# Application of the hypercomplex fractional integro-differential operators to the fractional Stokes equation*

M. Ferreira[†,‡], R.S. Kraußhar[§], M.M. Rodrigues[‡], N. Vieira[‡]

[†] School of Technology and Management
Polytechnic Institute of Leiria
P-2411-901, Leiria, Portugal.
Email: milton.ferreira@ipleiria.pt

[§] Fachgebiet Mathematik
Erziehungswissenschaftliche Fakultät, Universität Erfurt
Nordhäuserstr. 63, 99089 Erfurt, Germany.
E-mail: soeren.krausshar@uni-erfurt.de

[‡] CIDMA - Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal.
E-mails: mferreira@ua.pt, mrodrigues@ua.pt, nloureirovieira@gmail.com

**Abstract**

We present a generalization of several results of the classical continuous Clifford function theory to the context of fractional Clifford analysis. The aim of this paper is to show how the fractional integro-differential hypercomplex operator calculus can be applied to a concrete fractional Stokes problem in arbitrary dimensions which has been attracting recent interest (cf. [1,6]).

## 1 Basics on fractional calculus and special functions

For $a, b \in \mathbb{R}$ with $a < b$ and $\alpha > 0$, the left and right Riemann-Liouville fractional integrals $I^\alpha_{a^+}$ and $I^\alpha_{b^-}$ of order $\alpha$ are defined by (see [5])

$$\left(I^\alpha_{a^+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad x > a, \qquad \left(I^\alpha_{b^-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\, dt, \quad x < b. \tag{1}$$

By ${}^{RL}D^\alpha_{a^+}$ and ${}^{RL}D^\alpha_{b^-}$ we denote the left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ on $[a, b] \subset \mathbb{R}$ (see [5]):

$$\left({}^{RL}D^\alpha_{a^+} f\right)(x) = \left(D^m I^{m-\alpha}_{a^+} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}}\, dt, \qquad x > a \tag{2}$$

$$\left({}^{RL}D^\alpha_{b^-} f\right)(x) = (-1)^m \left(D^m I^{m-\alpha}_{b^-} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(t-x)^{\alpha-m+1}}\, dt, \qquad x < b. \tag{3}$$

*The final version is published in *AIP Conference Proceedings - ICNAAM 2018*, **2116**, (2019), Article No. 160004, 4pp. It as available via the website:`https://aip.scitation.org/doi/pdf/10.1063/1.5114148`

Here, $m = [\alpha] + 1$ and $[\alpha]$ means the integer part of $\alpha$. The symbols ${}^{C}D^{\alpha}_{a^+}$ and ${}^{C}D^{\alpha}_{b^-}$ denote the left (resp. right) Caputo fractional derivative of order $\alpha > 0$:

$$\left({}^{C}D^{\alpha}_{a^+} f\right)(x) = \left(I^{m-\alpha}_{a^+} D^m f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}\, dt, \qquad x > a \tag{4}$$

$$\left({}^{C}D^{\alpha}_{b^-} f\right)(x) = (-1)^m \left(I^{m-\alpha}_{b^-} D^m f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b \frac{f^{(m)}(t)}{(t-x)^{\alpha-m+1}}\, dt, \qquad x < b. \tag{5}$$

We denote by $I^{\alpha}_{a^+}(L_1)$ the class of functions $f$ that are represented by the fractional integral (1) of a summable function, that is $f = I^{\alpha}_{a^+}\varphi$, with $\varphi \in L_1(a,b)$. The space $AC^m([a,b])$ contains all functions that are continuously differentiable over $[a,b]$ up to the order $m-1$ and $f^{(m-1)}$ is supposed to be absolutely continuous over $[a,b]$.

