilde{C}_i^h, B^h, B_{ij}^v = \tilde{B}^v D_i^h, C_{ij}^h = D_j^v\tilde{C}_i^h, C_j^v, D_{ij}\Big)$$

that corresponds to the input u and zero initial conditions $\big(x^h(0,t_2)=0, x^v(t_1,0)=0\big)$ is such that, for $\ell, m \in \mathbb{N}_0$, $i,j=0,1$,

$$\begin{aligned}
w(2\ell+i, 2m+j) &= D_j^v D_i^h v(2\ell+i, 2m+j)\\
&+ \textstyle\sum_{t_1\geq 1} C_{ij}^h (A^h)^{t_1-1} B^h u(2\ell+i-t_1, 2m+j)\\
&+ \sum_{t_2\geq 1} C_j^v (A^v)^{t_2-1} B_{ij}^v v(2\ell+i, 2m+j-t_2)\\
&+ \sum_{t_1,t_2\geq 1} C_j^v (A^v)^{t_2-1} A_{ij}^{vh} (A^h)^{t_1-1} B^h v(2\ell+i-t_1, 2m+j-t_2).
\end{aligned}$$

Let us now show that the codeword

$$\tilde{w} = \big(V^j(z_2)H^i(z_1)\big)u$$

equals w. For that, note that since $\Sigma_0^h = (A^h, B^h, \tilde{C}_0^h, D_0^h)$ is a realization of H^0 we have that

$$H^0(z_1) = D_0^h + \sum_{t_1\geq 1} \tilde{C}_0^h (A^h)^{t_1-1} B^h z_1^{t_1}.$$

In the same way

$$H^1(z_1) = D_1^h + \sum_{t_1\geq 1} \tilde{C}_1^h (A^h)^{t_1-1} B^h z_1^{t_1},$$

$$V^0(z_2) = D_0^v + \sum_{t_2\geq 1} C_0^v (A^v)^{t_2-1} \tilde{B}^v z_2^{t_2}.$$

and

$$V^1(z_2) = D_1^v + \sum_{t_2\geq 1} C_1^v (A^v)^{t_2-1} \tilde{B}^v z_2^{t_2}.$$

Thus

$$\tilde{w}(2\ell+i, 2m+j) = \sum_{\substack{0\leq t_1\leq 2k+i\\ 0\leq t_2\leq 2l+j}} G(i,j)u(2\ell+i-t_1, 2m+j-t_2)$$

where $G(i,j)$ is the coefficient of $z_1^i z_2^j$ of the polynomial matrix in z_1 and z_2, $V^j(z_2)H^i(z_1)$. It is not difficult to check that

$$\begin{aligned}
V^j(z_2)H^i(z_1) &= D_j^v D_i^h + \sum_{t_1\geq 1} D_j^v \tilde{C}_i^h (A^h)^{t_1-1} B^h z_1^{t_1}\\
&+ \textstyle\sum_{t_2\geq 1} C_j^v (A^v)^{t_2-1} \tilde{B}^v D_i^h z_2^{t_2}\\
&+ \textstyle\sum_{t_1,t_2\geq 1} C_j^v (A^v)^{t_2-1} \tilde{B}^v \tilde{C}_i^h (A^h)^{t_1-1} B^h z_1^{t_1} z_2^{t_2}.
\end{aligned}$$

Taking into account that $C_{ij}^h = D_j^v \tilde{C}_i^h$, $B_{ij}^v = \tilde{B}^v D_i^h$ and $A_{ij}^{vh} = \tilde{B}^v\tilde{C}_i^h$, this allows to conclude that $\tilde{w} = w$. ■

In order to study the minimality of the 2D state space realization (11) we start by defining its lifted version.

