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Positive Solutions for Parametric Nonlinear Nonhomogeneous Robin Problems

By

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Abstract. We consider a parametric nonlinear Robin problem driven by a nonhomogeneous differential operator. Using variational tools together with suitable truncation and perturbation techniques, we prove a bifurcation-type theorem describing the dependence of the set of positive solutions on the parameter.

Key Words and Phrases. Nonhomogeneous differential operator, Robin boundary condition, Nonlinear regularity theory, Nonlinear maximum principle, Bifurcation-type theorem.

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1. Introduction

Let $\Omega \subset \boldsymbol{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric Robin problem

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\operatorname{div} a(Du(z)) = f(z, u(z), \lambda) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z) u^{p-1} = 0 & \text{on } \partial\Omega,\ u > 0,\ 1 < p < \infty. \end{cases}$$

The aim of this work is to study the dependence of the set of positive solutions on the parameter $\lambda > 0$. In problem (P_λ) the map $a : \boldsymbol{R}^N \to \boldsymbol{R}^N$ involved in the definition of the differential operator is continuous, strictly monotone (hence maximal monotone too) and satisfies other regularity and growth conditions, listed in hypotheses $\mathbf{H}(a)$ below. These hypotheses are general enough to include into our framework many differential operators of interest such as the p-Laplacian. However, we stress that the differential operator in problem (P_λ) is not homogeneous and this fact is a source of technical difficulties which require different techniques. The reaction term $f(z, x, \lambda)$ is a Carathéodory function in $(z, x) \in \Omega \times \boldsymbol{R}$ for all $\lambda > 0$ (that is, for all $x \in \boldsymbol{R}$ and all $\lambda > 0$, $z \to f(z, x, \lambda)$ is measurable and for a.a. $z \in \Omega$ and all $\lambda > 0$, $x \to f(z, x, \lambda)$ is continuous), which exhibits strictly $(p-1)$-sublinear growth near $+\infty$, while it is $(p-1)$-superlinear near 0^+. Also, $\partial u/\partial n_a$ denotes the generalized normal derivative defined by $\partial u/\partial n_a = (a(Du), n)_{\boldsymbol{R}^N}$ where $n(.)$ is the outward unit normal on $\partial\Omega$ and β is a positive boundary function (cf. **H**(β) in Section 2). We show that there exists $\lambda^* > 0$ such that

- for all $\lambda > \lambda^*$ problem (P_λ) admits at least two positive solutions;
- for $\lambda = \lambda^*$ problem (P_λ) admits at least one positive solution;
- for all $\lambda \in (0, \lambda^*)$ there are no positive solutions of problem (P_λ).

Moreover, we show that for all $\lambda \geq \lambda^*$, problem (P_λ) has a smallest positive solution $\bar{u}_\lambda$ and we investigate the continuity and the monotonicity properties of the map $\lambda \to \bar{u}_\lambda$.

Such bifurcation-type results were proved primarily for problems driven by the Laplacian or the p-Laplacian with competing nonlinearities (concave-convex problems). We mention the works of Ambrosetti-Brezis-Cerami [6], Garcia Azorero-Manfredi-Peral Alonso [12], Guo-Zhang [16], Hu-Papageorgiou [18] (Dirichlet problems) and Motreanu-Motreanu-Papageorgiou [21] (Neumann problems). For such problems, the bifurcation occurs near zero (that is, for small values of $\lambda > 0$). Recently, Aizicovici-Papageorgiou-Staicu [4] (semilinear Dirichlet problems) and Papageorgiou-Radulescu [26] (nonlinear Robin problems driven by the p-Laplacian) examined superdiffusive type logistic equations and proved bifurcation-type results near $+\infty$ (that is, for large values of $\lambda > 0$). We also mention the relevant works on periodic scalar nonlinear equations of Aizicovici-Papageorgiou-Staicu [2], [3].

2. Mathematical Background

Let $(X, \|.\|)$ be a Banach space. By X^* we denote its topological dual and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X^*, X). We will use the symbol "$\xrightarrow{w}$" to designate weak convergence.

Suppose that $\varphi \in C^1(X)$. We say that φ satisfies the *Palais-Smale condition* (the *PS-condition*, for short), if the following is true:

"every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subseteq \boldsymbol{R}$ is bounded and

$$\varphi'(x_n) \to 0 \qquad \text{in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence."

This is a compactness-type condition on the functional φ, which leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. Prominent in that theory is the "mountain pass theorem" of Ambrosetti-Rabinowitz [7], which reads as follows:

Theorem 1. *If $\varphi \in C^1(X)$ satisfies the PS-condition, $u_0, u_1 \in X$ and $\rho > 0$ are such that $\|u_1 - u_0\| > \rho$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} =: m_\rho,$$

and

$$c = \inf_{\gamma \in \Gamma} \inf_{t \in [0,1]} \varphi(\gamma(t)) \qquad \textit{with } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\},$$

then $c \geq m_p$ *and* c *is a critical value of* φ *(i.e., there exists* $u^* \in X$ *such that* $\varphi'(u^*) = 0$ *and* $\varphi(u^*) = c$*).*

In the analysis of problem (P_λ), we will use the Sobolev space $W^{1,p}(\Omega)$ and by $\|.\|$ we will denote its norm defined by

$$\|u\| = [\|u\|_p^p + \|Du\|_p^p]^{1/p},$$

where $\|.\|_p$ stands for the L^p-norm. In addition, we will also use the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^r(\partial\Omega)$ $(1 \leq r \leq \infty)$.

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with an order (positive) cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \ \textit{for all} \ z \in \overline{\Omega}\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \{u \in C^1(\overline{\Omega}) : u(z) > 0 \ \textit{for all} \ z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(.)$. Having this measure on $\partial\Omega$, we can define in the usual way the Lebesgue spaces $L^r(\partial\Omega)$ $(1 \leq r \leq \infty)$.

The theory of Sobolev spaces says that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \qquad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, we can view the trace map as representing the "boundary values" of a Sobolev function $u \in W^{1,p}(\Omega)$.

The trace map γ_0 is compact from $W^{1,p}(\Omega)$ into $L^r(\partial\Omega)$ for all $r \in [1, (N-1)p/(N-p))$ when $p < N$ and into $L^r(\partial\Omega)$ for all $r \in [1, \infty)$ when $p \geq N$. In addition, we have

$$\operatorname{Im} \gamma_0 = W^{1/p',p}(\partial\Omega) \quad \text{with } \frac{1}{p} + \frac{1}{p'} = 1 \qquad \text{and} \qquad \ker \gamma_0 = W_0^{1,p}(\Omega).$$

For the sake of notational simplicity, in the sequel we drop the use of the trace map γ_0. All restrictions of Sobolev functions to the boundary $\partial\Omega$ are defined in the sense of traces.

Let $\theta \in C^1(0, \infty)$ be a function such that

$$(2.1) \qquad 0 < \hat{C} \le \frac{t\theta'(t)}{\theta(t)} \le C_0 \quad \text{and}$$

$$C_1 t^{p-1} \le \theta(t) \le C_2(1 + t^{p-1}) \quad \text{for all } t > 0$$

with $C_0, C_1, C_2 > 0$ and $1 < p < \infty$.

The hypotheses on the map $a(y)$ involved in the definition of the differential operator of (P_λ) are the following:

$(\mathbf{H}(a))$ $a(y) = a_0(|y|)y$, for all $y \in \boldsymbol{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and:

(i) $a_0 \in C^1(0, \infty)$, $t \to a_0(t)t$ is strictly increasing,

$$\lim_{t \to 0^+} a_0(t)t = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) for some $C_3 > 0$ and for all $y \in \boldsymbol{R}^N$

$$|\nabla a(y)| \le C_3 \frac{\theta(|y|)}{|y|} \quad \text{for all } y \in \boldsymbol{R}^N;$$

(iii) for all $y \in \boldsymbol{R}^N \backslash \{0\}$ and all $\xi \in \boldsymbol{R}^N$

$$(\nabla a(y)\xi, \xi)_{\boldsymbol{R}^N} \ge \frac{\theta(|y|)}{|y|} |\xi|^2.$$

Here and in what follows, $|y|$ denotes the $\boldsymbol{R}^N$ norm of $y \in \boldsymbol{R}^N$.

Remarks. These hypotheses are motivated by the nonlinear regularity theory of Lieberman [20] and the nonlinear maximum principle of Pucci-Serrin [30] (see also Uhlenbeck [32]). They imply that the primitive

$$G_0(t) = \int_0^t a_0(s)s\, ds, \quad \text{for } t \ge 0$$

is strictly increasing and strictly convex. We set $G(y) = G_0(|y|)$ for all $y \in \boldsymbol{R}^N$. Then $G(.)$ is convex and continuously differentiable. More precisely, we have $\nabla G(0) = 0$ and

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \boldsymbol{R}^N \backslash \{0\}.$$

So, $G(.)$ is the primitive of the map $a(.)$. The convexity of $G(.)$ implies that

$$(2.2) \qquad G(y) \le (a(y), y)_{\boldsymbol{R}^N} \quad \text{for all } y \in \boldsymbol{R}^N.$$

The next lemma summarizes the main properties of the map $a(.)$ and it is an easy consequence of the hypotheses $(\mathbf{H}(a))$.