To explicitly describe the integral kernels that are used in the sequel we need to introduce the two-parameter Mittag-Leffler function $E_{\mu,\nu}(z)$ (cf [3]) as $E_{\mu,\nu}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\mu k+\nu)}$, $\mu > 0$, $\nu \in \mathbb{R}$, $z \in \mathbb{C}$. Let us now turn to the treatment of the higher dimensional setting. We consider bounded open rectangular domains in $\mathbb{R}^n$ of the form $\Omega = \prod_{i=1}^n ]a_i, b_i[$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in ]0,1]$, $i = 1, \ldots, n$. The $n$-parameter fractional Laplace operators ${}^{RL}\Delta^{\alpha}_{a^+}$ and ${}^{C}\Delta^{\alpha}_{a^+}$, and the associated fractional Dirac operators ${}^{RL}\mathcal{D}^{\alpha}_{a^+}$ and ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ acting on the variables $(x_1, \cdots, x_n) \in \mathbb{R}^n$ are defined over $\Omega$ by

$${}^{RL}\Delta^{\alpha}_{a^+} = \sum_{i=1}^n {}^{RL}_{a_i^+}\partial^{1+\alpha_i}_{x_i}, \quad {}^{C}\Delta^{\alpha}_{a^+} = \sum_{i=1}^n {}^{C}_{a_i^+}\partial^{1+\alpha_i}_{x_i}, \quad {}^{RL}\mathcal{D}^{\alpha}_{a^+} = \sum_{i=1}^n e_i\, {}^{RL}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}, \quad {}^{C}\mathcal{D}^{\alpha}_{a^+} = \sum_{i=1}^n e_i\, {}^{C}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}. \tag{6}$$

For $i = 1, \ldots, n$ the partial derivatives ${}^{RL}_{a_i^+}\partial^{1+\alpha_i}_{x_i}$, ${}^{RL}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}$, ${}^{C}_{a_i^+}\partial^{1+\alpha_i}_{x_i}$ and ${}^{C}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}$ are the left Riemann-Liouville and Caputo fractional derivatives (2) and (4) of orders $1+\alpha_i$ and $\frac{1+\alpha_i}{2}$, with respect to the variable $x_i \in ]a_i, b_i[$. Under certain conditions we have that ${}^{RL}\Delta^{\alpha}_{a^+} = -{}^{RL}\mathcal{D}^{\alpha}_{a^+}\, {}^{RL}\mathcal{D}^{\alpha}_{a^+}$ and ${}^{C}\Delta^{\alpha}_{a^+} = -\, {}^{C}\mathcal{D}^{\alpha}_{a^+}\, {}^{C}\mathcal{D}^{\alpha}_{a^+}$ (see [2]). Due to the nature of the eigenfunctions and the fundamental solution of these operators we additionally need to consider the variable $\widehat{x} = (x_2, \ldots, x_n) \in \widehat{\Omega} = \prod_{i=2}^n ]a_i, b_i[$, and the fractional Laplace and Dirac operators acting on $\widehat{x}$ defined by:

$${}^{RL}\widehat{\Delta}^{\alpha}_{a^+} = \sum_{i=2}^n {}^{RL}_{a_i^+}\partial^{1+\alpha_i}_{x_i}, \quad {}^{C}\widehat{\Delta}^{\alpha}_{a^+} = \sum_{i=2}^n {}^{C}_{a_i^+}\partial^{1+\alpha_i}_{x_i}, \quad {}^{RL}\widehat{\mathcal{D}}^{\alpha}_{a^+} = \sum_{i=2}^n e_i\, {}^{RL}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}, \quad {}^{C}\widehat{\mathcal{D}}^{\alpha}_{a^+} = \sum_{i=2}^n e_i\, {}^{C}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}. \tag{7}$$

Next recalling from [2] we know that a family of fundamental solutions of the fractional Dirac operator ${}^{C}\mathcal{D}^{\alpha}_{a^+}$ can be represented in the way ${}^{C}\mathcal{G}^{\alpha}_{+}(x) = \sum_{i=1}^n e_i \left({}^{C}\mathcal{G}^{\alpha}_{+}\right)_i(x)$, where

$$\begin{aligned}\left({}^{C}\mathcal{G}^{\alpha}_{+}\right)_1(x) = {} & (x_1-a_1)^{-\frac{1+\alpha_1}{2}} E_{1+\alpha_1, \frac{1-\alpha_1}{2}}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{C}\widehat{\Delta}^{\alpha}_{a^+}\right) g_0(\widehat{x}) \\ & + (x_1-a_1)^{\frac{1-\alpha_1}{2}} E_{1+\alpha_1, \frac{3-\alpha_1}{2}}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{C}\widehat{\Delta}^{\alpha}_{a^+}\right) g_1(\widehat{x}),\end{aligned} \tag{8}$$

