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Abstract

A system is considered, which is deteriorating over time according to a non homogeneous gamma process with unknown parameters. The system is subject to periodic and instantaneous imperfect maintenance actions (repairs). Each imperfect repair removes a proportion ρ of the accumulated degradation since the previous repair. The parameter ρ hence appears as a measure for the maintenance efficiency. This model is called arithmetic reduction of degradation of order 1. The system is inspected right before each maintenance action, thus providing some multivariate measurement of the successively observed deterioration levels. Based on these data, a semiparametric estimator of ρ is proposed, considering the parameters of the underlying gamma process as nuisance parameters. This estimator is mainly based on the range of admissible ρ's, which depends on the data. Under technical assumptions, consistency results are obtained, with surprisingly high convergence rates (up to exponential). The case where several i.i.d. systems are observed is next envisioned. Consistency results are obtained for the efficiency estimator, as the number of systems tends to infinity, with a convergence rate that can be higher or lower than the classical square root rate. Finally, the performances of the estimators are illustrated on a few numerical examples.

Keywords: Reliability, Deterioration, Imperfect repair, Gamma process, Arithmetic reduction of degradation, Maintenance efficiency, Hyper convergent estimator

Acronyms:		
	p.d.f	probability density function
	i.i.d.	independent and identically distributed
	a.s.	almost surely
	CBM	Condition-Based Maintenance
	ARD_1	Arithmetic Reduction of Degradation of order 1
	ARD_m	Arithmetic Reduction of Degradation of order m
	ARD_∞	Arithmetic Reduction of Degradation of order ∞
	ARA_1	Arithmetic Reduction of Age of order 1
	ARI_1	Arithmetic Reduction of Intensity of order 1
	ECR	Exponential Convergence Rate
	S-ECR	Sub-Exponential Convergence Rate
	EB	Empirical Bias
	PNEE	Proportion of Numerically Exact Estimates

1. Introduction

Safety and dependability are crucial issues in many industries (such as, e.g., railways, aircraft engines or nuclear power plants), which have lead to the development of the so-called reliability theory. For many years, only lifetime data were available and the first reliability studies were focused on lifetime data analysis (see, e.g., [23]), which still remains of interest in many cases. In that context and in case of repairable systems with instantaneous repairs, successive failure (or repair) times appear as the arrival points of a counting process, and failures hence correspond to recurrent events. As for the type of possible repairs, typical classical models are perfect (As-Good-As-New) and minimal (As-Bad-As-Old) repairs, leading to renewal and non homogeneous Poisson processes as underlying counting processes, respectively (see [4]). The reality often lies in-between, leading to the class of imperfect repairs. Many models have been envisioned in the literature for their modeling, such as, e.g., virtual age models introduced by Kijima [18] and further studied in [7, 10], geometric processes [19] (extended in [5]) or models based on reduction of failure intensity [7, 10]. See, e.g., [11] for a recent account and extensions of such models. See also [27] for more references and other models.

Nowadays, the development of online monitoring and the increasing use of sensors for safety assessment make it possible to get specific information on the health of a system and on its effective evolution over time, without waiting for the system failure. This information is often synthesized into a scalar indicator, which can for instance stand for the length of a crack, the thickness of a cable, the intensity of vibrations, corrosion level, ... This scalar indicator can be considered as a measurement of the deterioration level of the system. The evolution of this deterioration indicator over time is nowadays commonly modeled through a continuous-time and continuous-state stochastic process, which is often considered to have an increasing trend. Classical models include inverse Gaussian [31] or Wiener processes with trend [13, 21, 32], which are also quite common in many other fields out of reliability theory, such as finance, insurance or epidemiology. This paper focuses on gamma processes, which are widely used since they were introduced in the reliability field by Çinlar [15]. See [29] and its references for a large overview.

In order to mitigate the degradation of the system over time and extends its lifetime, preventive maintenance actions can be considered, in addition to corrective repairs which are performed at failure. In the context of deteriorating systems, many preventive maintenance policies from the literature consider condition-based maintenance (CBM) actions, where the preventive repair is triggered by the reaching of a preventive maintenance threshold by the deterioration level. In that context, "most of the existing CBM models have been limited to perfect maintenance actions", as noted by [3] (see also [32]). Some imperfect repair models are however emerging in the latest reliability literature, in this new context of deteriorating systems, see [3] for a recent review. Some models are based on the notion of virtual age previously introduced in the context of recurrent events (see, e.g., [12, 24]), where the system is rejuvenated by a maintenance action. Other models consider that an imperfect repair reduces the deterioration level of the system, such as [17, 20, 26, 28, 30], which can be accompanied by some increase in the deterioration rate, as in [9]. Also, some papers consider that the efficiency decreases with the number of repair (see, e.g., [22, 33]), and further studies, as in [14], deal with imperfect maintenance models such that (i) repairs have a random efficiency (ii) the deterioration rate increases with the number of repairs. In all these papers however, the main point mostly is on the optimization of a maintenance policy, including these imperfect maintenance actions together with perfect repairs (replacements). Up to our knowledge, very few papers from the literature deal with statistical issues concerning imperfect repair models for deteriorating systems, except from [32], where the authors suggest a maximum likelihood method for estimating the parameters of the Wiener process together with an iterative procedure based on a Kalman filter for the different factors implied in successive imperfect repairs. This estimation procedure is developed in a fully parametric context and validated on simulated data, without any study of the asymptotic properties of the estimators.

The evaluation of the maintenance actions efficiency is mainly used for maintenance policies optimization. Once the repair efficiency has been estimated, the future behavior of the maintained system can be predicted, which allows to adapt (optimize) the periodicity of the maintenance actions and efficiently plan a general overhaul. From a safety point of view, the principal inquiry is to ensure that the maintenance actions are effective enough to keep with a high probability the degradation level below a fixed threshold (safety level). As long as this safety level is not reached, the maintenance actions must be adjusted, either by adapting their periodicity or by improving their efficiency (if possible). Of course, apart from the previous safety concern, the maintenance costs are another issue. As an example, in [30], the costs minimization is based on the monitoring time and on the imperfect maintenance efficiency. In [14], the author considers a threshold for the degradation, beyond which an imperfect maintenance is performed. The optimization is made with respect to this threshold, the inspections periodicity and the repairs efficiency. See, e.g., both papers cited above and their reference for an overview on maintenance policies optimization.

This paper focuses on a specific imperfect repair model, where each maintenance action reduces the deterioration level of the system. The model was first introduced in [6] and further studied in [25], where it was called Arithmetic Reduction of Degradation model of order 1 (ARD_1). Mimicking Arithmetic Reduction of Age (ARA_1) and Arithmetic Reduction of Intensity (ARI_1) models of order 1 developed by [10] in the context of recurrent events, the idea of an ARD_1 model is that a maintenance action removes a proportion ρ of the degradation accumulated by the system from the last maintenance action (where $\rho \in [0, 1)$). The parameter ρ appears as a measure of the maintenance efficiency, which lies between As-Good-As-New when $\rho \to 1^-$ and As-Bad-As-Old when $\rho = 0$. Along the same lines as [6, 10, 25], the maintenance actions efficiency is here assumed to be fixed and independent of the intrinsic degradation.

This paper is concerned with the development and study of an estimation procedure for the maintenance efficiency parameter ρ, in the context of a gamma deteriorating system subject to periodic ARD_1 imperfect repairs. Observations are lead on just before each maintenance action. Considering n successive repairs, this leads to multivariate data, from where an estimator of ρ is proposed. The idea of this estimator has come from a preliminary study in a parametric framework based on the maximum likelihood method, where we have observed that the minimum of admissible ρ's has quite an interesting behavior, getting quickly very close to the unknown efficiency parameter when n increases. This has lead to the proposition of an original estimator for ρ, which depends only on the data, and not on the shape function and rate parameter of the gamma process, leading to a semiparametric framework. Under technical assumptions, the strong consistency of this new estimator is shown, as the number n of repairs tends towards infinity. Also, the convergence rate is proved to be surprisingly high, and can even reach an exponential speed in some cases. This estimator hence appears to be super consistent (under specific conditions). This is illustrated on simulated data at the end of the paper, where we provide two examples for which we observe that roughly 95% of the estimates are exact at the machine precision level (6.10^{-17}) as soon as $n > 40$ and $n > 88$, respectively, with a mean error below 10^{-15} in both cases. The study is next extended to the case where s independent and identical systems are observed (n times each). A similar semiparametric estimator is proposed for the (common) maintenance efficiency and the strong consistency is proved to hold as s tends towards infinity, no matter the fixed value of n and out of any technical condition requirement. The convergence rate is studied, which is shown to depend on the shape function of the gamma process and on the maintenance period, leading to a speed that can be either slower or faster than $\sqrt{s}$, according to the case.

The outline of this paper is as follows. The framework is specified in Section 2, which covers the gamma deterioration process, the ARD_1 imperfect repair model and the observation scheme. Section 3 is devoted to the study of the semiparametric estimator in the case where one single system is observed, which includes its asymptotic properties when the number of repairs tends towards infinity. Section 4 deals with the extension to several systems and considers the asymptotic properties with respect to the number of observed systems. Some illustrations of the estimator performances are provided in Section 5 and conclusions are formulated in Section 6.

2. Framework

2.1. Intrinsic deterioration

Let us first recall that a random variable X is said to be gamma distributed with a and b as shape and rate parameters, respectively ($X \sim \Gamma(a,b)$ with $a,b>0$), if its distribution admits the following probability density function (p.d.f.):

$$f_X(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \ \mathbf{1}_{\mathbb{R}^+}(x)$$

with respect to Lebesgue measure. Its mean, variance and Laplace transform are provided by

$$\mathbb{E}\left(X\right) = \frac{a}{b},\ \mathbb{V}\left(X\right) = \frac{a}{b^2},\ \mathcal{L}_X\left(t\right) = \mathbb{E}\left(e^{-tX}\right) = \left(\frac{b}{b+t}\right)^a,\ \forall t \geq 0,$$

respectively. Moreover, $cX \sim \Gamma(a, b/c)$ for any $c>0$, and the sum of n independent random variables $X_i \sim \Gamma(a_i, b)$ (with $1 \leq i \leq n$) is also gamma distributed with $X_1 + \cdots + X_n \sim \Gamma\left(a_1 + \cdots + a_n, b\right)$.

Now, let $a(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a continuous and non decreasing function such that $a(0)=0$ and let $b>0$. Also let $(X_t)_{t\geq 0}$ be a right-continuous process with left-side limits. Then, we recall that $(X_t)_{t\geq 0}$ is a non homogeneous gamma process with shape function $a(\cdot)$ and rate parameter b, as soon as

- $X_0 = 0$ almost surely (a.s.),
- $(X_t)_{t\geq 0}$ has independent increments,
- each increment is gamma distributed: for all $0 \leq s < t$, we have $X_t - X_s \sim \Gamma(a(t)-a(s), b)$,

(see, e.g., [1]).

In all the sequel, the intrinsic deterioration of the system (that is out of repairs) is assumed to modeled by a non homogeneous gamma process $(X_t)_{t\geq 0}$ with shape function $a(\cdot)$ and rate parameter b.

2.2. The imperfect repair model

In order to lower the deterioration level, instantaneous and periodic imperfect repairs are carried out on the system every T units of time ($T>0$). Following [6, 25], an Arithmetic Reduction Degradation model of order 1 (ARD_1) is considered, where a maintenance action removes a proportion $\rho \in [0,1)$ of the deterioration accumulated since the last maintenance action (or from time $t=0$). The model used in the present paper is just the same as that used in [25], which we now recall, for sake of completeness.

