NEW CONVOLUTIONS WEIGHTED BY HERMITE FUNCTIONS AND THEIR APPLICATIONS[†]

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Abstract. We introduce eight new convolutions weighted by multi-dimensional Hermite functions, prove two Young-type inequalities, and exhibit their applications in different subjects. One application consists in the study of the solvability of a very general class of integral equations whose kernel depends on four different functions. Necessary and sufficient conditions for the unique solvability of such integral equations are here obtained.

1. Introduction

We start by recalling that the classical Hermite functions satisfy the differential equation

$$\psi_n''(x) + (2n+1-x^2)\psi_n(x) = 0,$$

which is equivalent to the Schrödinger equation for a harmonic oscillator in Quantum Mechanics, as these functions are their eigenfunctions. More generally, the multi-dimensional Hermite functions are defined by

$$\Phi_\alpha(x) := (-1)^{|\alpha|} e^{\frac{1}{2}|x|^2} D_x^\alpha e^{-|x|^2}, \quad x \in \mathbb{R}^n,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers $\alpha_k, k = 1, \ldots, n$, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. The Hermite functions form an orthonormal basis of $L^2(\mathbb{R})$, and they are closely related to the Whittaker function. Theoretically, Hermite functions can be seen as essential particular components of Functional Analysis by which many objects have been developed (see [12] and references therein). In practice, the Gaussian functions are widely known for describing the normal distributions in Statistics, defining Gaussian filters in image and signal processing, solving heat and diffusion equations, generating the Weierstrass transform, etc., while the Hermite functions are related to the parabolic cylinder and essential functions in Harmonic Analysis. All of

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those are concerned to the theory of Integral Equations. General integral equations are key objects both in theoretical and applied research, showing a long-term steady number of publications (for instance, for the above sense, see [1, 2, 3, 4, 5, 7, 8, 11, 14, 15] and references therein).

Motivated by the above references, the main aim of this work is to introduce new convolutions which will help us to analyse e.g. some "classic" integral equations in a way, or for conditions, not previously known. A very general class of integral equations, to which we will be able to apply our new convolutions and derive new results, has the form

$$\lambda \varphi(x) + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big[k_1(u)\Phi_\alpha(x-u-v) + k_2(u)\Phi_\alpha(x-u+v) + k_3(u)\Phi_\alpha(x+u-v) + k_4(u)\Phi_\alpha(x+u+v) \Big] \varphi(v)\, du dv = h(x), \tag{1.1}$$

where the kernel is formed by the sum of the just exhibited four components, $\lambda \in \mathbb{C}$, $k_1, k_2, k_3, k_4, h \in L^1(\mathbb{R}^n)$ are given, and φ is unknown in $L^1(\mathbb{R}^n)$. We stress that this is not a convolution equation. Anyway, with respect to practical applications of such kind of equations in image and digital signal processing, we would like to point out that the second term in the left-hand side of (1.1) can be seen as an arbitrary combination of convolutions and cross-correlations of the Hermite filtering Φ_α with the arbitrary L^1-function coefficients k_1, k_2, k_3, k_4. The class of such equations is enough large for our consideration. Therefore, the main idea to treat the equation (1.1) will be based on the use of a set of eight new convolutions associated with an integral operator (and its inverse) and Hermite functions. In particular, this will exhibit the effective use of such convolutions. In addition, further functional characteristics of those convolutions will be investigated.

This work is divided into four sections and organized as follows. In the next section – which may be seen as an auxiliary one – we recall an integral operator and its essential properties which are helpful for the proofs in the forthcoming sections. Section 3 is a crucial part of the paper and there we provide two sets of convolutions presented in Definitions 3.1 and 3.2, and prove some fundamental properties of such convolutions in Theorems 6 and 7. Propositions 1 and 2 show that the eight constructed convolutions are barely sufficient and helpful for our applied purposes. In this third section, we also deal with Young-type inequalities and Wiener algebras associated with the presented convolutions. Section 4 is devoted to the application of the introduced convolutions in the analysis of the integral equation exhibited above. Indeed, we prove there a sufficient and necessary condition for the solvability of the equation (1.1) in $L^1(\mathbb{R}^n)$ and derive the explicit solution. In particular, each one of the functional identities (4.6), (4.7), and (4.8) in Theorem 10 can be called *solvable conditions* of the considered equation within the Banach space $L^1(\mathbb{R}^n)$, in some sense. At the end of this section, some examples of integral equations are investigated by the convolution approach.

2. Auxiliary Machinery

As already explained, this section has an auxiliary nature. We will introduce here some auxiliary objects which will help us on the forthcoming developments. Namely, in what follows we will identify a particular oscillatory integral operator (that the authors already analysed in [6] in view to obtain Heisenberg uncertainty principles) whose properties will be intrinsically associated with the convolutions to be presented in the next section.

DEFINITION 2.1. (see [6]) Consider the integral transform defined by

$$(Tf)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} [2\cos(xy) + i\sin(xy)]\, f(y)\, dy. \tag{2.1}$$

We see that $T = 2T_c + iT_s$, where

$$(T_c f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \cos(xy) f(y) dy, \quad (T_s f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sin(xy) f(y) dy \tag{2.2}$$

are the Fourier cosine and Fourier sine integral transforms, respectively. Let us recall the Fourier and inverse Fourier transforms:

$$(F^{\pm 1} f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\mp ixy} f(y) dy. \tag{2.3}$$

In fact, the operator T can also be rewritten in terms of the Fourier and inverse Fourier transforms, as $2T = F + 3F^{-1}$.

Let us recall some lemmas and theorems by which we are able to complete some proofs in the next sections. Namely, upper bounds for some norms of that operator and invertibility results are already known in the following sense.

THEOREM 1. (Riemann-Lebesgue Lemma; see [6]) *T is a bounded linear operator from $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$. Namely, if $f \in L^1(\mathbb{R}^n)$, then $Tf \in C_0(\mathbb{R}^n)$ and*

$$\|Tf\|_\infty \leq \frac{2}{(2\pi)^{\frac{n}{2}}} \|f\|_1, \tag{2.4}$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are the usual supremum and L^1-norms, respectively.

Let $\mathscr{S}$ denote the Schwartz space and W the reflection operator, this is, $(W\varphi)(x) := \tilde{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}^n$.

THEOREM 2. (see [6]) *T is a continuous linear operator of $\mathscr{S}$ into itself, and fulfills the following reflection and polynomial identities:*

$$T^2 = \frac{3}{2}I + \frac{5}{2}W, \quad T^4 - 3T^2 - 4I = 0, \tag{2.5}$$

where I is the identity operator. Moreover, it is invertible in $\mathscr{S}$.

As a remark, we point out that the above identities exhibit a significant difference between our operator T and the Fourier transform. Indeed, for the Fourier transform it is well-known that $F^2 = W$ and $F^4 = I$.

COROLLARY 1. *If $f \in L^1(\mathbb{R}^n)$ and $Tf \in L^1(\mathbb{R}^n)$, then the identities (2.5) hold. Namely,*

$$(T^2 f)(x) = \frac{3}{2} f(x) + \frac{5}{2} f(-x), \quad \text{and} \quad (T^4 f)(x) - 3(Tf^2)(x) - 4f(x) = 0, \tag{2.6}$$

for almost every $x \in \mathbb{R}^n$.

Corollary 1 is a direct consequence of Theorem 2 by invoking the fact that $\mathscr{S}$ is dense in L^1 (and also in L^2).

THEOREM 3. (Inversion Theorem; see [6]) *If $f \in L^1(\mathbb{R}^n)$, and if $Tf \in L^1(\mathbb{R}^n)$, then*

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left[\frac{1}{2} \cos(xy) - i \sin(xy) \right] (Tf)(y)\, dy = f(x), \tag{2.7}$$

for almost every $x \in \mathbb{R}^n$.

We denote by T^{-1} the inverse operator of T.

THEOREM 4. (Riemann-Lebesgue Lemma for T^{-1}) *T^{-1} is a bounded linear operator from $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$. Namely, if $f \in L^1(\mathbb{R}^n)$, then $T^{-1} f \in C_0(\mathbb{R}^n)$ and $\|T^{-1} f\|_\infty \leq (2\pi)^{\frac{-n}{2}} \|f\|_1$.*

Indeed, we have

$$\begin{aligned}
\sup_{x \in \mathbb{R}^n} \left| (T^{-1} f)(x) \right| &= \frac{1}{(2\pi)^{\frac{n}{2}}} \sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} \left[\frac{1}{2} \cos(xy) - i \sin(xy) \right] f(y) dy \right| \\
&\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \sup_{x \in \mathbb{R}^n} \left| \frac{1}{2} \cos(xy) - i \sin(xy) \right| \|f\|_1 \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \left[\frac{1}{4} \cos^2(xy) + \sin^2(xy) \right]^{\frac{1}{2}} \|f\|_1 \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \|f\|_1 .
\end{aligned}$$

We also have the following corollary for T^{-1}.

COROLLARY 2. *Let $f \in L^1(\mathbb{R}^n)$ be given. If $T^{-1} f \in L^1(\mathbb{R}^n)$, then*

$$\begin{gathered}
(T^{-2} f)(x) = -\frac{3}{8} f(x) + \frac{5}{8} f(-x) \ \text{ and} \\
4(T^{-4} f)(x) + 3(T^{-2} f)(x) - f(x) = 0,
\end{gathered} \tag{2.8}$$

for almost every $x \in \mathbb{R}^n$.

Let us ignore the proof of this corollary as it is similar to the proof of Corollary 1. An integration of Corollaries 1 and 2 is given in Remark 4.1.

THEOREM 5. (see [6]) *The Hermite functions are eigenfunctions with eigenvalues* $\pm 2, \pm i$ *of the operator* T*:*

$$T\Phi_\alpha = \begin{cases} (-1)^{\frac{|\alpha|}{2}} 2\Phi_\alpha, & if\ |\alpha| \equiv 0,2 \pmod 4 \\ (-1)^{\frac{|\alpha|-1}{2}} i\Phi_\alpha, & if\ |\alpha| \equiv 1,3 \pmod 4). \end{cases}$$

We can now state the following corollary.

