Picard group of unipotent groups, restricted Picard functor

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The Picard group of unipotent groups

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Abstract

Let k be a field. In this article, we study the Picard group of the smooth connected unipotent k-algebraic groups, and more generally the Picard group of the forms of the affine n-space $\mathbb{A}_k^n$.

To study the Picard group of a form of the affine n-space with geometric methods, we define a restricted Picard functor. First, we prove that if a form of the affine n-space X admits a regular completion, then the restricted Picard functor of X is representable by a smooth unipotent k-algebraic group. Then, we generalise a result of B. Totaro: if k is separably closed and if the Picard group of a smooth connected unipotent k-algebraic group is nontrivial then it admits a nontrivial extension by the multiplicative group. Moreover, we obtain that the Picard group of a unirational form of $\mathbb{A}_k^n$ is finite.

Keywords

Picard group, Picard functor, Unipotent group, Form of the affine space, Imperfect field, Unirational.

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Contents

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Introduction

Let k be a field. In this article, we study smooth connected k-group schemes of finite type, we call them *k-group.*

Background

Over a perfect field, any unipotent k-group is isomorphic as a k-scheme to the affine n-space $\mathbb{A}^n_k$ (where $n = \dim(U)$, see [DG70, Th. IV.4.4.1, and Cor. IV.2.2.10]). Thus, over a perfect field, the Picard group of a unipotent k-group is trivial; and, the only unipotent k-group of dimension 1 is the additive group $\mathbb{G}_{a,k}$.

Over a non perfect field, none of the above affirmations are true. From now on, we assume that k is non perfect of characteristic p.

Example. We consider $t \in k \setminus k^p$, and G the k-subgroup of $\mathbb{G}^2_{a,k}$ defined as

$$G := \{(x, y) \in \mathbb{G}^2_{a,k} \mid y^p = x + tx^p\}.$$

Then G is a unipotent k-group of dimension 1, such that $G_{k(t^{1/p})} \cong \mathbb{G}_{a,k(t^{1/p})}$. We denote the regular completion of G by C, then

$$C = \{[x : y : z] \in \mathbb{P}^2_k \mid y^p = xz^{p-1} + tx^p\}.$$

As a set $C \setminus G$ is a unique point of residue field $k(t^{1/p}) \neq k$. Thus, G is not isomorphic to $\mathbb{G}_{a,k}$. Moreover, the degree morphism $deg : \mathrm{Pic}(C) \to \mathbb{Z}$ induces a morphism

$$\overline{deg} : \mathrm{Pic}(G) \to \mathbb{Z}/p\mathbb{Z},$$

that is surjective [Ach17, (2.1.3)]. Hence $\mathrm{Pic}(G) \neq \{0\}$.

A unipotent k-group U is said *k-wound* if U does not admit a closed k-subgroup isomorphic to $\mathbb{G}_{a,k}$. The group G of the example above is a k-wound unipotent k-group. The k-wound unipotent k-group have been studied by J. Tits [Tit67], P. Russell [Rus70], T. Kambayashi, M. Miyanishi and M. Takeuchi [KMT74, KM77] and J. Oesterlé [Oes84].

Recently, a major contribution to the subject of linear k-groups has been made with the study of the structure of pseudo-reductive groups by B. Conrad, O. Gabber, and G. Prasad [CGP15, CP16]. For any linear k-group G, we denote by $\mathcal{R}_{u,k}(G)$ the *k-unipotent radical* of G i.e. $\mathcal{R}_{u,k}(G)$ is the maximal unipotent normal k-subgroup of G. Then, G is said to be *k-pseudo-reductive* if $\mathcal{R}_{u,k}(G) = \{1\}$. If the base field is perfect, k-pseudo-reductive groups are reductive groups; over a non-perfect field the notion of k-pseudo-reductive group generalizes that of reductive group.

In characteristic $p \geqslant 5$ every k-pseudo-reductive group is obtain via the *standard construction* (see [CGP15, Def. 1.4.4, and Th. 5.1.1]); in characteristic 2 and 3, the situation is more complicated. The standard construction essentially reduces the classification of the k-pseudo-reductive groups to the particular case of the commutative k-pseudo-reductive groups, which seems intractable.

For any linear k-group G, there is an exact sequence:

$$1 \to \mathcal{R}_{u,k}(G) \to G \to G/\mathcal{R}_{u,k}(G) \to 1,$$

where $\mathcal{R}_{u,k}(G)$ is unipotent and $G/\mathcal{R}_{u,k}(G)$ is k-pseudo-reductive. As any linear k-group is the extension of a k-pseudo-reductive group by a unipotent k-group, there is a renewed interest in the structure of unipotent k-group over non perfect-field.

Hence, there are two main motivations to study the Picard group of unipotent k-group. First, the study of the Picard group of the unipotent k-group is a necessary step to study the Picard group of the linear k-group. The second motivation is linked to the structure of k-pseudo-reductive group: commutative k-pseudo-reductive groups are extensions of a commutative unipotent k-group by a k-torus [DG70, Th. IV.3.1.1]. B. Totaro prove that, if U is a commutative unipotent k-group, then $\mathrm{Ext}^1(U, \mathbb{G}_{m,k})$ identifies with the subgroup of the translation invariant elements of $\mathrm{Pic}(U)$ [Tot13, Lem. 9.2]. Then, B. Totaro deduce that if U is a k-wound unipotent k-group of dimension 1, then $\mathrm{Ext}^1(U_{k_s}, \mathbb{G}_{m,k_s}) \neq \{0\}$ [Tot13, Lem. 9.4]. And finally, B. Totaro obtain a classification of the 2 dimensional commutative k-pseudo-reductive groups [Tot13, Cor. 9.5].

Main results and outline of the article

Definition. let X, and Y be k-schemes (resp. k-group schemes). We call X *a form* of Y if there is a field extension K/k such that X_K is isomorphic as a K-scheme (resp. K-group scheme) to Y_K.

In [Ach17], we studied the Picard group of the forms of the additive group $\mathbb{G}_{a,k}$. More generally, we used geometric methods to study simultaneously the Picard group of the forms of the additive group $\mathbb{G}_{a,k}$ and of the affine line $\mathbb{A}^1_k$.

Let X be a regular k-algebraic varieties. We call a regular proper k-algebraic variety $\overline{X}$ a *regular completion* of X, if there is an open dominant immersion $X \to \overline{X}$. The (canonical) regular completion C of a form X of $\mathbb{A}^1_k$ is an important invariant, as the Picard group of C is related to the Picard group of X [Ach17, §2.1]. Moreover, we have a powerful tool to study the Picard group of C: the fppf Picard functor $\mathrm{Pic}_{C/k}$. It is a representable functor [BLR90, 8.2 Th. 3] that can be study with geometric methods.