Between repairs, the system is assumed to evolve according to independent and identically distributed (i.i.d.) copies $\left(X_t^{(i)}\right)_{t\geq 0}$, $i=1,2,\ldots$ of the gamma process $(X_t)_{t\geq 0}$, where exponent (i) refers to the i-th between-repair period $[(i-1)T, iT)$ (where time 0 is considered as a repair time). We set $(Y_t)_{t\geq 0}$ to describe the overall deterioration level of the maintained system, as a result of the intrinsic deterioration and of the imperfect periodic repairs.

On the first time interval $[0,T)$, there is no repair and

$$Y_t = X_t^{(1)} \text{ for all } t \in [0,T).$$

This implies that $Y_{T^-} = X_{T^-}^{(1)} = X_T^{(1)}$ a.s., based on the almost sure continuity of a gamma process.

At time T, the deterioration level is reduced of $\rho X_T^{(1)}$ so that $Y_T = (1-\rho) X_T^{(1)}$ a.s.

On the second time interval $[T, 2T)$, we now have:

$$Y_t = Y_T + X_t^{(2)} - X_T^{(2)} \text{ for all } t \in [T, 2T),$$

which leads to

$$Y_{2T^-} = Y_T + X_{2T^-}^{(2)} - X_T^{(2)} = (1-\rho) X_T^{(1)} + X_{2T}^{(2)} - X_T^{(2)} \text{ a.s.}$$

and

$$Y_{2T} = (1-\rho) X_T^{(1)} + (1-\rho)\left(X_{2T}^{(2)} - X_T^{(2)}\right) \text{ a.s.}$$

More generally, on the j-th time interval (with $j \in \mathbb{N}$), the effective degradation level can be expressed as

$$Y_t = Y_{jT} + \left(X_t^{(j+1)} - X_{jT}^{(j+1)}\right) \text{ for all } t \in [jT, (j+1)T),$$

which leads to

$$Y_{jT^-} = (1-\rho) \sum_{p=1}^{j-1} \left(X_{pT}^{(p)} - X_{(p-1)T}^{(p)}\right) + \left(X_{jT}^{(j)} - X_{(j-1)T}^{(j)}\right) \tag{1}$$

and

$$Y_{jT} = (1-\rho) \sum_{p=1}^{j} \left(X_{pT}^{(p)} - X_{(p-1)T}^{(p)}\right)$$

(with the convention that an empty sum is zero).

An example of trajectory of $(Y_t)_{t\geq 0}$ is given in Figure 1 for $a(t) = t^{1.5}$, $b = 1$, $T = 1$ and $\rho = 0.5$, together with the corresponding trajectories of the $(X_t^{(j)})_{t\geq 0}, j = 1, 2, \ldots$

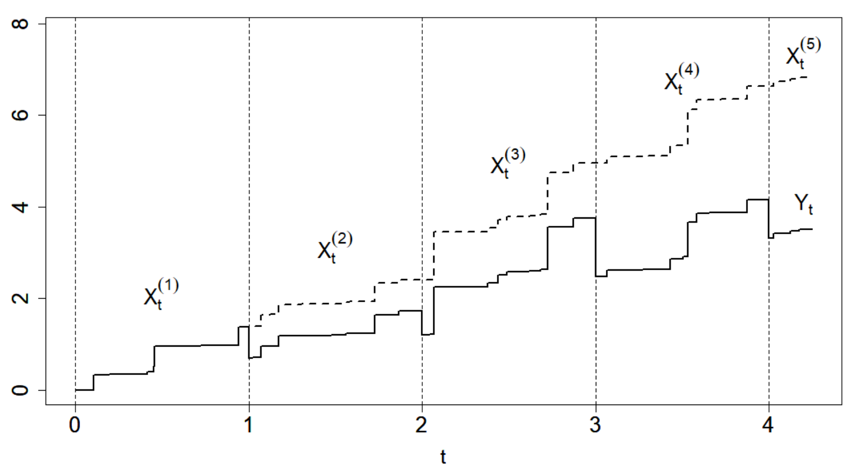

Figure 1: An example of simulated trajectories of $\left(X_t^{(j)}\right)_{t\geq 0}, j = 1, \ldots, 5$ and of $(Y_t)_{t\geq 0}$ (intrinsic and overall degradation levels, respectively) with parameters $a(t) = t^{1.5}$, $b = 1$, $T = 1$ and $\rho = 0.5$.

Note that out of maintenance times ($t \notin \{jT, j = 1, 2, \ldots\}$), the random variable Y_t is the sum of two gamma random variables which do not share the same rate parameter (except if $\rho = 0$). Hence, it is not gamma distributed. Please see [25] for more details on this model.

The periodicity T is assumed to be known and the previous model is called ARD_1 model with parameter $(a(\cdot), b, \rho)$ in the following (with T omitted).

As known from the introduction, our focus is on the development of an estimation procedure for the maintenance efficiency parameter ρ. We now specify the observation scheme and derive some first consequences.

2.3. Observation scheme and first consequences

The deterioration level of the maintained system is assumed to be (perfectly) measured n times ($n \in \mathbb{N}^*$), right before the n first maintenance actions, that is at times $T^-, \ldots, nT^-$. The data hence is an observation of $(Y_{T^-}, \ldots, Y_{nT^-})$, where Y_{jT^-} is provided by (1).

For the sake of simplicity, we set

$$\begin{aligned} Y_j &= Y_{jT^-}, \\ U_j &= X_{jT}^{(j)} - X_{(j-1)T,}^{(j)} \\ a_j &= a(jT) - a\big((j-1)T\big) \end{aligned}$$

for all $j = 1, \ldots, n$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$.

With the previous notation and based on the independent increments of a gamma process, the random variables U_j's can be seen to be independent with $U_j \sim \Gamma(a_j, b)$ for all $j = 1, \ldots, n$. Also, for an ARD$_1$ model with parameter $(a(\cdot), b, \rho)$, Equation (1) can now be written as:

$$Y_j = (1-\rho) \sum_{p=1}^{j-1} U_p + U_j \tag{2}$$

for all $1 \leq j \leq n$.

We first check the theoretical identifiability of the model, considering the parameters of the underlying gamma process as nuisance parameters (and T fixed).

Proposition 1. (Identifiability) *Let* $\mathbf{Y} = \left(Y_1, \ldots, Y_n\right)$ *and* $\tilde{\mathbf{Y}} = \left(\tilde{Y}_1, \ldots, \tilde{Y}_n\right)$ *be two random vectors based on the ARD$_1$ repair model with parameters* $(a(\cdot), b, \rho)$ *and* $\left(\tilde{a}(\cdot), \tilde{b}, \tilde{\rho}\right)$*, respectively (and the same period T). Assume that there are at least two observations ($n \geq 2$). Then if* $\mathbf{Y}$ *and* $\tilde{\mathbf{Y}}$ *are identically distributed (denoted by* $\mathbf{Y} \stackrel{\mathcal{D}}{=} \tilde{\mathbf{Y}}$ *), necessarily* $\rho = \tilde{\rho}$.

Proof. Assume that $\mathbf{Y} \stackrel{\mathcal{D}}{=} \tilde{\mathbf{Y}}$ and $n \geq 2$. Then Y_1 and $\tilde{Y}_1$ are identically distributed, with $Y_1 = U_1 \sim \Gamma\left(a_1, b\right)$ and $\tilde{Y}_1 = \tilde{U}_1 \sim \Gamma\left(\tilde{a}_1, \tilde{b}\right)$. This implies that $a_1 = \tilde{a}_1$ and $b = \tilde{b}$.

Also, $Y_2 = (1-\rho)U_1 + U_2$ and $\tilde{Y}_2 = (1-\tilde{\rho})\tilde{U}_1 + \tilde{U}_2$ must share the same distribution, and hence the same Laplace transform.

Based on the independence between U_1 and U_2, the Laplace transform of Y_2 is

$$\mathcal{L}_{Y_2}(t) = \mathcal{L}_{(1-\rho)U_1}(t)\, \mathcal{L}_{U_2}(t) = \left(\frac{b}{b+(1-\rho)t}\right)^{a_1} \left(\frac{b}{b+t}\right)^{a_2},$$

with a similar expression for $\tilde{Y}_2$. Remembering that $a_1 = \tilde{a}_1$ and $b = \tilde{b}$, this leads to

$$\left(\frac{b}{b+(1-\rho)t}\right)^{a_1} \left(\frac{b}{b+t}\right)^{a_2} = \left(\frac{b}{b+(1-\tilde{\rho})t}\right)^{a_1} \left(\frac{b}{b+t}\right)^{\tilde{a}_2},$$

for all $t \geq 0$, which can be simplified into

$$\left(\frac{1+(1-\tilde{\rho})u}{1+(1-\rho)u}\right)^{a_1} = \frac{1}{(1+u)^{\tilde{a}_2 - a_2}}$$

for all $u \geq 0$, setting $u = t/b$. A first order series expansion at point 0 induces that $a_1(\tilde{\rho} - \rho) = \tilde{a_2} - a_2$ and next that

$$\frac{1+(1-\tilde{\rho})u}{1+(1-\rho)u} = \frac{1}{(1+u)^{\tilde{\rho}-\rho}}$$

for all $u \geq 0$. Taking the limit when $u \to +\infty$ in the previous equation, we get

$$\frac{1-\tilde{\rho}}{1-\rho} = \lim_{u\to+\infty} \frac{1}{(1+u)^{\tilde{\rho}-\rho}} = \begin{cases} 0 \text{ if } \tilde{\rho} > \rho, \\ 1 \text{ if } \tilde{\rho} = \rho, \\ \infty \text{ if } \tilde{\rho} < \rho. \end{cases}$$

This is possible only if $\tilde{\rho} = \rho$ since ρ and $\tilde{\rho}$ belong to $[0, 1)$, which achieves the proof. □

The identifiability hence holds as soon as two observations are available.

From now on, we assume that the true maintenance efficiency parameter is $\rho_0 \in [0, 1)$. The Y_j's and the U_j's hence correspond to an ARD$_1$ model with parameters $(a(\cdot), b, \rho_0)$. A first link between the Y_j's and the U_j's ($j = 1, \cdots, n$) has been provided in Equation (2). We now invert this system of equations, thus providing an expression of the U_j's with respect to the Y_j's, that will be used in the sequel.

Lemma 1. *For each $j \in \{1, \cdots, n\}$, the increment U_j can be expressed with respect to the observations $Y_1, \ldots, Y_j$ and to the maintenance efficiency parameter ρ_0 as follows:*

$$U_j = \sum_{p=1}^{j} \rho_0^{j-p} (Y_p - Y_{p-1}), \tag{3}$$

where we set $Y_0 = 0$.

Proof. This result is proved by induction on j. For $j = 1$, the ARD$_1$ model definition provides $Y_1 = U_1$.

Now, assume that (3) is true for some fixed $1 \leq j \leq n-1$. Observe from (2) that

$$\begin{aligned} Y_{j+1} - Y_j &= (1-\rho_0)\sum_{p=1}^{j} U_p + U_{j+1} - \left[(1-\rho_0)\sum_{p=1}^{j-1} U_p + U_j\right] \\ &= U_{j+1} - \rho_0 U_j, \end{aligned}$$

or equivalently that $U_{j+1} = Y_{j+1} - Y_j + \rho_0 U_j$.