and for $i = 2, \ldots, n$

$$\begin{aligned}\left({}^{C}\mathcal{G}^{\alpha}_{+}\right)_i(x) = {} & \left(E_{1+\alpha_1,1}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{C}\widehat{\Delta}^{\alpha}_{a^+}\right) {}^{C}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}\right) g_0(\widehat{x}) \\ & + (x_1-a_1)\left(E_{1+\alpha_1,2}\left(-(x_1-a_1)^{1+\alpha_1}\ {}^{C}\widehat{\Delta}^{\alpha}_{a^+}\right) {}^{C}_{a_i^+}\partial^{\frac{1+\alpha_i}{2}}_{x_i}\right) g_1(\widehat{x}),\end{aligned} \tag{9}$$

with $g_0(\widehat{x}) = v(a_1, \widehat{x})$ and $g_1(\widehat{x}) = v'_{x_1}(a_1, \widehat{x})$. The functions $v$ and $v'_{x_1}$ are defined in Corollary 3.5 of [2].

# 2 Fractional Hypercomplex Integral Operators

In this section we recall the definitions and the main properties of the fractional versions of the Teodorescu and Cauchy-Bitsadze operators that are going to be used in the sequel to treat the fractional Stokes problem. For all the detailed proofs and calculations we refer to our paper [2]. First we recall the following fractional Stokes formula

**Theorem 2.1** *Let $f, g \in C\ell_{0,n}(\Omega) \cap AC^1(\Omega) \cap AC(\overline{\Omega})$ Then we have*

$$\int_\Omega \left[ -\left(f \ {}^C\mathcal{D}^\alpha_{b^-}\right)(x)\, g(x) + f(x)\ \left({}^{RL}\mathcal{D}^\alpha_{a^+} g\right)(x)\right] dx = \int_{\partial\Omega} f(x)\, d\sigma(x)\ \left(I^\alpha_{a^+} g\right)(x), \tag{10}$$

*where $d\sigma(x) = n(x)\, d\Omega$, with $n(x)$ being the outward pointing unit normal vector at $x \in \partial\Omega$, where $d\Omega$ is the classical surface element, and where $dx$ represents the $n$-dimensional volume element.*

Replacing $f$ by ${}^C\mathcal{G}^\alpha_+(x-y)$ in (10) we now may obtain the following fractional Borel-Pompeiu formula (a detailed proof is presented in [2]).

**Theorem 2.2** *Let $g \in C\ell_{0,n}(\Omega) \cap AC^1(\Omega) \cap AC(\overline{\Omega})$. Then the following fractional Borel-Pompeiu formula holds*

$$-\int_\Omega {}^C\mathcal{G}^\alpha_+(x+a-y)\ \left({}^{RL}\mathcal{D}^\alpha_{a^+} g\right)(y)\, dy + \int_{\partial\Omega} {}^C\mathcal{G}^\alpha_+(x+a-y)\, d\sigma(y)\ \left(I^\alpha_{a^+} g\right)(y) = g(x). \tag{11}$$

From (11) we may introduce the following definition.

**Definition 2.3** *Let $g \in AC^1(\Omega)$. Then the linear integral operators*

$$\left(T^\alpha g\right)(x) = -\int_\Omega {}^C\mathcal{G}^\alpha_+(x+a-y)\, g(y)\ dy, \qquad \left(F^\alpha g\right)(x) = \int_{\partial\Omega} {}^C\mathcal{G}^\alpha_+(x+a-y)\, d\sigma(y)\, \left(I^\alpha_{a^+} g\right)(y) \tag{12}$$

*are called the fractional Teodorescu and Cauchy-Bitsadze operator, respectively.*

The previous definition allows us to rewrite (11) in the alternative form $\left(T^\alpha\, {}^{RL}\mathcal{D}^\alpha_{a^+} g\right)(x) + \left(F^\alpha g\right)(x) = g(x)$, with $x \in \Omega$. For the regularity and mapping properties of (12) we refer to [2]. Again, in [2] we proved the following result:

**Theorem 2.4** *The fractional operator $T^\alpha$ is the right inverse of ${}^C\mathcal{D}^\alpha_{a^+}$, i.e., for $g \in L_p(\Omega)$, with $p \in \left]1, \frac{2}{1-\alpha^*}\right[$ and $\alpha^* = \min_{1\le i\le n}\{\alpha_i\}$, we have $\left({}^C\mathcal{D}^\alpha_{a^+}\, T^\alpha g\right)(x) = g(x)$.*

