Modeling ophthalmic surfaces using Zernike, Bessel and Chebyshev type functions*

M.M. Rodrigues[‡], A. Rosa[†], N. Vieira[‡] and J. N. Murta[†]

[†] Ophthalmology Unit, Centro Hospitalar e Universitário de Coimbra, Coimbra
Portugal, Centro Cirúrgico de Coimbra
Coimbra, Portugal, Faculty of Medicine, University of Coimbra, Coimbra.

[‡] CIDMA - Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal.
E-mails: mrodrigues@ua.pt, nloureirovieira@gmail.com

Abstract

In this paper the application of Zernike, Bessel and Chebyshev functions is studied and the results are compared when modeling ophthalmic surfaces in visual optics. The total RMS error is presented when addressing capability of these functions in fitting with different surfaces. It is shown that Chebyshev polynomials could be appropriate alternatives of the Zernike polynomials to represent complete anterior corneal surfaces.

1 Introduction

The visual system of the human eye is a part of the central nervous system by which the human body sees and interprets the information provided by the visible light in order to build a representation of the world around. During the propagations of the light through the eye, the retinal image can be deteriorated by diseases and disorders. For retinal images, the most important sources of images quality degradation are diffraction and optical aberrations. In order to measure and correct aberrations, there is a number of surfaces related to the anatomy and physiology of the eye. It is important to measure and mathematically model these surfaces to study their properties.

Orthogonal polynomials are one of the most important families of special functions. Their properties are very important for the resolution of several problems of mathematics and physics. For instance, ocular aberrations are commonly described in terms of a series of Zernike polynomials that offer distinct advances due to their normalization on a circular pupil. However, in certain cases with slow convergence, they may not be the most appropriate choice. This case of slow convergence was analyzed by Trevino et al. in [1], where the authors found improved fitting accuracy of circular Bessel functions for abrupt variations of post-surgical corneal surfaces. Bessel functions are also explored in the work by Lambert et al. in [2], where the authors examined the use of a Bessel beacon generated with a spatial light modulator as a fixation target for ophthalmic adaptive optics systems, rather than a conventional point-spread-function. Moreover, in [2] it was provided with an evidence of an increased immunity to defocus fluctuations and examined power spectral density variations of individual Zernike terms. In [3] the authors concluded that the Zernike polynomials excel in extracting the low-order optical characteristics of visual optics. Zernike's polynomials accurately represent both low- and high-order aberrations in normal eyes where high-order aberrations are clinically insignificant. For eyes after corneal surgery or eyes

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with corneal pathology such as keratoconus that have significant higher-order aberrations, the Zernike method fails to capture all clinically significant higher-order aberrations.

The Bessel functions arise naturally in many two dimensional problems with cylindrical symmetry. These functions have been applied in different fields that range from modelling impact crater surface elevation to pattern recognition (see [4, 5]). The Bessel functions, due to their more uniform and radial quasi-periodic behaviour, have advantages over the Zernike polynomials when approximating surfaces with high-frequency content. This comparison could be done based on Sturm-Liouville (S-L) theory (see [6,7]). Within the framework of this theory, the set of polynomials that has the two main important properties, namely the orthogonality and completeness (on a circular pupil), that can be used to approximate any surface defined in that domain.

The Chebyshev polynomials have been used in many applications due to its ability for approximate general classes of functions (see [8]). We aim to investigate the applicability of Chebyshev polynomials in modelling ophthalmic surfaces. To be more precise, the goal of this study is analyzing and evaluating Chebyshev polynomials representation with respect to those of Zernike and Bessel functions. The detection of RMS error is used as indicator for the comparison of those polynomials.

2 Methods and Analysis

A ophthalmic surface can be modeled by

$$C(r,\phi) = \sum_{p=1}^{P} a_p \psi_p(r,\phi) + \varepsilon_p(r,\phi), \tag{1}$$

where the index p is a polynomial ordering number, $\psi_p(r,\phi)$, with $p = 1,\dots,P$, is the pth polynomial, a_p, with $p = 1,\dots,P$, is the coefficient associated with $\psi_p(r,\phi)$, P is the order, r is the normalized distance from the origin, ϕ is the angle and $\varepsilon_p(r,\phi)$ represents the modeling error. Throughout this work, we choose the polar coordinate system for convenience. It is often a requirement that the polynomials used in the modeling are orthogonal and have complete set of modes for representing the surface. Using a set of such orthogonalized discrete polynomials, we can form a linear model

$$C = \psi a + \varepsilon, \tag{2}$$

where C is a D-element column vector of surface evaluated at discrete points (r_d, ϕ_d), $d = 1,...,D$, ψ is a $(D \times P)$ matrix of discrete orthogonal polynomials $\psi_p(r_d, \phi_d)$, a is a P-element column vector of coefficients, and ε represents a D-element column vector of the measurement and modeling error. For the model (2), the coefficient vector a can be estimated using the method of least-squares, i.e.,

$$\widehat{a} = (\psi^T \psi)^{-1} \psi^T C, \tag{3}$$

where T denotes the transposition, provided that the inverse exists. The RMS error is given by

$$RMS_{error} = \frac{\sqrt{\sum_{p=1}^{P} (\varepsilon_p(r,\phi))^2}}{P}. \tag{4}$$