Using the induction assumption, we easily derive that

$$\begin{aligned} U_{j+1} &= Y_{j+1} - Y_j + \rho_0 \sum_{p=1}^{j} \rho_0^{j-p} (Y_p - Y_{p-1}) \\ &= \sum_{p=1}^{j+1} \rho_0^{j+1-p} (Y_p - Y_{p-1}). \end{aligned}$$

Hence, Equation (3) holds for $j + 1$, which achieves the proof. □

For each $1 \leq j \leq n$, let us now define the function $g_j(\rho, \mathbf{Y})$ by

$$g_j(\rho, \mathbf{Y}) = \sum_{p=1}^{j} \rho^{j-p} (Y_p - Y_{p-1}), \ \forall \rho \in [0,1), \tag{4}$$

where we recall that the Y_j's refer to the true maintenance efficiency parameter ρ_0.

Lemma 1 ensures that $g_j(\rho_0, \mathbf{Y})$ matches with the increment U_j, that is

$$g_j(\rho_0, \mathbf{Y}) = U_j, \tag{5}$$

for each $1 \leq j \leq n$. As these increments are gamma distributed, they necessarily are non negative. Hence the true parameter ρ_0 fulfils the condition $g_j(\rho_0, \mathbf{Y}) \geq 0$ for each $j \in \{1, \dots, n\}$. An important consequence is that the range for the admissible ρ's can be restricted to the set

$$D_n = \{\rho \in [0,1) : g_j(\rho, \mathbf{Y}) \geq 0 \text{ for all } j \in \{1, \dots, n\}\}.$$

For a better understanding of what the D_n's are, we now look at an example, based on simulated data, where we consider the successive $D_1, \dots, D_n$ (where the D_j's, $j = 1, \dots, n$, are defined in a similar way as D_n).

Example 1. *Two sets of parameters are considered, with $\rho_0 = 0.5$, $T = 1$ and $b = 1$ for both, $a(t) = \sqrt{t}$ (concave function) for the first set and $a(t) = t^{1.5}$ (convex function) for the second one. An observation of $\mathbf{Y}$ is generated for each of the two parameter sets, and the corresponding observations of the D_j's are next computed. They are plotted in the left and right plots of Figure 2 for the concave and convex cases, respectively. The range for n is $\{1, \cdots, 30\}$ for the left plot (concave case) and $\{1, \cdots, 10^6\}$ for the right plot (convex case). Also, the parameter ρ_0 is highlighted by a vertical blue line on each plot. We can observe that in both cases, the sets D_j's are intervals of the shape $[M_j, 1)$ and that $M_1 \leq M_2 \leq \cdots \leq M_n \leq \rho_0$. (Please note that the M_j's are indicated by blue crosses on the graphs). As can be seen on the left plot, it seems that, in case of a concave shape function, the sequence (M_j) converges very quickly towards ρ_0 when j increases. When the shape function is convex, it might also be convergent towards ρ_0 (?), but if so, it can only be at a very slow rate.*

From the previous example, it seems that M_n could be a very good estimator for ρ_0 in the concave case. However, in the convex case, even if the sequence (M_j) happened to converge towards ρ_0 when j increases (which we do not know), the rate of convergence would apparently be far below the classical square-root speed that could be obtained, e.g., with a maximum likelihood estimator. There hence is no interest in pursuing on this way in the convex case.

As a summary, from the previous observations, we suggest to use M_n as an estimator of the maintenance efficiency parameter ρ_0, that we hope to be convergent at a very high speed in the case of a concave shape function. Note that it is a semiparametric estimator of ρ_0 since the parameters of the gamma process are unknown and not restricted to a parametric family (the shape function $a(\cdot)$ is unknown).

3. The semiparametric estimator and its asymptotic properties

This section is devoted to the formal definition of the semiparametric estimator (Subsection 3.1), together with the study of its asymptotic properties (Subsection 3.3), when the number of imperfect repairs n tends to infinity, with one single system observed.

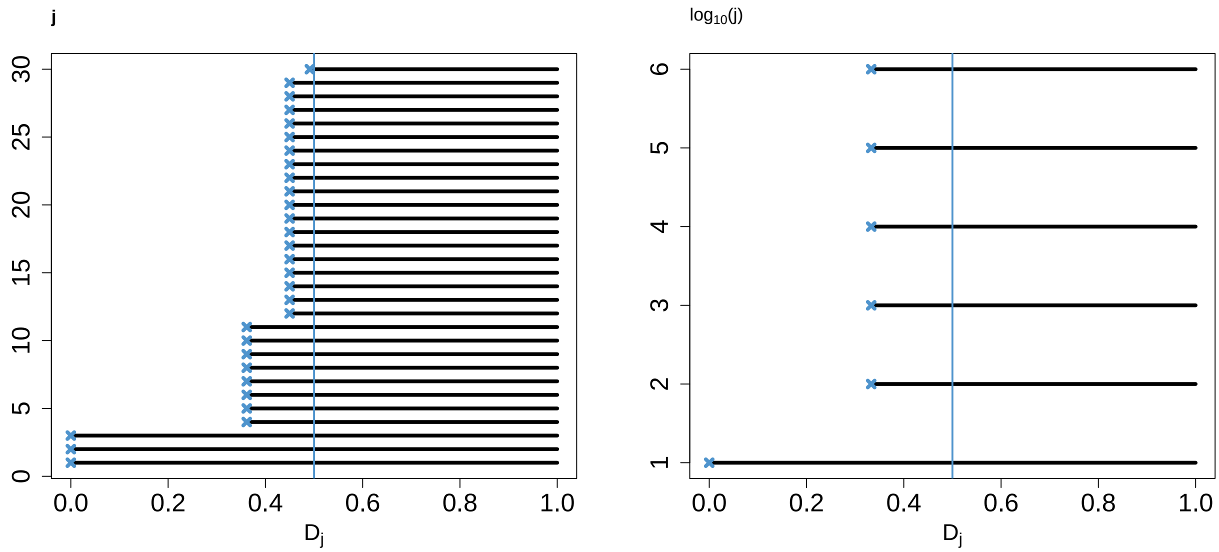

Figure 2: Representation of the sets $D_j, j = 1, \ldots, n$ (black horizontal segments) for $\rho_0 = 0.5$ (vertical blue line) and $b = 1$, with $a(t) = \sqrt{t}$ (concave function), $n = 30$ for the left plot, and $a(t) = t^{1.5}$ (convex function), $n = 10^6$ for the right one, where the lower bounds M_j's of the D_j's are highlighted by blue crosses, Example 1.

3.1. Definition and first properties

Let us first recall from the previous section that

$$D_j = \{\rho \in [0,1) : g_k(\rho, \mathbf{Y}) \geq 0 \text{ for all } k \in \{1, \ldots, j\}\}$$

for all $1 \leq j \leq n$. Note that each set D_j is non empty, because $g_k(1^-, \mathbf{Y}) = Y_k \geq 0$ for all $k \in \{1, \ldots, j\}$.

Also, $g_1(\rho, \mathbf{Y}) = Y_1 \geq 0$ for all $\rho \in [0,1)$, which implies that $D_1 = [0,1)$. Finally, it is readily seen that

$$D_{j+1} = \{\rho \in D_j : g_{j+1}(\rho, \mathbf{Y}) \geq 0\}$$

and that $D_{j+1} \subset D_j$ for all $1 \leq j \leq n-1$.

Let us set

$$M_j = \inf(D_j)$$

for all $1 \leq j \leq n$.

Proposition 2. *The function $g_j(\rho, \mathbf{Y})$ and the sequence $(M_j)_{1 \leq j \leq n}$ (almost surely) satisfy the following properties:*

1. *$M_j \leq M_{j+1}$ for all $1 \leq j \leq n-1$;*
2. *$\rho \mapsto g_j(\rho, \mathbf{Y})$ is non negative on D_j for all $1 \leq j \leq n$;*
3. *$\rho \mapsto g_j(\rho, \mathbf{Y})$ is non decreasing on D_{j-1} for all $1 \leq j \leq n$ (where we set $D_0 = D_1 = [0,1)$ for convenience);*
4. *$D_j = [M_j, 1)$ for all $1 \leq j \leq n$;*
5. *$M_j \leq \rho_0$ for all $1 \leq j \leq n$.*

Proof. Points 1 and 2 are clear due to $D_{j+1} \subset D_j$ and to the definition of D_j.

Let us show the three following points (Points 3-5) all together by induction on j.

At first, we have $D_0 = D_1 = [0, 1)$, $M_1 = 0 \le \rho_0$ and $g_1(\rho, \mathbf{Y}) = Y_1$ for all $\rho \in [0, 1)$. Hence Points 3-5 are true for $j = 1$.

Now, assume Points 3-5 to be true for some $j \in \{1, \cdots, n-1\}$ and, to begin with, let us note that Equation (4) implies that

$$\begin{aligned} g_{j+1}(\rho, \mathbf{Y}) &= Y_{j+1} - Y_j + \rho \sum_{p=1}^{j} \rho^{j-p} (Y_p - Y_{p-1}) \\ &= Y_{j+1} - Y_j + \rho \, g_j(\rho, \mathbf{Y}) \end{aligned} \tag{6}$$

for all $1 \le j \le n-1$ (where $Y_{j+1} - Y_j$ might be negative).

By the induction assumption, $g_j(\rho, \mathbf{Y})$ is non decreasing on D_{j-1}, and hence also on D_j (as $D_j \subset D_{j-1}$). Based on the previous recursion formula (6) and as $g_j(\rho, \mathbf{Y})$ also is non negative on D_j, this implies that $g_{j+1}(\rho, \mathbf{Y})$ is non decreasing on D_j.

As $D_j = [M_j, 1)$ by the induction assumption, we now have

$$M_{j+1} = \inf\{\rho \in [M_j, 1) : g_{j+1}(\rho, \mathbf{Y}) \ge 0\},$$

where g_{j+1} is non decreasing and continuous on $[M_j, 1)$, with $g_{j+1}(1^-, \mathbf{Y}) = Y_{j+1} > 0$ (almost surely). This implies that D_{j+1} is an interval and $D_{j+1} = [M_{j+1}, 1)$.

Finally, from Equation (5), we have $g_{j+1}(\rho_0, \mathbf{Y}) = U_{j+1} \ge 0$. As $M_j \le \rho_0$ by the induction assumption and $g_{j+1}(\rho, \mathbf{Y})$ is known to be non decreasing on $[M_j, 1)$, we necessarily have $M_{j+1} \le \rho_0$. Hence Points 2-5 are true for $j + 1$, and this achieves the proof. □

Based on the previous results, we can see that the sequence $(M_j)_{1 \le j \le n}$ can be alternatively defined through

$$\begin{cases} M_1 = 0, \\ M_{j+1} = \inf\{\rho \in [M_j, 1) \, : \, g_{j+1}(\rho, \mathbf{Y}) \ge 0\}, \quad \text{for all } 1 \le j \le n-1. \end{cases} \tag{7}$$

Also, considering a possibly infinite number of observations, the sequence $(M_n)_{n \ge 1}$ is non decreasing and upperly bounded by ρ_0. It hence is an almost sure convergent sequence. It remains to prove that it converges towards ρ_0, which is done in Subsection 3.3, under specific technical assumptions (among with, concavity of the shape function of the gamma process). With that aim, some technical results have first to be established, which is done in the next subsection.

