Available online: January 15, 2019

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 1186–1196 (2019)
DOI: 10.31801/cfsuasmas.513107
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

STRUCTURAL DERIVATIVES ON TIME SCALES

BENAOUMEUR BAYOUR AND DELFIM F. M. TORRES

Abstract. We introduce the notion of structural derivative on time scales. The new operator of differentiation unifies the concepts of fractal and fractional order derivative and is motivated by lack of classical differentiability of some self-similar functions. Some properties of the new operator are proved and illustrated with examples.

1. Introduction

In the past few years, several operators of differentiation have been investigated by researchers, from almost all branches of sciences, technology, and engineering, due to their capabilities of better modeling and predict complex systems [8, 12, 13]. In [5], the concept of *Hausdorff derivative* of a function $f(t)$ with respect to a fractal measure of t is introduced:

$$\frac{df(t)}{dt^\alpha} = \lim_{s \mapsto t} \frac{f(t) - f(s)}{t^\alpha - s^\alpha}. \tag{1}$$

In order to describe a rather large number of experimental results in Biomedicine, related to the structure of the diffusion of magnetic resonance imaging signals in human brain regions, the *structural derivative* is defined in [13] as

$$\frac{df(t)}{d_p t} = \lim_{s \mapsto t} \frac{f(t) - f(s)}{p(t) - p(s)}, \tag{2}$$

where $p(\cdot)$ is the structural function. When $p(t) = t^\alpha$, the structural derivative (2) coincides with the *fractal derivative* (1), as called in [1]. It is important to emphasize that the structural function $p(t)$ is not necessarily a power function. Examples in the literature can be found where $p(t)$ is the inverse Mittag–Leffler function, the probability density function, or the stretched exponential function [5]. Compared with classical nonlinear models, structural differential equations require

Received by the editors: Received: May 14, 2018; Revised: November 18, 2018.

2010 *Mathematics Subject Classification.* 26A33; 26E70.

Key words and phrases. Hausdorff derivative of a function with respect to a fractal measure; structural and fractal derivatives; self-similarity; time scales; Hilger derivative of non-integer order.

Submitted via ICCDS2018.

fewer parameters and lower computational costs in detecting causal relationships between mesoscopic time-space structures and certain physical behaviors [6].

Here we generalize the important notion of structural derivative to an arbitrary time scale $\mathbb{T}$. As particular cases, we get the fractional order derivative [7] and the fractional derivative on time scales recently introduced in [2]. Moreover, we claim that the new *structural derivative on time scales* is more than a mathematical generalization, allowing to deal with important concepts that may appear in complex systems, such as self-similarity and non-differentiability (see Example 15).

The need for the structural derivative notion, on different time scales than the set of real numbers, appears naturally in complex coarse-graininess structures, for instance, in anomalous radiation absorption where a tumor tissue interacts with the media and the radiation [11]. In such coarse-grained spaces, a point is not infinitely thin, and this feature is better modeled by means of our time-scale structural derivative. Indeed, in our approach we include a scale in time, allowing to consider the effects of internal times on the systems. For examples on the usefulness of structural derivatives on the quantum time scale, to model complex systems on life, medical, and biological sciences, see also [10]. Our structural derivative on time scales allows to unify different structural derivatives found in the literature in specific time scales, as in the continuous, the discrete, and the quantum scales.

The paper is organized as follows. In Section 2, we briefly recall the necessary concepts from the time-scale calculus. Then, in Section 3, we introduce the new structural derivative on time scales and prove its main proprieties. Illustrative examples are given along the text. In Section 4, we remark that a function can be structural differentiable on a general time scale without being differentiable. We end with Section 5 of conclusions and possible future work.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. Then, one defines the graininess function $\mu : \mathbb{T} \to [0, +\infty[$ by $\mu(t) = \sigma(t) - t$. If $\sigma(t) > t$, then we say that t is right-scattered; if $\rho(t) < t$, then t is left-scattered. Moreover, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum m, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$, then $f^\sigma : \mathbb{T} \to \mathbb{R}$ is given by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Definition 1 (The Hilger derivative [4]). *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. We define $f^\Delta(t)$ to be the number, provided it exists, with the property that given any $\epsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the Hilger (or the time-scale) derivative of f at t.

For more on the calculus on time scales, we refer the reader to the books [3, 4].