For the Zernike polynomials, the most accepted representation of their radial function is (see [9])

$$R_n^{|m|}(r) = \sum_{s=0}^{(n-|m|)/2} \frac{(-1)^s (n-s)!}{s!(\frac{n-|m|}{2}-s)!(\frac{n+|m|}{2}-s)!} r^{n-2s}, \tag{5}$$

where $n \geq m$ so the parity of a polynomial is the same as the corresponding n. The Zernike polynomials are given in the normalized form by:

$$Z_n^m(r,\phi) = \sqrt{\frac{2(n+1)}{1+\delta_{m,0}}}\, R_n^{|m|}(r)\, \cos(m\phi), \quad \text{for even m}$$

$$Z_n^{-m}(r,\phi) = \sqrt{\frac{2(n+1)}{1+\delta_{m,0}}}\, R_n^{|m|}(r)\, \sin(m\phi), \quad \text{for odd m}. \tag{6}$$

The Bessel functions of first kind have the following series representation

$$J_m(c_{mk}r) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(m+s)!} \left(\frac{c_{mk}r}{2}\right)^{m+2s}, \tag{7}$$

and its normalized expression given by (see [10])

$$B_k^m(r,\phi) = \sqrt{\frac{2}{1+\delta_{m,0}}} \frac{1}{J_{m+1}(c_{mk})} J_m(c_{mk}r) \cos(m\phi), \quad \text{for even m} \tag{8}$$

$$B_k^{-m}(r,\phi) = \sqrt{\frac{2}{1+\delta_{m,0}}} \frac{1}{J_{m+1}(c_{mk})} J_m(c_{mk}r) \sin(m\phi), \quad \text{for odd m.} \tag{9}$$

In (7), (8) and (9) $c_{mk} = n(n+2)$ are the eigenvalues of the Bessel function which are found by k of zeros in the domain of J_m. Therefore, each Bessel function is scaled to its k-th zero. This scaling is related to the orthogonality of the Bessel functions (see [10]).

The consideration of Chebyshev polynomials on a circular disk is given in terms of Cylindrical Robert functions by (see [11]):

$$\begin{aligned} Q_n^m(r,\phi) &= r^m\, T_n(r) \cos(m\phi), \quad \text{for even } m \\ Q_n^{-m}(r,\phi) &= r^m\, T_n(r) \sin(m\phi), \quad \text{for odd } m, \end{aligned} \tag{10}$$

where $T_n^m(r)$ is the associated nth Chebyshev polynomial.

3 Evaluation of RMS error for different polynomials

In order to study behaviour of above-mentioned polynomials, we used 36 modes to model three surfaces and we calculated the RMS error. The considered surfaces were the Gaussian surface, surface with rings and total anterior eye. Concerning the Zernike and the Chebyshev polynomials the 36 modes are the first ones of the standard pyramides. In case of Bessel functions we observe that the standard pyramid has only 30 modes (see [1]), however, we repeated 6 modes in order to have the same number of modes.

4 Results and Discussion

We start with the study of the fitting capability of Zernike polynomials in Gaussian surfaces. The Gaussian surface is given by:

$$G(x,y) = \exp\left\{\frac{(y-k)^2 - (x-h)^2}{W^2}\right\}. \tag{11}$$

The Gaussian functions have been frequently used in modelling of influence functions of adaptive optics systems and keratoconic corneas, since there is a protuberance on their anterior surface (see [12, 13]). Figure 1 shows three Gaussian surfaces with various widths at the same off axis position within the unit disk and the evaluation of RMS error for the Zernike polynomials when fitting these surfaces. As the results show, for the narrowest Gaussian function ($W = 0.2$) the RMS error of Zernike reached to 0.028 μm. When the Gaussian function gets wider ($W = 0.5$ and $W = 0.75$) the Zernike polynomials give a better fitting error (0.022 μm) because the surface gets flatter. The experimental validation of the last conclusion was performed in [14]. We considered different number of Chebyshev modes in evaluation of RMS fitting error with eye model. Figure 2 depicts the results for this analysis. As this figure shows, with increasing the number of modes we can reach to a more stable RMS error around 3.07 μm, which seems to more reliable for this kind of studies (detection of RMS error). Since the number of modes is enough to cover all point of desired surface, keeping fixed RMS error in this analysis shows the optimum situation for valuation of fitting error.