COROLLARY 3. *The Hermite functions are eigenfunctions with eigenvalues* $\pm 1/2$, $\pm i$ *of* T^{-1}*:*

$$T^{-1}\Phi_\alpha = \begin{cases} (-1)^{\frac{|\alpha|}{2}} \frac{1}{2}\Phi_\alpha, & if\ |\alpha| \equiv 0,2 \pmod 4 \\ (-1)^{\frac{|\alpha|+1}{2}} i\Phi_\alpha, & if\ |\alpha| \equiv 1,3 \pmod 4). \end{cases}$$

As consequence, the identity $T^n\Phi_\alpha = \Phi_\alpha$ cannot hold true for any $n \in \mathbb{Z}$, while $F^4\Phi_\alpha = \Phi_\alpha$ holds for the Fourier case. As emphasized in [6], T and T^{-1} are not unitary operators in the Hilbert space $L^2(\mathbb{R}^n)$, and satisfy $\|T\|_2 = 2,\ \|T^{-1}\|_2 = 1/2$.

We can also write the identity of Theorem 5 as

$$\Phi_\alpha = \begin{cases} \frac{(-1)^{\frac{|\alpha|}{2}}}{2} T\Phi_\alpha, & if\ |\alpha| \equiv 0,2 \pmod 4 \\ (-1)^{\frac{|\alpha|+1}{2}} iT\Phi_\alpha, & if\ |\alpha| \equiv 1,3 \pmod 4). \end{cases}$$

3. New Convolutions

In this section we present two sets of four new convolutions associated with the operators T and T^{-1}, respectively, and the Hermite functions. The convolutions constructed here are the main objects of this work. Their applicability will be mostly exhibited when analysing the solvability of the integral equation previously presented.

Let

$$C(\alpha) := \begin{cases} \frac{(-1)^{\frac{|\alpha|}{2}}}{2}, & if\ |\alpha| \equiv 0,2 \pmod 4 \\ (-1)^{\frac{|\alpha|+1}{2}} i, & if\ |\alpha| \equiv 1,3 \pmod 4). \end{cases} \tag{3.1}$$

DEFINITION 3.1. We define four convolution multiplications in $L^1(\mathbb{R}^n)$, for any two elements $f, g \in L^1(\mathbb{R}^n)$, as follows:

$$\begin{aligned}(f \underset{(1)}{*} g)(x) :=& \frac{C(\alpha)}{4(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[(10 + (-1)^{|\alpha|}3)\Phi_\alpha(x-u-v) + 3\Phi_\alpha(x-u+v) \\ &+ 3\Phi_\alpha(x+u-v) - (-1)^{|\alpha|}3\Phi_\alpha(x+u+v)\Big] f(u)g(v)\,dudv; \end{aligned} \tag{3.2}$$

$$\begin{aligned}(f \underset{(2)}{*} g)(x) :=& \frac{C(\alpha)}{32(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[-(15 + (-1)^{|\alpha|}9)\Phi_\alpha(x-u-v) \\ &+ (41 + (-1)^{|\alpha|}15)\Phi_\alpha(x-u+v) - (9 + (-1)^{|\alpha|}15)\Phi_\alpha(x+u-v) \\ &+ (15 + (-1)^{|\alpha|}9)\Phi_\alpha(x+u+v)\Big] f(u)g(v)\,dudv; \end{aligned} \tag{3.3}$$

$$(f \underset{(3)}{*} g)(x) := \frac{C(\alpha)}{16(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big[-(-1)^{|\alpha|} 3\Phi_\alpha(x-u-v) - 3\Phi_\alpha(x-u+v) - 3\Phi_\alpha(x+u-v) + (10+(-1)^{|\alpha|}3)\Phi_\alpha(x+u+v) \Big] f(u)g(v)\,dudv; \tag{3.4}$$

$$(f \underset{(4)}{*} g)(x) := \frac{C(\alpha)}{32(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big[-(15+(-1)^{|\alpha|}9)\Phi_\alpha(x-u-v) - (9+(-1)^{|\alpha|}15)\Phi_\alpha(x-u+v) + (41+(-1)^{|\alpha|}15)\Phi_\alpha(x+u-v) + (15+(-1)^{|\alpha|}9)\Phi_\alpha(x+u+v) \Big] f(u)g(v)\,dudv. \tag{3.5}$$

The next theorem clarifies the relation of those convolutions with the previously considered operators.

THEOREM 6. *If $f,g \in L^1(\mathbb{R}^n)$, then each one of the multiplications introduced in Definition* 3.1 *has its factorization identity associated with the integral operators T, T^{-1} and the function $\Phi_\alpha(x)$ and the norm inequality:*

$$(T(f \underset{(1)}{*} g))(x) = \Phi_\alpha(x)(Tf)(x)(Tg)(x), \tag{3.6}$$
$$\|f \underset{(1)}{*} g\|_1 \leq \frac{11|C(\alpha)|}{2(2\pi)^n} \|f\|_1 \|g\|_1 \|\Phi_\alpha\|_1;$$

$$(T(f \underset{(2)}{*} g))(x) = \Phi_\alpha(x)(Tf)(x)(T^{-1}g)(x), \tag{3.7}$$
$$\|f \underset{(2)}{*} g\|_1 \leq \frac{4|C(\alpha)|}{(2\pi)^n} \|f\|_1 \|g\|_1 \|\Phi_\alpha\|_1;$$

$$(T(f \underset{(3)}{*} g))(x) = \Phi_\alpha(x)(T^{-1}f)(x)(T^{-1}g)(x), \tag{3.8}$$
$$\|f \underset{(3)}{*} g\|_1 \leq \frac{11|C(\alpha)|}{8(2\pi)^n} \|f\|_1 \|g\|_1 \|\Phi_\alpha\|_1;$$

$$(T(f \underset{(4)}{*} g))(x) = \Phi_\alpha(x)(T^{-1}f)(x)(Tg)(x), \tag{3.9}$$
$$\|f \underset{(4)}{*} g\|_1 \leq \frac{4|C(\alpha)|}{(2\pi)^n} \|f\|_1 \|g\|_1 \|\Phi_\alpha\|_1.$$

Proof. We give a very short proof of these inequalities, and take also the opportunity to introduce some notation that is needed for the next procedures. For short, let us write:

$$P(x) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_\alpha(x-u-v) f(u)g(v)\,dudv; \tag{3.10}$$
$$Q(x) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_\alpha(x-u+v) f(u)g(v)\,dudv; \tag{3.11}$$
$$R(x) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_\alpha(x+u-v) f(u)g(v)\,dudv; \tag{3.12}$$

$$S(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi_\alpha(x+u+v) f(u) g(v)\, dudv. \tag{3.13}$$

Then, P, Q, R and S define L^1-functions. Indeed, noticing that

$$\int_{\mathbb{R}^n} \Phi_\alpha(x \pm u \pm v) dx = \int_{\mathbb{R}^n} \Phi_\alpha(s) ds = \|\Phi_\alpha\|_1, \quad \text{for any} \quad u, v \in \mathbb{R}^n,$$

we have

$$\begin{aligned}\int_{\mathbb{R}^n} |P(x)|\, dx &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\Phi_\alpha(x-u-v)|\, |f(u)||g(v)| dxdudv \\ &\leq \int_{\mathbb{R}^n} |f(u)| du \int_{\mathbb{R}^n} |g(v)| dv \int_{\mathbb{R}^n} |\Phi_\alpha(x-u-v)| dx \leq \|f\|_1 \|g\|_1 \|\Phi_\alpha\|_1 < \infty,\end{aligned}$$

which gives that $P \in L^1(\mathbb{R}^n)$ and $\|P\|_1 \leq \|f\|_1\|g\|_1\|\Phi_\alpha\|_1$. Analogously,

$$\|Q\|_1 \leq \|f\|_1\|g\|_1\|\Phi_\alpha\|_1, \ \|R\|_1 \leq \|f\|_1\|g\|_1\|\Phi_\alpha\|_1, \ \text{ and } \ \|S\|_1 \leq \|f\|_1\|g\|_1\|\Phi_\alpha\|_1$$

which together imply that the norms of the L^1-functions P, Q, R, S are bounded by the same number $\|f\|_1\|g\|_1\|\Phi_\alpha\|_1$. By this and some straightforward computation, we reach to the norm inequalities in (3.6)-(3.9).

Let us prove the identity (3.6). We perform the direct computation

$$\begin{aligned}\Phi_\alpha(x)(Tf)(x)(Tg)(x) &= \frac{\Phi_\alpha(x)}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [2\cos(xu) + i\sin(xu)] \\ &\qquad [2\cos(xv) + i\sin(xv)]\, f(u) g(v)\, dudv \\ &= \frac{\Phi_\alpha(x)}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big\{ \frac{13}{8}\big[2\cos(x(u+v)) + i\sin(x(u+v))\big] \\ &\qquad + \frac{3}{8}\big[2\cos(x(u-v)) + i\sin(x(u-v))\big] \\ &\qquad + \frac{3}{8}\big[2\cos(x(-u+v)) + i\sin(x(-u+v))\big] \\ &\qquad - \frac{3}{8}\big[2\cos(x(-u-v)) + i\sin(x(-u-v))\big] \Big\} f(u) g(v) dudv.\end{aligned} \tag{3.14}$$

Using the identity (3.1), we have

$$\begin{aligned}\Phi_\alpha(x)(Tf)(x)(Tg)(x) = &\frac{C(\alpha)}{(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (2\cos(xt) + i\sin(xt)) \\ &\Big\{ \frac{13}{8}\big[2\cos(x(u+v)) + i\sin(x(u+v))\big] \\ &+ \frac{3}{8}\big[2\cos(x(u-v)) + i\sin(x(u-v))\big] \\ &+ \frac{3}{8}\big[2\cos(x(-u+v)) + i\sin(x(-u+v))\big] \\ &- \frac{3}{8}\big[2\cos(x(-u-v)) + i\sin(x(-u-v))\big] \Big\} \Phi_\alpha(t) f(u) g(v) dudvdt,\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&\Phi_\alpha(x)(Tf)(x)(Tg)(x)\\
&=\frac{C(\alpha)}{4(2\pi)^{3n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[10\,(2\cos(x(t+u+v))+i\sin(x(t+u+v)))\\
&\quad+3\,(2\cos(x(-t+u+v))+i\sin(x(-t+u+v)))\\
&\quad-3\,(2\cos(x(-t-u-v))+i\sin(x(-t-u-v)))\\
&\quad+3\,(2\cos(x(t+u-v))+i\sin(x(t+u-v)))\\
&\quad+3\,(2\cos(x(t-u+v))+i\sin(x(t-u+v)))]\,\Phi_\alpha(t)f(u)g(v)\,dudvdt\\
&=\frac{C(\alpha)}{4(2\pi)^{3n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(2\cos(xs)+i\sin(xs))\,[10\Phi_\alpha(s-u-v)\\
&\quad+3\Phi_\alpha(-s+u+v)-3\Phi_\alpha(-s-u-v)+3\Phi_\alpha(s-u+v)+3\Phi_\alpha(s+u-v)]\\
&\quad f(u)g(v)dudvds
\end{aligned}$$

Having in mind that $\Phi_\alpha(-x)=(-1)^{|\alpha|}\Phi_\alpha(x)$, for $|\alpha|\equiv 0,1,2,3 \pmod 4$, we have

$$\begin{aligned}
&\Phi_\alpha(x)(Tf)(x)(Tg)(x)\\
&=\frac{C(\alpha)}{4(2\pi)^{3n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(2\cos(xs)+i\sin(xs))\Big[(10+(-1)^{|\alpha|}3)\Phi_\alpha(s-u-v)\\
&\quad+3\Phi_\alpha(s-u+v)+3\Phi_\alpha(s+u-v)-(-1)^{|\alpha|}3\Phi_\alpha(s+u+v)\Big]f(u)g(v)dudvds\\
&=\frac{C(\alpha)}{4(2\pi)^n}\Big[T\Big((10+(-1)^{|\alpha|}3)P+3Q+3R-(-1)^{|\alpha|}3S\Big)\Big](x)=(T(f\underset{(1)}{*}g))(x),
\end{aligned}$$

which proves identity (3.6).