Lemma 1. *If hypotheses* $(\mathbf{H}(a))$ *hold, then*:

(a) *the map* $y \to a(y)$ *is continuous and strictly monotone, hence maximal monotone too*;

(b) $|a(y)| \le C_4(1 + |y|^{p-1})$ *for all* $y \in \boldsymbol{R}^N$ *and some* $C_4 > 0$;

(c) $(a(y), y)_{\boldsymbol{R}^N} \ge (C_1/(p-1))|y|^p$ *for all* $y \in \boldsymbol{R}^N$.

This lemma together with (2.1) and (2.2) leads to the following growth estimates for the primitive $G(.)$:

Corollary 1. *If hypotheses* $(\mathbf{H}(a))$ *hold, then*

$$\frac{C_1}{p(p-1)}|y|^p \le G(y) \le C_5(1 + |y|^p) \qquad \text{for all } y \in \boldsymbol{R}^N, \text{ some } C_5 > 0.$$

Examples. The following maps $a(.)$ satisfy hypotheses $(\mathbf{H}(a))$:

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$.

This map corresponds to the p-Laplacian differential operator defined by

$$\triangle_p u = \operatorname{div}(|Du|^{p-2}Du), \qquad \text{for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$.

This map corresponds to the (p, q)-Laplacian differential operator defined by

$$\triangle_p u + \triangle_q u, \qquad \text{for all } u \in W^{1,p}(\Omega).$$

Such operators arise in problems of mathematical physics (see, Benci-D'Avenia-Fortunato-Pisani [8] (quantum physics) and Cherfils-Ilyasov [10] (plasma physics).)

Recently there have been some existence and multiplicity results for equations driven by such operators. We refer to the papers of Aizicovici-Papageorgiou-Staicu [5], Cingolani-Degiovanni [11], Carmona-Cingolani-Martinez Aparicio-Vannella [9], Mugnai-Papageorgiou [23], Papageorgiou-Radulescu [24], Papageorgiou-Winkert [29], Sun-Zhang-Su [31].

(c) $a(y) = (1 + |y|^2)^{(p-2)/2}y$, with $1 < p < \infty$.

This map corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}((1 + |Du|^2)^{(p-2)/2}Du), \qquad \text{for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2}y(1 + 1/(1 + |y|^p))$ with $1 < p < \infty$.

This map corresponds to the following perturbation of the p-Laplacian

$$\triangle_p u + \operatorname{div}\left(\frac{|Du|^{(p-2)/2}Du}{1 + |Du|^p}\right), \qquad \text{for all } u \in W^{1,p}(\Omega).$$

The assumptions on the boundary coefficient function $\beta(.)$ are the following: $(\mathbf{H}(\beta))$ $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$, $\beta(z) > 0$ for all $z \in \partial\Omega$.

Remark. The above hypothesis excludes Neumann problems (they correspond to $\beta \equiv 0$).

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\boldsymbol{R}^N} dz \qquad \text{for all } u, h \in W^{1,p}(\Omega). \tag{2.3}$$

From Papageorgiou-Rocha-Staicu [28] we have:

Proposition 1. *If hypotheses* $(\mathbf{H}(a))$ *hold, then the nonlinear map* $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *defined by* (2.3) *is continuous, monotone, hence maximal monotone, too, and of type* $(S)_+$, *that is, if* $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ *is such that* $u_n \xrightarrow{w} u$ *in* $W^{1,p}(\Omega)$ *and*

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0,$$

then $u_n \to u$ *in* $W^{1,p}(\Omega)$ *as* $n \to \infty$.

Let $f_0 : \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ be a Carathéodory function such that

$$|f_0(z,x)| \leq \alpha_0(z)(1 + |x|^{r-1}) \qquad \text{for a.a. } z \in \Omega, \text{ all } x \in \boldsymbol{R}$$

with $\alpha_0 \in L^\infty(\Omega)_+ := \{\alpha \in L^\infty(\Omega) : \alpha(z) \geq 0 \textit{ for a.a. } z \in \Omega\}$ and $r \in (1, p^*)$ with

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases} \quad \text{(the critical Sobolev exponent for } p\text{)}.$$

Moreover, let $k_0 \in C^{0,\eta}(\partial\Omega \times \boldsymbol{R})$ with $\eta \in (0,1)$ be such

$$|k_0(t,x)| \leq C_6|x|^\tau \qquad \text{for all } (z,x) \in \partial\Omega \times \boldsymbol{R},$$

with $C_6 > 0$ and $\tau \in (1,p]$. We set

$$F_0(z,x) = \int_0^x f_0(z,s)ds \qquad \text{and} \qquad K_0(z,x) = \int_0^x k_0(z,s)ds$$

and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \boldsymbol{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du(z))dz + \int_{\partial\Omega} K_0(z,u(z))d\sigma - \int_\Omega F_0(z,u(z))dz \qquad \text{for all } u \in W^{1,p}(\Omega).$$

From Papageorgiou-Radulescu [25] (see also Gasinski-Papageorgiou [14] for the Dirichlet case), we have

Proposition 2. *If hypotheses* $(\mathbf{H}(a))$ *hold and* $u_0 \in W^{1,p}(\Omega)$ *is a local* $C^1(\overline{\Omega})$*-minimizer of* φ_0*, that is, there exists* $\rho_0 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \qquad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$ *and* u_0 *is a local* $W^{1,p}(\Omega)$*-minimizer of* φ_0*, that is, there exists* $\rho_1 > 0$ *such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \qquad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\| \leq \rho_1.$$

Remark. The result remains true even if $f_0(z,.)$ has critical growth (that is, if $r = p^*$). We refer to Papageorgiou-Radulescu [27] for this generalization.

The next strong comparison theorem is related to Proposition 2.2 of Guedda-Veron [15].

Proposition 3. *If hypotheses* $(\mathbf{H}(a))$ *hold,* $\xi \in L^\infty(\Omega)_+$, $h_1, h_2 \in L^\infty(\Omega)$ *satisfy* $h_1 \prec h_2$*, that is there exists* $C_7 > 0$ *such that*

$$0 < C_7 \leq h_2(z) - h_1(z) \qquad \text{for all } z \in \Omega,$$

$u \in C^1(\overline{\Omega}) \setminus \{0\}$, $v \in \operatorname{int} C_+$, $u \leq v$,

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0 \qquad \text{or} \qquad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} < 0$$

and

$$\begin{cases} -\operatorname{div} a(Du(z)) + \xi(z)|u(z)|^{p-2}u(z) = h_1(z) \text{ for a.a. } z \in \Omega, \\ -\operatorname{div} a(Dv(z)) + \xi(z)v(z)^{p-2}v(z) = h_2(z) \text{ for a.a. } z \in \Omega, \end{cases}$$

then

$$(v-u)(z) > 0 \qquad \text{for all } z \in \Omega$$

and

$$\frac{\partial(v-u)}{\partial n}(z_0) < 0 \qquad \text{for all } z_0 \in \Sigma_0 := \{z \in \partial\Omega : u(z) = v(z)\}.$$

Proof. By hypothesis, we have

$$(2.4) \quad -\operatorname{div}(a(Dv(z)) - a(Du(z))) = h_2(z) - h_1(z) - \xi(z)(v(z)^{p-1} - |u(z)|^{p-2}u(z)) \qquad \text{for a.a. } z \in \Omega.$$

If $a = (a_k)_{k=1}^N$ with $a_k : \boldsymbol{R}^N \to \boldsymbol{R}$ for all $k \in \{1, 2, \dots, N\}$, then using the chain rule, we have

$$a_k(y) - a_k(y') = \sum_{i=1}^N \int_0^1 \frac{\partial a_k}{\partial y_i}(y' + t(y - y'))(y_i - y_i')dt$$

for all $y = (y_i)_{i=1}^N$ and $y' = (y_i')_{i=1}^N \in \boldsymbol{R}^N$ and $k \in \{1, 2, \dots, N\}$.

We introduce the following functions defined on Ω:

$$\theta_{k,i}(z) = \int_0^1 \frac{\partial a_k}{\partial y_i}(Du(z) + t(Dv(z) - Du(z)))dt \qquad \text{for all } z \in \Omega.$$

Using these functions, we define the following linear differential operator

$$\begin{aligned} L(w) &= -\mathrm{div}\left(\sum_{i=1}^N \theta_{k,i}(z)\frac{\partial w}{\partial z_i}\right) \\ &= -\sum_{k,i=1}^N \frac{\partial}{\partial z_k}\left(\theta_{k,i}(z)\frac{\partial w}{\partial z_i}\right), \qquad \forall w \in W^{1,p}(\Omega). \end{aligned}$$

Setting $y = v - u \in C_+ \backslash \{0\}$, from (2.4) we have

$$L(y) = h_2(z) - h_1(z) - \xi(z)(v^{p-1} - |u|^{p-2}u) \tag{2.5}$$

Let

$$E = \{z \in \Omega : u(z) = v(z)\} \qquad \text{and} \qquad E_0 = \{z \in \Omega : Du(z) = Dv(z) = 0\}$$

Claim $E \subseteq E_0$.