3. Structural derivatives on time scales

We introduce the definition of structural derivative on time scales. Here we follow the delta/forward approach. However, it should be mentioned that such choice is not fundamental for our structural derivative notion on time scales. In particular, the nabla/backward approach is also possible, with the properties we prove here for the delta derivative calculus being easily mimicked to the nabla case, where instead of using the forward $\sigma(t)$ operator we use the backward $\rho(t)$ operator of time scales.

Definition 2 (The time-scale structural derivative)**.** *Assume $f, p : \mathbb{T} \to \mathbb{R}$ with $\mathbb{T}$ a time scale. Let $t \in \mathbb{T}^\kappa$ and $\lambda > 0$. We define $f^{\Delta_p^\lambda}(t)$ to be the number, provided it exists, with the property that given any $\epsilon > 0$ there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$\left|[f^\lambda(\sigma(t)) - f^\lambda(s)] - f^{\Delta_p^\lambda}(t)[p(\sigma(t)) - p(s)]\right| \le \epsilon \left|p(\sigma(t)) - p(s)\right|$$

for all $s \in U$. We call $f^{\Delta_p^\lambda}(t)$ the structural derivative of f at t (associated with λ and $p(\cdot)$). Moreover, we say that f is structural differentiable on $\mathbb{T}^\kappa$ (or Δ_p^λ-differentiable), provided $f^{\Delta_p^\lambda}(t)$ exists for all $t \in \mathbb{T}^\kappa$.

It is clear that if $\lambda = 1$ and $p(t) = t$, then the new derivative coincides with the standard Hilger derivative (i.e., Definition 2 reduces to Definition 1). Our first result shows, in particular, that for $\mathbb{T} = \mathbb{R}$ and $\lambda = 1$, we obtain from Definition 2 the structural derivative (2).

Theorem 3. *Assume $f, p : \mathbb{T} \to \mathbb{R}$ with $\mathbb{T}$ a time scale. Let $t \in \mathbb{T}^\kappa$ and $\lambda \in \mathbb{R}$, $\lambda > 0$. Then the following proprieties hold:*

(1) *If f is continuous at t and t is right-scattered, then f is Δ_p^λ-differentiable at t with*
$$f^{\Delta_p^\lambda}(t) = \frac{f^\lambda(\sigma(t)) - f^\lambda(t)}{p(\sigma(t)) - p(t)}. \tag{3}$$

(2) *If t is right-dense, then f is structural differentiable at t if and only if the limit*
$$\lim_{s \to t} \frac{f^\lambda(t) - f^\lambda(s)}{p(t) - p(s)}$$
exists as a finite number. In this case,
$$f^{\Delta_p^\lambda}(t) = \lim_{s \to t} \frac{f^\lambda(t) - f^\lambda(s)}{p(t) - p(s)}. \tag{4}$$

(3) *If f is structural differentiable at t, then*
$$f^\lambda(\sigma(t)) = f^\lambda(t) + (p(\sigma(t)) - p(t)) f^{\Delta_p^\lambda}(t).$$

Proof. (1) Assume f is continuous at t with t right-scattered. By continuity,

$$\lim_{s\to t}\frac{f^\lambda(\sigma(t))-f^\lambda(s)}{p(\sigma(t))-p(s)}=\frac{f^\lambda(\sigma(t))-f^\lambda(t)}{p(\sigma(t))-p(t)}.$$

Hence, given $\epsilon>0$, there is a neighborhood U of t such that

$$\left|\frac{f^\lambda(\sigma(t))-f^\lambda(s)}{p(\sigma(t))-p(s)}-\frac{f^\lambda(\sigma(t))-f^\lambda(t)}{p(\sigma(t))-p(t)}\right|\le\epsilon$$

for all $s\in U$. It follows that

$$\left|f^\lambda(\sigma(t))-f^\lambda(s)-\frac{f^\lambda(\sigma(t))-f^\lambda(t)}{p(\sigma(t))-p(t)}[p(\sigma(t))-p(s)]\right|\le\epsilon\,|p(\sigma(t))-p(s)|$$

for all $s\in U$. Hence, we get the desired equality (3).