To prove the factorization identities (3.7)–(3.9), we should decompose the corresponding kernels and use the same technique. We will not present those details in here. □

PROPOSITION 1. *The convolutions presented in Definition* 3.1 *are linearly independent.*

Proof. As realized in the proof of Theorem 6, P, Q, R and S defined by (3.10)–(3.13) are L^1-functions. In fact, they are linearly independent in $L^1(\mathbb{R}^n)$. Indeed, suppose that

$$aP(x)+bQ(x)+cR(x)+dS(x)=0 \text{ (in } L^1(\mathbb{R}^n) \text{ for some } a,b,c,d\in\mathbb{C}).$$

Considering $\tilde{f}(x)=f(-x)$ and the Fourier transform $\hat{f}(x)=(Ff)(x)$, for $f\in L^1(\mathbb{R}^n)$, we split this equality as

$$a(\Phi_\alpha*f*g)(x)+b(\Phi_\alpha*f*\tilde{g})(x)+c(\Phi_\alpha*\tilde{f}*g)(x)+(\Phi_\alpha*\tilde{f}*\tilde{g})(x)=0$$

in $L^1(\mathbb{R}^n)$, where $*$ stands for the Fourier convolution. Acting the Fourier transform F to both sides of this identity, we obtain

$$(-i)^{|\alpha|}\Phi_\alpha(x)\left[a\widehat{f}(x)\widehat{g}(x)+b\widehat{f}(x)\widehat{\tilde{g}}(x)+c\widehat{\tilde{f}}(x)\widehat{g}(x)+d\widehat{\tilde{f}}(x)\widehat{\tilde{g}}(x)\right]=0.$$

Note that Φ_α is an Hermite function, and this identification is in the Wiener's algebra $\mathscr{W}_F := F(L^1(\mathbb{R}^n))$ equipped with the pointwise multiplication. It follows that

$$a\widehat{f}(x)\widehat{g}(x)+b\widehat{f}(x)\widehat{\tilde{g}}(x)+c\widehat{\tilde{f}}(x)\widehat{g}(x)+d\widehat{\tilde{f}}(x)\widehat{\tilde{g}}(x)=0. \tag{3.15}$$

In (3.15), choosing $f=\Phi_\beta, g=\Phi_\gamma$ and using

$$F\Phi_\alpha = (-i)^{|\alpha|}\Phi_\alpha, \quad F^{-1}\Phi_\alpha = i^{|\alpha|}\Phi_\alpha, \tag{3.16}$$

$$\Phi_\alpha = i^{|\alpha|}F\Phi_\alpha, \quad \Phi_\alpha = (-i)^{|\alpha|}F^{-1}\Phi_\alpha, \tag{3.17}$$

we obtain

$$\begin{cases} a+b+c+d=0, & \text{if} \quad |\beta|=0,2 \text{ and } |\gamma|=0,2 \pmod 4,\\ a-b+c-d=0, & \text{if} \quad |\beta|=0,2 \text{ and } |\gamma|=1,3 \pmod 4,\\ a+b-c-d=0, & \text{if} \quad |\beta|=1,3 \text{ and } |\gamma|=0,2 \pmod 4,\\ a-b-c+d=0, & \text{if} \quad |\beta|=1,3 \text{ and } |\gamma|=1,3 \pmod 4. \end{cases}$$

This implies $a=b=c=d=0$, as desired. Now, we suppose that

$$\theta_1(f \underset{(1)}{*} g)+\theta_2(f \underset{(2)}{*} g)+\theta_3(f \underset{(3)}{*} g)+\theta_4(f \underset{(4)}{*} g)=0$$

for some complex numbers $\theta_1,\theta_2,\theta_3,\theta_4 \in \mathbb{C}$. This is equivalent to

$$\begin{aligned} &[8(10+(-1)^{|\alpha|}3)\theta_1-(15+(-1)^{|\alpha|}9)\theta_2-(-1)^{|\alpha|}6\theta_3-(15+(-1)^{|\alpha|}9)\theta_4\Big]P\\ &\quad+\Big[24\theta_1+(41+(-1)^{|\alpha|}15)\theta_2-6\theta_3-(9+(-1)^{|\alpha|}15)\theta_4\Big]Q\\ &\quad+\Big[24\theta_1-(9+(-1)^{|\alpha|}15)\theta_2-6\theta_3+(41+(-1)^{|\alpha|}15)\theta_4\Big]R\\ &\quad+\Big[-(-1)^{|\alpha|}24\theta_1+(15+(-1)^{|\alpha|}9)\theta_2+2(10+(-1)^{|\alpha|}3)\theta_3\\ &\quad+(15+(-1)^{|\alpha|}9)\theta_4\Big]S=0. \end{aligned}$$

As proved above, we obtain the system of equations

$$\begin{cases} 8(10+(-1)^{|\alpha|}3)\theta_1-(15+(-1)^{|\alpha|}9)\theta_2-(-1)^{|\alpha|}6\theta_3-(15+(-1)^{|\alpha|}9)\theta_4 &=0\\ 24\theta_1+(41+(-1)^{|\alpha|}15)\theta_2-6\theta_3-(9+(-1)^{|\alpha|}15)\theta_4 &=0\\ 24\theta_1-(9+(-1)^{|\alpha|}15)\theta_2-6\theta_3+(41+(-1)^{|\alpha|}15)\theta_4 &=0\\ -(-1)^{|\alpha|}24\theta_1+(15+(-1)^{|\alpha|}9)\theta_2+2(10+(-1)^{|\alpha|}3)\theta_3+(15+(-1)^{|\alpha|}9)\theta_4 &=0. \end{cases}$$

To simplify the computations, we will compute the determinant of the system when $|\alpha| \equiv 0,2 \pmod 4$ and $|\alpha| \equiv 1,3 \pmod 4$, respectively. For the first case ($|\alpha| \equiv 0,2 \pmod 4$), the determinant is

$$\begin{vmatrix} 104 & -24 & -6 & -24 \\ 24 & 56 & -6 & -24 \\ 24 & -24 & -6 & 56 \\ -24 & 24 & 26 & 24 \end{vmatrix} = -10240000.$$

For the second case ($|\alpha| \equiv 1,3 \pmod 4$), we have

$$\begin{vmatrix} 56 & -6 & 6 & -6 \\ 24 & 26 & -6 & 6 \\ 24 & 6 & -6 & 26 \\ 24 & 6 & 14 & 6 \end{vmatrix} = -640000.$$

So, the solution is trivial, i.e., $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$. □

The second set of four convolutions is given by Definition 3.2.

DEFINITION 3.2. We introduce new convolution multiplications in $L^1(\mathbb{R}^n)$, for any two elements $f,g \in L^1(\mathbb{R}^n)$, as follows:

$$\begin{aligned}(f \underset{(1)}{\star} g)(x) = &\frac{C(\alpha)}{8(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[(15+(-1)^{|\alpha|}9)\Phi_\alpha(x-u-v) \\ &+(9+(-1)^{|\alpha|}15)\Phi_\alpha(x-u+v)+(9+(-1)^{|\alpha|}15)\Phi_\alpha(x+u-v) \\ &+(15+(-1)^{|\alpha|}41)\Phi_\alpha(x+u+v)\Big] f(u)g(v)\,dudv; \end{aligned} \tag{3.18}$$

$$\begin{aligned}(f \underset{(2)}{\star} g)(x) = &\frac{C(\alpha)}{4(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[(-1)^{|\alpha|}3\Phi_\alpha(x-u-v)+3\Phi_\alpha(x-u+v) \\ &+(3+(-1)^{|\alpha|}10)\Phi_\alpha(x+u-v) \\ &-(-1)^{|\alpha|}3\Phi_\alpha(x+u+v)\Big] f(u)g(v)\,dudv; \end{aligned} \tag{3.19}$$

$$\begin{aligned}(f \underset{(3)}{\star} g)(x) = &\frac{C(\alpha)}{32(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[(15+(-1)^{|\alpha|}41)\Phi_\alpha(x-u-v) \\ &-(9+(-1)^{|\alpha|}15)\Phi_\alpha(x-u+v)-(9+(-1)^{|\alpha|}15)\Phi_\alpha(x+u-v) \\ &+(15+(-1)^{|\alpha|}9)\Phi_\alpha(x+u+v)\Big] f(u)g(v)\,dudv; \end{aligned} \tag{3.20}$$

$$\begin{aligned}(f \underset{(4)}{\star} g)(x) = &\frac{C(\alpha)}{4(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[(-1)^{|\alpha|}3\Phi_\alpha(x-u-v) \\ &+(3+(-1)^{|\alpha|}10)\Phi_\alpha(x-u+v)+3\Phi_\alpha(x+u-v) \\ &-(-1)^{|\alpha|}3\Phi_\alpha(x+u+v)\Big] f(u)g(v)\,dudv. \end{aligned} \tag{3.21}$$