A unipotent k-group of dimension n is a form of the affine n-space [DG70, Th. IV.4.4.1]. As in the dimension 1 case, we would like to use geometric methods to study the Picard group of the forms of $\mathbb{A}^n_k$. If $n > 1$, there is no canonical regular completion. And worst, the existence of a regular completion is only proved yet if $n \leqslant 3$ [CP14, Th. 1.1].

To obtain an object that replace the Picard functor $\mathrm{Pic}_{C/k}$, we are going to follow an idea of M. Raynaud and consider a Picard functor restricted to smooth schemes.

Definition. Let X be a k-scheme, we consider the contravariant functor

$$\begin{array}{cccc} \mathrm{Pic}^+_{X/k}: & (\text{Smooth Scheme}/k)^\circ & \to & (\text{Group}) \\ & T & \mapsto & \dfrac{\mathrm{Pic}(X \times_k T)}{p_2^* \mathrm{Pic}(T)}, \end{array}$$

where (Smooth Scheme/k) denotes the category of smooth k-scheme, (Group) the category of "abstract" group, and $p_2 : X \times_k T \to T$ is the second projection.

We call $\mathrm{Pic}^+_{X/k}$ the *restricted Picard functor* of X.

One of the main result of this article is the following representability Theorem:

Theorem. *1.1*

We consider a form X of $\mathbb{A}^d_k$ which admits a regular completion.

Then, the restricted Picard functor $\mathrm{Pic}^+_{X/k}$ is represented by a smooth commutative unipotent k-algebraic group whose neutral component is k-wound.

This Theorem is proved in Section 1. The proof is quite technical, but the idea behind it is rather intuitive. If $\overline{X}$ is a regular completion of X, then there is an exact sequence:

$$0 \to D \to \mathrm{Pic}\left(\overline{X}\right) \to \mathrm{Pic}(X) \to 0, \qquad (*)$$

where D is the free $\mathbb{Z}$-module generated by the divisors in $\overline{X} \setminus X$.

The idea is to generalize this exact sequence. The fppf Picard functor $\mathrm{Pic}_{\overline{X}/k}$ is representable by a locally algebraic group [BLR90, 8.2 Th. 3], we still denote it $\mathrm{Pic}_{\overline{X}/k}$. We are going to use $\mathrm{Pic}_{\overline{X}/k}$ as a middleman to show that $\mathrm{Pic}^{+}_{X/k}$ is representable. More precisely, inspire by P. Deligne definition of 1-motifs [Del74, §10.1] and by the exact sequence $(*)$ we are going to look at the quotient of $\mathrm{Pic}_{\overline{X}/k}$ by the constant k-locally algebraic group generated by the divisors in $\overline{X} \setminus X$.

Let us present two consequences of Theorem 1.1. First, in Section 3 we study the group $\mathrm{Ext}^1(U, \mathbb{G}_{m,k})$ of extensions of a commutative unipotent k-group U by the multiplicative group $\mathbb{G}_{m,k}$. In Subsection 3.1, we give a summary of some results of [Tot13, §9] and [Ros18]. In Subsection 3.2, we define an action of U on $\mathrm{Pic}^{+}_{U/k}$. In Subsection 3.3, we consider a fixed point functor $\mathrm{Pic}^{+\,U}_{U/k}$, and we use the representability of $\mathrm{Pic}^{+}_{U/k}$ to prove the representability of $\mathrm{Pic}^{+\,U}_{U/k}$ (Proposition 3.4). Then, we obtain the following generalization of Lemma [Tot13, Lem. 9.4].

Theorem. *3.6*

We consider a unipotent k-group U which admits a regular completion. Then:

(i) if $\mathrm{Pic}(U_{k_s})$ *is finite, then* $\mathrm{Ext}^1(U, \mathbb{G}_{m,k}) = \mathrm{Pic}(U)$*;*

(ii) if $\mathrm{Pic}(U_{k_s})$ *is infinite, then* $\mathrm{Ext}^1(U_{k_s}, \mathbb{G}_{m,k_s})$ *is likewise infinite;*

(iii) if $\mathrm{Pic}(U_{k_s}) \neq \{0\}$*, then* $\mathrm{Ext}^1(U_{k_s}, \mathbb{G}_{m,k_s}) \neq \{0\}$*.*

The full statement of Theorem 3.6 includes a result on $\mathrm{Pic}^{+\,U}_{U/k}$. Finally in Subsection 3.5, we give an *ad hoc* criteria for the commutative unipotent k-groups that are quotients of a commutative k-pseudo-reductive k-group.

The second consequence of Theorem 1.1 concerns the unirational forms of $\mathbb{A}^n_k$. Let us recall that an integral k-variety X is called *unirational* if there is a dominant rational map $\mathbb{P}^d_k \dashrightarrow X$ (for some $d \in \mathbb{N}$). In Section 4, we study the subtle relationship between the notion of unirationality and the unipotent k-groups. The k-wound unipotent k-groups have strange properties: while some are unirational, others do not contain a nontrivial unirational k-subgroup. In Subsection 4.1, we study an example of k-wound unirational unipotent k-group. In Subsection 4.2, we make a *dévissage* of the commutative k-group between unirational k-group and strongly-wound k-group (see Definition 4.6).

The main result of Section 4 is the following:

Theorem. *4.11*

Let X be a unirational form of $\mathbb{A}^n_k$ which admits a regular completion. Then:

(i) the unipotent k-algebraic group $\mathrm{Pic}^{+}_{X/k}$ *is étale;*

(ii) the groups $\mathrm{Pic}(X)$ *and* $\mathrm{Pic}(X_{k_s})$ *are finite.*

Theorem 4.11 is proved in Subsection 4.3, it is a straightforward consequence of Theorem 1.1. In Subsection 4.4, we study the torsion of the Picard group and of the restricted Picard functor of a form of $\mathbb{A}^n_k$.

The author hopes that the consequences of Theorem 1.1, and the relative simplicity of their demonstrations will convince the reader that the restricted Picard functor is the right object to look at. Some questions (questions 3.16, 4.2, and 4.9) will require more work to have a satisfying answers.

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Conventions

We consider a field k, unless explicitly stated, k is a non perfect field of characteristic $p > 0$. We fix an algebraic closure $\overline{k}$ of k, and we denote by $k_s \subset \overline{k}$ the separable closure of k in $\overline{k}$. For any non-negative integer n, we denote the field $\{x \in \overline{k}$ such that $x^{p^n} \in k\}$ by $k^{p^{-n}}$.

All schemes are assumed to be separated and locally noetherian. For every scheme X, we denote the structural sheaf of X by $\mathcal{O}_X$. We denote the ring of regular functions on X by $\mathcal{O}(X)$, and the multiplicative group of invertible regular functions on X by $\mathcal{O}(X)^*$. For every $x \in X$, we denote the stalk of $\mathcal{O}_X$ at x by $\mathcal{O}_{X,x}$, and the residue field of $\mathcal{O}_{X,x}$ by $\kappa(x)$.