(2) Assume f is structural differentiable at t and t is right-dense. Let $\epsilon>0$ be given. Since f is differentiable at t, there is a neighborhood U of t such that

$$|\,[f^\lambda(\sigma(t))-f^\lambda(s)]-f^{\Delta_p^\lambda}(t)\,[p(\sigma(t))-p(s)]\,|\le\epsilon\mid p(\sigma(t))-p(s)\mid$$

for all $s\in U$. Moreover, because $\sigma(t)=t$, we have that

$$|\,[f^\lambda(t)-f^\lambda(s)]-f^{\Delta_p^\lambda}(t)[p(t)-p(s)]\,|\le\epsilon\mid p(t)-p(s)\mid$$

for all $s\in U$. It follows that $\left|\frac{f^\lambda(t)-f^\lambda(s)}{p(t)-p(s)}-f^{\Delta_p^\lambda}(t)\right|\le\epsilon$ for all $s\in U$, $s\neq t$, and we get equality (4). Assume $\lim_{s\to t}\frac{f^\lambda(t)-f^\lambda(s)}{p(t)-p(s)}$ exists and is equal to ξ and $\sigma(t)=t$. Let $\epsilon>0$. Then there is a neighborhood U of t such that

$$\left|\frac{f^\lambda(\sigma(t))-f^\lambda(s)}{p(t)-p(s)}-\xi\right|\le\epsilon$$

for all $s\in U$. Because $|\ f^\lambda(\sigma(t))-f^\lambda(s)-\xi(p(t)-p(s))\ |\le\epsilon|p(t)-p(s)|$ for all $s\in U$,

$$f^{\Delta_p^\lambda}(t)=\xi=\lim_{s\to t}\frac{f^\lambda(t)-f^\lambda(s)}{p(t)-p(s)}.$$

(3) If $\sigma(t)=t$, then $p(\sigma(t))-p(t)=0$ and

$$f^\lambda(\sigma(t))=f^\lambda(t)=f^\lambda(t)+(p(\sigma(t))-p(t))f^{\Delta_p^\lambda}(t).$$

On the other hand, if $\sigma(t)>t$, then by item 1

$$\begin{aligned}f^\lambda(\sigma(t))&=f^\lambda(t)+(p(\sigma(t))-p(t))\frac{f^\lambda(\sigma(t))-f^\lambda(t)}{p(\sigma(t))-p(t)}\\&=f^\lambda(t)+(p(\sigma(t))-p(t))f^{\Delta_p^\lambda}(t)\end{aligned}$$

and the proof is complete. ∎

Example 4. *If $\mathbb{T}=\mathbb{R}$ and $p(t)=t^{\alpha}$, then it follows from Theorem 3 that our structural derivative on time scales reduces to the generalized fractal-fractional derivative of f of order α in* [1]*:*

$$f^{\Delta_p^\lambda}(t)=\lim_{s\to t}\frac{f^{\lambda}(t)-f^{\lambda}(s)}{t^{\alpha}-s^{\alpha}}.$$

Example 5. *If $\mathbb{T}=\mathbb{R}$, $p(t)=t^{\alpha}$, and $\lambda=\alpha$, then item 2 of Theorem 3 yields that $f:\mathbb{R}\to\mathbb{R}$ is structural differentiable at $t\in\mathbb{R}$ if, and only if,*

$$f^{\Delta_p^\lambda}(t)=\lim_{s\to t}\frac{f^{\alpha}(t)-f^{\alpha}(s)}{t^{\alpha}-s^{\alpha}}$$

exists. In this case, we get the fractional order derivative $f^{(\alpha)}$ of [7].

Example 6. *If $\mathbb{T}=\mathbb{Z}$, then item 1 of Theorem 3 yields that $f:\mathbb{Z}\to\mathbb{R}$ is structural differentiable at $t\in\mathbb{Z}$ with*

$$f^{\Delta_p^\lambda}(t)=\frac{f^{\lambda}(\sigma(t))-f^{\lambda}(t)}{p(\sigma(t))-p(t)}=\frac{f^{\lambda}(t+1)-f^{\lambda}(t)}{p(t+1)-p(t)}.$$

Example 7. *If $f:\mathbb{T}\to\mathbb{R}$ is defined by $f(t)\equiv\gamma\in\mathbb{R}$, then $f^{\Delta_p^\lambda}(t)\equiv 0$. Indeed, if t is right-scattered, then by item 1 of Theorem 3 we get*

$$f^{\Delta_p^\lambda}(t)=\frac{f^{\lambda}(\sigma(t))-f^{\lambda}(t)}{p(\sigma(t))-p(t)}=\frac{\gamma^{\lambda}-\gamma^{\lambda}}{p(\sigma(t))-p(t)}=0;$$

if t is right-dense, then by (4) *we get*

$$f^{\Delta_p^\lambda}(t)=\lim_{s\to t}\frac{\gamma^{\lambda}-\gamma^{\lambda}}{p(t)-p(s)}=0.$$