THEOREM 7. *If $f,g \in L^1(\mathbb{R}^n)$, then each one of the multiplications given by Definition 3.2 has its factorization identity associated with the transforms T and T^{-1}, and the function $\Phi_\alpha(x)$, together with the norm inequality:*

$$(T^{-1}(f \underset{(1)}{\star} g))(x) = \Phi_\alpha(x)(Tf)(x)(Tg)(x), \tag{3.22}$$

$$\|f \underset{(1)}{\star} g\|_1 \leq \frac{16|C(\alpha)|}{(2\pi)^n}\|f\|_1\|g\|_1\|\Phi_\alpha\|_1;$$

$$(T^{-1}(f \underset{(2)}{\star} g))(x) = \Phi_\alpha(x)(Tf)(x)(T^{-1}g)(x), \tag{3.23}$$

$$\|f \underset{(2)}{\star} g\|_1 \leq \frac{11|C(\alpha)|}{2(2\pi)^n}\|f\|_1\|g\|_1\|\Phi_\alpha\|_1;$$

$$(T^{-1}(f \underset{(3)}{\star} g))(x) = \Phi_\alpha(x)(T^{-1}f)(x)(T^{-1}g)(x) \tag{3.24}$$

$$\|f \underset{(3)}{\star} g\|_1 \leq \frac{4|C(\alpha)|}{(2\pi)^n}\|f\|_1\|g\|_1\|\Phi_\alpha\|_1;$$

$$(T^{-1}(f \underset{(4)}{\star} g))(x) = \Phi_\alpha(x)(T^{-1}f)(x)(Tg)(x), \tag{3.25}$$

$$\|f \underset{(4)}{\star} g\|_1 \leq \frac{11|C(\alpha)|}{2(2\pi)^n}\|f\|_1\|g\|_1\|\Phi_\alpha\|_1.$$

Proof. The proof of this theorem is similar to that of Theorem 6 and so it is here omitted. □

PROPOSITION 2. *The convolutions presented in Definition* 3.2 *are linearly independent.*

We also omit the proof of Proposition 2, since it can be performed similarly to that one of Proposition 1, after realizing a consequent system. For the even case, we have the following determinant

$$\begin{vmatrix} 96 & 24 & 56 & 24 \\ 96 & 24 & -24 & 104 \\ 96 & 104 & -24 & 24 \\ 224 & -24 & 24 & -24 \end{vmatrix} = -163840000.$$

For the odd case, we have the following determinant

$$\begin{vmatrix} 24 & -24 & -26 & -24 \\ -24 & 24 & 6 & -56 \\ -24 & -56 & 6 & 24 \\ -104 & 24 & 6 & 24 \end{vmatrix} = -10240000.$$

Although the four convolutions in Definition 3.1 as well as those in Definition 3.2 are linear independent, we still have

$$f \underset{(2)}{*} g = g \underset{(4)}{*} f \quad \text{and} \quad f \underset{(2)}{\star} g = g \underset{(4)}{\star} f.$$

The presented convolutions have their Young-type inequalities, some of which are better and more flexible than that of the Fourier case. Due to the number of convolutions being here considered, let us use the same generic symbol $\star$ for the convolutions (3.2)-(3.5) and (3.18)-(3.21), just to shorten the notation.

THEOREM 8. *Let* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$, *with* $1\le p,q,r\le\infty$. *If* $f\in L^p(\mathbb{R}^n)$, $g\in L^q(\mathbb{R}^n)$, *then*

$$\|f\star g\|_r\le C_1\|f\|_p\,\|g\|_q,\ \text{where } C_1 \text{ is some positive constant.} \tag{3.26}$$

If $p=q=1$, *then a further inequality holds*

$$\|f\star g\|_s\le C_2\|f\|_1\,\|g\|_1 \text{ for any } s\ge 1 \text{ with some } C_2>0. \tag{3.27}$$

Proof. In first place, we will prove (3.26). In this situation, we have two different cases associated with the parameter r.
Case 1: $1\le r<\infty$. We will consider P,Q,R,S defined in (3.10)–(3.13). Each convolution has four terms P,Q,R,S and each one of those is scaled by a constant. By the Minkowski inequality it suffices to prove the Young inequality for each term. Indeed, if $f\in L^p(\mathbb{R}^n)$, $g\in L^q(\mathbb{R}^n)$ and replacing $x-u=u$, we have

$$\begin{aligned}
Q(x)&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi_\alpha(x-u+v)f(u)g(v)dudv\\
&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi_\alpha(u+v)f(x-u)g(v)dudv\\
&=\int_{\mathbb{R}^n}f(x-u)\left[\int_{\mathbb{R}^n}\Phi_\alpha(u+v)g(v)dv\right]du\\
&=\int_{\mathbb{R}^n}f(x-u)\left[\int_{\mathbb{R}^n}\Phi_\alpha(u-v)\widetilde{g}(v)dv\right]du\\
&=\left(f\overset{F}{*}\left(\Phi_\alpha\overset{F}{*}\widetilde{g}\right)\right)(x),
\end{aligned}$$

where $(\cdot\overset{F}{*}\cdot)$ stands for the usual Fourier convolution.

Clearly, $\widetilde{g}\in L^q(\mathbb{R}^n)$ and $\|\widetilde{g}\|_q=\|g\|_q$ for any $q\ge 1$. By the known Young inequality for the Fourier transform,

$$\|Q(x)\|_r=\|f\overset{F}{*}\left(\Phi_\alpha\overset{F}{*}\widetilde{g}\right)\|_r\le\|f\|_p\|\Phi_\alpha\overset{F}{*}\widetilde{g}\|_q \tag{3.28}$$

We know that $\Phi_\alpha\in L^1(\mathbb{R}^n)$ and so, applying again the Young inequality, we have

$$\|\Phi_\alpha\overset{F}{*}\widetilde{g}\|_q\le\|\Phi_\alpha\|_1\|g\|_q. \tag{3.29}$$

So, combining (3.28) and (3.29), we obtain that

$$\|Q(x)\|_r\le\|\Phi_\alpha\|_1\|f\|_p\|g\|_q.$$

Similarly, we have

$$\|P(x)\|_r = \| \left(f \overset{F}{*} \left(\Phi_\alpha \overset{F}{*} g \right) \right)(x) \|_r \leq \|\Phi_\alpha\|_1 \|f\|_p \|g\|_q;$$
$$\|R(x)\|_r = \| \left(\widetilde{f} \overset{F}{*} \left(\Phi_\alpha \overset{F}{*} g \right) \right)(x) \|_r \leq \|\Phi_\alpha\|_1 \|f\|_p \|g\|_q;$$
$$\|S(x)\|_r = \| \left(\widetilde{f} \overset{F}{*} \left(\Phi_\alpha \overset{F}{*} \widetilde{g} \right) \right)(x) \|_r \leq \|\Phi_\alpha\|_1 \|f\|_p \|g\|_q;$$

Case 2: $r = \infty$. By the Hölder's inequality with $1/p + 1/q = 1$, we have

$$\begin{aligned} \|Q(x)\|_\infty &\leq \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi_\alpha(x-u+v)||f(u)||g(v)|\, dudv \\ &\leq \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|\Phi_\alpha(x-u+v)\|_\infty \|f(u)\|_p \|g(v)\|_q = \|\Phi_\alpha\|_\infty \|f\|_p \|g\|_q. \end{aligned}$$

For the function P, R, S, we proceed in a similar way.

The result follows from the Minkowski inequality.

We now prove (3.27). Remind that $\Phi_\alpha \in L^s(\mathbb{R}^n)$, for any $s \geq 1$, and the Minkowski inequality for integrals

$$\left[\int_{\Theta_2} \left| \int_{\Theta_1} F(x,y)\, d\mu_1(x) \right|^s d\mu_2(y) \right]^{\frac{1}{s}} \leq \int_{\Theta_1} \left(\int_{\Theta_2} |F(x,y)|^s d\mu_2(y) \right)^{\frac{1}{s}} d\mu_1(x), \quad (3.30)$$

where (Θ_1, μ_1) and (Θ_2, μ_2) are two measure spaces, $F(\cdot,\cdot) : \Theta_1 \times \Theta_2 \longrightarrow \mathbb{C}$ is a measurable function and $s \geq 1$.

$$\left(\int_{\mathbb{R}^n} |\Phi_\alpha(x \pm u \pm v)|^s dx \right)^{\frac{1}{s}} = \|\Phi_\alpha\|_s \quad (u, v \text{ are fixed in } \mathbb{R}^n).$$

We apply (3.30) to receive

$$\begin{aligned} &\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_\alpha(x \pm u \pm v) f(u) g(v) dudv \right|^s dx \right]^{\frac{1}{s}} \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\Phi_\alpha(x \pm u \pm v)|^s\, |f(u)|^s |g(v)|^s dx \right)^{\frac{1}{s}} dudv \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\Phi_\alpha(x \pm u \pm v)|^s\, dx \right)^{\frac{1}{s}} |f(u)|\, |g(v)| dudv \\ &= \|\Phi_\alpha\|_s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(u)|\, |g(v)| dudv = \|\Phi_\alpha\|_s \|f\|_1 \|g\|_1. \end{aligned}$$

We thus obtain (3.27), by the Minkowski inequality (3.30). □

REMARK 3.1. (i) Using a "direct" and simple notation, we may write the last result in the form:

$$L^p(\mathbb{R}^n) \star L^q(\mathbb{R}^n) \subseteq L^r(\mathbb{R}^n), \text{ where } \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1; \quad (3.31)$$

$$L^1(\mathbb{R}^n) \star L^1(\mathbb{R}^n) \subseteq L^s(\mathbb{R}^n), \text{ for any } \quad s \geq 1. \tag{3.32}$$

Letting $s=1$ in (3.32), we retrieve the norm inequalities addressed in Theorems 6 and 7 with the explicit constants shown there.

(ii) Choosing $s=2$ in (3.27), we see that if $f,g \in L^1(\mathbb{R}^n)$, then the convolution defines a function in the space $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. On one side, this result is in accordance with the known circumstance that a convolution $f * g$ inherits the best properties of both f and g, since a convolution can be seen as a filtering, an averaging or an inner product which acts smoothly. On the other side, the introduced convolutions possess a striking product feature in some sense: we need only to assume that $f,g \in L^1(\mathbb{R}^n)$, that we will obtain $f \star g \in L^s(\mathbb{R}^n)$ for every $s \geq 1$. Comparing with other known corresponding situations, (3.27) exhibits a remarkable property of the proposed convolutions. That inequality allows us to realize that the image spaces of the constructed convolutions are more flexible and larger than those of previously known convolutions. From our point of view, the contribution of Hermite functions in those convolution kernels plays an important role in this issue. In fact, such Young-type inequality (3.27) can be considered as an evidence of the very different structure of our convolutions.

