Advances in Mathematics of Communications
Volume 10, No. 1, 2016, 163–177
DOI:10.3934/amc.2016.10.163

COMPOSITION CODES

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Abstract. In this paper we introduce a special class of 2D convolutional codes, called composition codes, which admit encoders $G(d_1, d_2)$ that can be decomposed as the product of two 1D encoders, i.e., $G(d_1, d_2) = G_2(d_2)G_1(d_1)$. Taking into account this decomposition, we obtain syndrome formers of the code directly from $G_1(d_1)$ and $G_2(d_2)$, in case $G_1(d_1)$ and $G_2(d_2)$ are right prime. Moreover we consider 2D state-space realizations by means of a separable Roesser model of the encoders and syndrome formers of a composition code and we investigate the minimality of such realizations. In particular, we obtain minimal realizations for composition codes which admit an encoder $G(d_1, d_2) = G_2(d_2)G_1(d_1)$ with $G_2(d_2)$ a systematic 1D encoder. Finally, we investigate the minimality of 2D separable Roesser state-space realizations for syndrome formers of these codes.

1. Introduction

In this paper we define a new class of two-dimensional (2D) convolutional codes, called composition codes. These codes admit encoders $G(d_1, d_2)$ that can be obtained from the series connection of two one-dimensional (1D) encoders $G_1(d_1)$ and $G_2(d_2)$, i.e., as $G(d_1, d_2) = G_2(d_2)G_1(d_1)$. This decomposition allows us to apply the well-developed theory of 1D convolutional codes to the study of composition codes. It is our conviction that the special structure of composition codes can be exploited to construct 2D convolutional codes with good distance properties based on 1D results. Moreover, we think that it will allow developing a decoding algorithm based on a sequencial application of 1D decoding procedures. This would be a great advantage since there are no decoding algorithms for 2D convolutional codes.

2010 *Mathematics Subject Classification:* Primary: 94B10, 93C35; Secondary: 93B25, 93B20.

Key words and phrases: Encoders and syndrome formers, 2D composition codes, minimal 2D state-space models.

The second and third authors were supported by Portuguese funds through the Center for Research and Development in Mathematics and Applications (CIDMA), and The Portuguese Foundation for Science and Technology (FCT - Fundação para a Ciência e a Tecnologia), within project UID/MAT/04106/2013.

Restricting to these codes we first obtain syndrome formers directly from $G_1(d_1)$ and $G_2(d_2)$.

We then focus on state-space realizations of encoders and syndrome formers of composition codes. For that, we consider 2D state-space realizations by means of separable Roesser models. Such models admit a characterization of minimality of the dimension of realizations in terms of the corresponding matrices, which does not happen if we consider other models [1, 2, 12]. Moreover, the problem of 2D state-space realization by means of separable Roesser models can be reduced to two sequential 1D realization problems.

We investigate how to obtain minimal state-space realizations (realizations with minimal dimension) of composition codes for code generation and code verification. This question has been solved for 1D convolutional codes [3, 5, 6] by means of characterizing the encoders and syndrome formers with realizations of minimal dimension among all the encoders and syndrome formers of the code, respectively. These encoders and syndrome formers are called minimal. However, the characterization of minimal encoders and syndrome formers is still an open problem for the 2D case.

A characterization of minimal encoders for general 2D convolutional codes of rate $1/n$ was obtained in [9, 10]. However, the generalization of the results in [9, 10] for 2D convolutional codes of rate k/n, with $k > 1$, appears to be very difficult. However the problem becomes easier to handle if we restrict our study to the classes of composition codes, and in particular to those which admit an encoder $G(d_1, d_2) = G_2(d_2)G_1(d_1)$ where $G_2(d_2)$ is systematic. Here, we obtain minimal encoders for such codes and study the minimality of realizations of their syndrome formers.

This paper is organized as follows. In the next section we present some preliminaries on 1D and 2D polynomial matrices. In section 3 we give the basic notions on 2D convolutional codes. In section 4 we introduce the composition codes and give a construction of syndrome formers for composition codes which admit encoders $G(d_1, d_2) = G_2(d_2)G_1(d_1)$ where $G_1(d_1)$ and $G_2(d_2)$ are right prime. Such construction is obtained from $G_1(d_1)$ and $G_2(d_2)$. State-space realizations by means of separable Roesser models of encoders and syndrome formers of a 2D convolutional code are presented in section 5. Finally, in section 6 composition codes which admit encoders $G(d_1, d_2) = G_2(d_2)G_1(d_1)$ where $G_2(d_2)$ is systematic are considered and minimal encoders of such codes are obtained. Minimal syndrome formers among a class of syndrome formers of such codes are also obtained.

2. Preliminaries

In this paper we adopt the usual notation of $\mathbb{F}[d]$, $\mathbb{F}^{n\times k}[d]$, $\mathbb{F}[d_1, d_2]$ and $\mathbb{F}^{n\times k}[d_1, d_2]$ to denote the ring of 1D polynomials in the indeterminate d, the set of matrices of size $n \times k$ with elements in $\mathbb{F}[d]$, the ring of 2D polynomials in the indeterminates d_1 and d_2 and the set of matrices of size $n \times k$ with elements in $\mathbb{F}[d_1, d_2]$, respectively, over an arbitrary field $\mathbb{F}$.

In this section we summarize some results on polynomial matrices over $\mathbb{F}[d]$ and over $\mathbb{F}[d_1, d_2]$ for future reference.

Definition 2.1. A matrix $G(d) \in \mathbb{F}^{n\times k}[d]$ is:

- unimodular if $n = k$ and it has polynomial inverse;

- right prime if it has full column rank and for every factorization $G(d) = \bar{G}(d)T(d)$, for $\bar{G}(d) \in \mathbb{F}^{n\times k}[d]$ and $T(d) \in \mathbb{F}^{k\times k}[d]$, $T(d)$ is unimodular;

A matrix $U(d) \in \mathbb{F}^{k\times k}[d]$ is unimodular if and only if $det\, G \in \mathbb{F}\backslash\{0\}$ and a full column rank matrix $G(d) \in \mathbb{F}^{n\times k}[d]$ is right prime if and only if there exists a matriz $L(d) \in \mathbb{F}^{k\times n}[d]$ such that $L(d)G(d) = I$, or equivalently if and only if the ideal generated by the maximal order minors of $G(d)$ is the ring $\mathbb{F}[d]$.

All statements on "column" and "right" factors can be couched in "row" and "left" terms, upon taking transposes.

Let $G(d) \in \mathbb{F}^{n\times k}[d]$ and $H(d) \in \mathbb{F}^{(n-k)\times n}[d]$. The maximal order minor of $G(d)$ constituted by the rows $1 \leq r_1 < r_2 < \cdots < r_k \leq n$ and the maximal order minor of $H(d)$ constituted by the rows $1 \leq s_1 < s_2 < \cdots < s_{n-k} \leq n$ are said to be corresponding if $\{r_1, r_2, \ldots, r_k, s_1, s_2, \ldots, s_{n-k}\} = \{1, \ldots, n\}$.