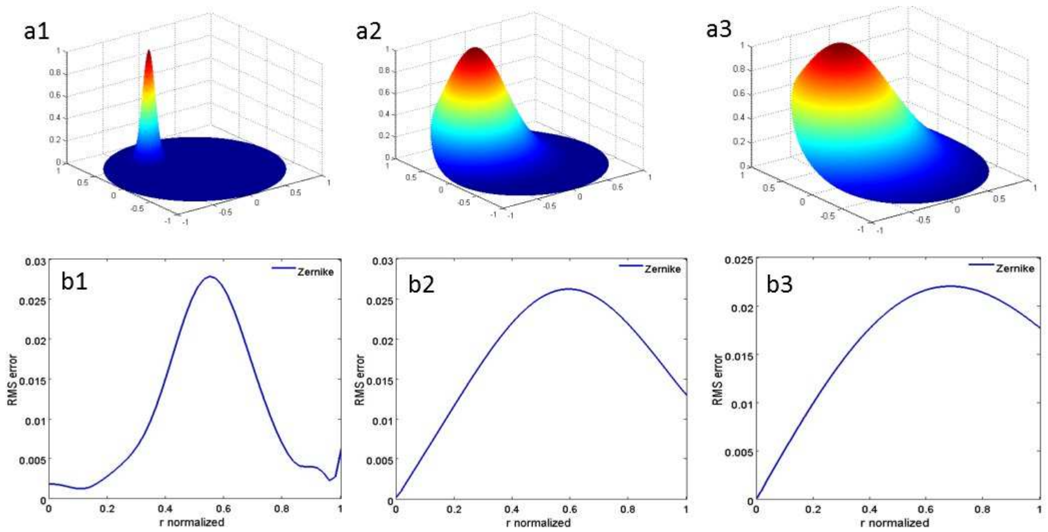


Figure 1: Evolution of RMS error for Gaussian functions with different widths (a1) $W = 0.2$, (a2) $W = 0.5$ and (a3) $W = 0.75$ on a disk of $r = 1$ when $h = -0.25$ and $k = 0.5$.

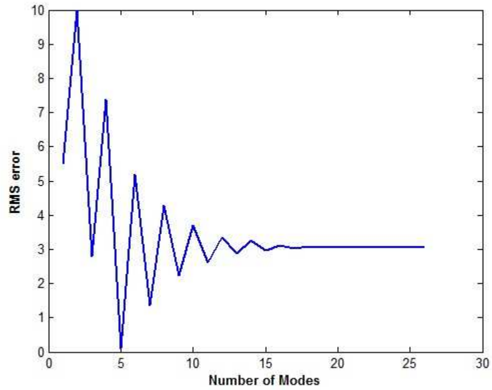


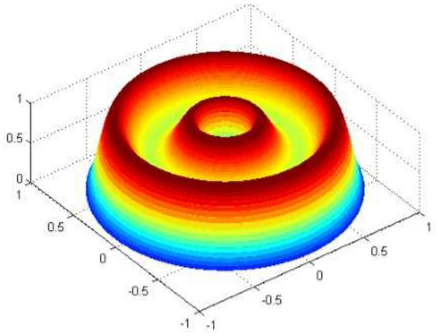
Figure 2: Evaluation of RMS error for different Chebyshev modes.

Now we pass to anther type of surface. A ring structure can be obtained by a radial Gaussian function centred at a radius r_0 and a constant angular factor,

$$C(r, \phi) = \exp\left\{\frac{-(r - r_0)^2 \times (r - r_1)^2}{W^2}\right\}. \tag{12}$$

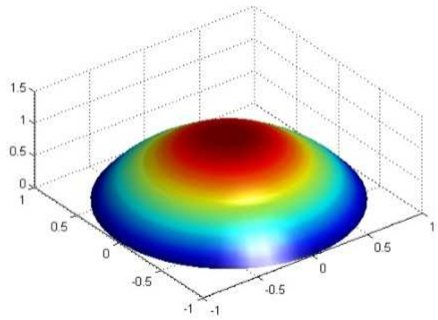
This type of surface could represent the effects of wearing a certain type of contact lenses. A multiple ringed surface, for instance a sum of two or more similar functions, could also represent a wavefront generated by a multifocal lens (see [15]). We performed the comparison 36 modes of Zernike, Bessel and Chebyshev polynomials in the modelling of a surface with rings. The evaluation of RMS error for each polynomial is shown in Figure 3. We observe that Chebyshev and Zernike polynomials have the less fitting error in comparison with Bessel function. In this type of surfaces the Bessel polynomials exhibit the maximum fitting error. The fitting error for Chebyshev and Zernike polynomials is similar, however, the Chebyshev polynomials show slightly less error.