THEOREM 9. *The space* $\mathscr{X} := L^1(\mathbb{R}^n)$, *equipped with each one of the convolution multiplications presented in Definitions* 3.1 *and* 3.2, *becomes a normed ring having no unit. Moreover:*

(a) *for the convolutions* (3.2), (3.4), (3.18) *and* (3.20), $\mathscr{X}$ *is commutative;*

(b) *for the convolutions* (3.3), (3.5), (3.19) *and* (3.21), $\mathscr{X}$ *is non-commutative.*

Proof. The proof is divided into two steps.
Step 1: $\mathscr{X}$ *has a normed ring structure.* It is clear that $\mathscr{X}$, equipped with each one of the convolutions listed above, has a normed ring structure with the multiplicative inequality as showed in Theorems 6 and 7.
Step 2: $\mathscr{X}$ *has no unit.* For the briefness of the proof, let us use the common symbol $\star$ for all the above-mentioned convolutions, and the notations:

$$T^+ := T, \;\; T^- := T^{-1}, \;\; 2^{\pm(0,1,2)} \in \{1,2,4,\frac{1}{2},\frac{1}{4}\}.$$

Suppose that there exits an element $e \in \mathscr{X}$ such that $f = f \star e = e \star f$ for every $f \in \mathscr{X}$. Choosing the Hermite function Φ_0 as f and applying the factorization identities of those convolutions, we obtain $T^{\pm}(\Phi_0) = \Phi_\alpha T^{\pm}(\Phi_0) T^{\pm}(e)$ in which the signs $+$ or $-$ are separated, this is, they depend on the considered convolution among (3.2)-(3.5) and (3.18)-(3.21). By Theorem 5 and Corollary 3 we have $T^{\pm}\Phi_0 = 2^{\pm 1}\Phi_0$. Inserting this into the above identity we find $2^{\pm 1}\Phi_0 = 2^{\pm 1}\Phi_\alpha \Phi_0 T^{\pm}(e)$. Since $\Phi_0(x) \neq 0$ for every $x \in \mathbb{R}^n$, we derive that $\Phi_\alpha(x)(T^{\pm}(e))(x) = 2^{\pm(0,1,2)}$ for every $x \in \mathbb{R}^n$. But, this contradicts Theorems 1 and 4, which state that $\lim\limits_{|x| \to \infty} \Phi_\alpha(x)(T^{\pm}(e))(x) = 0$. Hence, $\mathscr{X}$ has no unit. Evidently, the convolutions (3.2), (3.4), (3.18) and (3.20) are commutative. For instance, we have

$$(T(f \underset{(1)}{*} g))(x) = \Phi_\alpha(x)(Tf)(x)(Tg)(x)$$

$$= \Phi_\alpha(x)(Tg)(x)(Tf)(x) = (T(g \underset{(1)}{*} f))(x),$$

which follows that $f \underset{(1)}{*} g = g \underset{(1)}{*} f$ in $L^1(\mathbb{R}^n)$ for every $f, g \in L^1(\mathbb{R}^n)$, by the uniqueness theorem of T.

It suffices to prove the non-commutativity for (3.3), as that for the others can be proved analogously. Suppose that $(f \underset{(2)}{*} g)(x) = (g \underset{(2)}{*} f)(x)$, for all $f, g \in L^1(\mathbb{R}^n)$. We have

$$\begin{aligned}(T(f \underset{(2)}{*} g))(x) &= \Phi_\alpha(x)(Tf)(x)(T^{-1}g)(x)\\ &= \Phi_\alpha(x)(T^{-1}g)(x)(Tf)(x) = (T(g \underset{(4)}{*} f))(x).\end{aligned}$$

Due to the uniqueness theorem, $g \underset{(4)}{*} f = f \underset{(2)}{*} g$ in $L^1(\mathbb{R}^n)$ for all $f, g \in L^1(\mathbb{R}^n)$, which contradicts Proposition 1, that states that the convolutions $(\cdot \underset{(2)}{*} \cdot)$ and $(\cdot \underset{(4)}{*} \cdot)$ are independent of each other. Thus, $\mathscr{X}$ is non-commutative when endowed with (3.3). □

In comparison with the well-known Wiener's algebra $F\left(L^1\mathbb{R}^n)\right)$, we may also call $T^{\pm 1}\left(L^1(\mathbb{R}^n)\right)$ Wiener's algebras, by considering the convolutions (3.2), (3.4), (3.18) and (3.20).

4. Application: Solvability of an Integral Equation

In this section, as an application of the convolutions introduced in the previous section, we will work out necessary and sufficient conditions for which equation (1.1) has a unique solution in $L^1(\mathbb{R}^n)$. The equation (1.1) can be equivalently rewritten in a shorter form as:

$$\lambda\varphi(x) + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big[K_1(u)\Phi_\alpha(x-u-v) + K_2(u)\Phi_\alpha(x+u+v)\Big]\varphi(v)dudv = h(x). \tag{4.1}$$

Indeed, by a direct substitution of the following identities:

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_3(u)\Phi_\alpha(x+u-v)\,dudv &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_3(-u)\Phi_\alpha(x-u-v)\,dudv,\\ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_2(u)\Phi_\alpha(x-u+v)\,dudv &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_2(-u)\Phi_\alpha(x+u+v)\,dudv,\end{aligned}$$

we see that equation (1.1) turns out to be (4.1) with $K_1(u) := k_1(u) + k_3(-u)$ and $K_2(u) := k_4(u) + k_2(-u)$. In what follows, we shall consider (4.1), where $\lambda \in \mathbb{C}$, $K_1, K_2, h \in L^1(\mathbb{R}^n)$ are predetermined, and $\varphi \in L^1(\mathbb{R}^n)$ is unknown. It is worth saying that the inequalities in Theorems 6 and 7 are essential for our study in this section, since by those we realize that the equation (4.1) makes sense in the space $L^1(\mathbb{R}^n)$. In other words, the left-hand side of (4.1) well defines a continuous linear operator in $L^1(\mathbb{R}^n)$ for any $K_1, K_2 \in L^1(\mathbb{R}^n)$ and every Hermite function Φ_α.

We start by defining some functions which are needed to prove what follows. Namely, let us write:

$$
\begin{aligned}
E(x):=&\ \lambda+\frac{(2\pi)^n}{C(\alpha)25(5+(-1)^{|\alpha|}3)}\Phi_\alpha(x)\Big[(41+(-1)^{|\alpha|}15)(TK_1)(x)\\
&-24(T^{-1}K_1)(x)+3(3+(-1)^{|\alpha|}5)(TK_2)(x)+24(T^{-1}K_2)(x)\Big];\\
F(x):=&\ \frac{(2\pi)^n}{C(\alpha)25(5+(-1)^{|\alpha|}3)}\Phi_\alpha(x)\,[-24(TK_1)(x)\\
&+12(3+(-1)^{|\alpha|}5)(T^{-1}K_1)(x)+24(TK_2)(x)\\
&+4(41+(-1)^{|\alpha|}15)(T^{-1}K_2)(x)\Big];\\
G(x):=&\ \frac{(2\pi)^n}{C(\alpha)25(15+(-1)^{|\alpha|}17)}\Phi_\alpha(x)\Big[-3(3+(-1)^{|\alpha|}5)(TK_1)(x)\\
&+3(17+(-1)^{|\alpha|}15)(T^{-1}K_1)(x)+(59+(-1)^{|\alpha|}45)(TK_2)(x)\\
&-3(17+(-1)^{|\alpha|}15)(T^{-1}K_2)(x)\Big];\\
H(x):=&\ \lambda+\frac{(2\pi)^n}{C(\alpha)25(15+(-1)^{|\alpha|}17)}\Phi_\alpha(x)\Big[3(17+(-1)^{|\alpha|}15)(TK_1)(x)\\
&+4(59+(-1)^{|\alpha|}45)(T^{-1}K_1)(x)-3(17+(-1)^{|\alpha|}15)(TK_2)(x)\\
&-12(3+(-1)^{|\alpha|}5)(T^{-1}K_2)(x)\Big];\\
D(x):=&\ E(x)H(x)-F(x)G(x);\\
\Delta(x):=&\ \frac{H(x)(T^{-1}h)(x)-F(x)(Th)(x)}{D(x)};\\
\Gamma(x):=&\ \frac{E(x)(T^{-1}h)(x)-G(x)(Th)(x)}{D(x)}.
\end{aligned}
$$

We need some technical lemmas.

LEMMA 1. (a) *If $\lambda \neq 0$, then $D(x) \neq 0$ for every x outside a ball with a finite radius.*

(b) *Let $h \in L^1(\mathbb{R}^n)$ be given. Then, $Th \in L^1(\mathbb{R}^n)$ if and only if $T^{-1}h \in L^1(\mathbb{R}^n)$.*

(c) *Assume that $\lambda \neq 0$ and $D(x) \neq 0$ for every $x \in \mathbb{R}^n$. If $Th \in L^1(\mathbb{R}^n)$, then $\Delta, \Gamma \in L^1(\mathbb{R}^n)$.*

Proof. (a) By hypothesis, having in mind the fact that $\Phi_\alpha(x)$ is an Hermite function and by Theorems 1 and 4, we have $\lim_{|x|\to\infty} D(x) = \lambda^2 \neq 0$. Combining this with the uniform continuity of the function D, we deduce that there is an $R > 0$ such that $D(x) \neq 0$ for all $x \in \mathbb{R}^n$ such that $|x| \geq R$.

(b) Note that $f \in L^1(\mathbb{R}^n)$ if and only if $\tilde{f} \in L^1(\mathbb{R}^n)$ and $\|f\|_1 = \|\tilde{f}\|_1$. Suppose that $Th \in L^1(\mathbb{R}^n)$. Therefore,

$$(Th)(x) = 2(T_c h)(x) + i(T_s h)(x) \in L^1(\mathbb{R}^n),$$
$$(Th)(-x) = 2(T_c h)(x) - i(T_s h)(x) \in L^1(\mathbb{R}^n).$$

This implies that $T_c h, T_s h \in L^1(\mathbb{R}^n)$ and so, $T^{-1}h = \frac{1}{2}T_c h - iT_s h \in L^1(\mathbb{R}^n)$. The proof is completed, because T and T^{-1} are the inverse integral operators of each other. In fact, the conclusion of item (b) follows directly from Corollary 2.
(c) As showed above, D is a uniformly bounded and continuous function on $\mathbb{R}^n$ with $\lim_{|x|\to\infty} D(x) = \lambda^2 \neq 0$. This implies $D \in L^\infty(\mathbb{R}^n)$. Since $D(x) \neq 0$ for every $x \in \mathbb{R}^n$, and it is uniformly continuous, we derive that $\inf_{x\in\mathbb{R}^n} |D(x)| = \delta_0 > 0$, which implies that $D^{-1} \in L^\infty(\mathbb{R}^n)$. By the assumptions and Item (b), $Th, T^{-1}h \in L^1(\mathbb{R}^n)$. Since E, F, G, H are uniformly bounded and continuous functions on $\mathbb{R}^n$ and $D^{-1} \in L^\infty(\mathbb{R}^n)$, we deduce that $\Delta, \Gamma \in L^1(\mathbb{R}^n)$. □

LEMMA 2. *Let $\lambda \neq 0$. Assume that $D(x) \neq 0$ for every $x \in \mathbb{R}^n$, and $Th \in L^1(\mathbb{R}^n)$. Then, the following conditions are equivalent:*

(a) $(T^{-1}\Delta)(x) = (T\Gamma)(x) \in L^1(\mathbb{R}^n)$.