3.2. Technical results

Lemma 2. *Let $\rho \in [0, 1)$. Then, $g_j(\rho, \mathbf{Y}) \ge 0$ implies that $\rho_0 - \rho \le \dfrac{U_j}{U_{j-1}}$ for each $2 \le j \le n$.*

Proof. Let us first prove by induction that

$$g_j(\rho, \mathbf{Y}) = (\rho - \rho_0) \sum_{p=1}^{j-1} \rho^{j-1-p} U_p + U_j.$$

For $j = 1$, the result is clear because $g_1(\rho, \mathbf{Y}) = Y_1 = U_1$. Assume that the result holds for some $j \in \{1, \cdots, n-1\}$. Based on (6) and (5), we know that

$$\begin{aligned} g_{j+1}(\rho, \mathbf{Y}) &= Y_{j+1} - Y_j + \rho \, g_j(\rho, \mathbf{Y}), \\ U_{j+1} &= Y_{j+1} - Y_j + \rho_0 \, U_j, \end{aligned}$$

(taking $\rho = \rho_0$ in the first line to derive the second one), which provides

$$g_{j+1}(\rho, \mathbf{Y}) = U_{j+1} - \rho_0\, U_j + \rho\, g_j(\rho, \mathbf{Y}).$$

Using the induction assumption, $g_{j+1}(\rho, \mathbf{Y})$ can now be expressed as follows:

$$\begin{aligned} g_{j+1}(\rho, \mathbf{Y}) &= U_{j+1} - \rho_0 U_j + \rho \left((\rho - \rho_0) \sum_{p=1}^{j-1} \rho^{j-1-p} U_p + U_j \right) \\ &= (\rho - \rho_0) \sum_{p=1}^{j} \rho^{j-p} U_p + U_{j+1} \end{aligned}$$

where the last equality results from straightforward calculations. Thus we obtain the first result.

Next we note that $g_j(\rho, \mathbf{Y}) \geq 0$ is true as soon as

$$(\rho - \rho_0) \sum_{p=1}^{j-1} \rho^{j-1-p} U_p + U_j \geq 0,$$

or equivalently

$$\rho_0 - \rho \leq \frac{U_j}{U_{j-1} + \sum_{p=1}^{j-2} \rho^{j-1-p} U_p}.$$

This implies the result since $\displaystyle\sum_{p=1}^{j-2} \rho^{j-1-p} U_p \geq 0$. □

In the following, we will have to control quantities of the shape $\mathbb{P}(\rho_0 - \rho > \varepsilon)$, which will be done by controlling quantities of the shape $\mathbb{P}(U_j / U_{j-1} > \varepsilon)$, using arguments based on the previous lemma. This will be achieved through the use of the following Remark and Lemma.

Remark 1. *For each $j \geq 2$, the random variables U_{j-1} and U_j are known to be independent and gamma distributed $\Gamma(a_{j-1}, b)$ and $\Gamma(a_j, b)$, respectively. It follows that, for all $\varepsilon \geq 0$,*

$$\mathbb{P}\left(\frac{U_j}{U_{j-1}} > \varepsilon \right) = \mathbb{P}\left(\frac{U_j}{U_{j-1} + U_j} > \frac{\varepsilon}{1 + \varepsilon} \right)$$

where the random variable $U_j / (U_{j-1} + U_j)$ is beta distributed $\mathcal{B}(a_j, a_{j-1})$ (standard property of gamma distributions), with p.d.f.

$$f_{a_j, a_{j-1}}(t) = \frac{1}{\mathcal{B}(a_j, a_{j-1})} t^{a_j - 1} (1 - t)^{a_{j-1} - 1}, \; \forall t \in [0, 1]. \tag{8}$$

Lemma 3. *Let us denote by $I_x(\alpha_1, \alpha_2)$ the cumulative density function of the beta distribution $\mathcal{B}(\alpha_1, \alpha_2)$ with positive parameters α_1 and α_2 (which is also called the regularized incomplete beta function). For all $x \in [0, 1]$,*

$$I_x(\alpha_1, \alpha_2) \geq \frac{x^{\alpha_1} (1 - x)^{\alpha_2}}{1 + \frac{\alpha_1}{\alpha_2}}. \tag{9}$$

Proof. Let us first show that

$$I_x(\alpha_1,\alpha_2) \geq \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{\alpha_1 \mathcal{B}(\alpha_1,\alpha_2)} \tag{10}$$

for all $x \in [0,1]$.

Note that Inequality (10) can be seen as a direct consequence of [8, Eq. 8.17.20]. As it is stated without proof in the quoted reference, we prefer to propose some details here.

For $(\alpha_{1,}\alpha_2)$ fixed, let us set

$$g\left(x\right) = I_x(\alpha_1,\alpha_2) - \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{\alpha_1 \mathcal{B}(\alpha_1,\alpha_2)}, \forall x \in \left[0,1\right].$$

Based on the p.d.f. of a beta distribution recalled in (8), it is easy to check that

$$g'\left(x\right) = \frac{x^{\alpha_1}\left(1-x\right)^{\alpha_2-1}}{\mathcal{B}(\alpha_1,\alpha_2)}\left(1+\frac{\alpha_2}{\alpha_1}\right) \geq 0.$$

Thus $g\left(x\right)$ is non decreasing with respect to x. As $g\left(0\right)=0$, we derive that $g\left(x\right) \geq 0$ for all $x \in [0,1]$ and Inequality (10) is true.

It remains to show that

$$\frac{1}{\alpha_1 \mathcal{B}(\alpha_1,\alpha_2)} \geq \frac{1}{1+\frac{\alpha_1}{\alpha_2}} \tag{11}$$

to derive (9). Now, [2, Eq. 6.1.3 p. 255] states that for all positive real number α, the inverse of $\Gamma(\alpha)$ can be expressed as

$$\Gamma(\alpha)^{-1} = \alpha \exp\left(\gamma\alpha\right) \prod_{m\geq 1} \left[\left(1+\frac{\alpha}{m}\right)\exp\left(-\frac{\alpha}{m}\right)\right],$$

where γ is Euler's constant. By definition of the Beta function, we hence have

$$\begin{aligned}\mathcal{B}(\alpha_1,\alpha_2) &= \frac{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}{\Gamma\left(\alpha_1+\alpha_2\right)} \\ &= \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}\left[\frac{\prod_{m\geq 1}\left[\left(1+\frac{\alpha_1+\alpha_2}{m}\right)\exp\left(-\frac{\alpha_1+\alpha_2}{m}\right)\right]}{\prod_{m\geq 1}\left[\left(1+\frac{\alpha_1}{m}\right)\exp\left(-\frac{\alpha_1}{m}\right)\right]\prod_{m\geq 1}\left[\left(1+\frac{\alpha_2}{m}\right)\exp\left(-\frac{\alpha_2}{m}\right)\right]}\right].\end{aligned}$$

As the products are convergent, this can be simplified into

$$\begin{aligned}\mathcal{B}(\alpha_1,\alpha_2) &= \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}\prod_{m\geq 1}\left[\frac{\left(1+\frac{\alpha_1+\alpha_2}{m}\right)}{\left(1+\frac{\alpha_1}{m}\right)\left(1+\frac{\alpha_2}{m}\right)}\right] \\ &\leq \frac{1}{\alpha_1}+\frac{1}{\alpha_2},\end{aligned}$$

which provides (11) and the result. □

Corollary 1. *Let $2 \leq j \leq n$. Then $\rho_0 - M_j \leq U_j/U_{j-1}$ and*

$$\mathbb{P}\left(\rho_0 - M_j > \varepsilon\right) \leq \mathbb{P}\left(\frac{U_j}{U_{j-1}} > \varepsilon\right) \leq 1 - \frac{\tilde{\varepsilon}^{a_j}\left(1-\tilde{\varepsilon}\right)^{a_{j-1}}}{1+\frac{a_j}{a_{j-1}}}$$

for all $\varepsilon \in (0,1)$, *with* $\tilde{\varepsilon} = \varepsilon/(1+\varepsilon)$.

Proof. By definition of M_j, we know that $g_j(M_j, \mathbf{Y}) \geq 0$. Based on Lemma 2, we derive that $\rho_0 - M_j \leq U_j/U_{j-1}$. Hence:

$$\mathbb{P}(\rho_0 - M_j > \varepsilon) \leq \mathbb{P}\left(\frac{U_j}{U_{j-1}} > \varepsilon\right).$$

Now, a direct consequence from Remark 1 and Lemma 3 can be written as follows:

$$\mathbb{P}\left(\frac{U_j}{U_{j-1}} > \varepsilon\right) = 1 - I_{\tilde{\varepsilon}}(a_j, a_{j-1}) \leq 1 - \frac{\tilde{\varepsilon}^{a_j}(1-\tilde{\varepsilon})^{a_{j-1}}}{1 + \frac{a_j}{a_{j-1}}}$$

with $\tilde{\varepsilon} = \varepsilon/(1+\varepsilon)$, which allows to conclude. □

All the previous results are valid without any assumption on the shape function $a(\cdot)$. We now come to specific technical results, which requires some concavity assumption for $a(\cdot)$ to hold. To be more precise, our main assumption states as follows.

Assumption (H) The shape function $a(\cdot)$ of the gamma process is concave and differentiable on $\mathbb{R}^+$, and such that $\lim_{t\to\infty} a'(t) = 0$.

Remark 2. *All the asymptotic results of the paper remain valid if the concavity and differentiability properties for* $a(\cdot)$ *only hold from one point* t_0 *(that is on a set* $[t_0, +\infty)$*), and not on the whole half-line* $\mathbb{R}_+$.

Remark 3. *Many classical concave shape functions fulfills Assumption (H), such as*

$$a_1(t) = \alpha t^\beta,\ a_2(t) = \log\left(1 + \alpha t^\beta\right) \ \text{ or } \ a_3(t) = 1 - \exp\left(-\alpha t^\beta\right),$$

with $\alpha > 0$ *and* $0 < \beta < 1$ *for the first case and* $0 < \beta \leq 1$ *in the other two cases.*

When $\beta > 1$*, both shape functions* $a_2(\cdot)$ *and* $a_3(\cdot)$ *are concave only from the point* t_0*, with* $t_0 = [(\beta-1)/\alpha]^{1/\beta}$ *and* $t_0 = [(\beta-1)/(\alpha\beta)]^{1/\beta}$*, respectively. Hence, as stated in Remark 2, all the asymptotic results remain valid for* $a(\cdot) = a_2(\cdot)$ *or* $a_3(\cdot)$ *and for all* $\beta > 0$.

Nevertheless, for the sake of simplicity, the shape function $a(\cdot)$ *is assumed to be concave and differentiable from the initial time, in all the results requiring Assumption (H) to hold.*

Lemma 4. *Suppose Assumption (H) to hold. Then the sequence* $(a_n)_{n\in\mathbb{N}^*}$ *is non increasing, and tends to 0 when* n *tends to* ∞.

Proof. By the mean value theorem, there exists $c(n)$ in $((n-1)T\,,\,nT)$ such that

$$a_n = a(nT) - a((n-1)T) = T\,a'(c(n)).$$

As $\lim_{t\to\infty} a'(t) = 0$ by assumption and $\lim_{n\to\infty} c(n) = \infty$, it follows that $\lim_{t\to\infty} a'(c(n)) = 0$, which induces the convergence of $(a_n)_{n\in\mathbb{N}^*}$ towards 0. Finally, $a_n \geq a_{n+1}$ for all $n \geq 0$ is a direct consequence of the concavity of $a(\cdot)$. □

Lemma 5. *Suppose Assumption (H) to hold. Then,*

$$\mathbb{P}(\rho_0 - M_{4n} > \varepsilon) \leq \left(1 - \frac{\varepsilon^{a_{2(n+1)}}}{2^{1+a_{2n+1}+a_{2(n+1)}}}\right)^n$$

for all $n \geq 1$ and all $\varepsilon \in (0,1)$.