Let $z_0 \in E$. Then y attains its infimum at z_0, hence $Dy(z_0) = 0$, therefore

$$Du(z_0) = Dv(z_0).$$

If $z_0 \notin E_0$, then we can find $\rho > 0$ small such that

$$|Du(z)| > 0, \qquad |Dv(z)| > 0, \qquad (Du(z), Dv(z))_{\boldsymbol{R}^N} > 0 \qquad \text{for all } z \in \overline{B}_\rho(z_0).$$

Here $B_\rho(z_0)$ denotes the open ball centered at z_0 of radius ρ.

Taking $\rho > 0$ even smaller if necessary we can have

$$L \text{ is strictly elliptic on } \overline{B}_\rho(z_0)$$

and

$$L(y(z)) > 0 \qquad \text{for all } z \in \overline{B}_\rho(z_0),$$

(see (2.5) and recall that $h_1 \prec h_2$). Then by the strong maximum principle we have

$$y(z) > 0 \qquad \text{for all } z \in \bar{B}_\rho(z_0),$$

a contradiction (let $z = z_0$). This proves the Claim.

Note that $E_0 \subset \Omega$ is compact. Hence so is E and so we can find $\Omega_1 \subset \Omega$ open with a C^2-boundary such that

$$E \subseteq \Omega_1 \subseteq \bar{\Omega}_1 \subset \Omega.$$

Let $\varepsilon > 0$ be such that

$$u(z) + \varepsilon < v(z) \quad \text{for all } z \in \partial\Omega_1 \qquad \text{and} \qquad h_1(z) + \varepsilon < h_2(z) \quad \text{for a.a. } z \in \bar{\Omega}_1.$$

Using these facts and the weak comparison principle we can easily check that for $\delta > 0$ small $u + \delta \le v$ on Ω_1. Hence $E = \emptyset$ and $(v - u)(z) > 0$ for all $z \in \Omega$. Also from the boundary point theorem (see [22], p. 217 and [30], p. 120) we have

$$\left.\frac{\partial(v-u)}{\partial n}\right|_{\Sigma_0} < 0. \qquad \square$$

Remark. Consider the following order cone in $C^1(\bar{\Omega})$

$$\hat{C}_+ = \left\{ y \in C^1(\bar{\Omega}) : y(z) \ge 0 \textit{ for all } z \in \bar{\Omega}, \frac{\partial y}{\partial n}(z) \le 0 \textit{ for all } z \in \Sigma_0 \right\},$$

where $\Sigma_0 := \{z \in \partial\Omega : y(z) = 0\}$. This cone has a nonempty interior given by

$$\operatorname{int} \hat{C}_+ = \left\{ y \in \hat{C}_+ : y(z) > 0 \textit{ for all } z \in \Omega, \frac{\partial y}{\partial n}(z) < 0 \textit{ for all } z \in \Sigma_0 \right\}.$$

Then Proposition 3 says that $v - u \in \operatorname{int} \hat{C}_+$.

Consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\triangle_p u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \tilde{\beta}(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

where $1 < p < \infty$ and $\tilde{\beta} \in L^\infty(\partial\Omega)$, $\tilde{\beta} \ge 0$, $\tilde{\beta} \ne 0$.

Recall that $\triangle_p$ $(1 < p < \infty)$ denotes the p-Laplacian differential operator defined by

$$\triangle_p u = \operatorname{div}(|Du|^{p-2}Du), \qquad \text{for all } u \in W^{1,p}(\Omega).$$

In this case $a(y) = |y|^{p-2}y$ for all $y \in \boldsymbol{R}^N$, and so the generalized directional derivative $\partial u/\partial n_p$ on $\partial\Omega$ is defined by

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\boldsymbol{R}^N} \qquad \text{for all } u \in W^{1,p}(\Omega),$$

where $n(.)$ is the outward unit normal on $\partial\Omega$.

From Papageorgiou-Radulescu [25], we know that problem (2.6) has a smallest eigenvalue $\hat{\lambda}_1(\tilde{\beta})$ which has the following properties:

- $\hat{\lambda}_1(\tilde{\beta}) > 0$ and it is isolated in the spectrum of (2.6);
- $\hat{\lambda}_1(\tilde{\beta})$ is simple;
-
$$\hat{\lambda}_1(\tilde{\beta}) = \inf\left\{\frac{\theta(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega),\, u \neq 0\right\}, \tag{2.7}$$

  where $\theta : W^{1,p}(\Omega) \to \boldsymbol{R}$ is the C^1-functional defined by

$$\theta(u) = \|Du\|_p^p + \int_{\partial\Omega} \tilde{\beta}(z)|u|^p d\sigma \qquad \text{for all } u \in W^{1,p}(\Omega).$$

The infimum in (2.7) is achieved on the corresponding one dimensional eigenspace. From (2.7) it is clear that the elements of this eigenspace do not change sign. In what follows, by $\hat{u}_1(\tilde{\beta})$ we denote the positive L^p-normalized eigenfunction (that is, $\|\hat{u}_1(\tilde{\beta})\|_p = 1$) corresponding to $\hat{\lambda}_1(\tilde{\beta}) > 0$.

The nonlinear regularity theory of Lieberman [20] and the nonlinear maximum principle of Pucci-Serrin [30] imply that $\hat{u}_1(\tilde{\beta}) \in \operatorname{int} C_+$. These properties lead easily to the following simple but useful lemma (see Papageorgiou-Radulescu [25]).

Lemma 2. *If $\xi \in L^\infty(\Omega)_+$, $\xi(z) \leq \hat{\lambda}_1(\tilde{\beta})$ for a.a. $z \in \Omega$ and $\xi \neq \hat{\lambda}_1(\tilde{\beta})$, then there exists $C_8 > 0$ such that*

$$\theta(u) - \int_\Omega \xi(z)|u(z)|^p dz \geq C_8\|u\|^p \qquad \textit{for all } u \in W^{1,p}(\Omega).$$

Finally, we comment on our notation throughout the remainder of the paper. By $|.|_N$ we denote the Lebesgue measure on $\boldsymbol{R}^N$. If $x \in \boldsymbol{R}$ then $x^\pm = \max\{\pm x, 0\}$. Given $u \in W^{1,p}(\Omega)$, we define $u^\pm(.) = u(.)^\pm$ and we have

$$u^\pm \in W^{1,p}(\Omega), \qquad u = u^+ - u^- \qquad \text{and} \qquad |u| = u^+ + u^-,$$

3. A bifurcation-type theorem

In this section we prove a bifurcation-type theorem decribing the behavior of the set of positive solutions of problem (P_λ) as the parameter $\lambda > 0$ varies.

The hypotheses on the reaction term $f(z,x,\lambda)$ are the following:

($\mathbf{H}_1$): $f : \Omega \times \boldsymbol{R} \times (0,\infty) \to \boldsymbol{R}$ is a function such that for all $\lambda > 0$, $(z,x) \to f(z,x,\lambda)$ is a Carathéodory function, $f(z,x,\lambda) \geq 0$ for a.a. $z \in \Omega$ and all $x \geq 0$, $f(z,0,\lambda) = 0$ for a.a. $z \in \Omega$ and:

(i) for every $\rho > 0$ and $I \subseteq (0,\infty)$ bounded, we can find $a_\rho^I \in L^\infty(\Omega)_+$ such that

$$0 \leq f(z,x,\lambda) \leq a_\rho^I(z) \qquad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho, \text{ all } \lambda \in I;$$

(ii) for every $\lambda > 0$,

$$\lim_{x \to +\infty} \frac{f(z,x,\lambda)}{x^{p-1}} = 0 \qquad \text{uniformly for a.a. } z \in \Omega;$$

(iii) for every $\lambda > 0$,

$$\lim_{x \to 0^+} \frac{f(z,x,\lambda)}{x^{p-1}} = 0 \qquad \text{uniformly for a.a. } z \in \Omega;$$

(iv) there exists $\tilde{h} \in L^p(\Omega)$ such that

$$\int_\Omega F(z,\tilde{h}(z),\lambda)dz > 0 \qquad \text{for all } \lambda \geq \lambda_0 > 0$$

where

$$F(z,x,\lambda) = \int_0^x f(z,s,\lambda)ds;$$

(v) for a.a. $z \in \Omega$ and all $x > 0$, $\lambda \to f(z,x,\lambda)$ is strictly increasing; for each $s > 0$, we can find $\eta_s > 0$ such that

$$0 < \eta_s \leq f(z,x,\tau) - f(z,x,\lambda) \qquad \text{for a.a. } z \in \Omega,$$
$$\text{all } x \geq s > 0, \text{ and all } \tau > \lambda > 0,$$
$$f(z,x,\lambda) \to 0^+ \qquad \text{as } \lambda \to 0^+ \text{ uniformly for a.a. } z \in \Omega,$$
$$\text{all } x \in K \subseteq \boldsymbol{R} \text{ compact}$$

and

$$f(z,x,\lambda) \to +\infty \qquad \text{as } \lambda \to +\infty \text{ uniformly for a.a. } z \in \Omega.$$

Remarks. Since we are interested in positive solutions and all the above conditions on the reaction $f(z,.,\lambda)$, concern the positive semiaxis $(0,\infty)$, without any loss of generality, we may assume that

$$f(z,x,\lambda) = 0 \qquad \text{for a.a. } z \in \Omega, \text{ all } x \leq 0, \text{ all } \lambda > 0.$$

Hypotheses ($\mathbf{H}_1$) (ii), (iii) imply that $f(z,.,\lambda)$ is strictly $(p-1)$-sublinear both near $+\infty$ and near to 0^+.