Example 8. *If $f:\mathbb{T}\to\mathbb{R}$ is given by $f(t)=t$, then $f^{\Delta_p^\lambda}(t)\neq 1$ because if $\sigma(t)>t$ (i.e., t is right-scattered), then*

$$f^{\Delta_p^\lambda}(t)=\frac{f^{\lambda}(\sigma(t))-f^{\lambda}(t)}{p(\sigma(t))-p(t)}=\frac{\sigma^{\lambda}(t)-t^{\lambda}}{p(\sigma(t))-p(t)}\neq 1;$$

if $\sigma(t)=t$ (i.e., t is right-dense), then

$$f^{\Delta_p^\lambda}(t)=\lim_{s\to t}\frac{f^{\lambda}(t)-f^{\lambda}(s)}{p(t)-p(s)}=\lim_{s\to t}\frac{t^{\lambda}-s^{\lambda}}{p(t)-p(s)}\neq 1.$$

Example 9. *Let $g:\mathbb{T}\to\mathbb{R}$ be given by $g(t)=\frac{1}{t}$. We have*

$$g^{\Delta_p^\lambda}(t)=-\frac{(t)^{\Delta_p^\lambda}}{(\sigma(t)t)^{\lambda}}.$$

Indeed, if $\sigma(t)=t$, then $g^{\Delta_p^\lambda}(t)=-\frac{(t)^{\Delta_p^\lambda}}{\sigma^{\lambda}(t)t^{\lambda}}$; if $\sigma(t)>t$, then

$$g^{\Delta_p^\lambda}(t)=\frac{g^{\lambda}(\sigma(t))-g^{\lambda}(t)}{p(\sigma(t))-p(t)}=\frac{\left(\frac{1}{\sigma(t)}\right)^{\lambda}-\left(\frac{1}{t}\right)^{\lambda}}{p(\sigma(t))-p(t)}=\frac{\frac{t^{\lambda}-\sigma^{\lambda}(t)}{t^{\lambda}\sigma^{\lambda}(t)}}{t^{\lambda}-\sigma^{\lambda}(t)}=-\frac{(t)^{\Delta_p^\lambda}}{t^{\lambda}\sigma^{\lambda}(t)}.$$

Example 10. *Let $h : \mathbb{T} \to \mathbb{R}$ be defined by $h(t) = t^2$. We have*

$$h^{\Delta_p^\lambda}(t) = (t)^{\Delta_p^\lambda}(\sigma(t) + t).$$

Indeed, if t is right-dense, then $h^{\Delta_p^\lambda}(t) = \lim_{s\to t} \frac{t^{2\lambda}-s^{2\lambda}}{p(t)-p(s)} = (t)^{\Delta_p^\lambda}(\sigma(t) + t)$; if t is right-scattered, then

$$h^{\Delta_p^\lambda}(t) = \frac{h^\lambda(\sigma(t)) - h^\lambda(t)}{p(\sigma(t)) - p(t)} = \frac{\sigma^{2\lambda}(t) - t^{2\lambda}}{p(\sigma(t)) - p(t)} = (t)^{\Delta_p^\lambda}(\sigma(t) + t).$$

Example 11. *Consider the time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Let f be the function defined by $f : h\mathbb{Z} \to \mathbb{R}$, $t \mapsto (t-c)^2$, $c \in \mathbb{R}$. The time-scale structural derivative of f at t is*

$$\begin{aligned} f^{\Delta_p^\lambda}(t) &= \frac{f^\lambda(\sigma(t)) - f^\lambda(t)}{p(\sigma(t)) - p(t)} = \frac{((\sigma(t) - c)^2)^\lambda - ((t - c)^2)^\lambda}{p(\sigma(t)) - p(t)} \\ &= \frac{(t + h - c)^{2\lambda} - (t - c)^{2\lambda}}{p(t + h) - p(t)}. \end{aligned}$$

Remark 12. *Our examples show that, in general, $f^{\Delta_p^\lambda}(t)$ is a complex number (for instance, choose $\lambda = \frac{1}{2}$ and $t < 0$ in Example 8).*

Our second theorem shows that it is possible to develop a calculus for the time-scale structural derivative.