The morphisms considered between two k-schemes are morphisms over k. An *algebraic variety* is a scheme of finite type over $\mathrm{Spec}(k)$. In order to lighten our notation, we will denote the product $X \times_{\mathrm{Spec}(k)} Y$ for X and Y two k-schemes by $X \times_k Y$. And for any field extension K/k, we denote the base change $X \times_k \mathrm{Spec}(K)$ by X_K. We denote the function field of an integral variety X by $\kappa(X)$.

A k-scheme is said to be *smooth* if it is formally smooth [EGAIV4, Def. 17.1.1], separated and locally of finite type over $\mathrm{Spec}(k)$. A group scheme locally of finite type over k will be called a *k-locally algebraic group.* A group scheme of finite type over k will be called a *k-algebraic group.* A smooth connected k-algebraic group will be called a *k-group.* A unipotent k-group U is said to be *k-split* if U has a central composition series with successive quotients isomorphic to $\mathbb{G}_{a,k}$. A unipotent k-group U over k is said to be *k-wound* if every morphism of k-scheme $\mathbb{A}^1_k \to U$ is constant (with image a point of $U(k)$); an equivalent definition of k-wound is that U does not have a central subgroup isomorphic to $\mathbb{G}_{a,k}$ [CGP15, Pro. B.3.2].

1 Representability of the restricted Picard functor

This section is dedicated to the proof of the following theorem:

Theorem 1.1. *We consider a form X of $\mathbb{A}^d_k$ which admits a regular completion.*

Then, the restricted Picard functor $\mathrm{Pic}^+_{X/k}$ is represented by a smooth commutative unipotent k-algebraic group whose neutral component is k-wound.

In Subsection 1.1, we gather some preliminary results. In Subsection 1.2, we prove Theorem 1.1 assuming k is separably closed. Finally, in Subsection 1.3, we use a Galois descent argument to finish the proof of Theorem 1.1.

1.1 Picard functor of a normal completion of a form of the affine space

In this Subsection, we consider a form X of $\mathbb{A}^d_k$ ($d \geqslant 1$) and Y a normal completion of X (i.e. a normal proper k-variety Y such that there is a dominant open immersion $X \to Y$). We are going to describe the fppf Picard functor $\mathrm{Pic}_{Y/k}$. First, we have some preliminary lemmas.

Lemma 1.2. *Let X be a form of $\mathbb{A}^d_k$, then $X_{\overline{k}}$ is $\overline{k}$-isomorphic to $\mathbb{A}^d_{\overline{k}}$.*

Proof. The proof use a standard argument: there is a field extension K/k such that X_K is K-isomorph to $\mathbb{A}^d_K$. We can assume that K is algebraically closed, and that K is an extension of $\overline{k}$. Then, K is the inductive limits of its finite type sub-$\overline{k}$-algebra; thus, there is a finite $\overline{k}$-algebra A such that X_A is isomorphic to the A-scheme $\mathbb{A}^d_A$ [EGAIV3, Th. 8.8.2].

Moreover, as A is a finite type $\overline{k}$-algebra and $\overline{k}$ is algebraically closed, there are non trivial morphism of $\overline{k}$-algebra $A \to \overline{k}$. Hence, $X_{\overline{k}}$ is $\overline{k}$-isomorphic to $\mathbb{A}^d_{\overline{k}}$. ☐

Recall that the fppf Picard functor $\mathrm{Pic}_{Y/k}$ is representable by a k-locally algebraic group [BLR90, 8.2 Th. 3].

Lemma 1.3. *We consider a normal, proper, geometrically integral k-algebraic variety Y, and V an open of $\mathbb{A}^n_k$ ($n \in \mathbb{N}$). Then, any morphism $V \to \mathrm{Pic}_{Y/k}$ is constant (of image a k-rational point).*

Proof. We can assume that $k = k_s$. We are going to show that any morphism $V \to \mathrm{Pic}_{Y/k}$, whose image contains the identity element e of $\mathrm{Pic}_{Y/k}$, is constant of image e.

Let $p_2 : Y \times_k V \to V$ and $p_1 : Y \times_k V \to Y$ be the two projections. According to Proposition [BLR90, 8.1 Pro. 4],

$$\mathrm{Pic}_{Y/k}(V) = \frac{\mathrm{Pic}(Y \times_k V)}{p_2^* \mathrm{Pic}(V)}.$$

Moreover, $\mathrm{Pic}(V)$ is trivial, so $\mathrm{Pic}_{Y/k}(V) = \mathrm{Pic}(Y \times_k V)$.

In addition, the group morphism:

$$p_1^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(Y \times_k V)$$

is an isomorphism. Indeed, any k-rational point a of V defined a section of p_1^*, so it is injective. And p_1^* is surjective [EGAIV4, Cor. 21.4.11].

Let a be a k-rational point of V, the point a induces a k-group morphism:

$$a^* : \mathrm{Pic}_{Y/k}(V) \to \mathrm{Pic}_{Y/k}(k) = \mathrm{Pic}(Y).$$

The elements of the kernel of a^* are the k-scheme morphisms $f : V \to \mathrm{Pic}_{Y/k}$ such that $f(a) = e$. So there is an isomorphism

$$\ker(a^*) \cong \frac{\mathrm{Pic}(Y \times_k V)}{p_1^* \mathrm{Pic}(Y) \times p_2^* \mathrm{Pic}(V)}.$$

Indeed, if x is a k-rational point of Y (such a point always exists if $k = k_s$ [Liu06, Pro. 3.2.20]), then

$$a^* \times x^* : \mathrm{Pic}(Y \times_k V) \to \mathrm{Pic}(Y) \times \mathrm{Pic}(V)$$

is a retraction of

$$p_1^* \times p_2^* : \mathrm{Pic}(Y) \times \mathrm{Pic}(V) \to \mathrm{Pic}(Y \times_k V).$$

□

Remark 1.4. Let T be a k-torus. With the hypothesis of Lemma 1.3, any morphism of k-scheme $T \to \mathrm{Pic}_{Y/k}$ is constant of image a k-rational point of $\mathrm{Pic}_{Y/k}$.

Indeed, we can assume that $k = k_s$, so that T is a split torus. According to Lemma 1.3, any morphism $\mathbb{G}_{m,k_s}^{\dim(T)} \to \mathrm{Pic}_{Y_{k_s}/k_s}$ is constant.