Examples. The following functions satisfy hypotheses ($\mathbf{H}_1$). For the sake of simplicity we drop the $\Omega \ni z$-dependence.

$$f_1(x,\lambda) = \begin{cases} \lambda(x^{r-1} - x^{\tau-1}) & \text{if } 0 \le x \le 1 \\ \lambda x^{q-1} \ln x & \text{if } 1 < x \end{cases} \quad \text{with } 1 < q < p < \tau < r;$$

$$f_2(x,\lambda) = \begin{cases} \lambda x^{r-1} & \text{if } 0 \le x \le 1 \\ \lambda x^{q-1} & \text{if } 1 < x \end{cases} \quad \text{with } 1 < q < p < r;$$

$$f_3(x,\lambda) = \begin{cases} \lambda(x^{r-1} + x^{\tau-1}) & \text{if } 0 \le x \le 1 \\ 2\lambda x^{q-1} & \text{if } 1 < x \end{cases} \quad \text{with } 1 < q < p < r, \tau;$$

$$f_4(x,\lambda) = \begin{cases} \lambda x^{r-1} & \text{if } 0 \le x \le \rho(\lambda) \\ x^{q-1} + \sigma(\lambda) & \text{if } \rho(\lambda) < x \end{cases}$$

with $\rho(\lambda) \in (0,1]$, $\rho(\lambda) \to 0^+$ as $\lambda \to 0^+$, $\rho(.)$ is strictly increasing,

$$\sigma(\lambda) = (\lambda \rho(\lambda)^{r-q} - 1)\rho(\lambda), \qquad \text{and} \qquad 1 < q < p < r.$$

We introduce the following two sets

$$\mathscr{L} = \{\lambda > 0 : \textit{problem } (\mathrm{P}_\lambda) \textit{ admits a positive solution}\},$$

and

$$\mathscr{S}(\lambda) = \text{the set of positive solutions of } (\mathrm{P}_\lambda).$$

We set

$$\lambda^* = \inf \mathscr{L}$$

(if $\mathscr{L} = \varnothing$ then $\lambda^* = +\infty$).

Proposition 4. *If hypotheses* ($\mathbf{H}(a)$), ($\mathbf{H}(\beta)$), ($\mathbf{H}_1$) *hold, then for all* $\lambda > 0$ *we have* $\mathscr{S}(\lambda) \subseteq \operatorname{int} C_+$ *and* $\lambda^* > 0$.

Proof. Assume that $\mathscr{L} \neq \varnothing$ and let $\lambda \in \mathscr{L}$. Then there exists $u \in \mathscr{S}(\lambda)$ such that

$$\text{(3.1)} \qquad \begin{cases} -\operatorname{div} a(Du(z)) = f(z,u(z),\lambda) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a} + \beta(z) u^{p-1} = 0 & \text{on } \partial\Omega \end{cases}$$

(see Papageorgiou-Radulescu [25]). From (3.1) it follows that $u \in L^\infty(\Omega)$ (see Papageorgiou-Radulescu [27]). Then using the nonlinear regularity theory of Lieberman ([20], p. 320) we have that $u \in C_+ \backslash \{0\}$. Hypotheses ($\mathbf{H}_1$) (i), (ii),

(iii) imply that given $\rho > 0$, we can find $\tilde{\xi}_\rho > 0$ such that

$$f(z, x, \lambda) + \tilde{\xi}_\rho x^{p-1} \geq 0 \qquad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho. \tag{3.2}$$

Let $\rho = \|u\|_\infty$ and let $\tilde{\xi}_\rho > 0$ be as postulated by (3.2) above. We have

$$\begin{aligned} -\text{div}\, a(Du(z)) + \tilde{\xi}_\rho u(z)^{p-1} &= f(z, u(z), \lambda) + \tilde{\xi}_\rho u(z)^{p-1} \\ &\geq 0 \qquad \text{for a.a. } z \in \Omega \text{ (see (3.2))}, \end{aligned}$$

hence

$$\text{div}\, a(Du(z)) \leq \tilde{\xi}_\rho u(z)^{p-1} \qquad \text{for a.a. } z \in \Omega.$$

Let

$$\eta(t) = a_0(t)t \qquad \text{for all } t > 0.$$

Hypothesis ($\mathbf{H}(a)$) (iii) (unidimensional version) and (2.1) imply that

$$\eta'(t)t = a_0'(t)t^2 + a_0(t)t \geq C_1 t^{p-1}.$$

Performing an integration by parts, we have

$$\begin{aligned} \int_0^t \eta'(s)s\, ds &= \eta(t)t - \int_0^t \eta(s)ds \\ &= \eta(t)t - G_0(t) \\ &\geq \frac{C_1}{p} t^p, \qquad \text{for all } t > 0. \end{aligned} \tag{3.3}$$

Let $H(t) = a_0(t)t^2 - G_0(t)$ and $H_0(t) = (C_1/p)t^p$, for all $t > 0$ and consider the sets

$$\begin{aligned} D_1 &= \{t \in (0,1) : H(t) \geq \mu\} \qquad \text{and} \\ D_2 &= \{t \in (0,1) : H_0(t) \geq \mu\} \qquad \text{with } \mu > 0. \end{aligned}$$

From (3.3) we have $H_0 \leq H$, hence $D_2 \subseteq D_1$, therefore

$$\inf D_1 \leq \inf D_2.$$

From Leoni ([19], p. 6) we have

$$H^{-1}(\mu) \leq H_0^{-1}(\mu),$$

hence

$$\begin{aligned} \int_0^\delta \frac{1}{H^{-1}\left(\frac{\tilde{\xi}_\rho}{p}\mu^p\right)}\, d\mu &\geq \int_0^\delta \frac{1}{H_0^{-1}\left(\frac{\tilde{\xi}_\rho}{p}\mu^p\right)}\, d\mu \\ &= \frac{\tilde{\xi}_\rho}{p} \int_0^\delta \frac{d\mu}{\mu} = +\infty. \end{aligned}$$

So, we can use the strong maximum principle of Pucci-Serrin ([30], p. 111) and infer that $u(z) > 0$ for all $z \in \Omega$. Using the boundary point theorem of Pucci-Serrin ([30], p. 120) we conclude that $u \in \operatorname{int} C_+$. Therefore

$$\mathscr{S}(\lambda) \subseteq \operatorname{int} C_+ \qquad \text{for all } \lambda > 0.$$

Hypotheses $(\mathbf{H}_1)$ (ii), (iii), (v) imply that we can find $\bar{\lambda} > 0$ such that

$$f(z, x, \bar{\lambda}) \leq \frac{C_1}{p-1} \hat{\lambda}_1(\hat{\beta}) x^{p-1} \qquad \text{for a.a. } z \in \Omega, \text{ all } x \in \boldsymbol{R} \tag{3.4}$$

(here $\hat{\beta} = ((p-1)/C_1)\beta \in L^\infty(\Omega)$). Choose $\lambda \in (0, \bar{\lambda})$ and assume that $\lambda \in \mathscr{L}$. Then we can find $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ such that

$$\begin{aligned} \langle A(u_\lambda), h \rangle &+ \int_{\partial\Omega} \beta(z) u_\lambda(z)^{p-1} h \, d\sigma \\ &= \int_\Omega f(z, u_\lambda(z), \lambda) h \, dz \qquad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.5}$$

In (3.5) we choose $h = u_\lambda \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\frac{C_1}{p-1} \|Du_\lambda\|_p^p + \int_{\partial\Omega} \beta(z) u_\lambda^p \, d\sigma \\ &\quad \leq \int_\Omega f(z, u_\lambda(z), \lambda) u_\lambda \, dz \qquad \text{(see Lemma 1)} \\ &\quad < \frac{C_1}{p-1} \hat{\lambda}_1(\hat{\beta}) \|u_\lambda\|_p^p \qquad \text{(see hypothesis } (\mathbf{H}_1) \text{ (v) and use (3.5))}, \end{aligned}$$

hence

$$\frac{C_1}{p-1} \left[\|Du_\lambda\|_p^p + \int_{\partial\Omega} \hat{\beta}(z) u_\lambda^p \, d\sigma \right] < \frac{C_1}{p-1} \hat{\lambda}_1(\hat{\beta}) \|u_\lambda\|_p^p,$$

a contradiction. This proves that

$$\lambda^* \geq \bar{\lambda} > 0.$$

□

Proposition 5. *If hypotheses* $(\mathbf{H}(a))$, $(\mathbf{H}(\beta))$, $(\mathbf{H}_1)$ *hold, then* $\mathscr{L} \neq \emptyset$, *and so* $\lambda^* \in (0, +\infty)$.