Theorem 13. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are continuous and structural differentiable at $t \in \mathbb{T}^\kappa$. Then the following proprieties hold:*

(1) *For any constant γ, function $\gamma f : \mathbb{T} \to \mathbb{R}$ is structural differentiable at t with $(\gamma f)^{\Delta_p^\lambda}(t) = \gamma^\lambda f^{\Delta_p^\lambda}(t)$.*

(2) *The product $fg : \mathbb{T} \to \mathbb{R}$ is structural differentiable at t with*

$$\begin{aligned} (fg)^{\Delta_p^\lambda}(t) &= f^{\Delta_p^\lambda}(t)g^\lambda(t) + f^\lambda(\sigma(t))g^{\Delta_p^\lambda}(t) \\ &= f^{\Delta_p^\lambda}(t)g^\lambda(\sigma(t)) + f^\lambda(t)g^{\Delta_p^\lambda}(t). \end{aligned}$$

(3) *If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is structural differentiable at t with*

$$\left(\frac{1}{f}\right)^{\Delta_p^\lambda}(t) = \frac{-f^{\Delta_p^\lambda}(t)}{f^\lambda(\sigma(t))f^\lambda(t)}.$$

(4) *If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is structural differentiable at t with*

$$\left(\frac{f}{g}\right)^{\Delta_p^\lambda}(t) = \frac{f^{\Delta_p^\lambda}(t)g^\lambda(t) - f^\lambda(t)g^{\Delta_p^\lambda}(t)}{g^\lambda(\sigma(t))g^\lambda(t)}.$$

Proof. (1) Let $\epsilon \in (0, 1)$. Define $\epsilon^* = \frac{\epsilon}{|\gamma|^\lambda} \in (0, 1)$. Then there exists a neighborhood U of t such that

$$|f^\lambda(\sigma(t)) - f^\lambda(s) - f^{\Delta_p^\lambda}(t)(p(\sigma(t)) - p(s))| \leq \epsilon^*|p(\sigma(t)) - p(s)|$$

for all $s \in U$. It follows that

$$\begin{aligned}
&|(\gamma f)^\lambda(\sigma(t)) - (\gamma f)^\lambda(s) - \gamma^\lambda f^{\Delta_p^\lambda}(t)(p(\sigma(t) - p(s))| \\
&\quad = |\gamma|^\lambda \mid f^\lambda(\sigma(t)) - f^\lambda(s) - f^{\Delta_p^\lambda}(t)(p(\sigma(t)) - p(s)) \mid \\
&\quad \leq \epsilon^*|\gamma|^\lambda |p(\sigma(t)) - p(s)| \\
&\quad \leq \frac{\epsilon}{|\gamma|^\lambda}|\gamma|^\lambda |p(\sigma(t)) - p(s)| \\
&\quad = \epsilon|p(\sigma(t)) - p(s)|
\end{aligned}$$

for all $s \in U$. Thus, $(\gamma f)^{\Delta_p^\lambda}(t) = \gamma^\lambda f^{\Delta_p^\lambda}(t)$ holds.

(2) Let $\epsilon \in (0,1)$. Define

$$\epsilon^* = \epsilon\left[1 + \left|f^\lambda(t)\right| + \left|g^\lambda(\sigma(t))\right| + \left|g^{\Delta_p^\lambda}(\sigma(t))\right|\right]^{-1}.$$

Then $\epsilon^* \in (0,1)$ and there exist neighborhoods U_1, U_2 and U_3 of t such that

$$|f^\lambda(\sigma(t)) - f^\lambda(s) - f^{\Delta_p^\lambda}(t)(p(\sigma(t)) - p(s))| \leq \epsilon^*|p(\sigma(t)) - p(s)|$$

for all $s \in U_1$,

$$\left|g^\lambda(\sigma(t)) - g^\lambda(s) - g^{\Delta_p^\lambda}(t)\left(p(\sigma(t)) - p(s)\right)\right| \leq \epsilon^*\left|p(\sigma(t)) - p(s)\right|$$

for all $s \in U_2$, and (f is continuous) $|f(t) - f(s)| \leq \epsilon^*$ for all $s \in U_3$. Define $U = U_1 \cap U_2 \cap U_3$ and let $s \in U$. It follows that