In general, the Picard schemes of a k-projective variety is a non-necessary smooth k-locally algebraic group. In order to deal with the lack of smoothness, we will use the following "smoothification" lemma:

Lemma ([CGP15, Lem. C.4.1])**.**

Let X be a k-scheme locally of finite type. There is a unique geometrically reduced closed subscheme X^+ of X such that $X^+(K) = X(K)$ for all separable extension fields K/k (no hypothesis of finiteness here). The formation of X^+ is functorial in X and commutes with the formation of products over k and separable extension of the ground field. In particular, if G is a k-locally algebraic group then G^+ is a smooth k-locally algebraic group.

We call G^+ the *smoothification* of G. We remark that if G is a k-locally algebraic group and T is a smooth k-schemes, then we have the equality $G(T) = G^+(T)$.

Proposition 1.5. *Let X be a form of $\mathbb{A}_k^d$, and let Y be a normal completion of X. Then, the Picard functor $\mathrm{Pic}_{Y/k}$ is representable by a commutative k-locally algebraic group, and $\mathrm{Pic}_{Y/k}^{+\,0}$ is a k-wound unipotent k-group.*

Proof. The Picard functor $\mathrm{Pic}_{Y/k}$ is representable by a commutative k-locally algebraic group [BLR90, 8.2 Th. 3]. So $\mathrm{Pic}_{Y/k}^{+}$ is a smooth commutative k-locally algebraic group and the neutral component $\mathrm{Pic}_{Y/k}^{+\,0}$ of $\mathrm{Pic}_{Y/k}^{+}$ is a commutative k-group.

The construction of the neutral component and the smoothification process commute with separable extension. Moreover, a unipotent k-group U is k-wound if and only if U_{k_s} is k_s-wound [CGP15, Pro. B.3.2]. Thus, we can suppose that $k = k_s$ (and then, $X(k) \neq \emptyset$).

Let A and B be two k-groups. We denote $\mathrm{Hom}_{pt.}(A, B)$ the set of k-scheme morphisms from A into B such that the image of the identity element of A is the identity element of B, and by $\mathrm{Hom}_{grp.}(A, B)$ the set of k-group morphisms from A into B.

We consider A a k-abelian variety. According to Proposition [Bri17, Pro. 3.3.4]:

$$\mathrm{Hom}_{grp.}\left(A, \mathrm{Pic}^{+\,0}_{Y/k}\right) = \mathrm{Hom}_{pt.}\left(A, \mathrm{Pic}^{+\,0}_{Y/k}\right).$$

And, as A is smooth, we have the following equality:

$$\mathrm{Hom}_{pt.}\left(A, \mathrm{Pic}^{+\,0}_{Y/k}\right) = \mathrm{Hom}_{pt.}\left(A, \mathrm{Pic}^{0}_{Y/k}\right).$$

Moreover

$$\begin{aligned}\mathrm{Hom}_{pt.}\left(A, \mathrm{Pic}^{0}_{Y/k}\right) = \mathrm{Hom}_{pt.}\left(A, \mathrm{Pic}_{Y/k}\right) &\cong \frac{\mathrm{Pic}\left(Y \times_k A\right)}{p_1^*\mathrm{Pic}(Y) \times p_2^*\mathrm{Pic}(A)} \\ &\cong \mathrm{Hom}_{pt.}\left(Y, \mathrm{Pic}_{A/k}\right) = \mathrm{Hom}_{pt.}\left(Y, \mathrm{Pic}^{0}_{A/k}\right).\end{aligned}$$

And

$$\mathrm{Hom}_{pt.}\left(Y, \mathrm{Pic}^{0}_{A/k}\right) \subseteq \mathrm{Hom}_{pt.}\left(X, \mathrm{Pic}^{0}_{A/k}\right).$$

As $X_{\overline{k}} \cong \mathbb{A}^d_{\overline{k}}$, the morphisms from $X_{\overline{k}}$ into $\mathrm{Pic}^0_{A_{\overline{k}}/\overline{k}}$ are constant (Lemma 1.3). And, as $\mathrm{Pic}^0_{A/k} \times_k \overline{k} = \mathrm{Pic}^0_{A_{\overline{k}}/\overline{k}}$, the set $\mathrm{Hom}_{pt.}\left(X, \mathrm{Pic}^0_{A/k}\right)$ is reduced to the constant morphism, and finally $\mathrm{Hom}_{grp.}\left(A, \mathrm{Pic}^{+\,0}_{Y/k}\right) = \{0\}$.

A *k-semi-abelian variety* is a k-algebraic group obtained as the extension of a k-abelian variety by a k-torus. Any commutative k-algebraic group G has a largest k-semi-abelian subvariety G_{sab} [Bri17, Lem. 5.6.1]. Moreover, if G is smooth and connected, then G/G_{sab} is unipotent [Bri17, Th. 5.6.3].

Let us denote $\left(\mathrm{Pic}^{+\,0}_{Y/k}\right)_{sab}$ as H. According to Remark 1.4, H is a k-abelian variety. Moreover, we just show that there is no nonconstant morphism from an abelian variety into $\mathrm{Pic}^{+\,0}_{Y/k}$. Thus $H = \{0\}$, and $\mathrm{Pic}^{+\,0}_{Y/k}$ is a unipotent k-group. Finally, according to Lemma 1.3, it is k-wound. □

1.2 Separably closed case

In this Subsection, we prove Theorem 1.1 assuming k is separably closed. We denote by $\overline{X}$ a regular completion of X.

Let T be a smooth k-scheme, then we note:

$$T = \coprod_{i \in I} T_i,$$

where the T_i are the open irreducible component of T. Then for all i, the k-scheme $\overline{X} \times_k T_i$ and $X \times_k T_i$ are regular [EGAIV2, Pro. 6.8.5 (i)] and irreducible [EGAIV2, Cor. 4.5.8 (i)]. Thus, we can identify the class group of $X \times_k T_i$ with its Picard group, and likewise for $\overline{X} \times_k T_i$ [Liu06, Pro. 7.2.16].

Moreover, we denote $D = \overline{X} \setminus X$. Then, D is pure of codimension 1 in $\overline{X}$ [EGAIV4, Cor. 21.12.7]. Thus, D the union of a finite number of divisors of $\overline{X}$; we denote:

$$D = \overline{X} \setminus X = \bigcup_{j=1}^{n} D_j.$$

We consider D and the D_j with their structure of reduced closed subscheme.

As $k = k_s$, the D_j are geometrically irreducible. For any i, we have an exact sequence:

$$\bigoplus_{j=0}^{n} \mathbb{Z}[D_j \times_k T_i] \to \mathrm{Cl}\left(\overline{X} \times_k T_i\right) \to \mathrm{Cl}\left(X \times_k T_i\right) \to 0.$$

Indeed, the morphism $\mathrm{Cl}\left(\overline{X} \times_k T_i\right) \to \mathrm{Cl}\left(X \times_k T_i\right)$ is the restriction of Weil divisor of $\overline{X} \times_k T_i$ to $X \times_k T_i$, so it is surjective. Its kernel is generated by the class of the irreducible divisor of $\overline{X} \times_k T_i \setminus X \times_k T_i = D \times_k T_i$, so by the $[D_j \times_k T_i]$ for $1 \leqslant j \leqslant n$.