Proof. Let $2 \leq k \leq n$. Based on Corollary 1, we know that $\rho_0 - M_k \leq U_k/U_{k-1}$. As $(M_k)_{2\leq k\leq n}$ is non decreasing, we get that

$$\rho_0 - M_n = \min_{2\leq k\leq n} (\rho_0 - M_k) \leq \min_{2\leq k\leq n} \frac{U_k}{U_{k-1}}.$$

Now, in order to boil down to independent random variables U_k/U_{k-1}, let us consider only the even terms $k = 2j$. Also, for sake of simplification, let us substitute n by $4n$. We get

$$\rho_0 - M_{4n} \leq \min_{2\leq k\leq 4n} \frac{U_k}{U_{k-1}} \leq \min_{1\leq j\leq 2n} \frac{U_{2j}}{U_{2j-1}}.$$

This induces

$$\mathbb{P}(\rho_0 - M_{4n} > \varepsilon) \leq \mathbb{P}\left(\min_{1\leq j\leq 2n} \frac{U_{2j}}{U_{2j-1}} > \varepsilon\right) = \prod_{j=1}^{2n} \mathbb{P}\left(\frac{U_{2j}}{U_{2j-1}} > \varepsilon\right)$$

since the ratios are independent random variables. Based on Corollary 1 again, we derive that

$$\mathbb{P}(\rho_0 - M_{4n} > \varepsilon) \leq \prod_{j=1}^{2n} \left[1 - \frac{\tilde{\varepsilon}^{\,a_{2j}}(1-\tilde{\varepsilon})^{a_{2j-1}}}{1 + \frac{a_{2j}}{a_{2j-1}}}\right] \tag{12}$$

where $\tilde{\varepsilon} = \varepsilon/(1+\varepsilon)$. Under Assumption (H), we know from Lemma 4 that $a_{2j} \leq a_{2j-1}$ for all $1 \leq j \leq 2n$. Thus

$$\frac{1}{1 + \frac{a_{2j}}{a_{2j-1}}} \geq \frac{1}{2}$$

and

$$\begin{aligned}\prod_{j=1}^{2n} \left[1 - \frac{\tilde{\varepsilon}^{\,a_{2j}}(1-\tilde{\varepsilon})^{a_{2j-1}}}{1 + \frac{a_{2j}}{a_{2j-1}}}\right] &\leq \prod_{j=1}^{2n} \left[1 - \frac{1}{2}\tilde{\varepsilon}^{\,a_{2j}}(1-\tilde{\varepsilon})^{a_{2j-1}}\right] \\ &\leq \prod_{j=n+1}^{2n} \left[1 - \frac{1}{2}\tilde{\varepsilon}^{\,a_{2j}}(1-\tilde{\varepsilon})^{a_{2j-1}}\right] \end{aligned} \tag{13}$$

(as each term is smaller than 1 in the product).

Using again the non increasingness of $(a_j)_{n+1\leq j\leq 2n}$, we get that $\tilde{\varepsilon}^{a_{2j}} \geq \tilde{\varepsilon}^{a_{2(n+1)}}$ and $(1-\tilde{\varepsilon})^{a_{2j-1}} \geq (1-\tilde{\varepsilon})^{a_{2n+1}}$ for all $j \in \{n+1, n+2, \ldots, 2n\}$, since $\tilde{\varepsilon} \in (0,1)$. This implies

$$\prod_{j=n+1}^{2n} \left[1 - \frac{1}{2}\tilde{\varepsilon}^{\,a_{2j}}(1-\tilde{\varepsilon})^{\,a_{2j-1}}\right] \leq \left(1 - \frac{1}{2}\tilde{\varepsilon}^{\,a_{2(n+1)}}(1-\tilde{\varepsilon})^{\,a_{2n+1}}\right)^n. \tag{14}$$

Putting together (12), (13) and (14) leads to

$$\begin{aligned}\mathbb{P}(\rho_0 - M_{4n} > \varepsilon) &\leq \left(1 - \frac{1}{2}\tilde{\varepsilon}^{\,a_{2(n+1)}}(1-\tilde{\varepsilon})^{\,a_{2n+1}}\right)^n \\ &= \left(1 - \frac{\varepsilon^{\,a_{2(n+1)}}}{2(1+\varepsilon)^{\,a_{2n+1}+a_{2(n+1)}}}\right)^n\end{aligned}$$

by definition of $\tilde{\varepsilon}$. Finally, because $1/(1+\varepsilon) > 1/2$ (as $\varepsilon < 1$), we have

$$\mathbb{P}(\rho_0 - M_{4n} > \varepsilon) \leq \left(1 - \frac{\varepsilon^{a_{2(n+1)}}}{2^{1+a_{2n+1}+a_{2(n+1)}}}\right)^n,$$

which finishes the proof. □

We are now ready to state our main results, which is done in next subsection.

3.3. Consistency and convergence rates

Theorem 1. *Suppose Assumption (H) to hold. Then M_n is a strongly consistent estimator of ρ_0 ($M_n \longrightarrow \rho_0$ almost surely) as the number of repairs n tends to infinity.*

Proof. First, because $(M_n)_{n\geq 1}$ is non decreasing, it is sufficient to prove the almost sure convergence of the subsequence $(M_{4n})_{n\geq 1}$ towards ρ_0. (The same remark is valid for the convergence rate, hereafter). Let $\varepsilon \in (0,1)$. From Lemma 5, we know that

$$\sum_{n\geq 1} \mathbb{P}(\rho_0 - M_{4n} > \varepsilon) \leq \sum_{n\geq 1} \left(1 - \frac{\varepsilon^{a_{2(n+1)}}}{2^{1+a_{2n+1}+a_{2(n+1)}}}\right)^n. \tag{15}$$

Hence, it is enough to show the convergence of the right-side series in the previous inequality, to show the strong consistency.

Under Assumption (H) and by Lemma 4, we have $\lim_{n\to+\infty} a_{2(n+1)} = \lim_{n\to+\infty} a_{2n+1} = 0$. Then

$$\lim_{n\to+\infty} \sqrt[n]{\left(1 - \frac{\varepsilon^{a_{2(n+1)}}}{2^{1+a_{2n+1}+a_{2(n+1)}}}\right)^n} = \lim_{n\to+\infty} \left(1 - \frac{\varepsilon^{a_{2(n+1)}}}{2^{1+a_{2n+1}+a_{2(n+1)}}}\right) = \frac{1}{2} < 1.$$

The root test ensures the convergence of the right-side series in (15), which allows to conclude. □

We now look at the convergence rate, which reveals itself to be very high (at least sub-exponential, or even exponential).

Theorem 2. *Suppose Assumption (H) to hold. Then we have:*

1. *The almost sure convergence rate of the estimator M_n is at least sub-exponential (that is at least polynomial of order k, for any $k > 0$) as soon as $a_{2n} = O\left((\log n)^{-1}\right)$.*
2. *The almost sure convergence rate is at least exponential as soon as $a_{2n} = O\left(n^{-1}\right)$.*
3. *The convergence rate in probability is at least exponential as soon as $a_{2n} = o\left(n^{-1} \log\, n\right)$.*

Proof. Let $\varepsilon_n \in (0,1)$ for all $n \in \mathbb{N}^*$. Based on Lemma 5, we have

$$\mathbb{P}(\rho_0 - M_{4n} > \varepsilon_{4n}) \leq \left(1 - \frac{\varepsilon_{4n}^{a_{2(n+1)}}}{2^{1+a_{2n+1}+a_{2(n+1)}}}\right)^n.$$

As $a_{2n+1} \leq a_{2(n+1)} \leq a_1$ from Lemma 4, we get that $2^{1+a_{2n+1}+a_{2(n+1)}} \leq 2^{1+2a_1}$ and hence

$$\mathbb{P}(\rho_0 - M_{4n} > \varepsilon_{4n}) \leq (1-u_n)^n \tag{16}$$

with $u_n = C\ \varepsilon_{4n}^{a_{2(n+1)}}$ and $C = 1/\left(2^{1+2a_1}\right)$. From the root test, we know that the series with generic term $(1-u_n)^n$ is convergent as soon as $\limsup_{n\to+\infty}(1-u_n) < 1$, or equivalently as soon as

$\liminf_{n\to\infty} u_n > 0$. Hence, the series with generic term the left-side expression in (16) is convergent as soon as $\liminf_{n\to\infty} u_n > 0$.

Let us now look at the three different points of the theorem. Assume first that $\varepsilon_n = \varepsilon/n^k$ with $k > 0$ and $\varepsilon \in (0,1)$. This provides

$$u_n = C \left(\frac{\varepsilon}{(4n)^k}\right)^{a_{2(n+1)}} = C \ \exp\left\{a_{2(n+1)}\left[\log(\varepsilon) - k\log(4) - k\log(n)\right]\right\}.$$

Assume further that $a_{2n} = O\left((\log n)^{-1}\right)$, or equivalently that $a_{2(n+1)} = O\left((\log n)^{-1}\right)$. Then, there exists $K > 0$ such that $a_{2(n+1)}\log(n) < K$, from where we derive that

$$u_n > C \ \exp\left\{a_{2(n+1)}\left(\log(\varepsilon) - k\log(4)\right) - k \ K\right\}.$$

Hence

$$\liminf_{n\to\infty} u_n \geq C \liminf_{n\to\infty} \ \exp\{-k \ K\} > 0$$

because $a_{2(n+1)}$ converges towards 0 (see Lemma 4).

This shows that the series with generic term the left-side expression in (16) is convergent for $\varepsilon_n = \varepsilon/n^k$ and any (k,ε), which means that M_n almost surely converges towards ρ_0 at speed at least n^{-k} for any $k > 0$, namely the convergence rate is at least sub-exponential, which proves the first point.

Now let us set $\varepsilon_n = \varepsilon\exp(-kn)$ with $k > 0$ and $\varepsilon \in (0,1)$.

We have

$$\begin{aligned} u_n &= C \ \varepsilon^{a_{2(n+1)}} \exp\left(-4kna_{2(n+1)}\right) \\ &= C \ \exp\left(na_{2(n+1)}\left(\frac{\log(\varepsilon)}{n} - 4k\right)\right). \end{aligned}$$

Assume that $a_{2n} = O\left(n^{-1}\right)$. Then, there exists $K > 0$ such that $na_{2(n+1)} < K$, from where we derive that

$$u_n > C \ \exp\left(K\left(\frac{\log(\varepsilon)}{n} - 4k\right)\right).$$

Hence

$$\liminf_{n\to\infty} u_n \geq C \ \exp(-4Kk) > 0,$$

which allows to conclude for the second point.

Finally, assume that $a_{2n} = o\left(n^{-1}\log n\right)$. The point here is to show the convergence in probability. Based on (16), it is sufficient to show that $\lim_{n\to+\infty}(1-u_n)^n = 0$.