4.3. Lifted 2D realization. Following the ideas of (Aleixo and Rocha, 2017; Aleixo and Rocha, 2018), consider the (2,2)-periodic 2D separable Roesser state space system Σ_p^{2D} given by (11) and define the lifted versions of the horizontal and vertical states as:

$$X^h(\ell,m) = \begin{bmatrix} x^h(2\ell,2m)\\ x^h(2\ell,2m+1)\end{bmatrix}$$

and

$$X^v(\ell,m) = \begin{bmatrix} x^v(2\ell,2m)\\ x^v(2\ell+1,2m)\end{bmatrix},$$

respectively; define also the lifted versions of the input and the output, respectively, as

$$u^L(\ell,m) = \begin{bmatrix} u(2\ell,2m)\\ u(2\ell+1,2m)\\ u(2\ell,2m+1)\\ u(2\ell+1,2m+1)\end{bmatrix}$$

and

$$w^L(\ell,m) = \begin{bmatrix} w(2\ell,2m)\\ w(2\ell+1,2m)\\ w(2\ell,2m+1)\\ w(2\ell+1,2m+1)\end{bmatrix}.$$

This yields the following 2D invariant separable Roesser model

$$\begin{aligned}
\begin{bmatrix} X^h(\ell+1,m)\\ X^v(\ell,m+1)\end{bmatrix} &= P\begin{bmatrix} X^h(\ell,m)\\ X^v(\ell,m)\end{bmatrix} + Qu^L(\ell,m)\\
w^L(\ell,m) &= R\begin{bmatrix} X^h(\ell,m)\\ X^v(\ell,m)\end{bmatrix} + Su^L(\ell,m)
\end{aligned}\qquad (12)$$

where the matrices P, Q, R and S are constant and can be decomposed as follows

$$P = \begin{bmatrix} P^h & 0 \\ P^{vh} & P^v \end{bmatrix}, \ Q = \begin{bmatrix} Q^h \\ Q^v \end{bmatrix}, \ R = \begin{bmatrix} R^h & R^v \end{bmatrix} \quad (13)$$

where the dimensions of the blocks are determined by the dimensions of X^h and X^v and, moreover,

$$P^h = \begin{bmatrix} A_1^h A_0^h & 0 \\ 0 & A_1^h A_0^h \end{bmatrix}, \ P^v = \begin{bmatrix} A_1^v A_0^v & 0 \\ 0 & A_1^v A_0^v \end{bmatrix},$$
$$P^{vh} = \begin{bmatrix} A_1^v \tilde{B}_0^v \tilde{C}_0^h & \tilde{B}_1^v \tilde{C}_0^h \\ A_1^v \tilde{B}_0^v \tilde{C}_1^h A_0^h & \tilde{B}_1^v \tilde{C}_1^h A_0^h \end{bmatrix} \quad (14)$$

$$Q^h = \begin{bmatrix} A_1^h B_0^h & B_1^h & 0 & 0 \\ 0 & 0 & A_1^h B_0^h & B_1^h \end{bmatrix}$$
$$Q^v = \begin{bmatrix} A_1^v \tilde{B}_0^v D_0^h & 0 & \tilde{B}_1^v D_0^h & 0 \\ A_1^v \tilde{B}_0^v \tilde{C}_1^h B_0^h & A_1^v \tilde{B}_0^v D_1^h & \tilde{B}_1^v \tilde{C}_1^h B_0^h & \tilde{B}_1^v D_1^h \end{bmatrix} \quad (15)$$

$$R^h = \begin{bmatrix} D_0^v \tilde{C}_0^h & 0 \\ D_0^v \tilde{C}_1^h A_0^h & 0 \\ C_1^v \tilde{B}_0^v \tilde{C}_0^h & D_1^v \tilde{C}_0^h \\ C_1^v \tilde{B}_0^v \tilde{C}_1^h A_0^h & D_1^v \tilde{C}_1^h A_0^h \end{bmatrix}, R^v = \begin{bmatrix} C_0^v & 0 \\ 0 & C_0^v \\ C_1^v A_0^v & 0 \\ 0 & C_1^v A_0^v \end{bmatrix} \quad (16)$$

$$S = \begin{bmatrix} D_0^v D_0^h & 0 & 0 & 0 \\ D_0^v \tilde{C}_1^h B_0^h & D_0^v D_1^h & 0 & 0 \\ C_1^v \tilde{B}_0^v D_0^h & 0 & D_1^v D_0^h & 0 \\ C_1^v \tilde{B}_0^v \tilde{C}_1^h B_0^h & C_1^v \tilde{B}_0^v D_1^h & D_1^v \tilde{C}_1^h B_0^h & D_1^v D_1^h \end{bmatrix} \quad (17)$$

We denote this 2D invariant lifted model by $\Sigma^L = (P, Q, R, S)$.