$$\begin{aligned}
&\left|(fg)^\lambda(\sigma(t)) - (fg)^\lambda(s) - \left[g^{\Delta_p^\lambda}(t)f^\lambda(t) + g^\lambda(\sigma(t))f^{\Delta_p^\lambda}(t)\right][p(\sigma(t)) - p(s)]\right| \\
&\quad = \Big|[f^\lambda(\sigma(t)) - f^\lambda(s) - f^{\Delta_p^\lambda}(t)(p(\sigma(t)) - p(s))]g^\lambda(t) + g^\lambda(\sigma(t))f^\lambda(s) \\
&\qquad + g^\lambda(\sigma(t))f^{\Delta_p^\lambda}(t)(p(\sigma(t) - p(s)) - f^\lambda(s)g^\lambda(s) \\
&\qquad - \left[g^{\Delta_p^\lambda}(t)f^\lambda(t) + g^\lambda(\sigma(t))f^{\Delta_p^\lambda}(t)\right][p(\sigma(t)) - p(s)]\Big| \\
&\quad = \Big|[f^\lambda(\sigma(t)) - f^\lambda(s) - f^{\Delta_p^\lambda}(t)(p(\sigma(t)) - p(s))](g^\lambda(\sigma(t))) \\
&\qquad + \left[g^\lambda(\sigma(t)) - g^\lambda(s) - g^{\Delta_p^\lambda}(t)(p(\sigma(t)) - p(s))\right]f^\lambda(s) \\
&\qquad + \left[g^\lambda(\sigma(t)) - g^\lambda(s) - g^{\Delta_p^\lambda}(t)(p(\sigma(t)) - p(s))\right]\left(f^\lambda(s) - f^\lambda(t)\right) + f^\lambda(s)g^\lambda(s) \\
&\qquad + g^{\Delta_p^\lambda}(t)f^\lambda(s)(p(\sigma(t)) - p(s)) + g^\lambda(\sigma(t))f^{\Delta_p^\lambda}(t)(p(\sigma(t)) - p(s)) - g^\lambda(s)f^\lambda(s) \\
&\qquad - \left[g^{\Delta_p^\lambda}(t)f^\lambda(t) + g^\lambda(\sigma(t))f^{\Delta_p^\lambda}(t)\right][p(\sigma(t)) - p(s)]\Big| \\
&\quad \leq \left|f^\lambda(\sigma(t)) - f^\lambda(s) - f^{\Delta_p^\lambda}(t)(p(\sigma(t)) - p(s))\right|\left|g^\lambda(\sigma(t))\right| \\
&\qquad + \left|g^\lambda(\sigma(t)) - g^\lambda(s) - g^{\Delta_p^\lambda}(t)(p(\sigma(t)) - p(s))\right|\left|f^\lambda(t)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|g^{\lambda}(\sigma(t))-g^{\lambda}(s)-g^{\Delta_p^{\lambda}}(t)(p(\sigma(t))-p(s))\right|\left|f^{\lambda}(s)-f^{\lambda}(t)\right| \\
&+\left|g^{\Delta_p^{\lambda}}(t)\right|\left|f^{\lambda}(t)-f^{\lambda}(s)\right||p(\sigma(t))-p(s)| \\
=\ &\epsilon^*\left|g^{\lambda}(\sigma(t))\right||p(\sigma(t))-p(s)| \\
&+\epsilon^*\left|f^{\lambda}(t)\right||p(\sigma(t))-p(s)|+\epsilon^*|p(\sigma(t))-p(s)|\epsilon^*+\epsilon^*\left|g^{\Delta_p^{\lambda}}(t)\right||p(\sigma(t))-p(s)| \\
\leq\ &\epsilon^*|p(\sigma(t))-p(s)|\left(\epsilon^*+\left|f^{\lambda}(t)\right|+\left|g^{\lambda}(t)\right|+\left|g^{\Delta_p^{\lambda}}(t)\right|\right) \\
\leq\ &\epsilon^*|p(\sigma(t))-p(s)|\left(1+\left|f^{\lambda}(t)\right|+\left|g^{\lambda}(t)\right|+\left|g^{\Delta_p^{\lambda}}(t)\right|\right)=\epsilon\,|p(\sigma(t))-p(s)|\,.
\end{aligned}
$$

Thus $(fg)^{\Delta_p^{\lambda}}(t)=f^{\lambda}(t)g^{\Delta_p^{\lambda}}(t)+f^{\Delta_p^{\lambda}}(t)g^{\lambda}(\sigma(t))$ holds at t. The other product rule follows from this last equality by interchanging functions f and g.