Proposition 2.2 ([5]). *Let $G(d) \in \mathbb{F}^{n\times k}[d]$ and $H(d) \in \mathbb{F}^{(n-k)\times n}[d]$ be right prime and left prime matrices, respectively, such that $H(d)G(d) = 0$. Then the corresponding maximal order minors of $G(d)$ and $H(d)$ are equal, up to a unit of $\mathbb{F}[d]$.*

Given a polynomial matrix $G(d) \in \mathbb{F}^{n\times k}[d]$. The degree of a column $G(d)$ is defined as the maximum degrees of its entries.

Definition 2.3. Let $G(d) \in \mathbb{F}^{n\times k}[d]$ with columns degrees $\ell_1, \ell_2, \ldots, \ell_k$.

- The external degree of $G(d)$, $\text{extdeg}(G)$, is the sum of its column degrees, i.e., $\text{extdeg}(G) = \sum_{i=1}^{k} \ell_i$;
- The internal degree of of $G(d)$, $\text{intdeg}(G)$, is the maximum degree of its full size minors.
- $G(d)$ is column reduced if $\text{extdeg}(G) = \text{intdeg}(G)$.

Next we consider 2D polynomial matrices. Concerning matrix factorization, there exist two notions of primeness for such matrices.

Definition 2.4. A matrix $G(d_1, d_2) \in \mathbb{F}^{n\times k}[d_1, d_2]$ is:

- unimodular if $n = k$ and it has polynomial inverse;
- right factor prime (rFP) if it has full column rank and for every factorization $G(d_1, d_2) = \bar{G}(d_1, d_2)T(d_1, d_2)$, for $\bar{G}(d_1, d_2) \in \mathbb{F}^{n\times k}[d_1, d_2]$ and $T(d_1, d_2) \in \mathbb{F}^{k\times k}[d_1, d_2]$, $T(d_1, d_2)$ is unimodular;
- right zero prime (rZP) if it has full column rank and the ideal generated by the maximal order minors of $G(d_1, d_2)$ is the ring $\mathbb{F}[d_1, d_2]$.

A matrix $U(d_1, d_2) \in \mathbb{F}^{k\times k}[d_1, d_2]$ is unimodular if and only if $det\, U \in \mathbb{F}\backslash\{0\}$ and a matrix $G(d_1, d_2) \in \mathbb{F}^{n\times k}[d_1, d_2]$ is rZP if and only if there exists a matrix $L(d_1, d_2) \in \mathbb{F}^{k\times n}[d_1, d_2]$ such that $L(d_1, d_2)G(d_1, d_2) = I$.

As happens in the 1D case, analogous results can be defined for left factorization. Moreover, Proposition 2.2 also holds for the 2D case.

Proposition 2.5 ([4]). *Let $G(d_1, d_2) \in \mathbb{F}^{n\times k}[d_1, d_2]$ and $H(d_1, d_2) \in \mathbb{F}^{(n-k)\times n}[d_1, d_2]$ be right factor prime and left factor prime matrices, respectively, such that*

$$H(d_1, d_2)G(d_1, d_2) = 0.$$

Then the corresponding maximal order minors of $G(d_1, d_2)$ and $H(d_1, d_2)$ are equal, up to a unit of $\mathbb{F}[d_1, d_2]$.

3. 2D convolutional codes

In this paper we consider 2D convolutional codes constituted by sequences indexed by $\mathbb{Z}^2$ and taking values in $\mathbb{F}^n$, where $\mathbb{F}$ is a field. Such sequences $\{\mathbf{w}(i,j)\}_{(i,j)\in\mathbb{Z}^2}$ can be represented by bilateral formal power series

$$\hat{\mathbf{w}}(d_1,d_2) = \sum_{(i,j)\in\mathbb{Z}^2} \mathbf{w}(i,j) d_1^i d_2^j.$$

In the sequel we shall use the sequence and the corresponding series interchangeably, depending on the problem we are dealing with.

For $n \in \mathbb{N}$, the set of bilateral formal power series over $\mathbb{F}^n$ is denoted by $\mathcal{F}_{2D}^n$. This set is a module over the ring $\mathbb{F}[d_1,d_2]$.

Definition 3.1. A 2D convolutional code $\mathcal{C}$ is a submodule of $\mathcal{F}_{2D}^n$ which coincides with the image of $\mathcal{F}_{2D}^k$ (for some $k \in \mathbb{N}$) by a polynomial operator $G(d_1,d_2)$, i.e.,

$$\begin{aligned} \mathcal{C} &= \operatorname{Im} G(d_1,d_2) \\ &= \{\hat{\mathbf{w}}(d_1,d_2) = G(d_1,d_2)\hat{\mathbf{u}}(d_1,d_2),\ \hat{\mathbf{u}}(d_1,d_2) \in \mathcal{F}_{2D}^k\}. \end{aligned}$$

It follows, as a consequence of [Theorem 2.2, [7]], that a 2D convolutional code can always be given as the image of a full column rank polynomial operator $G(d_1,d_2) \in \mathbb{F}^{n\times k}[d_1,d_2]$. Such polynomial matrix is called an *encoder* of $\mathcal{C}$. A code with encoders of size $n \times k$ is said to have *rate* k/n.

Note that this definition of code differs from the definition in [13, 14], where only finite support codewords are considered. Moreover it also differs from the one in [4] where non full column rank 2D polynomial matrices are allowed as encoders. However, our definition is motivated by the fact that only full column rank encoders are relevant for the purpose of obtaining minimal realizations of a code.

Two encoders, $G_1(d_1,d_2) \in \mathbb{F}^{n\times k}[d_1,d_2]$ and $G_2(d_1,d_2) \in \mathbb{F}^{n\times k}[d_1,d_2]$ are said to be *equivalent* if they generate the same code $\mathcal{C}$. If $G_1(d_1,d_2)$ and $G_2(d_1,d_2)$ are equivalent encoders, there exist two square non-singular matrices over $\mathbb{F}[d_1,d_2]$, $P_1(d_1,d_2)$ and $P_2(d_1,d_2)$, such that

$$G_1P_1 = G_2P_2.$$

This implies that

$$G_1 = G_2U_2 \quad \text{and} \quad G_2 = G_1U_1,$$

with $U_2 = P_2P_1^{-1}$ and $U_1 = P_1P_2^{-1}$, i.e., the convolutional encoders are unique up to the post-multiplication by a square nonsingular 2D rational matrix.

If $G_1(d_1,d_2)$ is right factor prime and $G_2(d_1,d_2)$ is equivalent to $G_1(d_1,d_2)$ then

$$G_2 = G_1P,$$

for some square 2D polynomial matrix $P(d_1,d_2)$. In case $G_1(d_1,d_2)$ and $G_2(d_1,d_2)$ are both right factor prime then

$$G_2 = G_1U,$$

for some 2D unimodular polynomial matrix $U(d_1,d_2) \in \mathbb{F}^{k\times k}[d_1,d_2]$. Thus, if $\mathcal{C}$ admits a rZP encoder, then all its rFP encoders are rZP.