Proof. Hypotheses $(\mathbf{H}_1)$ (i), (ii) imply that given $\varepsilon > 0$ and $\lambda > 0$ we can find $C_9 = C_9(\varepsilon, \lambda) > 0$ such that

$$F(z, x, \lambda) \leq \frac{\varepsilon}{p} x^p + C_9 \qquad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.6}$$

We introduce the following Carathéodory function

$$(3.7)\quad \hat{f}(z,x,\lambda) = f(z,x,\lambda) + \frac{1}{p-1}(x^+)^p \qquad \text{for all } (z,x,\lambda) \in \Omega \times \boldsymbol{R} \times (0,+\infty).$$

We set $\hat{F}(z,x,\lambda) = \int_0^x \hat{f}(z,s,\lambda)ds$ and introduce the C^1-functional $\hat{\varphi}_\lambda : W^{1,p}(\Omega) \to \boldsymbol{R}$ by

$$\hat{\varphi}_\lambda(u) = \int_\Omega G(Du(z))dz + \frac{1}{p(p-1)}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \hat{F}(z,u,\lambda)dz \qquad \text{for all } u \in W^{1,p}(\Omega).$$

Using Corollary 1, (3.7) and choosing $\varepsilon > 0$ small, we have

$$\begin{aligned}
\hat{\varphi}_\lambda(u) &\geq \frac{C_1}{p(p-1)}\left[\|Du^+\|_p^p + \int_{\partial\Omega} \hat{\beta}(z)(u^+)^p d\sigma\right] - \int_\Omega F(z,u^+,\lambda)dz \\
&\quad + \frac{C_1}{p(p-1)}\left[\|Du^-\|_p^p + \|u^-\|_p^p + \int_{\partial\Omega} \hat{\beta}(z)(u^-)^p d\sigma\right] \\
&\geq \frac{C_1}{p(p-1)}\left[\|Du^+\|_p^p + \int_{\partial\Omega} \hat{\beta}(z)(u^+)^p d\sigma - \frac{\varepsilon(p-1)}{C_1}\|u^+\|^p\right] \\
&\quad + \frac{C_1}{p(p-1)}\|u^-\|^p - C_9|\Omega|_N \qquad \text{(see (3.6))} \\
&\geq C_{10}\|u\|^p - C_9|\Omega|_N \qquad \text{for some } C_{10} > 0 \text{ (see Lemma 2)},
\end{aligned}$$

hence $\hat{\varphi}_\lambda$ is coercive for all $\lambda > 0$.

Also using the Sobolev embedding theorem and the compactness of the trace map, we can see that $\hat{\varphi}_\lambda$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$(3.8)\qquad \hat{\varphi}_\lambda(u_\lambda) = \inf\{\hat{\varphi}_\lambda(u) : u \in W^{1,p}(\Omega)\}.$$

Hypotheses $(\mathbf{H}_1)$ (i), (ii), (iii) imply that for every $\lambda > 0$, we can find $C_{11} = C_{11}(\lambda) > 0$ such that

$$(3.9)\qquad 0 \leq F(z,x,\lambda) \leq C_{11}x^p \qquad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0.$$

Consider the integral functional $J_\lambda : L^p(\Omega) \to \boldsymbol{R}$ defined by

$$J_\lambda(h) = \int_\Omega F(z,h(z),\lambda)dz \qquad \text{for all } h \in L^p(\Omega).$$

Then, from (3.9) and the dominated convergence theorem, it follows that $J_\lambda(.)$ is continuous. Moreover, due to hypothesis $(\mathbf{H}_1)$ (iv), we have

$$J_\lambda(\tilde{h}) > 0 \qquad \text{for all } \lambda \geq \lambda_0.$$

Exploiting the density of $W^{1,p}(\Omega)$ into $L^p(\Omega)$, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$J_\lambda(\tilde{u}) > 0 \qquad \text{for all } \lambda \geq \lambda_1 \geq \lambda_0.$$

Moreover, since $F(z,u,\lambda) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$ and all $\lambda > 0$, we can replace $\tilde{u}$ by $\tilde{u}^+ \in W^{1,p}(\Omega)$, and so, without any loss of generality, we may assume that $\tilde{u} \geq 0$.

Hypothesis $(\mathbf{H}_1)$ (v) and Fatou's lemma imply that

$$\lim_{\lambda\to\infty} \int_\Omega F(z,\tilde{u}(z),\lambda)dz = +\infty.$$

So, using Corollary 1 and hypothesis $(\mathbf{H}(\beta))$, we see that we can find $\lambda > 0$ large enough such that

$$\hat{\varphi}_\lambda(\tilde{u}) < 0.$$

Then

$$\hat{\varphi}_\lambda(u_\lambda) < 0 = \hat{\varphi}_\lambda(0) \qquad (\text{see } (3.8)),$$

hence $u_\lambda \neq 0$. From (3.8), we have

$$\hat{\varphi}_\lambda'(u_\lambda) = 0$$

hence

$$\text{(3.10)} \qquad \langle A(u_\lambda),h\rangle + \frac{1}{p-1}\int_\Omega |u_\lambda|^{p-2}u_\lambda h\,dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h\,d\sigma$$
$$= \int_\Omega \hat{f}(z,u_\lambda,\lambda)h\,dz \qquad \text{for all } h \in W^{1,p}(\Omega).$$

In (3.10) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\frac{1}{p-1}[C_1\|Du_\lambda^-\|_p^p + \|u_\lambda^-\|_p^p] \leq 0$$

(see Lemma 1, hypothesis $(\mathbf{H}(\beta))$ and (3.7)), hence

$$u_\lambda \geq 0, \qquad u_\lambda \neq 0.$$

Then equation (3.10) becomes

$$\langle A(u_\lambda), h\rangle + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1} h \, d\sigma = \int_\Omega f(z, u_\lambda, \lambda) h \, dz \qquad \text{for all } h \in W^{1,p}(\Omega),$$

therefore $u_\lambda \in \mathscr{S}(\lambda) \subseteq \operatorname{int} C_+$ for $\lambda > 0$ large, and we get $\lambda \in \mathscr{L}$ and $\mathscr{L} \neq \emptyset$. $\square$

Proposition 6. *If hypotheses* $(\mathbf{H}(a))$, $(\mathbf{H}(\beta))$, $(\mathbf{H}_1)$ *hold and* $\lambda \in \mathscr{L}$, *then* $[\lambda, +\infty) \subseteq \mathscr{L}$.

Proof. Let $\mu > \lambda$. Since by hypothesis $\lambda \in \mathscr{L}$, we can find $u_\lambda \in \mathscr{S}(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 4). Using u_λ we introduce the following truncation-perturbation of the reaction term for problem (P_μ):

$$\eta_\mu(z, x) = \begin{cases} f(z, u_\lambda(z), \mu) + u_\lambda(z)^{p-1} & \text{if } x \leq u_\lambda(z) \\ f(z, x, \mu) + x^{p-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.11}$$

This is a Carathéodory function. We set $H_\mu(z, x) = \int_0^x \eta_\mu(z, s) ds$ and introduce the C^1-functional $\psi_\lambda : W^{1,p}(\Omega) \to \boldsymbol{R}$ defined by

$$\psi_\mu(u) = \int_\Omega G(Du(z)) dz + \frac{1}{p} \|u\|_p^p + \frac{1}{p} \int_{\partial\Omega} \beta(z) |u|^p d\sigma - \int_\Omega H_\mu(z, u) dz \qquad \text{for all } u \in W^{1,p}(\Omega).$$

As before (see the proof of Proposition 5), using hypotheses $(\mathbf{H}_1)$ (i), (ii), we can see that the functional ψ_μ is coercive and sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\psi_\mu(u_\mu) = \inf\{\psi_\mu(u) : u \in W^{1,p}(\Omega)\},$$

hence

$$\psi_\mu'(u_\mu) = 0,$$

and this implies

$$\langle A(u_\mu), h\rangle + \int_\Omega |u_\mu|^{p-2} u_\mu h \, dz + \int_{\partial\Omega} \beta(z) |u_\lambda|^{p-2} u_\lambda h \, d\sigma = \int_\Omega \eta_\mu(z, u_\mu) h \, dz \qquad \text{for all } h \in W^{1,p}(\Omega). \tag{3.12}$$