Moreover, the kernel of the morphism

$$\bigoplus_{j=0}^{n} \mathbb{Z}[D_j \times_k T_i] \to \mathrm{Cl}\left(\overline{X} \times_k T_i\right)$$

is generated by the principal divisor of $\overline{X} \times_k T_i$ without zero, nor pole outside of $D \times_k T_i$, so by the image of the morphism

$$\mathrm{div} : \kappa\left(\overline{X} \times_k T_i\right) \to \mathrm{Div}(\overline{X} \times_k T_i)$$

restricted to $\mathcal{O}\left(X \times_k T_i\right)^*$. Let us show that $\mathcal{O}\left(X \times_k T_i\right)^* = \mathcal{O}\left(T_i\right)^*$. Indeed,

$$\mathcal{O}\left(T_i\right)^* \subset \mathcal{O}\left(X \times_k T_i\right)^* \subset \mathcal{O}\left(X_{\overline{k}} \times_{\overline{k}} T_{i\overline{k}}\right)^*.$$

But $X_{\overline{k}} \cong \mathbb{A}^d_{\overline{k}}$, and we can assume that T_i is affine. As $R^* = R[x_1, \ldots, x_n]^*$ for any integral domain R, then $\mathcal{O}\left(X_{\overline{k}} \times_{\overline{k}} T_{i\overline{k}}\right)^* = \mathcal{O}\left(T_{i\overline{k}}\right)^*$. So $\bigoplus_{j=0}^{n} \mathbb{Z}[D_j \times_k T_i] \to \mathrm{Cl}\left(X \times_k T_i\right)$ is an injective group morphism.

Hence, the sequence

$$0 \to \mathbb{Z}^n \xrightarrow{f} \mathrm{Cl}\left(\overline{X} \times_k T_i\right) \to \mathrm{Cl}\left(X \times_k T_i\right) \to 0$$

is exact, where f is the application which map the j-th element of the canonical base of $\mathbb{Z}^n$ to the class of the Weil divisor $[D_j \times_k T_i]$.

We identify the Weil class group and the Picard group, and obtain the exact sequence:

$$0 \to \mathbb{Z}^n \to \mathrm{Pic}\left(\overline{X} \times_k T_i\right) \to \mathrm{Pic}\left(X \times_k T_i\right) \to 0.$$

Moreover, by combining all these sequences for $i \in I$, we obtain the exact sequence:

$$0 \to \prod_{i \in I} \mathbb{Z}^n \to \mathrm{Pic}\left(\overline{X} \times_k T\right) \to \mathrm{Pic}\left(X \times_k T\right) \to 0.$$

We denote $p_2 : X \times_k T \to T$ and $q_2 : \overline{X} \times_k T \to T$ the second projections. The intersection of the image of $q_2^* : \mathrm{Pic}(T) \to \mathrm{Pic}\left(\overline{X} \times_k T\right)$ with the image of

$$\prod_{i \in I} \mathbb{Z}^n \to \mathrm{Pic}\left(\overline{X} \times_k T\right)$$

is trivial. Thus the sequence:

$$0 \to \prod_{i \in I} \mathbb{Z}^n \to \frac{\mathrm{Pic}\left(\overline{X} \times_k T\right)}{q_2^* \mathrm{Pic}(T)} \to \frac{\mathrm{Pic}\left(X \times_k T\right)}{p_2^* \mathrm{Pic}(T)} \to 0 \tag{1.1}$$

is exact.

By hypothesis $k = k_s$, so X has a k-rational point and likewise $\overline{X}$. According to Proposition [BLR90, 8.4 Pro. 1],

$$\mathrm{Pic}_{\overline{X}/k}\left(T\right) = \frac{\mathrm{Pic}\left(\overline{X} \times_k T\right)}{q_2^* \mathrm{Pic}\left(T\right)}.$$

We denote $\mathbb{Z}_k^n$ the constant k-locally algebraic group associated to the formal group $\mathbb{Z}^n$. For any j, the divisor $[D_j]$ defined a k-rational point of $\mathrm{Pic}_{\overline{X}/k}$. The morphism $\mathbb{Z}_k^n \to \mathrm{Pic}_{\overline{X}/k}$ induced by the D_j is a monomorphism between two k-locally algebraic groups, thus it is a closed immersion [SGAIII1, VIB Cor. 1.4.2].

We can rewrite the sequence (1.1) as:

$$0 \to \mathbb{Z}_k^n(T) \to \mathrm{Pic}_{\overline{X}/k}(T) \to \mathrm{Pic}^+_{X/k}(T) \to 0.$$

We now consider $\mathrm{Pic}^+_{\overline{X}/k}$ the smoothification of $\mathrm{Pic}_{\overline{X}/k}$. As $\mathbb{Z}_k^n$ is smooth, the morphism $\mathbb{Z}_k^n \to \mathrm{Pic}_{\overline{X}/k}$ factorizes as

$$\begin{array}{ccc} \mathbb{Z}_k^n & \longrightarrow & \mathrm{Pic}_{\overline{X}/k} \\ \downarrow & \nearrow & \\ \mathrm{Pic}^+_{\overline{X}/k}. & & \end{array}$$

As $\mathbb{Z}_k^n \to \mathrm{Pic}_{\overline{X}/k}$ is a closed immersion, $\mathbb{Z}_k^n \to \mathrm{Pic}^+_{\overline{X}/k}$ is also a closed immersion. The fppf quotient $\mathrm{Pic}^+_{\overline{X}/k}/\mathbb{Z}_k^n$ is representable by a k-locally algebraic group [SGAIII1, VIA Th. 3.3.2]; we denote G this quotient.

We are going to show that $\mathbb{Z}_k^n \cap \mathrm{Pic}^{+\,0}_{\overline{X}/k} = \{0\}$. Indeed, $\mathrm{Pic}^{+\,0}_{\overline{X}/k}$ is a unipotent k-group (Proposition 1.5), so it is of p^m-torsion for some $m > 0$, and the closed points of $\mathbb{Z}_k^n \cap \mathrm{Pic}^{+\,0}_{\overline{X}/k}$ are of p^m-torsion. But $\mathbb{Z}_k^n \to \mathrm{Pic}^+_{\overline{X}/k}$ is a monomorphism, thus $\mathbb{Z}_k^n \cap \mathrm{Pic}^{+0}_{\overline{X}/k} = \{0\}$.