A 2D convolutional code $\mathcal{C}$ of rate k/n can also be represented as the kernel of a $(n-k)\times n$ left factor prime polynomial matrix.

Definition 3.2. Let $\mathcal{C}$ be a 2D convolutional code of rate k/n. A left factor prime matrix $H(d_1,d_2) \in \mathbb{F}^{(n-k)\times n}[d_1,d_2]$ such that

$$\mathcal{C} = \text{Ker } H(d_1,d_2),$$

is called a syndrome former of $\mathcal{C}$.

Note that w is in $\mathcal{C}$ if and only if $H(d_1,d_2)w = 0$. Given a right factor prime encoder of $\mathcal{C}$, a syndrome former of $\mathcal{C}$ can be obtained by constructing a $(n-k)\times n$ left-factor prime matrix $H(d_1,d_2)$ such that

$$H(d_1,d_2)G(d_1,d_2) = 0.$$

Moreover all syndrome formers of $\mathcal{C}$ are of the form $U(d_1,d_2)H(d_1,d_2)$, where $U(d_1,d_2) \in \mathbb{F}^{(n-k)\times(n-k)}[d_1,d_2]$ is unimodular. This means that if a 2D convolutional code $\mathcal{C}$ admits a rZP encoder, then the corresponding syndrome formers are lZP (see Proposition 2.5).

1D convolutional codes and its encoders and their syndrome formers are defined in a similar way as for the 2D convolutional codes, but are in this case polynomial matrices in one indeterminate d (instead of d_1 and d_2) [3, 5, 6].

4. Composition codes

In this section we consider a particular class of 2D convolutional codes generated by 2D polynomial encoders that are obtained from the composition of two 1D polynomial encoders. Such encoders/codes will be called *composition encoders/codes*. The formal definition of composition encoders is as follows.

Definition 4.1. An encoder $G(d_1,d_2) \in \mathbb{F}^{n\times k}[d_1,d_2]$ such that

$$G(d_1,d_2) = G_2(d_2)G_1(d_1),$$

where $G_1(d_1) \in \mathbb{F}^{p\times k}[d_1]$ and $G_2(d_2) \in \mathbb{F}^{n\times p}[d_2]$ are 1D encoders, is said to be a composition encoder.

Note that the requirement that $G_i(d_i)$, for $i = 1, 2$, is a 1D encoder is equivalent to the condition that $G_i(d_i)$ is a full column rank matrix. Moreover this requirement clearly implies that $G_2(d_2)G_1(d_1)$ has full column rank, hence the composition G_2G_1 of two 1D encoders is indeed a 2D encoder.

The 2D code $\mathcal{C}$ associated with $G = G_2G_1$, given as

$$\begin{aligned}\mathcal{C} &= \text{Im } G(d_1,d_2) = G_2(d_2)(\text{Im } (G_1(d_1))) \\ &= \{\hat{\mathbf{w}}(d_1,d_2) \in \mathcal{F}_{2D}^q : \exists\ \hat{\mathbf{z}}(d_1,d_2) \in \text{Im } (G_1(d_1)) \\ &\quad\ \text{such that } \ \hat{\mathbf{w}}(d_1,d_2) = G_2(d_2)\hat{\mathbf{z}}(d_1,d_2)\},\end{aligned}$$

is called a composition code.

Note that every polynomial matrix $M(d_1,d_2) \in \mathbb{F}^{s\times r}[d_1,d_2]$ can be factorized as follows:

$$M(d_1,d_2) = M_2(d_2)M_1(d_1), \tag{1}$$

where $M_2(d_2) = \begin{bmatrix} I_n \mid \cdots \mid I_n d_2^{\ell_2}\end{bmatrix} N_2 \in \mathbb{F}^{s\times p}[d_2]$ and $M_1(d_1) = N_1 \begin{bmatrix} I_k & \dots & I_k d_1^{\ell_1}\end{bmatrix}^T$ is in $\mathbb{F}^{p\times r}[d_1]$, with N_2 and N_1 constant matrices. If N_2 has full column rank and N_1 has full row rank we say that (1) is an *optimal decomposition* of $M(d_1,d_2)$. Thus, if $G_2(d_2)G_1(d_1)$ is an optimal decomposition of a composition encoder $G(d_1,d_2)$, then $G_2(d_2)$ and $G_1(d_1)$ are full column rank matrices.

In the sequel we shall focus on the syndrome formers of composition codes. Since composition encoders can be written as a product of two 1D convolutional encoders, we use this property for constructing syndrome formers of the corresponding code based on 1D polynomial methods. For that purpose we shall concentrate on composition codes that admit an encoder $G(d_1, d_2) = G_2(d_2)G_1(d_1)$ where $G_1(d_1)$ and $G_2(d_2)$ are right factor prime. Note that, in this case, $G(d_1, d_2)$ has a 2D polynomial left inverse $G_1(d_1)^{-1}G_2(d_2)^{-1}$, where $G_1(d_1)^{-1}$ and $G_2(d_2)^{-1}$ are left inverses of $G_1(d_1)$ and $G_2(d_2)$, respectively. This means that $G(d_1, d_2)$ is rZP and therefore all rFP encoders of the code are rZP. Moreover, the corresponding syndrome formers are also lZP (see Proposition 2.5).

Since $G_2(d_2) \in \mathbb{F}^{n\times p}[d_2]$ is right prime there exists a unimodular matrix $U(d_2) \in \mathbb{F}^{n\times n}[d_2]$ such that

$$U(d_2)G_2(d_2) = \begin{bmatrix} I_p \\ 0 \end{bmatrix}.$$

We shall partition $U_2(d_2)$ as

$$U(d_2) = \begin{bmatrix} L_2(d_2) \\ H_2(d_2) \end{bmatrix}, \tag{2}$$

where $L_2(d_2)$ has p rows.

It is easy to check that, if $H_1(d_1) \in \mathbb{F}^{(p-k)\times p}[d_1]$ is a syndrome former of the 1D convolutional code Im $G_1(d_1)$ (i.e., $H_1(d_1)$ is left prime and is such that $H_1(d_1)G_1(d_1) = 0$), then

$$\begin{bmatrix} H_1(d_1)L_2(d_2) \\ H_2(d_2) \end{bmatrix} G_2(d_2)G_1(d_1) = 0. \tag{3}$$

This reasoning leads to the following proposition.