In (3.12) we choose $h = (u_\lambda - u_\mu)^+ \in W^{1,p}(\Omega)$. Then

$$
\begin{aligned}
(3.13)\qquad & \langle A(u_\mu), (u_\lambda - u_\mu)^+\rangle + \int_\Omega |u_\mu|^{p-2} u_\mu (u_\lambda - u_\mu)^+ dz \\
&\quad + \int_{\partial\Omega} \beta(z)|u_\mu|^{p-2} u_\mu (u_\lambda - u_\mu)^+ d\sigma \\
&= \int_\Omega \eta_\mu(z, u_\mu)(u_\lambda - u_\mu)^+ dz \\
&= \int_\Omega [f(z, u_\lambda, \mu) + u_\lambda^{p-1}](u_\lambda - u_\mu)^+ dz \qquad \text{(see (3.11))} \\
&\geq \int_\Omega [f(z, u_\lambda, \lambda) + u_\lambda^{p-1}](u_\lambda - u_\mu)^+ dz \qquad \text{(see hypothesis } (\mathbf{H}_1) \text{ (v))} \\
&= \langle A(u_\lambda), (u_\lambda - u_\mu)^+\rangle + \int_\Omega u_\lambda^{p-1}(u_\lambda - u_\mu)^+ dz \\
&\quad + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1}(u_\lambda - u_\mu)^+ d\sigma \qquad (\text{since } u_\lambda \in \mathscr{S}(\lambda)),
\end{aligned}
$$

hence

$$\langle A(u_\lambda) - A(u_\mu), (u_\lambda - u_\mu)^+\rangle + \int_\Omega (u_\lambda^{p-1} - |u_\mu|^{p-2}u_\mu)(u_\lambda - u_\mu)^+ dz \leq 0$$

(see hypothesis $(\mathbf{H}(\beta))$), therefore

$$|\{u_\lambda > u_\mu\}|_N = 0,$$

and we conclude that

$$u_\lambda \leq u_\mu.$$

Then, because of (3.11), equation (3.12) becomes

$$\langle A(u_\mu), h\rangle + \int_{\partial\Omega} \beta(z) u_\mu^{p-1} h \, d\sigma = \int_\Omega f(z, u_\mu, \mu) h \, dz \qquad \text{for all } h \in W^{1,p}(\Omega).$$

Then $u_\mu \in \mathscr{S}(\mu) \subseteq \operatorname{int} C_+$, and we conclude that $[\lambda, +\infty) \subseteq \mathscr{L}$. □

According to Proposition 6,

$$(\lambda^*, +\infty) \subseteq \mathscr{L}.$$

Next we show that for every $\lambda > \lambda^*$, problem (P_λ) admits at least two positive solutions. To have this multiplicity result, we need to stregthen a little the conditions on the reaction term $f(z, x, \lambda)$.

The new hypotheses are the following:

($\mathbf{H}_2$): $f : \Omega \times \mathbf{R} \times (0, \infty) \to \mathbf{R}$ is such that for all $\lambda > 0$, $(z, x) \to f(z, x, \lambda)$ is a Carathéodory function, $f(z, x, \lambda) \geq 0$ for a.a. $z \in \Omega$ and all $x \geq 0$, $f(z, 0, \lambda) = 0$ for a.a. $z \in \Omega$ and:

(i) for every $\rho > 0$ and every $I \subseteq (0, \infty)$ bounded, we can find $a_\rho^I \in L^\infty(\Omega)_+$ such that

$$0 \leq f(z, x, \lambda) \leq a_\rho^I(z) \qquad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \rho], \text{ all } \lambda \in I;$$

(ii) for every $\lambda > 0$,

$$\lim_{x \to +\infty} \frac{f(z, x, \lambda)}{x^{p-1}} = 0 \qquad \text{uniformly for a.a. } z \in \Omega;$$

(iii) for every $\lambda > 0$,

$$\lim_{x \to 0^+} \frac{f(z, x, \lambda)}{x^{p-1}} = 0 \qquad \text{uniformly for a.a. } z \in \Omega$$

(iv) there exists $\tilde{h} \in L^p(\Omega)$ such that

$$\int_\Omega F(z, \tilde{h}(z), \lambda) dz > 0 \qquad \text{for all } \lambda \geq \lambda_0 > 0$$

where

$$F(z, x, \lambda) = \int_0^x f(z, s, \lambda) ds;$$

(v) for a.a. $z \in \Omega$ and all $x > 0$, $\lambda \to f(z, x, \lambda)$ is strictly increasing; for each $s > 0$, we can find $\eta_s > 0$ such that

$$0 < \eta_s \leq f(z, x, \tau) - f(z, x, \lambda) \qquad \text{for a.a. } z \in \Omega,$$
$$\text{all } x \geq s > 0, \text{ and all } \tau > \lambda > 0,$$

$$f(z, x, \lambda) \to 0^+ \qquad \text{as } \lambda \to 0^+ \text{ uniformly for a.a. } z \in \Omega,$$
$$\text{all } x \in K \subseteq \mathbf{R} \text{ compact}$$

and

$$f(z, x, \lambda) \to +\infty \qquad \text{for a.a. } z \in \Omega \text{ as } \lambda \to +\infty;$$

(vi) for every $\rho > 0$ and every $I \subseteq (0, \infty)$ bounded, we can find $\tilde{\xi}_\rho^I > 0$ such that for a.a. $z \in \Omega$ and all $\lambda \in I$ the function

$$x \to f(z, x, \lambda) + \tilde{\xi}_\rho^I x^{p-1} \text{ is nondecreasing on } [0, \rho].$$

Remarks. If for a.a. $z \in \Omega$, $f(z, ., \lambda)$ is differentiable on $(0, +\infty)$ and the function $x \to f'(z, x, \lambda)$ has at most $(p-2)$-polynomial growth, uniformly for all $\lambda \in I \subseteq (0, +\infty)$ bounded, then hypothesis $(\mathbf{H}_2)$ (vi) holds. The examples given after the hypotheses $(\mathbf{H}_1)$ satisfy the new conditions.

Proposition 7. *If hypotheses* $(\mathbf{H}(a))$, $(\mathbf{H}(\beta))$, $(\mathbf{H}_2)$ *hold and* $\lambda \in (\lambda^*, +\infty)$, *then problem* (P_λ) *admits at least two positive solutions*

$$u_0,\ \hat{u} \in \operatorname{int} C_+.$$

Proof. Recall that $(\lambda^*, +\infty) \subseteq \mathscr{L}$. Let $\eta, \lambda, \mu \in (\lambda^*, +\infty)$ and assume that

$$\eta < \lambda < \mu.$$

From the proof of Proposition 6, we know that we can find $u_\eta \in \mathscr{S}(\eta) \subseteq \operatorname{int} C_+$ and $u_\mu \in \mathscr{S}(\mu) \subseteq \operatorname{int} C_+$ such that

$$u_\eta \leq u_\mu.$$

We introduce the following Carathéodory function

$$e_\lambda(z,x) = \begin{cases} f(z, u_\eta(z), \lambda) + u_\eta(z)^{p-1} & \text{if } x < u_\eta(z) \\ f(z, x, \lambda) + x^{p-1} & \text{if } u_\eta(z) \leq x \leq u_\mu(z). \\ f(z, u_\mu(z), \lambda) + u_\mu(z)^{p-1} & \text{if } u_\mu(z) < x. \end{cases} \tag{3.14}$$

We set $E_\lambda(z,x) = \int_0^x e_\lambda(z,s)ds$ and consider the C^1-functional $\gamma_\lambda : W^{1,p}(\Omega) \to \boldsymbol{R}$ defined by

$$\gamma_\lambda(u) = \int_\Omega G(Du)dz + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma$$
$$- \int_\Omega E_\lambda(z,u)dz \qquad \text{for all } u \in W^{1,p}(\Omega).$$

Corollary 1 and (3.14) imply that γ_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\gamma_\lambda(u_0) = \inf\{\gamma_\lambda(u) : u \in W^{1,p}(\Omega)\}.$$

Then

$$\gamma_\lambda'(u_0) = 0,$$

hence

$$\langle A(u_0), h\rangle + \int_\Omega |u_0|^{p-2}u_0 h\, dz + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0 h\, d\sigma \tag{3.15}$$
$$= \int_\Omega e_\lambda(z,u_0)h\, dz \qquad \text{for all } h \in W^{1,p}(\Omega).$$

In (3.15) we choose $h = (u_0 - u_\mu)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A(u_0), (u_0 - u_\mu)^+\rangle + \int_\Omega u_0^{p-1}(u_0 - u_\mu)^+ dz \\
&\quad + \int_{\partial\Omega} \beta(z) u_0^{p-1}(u_0 - u_\mu)^+ d\sigma \\
&= \int_\Omega e_\lambda(z, u_0)(u_0 - u_\mu)^+ dz \\
&= \int_\Omega [f(z, u_\mu, \lambda) + u_\mu^{p-1}](u_0 - u_\mu)^+ dz \qquad \text{(see (3.14))} \\
&\leq \int_\Omega [f(z, u_\mu, \mu) + u_\mu^{p-1}](u_0 - u_\mu)^+ dz \\
&\quad \text{(see } (\mathbf{H}_2) \text{ (v) and recall that } \lambda < \mu) \\
&= \langle A(u_\mu), (u_0 - u_\mu)^+\rangle + \int_\Omega u_\mu^{p-1}(u_0 - u_\mu)^+ dz \\
&\quad + \int_{\partial\Omega} \beta(z) u_\mu^{p-1}(u_0 - u_\mu)^+ d\sigma \qquad (\text{since } u_\mu \in \mathscr{S}(\mu)),
\end{aligned}$$

hence

$$\begin{aligned}
&\langle A(u_0) - A(u_\mu), (u_0 - u_\mu)^+\rangle + \int_\Omega (u_0^{p-1} - u_\mu^{p-1})(u_0 - u_\mu)^+ dz \\
&\quad + \int_{\partial\Omega} \beta(z)(u_0^{p-1} - u_\mu^{p-1})(u_0 - u_\mu)^+ d\sigma \leq 0,
\end{aligned}$$