There is a commutative diagram of k-locally algebraic group:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \mathrm{Pic}^{+\,0}_{\overline{X}/k} & \xrightarrow[\varphi]{\sim} & G^0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_k^n & \longrightarrow & \mathrm{Pic}^+_{\overline{X}/k} & \longrightarrow & G & \longrightarrow & 0 \\ & & \wr\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_k^n & \longrightarrow & \pi_0\left(\mathrm{Pic}^+_{\overline{X}/k}\right) & \longrightarrow & \pi_0(G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array}$$

where the three columns and the second line are exact. Moreover, the morphism φ is a closed immersion (it is a monomorphism) which is faithfully flat, thus φ is an isomorphism. So $G^0 \cong \mathrm{Pic}^{+0}_{\overline{X}/k}$ is a k-wound commutative unipotent k-group. As $k = k_s$, all the groups of the third line are constant groups, in order to show that the third line is an exact sequence it is enough to show that the sequence induce on the $\overline{k}$-points is exact, this is a consequence of the snake Lemma.

Next, we prove that G represents the restricted Picard functor. In order to prove that for all smooth k-schemes T, we have $G(T) = \mathrm{Pic}^+_{X/k}(T)$, we obtain the existence of a section of the quotient morphism $\mathrm{Pic}^+_{\overline{X}/k} \to G$. To choose such a section, one just need to choose a section of $\pi_0\left(\mathrm{Pic}^+_{\overline{X}/k}\right) \to \pi_0(G)$. Indeed, if s is such a section, the morphism $\mathrm{Pic}^+_{\overline{X}/k} \to G$ induces a morphism $f_\nu : \mathrm{Pic}^{+s(\nu)}_{\overline{X}/k} \to G^\nu$ where G^ν is the fiber in $\nu \in \pi_0$ of $G \to \pi_0(G)$ (likewise for $\mathrm{Pic}^{+s(\nu)}_{\overline{X}/k}$). As $k = k_s$, the $\mathrm{Pic}^{+\,0}_{\overline{X}/k}$-torsor $\mathrm{Pic}^{+s(\nu)}_{\overline{X}/k}$ is the trivial one, and likewise

G^ν is a trivial G^0-torsor. Thus, the morphism f_ν is an isomorphism. Hence, we obtain a morphism $\coprod f_\nu^{-1} : G \to \mathrm{Pic}^+_{\overline{X}/k}$ which is a section of $\mathrm{Pic}^+_{\overline{X}/k} \to G$. Finally, choosing a section of $\pi_0\left(\mathrm{Pic}^+_{\overline{X}/k}\right) \to \pi_0(G)$ is the same as, choosing a section of $\pi_0\left(\mathrm{Pic}^+_{\overline{X}/k}\right)(k) \to \pi_0(G)(k)$ [DG70, II.5.1.7] (as $k = k_s$ any section is equivariant for $\mathrm{Gal}(k_s/k) = \{0\}$).

The last thing to prove, is that G^+ is of finite type over k, or equivalently that $\pi_0(G^+)(k)$ is a finite set. There is a purely inseparable extension K of k such that $X_K \cong \mathbb{A}^n_K$; so $\mathrm{Pic}(X) = \mathrm{Pic}^+_{X/k}(k) = G(k)$ is of p^m-torsion for some $m \geqslant 0$ [Ach17, Pro. 2.6]. Moreover, there is a closed immersion $\mathrm{Pic}^+_{\overline{X}/k} \to \mathrm{Pic}_{\overline{X}/k}$. By universal property of the scheme of connected components, this closed immersion induces a morphism of constant groups $\pi_0\left(\mathrm{Pic}^+_{\overline{X}/k}\right) \to \pi_0(\mathrm{Pic}_{\overline{X}/k})$. The kernel of this morphism corresponds to connected component of $\mathrm{Pic}^+_{\overline{X}/k}$ which are closed subscheme of $\mathrm{Pic}^0_{\overline{X}/k}$; as $\mathrm{Pic}^0_{\overline{X}/k}$ is of finite type over k, this kernel is finite. According to Neron-Severi Theorem [SGAVI, XIII Th. 5.1], the abelian group $\pi_0\left(\mathrm{Pic}_{\overline{X}/k}\right)(k)$ is of finite type. Thus, $\pi_0\left(\mathrm{Pic}^+_{\overline{X}/k}\right)(k)$ is the extension of a commutative group of finite type by a finite commutative group, therefore it is a commutative group of finite type. Hence, $\pi_0(G)(k)$ is a commutative group of finite type of p^m-torsion, so it is a finite unipotent k-group [DG70, Exa. b) IV.2.2.2].

1.3 General case

In this subsection, we are using a Galois descent argument to show the representability of $\mathrm{Pic}^+_{X/k}$ without hypothesis on the base field.

First, we define an action of $\Gamma = \mathrm{Gal}(k_s/k)$ on $\mathrm{Pic}^+_{X_{k_s}/k_s}$. The argument is standard, any $\sigma \in \Gamma$ induces a k-automorphism $\mathrm{Spec}(k_s) \to \mathrm{Spec}(k_s)$ that we will also denote σ. Let T be a smooth k_s-scheme, we have a commutative diagram of k-scheme:

$$\begin{array}{ccc}
X_{k_s} \times_{k_s} T & \xrightarrow{id_{X\times T}\times\sigma} & X_{k_s} \times_{k_s} T \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2} \\
T & \xrightarrow{id_T\times\sigma} & T \\
\downarrow & & \downarrow \\
\mathrm{Spec}(k_s) & \xrightarrow[\sigma]{} & \mathrm{Spec}(k_s).
\end{array}$$

Then, we have the following commutative diagram of abelian group:

$$\begin{array}{ccc}
\mathrm{Pic}(X_{k_s} \times_{k_s} T) & \xrightarrow{(id_{X\times T}\times\sigma)^*} & \mathrm{Pic}(X_{k_s} \times_{k_s} T) \\
\uparrow{\scriptstyle p_2^*} & & \uparrow{\scriptstyle p_2^*} \\
\mathrm{Pic}(T) & \xrightarrow[(id_T\times\sigma)^*]{} & \mathrm{Pic}(T).
\end{array}$$

Thus, σ induces a group morphism denoted σ_T^*:

$$\sigma_T^* : \mathrm{Pic}^+_{X_{k_s}/k_s}(T) \to \mathrm{Pic}^+_{X_{k_s}/k_s}(T).$$

Hence, according to Yoneda Lemma, σ induces a k-morphism between k_s-groups denoted σ^*:

$$\sigma^* : \mathrm{Pic}^+_{X_{k_s}/k_s} \to \mathrm{Pic}^+_{X_{k_s}/k_s}.$$

As the k_s-group $\mathrm{Pic}^+_{X_{k_s}/k_s}$ is affine, there is a k_s-algebra of finite type A such that $\mathrm{Pic}^+_{X_{k_s}/k_s} = \mathrm{Spec}(A)$. Let $V \subset A$ be a k_s-vector space of finite dimension such that V generate A as k_s-algebra, then there is a k_s-vector space of finite dimension W such that $V \subset W \subset A$ and W is stable under the Γ-action (indeed, any element of a base of W is stabilized by a subgroup of Γ of finite index). Thus, there is a Γ-equivariant isomorphism between A and $\mathrm{Sym}(W)/I$ where I is a Γ-stable ideal of $\mathrm{Sym}(W)$. Moreover, $\mathrm{Sym}(W) \cong \mathcal{O}(W^\vee)$ (where $W^\vee$ denotes the dual of W), thus the k_s-scheme $\mathrm{Pic}^+_{X_{k_s}/k_s}$ identifies Γ-equivariantly to the closed space of the affine space $W^\vee$ defined by I. Finally, there is a finite separable extension L/k and a L-scheme Y such that

$$\mathrm{Pic}^+_{X_{k_s}/k_s} \cong Y \times_L \mathrm{Spec}(k_s)$$

is a Γ-equivariant isomorphism of k_s-schemes.