(b) $T\Delta \in L^1(\mathbb{R}^n)$, *and* $\Delta(x) = \dfrac{3}{2}\Gamma(x) + \dfrac{5}{2}\Gamma(-x)$ *(in the space $L^1(\mathbb{R}^n)$).*

(c) $T\Gamma \in L^1(\mathbb{R}^n)$, *and* $\Gamma(x) = \dfrac{-3}{8}\Delta(x) + \dfrac{5}{8}\Delta(-x)$.

Proof. By the assumptions $\lambda \neq 0$, $D(x) \neq 0$, and $Th \in L^1(\mathbb{R}^n)$. By using Item (c) of Lemma 1, we have $\Delta, \Gamma \in L^1(\mathbb{R}^n)$.

Suppose that (a) is true. By this and (b) in Lemma 1, we find that $T\Delta, T\Gamma \in L^1(\mathbb{R}^n)$. So, the functions Δ and Γ fulfill the assumptions of Corollary 1. The identity in (a) implies

$$\Delta(x) = (T^2\Gamma)(x) = \frac{3}{2}\Gamma(x) + \frac{5}{2}\Gamma(-x) \quad \text{(for almost every } x \in \mathbb{R}^n).$$

Conversely, suppose that (b) is true. Having in mind that $\left(T\tilde{\Gamma}\right)(x) = (T\Gamma)(-x)$, we have

$$\begin{cases} (T\Delta)(x) = \dfrac{3}{2}(T\Gamma)(x) + \dfrac{5}{2}(T\Gamma)(-x) \in L^1(\mathbb{R}^n); \\ (T\Delta)(-x) = \dfrac{5}{2}(T\Gamma)(x) + \dfrac{3}{2}(T\Gamma)(-x) \in L^1(\mathbb{R}^n). \end{cases}$$

By this and $T\Delta, T\tilde{\Delta} \in L^1(\mathbb{R}^n)$, we deduce that $T\Gamma \in L^1(\mathbb{R}^n)$, too. Applying Corollary 1 to the second identity in (b), it yields $\Delta = T^2\Gamma$, from which it follows condition (a) by the inversion theorem of T.

The equivalency of (c) and (a) is proved in the same way, by using (2.8). □

REMARK 4.1. By Corollaries 1, 2 and Item (b) of Lemma 2, we can conclude that *if* $f \in L^1(\mathbb{R}^n)$ *and if* $Tf \in L^1(\mathbb{R}^n)$, *then the operator identities* (2.6), (2.8) *hold true. Moreover,* $T^k f \in L^1(\mathbb{R}^n)$ *for every* $k \in \mathbb{Z}$, *by induction. In other words, the operation* T^k, *with* $k \in \mathbb{Z}$, *makes sense with* (2.6) *and* (2.8), *provided* $f, Tf \in L^1(\mathbb{R}^n)$.

We will now investigate how to use the constructed convolutions. In particular, we will show an essential characteristic of the pair of four convolutions (3.2)-(3.5) and (3.18)-(3.21), which will be helpful for proving Lemma 3. With the L^1-functions P, Q, R, S defined by (3.10)-(3.13), for $f, g \in L^1(\mathbb{R}^n)$, let us write:

$$\begin{aligned} X(x) &:= (TP)(x); & X'(x) &:= (T^{-1}P)(x);\\ Y(x) &:= (TQ)(x); & Y'(x) &:= (T^{-1}Q)(x);\\ Z(x) &:= (TR)(x); & Z'(x) &:= (T^{-1}R)(x);\\ W(x) &:= (TS)(x); & W'(x) &:= (T^{-1}S)(x). \end{aligned}$$

Considering the convolutions (3.2)-(3.5), we have

$$(10+(-1)^{|\alpha|}3)P+3Q+3R-(-1)^{|\alpha|}3S = \frac{4(2\pi)^n}{C(\alpha)}(f \underset{(1)}{*} g),$$

$$-(15+(-1)^{|\alpha|}9)P+(41+(-1)^{|\alpha|}15)Q-(9+(-1)^{|\alpha|}15)R+(15+(-1)^{|\alpha|}9)S = \frac{32(2\pi)^n}{C(\alpha)}(f \underset{(2)}{*} g),$$

$$-(-1)^{|\alpha|}3P-3Q-3R+(10+(-1)^{|\alpha|}3)S = \frac{16(2\pi)^n}{C(\alpha)}(f \underset{(3)}{*} g),$$

$$-(15+(-1)^{|\alpha|}9)P-(9+(-1)^{|\alpha|}15)Q+(41+(-1)^{|\alpha|}15)R+(15+(-1)^{|\alpha|}9)S = \frac{32(2\pi)^n}{C(\alpha)}(f \underset{(4)}{*} g).$$

This can be considered as a system of four unknown functions P, Q, R, S whose determinant is given by

$$\det(A) := \begin{vmatrix} 13 & 3 & 3 & -3\\ -24 & 56 & -24 & 24\\ -3 & -3 & -3 & 13\\ -24 & -24 & 56 & 24 \end{vmatrix} = -640000,$$

for $\alpha \equiv 0, 2 \pmod 4$.

For $\alpha \equiv 1, 3 \pmod 4$, the determinant is given by

$$\det(A) := \begin{vmatrix} 7 & 3 & 3 & 3\\ -6 & 26 & 6 & 6\\ 3 & -3 & -3 & 7\\ -6 & 6 & 26 & 6 \end{vmatrix} = -40000.$$

We would like to point out that, once again, we divide the computation of the determinant in that two cases just to simplify the computations.

Solving it, we obtain a unique solution given by

$$
\begin{aligned}
P =&\frac{(2\pi)^n}{C(\alpha)25(5+(-1)^{|\alpha|}3)}\Big[(41+(-1)^{|\alpha|}15)(f\underset{(1)}{*}g)-24(f\underset{(2)}{*}g)\\
&+12(3+(-1)^{|\alpha|}5)(f\underset{(3)}{*}g)-24(f\underset{(4)}{*}g)\Big],\\
Q =&\frac{(2\pi)^n}{C(\alpha)25(17+(-1)^{|\alpha|}15)}\Big[(51+(-1)^{|\alpha|}45)(f\underset{(1)}{*}g)+4(59+(-1)^{|\alpha|}45)(f\underset{(2)}{*}g)\\
&-4(51+(-1)^{|\alpha|}45)(f\underset{(3)}{*}g)+12(3+(-1)^{|\alpha|}5)(f\underset{(4)}{*}g)\Big],\\
R =&\frac{(2\pi)^n}{C(\alpha)25(17+(-1)^{|\alpha|}15)}\Big[(51+(-1)^{|\alpha|}45)(f\underset{(1)}{*}g)+12(3+(-1)^{|\alpha|}5)(f\underset{(2)}{*}g)\\
&-4(51+(-1)^{|\alpha|}45)(f\underset{(3)}{*}g)+4(59+(-1)^{|\alpha|}45)(f\underset{(4)}{*}g)\Big],\\
S =&\frac{(2\pi)^n}{C(\alpha)25(5+(-1)^{|\alpha|}3)}\Big[3(3+(-1)^{|\alpha|}5)(f\underset{(1)}{*}g)+24(f\underset{(2)}{*}g)\\
&+4(41+(-1)^{|\alpha|}15)(f\underset{(3)}{*}g)+24(f\underset{(4)}{*}g)\Big].
\end{aligned}
$$

Applying T to both sides of these identities, and using (3.6)-(3.9), gives

$$
\begin{aligned}
X =&\frac{(2\pi)^n}{C(\alpha)25(5+(-1)^{|\alpha|}3)}\Phi_\alpha\Big[(41+(-1)^{|\alpha|}15)(Tf)(Tg)-24(Tf)(T^{-1}g)\\
&+12(3+(-1)^{|\alpha|}5)(T^{-1}f)(T^{-1}g)-24(T^{-1}f)(Tg)\Big],\\
Y =&\frac{(2\pi)^n}{C(\alpha)25(17+(-1)^{|\alpha|}15)}\Phi_\alpha\Big[(51+(-1)^{|\alpha|}45)(Tf)(Tg)\\
&+4(59+(-1)^{|\alpha|}45)(Tf)(T^{-1}g)-4(51+(-1)^{|\alpha|}45)(T^{-1}f)(T^{-1}g)\\
&+12(3+(-1)^{|\alpha|}5)(T^{-1}f)(Tg)\Big],\\
Z =&\frac{(2\pi)^n}{C(\alpha)25(17+(-1)^{|\alpha|}15)}\Phi_\alpha\Big[(51+(-1)^{|\alpha|}45)(Tf)(Tg)\\
&+12(3+(-1)^{|\alpha|}5)(Tf)(T^{-1}g)-4(51+(-1)^{|\alpha|}45)(T^{-1}f)(T^{-1}g)\\
&+4(59+(-1)^{|\alpha|}45)(T^{-1}f)(Tg)\Big],\\
W =&\frac{(2\pi)^n}{C(\alpha)25(5+(-1)^{|\alpha|}3)}\Phi_\alpha\Big[3(3+(-1)^{|\alpha|}5)(Tf)(Tg)+24(Tf)(T^{-1}g)\\
&+4(41+(-1)^{|\alpha|}15)(T^{-1}f)(T^{-1}g)+24(T^{-1}f)(Tg)\Big].
\end{aligned}
\tag{4.2}
$$

Analogously, the four convolutions (3.18)-(3.21) can be rewritten in terms of the vector

(P,Q,R,S) whose determinant of the associated matrix, for $\alpha \equiv 0,2 \pmod 4$, is

$$\det(B) := \begin{vmatrix} 24 & 24 & 24 & 56 \\ 3 & 3 & 13 & -3 \\ 56 & -24 & -24 & 24 \\ 3 & 13 & 3 & -3. \end{vmatrix} = -640000.$$

For $\alpha \equiv 1,3 \pmod 4$, we have the following determinant

$$\det(B) := \begin{vmatrix} 6 & -6 & -6 & -26 \\ -3 & 3 & -7 & 3 \\ -26 & 6 & 6 & 6 \\ -3 & -7 & 3 & 3. \end{vmatrix} = -40000.$$