Therefore

$$|\{u_0 > u_\mu\}|_N = 0,$$

and we conclude that

$$u_0 \leq u_\mu.$$

If in (3.15) we choose $h = (u_\eta - u_0)^+ \in W^{1,p}(\Omega)$ and argue similarly, then we show that

$$u_\eta \leq u_0.$$

So, we have proved that

$$u_0 \in [u_\eta, u_\mu] = \{u \in W^{1,p}(\Omega) : u_\eta(z) \leq u_0(z) \leq u_\mu(z) \ \textit{for a.a. } z \in \Omega\},$$

therefore $u_0 \in \mathscr{S}(\lambda) \subseteq \operatorname{int} C_+$ (see (3.14)).

For $\delta > 0$, let $u_0^\delta = u_0 + \delta \in \operatorname{int} C_+$. Also, let $\rho = \|u_\mu\|_\infty$ and $I = (0, \|u_\mu\|_\infty)$. According to hypothesis $(\mathbf{H}_2)$ (vi), we can find $\tilde{\xi}_\rho^I > 0$ such that for a.a. $z \in \Omega$ and all $\lambda' \in I$, we have that

(3.16) $$x \to f(z, x, \lambda') + \tilde{\xi}_\rho^I x^{p-1} \text{ is nondecreasing on } [0, \rho].$$

We have

(3.17) $$\begin{aligned} &-\text{div}\, a(Du_0^\delta) + \tilde{\xi}_\rho^I (u_0^\delta)^{p-1} \\ &\quad \le -\text{div}\, a(Du_0) + \tilde{\xi}_\rho^I u_0^{p-1} + \chi(\delta) \qquad \text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\ &\quad = f(z, u_0, \lambda) + \tilde{\xi}_\rho^I u_0^{p-1} + \chi(\delta) \qquad (\text{since } u_0 \in \mathcal{S}(\lambda)) \\ &\quad \le f(z, u_\mu, \lambda) + \tilde{\xi}_\rho^I u_\mu^{p-1} + \chi(\delta) \\ &\qquad (\text{see (3.16) and recall that } u_0 \le u_\mu) \\ &\quad = f(z, u_\mu, \mu) + \tilde{\xi}_\rho^I u_\mu^{p-1} - [f(z, u_\mu, \mu) - f(z, u_\mu, \lambda)] + \chi(\delta). \end{aligned}$$

Since $u_\mu \in \text{int}\, C_+$, we can find $s > 0$ such that

$$0 < s \le u_\mu(z) \qquad \text{for all } z \in \overline{\Omega}.$$

Then hypothesis $(\mathbf{H}_2)$ (v) implies that

$$0 < \eta_s \le f(z, u_\mu(z), \mu) - f(z, u_\mu(z), \lambda).$$

Since $\chi(\delta) \to 0^+$ as $\delta \to 0^+$, for $\delta > 0$ small we have

$$0 < \frac{1}{2}\eta_s \le \eta_s - \chi(\delta).$$

Returning to (3.17), we obtain

(3.18) $$\begin{aligned} &-\text{div}\, a(Du_0) + \tilde{\xi}_\rho^I u_0^{p-1} \\ &\quad \le f(z, u_\mu, \mu) + \tilde{\xi}_\rho^I u_0^{p-1} - \frac{1}{2}\eta_s \\ &\quad \le -\text{div}\, a(Du_\mu) + \tilde{\xi}_\rho^I u_\mu^{p-1} \qquad \text{for a.a. } z \in \Omega. \end{aligned}$$

From (3.18) and Proposition 3 we have

(3.19) $$u_\mu - u_0 \in \text{int}\, \hat{C}_+ \qquad (\text{recall that } u_0 \le u_\mu).$$

In a similar fashion we show that

(3.20) $$u_0 - u_\eta \in \text{int}\, \hat{C}_+ \qquad (\text{recall that } u_\eta \le u_0).$$

From (3.19) and (3.20) it follows that

(3.21) $$u_0 \in \text{int}_{C^1(\overline{\Omega})}[u_\eta, u_\mu].$$

Let $\hat{\varphi}_\lambda \in C^1(W^{1,p}(\Omega))$ be the functional from the proof of Proposition 5. From (3.14), we have

$$\hat{\varphi}_\lambda|_{[u_\eta, u_\mu]} = \gamma_\lambda|_{[u_\eta, u_\mu]} + \xi_\lambda^* \qquad \text{for some } \xi_\lambda^* \in \boldsymbol{R}. \tag{3.22}$$

Then (3.21) and (3.22) imply that u_0 is a local $C^1(\overline{\Omega})$-minimzer of $\hat{\varphi}_\lambda$, hence

$$u_0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \hat{\varphi}_\lambda, \tag{3.23}$$

(see Proposition 2). Hypothesis $(\mathbf{H}_2)$ (iii) implies that given any $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon, \lambda) > 0$ such that

$$F(z, x, \lambda) \le \frac{\varepsilon}{p}(x^+)^p \qquad \text{for a.a. } z \in \Omega, \text{ all } |x| \le \delta. \tag{3.24}$$

Let $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \le \delta$. We have

$$\begin{aligned}
\hat{\varphi}_\lambda(u) &= \int_\Omega G(Du(z))dz + \frac{1}{p(p-1)}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \hat{F}(z, u, \lambda)dz \\
&\ge \frac{C_1}{p(p-1)}\left[\|Du^+\|_p^p + \int_{\partial\Omega} \hat{\beta}(z)(u^+)^p d\sigma - \frac{\varepsilon(p-1)}{C_1}\|u^+\|_p^p\right] \\
&\quad + \frac{C_1}{p(p-1)}[\|Du^-\|_p^p + \|u^-\|_p^p] \qquad \text{(see (3.24) and hypothesis } (\mathbf{H}(\beta))) \\
&\ge C_{12}\|u\|^p \qquad \text{for some } C_{12} > 0,
\end{aligned}$$

hence $u = 0$ is a local $C^1(\overline{\Omega})$-minimizer of $\hat{\varphi}_\lambda$, therefore $u = 0$ is a local $W^{1,p}(\Omega)$-minimizer of $\hat{\varphi}_\lambda$. Without any loss of generality, we may assume that

$$0 = \hat{\varphi}_\lambda(0) \le \hat{\varphi}_\lambda(u_0). \tag{3.25}$$

The reasoning is similar if the opposite inequality holds. Because of (3.23), we can find $\rho \in (0, 1)$ small such that

$$\hat{\varphi}_\lambda(u_0) < \inf\{\hat{\varphi}_\lambda(u) : \|u - u_0\| = \rho\} =: \hat{m}_\lambda, \qquad \|u_0\| > \rho \tag{3.26}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29). Recall that $\hat{\varphi}_\lambda$ is coercive (see the proof of Proposition 5). Then

$$\hat{\varphi}_\lambda \text{ satisfies the PS-condition,} \tag{3.27}$$

(see Papageorgiou-Winkert [29]). Then (3.25), (3.26) and (3.27) permit the use of mountain-pass theorem (see Theorem 1). So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda'(\hat{u}) = 0 \qquad \text{and} \qquad \hat{m}_\lambda \le \hat{\varphi}_\lambda(\hat{u}).$$

Clearly $\hat{u} \notin \{0, u_0\}$ and $\hat{u} \in \mathscr{S}(\lambda) \subseteq \operatorname{int} C_+$. □

Next we establish the admissibility of the critical parameter value $\lambda^* > 0$.