We can assume that L/k is a Galois extension. Thus, we have an action of $\mathrm{Gal}(L/k)$ on Y compatible with the action of $\mathrm{Gal}(L/k)$ on $\mathrm{Spec}(L)$. This action defines a descent datum on Y [BLR90, 6.2 Exa. B]. Moreover, as Y is an affine L-scheme (Y_{k_s} is affine, so Y is affine) the descent is effective [BLR90, 6.1 Th. 6]. Hence, we obtain a k-scheme Z such that $Z \times_k \mathrm{Spec}(L) \cong Y$.

We need to prove that for any smooth k-scheme of finite type T:

$$Z(T) = \left(\mathrm{Pic}^+_{X_{k_s}/k_s}(T_{k_s})\right)^\Gamma.$$

Let $f : T_{k_s} \to Z_{k_s}$ be a Γ-equivariant morphism. As T is a k-scheme of finite type and Z is an affine k-scheme of finite type, there is a finite separable extension M/k and a morphism $g : T_M \to Z_M$ such that the diagram

$$\begin{array}{ccc} T_{k_s} & \xrightarrow{f} & Z_{k_s} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ T_M & \xrightarrow[g]{} & Z_M \end{array}$$

is commutative [EGAIV2, Pro. 4.8.13], where $p : T_{k_s} \to T_M$ and $q : Z_{k_s} \to Z_M$ are the projections. As the morphism g is $\mathrm{Gal}(M/k)$-equivariant, there is a k-morphism $h : T \to Z$ such that the diagram

$$\begin{array}{ccc} T_M & \xrightarrow{g} & Z_M \\ \downarrow & & \downarrow \\ T & \xrightarrow[h]{} & Z \end{array}$$

is commutative [BLR90, Th. 6 (a)]. And, by construction of Z, we have

$$Z(T) \subset \left(\mathrm{Pic}^+_{X_{k_s}/k_s}(T_{k_s})\right)^\Gamma.$$

Moreover, the exact sequence

$$0 \to \mathrm{Pic}(T_{k_s}) \overset{p_2^*}{\to} \mathrm{Pic}(X_{k_s} \times_{k_s} T_{k_s}) \to \frac{\mathrm{Pic}(X_{k_s} \times_{k_s} T_{k_s})}{p_2^*\mathrm{Pic}(T_{k_s})} \to 0,$$

admits a Γ-equivariant retraction. Indeed, let $e \in X_{k_s}$, then

$$p_2^* : \mathrm{Pic}(T_{k_s}) \to \mathrm{Pic}(X_{k_s} \times_{k_s} T_{k_s})$$

admits

$$(e \times id_T)^* : \mathrm{Pic}(X_{k_s} \times_{k_s} T_{k_s}) \to \mathrm{Pic}(T_{k_s})$$

as a Γ-equivariant retraction. Hence

$$\mathrm{Pic}^+_{X_{k_s}/k_s}(T_{k_s})^\Gamma = \frac{\mathrm{Pic}(X_{k_s} \times_{k_s} T_{k_s})^\Gamma}{p_2^*\mathrm{Pic}(T_{k_s})^\Gamma}.$$

Next, we consider the exact sequence of low degree of Hochschild-Serre spectral sequence [SGAIV2, VIII Cor. 8.5]:

$$0 \to \mathrm{H}^1\left(k, \mathcal{O}(X_{k_s} \times_{k_s} T_{k_s})^*\right) \to \mathrm{Pic}(X \times_k T) \to \mathrm{Pic}(X_{k_s} \times_{k_s} T_{k_s})^\Gamma \to \mathrm{H}^2\left(k, \mathcal{O}(X_{k_s} \times_{k_s} T_{k_s})^*\right),$$

and

$$0 \to \mathrm{H}^1\left(k, \mathcal{O}(T_{k_s})^*\right) \to \mathrm{Pic}(T) \to \mathrm{Pic}(T_{k_s})^\Gamma \to \mathrm{H}^2\left(k, \mathcal{O}(T_{k_s})^*\right).$$

As $\mathcal{O}(X_{k_s})^* \cong k_s^*$, we have $\mathcal{O}(X_{k_s} \times_{k_s} T_{k_s})^* \cong \mathcal{O}(T_{k_s})^*$. Thus

$$\mathrm{Pic}^+_{X_{k_s}/k_s}(T_{k_s})^\Gamma = \frac{\mathrm{Pic}(X_{k_s} \times_{k_s} T_{k_s})^\Gamma}{p_2^*\mathrm{Pic}(T_{k_s})^\Gamma} = \frac{\mathrm{Pic}(X \times_k T)}{p_2^*\mathrm{Pic}(T)} = \mathrm{Pic}^+_{X/k}(T).$$

Finally, for any smooth k-scheme T of finite type,

$$Z(T) = \mathrm{Pic}^+_{X_{k_s}/k_s}(T_{k_s})^\Gamma = \mathrm{Pic}^+_{X/k}(T).$$

More generally, if T is a smooth k-scheme, then $T = \coprod_{i \in I} T_i$ where the T_i are smooth k-schemes of finite type. And,

$$Z(T) = \prod_{i \in I} Z(T_i) = \prod_{i \in I} \mathrm{Pic}^+_{X/k}(T_i) = \mathrm{Pic}^+_{X/k}(T).$$

Thus Z represents the group functor $\mathrm{Pic}^+_{X/k}$. This concludes the proof of Theorem 1.1.

2 First properties and examples

2.1 Restricted Picard functor and projective limit

In this subsection, we denote a projective limit (also call inverse limit) by $\varprojlim$ and an inductive limit (also call direct limit) by $\varinjlim$.