An interesting case is the model of the total anterior eye surface (Figure 4), which includes the anterior surface of the cornea, limbus and sclera. This was done by etching together two spherical surfaces of different radius. To produce this model we employed typical parameters for anterior corneal radius, visible iris and diameter of the eye. As shown in Figure 4, the fitting error for Chebyshev and Zernike functions his similar, however, the Chebyshev functions show the smallest error.



Polynomial	Total RMS error
Zernike	3.82 μm
Bessel	42.10 μm
Chebyshev	3.66 μm

Figure 3: The model of Gaussian rings ($r_0 = 0.8$ - $r_1 = 0.25$ - $W = 0.1$) and the RMS error for different functions of Zernke, Bessel and Chebyshev.



Polynomial	Total RMS error
Zernike	1.99 μm
Bessel	3.19 μm
Chebyshev	1.53 μm

Figure 4: Total eye model with typical parameters and the RMS error for different functions of Zernke, Bessel and Chebyshev.

5 Experimental Validation

In this part of the work we considered RMS error regarding real data from two patients R and F. The patient R has a normal eye and patient F suffers from keratoconus, a corneal disease. We analyzed separately 4 different corneal zones. Zone 1 is the central optical zone and corresponds to the central 3-4 mm. Zone 2 extends to 7-8 mm and is the paracentral midperipheral cornea. Zone 3 and 4 are the peripheral transitional and limbal zones and extend to 11 and 12 mm, respectively. As depicted in figure 5 analysis of 4 different zones of the cornea shows a maximum error for zone 3 and zone 4, equal to 2.13, and minimum error for zone 1, equal to 0.73. Based on real data from a patient with normal corneal eye (patient R), we modeled this surface and detected the RMS error. The results could be used as a validation of our work conclusion. The evaluation of RMS error shows the error of 2.67 μm for Chebyshev and 3.01 μm for the Zernike. As a result, the Chebyshev polynomial has a very good behaviour and proves to be a competitive candidate in modelling of eye surface.

6 Conclusion

This study presents the analysis of 36 modes of Zernike, Bessel and Chebyshev polynomials that are orthogonal and complete in the unit circle. It also investigates RMS error when fitting different surfaces: Gaussian surface, surface with rings and total anterior eye. In the case of Gaussian surfaces, the obtained results show that the best fitting error is provided by Zernike polynomials when the Gaussian surface gets flatter. We found out that the Chebyshev polynomials had less error in comparison with Zernike and Bessel functions when analyzing the surface with rings and the total anterior eye. Therefore, the Chebyshev polynomials has a very good behaviour and prove to be a competitive candidate in modelling these kind of surfaces.

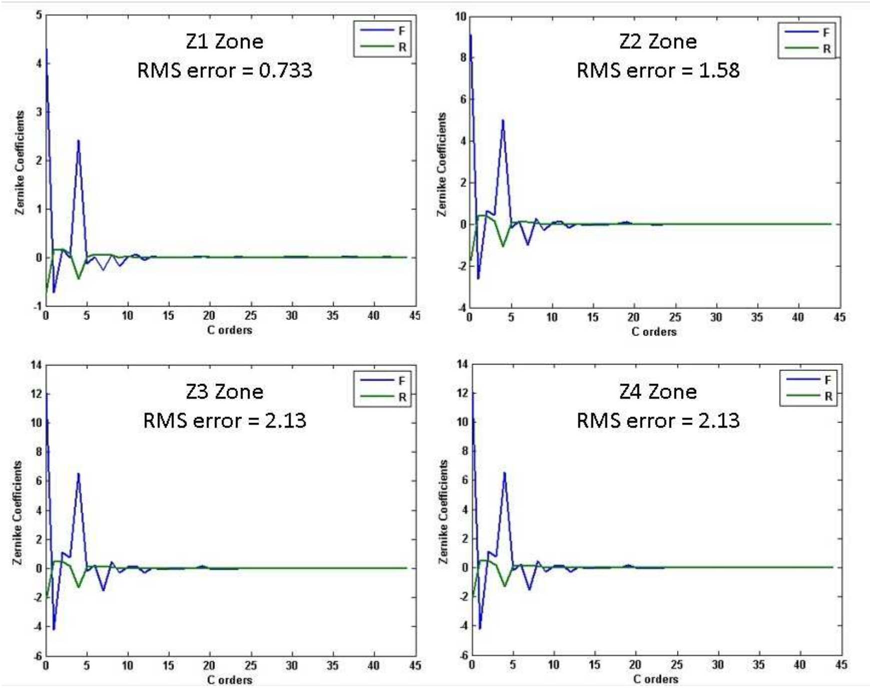


Figure 5: Evaluation of RMS error for different zones of Zernike Coefficients.

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