Proposition 4.2. *Let $\mathcal{C}$ = Im $G(d_1, d_2)$ be a composition code with $G(d_1, d_2) = G_2(d_2)G_1(d_1)$, where $G_2(d_2) \in \mathbb{F}^{n\times p}[d_2]$ and $G_1(d_1) \in \mathbb{F}^{p\times k}[d_1]$ are both right prime 1D encoders. Let further $H_1(d_1) \in \mathbb{F}^{(p-k)\times p}[d_1]$ be a 1D syndrome former of* Im $G_1(d_1)$ *and define $\begin{bmatrix} L_2(d_2) \\ H_2(d_2) \end{bmatrix}$ as in (2). Then*

$$H(d_1, d_2) = \begin{bmatrix} H_1(d_1)L_2(d_2) \\ H_2(d_2) \end{bmatrix}$$

is a syndrome former of

$$\mathcal{C} = \text{Im } G(d_1, d_2).$$

Proof. Since (3) is obviously satisfied and $H(d_1, d_2)$ has size $(n - k) \times n$, we only have to prove that $H(d_1, d_2)$ is left factor prime. Note that as $H_1(d_1)$ is left prime, there exists $R_1(d_1) \in \mathbb{F}^{p\times(p-k)}[d_1]$ such that $H_1(d_1)R_1(d_1) = I_{p-k}$. Now it is easy to see that

$$R(d_1, d_2) = U_2(d_2)^{-1} \begin{bmatrix} R_1(d_1) & 0 \\ 0 & I_{n-p} \end{bmatrix}.$$

constitutes a polynomial right inverse of $H(d_1, d_2)$. Consequently $H(d_1, d_2)$ is left zero prime which implies that it is left factor prime as we wish to prove. □

5. State-space realizations of encoders and syndrome formers

In this section we recall some fundamental concepts concerning 1D and 2D state-space realizations of transfer functions, having in mind the realizations of encoders and syndrome formers.

A 1D state-space model

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

denoted by $\Sigma^{1D}(A,B,C,D)$, where A, B, C and D are matrices of suitable dimensions and $x(t) \in \mathbb{F}^m$, is said to be a realization of dimension m of $M(d) \in \mathbb{F}^{s\times r}[d]$ if $M(d) = C(I_m - Ad)^{-1}Bd + D$. Moreover, it is a minimal realization if the size m of the state x is minimal among all the realizations of $M(d)$. The dimension of a minimal realization of $M(d)$ is called the *McMillan degree* of $M(d)$ and is given by $\mu(M) = \operatorname{int deg} \begin{bmatrix} M(d) \\ I_r \end{bmatrix}$ ([10]). Note that when considering a realization $\Sigma^{1D}(A,B,C,D)$ of an encoder $G(d)$ the input u is the information sequence and y the corresponding codeword, i.e., $w := y$; thus the sequence of inputs $\{u_i\}_{i\in\mathbb{Z}}$ will produce a sequence of outputs $\{w_i\}_{i\in\mathbb{Z}}$, such that $\hat{w}(d) = G(d)\hat{u}(d)$, where $\hat{u}(d) = \sum_{i\in\mathbb{Z}} u_i d^i$ and $\hat{w}(d) = \sum_{i\in\mathbb{Z}} w_i d^i$. On the other hand, when considering a realization $\Sigma^{1D}(A,B,C,D)$ of a syndrome former $H(d)$, the codewords w are the inputs u that yield zero output.

The encoders and syndrome formers of a 1D convolutional code $\mathcal{C}$ with minimal McMillan degree among all the encoders and all syndrome formers of $\mathcal{C}$, respectively, are said to be *minimal*. Minimal encoders and minimal syndrome formers of $\mathcal{C}$ have McMillan degree equal to the degree of the code $\mathcal{C}$, where the degree of $\mathcal{C}$ is defined as the external degree of the right prime and column reduced encoder of the code. Such encoders are called *canonical* and constitute a particular class of minimal encoders [3, 5].

Minimal encoders and syndrome formers of a 1D convolutional code were completely characterized in [3, 5, 6]. Such characterizations are given in terms of the properties of the encoders and syndrome formers as polynomial matrices. Another characterization of minimal encoders is given by the following theorem.

Theorem 5.1 ([15]). *Let $G(d) \in \mathbb{F}[d]^{n\times k}$ be an encoder of a 1D convolutional code $\mathcal{C}$ and $\Sigma^{1D}(A,B,C,D)$ be a minimal realization of $G(d)$. Then $G(d)$ is a minimal encoder of $\mathcal{C}$ if and only if the following conditions are satisfied.*

- *(i) $[B^T\ D^T]^T$ has full column rank;*
- *(ii) $[A\ B]$ has full row rank;*
- *(iii) ker $D \subseteq$ ker B (i.e., there exists a matrix L such that $B = LD$);*
- *(iv) Let L be as in (ii), and let Λ be a minimal left annihilator (mla) [1] of D. Then the pair $(A - LC, \Lambda C)$ is observable.*

In case $\Sigma^{1D}(A,B,C,D)$ is a minimal realization of a minimal encoder of a convolutional code $\mathcal{C}$ (i.e., A,B,C,D satisfy the condition of the theorem above), we say that $\Sigma^{1D}(A,B,C,D)$ is a minimal realization of $\mathcal{C}$ and define the McMillan degree of $\mathcal{C}$, $\mu(\mathcal{C})$, to be the dimension of a minimal realization of $\mathcal{C}$.

[1] A full row rank matrix Λ is a mla of D if $\Lambda D = 0$ and for all Λ^* such that $\Lambda^* D = 0$ there exists $\tilde{\Lambda}$ satisfying $\Lambda^* = \tilde{\Lambda}\Lambda$.

Considering the 2D case, there exist several types of state-space models [1, 2]. In our study we shall consider *separable Roesser models* [12]. These models have the following form:

$$(4)\qquad \begin{cases} x_1(i+1,j) = A_{11}x_1(i,j) + A_{12}x_2(i,j) + B_1u(i,j) \\ x_2(i,j+1) = A_{21}x_1(i,j) + A_{22}x_2(i,j) + B_2u(i,j) \\ y(i,j) = C_1x_1(i,j) + C_2x_2(i,j) + Du(i,j) \end{cases}$$

where A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2 and D are matrices over $\mathbb{F}$, with suitable dimensions, u is the input-variable, y is the output-variable, and $x = (x_1, x_2)$ is the state variable, where x_1 and x_2 are the horizontal and the vertical state-variables, respectively. The dimension of the system described by (4) is given by the size of x. Moreover either $A_{12} = 0$ or $A_{21} = 0$. The separable Roesser model corresponding to equations (4) with $A_{12} = 0$ is denoted by $\Sigma_{12}^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$, whereas the one with $A_{21} = 0$ is denoted by $\Sigma_{21}^{2D}(A_{11}, A_{12}, A_{22}, B_1, B_2, C_1, C_2, D)$.

The remaining considerations of this section can be stated both for cases when $A_{12} = 0$ or $A_{21} = 0$, however we just consider $A_{12} = 0$; the case $A_{21} = 0$ is completely analogous, with the obvious adaptations.