Proposition 8. *If hypotheses* $(\mathbf{H}(a))$, $(\mathbf{H}(\beta))$, $(\mathbf{H}_2)$ *hold, then*

$$\mathscr{L} = [\lambda^*, +\infty).$$

Proof. Let $\{\lambda_n\}_{n\geq 1} \subseteq (\lambda^*, +\infty)$ be such that $\lambda_n \to \lambda^*$ as $n \to \infty$. Then we can find $u_n \in \mathscr{S}(\lambda_n) \subseteq \operatorname{int} C_+$, for all $n \in \boldsymbol{N}$ and $\{u_n\}_{n\geq 1}$ is decreasing (see the proof of Proposition 7). We have

$$u_n \leq u_1 \quad \text{for all } n \in \boldsymbol{N}. \tag{3.28}$$

Then from (3.28) and hypotheses $(\mathbf{H}_2)$ (i), $(\mathbf{H}(\beta))$, we infer that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega), \quad \text{and} \quad u_n \to u_* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.29}$$

For every $n \in \boldsymbol{N}$, we have

$$\begin{cases} -\operatorname{div} a(Du_n(z)) = f(z, u_n(z), \lambda_n) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u_n}{\partial n_a} + \beta(z) u_n^{p-1} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.30}$$

(see Papageorgiou-Radulescu [25]). From (3.28), (3.30) and the nonlinear regularity theory of Lieberman [20] (p. 320), we know that we can find $\alpha \in (0,1)$ and $\hat{M} > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq \hat{M} \quad \text{for all } n \in \boldsymbol{N}. \tag{3.31}$$

Recalling that $C^{1,\alpha}(\overline{\Omega})$ is compactly embedded into $C^1(\overline{\Omega})$, from (3.29) and (3.31) we infer that

$$u_n \to u_* \quad \text{in } C^1(\Omega) \text{ as } n \to \infty. \tag{3.32}$$

Hypothesis $(\mathbf{H}_2)$ (iii) implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon, \lambda_1) > 0$ such that

$$f(z, x, \lambda_1) \leq \varepsilon x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \delta],$$

hence

$$f(z, x, \lambda_n) \leq \varepsilon x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in (0, \delta], \text{ all } n \in \boldsymbol{N} \tag{3.33}$$

(see hypotheses $(\mathbf{H}_2)$ (v)). Suppose that $u_* = 0$. Then from (3.32) we see that we can find $n_0 \in \boldsymbol{N}$ such that

$$u_n(z) \in (0, \delta] \quad \text{for } n \geq n_0,$$

therefore

$$f(z, u_n(z), \lambda_n) \leq \varepsilon u_n(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } n \geq n_0 \tag{3.34}$$

(see (3.33)). From (3.30) and (3.34) we have

$$-\operatorname{div} a(Du_n(z)) \leq \varepsilon u_n(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } n \geq n_0. \tag{3.35}$$

We multiply (3.35) with $u_n(z)$, integrate over Ω and use the nonlinear Green's identity (see Gasinski-Papageorgiou [13] (p. 320). We obtain

$$\int_\Omega (a(Du_n), Du_n)_{\mathbf{R}^N} dz + \int_{\partial\Omega} \beta(z) u_n^p \, d\sigma \leq \varepsilon \|u_n\|_p^p \quad \text{for all } n \geq n_0,$$

hence

$$\frac{C_1}{p-1}\left[\|Du_n\|_p^p + \int_{\partial\Omega} \hat{\beta}(z) u_n^p \, d\sigma\right] \leq \varepsilon \|u_n\|_p^p \quad \text{for all } n \geq n_0,$$

(see Lemma 1 and recall that $\hat{\beta} = ((p-1)/C_1)\beta \in L^\infty(\Omega))$, therefore

$$\frac{C_1}{p-1}\hat{\lambda}_1(\hat{\beta}) \leq \varepsilon \quad \text{(see (2.7))}.$$

But $\varepsilon > 0$ is arbitrary. We let $\varepsilon \to 0^+$ to conclude that

$$0 < \hat{\lambda}_1(\hat{\beta}) \leq 0,$$

a contradiction. (At this point we need the hypothesis $\beta \neq 0$, (see $\mathbf{H}(\beta)$) since it implies that $\hat{\lambda}_1(\hat{\beta}) > 0$). Therefore $u_* \neq 0$ and we have

$$u_* \in \mathscr{S}(\lambda^*) \subseteq \operatorname{int} C_+$$

hence

$$\lambda^* \in \mathscr{L} \quad \text{and so} \quad \mathscr{L} = [\lambda^*, +\infty). \qquad \square$$

Next we show that for all $\lambda \in \mathscr{L} = [\lambda^*, +\infty)$, problem (P_λ) has a smallest positive solution.

Proposition 9. *If hypotheses* $(\mathbf{H}(a))$, $(\mathbf{H}(\beta))$, $(\mathbf{H}_2)$ *hold and* $\lambda \in \mathscr{L} = [\lambda^*, +\infty)$, *then problem* (P_λ) *has a smallest positive solution* $\bar{u}_\lambda \in \operatorname{int} C_+$, *and the map* $\lambda \to \bar{u}_\lambda$ *from* $\mathscr{L}_0 = (\lambda^*, +\infty)$ *into* $C^1(\overline{\Omega})$ *is left continuous and strictly increasing in the sense that*

$$\lambda < \mu \Rightarrow \bar{u}_\mu - \bar{u}_\lambda \in \operatorname{int} \hat{C}_+.$$

Proof. Since our aim is to produce the smallest positive solution for problem (P_λ), from the proof of Proposition 6 we see that without any loss of generality, we may assume that

$$(3.36)\quad u \le u_\mu \quad \text{for all } u \in \mathscr{S}(\lambda), \text{ with } u_\mu \in \mathscr{S}(\mu) \subseteq \operatorname{int} C_+, \qquad \mu > \lambda.$$

From Lemma 3.10 of Hu-Papageorgiou ([17], p. 178), we know that we can find $\{u_n\}_{n\ge 1} \subseteq \mathscr{S}(\lambda)$ such that

$$\inf \mathscr{S}(\lambda) = \inf_{n\ge 1} u_n.$$

Using (3.36) and reasoning as in the proof of Proposition 7, we infer that

$$u_n \to \bar{u}_\lambda \qquad \text{in } C^1(\overline{\Omega}) \text{ with } \bar{u}_\lambda \in \mathscr{S}(\lambda) \subseteq \operatorname{int} C_+,$$

hence

$$\bar{u}_\lambda = \inf \mathscr{S}(\lambda).$$

Next we show the strict monotonicity of the map $\lambda \to \bar{u}_\lambda$. Let $\lambda < \mu$ and let $\bar{u}_\mu \in \mathscr{S}(\mu) \subseteq \operatorname{int} C_+$ be the minimal solution for problem (P_μ). From the proof of Proposition 7 we know that we can find $u_\lambda \in \mathscr{S}(\lambda) \subseteq \operatorname{int} C_+$ such that

$$\bar{u}_\mu - u_\lambda \in \operatorname{int} \hat{C}_+,$$

hence

$$\bar{u}_\mu - \bar{u}_\lambda \in \operatorname{int} \hat{C}_+,$$

therefore $\lambda \to \bar{u}_\lambda$ is strictly $\hat{C}_+$-increasing from $\mathscr{L}_0$ into $C^1(\overline{\Omega})$.

Finally we show the left continuity of $\lambda \to \bar{u}_\lambda$. So, we suppose that $\{\lambda_n\}_{n\ge 1} \subseteq \mathscr{L}_0$ and assume that $\lambda_n \to \lambda$ as $n \to \infty$. Then using (3.36) (with $\mu \ge \lambda_1$) and reasoning as above, we can show that

$$(3.37)\qquad \bar{u}_{\lambda_n} \to \tilde{u}_\lambda \qquad \text{in } C^1(\overline{\Omega}) \text{ with } \tilde{u}_\lambda \in \mathscr{S}(\lambda) \subseteq \operatorname{int} C_+.$$

Suppose that $\tilde{u}_\lambda \ne \bar{u}_\lambda$. Then we can find $z_0 \in \overline{\Omega}$ such that

$$\tilde{u}_\lambda(z_0) < \bar{u}_\lambda(z_0),$$

therefore

$$\bar{u}_\lambda(z_0) < \bar{u}_{\lambda_n}(z_0) \qquad \text{for all } n \ge n_0 \text{ (see (3.37))},$$

which contradicts the monotonicity of $\lambda \to \bar{u}_\lambda$ established earlier. Hence $\tilde{u}_\lambda = \bar{u}_\lambda$ and this proves the left continuity of $\lambda \to \bar{u}_\lambda$ from $\mathscr{L}_0$ into $C^1(\overline{\Omega})$. □

Summarizing the situation for problem (P_λ), we can state the following bifurcation-type result.

Theorem 2. *If hypotheses* $(\mathbf{H}(a))$, $(\mathbf{H}(\beta))$, $(\mathbf{H}_2)$ *hold, then there exists* $\lambda^* > 0$ *such that*

(a) *for all* $\lambda > \lambda^*$ *problem* (P_λ) *has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+;$$

(b) *for* $\lambda = \lambda^*$ *problem* (P_λ) *admits at least one positive solution*

$$u_* \in \operatorname{int} C_+;$$

(c) *for all* $\lambda \in (0, \lambda^*)$ *problem* (P_λ) *has no positive solutions*;

(d) *for every* $\lambda \in \mathscr{L} = [\lambda^*, \infty)$ *problem* (P_λ) *has a smallest positive solution* $\bar{u}_\lambda \in \operatorname{int} C_+$ *and the map* $\lambda \to \bar{u}_\lambda$ *from* $\mathscr{L}_0$ *into* $C^1(\overline{\Omega})$ *is left continuous and strictly increasing in the sense that*

$$\lambda < \mu \Rightarrow \bar{u}_\mu - \bar{u}_\lambda \in \operatorname{int} \hat{C}_+.$$

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