We have

$$(1-u_n)^n = \exp\left\{n\log\left[1 - C \ \varepsilon^{a_{2(n+1)}}\exp\left(-4kna_{2(n+1)}\right)\right]\right\}.$$

As $\log(1-x) \leq -x$ for all $x \in (0,1)$, we get that

$$\begin{aligned} (1-u_n)^n &\leq \exp\left\{-C \ n\varepsilon^{a_{2(n+1)}}\exp\left(-4kna_{2(n+1)}\right)\right\} \\ &= \exp\left(-C \ v_n\right) \end{aligned} \tag{17}$$

with

$$v_n = \exp\left\{\log(n)\left[1 + \frac{na_{2(n+1)}}{\log(n)}\left(\frac{\log(\varepsilon)}{n} - 4k\right)\right]\right\}$$

Based on $a_{2n} = o\left(n^{-1}\log n\right)$, we have $\lim_{n\to+\infty} na_{2n}/\log(n) = 0$, which implies

$$\lim_{n\to+\infty}\left[1+\frac{na_{2(n+1)}}{\log(n)}\left(\frac{\log(\varepsilon)}{n}-4k\right)\right]=1$$

and hence v_n tends to ∞. We derive from (17) that $(1-u_n)^n$ converges towards 0, which allows to conclude. □

Example 2. *Let $a(t) = \alpha t^\beta$ with $0 < \alpha, \beta < 1$, which is already known to fulfill Assumption (H) from Remark 3. Also, we have*

$$\begin{aligned} a_{2n} &= \alpha T^\beta\left((2n)^\beta-(2n-1)^\beta\right) \\ &= \alpha(2nT)^\beta\left(1-\left(1-\frac{1}{2n}\right)^\beta\right) \\ &\underset{n\to+\infty}{\sim} C\frac{1}{n^{1-\beta}} \end{aligned}$$

where $C = \alpha\beta 2^{\beta-1}T^\beta$. It is easy to check that the condition $a_{2n} = O\left((\log n)^{-1}\right)$ from Point 1 in Theorem 2 is satisfied (but not the conditions for the other points). Hence, we can conclude that the almost sure convergence holds with an at least sub-exponential rate.

Example 3. *Let $a(t) = \log\left(1+\alpha t^\beta\right)$ with $\alpha > 0$, $0 < \beta \leq 1$, which is already known to fulfill Assumption (H) from Remark 3. Also, based on $\log(x) \sim x-1$when $x \to 1$ for the second line, we have*

$$\begin{aligned} a_{2n} &= \log\left(\frac{1+\alpha T^\beta(2n)^\beta}{1+\alpha T^\beta(2n-1)^\beta}\right) \\ &\underset{n\to+\infty}{\sim} \alpha T^\beta\frac{(2n)^\beta-(2n-1)^\beta}{1+\alpha T^\beta(2n-1)^\beta} \\ &\underset{n\to+\infty}{\sim} \frac{1}{(1-\frac{1}{2n})^\beta}-1 \\ &\underset{n\to+\infty}{\sim} \frac{\beta}{2n}. \end{aligned}$$

Hence, the condition $a_{2n} = O\left(n^{-1}\right)$ is satisfied (strongest condition in Theorem 2), and the almost sure convergence holds with an at least exponential rate. Note that this result would remain valid for $\beta > 1$ as the shape function is concave from point $t_0 = \left[(\beta-1)/\alpha\right]^{1/\beta}$ (see Remark 3).

Example 4. *Let $a(t) = 1-\exp\left(-\alpha t^\beta\right)$ with $\alpha > 0$, $0 < \beta \leq 1$, which is already known to fulfill Assumption (H) from Remark 3. We have*

$$a_{2n} = \exp\left(-\alpha\left(2nT\right)^\beta\right)-\exp\left(-\alpha\left(2n-1\right)T^\beta\right),$$

which clearly implies $a_{2n} = O\left(n^{-1}\right)$. Hence the almost sure convergence holds with an at least exponential rate. Note that, here again, the result would remain valid for $\beta > 1$ as the shape function is concave from point $t_0 = \left[(\beta-1)/(\alpha\beta)\right]^{1/\beta}$ (see Remark 3).

Up to here, it was supposed that one single system is observed. In the next section, we now envision the possibility of observing several systems.

4. Extension to the case where several systems are observed

4.1. Extended semiparametric estimator

In this section, s identical and independent systems are considered. They share the same intrinsic deterioration and ARD_1 repair model with parameter $(a(\cdot), b, \rho_0)$ and they are all observed at times T^-, $2T^-$, ..., nT^-, as described in Section 2. For each $i \in \{1, \dots, s\}$, we add exponent (i) to each quantity referring to the i-th system. For instance, $\mathbf{Y^{(i)}} = \left(Y_1^{(i)}, \dots, Y_n^{(i)}\right)$ stands for the multivariate observation of the i-th system at times T^-, $2T^-$, ..., nT^-. Also, the sequence $\left(M_j^{(i)}\right)_{1 \le j \le n}$ is defined by

$$\begin{cases} M_1^{(i)} = 0 \\ M_j^{(i)} = \inf \left\{ \rho \in \left[M_{j-1}^{(i)}, 1 \right) \, : \, g_j \left(\rho, \mathbf{Y^{(i)}} \right) \ge 0 \right\} \quad \text{for all } 2 \le j \le n \end{cases}$$

in a similar way as in (7).

The extended semiparametric estimator is defined as

$$M_{s,n} = \max_{1 \le i \le s} \left(M_n^{(i)} \right)$$

for all $n \le 1$ and $s \ge 1$.

The asymptotic properties of each sequence $\left(M_n^{(i)}\right)_{n \in \mathbb{N}^*}$ (with i fixed) has been studied in the previous section. Clearly, similar results are valid for the sequence $(M_{s,n})_{n \in \mathbb{N}^*}$ with s fixed (with an even higher rate of convergence as $M_n^{(i)} \le M_{s,n} \le \rho_0$ for each i). We hence focus on the asymptotic properties of $(M_{s,n})_{s \in \mathbb{N}^*}$ with n fixed in the sequel of this section. We take $n \ge 2$, which ensures the identifiability, based on Proposition 1.

4.2. Consistency and convergence rates according to the number of observed systems

Theorem 3. *Let $n \ge 2$. Then $M_{s,n}$ is a strongly consistent estimator of ρ_0 as the number of observed systems s tends to infinity.*

Proof. From Proposition 2, we know that $M_2^{(i)} \le M_n^{(i)} \le \rho_0$ for each $1 \le i \le s$, which implies that $M_{s,2} \le M_{s,n} \le \rho_0$. Hence, it is enough to prove that $M_{s,2}$ is a strongly consistent estimator of ρ_0. From the definition of $M_{s,2}$, we have

$$\rho_0 - M_{s,2} = \min_{1 \le i \le s} \left(\rho_0 - M_2^{(i)} \right)$$

and as the systems are i.i.d., this provides

$$\mathbb{P}\left(\rho_0 - M_{s,2} > \varepsilon\right) = \mathbb{P}\left(\rho_0 - M_2^{(1)} > \varepsilon\right)^s \tag{18}$$

for all $\varepsilon > 0$. Now, Corollary 1 leads to

$$\mathbb{P}\left(\rho_0 - M_{s,2} > \varepsilon\right) \le \mathbb{P}\left(\frac{U_2^{(1)}}{U_1^{(1)}} > \varepsilon \right)^s$$

and because $\mathbb{P}\left(U_2^{(1)}/U_1^{(1)} > \varepsilon\right) < 1$, we get

$$\sum_{s\geq 1} \mathbb{P}\left(\frac{U_2^{(1)}}{U_1^{(1)}} > \varepsilon\right)^s < \infty.$$

Thus, $M_{s,2}$ tends towards ρ_0 almost surely, which proves the result. □

Theorem 4. *Let $n \geq 2$. The almost sure convergence rate of the estimator $M_{s,n}$ (with respect to s) is at least s^{-k}, for all positive real number k such that*

$$k < \frac{1}{\min\limits_{2\leq p\leq n} a_p}.$$

Proof. Let us set $\varepsilon_s = \varepsilon/s^k$, with $k > 0$ and $\varepsilon \in (0,1)$. The point is to show that the series with generic term $\mathbb{P}(\rho_0 - M_{s,n} > \varepsilon_s)$ converges for all $k < 1/\min_{2\leq p\leq n} a_p$. Using a similar procedure as for Equation (18) and based on Corollary 1, we have

$$\mathbb{P}(\rho_0 - M_{s,n} > \varepsilon_s) = \mathbb{P}\left[\min_{1\leq i\leq s}\left(\rho_0 - M_n^{(i)}\right) > \varepsilon_s\right] \leq \left(\prod_{p=2}^{n} P_p(s)\right)^s \tag{19}$$

where

$$P_p(s) = 1 - \frac{\varepsilon_s^{\,a_p}}{\left(1 + \frac{a_p}{a_{p-1}}\right)(1+\varepsilon_s)^{a_{p-1}+a_p}}.$$

Note that $P_p(s) \in (0,1)$ for any $s \geq 1$ and $2 \leq p \leq n$ so that the product in (19) is smaller than each of its term. Hence, keeping only the $2p$-th term, we get

$$\mathbb{P}(\rho_0 - M_{s,2n} > \varepsilon_s) \leq (P_{2p}(s))^s$$

for all $p \geq 1$ such that $2p \leq n$. Then, using that $\varepsilon_s = \varepsilon s^{-k}$ and $1/(1+\varepsilon_s) > 1/2$, we obtain

$$(P_{2p}(s))^s \leq \left(1 - \frac{s^{-ka_{2p}}}{\left(1 + \frac{a_{2p}}{a_{2p-1}}\right)2^{\,a_{2p-1}+a_{2p}}}\varepsilon^{\,a_{2p}}\right)^s.$$

Now, using that $\log(1-x) \leq -x$ for all x in $[0,1)$, it follows that $(P_{2p}(s))^s \leq u_s$, where

$$u_s = \exp\left(-\,C_p\, s^{1-ka_{2p}}\right) \text{ and } C_p = \varepsilon^{\,a_{2p}}\left(\left(1 + \frac{a_{2p}}{a_{2p-1}}\right)2^{\,a_{2p-1}+a_{2p}}\right)^{-1}.$$

Gathering the previous inequalities, we now have $\mathbb{P}(\rho_0 - M_{s,n} > \varepsilon_s) \leq u_s$ and the point is to study the convergence of the series with generic term u_s. If $1 - ka_{2p} \leq 0$, then u_s converges towards 1 or $\exp(-C_p)$ (if $ka_{2p} = 1$), and the series is divergent. If $1 - ka_{2p} > 0$, then

$$\lim_{s\to+\infty} s^2 u_s = 0$$

and $u_s \underset{s\to+\infty}{=} o\left(\frac{1}{s^2}\right)$, which entails that the series with generic term u_s is convergent, and hence also the series with generic term $\mathbb{P}(\rho_0 - M_{s,n} > \varepsilon_s)$. This allows to conclude that the almost sure convergence rate of the estimator $M_{s,n}$ is at least s^{-k}, for any $k < a_{2p}^{-1}$ and any $p \geq 1$ such that $2p \leq n$.