Definition 5.2. $\Sigma_{12}^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$ is said to be a realization of the 2D polynomial matrix $M(d_1, d_2) \in \mathbb{F}^{s\times r}[d_1, d_2]$ if

$$M(d_1,d_2) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} I_{m_1} - A_{11}d_1 & 0 \\ -A_{21}d_2 & I_{m_2} - A_{22}d_2 \end{bmatrix}^{-1} \left(\begin{bmatrix} B_1 \\ 0 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} d_2 \right) + D.$$

Realizations of $M(d_1, d_2)$ with minimal dimension are called *minimal*. The *Roesser McMillan degree* of $M(d_1, d_2)$, $\mu_R(M)$, is defined as the dimension of a minimal realization of $M(d_1, d_2)$.

Considering a factorization $M(d_1, d_2) = M_2(d_2)M_1(d_1)$ where $M_2(d_2) \in \mathbb{F}^{s\times p}[d_2]$ and $M_1(d_1) \in \mathbb{F}^{p\times r}[d_1]$ and

$$\begin{cases} x_1(i+1,j) = A_{11}x_1(i,j) + B_1u_1(i,j) \\ y_1(i,j) = \bar{C}_1x_1(i,j) + \bar{D}_1u_1(i,j) \end{cases}$$

a realization of $M_1(d_1)$ and

$$\begin{cases} x_2(i,j+1) = A_{22}x_2(i,j) + \bar{B}_2u_2(i,j) \\ y_2(i,j) = C_2x_2(i,j) + \bar{D}_2u_2(i,j) \end{cases}$$

a realization of $M_2(d_2)$. Then the 2D system obtained as the series concatenation of these two realizations (by considering $u_2(i,j) := y_1(i,j)$) is a realization of $M(d_1, d_2)$ given by

$$\begin{cases} x_1(i+1,j) = A_{11}x_1(i,j) + B_1u_1(i,j) \\ y_1(i,j) = \bar{C}_1x_1(i,j) + \bar{D}_1u_1(i,j) \\ x_2(i,j+1) = A_{22}x_2(i,j) + \bar{B}_2y_1(i,j) \\ y_2(i,j) = C_2x_2(i,j) + \bar{D}_2y_1(i,j) \end{cases}$$

or equivalently

$$\begin{cases} x_1(i+1,j) = A_{11}x_1(i,j) + B_1u(i,j) \\ x_2(i,j+1) = A_{21}x_1(i,j) + A_{22}x_2(i,j) + B_2u(i,j) \\ y(i,j) = C_1x_1(i,j) + C_2x_2(i,j) + Dy_1(i,j) \end{cases},$$

with $A_{21} = \bar{B}_2\bar{C}_1$, $B_2 = \bar{B}_2\bar{D}_1$, $C_1 = \bar{D}_2\bar{C}_1$ and $D = \bar{D}_2\bar{D}_1$.

As shown in [8, 9], if $M(d_1, d_2) = M_2(d_2)M_1(d_1)$ is an optimal decomposition, $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ is a minimal realization of $M_1(d_1)$ (of dimension $\mu(M_1)$) and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ is a minimal realization of $M_2(d_2)$ (of dimension $\mu(M_2)$) then the 2D system $\Sigma_{12}^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$ obtained above, is a minimal realization of $M(d_1, d_2)$ of dimension $\mu_R(M) = \mu(M_1) + \mu(M_2)$. A similar reasoning can be made if we factorize $M(d_1, d_2) = \bar{M}_1(d_1)\bar{M}_2(d_2)$, where $\bar{M}_1(d_1) \in \mathbb{F}^{s\times\bar{p}}[d_1]$ and $\bar{M}_2(d_2) \in \mathbb{F}^{\bar{p}\times r}[d_2]$, for some $\bar{p} \in \mathbb{N}$, to obtain a minimal realization $\Sigma_{21}^{2D}(A_{11}, A_{12}, A_{22}, B_1, B_2, C_1, C_2, D)$ of $M(d_1, d_2)$.

Note that, since both encoders and syndrome formers are (2D) polynomial matrices, they both can be realized by means of (4). However, as already mentioned, when considering realizations of an encoder $G(d_1, d_2) = G_2(d_2)G_1(d_1)$ we shall take $A_{12} = 0$ and $y = w$; on the other hand when considering realizations of a syndrome former $H(d_1, d_2) = H_1(d_1)H_2(d_2)$, we shall take $A_{21} = 0$, $u = w$ and $y = 0$. This means that when considering encoders we are interested in the input/output behavior of a 2D state-space model of the form (4), with $A_{12} = 0$, whereas when we consider syndrome formers we are interested in the output-nulling inputs of a 2D state-space model of the form (4), with $A_{21} = 0$.

As happens in the 1D case, we say that an encoder and a syndrome former of a 2D convolutional code $\mathcal{C}$ are *minimal* if they have minimal Roesser McMillan degree among all encoders and syndrome formers of $\mathcal{C}$, respectively. However, contrary to what happens in the 1D case it seems hard to obtain a characterization for minimal encoders and for minimal syndrome formers. In [11] sufficient conditions were established that guarantee the minimality of an encoder of a code. These sufficient conditions are given in the following result.

Theorem 5.3. *Let $\mathcal{C}$ be a 2D convolutional code and $G(d_1, d_2) \in \mathbb{F}^{n\times k}[d_1, d_2]$ be an encoder of $\mathcal{C}$ with minimal realization $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$. Suppose that $\Sigma^{1D}\left(A_{11}, B_1, \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}, \begin{bmatrix} B_2 \\ D \end{bmatrix}\right)$ and $\Sigma^{1D}(A_{22}, [A_{21}\ \ B_2], C_2, [C_1\ \ D])$ satisfy the conditions of Theorem 5.1. Then $G(d_1, d_2)$ is a minimal encoder of $\mathcal{C}$.*

As we shall see, the question of minimal realization seems less hard to handle for composition codes.

6. Minimal realizations of composition codes

In this section we restrict our study to 2D composition encoders that admit a special structure, namely, in which $G(d_1, d_2) = G_2(d_2)G_1(d_1)$, where $G_2(d_2)$ is a systematic encoder.

Definition 6.1. $G(d) \in \mathbb{F}^{n\times k}[d]$ is a systematic encoder if

$$G(d) = T \begin{bmatrix} \bar{G}(d) \\ I_k \end{bmatrix}, \tag{5}$$

where $T \in \mathbb{F}^{n\times n}$ is an invertible constant matrix and $\bar{G}(d) \in \mathbb{F}^{(n-k)\times k}[d]$.

Example 6.2. In $\mathbb{Z}_2$, consider the polynomial encoder given by

$$G(d) = \begin{bmatrix} d & 1 & d & 0 \\ 0 & d^2 & 0 & d^2 \\ d+1 & 0 & d+1 & 0 \\ 0 & d^2+1 & 0 & d^2+1 \\ 1 & 1 & 0 & 0 \\ d & d^2 & d & d^2 \end{bmatrix}.$$

$G(d)$ is a systematic encoder since

$$G(d) = T \begin{bmatrix} \bar{G}(d) \\ I_4 \end{bmatrix},$$

with $T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ invertible and $\bar{G}(d) = \begin{bmatrix} d & 0 & d & 0 \\ 0 & d^2 & 0 & d^2 \end{bmatrix}$.