(3) We use the structural derivative of a constant (Example 7). Since

$$\left(f\cdot\frac{1}{f}\right)^{\Delta_p^{\lambda}}(t)=0,$$

it follows from item 2 that

$$\left(\frac{1}{f}\right)^{\Delta_p^{\lambda}}(t)f^{\lambda}(\sigma(t))+f^{\Delta_p^{\lambda}}(t)\frac{1}{f^{\lambda}(t)}=0.$$

Because we are assuming $f(t)f(\sigma(t))\neq 0$, one has

$$\left(\frac{1}{f}\right)^{\Delta_p^{\lambda}}(t)=\frac{-f^{\Delta_p^{\lambda}}(t)}{f^{\lambda}(\sigma(t))f^{\lambda}(t)}.$$

(4) For the quotient formula we use items 2 and 3 to compute

$$
\begin{aligned}
\left(\frac{f}{g}\right)^{\Delta_p^{\lambda}}(t)&=\left(f\cdot\frac{1}{g}\right)^{\Delta_p^{\lambda}}(t) \\
&=f^{\lambda}(t)\left(\frac{1}{g}\right)^{\Delta_p^{\lambda}}(t)+f^{\Delta_p^{\lambda}}(t)\frac{1}{g^{\lambda}(\sigma(t))} \\
&=-f^{\lambda}(t)\frac{g^{\Delta_p^{\lambda}}(t)}{g^{\lambda}(\sigma(t))g^{\lambda}(t)}+f^{\Delta_p^{\lambda}}(t)\frac{1}{g^{\lambda}(\sigma(t))} \\
&=\frac{f^{\Delta_p^{\lambda}}(t)g^{\lambda}(t)-f^{\lambda}(t)g^{\Delta_p^{\lambda}}(t)}{g^{\lambda}(\sigma(t))g^{\lambda}(t)}.
\end{aligned}
$$

This concludes the proof. □

Remark 14. *The structural derivative of the sum $f+g:\mathbb{T}\to\mathbb{R}$ does not satisfy the usual property, that is, in general $(f+g)^{\Delta_p^{\lambda}}(t)\neq f^{\Delta_p^{\lambda}}(t)+g^{\Delta_p^{\lambda}}(t)$. For instance,* *let $\mathbb{T}$ be an arbitrary time scale and $f, g : \mathbb{T} \to \mathbb{R}$ be functions defined by $f(t) = t$ and $g(t) = 2t$. One can easily find that*

$$(f+g)^{\Delta_p^\lambda}(t) = 3^\lambda \left(\frac{\sigma^\lambda(t) - t^\lambda}{p(\sigma(t)) - p(t)} \right) \neq f^{\Delta_p^\lambda}(t) + g^{\Delta_p^\lambda}(t) = (1 + 2^\lambda) \frac{\sigma^\lambda(t) - t^\lambda}{p(\sigma(t)) - p(t)}.$$

4. A remark on self-similarity and nondifferentiabilty

In this section, we provide an example where it is natural to define structural derivatives on time scales. Precisely, we consider a function that is structural differentiable on a general time scale without being differentiable in the classical sense. This possibility, to differentiate nonsmooth functions, is very important in real world applications, e.g., to deal with models of hydrodynamics continuum flows in fractal coarse-grained (fractal porous) spaces, which are discontinuous in the embedding Euclidean space [9].

A self-similar function is a function that exhibits similar patterns when one changes the scale of observation: the patterns generated by $f(t)$ and $f(at)$, $a > 0$, looks the same. Formally, f is a self-similar function of order β if it satisfies

$$f(at) = a^\beta f(t), \quad a > 0, \quad \beta > 0,$$

which is interpreted as saying that in the vicinity of t and at the function looks the same. A self-similar function f obviously satisfies $f(0) = 0$ and, furthermore,

$$f(t) = ct^\beta, \quad c = f(1).$$

Let $0 < \beta < 1$. Then $f(t)$ is clearly not differentiable at $t = 0$. Example 15 shows, however, that $f(t)$ can be structurally differentiable at $t = 0$, in the sense of our Definition 2, on a general time scale.