Now, keeping only the $2p+1$-th term in the product in (19) provides

$$\mathbb{P}(\rho_0 - M_{s,2n} > \varepsilon_s) \leq (P_{2p+1}(s))^s$$

for any p such that $2p+1 \leq n$. Similar arguments as above allow to derive that the convergence rate it at least s^{-k}, for any $k < a_{2p+1}^{-1}$ and any $p \geq 1$ such that $2p+1 \leq n$. Finally, it is hence true for any $k < a_p^{-1}$ and any p such that $2 \leq p \leq n$, and consequently for any $k < \max_{2 \leq p \leq n} \left(a_p^{-1}\right)$, which allows to conclude. □

The real number a_p corresponds to the increment of the shape function over the time interval $[(p-1)T, pT)$. The overall convergence rate obtained in the previous theorem corresponds to the smallest increment. Hence the smaller this increment, the slower the degradation and the higher the convergence rate. More precisely, when the shape function is concave, the increments decrease over time and the convergence rate is at least s^{-1/a_n} because the smallest increment is the last one. On the other hand, the increments increase over time when the shape function is convex, hence the smallest increment is the second one and the convergence rate is at least s^{-1/a_2}. Then the convergence rate is higher than the standard square-root speed as soon as the smallest increment is less than 2. Note that this condition depends on both the shape function and the period T, as illustrated in the next example.

Example 5. *Let $a(t) = \alpha t$ with $\alpha > 0$, hence $a_p = \alpha T$ for each $2 \leq p \leq n$ and the almost sure convergence rate of $M_{s,n}$ with respect to s is at least $s^{-1/\alpha T}$. In comparison with the classical rate $s^{-1/2}$, it is higher if $\alpha T < 2$ and lower if $\alpha T > 2$.*

5. Empirical illustration based on simulated data

The aim of this section is to illustrate our most significant results, that is the fast convergence rates obtained in Section 3 in the case where a single system is observed. In that case asymptotic results are obtained with respect to an increasing number of repairs n. The point hence is to observe the empirical behavior of the semiparametric estimator M_n of ρ_0, which from Theorems 1 and 2 is known to be strongly consistent, with a convergence rate that can be either exponential (ECR) or sub-exponential (S-ECR) with respect to n (considering either almost sure convergence or convergence in probability).

These illustrations are based on two simulated datasets, considering $a(t) = \log(1+t)$ (first case) and $a(t) = \sqrt{t}$ (second case) as shape functions, which from Examples 3 and 2 provide exponential and sub-exponential almost sure convergence rates respectively. The first (resp. second) case will hence be referred to as ECR (resp. S-ECR) in the sequel.

Data simulation and empirical bias

To be able to compare results, we place ourselves within the same framework for both cases. First, the model parameters as well as the observation characteristics of a maintained system are the following:

- Shape function: $a(t) = \log(1+t)$ (ECR) and $a(t) = \sqrt{t}$ (S-ECR);
- Scale parameter: $b = 1$;
- Maintenance efficiency parameter: $\rho_0 = 0.5$;
- Period for repairs: $T = 1$;

- Observation times: $\{nT^-; 1 \leq n \leq 250\}$ because the system is maintained 250 times.

Thus for a single maintained system simulated over the time interval [0,250], that is a degradation trajectory over the time interval [0,250], M_j is computed for each observation time (right before the repair time) providing a realization $(m_1, \ldots, m_{250})$ of the random vector $(M_1, \ldots, M_{250})$. We generate 250 000 i.i.d trajectories, which leads to 250 000 i.i.d. realizations $\left\{\left(m_1^{(i)}, \ldots, m_{250}^{(i)}\right); 1 \leq i \leq 250\ 000\right\}$ of $(M_1, \ldots, M_{250})$. In other words, we have 250 000 estimations of ρ_0 at each observation time $T^-, 2T^-, \ldots, 250T^-$. Then the Empirical Bias (EB) is computed at each observation time nT^-as follows

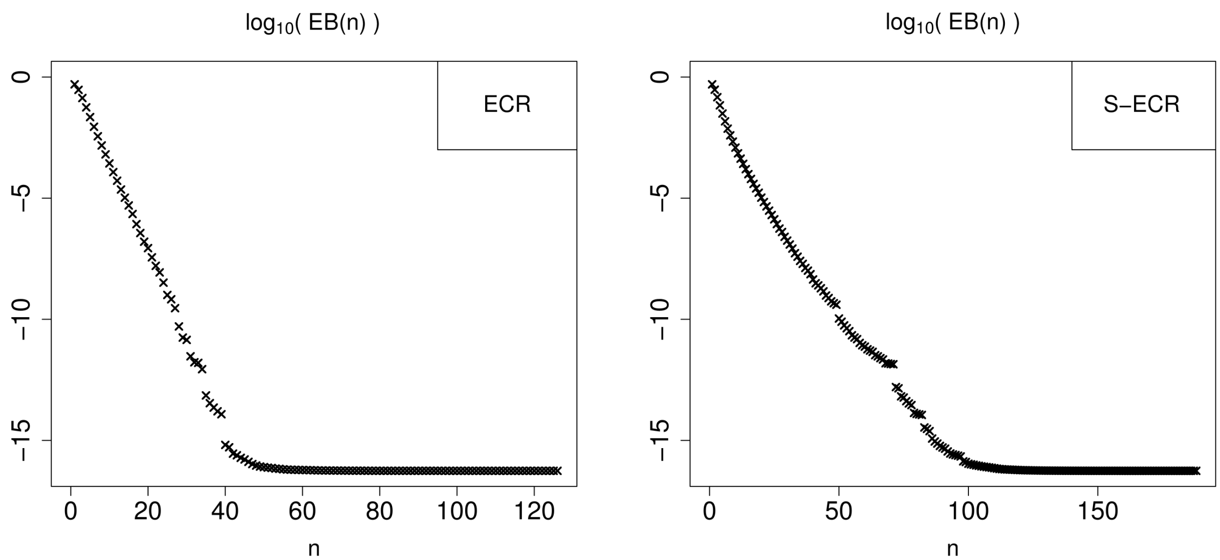

Figure 3: Plots of the common logarithm of the Empirical Bias versus n, for the ECR case on the left and the S-ECR case on the right.

$$\text{EB}(n) = \frac{1}{250\ 000} \sum_{i=1}^{250\ 000} \left(\rho_0 - m_n^{(i)}\right),$$

and the common logarithm of EB(n) for both the ECR and S-ECR cases are reported on Figure 3. In both plots, we can observe that

- the empirical bias tends towards a non-zero constant as n increases;
- viewed as a function of n, the empirical bias has jumps.

Our point now is to explain these two particularities, which are induced by computing limitations, as is discussed hereafter.

Let M_{num} be the largest positive real number such that $1 - M_{\text{num}} = 1$ (numerically negligible), which is roughly equal to 1.11×10^{-16} in our case. This entails that $(1 - M_{\text{num}})/2 = 1/2$, and hence

$$\rho_0 - \frac{M_{\text{num}}}{2} = \rho_0,$$

based on $\rho_0 = 1/2$. Then, when we obtain an estimate $m_n^{(i)} = 0.5$ for some $1 \leq i \leq 250\ 000$ and $1 \leq n \leq 250$, it only means that $\rho_0 - m_n^{(i)} \leq M_{\text{num}}/2$ and not that $m_n^{(i)} = \rho_0$. Thus, the numerical estimate of the bias is correct only if $\rho_0 - m_n^{(i)} \geq M_{\text{num}}/2$, otherwise it is underestimated.

This leads us to introduce the Proportion of Numerically Exact Estimates (PNEE) as a function of n

$$\mathrm{PNEE}(n) = \frac{\#\left\{1 \leq i \leq 250\ 000 :\ \rho_0 - m_n^{(i)} = \frac{M_{\mathrm{num}}}{2}\right\}}{250\ 000}.$$

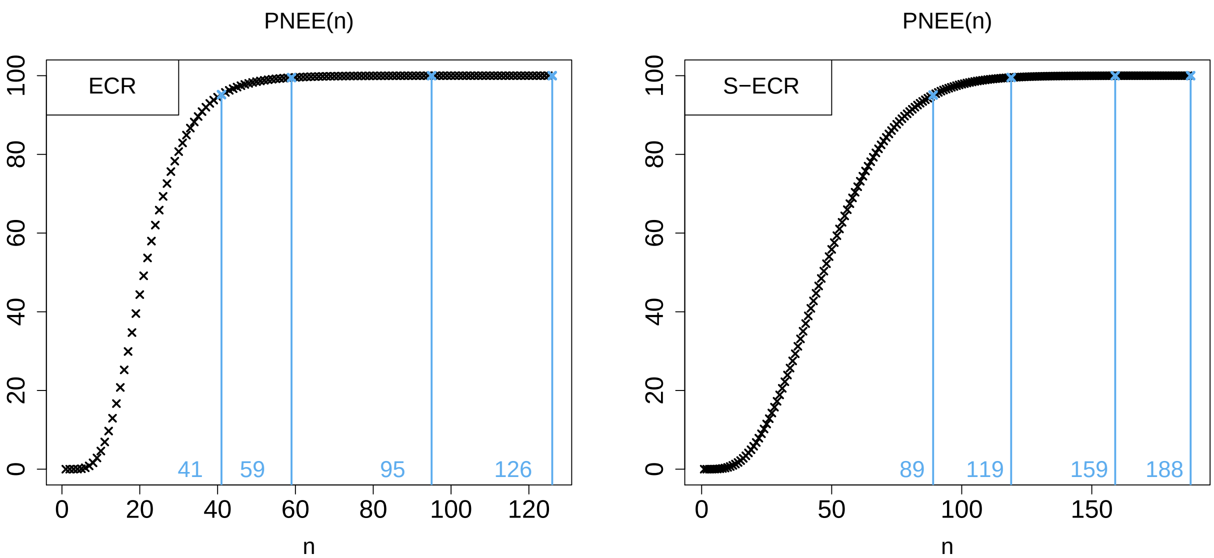

Figure 4: Plots of the PNEE as a function of n, for the ECR case on the left and the S-ECR case on the right. Blue vertical lines point to the smallest repair numbers n for which $\mathrm{PNEE}(n)$ is larger or equal than 95%, 99.5%, 99.99% and 100% from left to right.

This figure shows that the larger n, the more there are estimates equal to $M_{\mathrm{num}}/2$, which explains why $\mathrm{EB}(n)$ tends towards $M_{\mathrm{num}}/2$ instead of 0. Moreover we have $M_{\mathrm{num}}/2 \approx 5.55 \times 10^{-17}$, and so $\log_{10}(M_{\mathrm{num}}/2) \approx -16.25$, which matches the numerical results of Figure 3. Note that in both ECR and S-ECR cases, all the estimates are numerically exact before the last observation time $250T^-$, which illustrates a very fast convergence rate of the estimator. Looking carefully at the distribution of the estimates $m_n^{(i)}, i = 1, \ldots, 250\ 000$, for each n, we have observed that there are a (very) few extreme values, which correspond to trajectories for which the convergence is slow when n increases. For each trajectory, the sequence of estimates $m_n^{(i)}, n = 1, \ldots, 250$, is piecewise constant and approximates ρ_0 by below. When a large proportion of sequences has already reached the numerical precision $M_{\mathrm{num}}/2$ (and hence remains constant when n increases), each jump in an extreme trajectory entails a negative jump in the empirical bias, which is not counterbalanced by any positive jump. This explains the negative jumps observed in Figure 3.

As a summary, both jumps and convergence to a non-zero constant in Figure 3 are due to numerical limitations and the corresponding points on the plots should not be considered for the further study of the empirical bias. Thus, in the sequel, we only focus on the first values of n for which there is no jump, that is on $1 \leq n \leq 26$ for the ECR case and $1 \leq n \leq 49$ for the S-ECR case.

Our aim now is to explore the convergence rate from an empirical point of view. This can be done through the study of the empirical bias, as is now explained.