Note that this definition is slightly different from the usual one (see for instance [3]) as T is any invertible matrix rather than a permutation matrix.

Proposition 6.3 ([3, 5])**.** *Let $G(d) \in \mathbb{F}^{n\times k}[d]$ be a polynomial encoder. If $G(d)$ is systematic then it is a minimal encoder of $\mathcal{C} = \text{Im}\, G(d)$.*

Let $\mathcal{C}$ be a composition code generated by a composition encoder $G(d_1, d_2) \in \mathbb{F}^{n\times k}[d_1, d_2]$ such that

$$G(d_1, d_2) = G_2(d_2)G_1(d_1), \tag{6}$$

where $G_2(d_2) \in \mathbb{F}^{n\times p}[d_2]$, for some $p \in \mathbb{N}$, is a systematic encoder, and $G_1(d_1) \in \mathbb{F}^{p\times k}[d_1]$ is a minimal encoder. Note that the minimality assumption on $G_1(d_1)$ is not restrictive. In fact, we can assume without loss of generality that $G_1(d_1)$ is right prime, and in case $G_1(d_1)$ is not minimal there exists a suitable 1D unimodular matrix $X_1(d_1) \in \mathbb{F}[d_1]^{k\times k}$ such that $\tilde{G}(d_1) = G_1(d_1)X(d_1)$ is a minimal encoder of the corresponding 1D convolutional code [3, 5], and moreover, $G(d_1, d_2)X_1(d_1) = G_2(d_2)\tilde{G}_1(d_1)$ is also an encoder of $\mathcal{C}$.

Let $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ be minimal realizations of $G_1(d_1)$ and $G_2(d_2)$, respectively. Observe that, since $G_1(d_1)$ is a minimal encoder of the 1D code $\mathcal{C}_1 = \text{Im}\ G_1(d_1)$ and $G_2(d_2)$ is a minimal encoder of the 1D code $\mathcal{C}_2 = \text{Im}\ G_2(d_2)$ (see Proposition 6.3), it follows that the realizations $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ satisfy Theorem 5.1.

As already shown, connecting in series $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ yields the following 2D realization of $G(d_1, d_2)$:

$$\begin{cases} x_1(i+1, j) = A_{11}x_1(i, j) + B_1u(i, j) \\ x_2(i, j+1) = A_{21}x_1(i, j) + A_{22}x_2(i, j) + B_2u(i, j) \\ w(i, j) = C_1x_1(i, j) + C_2x_2(i, j) + Du(i, j) \end{cases}, \tag{7}$$

where $A_{21} = \bar{B}_2\bar{C}_1$, $B_2 = \bar{B}_2\bar{D}_1$, $C_1 = \bar{D}_2\bar{C}_1$ and $D = \bar{D}_2\bar{D}_1$.

The next theorem shows that, under the technical condition that $\begin{bmatrix} \bar{C}_1 & \bar{D}_1 \end{bmatrix}$ is invertible, the realization $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$ given by (7) is a minimal realization of the composition code $\mathcal{C}$.

Theorem 6.4. *Let $G(d_1, d_2) \in \mathbb{F}^{n\times k}[d_1, d_2]$ be a composition encoder of a 2D convolutional code $\mathcal{C}$, such that*

$$G(d_1, d_2) = G_2(d_2)G_1(d_1),$$

where $G_2(d_2) \in \mathbb{F}^{n\times p}[d_2]$ is systematic and $G_1(d_1) \in \mathbb{F}^{p\times k}[d_1]$, for some $p \in \mathbb{N}$, is a minimal 1D encoder. Moreover, let $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ be two 1D minimal realizations of $G_2(d_2)$ and $G_1(d_1)$, respectively, and assume that $\begin{bmatrix} \bar{C}_1 & \bar{D}_1 \end{bmatrix}$ is square and invertible. Then $\Sigma^{2D}(A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D)$, *where $A_{21} = \bar{B}_2\bar{C}_1$, $B_2 = \bar{B}_2\bar{D}_1$, $C_1 = \bar{D}_2\bar{C}_1$ and $D = \bar{D}_2\bar{D}_1$ is a minimal realization of $\mathcal{C}$.*

Proof. Since $\Sigma^{1D}(A_{11}, B_1, \bar{C}_1, \bar{D}_1)$ and $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$ are 1D minimal realizations of Im $G_1(d_1)$ and Im $G_2(d_2)$, respectively, it follows, by Theorem 5.1 that they satisfy the following conditions.

Condition 1: $\bar{D}_1$ and $\bar{D}_2$ have full column rank.

Condition 2: (A_{11}, B_1) and $(A_{22}, \bar{B}_2)$ are both controllable pairs.

Condition 3: $Ker\bar{D}_1 \subseteq KerB_1$ and $Ker\bar{D}_2 \subseteq Ker\bar{B}_2$ (i.e, there exist matrices L_1 and L_2 such that $B_1 = L_1\bar{D}_1$ and $\bar{B}_2 = L_2\bar{D}_2$).

Condition 4: Let L_1 and L_2 be defined as in Condition 3, and let Λ_1 and Λ_2 be minimal left-annihilators (*mla*) of $\bar{D}_1$ and $\bar{D}_2$, respectively. Then the pairs $(A_{11} - L_1\bar{C}_1, \Lambda_1\bar{C}_1)$ and $(A_{22} - L_2C_2, \Lambda_2C_2)$ are both observable.

Let us now define

$$E = \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix} = \begin{bmatrix} \bar{B}_2 \\ \bar{D}_2 \end{bmatrix} \bar{C}_1, \quad F = \begin{bmatrix} B_2 \\ D \end{bmatrix} = \begin{bmatrix} \bar{B}_2 \\ \bar{D}_2 \end{bmatrix} \bar{D}_1$$

and

$$J = \begin{bmatrix} A_{21} & B_2 \end{bmatrix} = \bar{B}_2 \begin{bmatrix} \bar{C}_1 & \bar{D}_1 \end{bmatrix}, \quad H = \begin{bmatrix} C_1 & D \end{bmatrix} = \bar{D}_2 \begin{bmatrix} \bar{C}_1 & \bar{D}_1 \end{bmatrix}.$$

Firstly we show that the four conditions of Theorem 5.1 for the minimality of $\Sigma^{1D}(A_{11}, B_1, E, F)$ as a code realization are satisfied:

(i) Since Condition 1 and Condition 3 hold,

$$F = \begin{bmatrix} \bar{B}_2 \\ \bar{D}_2 \end{bmatrix} \bar{D}_1 = \left[\begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2 \right] \bar{D}_1$$

has full column rank as its factors $\bar{D}_1$, $\bar{D}_2$ and $\begin{bmatrix} L_2 \\ I \end{bmatrix}$ have full column rank.