Converting from the convolutions (3.18)-(3.21) to the vector (P,Q,R,S), we obtain

$$\begin{aligned} P =& \frac{(2\pi)^n}{C(\alpha)25(15+(-1)^{|\alpha|}17)} \Big[-3(3+(-1)^{|\alpha|}5)(f \underset{(1)}{\star} g) \\ &+3(17+(-1)^{|\alpha|}15)(f \underset{(2)}{\star} g)+4(59+(-1)^{|\alpha|}45)(f \underset{(3)}{\star} g) \\ &+3(17+(-1)^{|\alpha|}15)(f \underset{(4)}{\star} g)\Big], \\ Q =& \frac{(2\pi)^n}{C(\alpha)25(3+(-1)^{|\alpha|}5)} \Big[6(f \underset{(1)}{\star} g)-3(3+(-1)^{|\alpha|}5)(f \underset{(2)}{\star} g) \\ &-24(f \underset{(3)}{\star} g)+(41+(-1)^{|\alpha|}15)(f \underset{(4)}{\star} g)\Big], \\ R =& \frac{(2\pi)^n}{C(\alpha)25(3+(-1)^{|\alpha|}5)} \Big[6(f \underset{(1)}{\star} g)+(41+(-1)^{|\alpha|}15)(f \underset{(2)}{\star} g) \\ &-24(f \underset{(3)}{\star} g)-3(3+(-1)^{|\alpha|}5)(f \underset{(4)}{\star} g)\Big], \\ S =& \frac{(2\pi)^n}{C(\alpha)25(15+(-1)^{|\alpha|}17)} \Big[(59+(-1)^{|\alpha|}45)(f \underset{(1)}{\star} g) \\ &-3(17+(-1)^{|\alpha|}15)(f \underset{(2)}{\star} g)-12(3+(-1)^{|\alpha|}5)(f \underset{(3)}{\star} g) \\ &-3(17+(-1)^{|\alpha|}15)(f \underset{(4)}{\star} g)\Big]. \end{aligned}$$

Applying T^{-1} to both sides of these identities and using (3.22)-(3.25), it follows

$$\begin{aligned} X' =& \frac{(2\pi)^n}{C(\alpha)25(15+(-1)^{|\alpha|}17)} \Phi_\alpha \Big[-3(3+(-1)^{|\alpha|}5)(Tf)(Tg) \\ &+3(17+(-1)^{|\alpha|}15)(Tf)(T^{-1}g)+4(59+(-1)^{|\alpha|}45)(T^{-1}f)(T^{-1}g) \\ &+3(17+(-1)^{|\alpha|}15)(T^{-1}f)(Tg)\Big], \end{aligned}$$

$$
\begin{aligned}
Y' =& \frac{(2\pi)^n}{C(\alpha)25(3+(-1)^{|\alpha|}5)} \Phi_\alpha \Big[6(Tf)(Tg) - 3(3+(-1)^{|\alpha|}5)(Tf)(T^{-1}g) \\
& -24(T^{-1}f)(T^{-1}g) + (41+(-1)^{|\alpha|}15)(T^{-1}f)(Tg)\Big], \qquad (4.3) \\
Z' =& \frac{(2\pi)^n}{C(\alpha)25(3+(-1)^{|\alpha|}5)} \Phi_\alpha \Big[6(Tf)(Tg) + (41+(-1)^{|\alpha|}15)(Tf)(T^{-1}g) \\
& -24(T^{-1}f)(T^{-1}g) - 3(3+(-1)^{|\alpha|}5)(T^{-1}f)(Tg)\Big], \\
W' =& \frac{(2\pi)^n}{C(\alpha)25(15+(-1)^{|\alpha|}17)} \Phi_\alpha \Big[(59+(-1)^{|\alpha|}45)(Tf)(Tg) \\
& - 3(17+(-1)^{|\alpha|}15)(Tf)(T^{-1}g) - 12(3+(-1)^{|\alpha|}5)(T^{-1}f)(T^{-1}g) \\
& -3(17+(-1)^{|\alpha|}15)(T^{-1}f)(Tg)\Big].
\end{aligned}
$$

Analysing carefully the identities (4.2) and (4.3), it is interesting to realize that each one of X,Y,Z,W, as well as X',Y', Z',W', satisfy a certain decomposition into some "circle" combination of elements in the Wiener's algebra $T(L^1(\mathbb{R}^n))$ (see Theorem 9 for further details). The identities (4.2) and (4.3) are key steps in the process of reducing the initial equation to a system of linear equations, as we will see in the next lemma and theorem.

LEMMA 3. *Considering $K_1, K_2, \varphi \in L^1(\mathbb{R}^n)$, we have*

$$
T\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [K_1(u)\Phi(x-u-v) + K_2(u)\Phi(x+u+v)]\,\varphi(v)dudv\right) = [E(x)-\lambda]\,(T\varphi)(x) + F(x)(T^{-1}\varphi)(x), \qquad (4.4)
$$

$$
T^{-1}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [K_1(u)\Phi(x-u-v) + K_2(u)\Phi(x+u+v)]\,\varphi(v)dudv\right) = G(x)(T\varphi)(x) + [H(x)-\lambda]\,(T^{-1}\varphi)(x). \qquad (4.5)
$$

Proof. We first prove (4.4), which can be rewritten as

$$
T\,(P+S)\,(x) = (E(x)-\lambda)\,(T\varphi)(x) + F(x)(T^{-1}\varphi)(x),
$$

where P is given by (3.10) with $f = K_1$, $g = \varphi$, and S is determined by (3.13) with $f = K_2$, $g = \varphi$. It is worth saying that the expressions (4.2) of X, Y, Z, W, as well as (4.3) of X', Y', Z', W' hold for every pair $(f,g) \in L^1(\mathbb{R}^n)$. Then, from (4.2), we will use X with $f = K_1$, $g = \varphi$, and W with $f = K_2$, $g = \varphi$. In this way, by a simple computation, we obtain $T\,(P+S) = X + W = (E(x)-\lambda)\,(T\varphi)(x) + F(x)(T^{-1}\varphi)(x)$. Thus, we have (4.4). Similarly, from (4.3), we will use X' with $f = K_1$, $g = \varphi$, and W' with $f = K_2, g = \varphi$. So, we will have

$$
T^{-1}\,(P+S) = X' + W' = G(x)(T\varphi)(x) + (H(x)-\lambda)\,(T^{-1}\varphi)(x),
$$

which is (4.5). □

Now, we state the main theorem of this section.

THEOREM 10. *Assume that* $D(x) \neq 0$*, for every* $x \in \mathbb{R}^n$*, and one of the following conditions is satisfied:*

(a) $\lambda \neq 0$*, and* $Th \in L^1(\mathbb{R}^n)$*;*

(b) $\lambda = 0$*, and* $\frac{Th}{D}, \frac{T^{-1}h}{D} \in L^1(\mathbb{R}^n)$.

Then, the equation (4.1) *has a unique solution in* $L^1(\mathbb{R}^n)$ *if and only if one of the three following conditions holds:*

$$(T^{-1}\Delta)(x) = (T\Gamma)(x) \in L^1(\mathbb{R}^n); \tag{4.6}$$

$$T\Delta \in L^1(\mathbb{R}^n) \text{ and } \Delta(x) = \frac{3}{2}\Gamma(x) + \frac{5}{2}\Gamma(-x) \quad \text{(a.e. } x \in \mathbb{R}^n\text{)}; \tag{4.7}$$

$$T\Gamma \in L^1(\mathbb{R}^n) \text{ and } \Gamma(x) = \frac{-3}{8}\Delta(x) + \frac{5}{8}\Delta(-x). \tag{4.8}$$

If (4.6)*,* (4.7) *or* (4.8) *holds, then the solution is given by*

$$\varphi(x) = \left(T^{-1}\Delta\right)(x). \tag{4.9}$$

Proof. (a) Suppose that there is a $\varphi_0 \in L^1(\mathbb{R}^n)$ that fulfills (4.1), i.e.

$$\lambda \varphi_0(x) + P(x) + S(x) = h(x),$$

where P is determined by (3.10) with $f = K_1$, $g = \varphi_0$ and S is given by (3.13) with $f = K_2$, $g = \varphi_0$. Applying T to both sides of the equation (4.1) and using Lemma 3, we will obtain $E(x)(T\varphi_0)(x) + F(x)(T^{-1}\varphi_0)(x) = (Th)(x)$. Acting T^{-1} to both sides of (4.1) and using Lemma 3, we have $G(x)(T\varphi_0)(x) + H(x)(T^{-1}\varphi_0)(x) = (T^{-1}h)(x)$. Combining these two equations, we obtain the following system of equations:

$$\begin{cases} E(x)(T\varphi_0)(x) + F(x)(T^{-1}\varphi_0)(x) = (Th)(x) \\ G(x)(T\varphi_0)(x) + H(x)(T^{-1}\varphi_0)(x) = (T^{-1}h)(x). \end{cases} \tag{4.10}$$

Since $D(x) \neq 0$, for all $x \in \mathbb{R}^n$, this system has a unique solution

$$(T\varphi_0)(x) = \Delta(x); \quad (T^{-1}\varphi_0)(x) = \Gamma(x). \tag{4.11}$$

By hypothesis, $Th \in L^1(\mathbb{R}^n)$. If we apply Lemma 1, we derive that $\Delta, \Gamma \in L^1(\mathbb{R}^n)$. Due to the inversion formulas, we obtain $\varphi_0 = T^{-1}(\Delta) = T(\Gamma)$, which is (4.6), and also (4.7) and (4.8) by Lemma 2.

Conversely, suppose that (4.6) holds. Consider the L^1-function $\varphi := T^{-1}(\Delta) = T(\Gamma) \in L^1(\mathbb{R}^n)$. This implies that

$$\begin{cases} (T\varphi)(x) = \Delta(x) \\ (T^{-1}\varphi)(x) = \Gamma(x), \end{cases} \quad \text{or} \quad \begin{cases} E(x)(T\varphi)(x) + F(x)(T^{-1}\varphi)(x) = (Th)(x) \\ G(x)(T\varphi)(x) + H(x)(T^{-1}\varphi)(x) = (T^{-1}h)(x). \end{cases}$$

Equivalently,

$$\begin{cases} T\Big(\lambda\varphi(x)+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big[K_1(u)\Phi_\alpha(x-u-v) \\ \qquad\qquad +K_2(u)\Phi_\alpha(x+u+v)\big]\varphi(v)dudv\Big)=(Th)(x) \\ T^{-1}\Big(\lambda\varphi(x)+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big[K_1(u)\Phi_\alpha(x-u-v) \\ \qquad\qquad +K_2(u)\Phi_\alpha(x+u+v)\big]\varphi(v)dudv\Big)=(T^{-1}h)(x). \end{cases} \tag{4.12}$$

By the inversion formulas, we conclude that φ satisfies the equation (4.1) for almost every $x\in\mathbb{R}^n$.