Link between the bias and the exponential convergence rate

Let us show that the convergence rate of M_n is at least exponential whenever $\log_{10}\big(\rho_0 - \mathbb{E}(M_n)\big)$ decreases linearly. Indeed, assuming that $\log_{10}\big(\rho_0 - \mathbb{E}(M_n)\big)$ decreases linearly, there exist $\tilde{k} > 0$ and

$C \in \mathbb{R}$ such that

$$\log_{10}\big(\rho_0 - \mathbb{E}(M_n)\big) = C - \tilde{k}n.$$

Because $M_n \in [0, \rho_0]$ for any $n \geq 1$ with probability one, the random variable $\rho_0 - M_n$ is non negative and by the Markov's inequality we have

$$\mathbb{P}\big(\rho_0 - M_n > \varepsilon_n\big) \leq \frac{\rho_0 - \mathbb{E}(M_n)}{\varepsilon_n}.$$

Setting $\varepsilon_n = \varepsilon \exp(-kn)$ with $k \in (0, \tilde{k})$ and $\varepsilon > 0$, we have

$$\mathbb{P}\big(\rho_0 - M_n > \varepsilon_n\big) \leq \frac{\exp(C)}{\varepsilon} \exp\left(-\left(\tilde{k} - k\right) n\right) \underset{n\to\infty}{\longrightarrow} 0,$$

and

$$\sum_{n\geq 1} \mathbb{P}\big(\rho_0 - M_n > \varepsilon_n\big) \leq \sum_{n\geq 1} \frac{\exp(C)}{\varepsilon} \exp\left(-\left(\tilde{k} - k\right) n\right) < +\infty.$$

Therefore the two last results allow to conclude that the rate of convergence of M_n is at least exponential for the convergence in probability as well as for the almost sure convergence. If $\log_{10}\big(\rho_0 - \mathbb{E}(M_n)\big)$ decreases at a slower rate than the linear rate, we can not conclude to a sub-exponential convergence rate, however we observe that the empirical evidence of a slower convergence rate for $\log_{10}\big(\rho_0 - \mathbb{E}(M_n)\big)$ coincides with a slower convergence rate for $\rho_0 - M_n$ in Theorem 2.

Link between theoretical and empirical results

Because of the previous explanations on the behavior of EB(n), linear regressions are performed on $\{\text{EB}(n); 1 \leq n \leq 26\}$ for the ECR case and $\{\text{EB}(n); 1 \leq n \leq 49\}$ for the S-ECR case. We assume that the relationship between EB(n) and n is modelled by a linear regression either simple or quadratic. The results are summarized on Figure 5, Table 1 and Table 2.

Table 1: Linear regressions summary for the ECR case

(value / p-value)	Intercept coefficient	First degree coefficient	Second degree coefficient
1^{st} degree	0.10 / 6.12×10^{-4}	-0.36 / $< 2 \times 10^{-16}$	
2^{nd} degree	0.19 / 1.70×10^{-5}	-0.38 / $< 2 \times 10^{-16}$	6.9×10^{-4} / 4×10^{-3}

(Residual error)	Minimum	1^{st} quartile	Median	3^{rd} quartile	Maximum
1^{st} degree	-0.09	-0.05	-0.01	0.04	0.12
2^{nd} degree	-0.14	-0.03	3×10^{-3}	0.04	0.10

The tables provide the linear regressions outcomes, that is the coefficients of the first and second degree polynomial regression (and their related p-values and indicators about residual error (minimum, maximum, first, second and third quartiles). For the ECR case, both first (simple) and second (quadratic) degree polynomial regression fit well $\log_{10}(\text{EB}(n))$, with adjusted R^2 of 0.9994 and 0.9996, respectively. However, Table 1 shows that the second degree term is not significant neither in comparison with the other terms nor for improving the model quality (the adjusted R^2 increases and the residual error decreases). We conclude to a linear decrease of $\log_{10}(\text{EB}(n))$ which induces an at least exponential convergence rate. It is thus consistent with results of Theorem 2 as well as with the related plot in Figure

Table 2: Linear regressions summary for the S-ECR case

(value / p-value)	Intercept coefficient	First degree coefficient	Second degree coefficient
1^{st} degree	-0.95 / 4.79×10^{-13}	-0.19 / $< 2 \times 10^{-16}$	
2^{nd} degree	-0.21 / 4.90×10^{-6}	-0.27 / $< 2 \times 10^{-16}$	1.76×10^{-4} / $< 2 \times 10^{-16}$

(Residual error)	Minimum	1^{st} quartile	Median	3^{rd} quartile	Maximum
1^{st} degree	-0.31	-0.27	-0.14	0.21	0.84
2^{nd} degree	-0.15	-0.06	-5×10^{-3}	0.06	0.22

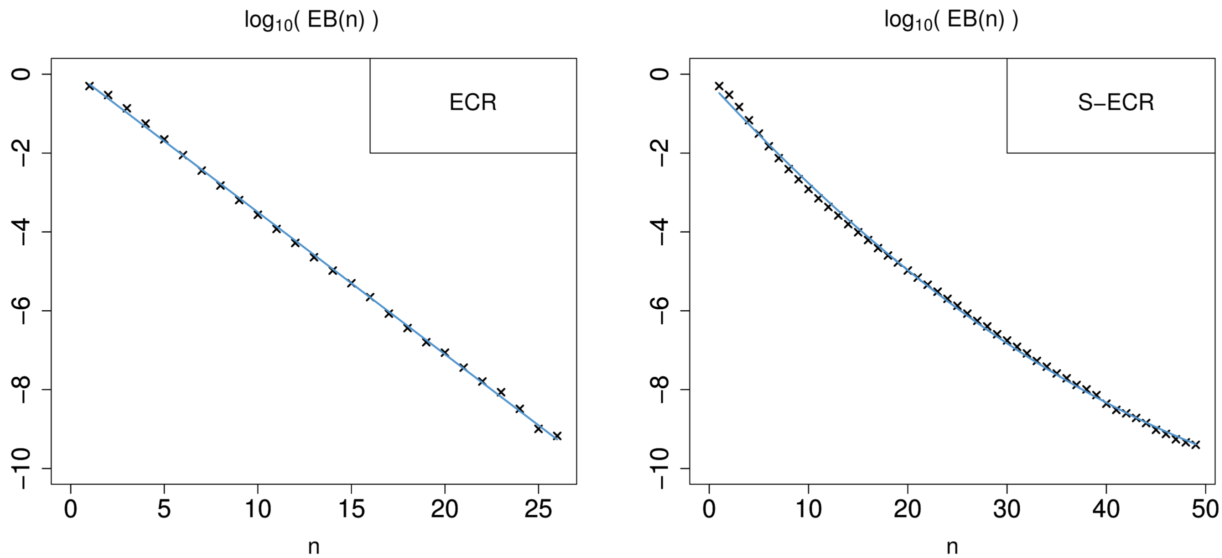

Figure 5: Partial plots of the common logarithm of the Empirical Bias versus n, for the ECR case on the left and the S-ECR case on the right. The blue lines correspond to the first degree polynomial regression for the ECR case and to the second degree polynomial regression for the S-ECR case.

5. Concerning the S-ECR case (see Table 2), the addition of the second degree coefficient improves the model quality, especially the adjusted R^2, which goes from 0.9845 for a linear polynomial to 0.9989 for a quadratic one. Hence the quadratic linear regression model is more relevant than the simple linear regression model, which coincides with the (at least) sub-exponential convergence rate mentioned in Theorem 2.

Regarding the optimality of our results, we recall that the condition of Theorem 2 to obtain an at least exponential convergence rate is $a_{2n} = o\left(n^{-1}\log n\right)$. Now repeating the study with $a_{2n} = n^{-1}\log n$ we see in Figure 6 that again an exponential convergence rate is expected because the bias decreases linearly. We conclude that the condition on a_{2n} in Theorem 2 is probably sufficient but not necessary.

6. Concluding remarks

In this paper we propose a semiparametric inference method for the maintenance efficiency parameter involved in the ARD_1 repair model for a Gamma deteriorating system. For a single system the

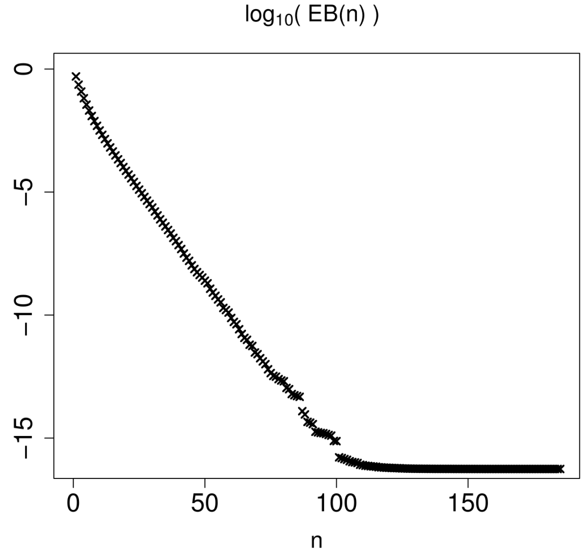

Figure 6: Plots of the common logarithm of the Empirical Bias versus n, with $a_{2n} = n^{-1} \log n$

main condition that insures the strong consistency of our semiparametric estimator of the maintenance efficiency parameter is the concavity of the shape function of the underlying Gamma deteriorating process. Two types of asymptotic results are obtained: either a single system is observed with the number of repairs tending to infinity, or it is the number of systems that tends to infinity. In the case of a single system the almost sure convergence rate of the estimator can be particularly fast, at least exponential for some particular cases. The simulation study illustrates the convergence rates obtained in case of a single trajectory. We observe that the theoretical convergence rates are consistent with the numerical simulation results. However it seems that it is still possible to refine the mathematical conditions under which the convergence rates are obtained. Thus improving the assumptions accuracy may constitute further work. Note that when several systems are considered the convergence rate of the estimator is slower but its strong consistency holds whatever the shape function is. Depending on the shape function, the convergence rate of the estimator may overcome the usual square root rate.

In the wake of this study we want to mention several lines of work that we consider as are important. First, the observation scheme could be decoupled from the scheme of maintenance actions. Indeed, it would be interesting, for instance for an ARD_1 model, to consider a system with scheduled times of maintenance actions for which the observation times are independent of the maintenance schedule. Also, there exist many other models that extend the ARD_1 model, such as for instance the ARD_m (resp. ARD_∞) for which the basic idea is that a maintenance action removes a proportion of the degradation accumulated by the system from the last m maintenance actions (resp. since the system was put into operation).

As explained in the introduction, there exist also alternatives to arithmetic reduction of degradation models such as those based on arithmetic reduction of age. For such models, instead of reducing the degradation level of the system, the maintenance action consists in reducing the age of the system. The use of these models is not restricted to gamma processes, and can be generalized to any non homogeneous Lévy process. As an example, [16] deals with both the ARD_1 and ARD_∞ models, as well as two arithmetic reduction of age models, considering a Wiener process based degradation. Nevertheless, the estimation procedure we developed highly relies on the non negativity of the gamma process and hence could not be adapted to a general non monotonous Lévy process. The adaptation of the estimation procedure of the present paper to another monotonous Lévy process than the gamma process would be interesting to study.

Hence there remain many estimation procedures to be developed for all these imperfect repair models for deteriorating systems, but the semiparametric estimation of the maintenance efficiency for the models mentioned above with various observation schemes, is probably the most challenging problems we aim at investigating in near future.

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