5. Minimality

Theorem 6. *Let $H^0(z_1), H^1(z_1) \in \mathbb{F}^{q\times k}[z_1]$ be two column reduced encoders with the same column degrees and $V^0(z_2), V^1(z_2) \in \mathbb{F}^{q\times n}[z_2]$ be also two column reduced encoders with the same column degrees. Let further $\Sigma_i^h = (A^h, B^h, \tilde{C}_i^h, D_i^h)$ and $\Sigma_j^v = (A^v, B^v, C_j^v, D_j^v)$ be, respectively, 1D state space realizations of H^i (of dimension δ_h) and V^j (of dimension δ_v), $i, j = 0, 1$, obtained as in Proposition 1. Define the 2D periodic Roesser separable model Σ_p^{2D} obtained from Σ_i^h and Σ_j^v as in (11). If the matrix*

$$M = \begin{bmatrix} \tilde{C}_0^h & D_0^h & 0 \\ \tilde{C}_1^h A^h & \tilde{C}_1^h B^h & D_1^h \end{bmatrix}$$

has full row rank and the matrix

$$N = \begin{bmatrix} D_0^v & 0 \\ C_1^v B^v & D_1^v \\ A^v B^v & B^v \end{bmatrix}$$

has full column rank, then Σ_p^{2D} is a minimal state space realization of the 2D periodic encoding map $\Phi_{V^0H^0, V^0H^1, V^1H^0, V^1H^1}$ given by (5).

Proof. The proof of this theorem is a direct consequence of the next four lemmas. ■

The next lemma follows immediately from the definition of the lifted system.

Lemma 3. *Σ_p^{2D} is a minimal state space realization if and only if Σ^L is minimal.*

As a consequence of the previous Lemma, to prove Theorem 6 we just need to prove that the lifted realization $\Sigma^L = (P, Q, R, S)$ given by (12) is separately locally controllable and separately locally observable. We start with the proof of the separate local controllability of Σ^L.

Lemma 4. *In the conditions of Theorem 6, the realization $\Sigma^L = (P, Q, R, S)$ given by (12) is separately locally controllable.*

Proof. By Lemma 1, we just have to prove that the matrices (P^h, Q^h) and $\left(P^v, \left[\ Q^v \mid P^{vh}\ \right]\right)$ are controllable, where the involved matrices are defined by (14) and (15) with $A_1^h = A_0^h = A^h$, $A_1^v = A_0^v = A^v$, $B_1^h = B_0^h = B^h$ and $B_1^v = \tilde{B}_0^v = \tilde{B}^v$.

By Proposition 2 we have that

$$(P^h, Q^h) \text{ controllable if and only if}$$
$$\text{rank}\left[\ Q^h \mid P^h Q^h \mid \cdots \mid (P^h)^{2\delta_h - 1} Q^h\ \right] = 2\delta_h$$

which is equivalent to

$$\text{rank}\left[\ \begin{bmatrix} A^h B^h & B^h \end{bmatrix} \mid \begin{bmatrix} (A^h)^3 B^h & (A^h)^2 B^h \end{bmatrix} \mid \cdots \mid \begin{bmatrix} (A^h)^{4\delta_h - 1} B^h & (A^h)^{4\delta_h - 2} B^h \end{bmatrix}\ \right] = \delta_h$$

and this last equality is true because (A^h, B^h) is controllable by Proposition 1, (note that the matrix in the expression contains all the column blocks of the controllability matrix $\mathcal{C}^h$ of (A^h, B^h)).

In order to prove the controllability of $\left(P^v, \left[\ Q^v \mid P^{vh}\ \right]\right)$ note that P^v, P^{vh} and Q^v are given by

$$P^v = \begin{bmatrix} (A^v)^2 & 0 \\ 0 & (A^v)^2 \end{bmatrix}, \quad P^{vh} = \begin{bmatrix} A^v B^v \tilde{C}_0^h & B^v \tilde{C}_0^h \\ A^v B^v \tilde{C}_1^h A^h & B^v \tilde{C}_1^h A^h \end{bmatrix},$$

$$Q^v = \begin{bmatrix} A^v B^v D_0^h & 0 & B^v D_0^h & 0 \\ A^v B^v \tilde{C}_1^h B^h & A^v B^v D_1^h & B^v \tilde{C}_1^h B^h & B^v D_1^h \end{bmatrix}.$$