(b) We can argue similarly to the proof of Item (a) to obtain the system (4.11), but with $\lambda=0$. In particular, we have

$$T\varphi_0=\Delta=H\frac{T^{-1}h}{D}-F\frac{Th}{D}, \tag{4.13}$$

$$T^{-1}\varphi_0=\Gamma=E\frac{T^{-1}h}{D}-G\frac{Th}{D}. \tag{4.14}$$

Having in mind that $E,F,G,H\in L^\infty(\mathbb{R}^n)$ and $\frac{T^{-1}h}{D},\frac{Th}{D}\in L^1(\mathbb{R}^n)$, we can see that the functions on the right-hand side of (4.13) and (4.14) are in $L^1(\mathbb{R}^n)$, from which it follows that $\Delta,\Gamma\in L^1(\mathbb{R}^n)$. Now, we can take the inverse formulas of (4.13) and (4.14) to obtain (4.6).

The sufficiency can be proved in the same way as in the Item (a). Let us omit the proof of it. The theorem is proved. □

REMARK 4.2. We would like to point out that the equivalency of the three conditions (4.6), (4.7) and (4.8), associated with the solvability of the equation (4.1), exhibits the nature of our approach by using simultaneously T and T^{-1}, together with the two sets of convolutions introduced in Definitions 3.1 and 3.2. In other words, the role of the two functions Δ and Γ may be interchangeable.

We shall work out some concrete cases to exemplify the power and effectiveness of our convolution approach. Two concrete cases of application of simple integral equations with Gaussian convolution kernels were studied in [9, 10] in which the Gaussian function is $\Phi_0(x)=e^{-\frac{1}{2}x^2}$.

EXAMPLE 1. Let us now consider the integral equation

$$\int_{\mathbb{R}}\int_{\mathbb{R}}[K_1(u)\Phi_0(x-u-v)+K_2(u)\Phi_0(x+u+v)]\varphi(v)dudv=h(x), \tag{4.15}$$

where $K_1(x)=e^{-\frac{1}{2}x^2}$, $K_2(x)=0$, $h(x)=\frac{1}{2}e^{-\frac{1}{2}x^2}\in L^1(\mathbb{R})$ and $\varphi\in L^1(\mathbb{R})$ is unknown. Using the above notation, we have

$$E(x)=2\pi e^{-x^2},\ F(x)=G(x)=0,\ H(x)=2\pi e^{-x^2},\ D(x)=4\pi^2e^{-2x^2}.$$

By Theorem 5 and Corollary 3, $(Th)(x)=e^{-\frac{1}{2}x^2}$ and $(T^{-1}h)(x)=\frac{1}{4}e^{-\frac{1}{2}x^2}$. Hence,

$$\Delta(x)=\frac{\frac{1}{2}\pi e^{-\frac{3}{2}x^2}}{4\pi^2 e^{-2x^2}}=\frac{1}{8\pi}e^{\frac{1}{2}x^2},\quad \Gamma(x)=\frac{\frac{1}{2}\pi e^{-\frac{3}{2}x^2}}{4\pi^2 e^{-2x^2}}=\frac{1}{8\pi}e^{\frac{1}{2}x^2},$$

that do not belong to $L^1(\mathbb{R})$. Proceeding similarly to the proof of Item (a) in Theorem 10, we conclude that the equation (4.15) has no L^1-solution. In fact, the assumption (b) in Theorem 10 is not fulfilled in this case. This reinforces both conditions of Theorem 10. In fact, when $\lambda=0$, the assumptions that $h, Th\in L^1(\mathbb{R}^n)$ do not imply neither $\Delta\in L^1(\mathbb{R}^n)$ nor $\Gamma\in L^1(\mathbb{R}^n)$, while they do in the case of $\lambda\neq 0$.

EXAMPLE 2. Consider the following integral equation

$$\begin{aligned}\lambda\varphi(x)+\frac{C(\alpha)}{8(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Big[&(10+(-1)^{|\alpha|}3)\Phi_\alpha(x-u-v)+3\Phi_\alpha(x-u+v)\\ &+3\Phi_\alpha(x+u-v)-(-1)^{|\alpha|}3\Phi_\alpha(x+u+v)\Big]k(u)\varphi(v)dudv=h(x),\end{aligned}\tag{4.16}$$

where $0\neq\lambda\in\mathbb{C}$, $k,h\in L^1(\mathbb{R}^n)$ are given elements, and $\varphi\in L^1(\mathbb{R}^n)$ is to find out. Equation (4.16) is a special case of the equation (4.1) with $K_1(x)=(10+(-1)^{|\alpha|}3)k(x)+3k(-x)$ and $K_2(x)=-(-1)^{|\alpha|}k(x)+3k(-x)$ for some $k\in L^1(\mathbb{R}^n)$. Hence, its solvability is achieved by Item (a) of Theorem 10 and its solution can be given by (4.9). Nevertheless, we can treat the equation (4.16) by another convolution approach, but to obtain an L^1-series solution. In particular, the equation becomes

$$\lambda\varphi(x)+(k\underset{(1)}{*}\varphi)(x)=h(x),\tag{4.17}$$

which may be called an integral equation simply generated by convolution (3.2). We thereby have a simpler and direct way to generate a series solution. Assume that $\lambda+\Phi_\alpha(x)(Tk)(x)\neq 0$, for every $x\in\mathbb{R}^n$. Suppose that $\varphi\in L^1(\mathbb{R}^n)$ is a solution of (4.16). Applying the factorization identity of convolution (3.2), we obtain

$$\lambda\left(T\varphi\right)(x)+\Phi_\alpha(x)(Tk)(x)(T\varphi)(x)=(Th)(x),\tag{4.18}$$

which is equivalent to

$$(T\varphi)(x)=\frac{(Th)(x)}{\lambda+\Phi_\alpha(x)(Tk)(x)},$$

or

$$\begin{aligned}(T\varphi)(x)&=\frac{(Th)(x)}{\lambda}\left[1-\frac{\Phi_\alpha(x)(Tk)(x)}{\lambda+\Phi_\alpha(x)(Tk)(x)}\right]\\ &=\frac{(Th)(x)}{\lambda}\left[1-\Phi_\alpha(x)(Tk)(x)\frac{1}{\lambda\left(1+\frac{\Phi_\alpha(x)(Tk)(x)}{\lambda}\right)}\right].\end{aligned}$$

Remark that the complex-valued function $\Phi_\alpha(x)(Tk)(x)$ is continuous and uniformly bounded on $\mathbb{R}^n$. Hence, further assuming that $|\Phi_\alpha(x)(Tk)(x)|<|\lambda|$, for all $x\in\mathbb{R}^n$, which ensures the absolute convergence of the below Taylor's series, we have

$$
\begin{aligned}
(T\varphi)(x) &= \frac{(Th)(x)}{\lambda}\left[1-\sum_{j\geq 0}(-1)^j\frac{\Phi_\alpha^{j+1}(x)(Tk)^{j+1}(x)}{\lambda^{j+1}}\right]\\
&= \frac{(Th)(x)}{\lambda}\left[1-\sum_{j\geq 0}(-1)^j\frac{\Phi_\alpha(x)\left(T(k)^{\overset{*}{(1)}j}\right)(x)}{\lambda^{j+1}}\right]\\
&= \frac{(Th)(x)}{\lambda}-\sum_{j\geq 0}(-1)^j\frac{\Phi_\alpha(x)\left[T\left((k)^{\overset{*}{(1)}j}\right)\right](x)(Th)(x)}{\lambda^{j+2}}\\
&= \frac{(Th)(x)}{\lambda}-\sum_{j\geq 0}(-1)^j\frac{\left[T\left((k)^{\overset{*}{(1)}j}\underset{(1)}{*}h\right)\right](x)}{\lambda^{j+2}}\\
&= T\left(\frac{h(x)}{\lambda}-\sum_{j\geq 0}(-1)^j\frac{\left[(k)^{\overset{*}{(1)}j}\underset{(1)}{*}h\right](x)}{\lambda^{j+2}}\right),
\end{aligned}
\tag{4.19}
$$

where $(k)^{\overset{*}{(1)}n}$ means $\underbrace{k\underset{(1)}{*}k\underset{(1)}{*}\cdots\underset{(1)}{*}k}_{n\text{ times}}$. By the uniqueness theorem of T, we obtain

$$
\varphi(x)=\frac{h(x)}{\lambda}-\sum_{j\geq 0}(-1)^j\frac{\left((k)^{\overset{*}{(1)}j}\underset{(1)}{*}h\right)(x)}{\lambda^{j+2}}. \tag{4.20}
$$

Note that φ given by (4.20) belongs to $L^1(\mathbb{R}^n)$ by the assumption $k,h\in L^1(\mathbb{R}^n)$ and by Theorem 6 for the convolution $(\cdot\underset{(1)}{*}\cdot)$. Moreover, φ given by (4.20) fulfills (4.19) which is equivalent to (4.18), or equation (4.17). Thus, φ given by (4.20) is the L^1-series solution of (4.16).

REMARK 4.3. (i) The assumption $|\Phi_\alpha(x)(Tk)(x)|<|\lambda|$ for all $x\in\mathbb{R}^n$ is always satisfied for L^1-functions k arbitrarily given, provided λ is large enough, since $\Phi_\alpha(x)(Tk)(x)$ is a uniformly bounded and continuous function on $\mathbb{R}^n$ and rapidly decreasing at infinity. In fact, this assumption is a necessary and sufficient condition for the analytic extension of the function $(\lambda+\Phi_\alpha(x)(Tk)(x))^{-1}$, corresponding to powers of $\Phi_\alpha(x)(Tk)(x)\in\mathbb{C}$, under which the series in (4.19) is convergent, for all $x\in\mathbb{R}^n$. In other words, that assumption ensures the existence of convergent L^1-series solution of some specific equations like (4.16). The assumption $D(x)\neq 0$, for every $x\in\mathbb{R}^n$, as in Item (a) of Theorem 10 is essential for more general cases, such general equations (4.1).

(ii) The integral equations simply generated by the convolutions (3.3)–(3.5) and (3.18)–(3.21) can be solved effectively in the same way and, so, we omit the corresponding presentation. This exemplifies the practical and flexible effectiveness of the convolution approach to integral equations here presented.

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