All these tools in hand allow us to obtain the following Hodge-type decomposition which is our key tool to treat boundary value problems related to the fractional Dirac operator, such as presented with a small example in the next section (see [2] for a detailed proof):

**Theorem 2.5** *Let $q = \frac{2p}{2-(1-\alpha^*)p}$, $p \in \left]1, \frac{2}{1-\alpha^*}\right[$, and $\alpha^* = \min_{1\le i\le n}\{\alpha_i\}$. The space $L_q(\Omega)$ admits the following direct decomposition*

$$L_q(\Omega) = L_q(\Omega) \cap \ker\left({}^C\mathcal{D}^\alpha_{a^+}\right) \ \oplus\ {}^C\mathcal{D}^\alpha_{a^+}\left(\overset{\circ}{W}{}^{\alpha,p}_{a^+}(\Omega)\right), \tag{13}$$

*where $\overset{\circ}{W}{}^{\alpha,p}_{a^+}(\Omega)$ is the space of functions $g \in W^{\alpha,p}_{a^+}(\Omega)$ such that $\mathrm{tr}(g) = 0$. Moreover, we can define the following projectors*

$$P^\alpha : L_q(\Omega) \to L_q(\Omega) \cap \ker\left({}^C\mathcal{D}^\alpha_{a^+}\right), \qquad Q^\alpha : L_q(\Omega) \to {}^C\mathcal{D}^\alpha_{a^+}\left(\overset{\circ}{W}{}^{\alpha,p}_{a^+}(\Omega)\right).$$

In the previous theorem the fractional Sobolev space $W^{\alpha,p}_{a^+}(\Omega)$ has the following norm:

$$\|f\|^p_{W^{\alpha,p}_{a^+}(\Omega)} := \|f\|^p_{L_p(\Omega)} + \sum_{k=1}^n \left\| {}^C_{a_k^+}\partial^{\frac{1+\alpha_k}{2}}_{x_k} f \right\|^p_{L_p(\Omega)},$$

where $\|\cdot\|_{L_p(\Omega)}$ is the usual $L_p$-norm in $\Omega$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_k \in ]0,1]$, $k = 1, \ldots, n$.

**Remark 2.6** *We would like to remark that our results coincide with the classical ones presented in [4] when considering the limit case of $\alpha = (1, \ldots 1)$. However, we can notice differences in the fractional setting, for instance in the expression of the fundamental solution and in the function spaces considered.*

## 3 A fractional Stokes problem

Recently fractional versions of the Stokes problem have attracted a fast growing interest (see for instance [6]). The application of the version of the Laplacian allows us to model sub-diffusion problems of (in our case incompressible) flows. The following system describes the simplest model of Stokes equation with sub (resp. super) dissipation. In the Riemann-Liouville case (the Caputo case is treated analogously) it has the form

$$\begin{aligned} -\,{}^{RL}\Delta^{\alpha}_{a^+} u + \operatorname{grad}^{\alpha}\mathfrak{p} &= F \quad \text{in } \Omega \\ \operatorname{div}^{\alpha} u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Here again we suppose that $\Omega$ is a rectangular domain, $F$ is given, $\mathfrak{p}$ is the unknown pressure of the flow and $u$ its unknown velocity. As in the continuous case treated in [4], the hypercomplex fractional calculus that we proposed in the previous section, now allows us to set up closed solution formulas for $u$ and $\mathfrak{p}$. To proceed in this direction, remember that following [2] the fractional Laplacian can be split in the form ${}^{RL}\Delta^{\alpha}_{a^+} = -{}^{RL}\mathcal{D}^{\alpha}_{a^+}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+}$. Applying the previously described inverse properties of the Teodorescu transform and the properties of the projector $Q^{\alpha}$ arising in the Hodge decomposition stated at the end of the previous section allows us to transform the first equation as follows:

$${}^{RL}\mathcal{D}^{\alpha}_{a^+}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+} u + {}^{RL}\mathcal{D}^{\alpha}_{a^+}\mathfrak{p} = F.$$