Applying block column permutations and defining the matrix

$$M = \begin{bmatrix} \tilde{C}_0^h & D_0^h & 0 \\ \tilde{C}_1^h A^h & \tilde{C}_1^h B^h & D_1^h \end{bmatrix}$$

the pair $\left(P^v, \left[\ Q^v \mid P^{vh}\ \right]\right)$ becomes:

$$\left(\begin{bmatrix} (A^v)^2 & 0 \\ 0 & (A^v)^2 \end{bmatrix}, \left[\begin{bmatrix} B^v & 0 \\ 0 & B^v \end{bmatrix} M \;\middle|\; \begin{bmatrix} A^v B^v & 0 \\ 0 & A^v B^v \end{bmatrix} M \right]\right). \tag{18}$$

Therefore, by Proposition 2 we have that $\left(P^v, \left[\ Q^v \mid P^{vh}\ \right]\right)$ is controllable if and only if

$$\operatorname{rank}\left[\begin{bmatrix} B^v & 0 \\ 0 & B^v \end{bmatrix} M \quad \begin{bmatrix} A^v B^v & 0 \\ 0 & A^v B^v \end{bmatrix} M \;\middle|\; \begin{bmatrix} (A^v)^2 B^v & 0 \\ 0 & (A^v)^2 B^v \end{bmatrix} M \quad \begin{bmatrix} (A^v)^3 B^v & 0 \\ 0 & (A^v)^3 B^v \end{bmatrix} M \;\middle|\; \cdots \right] = 2\delta_v$$

or, equivalently,

$$\operatorname{rank}\left[\left[\begin{bmatrix} B^v & 0 \\ 0 & B^v \end{bmatrix} \quad \begin{bmatrix} A^v B^v & 0 \\ 0 & A^v B^v \end{bmatrix} \quad \begin{bmatrix} (A^v)^2 B^v & 0 \\ 0 & (A^v)^2 B^v \end{bmatrix} \quad \begin{bmatrix} (A^v)^3 B^v & 0 \\ 0 & (A^v)^3 B^v \end{bmatrix} \quad \cdots \right] \operatorname{diag}(M)\right] = 2\delta_v,$$

which is clearly true since the matrix M has full row rank and (A^v, B^v) is controllable by Proposition 1. ■

To prove that the realization $\Sigma^L = (P, Q, R, S)$ given by (12) is separately locally observable we will first prove the next auxiliary Lemma.

Lemma 5. *Let $\nu_1, \nu_2, \ldots, \nu_k$ be nonnegative integers and define the matrix $A = \operatorname{diag}(A_1, A_2, \ldots, A_k)$ where*

$$A_i = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} \in \mathbb{F}^{\nu_i \times \nu_i}.$$

In the case where $\nu_i = 0$ the ith block of A is void.

Consider two matrices $C_0, C_1 \in \mathbb{F}^{n \times \overline{\nu}_k}$, with $\overline{\nu}_j = \sum_{i=1}^{j} \nu_i$, $j = 1, \ldots, k$, such that

$$[C_0]_{\overline{\nu}_1, \overline{\nu}_2, \ldots, \overline{\nu}_k} \quad \textit{and} \quad [C_1]_{\overline{\nu}_1, \overline{\nu}_2, \ldots, \overline{\nu}_k}$$

have full column rank, where $[C_i]_{\overline{\nu}_1, \overline{\nu}_2, \ldots, \overline{\nu}_k}$ represents the submatrix of C_i, $i = 0, 1$, with columns $\overline{\nu}_1, \overline{\nu}_2, \ldots, \overline{\nu}_k$.

Then the matrix

$$\begin{bmatrix} C_0 \\ C_1 A \\ C_0 A^2 \\ \vdots \\ C_r A^{\nu-1} \end{bmatrix},$$

where $\nu = \max \nu_i$, $i = 1, \ldots, k$ and $r = (\nu - 1) \bmod 2$, has full column rank.

Proof. Define $\overline{\nu}_j^{(n)} = \overline{\nu}_j - n$ if $\overline{\nu}_j - n \geq 0$ (otherwise $\overline{\nu}_j^{(n)}$ is not defined).