(ii) follows immediately from Condition 2.

(iii) Note that since $\bar{B}_2 = L_2\bar{D}_2$ and $\bar{D}_2$ has full column rank, $\begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2$ has also full column rank. Then there exists a matrix U such that

$$U \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2 = I. \tag{8}$$

On the other hand, Condition 3 implies that there exists a matrix L_1 such that $B_1 = L_1\bar{D}_1$. Then

$$B_1 = L_1U \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2\bar{D}_1 = L_1U \begin{bmatrix} \bar{B}_2 \\ \bar{D}_2 \end{bmatrix} \bar{D}_1 = \bar{L}_1F,$$

where $\bar{L}_1 = L_1U$.

(iv) Consider $\bar{L}_1 = L_1U$, as defined above. Note that

$$\Lambda_1UF = \Lambda_1U \begin{bmatrix} \bar{B}_2 \\ \bar{D}_2 \end{bmatrix} \bar{D}_1 = \Lambda_1U \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2\bar{D}_1 = \Lambda_1\bar{D}_1 = 0$$

due to (8) and to the fact that Λ_1 is, by definition, a *mla* of $\bar{D}_1$. This implies that a *mla* of F can be obtained by (if necessary) adding extra rows to Λ_1U.

Let then $\bar{\Lambda}_1 = \begin{bmatrix} \Lambda_1 U \\ T \end{bmatrix}$, for a suitable matrix T, be a *mla* of F. Now, the pair $(A_{11} - \bar{L}_1 E, \bar{\Lambda}_1 E)$ is given by

$$\left(A_{11} - \bar{L}_1 \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2 \bar{C}_1, \bar{\Lambda}_1 \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2 \bar{C}_1 \right),$$

which is equal to

$$\left(A_{11} - L_1 U \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2 \bar{C}_1, \begin{bmatrix} \Lambda_1 U \\ T \end{bmatrix} \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2 \bar{C}_1 \right),$$

or equivalently,

$$\left(A_{11} - L_1 \bar{C}_1, \begin{bmatrix} \Lambda_1 \bar{C}_1 \\ M \end{bmatrix} \right),$$

where $M = T \begin{bmatrix} L_2 \\ I \end{bmatrix} \bar{D}_2 \bar{C}_1$.

Since, by Condition 4, the pair $(A_{11} - L_1 \bar{C}_1, \Lambda_1 \bar{C}_1)$ is observable, then the pair

$$\left(A_{11} - L_1 \bar{C}_1, \begin{bmatrix} \Lambda_1 \bar{C}_1 \\ M \end{bmatrix} \right)$$

is also observable. In this way we conclude that $(A_{11} - \bar{L}_1 E, \bar{\Lambda}_1 E)$ is observable, as desired.

Therefore all the conditions of Theorem 5.1 are satisfied and $\Sigma^{1D}(A_{11}, B_1, E, F)$ is minimal as a code realization.

Finally, note that $\Sigma^{1D}(A_{22}, J, C_2, H)$ is given by

$$\Sigma^{1D} \left(A_{22}, \bar{B}_2 \begin{bmatrix} \bar{C}_1 & \bar{D}_1 \end{bmatrix}, C_2, \bar{D}_2 \begin{bmatrix} \bar{C}_1 & \bar{D}_1 \end{bmatrix} \right)$$

which corresponds to making an invertible input transformation, associated to $\begin{bmatrix} \bar{C}_1 & \bar{D}_1 \end{bmatrix}$, in $\Sigma^{1D}(A_{22}, \bar{B}_2, C_2, \bar{D}_2)$. Hence it is clear that the former model realizes the same code as the latter, with the same dimension. So $\Sigma^{1D}(A_{22}, J, C_2, H)$ is a minimal code realization.

Thus it follows by Theorem 5.3 that $\Sigma^{1D}(A_{11}, B_1, E, F)$ and $\Sigma^{1D}(A_{22}, J, C_2, H)$ is a minimal realization of $\mathcal{C}$. $\square$

Example 6.5. In $\mathbb{Z}_2$, consider the following composition encoder

$$G(d_1, d_2) = \begin{bmatrix} d_2 + d_1 d_2 & 1 \\ 0 & d_2^2 + d_1 d_2^2 \\ d_2 + d_1 d_2 + d_1 + 1 & 0 \\ 0 & d_2^2 + d_1 d_2^2 + d_1 + 1 \\ 1 & 1 \\ d_2 + d_1 d_2 & d_2^2 + d_1 d_2^2 \end{bmatrix}.$$

It is easy to factorize $G(d_1, d_2)$ as in (6) where

$$G_2(d_2) = \begin{bmatrix} d_2 & 1 & d_2 & 0 \\ 0 & d_2^2 & 0 & d_2^2 \\ d_2 + 1 & 0 & d_2 + 1 & 0 \\ 0 & d_2^2 + 1 & 0 & d_2^2 + 1 \\ 1 & 1 & 0 & 0 \\ d_2 & d_2^2 & d_2 & d_2^2 \end{bmatrix}$$

and

$$G_1(d_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ d_1 & 0 \\ 0 & d_1 \end{bmatrix},$$

which is canonical and therefore minimal. $\Sigma^{1D} = (A_{11}, B_1, \bar{C}_1, \bar{D}_1)$, where

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \bar{C}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \bar{D}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is a minimal realization of $G_1(d_1)$ with $\begin{bmatrix} \bar{C}_1 & \bar{D}_1 \end{bmatrix} = I_4$ invertible.

$G_2(d_2)$ is a systematic encoder ((see Example 6.2)) with minimal realization $\Sigma^{1D} = (A_{22}, \bar{B}_2, C_2, \bar{D}_2)$, where

$$A_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \bar{D}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, by Theorem 6.4,

$$\Sigma_{2D} = (A_{11}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D),$$

where

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

is a minimal realization of the 2D convolutional code generated by $G(d_1, d_2)$.