If we now apply the fractional $T^{\alpha}$-operator from the left to this equation we get

$$T^{\alpha}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+} u + T^{\alpha}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+}\mathfrak{p} = T^{\alpha}F.$$

Now we can apply our generalized fractional Borel-Pompeiu formula leading to

$${}^{RL}\mathcal{D}^{\alpha}_{a^+} u - F^{\alpha}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+} u + \mathfrak{p} - F^{\alpha}\mathfrak{p} = T^{\alpha}F.$$

Application of the projector $Q^{\alpha}$ arising the fractional version of the Hodge decomposition then leads to

$$Q^{\alpha}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+} u - Q^{\alpha}\,F^{\alpha}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+} u + Q^{\alpha}\mathfrak{p} - Q^{\alpha}\,F^{\alpha}\mathfrak{p} = T^{\alpha}F.$$

Since $F^{\alpha}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+} u$ and $F^{\alpha}\mathfrak{p}$ are in the kernel of ${}^{RL}\mathcal{D}^{\alpha}_{a^+}$, we get $Q^{\alpha}\,F^{\alpha}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+} u = 0$ and $Q^{\alpha}\,F^{\alpha}\mathfrak{p} = 0$ so that our original equation simplifies to

$$Q^{\alpha}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+} u + Q^{\alpha}\mathfrak{p} = Q^{\alpha}\,T^{\alpha}F.$$

Next we apply once more $T^{\alpha}$ to the left of the equation and use that $Q^{\alpha}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+} = {}^{RL}\mathcal{D}^{\alpha}_{a^+} u$ so that the latter equation is equivalent to

$$T^{\alpha}\,{}^{RL}\mathcal{D}^{\alpha}_{a^+} u + T^{\alpha}\,Q^{\alpha}\mathfrak{p} = T^{\alpha}\,Q^{\alpha}\,T^{\alpha}F$$

which in turn equals

$$u - \underbrace{F^{\alpha}u}_{=0} + T^{\alpha}\,Q^{\alpha}\mathfrak{p} = T^{\alpha}\,Q^{\alpha}\,T^{\alpha}F,$$

so that we finally get the following formula for the velocity of the flow

$$u = T^{\alpha}\,Q^{\alpha}\,T^{\alpha}F - T^{\alpha}\,Q^{\alpha}\mathfrak{p}.$$

The pressure then can be determined by the equation

$$\operatorname{Sc}(Q^{\alpha}\mathfrak{p}) = \operatorname{Sc}(T^{\alpha}\,Q^{\alpha}\,T^{\alpha}F)$$

resulting from the second equation.

## Acknowledgement

The work of M. Ferreira, M.M. Rodrigues and N. Vieira was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project UID/MAT/0416/2013.

The work of the authors was supported by the project *New Function Theoretical Methods in Computational Electrodynamics / Neue funktionentheoretische Methoden für instationäre PDE*, funded by Programme for Cooperation in Science between Portugal and Germany ("Programa de Ações Integradas Luso-Alemãs/2017" - Acção No. A-15/17 - DAAD-PPP Deutschland-Portugal, Ref: 57340281).

N. Vieira was also supported by FCT via the FCT Researcher Program 2014 (Ref: IF/00271/2014).

## References

[1] P.M. de Carvalho-Neto and G. Planas, *Mild solutions to the time fractional Navier-Stokes equations in* $\mathbb{R}^n$, J. Differ. Equations, **259**-No.7, (2015), 2948-2980.

[2] M. Ferreira, R.S. Kraußhar, M.M. Rodrigues and N. Vieira; *A higher dimensional fractional Borel-Pompeiu formula and a related hypercomplex fractional operator calculus*, submitted.

[3] R. Gorenflo, A.A. Kilbas, F. Mainardi and S. Rogosin, *Mittag-Leffler functions. Theory and applications*, Springer Monographs in Mathematics, Springer, Berlin, 2014.

[4] K. Gürlebeck and W. Sprößig, *Quaternionic and Clifford calculus for physicists and engineers*, Mathematical Methods in Practice, Wiley, Chichester, 1997.

[5] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies-Vol.204, Elsevier, Amsterdam, 2006.

[6] S. Lin, M. Azaïez and C. Xu, *Fractional Stokes equation and its spectral approximation*, Int. J. Numer. Anal. Model., **15**-No.1-2, (2018), 170-192.