By hypothesis, $[C_0]_{\overline{\nu}_1, \overline{\nu}_2, \ldots, \overline{\nu}_k}$ has full column rank. Note that the columns $\overline{\nu}_1, \overline{\nu}_2, \ldots, \overline{\nu}_k$ of A are zero, which implies that the columns of the same index of

$$\begin{bmatrix} C_1 A \\ C_0 A^2 \\ C_1 A^3 \\ \vdots \end{bmatrix}$$

are also zero. Moreover, $[C_1 A]_{\overline{\nu}_1^{(1)}, \overline{\nu}_2^{(1)}, \ldots, \overline{\nu}_k^{(1)}} = [C_1]_{\overline{\nu}_1, \overline{\nu}_2, \ldots, \overline{\nu}_k}$ has full column rank.

In the same way, the columns $\overline{\nu}_1, \overline{\nu}_1^{(1)} \overline{\nu}_2, \overline{\nu}_2^{(1)} \ldots, \overline{\nu}_k, \overline{\nu}_k^{(1)}$ of A^2 are also zero and therefore the columns of the same index of

$$\begin{bmatrix} C_0 A^2 \\ C_1 A^3 \\ \vdots \end{bmatrix}$$

are also zero and $\left[C_0 A^2\right]_{\overline{\nu}_1^{(2)}, \overline{\nu}_2^{(2)}, \ldots, \overline{\nu}_k^{(2)}} = [C_0]_{\overline{\nu}_1, \overline{\nu}_2, \ldots, \overline{\nu}_k}$ has full column rank.

Proceeding analogously it is easy to check that

$$\begin{bmatrix} C_0 \\ C_1 A \\ C_0 A^2 \\ \vdots \\ C_r A^{\nu-1} \end{bmatrix}$$

has full column rank. ■

Lemma 6. *In the conditions of Theorem 6, the realization* $\Sigma^L = (P, Q, R, S)$ *given by* (12) *is separately locally observable.*

Proof. By Lemma 2, we just have to prove that the matrices (P^v, R^v) and $\left(P^h, \begin{bmatrix} R^h \\ P^{vh} \end{bmatrix}\right)$ are observable, where the involved matrices are defined by (14) and (16) with $A_1^h = A_0^h = A^h$, $A_1^v = A_0^v = A^v$, $B_1^h = B_0^h = B^h$ and $B_1^v = \tilde{B}_0^v = \tilde{B}^v$.
The observability of (P^v, R^v) follows immediately by Proposition 2 and Lemma 5.
In order to prove the observability of $\left(P^h, \begin{bmatrix} R^h \\ P^{vh} \end{bmatrix}\right)$ note that this pair is equal to

$$\left(\begin{bmatrix} (A^h)^2 & 0 \\ 0 & (A^h)^2 \end{bmatrix}, \left[\begin{array}{cc} D_0^v \tilde{C}_0^h & 0 \\ D_0^v \tilde{C}_1^h A^h & 0 \\ C_1^v B^v \tilde{C}_0^h & D_1^v \tilde{C}_0^h \\ C_1^v B^v \tilde{C}_1^h A^h & D_1^v \tilde{C}_1^h A^h \\ \hline A^v B^v \tilde{C}_0^h & B^v \tilde{C}_0^h \\ A^v B^v \tilde{C}_1^h A^h & B^v \tilde{C}_1^h A^h \end{array}\right] \right). \tag{19}$$

Applying block row permutations and defining the matrix

$$N = \begin{bmatrix} D_0^v & 0 \\ C_1^v B^v & D_1^v \\ A^v B^v & B^v \end{bmatrix}$$

that has full column rank, by hypothesis, the pair (19) can be written as

$$\left(\begin{bmatrix} (A^h)^2 & 0 \\ 0 & (A^h)^2 \end{bmatrix}, \left[\begin{array}{c} N \begin{bmatrix} \tilde{C}_0^h & 0 \\ 0 & \tilde{C}_0^h \end{bmatrix} \\ \hline N \begin{bmatrix} \tilde{C}_1^h A^h & 0 \\ 0 & \tilde{C}_1^h A^h \end{bmatrix} \end{array}\right] \right). \tag{20}$$

The rest of the proof is analogous to the final part of the proof of Lemma 4 taking in account Lemma 5. ■

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