Example 15. *Let $0 < \beta < 1$, $\lambda > 0$, and $\alpha < \beta\lambda$. Let $\mathbb{T}$ be any time scale containing the origin, i.e., $0 \in \mathbb{T}$; $f : \mathbb{T} \to \mathbb{R}$ be the self-similar function $f(t) = ct^\beta$; and choose the structural function $p : \mathbb{T} \to \mathbb{R}$ to be $p(t) = t^\alpha$. It follows from Theorem 3 that if point $t = 0$ is right-dense, then*

$$\begin{aligned} f^{\Delta_p^\lambda}(0) &= \lim_{t \mapsto 0} \frac{f^\lambda(t) - f^\lambda(0)}{p(t) - p(0)} = \lim_{t \mapsto 0} \left[c^\lambda t^{\beta\lambda - \alpha} \right] \\ &= 0; \end{aligned}$$

if point $t = 0$ is right-scattered, then

$$\begin{aligned} f^{\Delta_p^\lambda}(0) &= \frac{f^\lambda(\sigma(0)) - f^\lambda(0)}{p(\sigma(0)) - p(0)} \\ &= c^\lambda \sigma(0)^{\beta\lambda - \alpha}. \end{aligned}$$

We conclude that f is always Δ_p^λ-differentiable at $t = 0$.

5. Conclusion

We introduced, for the first time, the notion of structural derivative on time scales. The developed calculus allows to unify and extend several decay models found in the literature. A nice mathematical example, showing the necessity to define structural derivatives on time scales, was given with respect to self-similar functions in Section 4. We claim that the new results here obtained may serve as key tools to model complex systems, for example in physics, life, and biological sciences. As future work, one shall develop such real world models, clearly showing the usefulness of the structural derivative on time scales. From the theoretical side, one can proceed with structural measures and structural Lebesgue integration.

Acknowledgements

This research was initiated during a one month visit of Bayour to the Department of Mathematics of University of Aveiro, Portugal, February and March 2018. The hospitality of the host institution and the financial support of University of Mascara, Algeria, are here gratefully acknowledged. Torres was supported by Portuguese funds through CIDMA and FCT, within project UID/MAT/04106/2019.

The authors are grateful to two anonymous referees, for several constructive remarks, suggestions, and questions.

References

[1] Atangana, A., Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system, *Chaos Solitons Fractals* **102** (2017), 396–406.

[2] Bayour, B. and Torres, D. F. M., Complex-valued fractional derivatives on time scales, in *Differential and difference equations with applications*, 79–87, Springer Proc. Math. Stat., 164, Springer, 2016.

[3] Bohner, M. and Peterson, A. Dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2001.

[4] Bohner, M. and Peterson, A., Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003.

[5] Chen, W., Time-space fabric underlying anomalous diffusion, *Chaos Solitons Fractals* **28** (2006), no. 4, 923–929.

[6] Chen, W., Liang, Y.-J. and Hei, X.-D., Local structural derivative and its applications, *Chin. J. Solid Mech.* **37** (2016), no. 5, 456–460.

[7] Karci, A. and Karadogan, A., Fractional order derivative and relationship between derivative and complex functions, *Math. Sci. Appl. E-Notes* **2** (2014), no. 1, 44–54.

[8] Strunin, D. V. and Suslov, S. A. Phenomenological approach to 3D spinning combustion waves: numerical experiments with a rectangular rod, *Int. J. Self Prop. High Temp. Synth.* **14** (2005), no. 1, 33–39.

[9] Tarasov, V. E., Fractional hydrodynamic equations for fractal media, *Ann. Physics* **318** (2005), no. 2, 286–307.

[10] Weberszpil, J., Lazo, M. J. and Helayel-Neto, J. A., On a connection between a class of q-deformed algebras and the Hausdorff derivative in a medium with fractal metric, *Physica A* **436** (2015), 399–404.

[11] Weberszpil, J. and Sotolongo-Costa, O., Structural derivative model for tissue radiation response, *J. Adv. Phys.* **13** (2017), no. 4, 4779–4785.

[12] Wio, H. S., Escudero, C., Revelli, J. A., Deza, R. R. and de la Lama, M. S., Recent developments on the Kardar-Parisi-Zhang surface-growth equation, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **369** (2011), no. 1935, 396–411.

[13] Yablonskiy, D. A., Bretthorst, G. L. and Ackerman, J. J. H., Statistical model for diffusion attenuated MR signal, *Mag. Res. Medicine* **50** (2003), no. 4, 664–669.

Current address: Benaoumeur Bayour: University of Mascara, B.P. 305, Mamounia Mascara, Mascara 29000, Algeria.

E-mail address: b.bayour@univ-mascara.dz

ORCID Address: http://orcid.org/0000-0003-3504-4655

Current address: Delfim F. M. Torres: University of Aveiro, Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, 3810-193 Aveiro, Portugal.

E-mail address: delfim@ua.pt

ORCID Address: http://orcid.org/0000-0001-8641-2505