Let us now focus on the syndrome formers of a composition code $\mathcal{C}$ which admits an encoder $G(d_1, d_2)$ of the form (6), where $G_1(d_1)$ has full row rank over $\mathbb{F}$. A construction of syndrome formers of $\mathcal{C}$ follows immediately from Proposition 4 as it is shown next. Indeed, define

$$H_1(d_1) = \begin{bmatrix} L_1(d_1) & 0 \\ 0 & I \end{bmatrix} \in \mathbb{F}^{(n-k)\times n}[d_1] \text{ and } H_2(d_2) = \begin{bmatrix} I & 0 \\ -\bar{G}_2(d_2) & I \end{bmatrix} T \in \mathbb{F}^{n\times n}[d_2],$$

where $L_1(d_1) \in \mathbb{F}^{(p-k)\times p}[d_1]$ and $\begin{bmatrix} -\bar{G}_2(d_2) & I \end{bmatrix} \in \mathbb{F}^{(n-p)\times n}[d_2]$ are 1D syndrome formers of the 1D convolutional codes Im $G_1(d_1)$ and Im $G_2(d_2)$, respectively. Let

$$H(d_1, d_2) = H_1(d_1)H_2(d_2) \tag{9}$$

$$= \begin{bmatrix} L_1(d_1) & 0 \\ -\bar{G}_2(d_2) & I \end{bmatrix} T. \tag{10}$$

It is easy to see that $H(d_1, d_2)$ is a syndrome former of $\mathcal{C}$. Moreover, it can be shown that it is possible to assume, without loss of generality, that (10) is an optimal decomposition of $H(d_1, d_2)$. Therefore:

$$\mu_R(H) = \mu(H_1) + \mu(H_2) = \mu(L_1) + \mu(-\bar{G}_2) = \mu(L_1) + \mu(G_2).$$

Note that since $L_1(d_1)$ is a syndrome former of the 1D convolutional code Im $G_1(d_1)$ and $G_1(d_1)$ is a minimal encoder of Im $G_1(d_1)$, it follows that $\mu(L_1) \geq \mu(G_1)$, [3, 5, 6], and hence $\mu_R(H) \geq \mu_R(G)$. Further, $\mu(L_1) = \mu(G_1)$ if $L_1(d_1)$ has minimal McMillan degree among all syndrome formers of Im $G_1(d_1)$, for instance, if $L_1(d_1)$ is row reduced, [3, 5, 6], (which can always be assumed without loss of generality, since otherwise pre-multiplication of $H(d_1, d_2)$ by a suitable unimodular matrix $U(d_1)$ yields another syndrome former for $\mathcal{C}$, with $L_1(d_1)$ row reduced); in this case $\mu_R(H) = \mu_R(G)$.

Therefore, given the encoder $G(d_1, d_2)$ we have constructed a syndrome former $H(d_1, d_2)$, as in Proposition 4.2, and based on the special properties of $G(d_1, d_2)$, we have shown that the minimal realizations of $H(d_1, d_2)$ have dimension $\mu_R(H) = \mu_R(G)$.

We next show that $\mu_R(H)$ is minimal among the McMillan degree of all syndrome formers of $\mathcal{C}$ with similar structure as $H(d_1, d_2)$. For this purpose we first state the following auxiliary result.

Lemma 6.6. *Let $G_1(d_1) \in \mathbb{F}^{p\times k}[d_1]$ be full row rank over $\mathbb{F}$. Then $X(d_2)G_1(d_1) = 0$ implies $X(d_2) = 0$ for all $X(d_2) \in \mathbb{F}^{\ell\times p}[d_2]$, where $\ell \in \mathbb{N}$.*

Proof. Assume that $X(d_2)G_1(d_1) = 0$ and write $X(d_2) = \sum_{i\geq 0} X_i d_2^i, \quad X_i \in \mathbb{F}^{\ell\times p}$. Then for all $i \geq 0$, $X_iG_1(d_1) = 0$. Since X_i is a matrix over $\mathbb{F}$ and $G_1(d_1)$ has full row rank over $\mathbb{F}$, this means that $X_i = 0$, for all $i \geq 0$, and therefore $X(d_2)$ is a null polynomial matrix. □

Theorem 6.7. *Let $\mathcal{C} = \text{Im } G(d_1, d_2)$ be a 2D composition code of the form (6), where $G(d_1)$ has full row rank over $\mathbb{F}$. Let further $\tilde{H}(d_1, d_2) = \begin{bmatrix} X_1(d_1) & 0 \\ X_{21}(d_2) & X_{22}(d_2)\end{bmatrix} T$ be a syndrome former of $\mathcal{C}$, where $X_1(d_1) \in \mathbb{F}^{(p-k)\times p}[d_1]$, $X_{21}(d_2) \in \mathbb{F}^{(n-p)\times p}[d_2]$, $X_{22}(d_2) \in \mathbb{F}^{(n-p)\times(n-p)}[d_2]$, and $T \in \mathbb{F}^{n\times n}$ is such that $TG_2(d_2) = \begin{bmatrix} I \\ \bar{G}_2(d_2)\end{bmatrix}$, $G_2(d_2) \in \mathbb{F}^{(n-p)\times n}[d_2]$. Then $\mu_R(\tilde{H}) \geq \mu_R(G)$.*

Proof. Note that $\tilde{H}(d_1, d_2)G(d_1, d_2) = 0$ if and only if

$$\begin{cases} X_1(d_1)G_1(d_1) = 0 \\ \left(X_{21}(d_2) + X_{22}(d_2)\bar{G}_2(d_2)\right) G_1(d_1) = 0. \end{cases}$$

Then $X_1(d_1)$ must be a syndrome former of the 1D convolutional code Im $G_1(d_1)$ and consequently $\mu(X_1) \geq \mu(G_1)$, [6]. On the other hand, since by assumption $G_1(d_1)$ has full row rank over $\mathbb{F}$, by Lemma 6.6, we have that $X_{21}(d_2) + X_{22}(d_2)\bar{G}_2(d_2) = 0$, which is equivalent to $\begin{bmatrix} X_{21}(d_2) & X_{22}(d_2)\end{bmatrix} \begin{bmatrix} I \\ \bar{G}_2(d_2)\end{bmatrix} = 0$, and therefore $\begin{bmatrix} X_{21}(d_2) & X_{22}(d_2)\end{bmatrix}$ is a syndrome former of the 1D convolutional code

$\begin{bmatrix} I \\ \bar{G}_2(d_2) \end{bmatrix}$. Hence $\mu\left(\begin{bmatrix} X_{21} & X_{22} \end{bmatrix}\right) \geq \mu\left(\begin{bmatrix} I \\ \bar{G}_2 \end{bmatrix}\right)$, since $\begin{bmatrix} I \\ \bar{G}_2(d_2) \end{bmatrix}$ is a minimal encoder of Im $\begin{bmatrix} I \\ \bar{G}_2(d_2) \end{bmatrix}$. Now, since $\tilde{H}(d_1, d_2) = \begin{bmatrix} X_1(d_1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ X_{21}(d_2) & X_{22}(d_2) \end{bmatrix} T$, it is not difficult to see that

$$\begin{aligned} \mu_R(\tilde{H}) &= \mu(X_1) + \mu\left(\begin{bmatrix} X_{21} & X_{22} \end{bmatrix}\right) \\ &\geq \mu(G_1) + \mu\left(\begin{bmatrix} I \\ \bar{G}_2 \end{bmatrix}\right) \\ &= \mu(G_1) + \mu\left(T^{-1}\begin{bmatrix} I \\ \bar{G}_2 \end{bmatrix}\right) \\ &= \mu_R(G) = \mu(\mathcal{C}). \end{aligned}$$

□

Corollary 1. *Using the notation and conditions of Theorem 6.7, the syndrome former of $\mathcal{C}$ given by (10) has minimal Roesser McMillan degree among all syndrome formers of the same structure.*

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