Proposition 2.1. *Let X be a form of $\mathbb{A}^n_k$ which admits a regular completion.*

Let $(T_i, f_{ij})_I$ be a projective system of smooth k-schemes of finite type such that for any $i \leqslant j \in I$, the k-scheme morphism $f_{ij} : T_i \to T_j$ is affine. Then:

(i) the projective limits $T = \varprojlim_i T_i$ exist in the category of k-schemes;

(ii) $\mathrm{Pic}(T) = \varinjlim_i \mathrm{Pic}(T_i)$ *and* $\mathrm{Pic}(X \times_k T) = \varinjlim_i \mathrm{Pic}(X \times_k T_i)$*;*

(iii) $\mathrm{Pic}^+_{X/k}(T) = \frac{\mathrm{Pic}(X \times_k T)}{p_2^*\mathrm{Pic}(T)}$.

Proof. (i) and (ii) are consequences of [Sta18, Tag 01YX] and [Sta18, Tag 0B8W].

Let us show (iii): first of all, the functor $\mathrm{Pic}^+_{X/k}$ is representable (Theorem 1.1). Thus $\mathrm{Pic}^+_{X/k}(T)$ is well defined and:

$$\mathrm{Pic}^+_{X/k}(T) = \varinjlim_i \mathrm{Pic}^+_{X/k}(T_i) = \varinjlim_i \frac{\mathrm{Pic}(X \times_k T_i)}{p_2^*\mathrm{Pic}(T_i)} = \frac{\varinjlim_i \mathrm{Pic}(X \times_k T_i)}{p_2^*\varinjlim_i \mathrm{Pic}(T_i)} = \frac{\mathrm{Pic}(X \times_k T)}{p_2^*\mathrm{Pic}(T)}.$$

The first equality come from [Sta18, Tag 01ZC], the second equality is the definition of $\mathrm{Pic}^+_{X/k}$, the third one is a consequence of the fact that an inductive limit of exact sequence is still exact, the last equality is (ii). □

Remark 2.2. Let X be a form of $\mathbb{A}^n_k$ with a regular completion.

(i) Recall that $k_s = \varinjlim_\lambda k_\lambda$ where k_λ are the finite separable extensions of k. So according to Proposition 2.1:

$$\mathrm{Pic}^+_{X/k}(k_s) = \mathrm{Pic}(X_{k_s}).$$

(*ii*) More generally, if F/k is any separable extension, then $F = \lim_{\rightarrow \lambda} k_\lambda$ where $F/k_\lambda/k$ are finite separable extension of k. So according to Proposition 2.1:

$$\mathrm{Pic}^+_{X/k}(F) = \mathrm{Pic}(X_F).$$

Remark 2.3. (*i*) A nontrivial form of $\mathbb{A}^1_k$ never becomes isomorphic to the affine line after a separable extension. T. Kambayashi proved the forms of $\mathbb{A}^2_k$ have the same properties [Kam75, Th. 3]. In the particular case of unipotent k-group, in any dimension, a unipotent k-group is k-wound if and only if it is L-wound (for any separable non necessarily algebraic field extension L/k) [CGP15, Pro. B.3.2].

But, in dimension 3, there is still no proof that $\mathbb{A}^3_{\mathbb{R}}$ is the only real form of $\mathbb{A}^3_{\mathbb{C}}$ [Kra95, Rem. 5.4].

(*ii*) If X is a form of $\mathbb{A}^n_k$ with a regular completion, then for any separable extension L/k, the group $\mathrm{Pic}(X)$ is a subgroup of $\mathrm{Pic}(X_L)$. Thus, if $\mathrm{Pic}^+_{X/k} \neq \{0\}$, then X does not become isomorphic to $\mathbb{A}^n_k$ after any separable extension.

2.2 Examples

Example 2.4. Let P_∞ be a point of $\mathbb{P}^1_k$ such that $\kappa(P_\infty)/k$ is a purely inseparable extension. We consider $X = \mathbb{P}^1_k \setminus \{P_\infty\}$, then X is a form of $\mathbb{A}^1_k$. And $\mathrm{Pic}^+_{X/k}$ is representable by the constant group $(\mathbb{Z}/[\kappa(P_\infty) : k]\mathbb{Z})_k$.

Example 2.5. Let X be a form of $\mathbb{A}^1_k$, we denote by $\overline{X}$ its (canonical) regular completion. Then $\mathrm{Pic}^+_{X/k}$ is representable by a unipotent k-group.

Moreover, if X has a k-rational point, then

$$0 \to \mathrm{Pic}^0_{\overline{X}/k} \to \mathrm{Pic}^+_{X/k} \to (\mathbb{Z}/[\kappa(P_\infty) : k]\mathbb{Z})_k \to 0.$$

is an exact sequence of algebraic group (where $\{P_\infty\} = \overline{X} \setminus X$, see [Rus70, Lem. 1.1]). Thus, the neutral component of $\mathrm{Pic}^+_{X/k}$ is $\mathrm{Pic}^0_{\overline{X}/k}$ and the group of irreducible component of $\mathrm{Pic}^+_{X/k}$ is the constant k-group $(\mathbb{Z}/[\kappa(P_\infty) : k]\mathbb{Z})_k$.

Example 2.6. We consider $k = \mathbb{F}_3(a)$. We denote G^t the form of $\mathbb{G}_{a,k}$ define as a subgroup of $\mathbb{G}^2_{a,k}$ by the equation $y^3 = x + tx^3$. We consider the regular completion $\overline{G^t}$ of G^t.

If t is not in k^3, then there is a closed immersion $G^t \to \mathrm{Pic}^0_{\overline{G^t}/k}$ [KMT74, Th. 6.7.9]. Moreover, in this case $\dim \mathrm{Pic}^0_{\overline{G^t}/k} = 1$, so $G^t \cong \mathrm{Pic}^0_{\overline{G^t}/k} = \mathrm{Pic}^{+\,0}_{G^t/k}$.

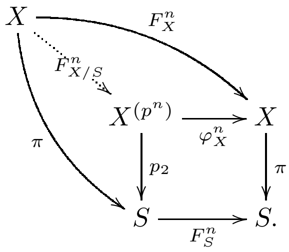
If $t = a$, then $G^t(k) = \{0\}$, so $\mathrm{Pic}(G^t) = \mathbb{Z}/3\mathbb{Z}$. Let n be an integer, we consider $q = p^n$ and $t = a + a^{q+2} + a^{2q+3} + \cdots + a^{(p-2)q+p-1}$. Then $G^t(k)$ is a finite group of cardinal greater than $p^{2^n/2n}$ [AV96, Th. 4.1].

As $\mathrm{Pic}\,(G^t)$ is an extension of $\mathbb{Z}/3\mathbb{Z}$ by $G^t(k)$, the group $\mathrm{Pic}\,(G^t)$ is finite of cardinal as large as one may wish.

Thus, we have obtain a family of forms of $\mathbb{G}_{a,k}$ with finite Picard group of unbounded cardinal.

We recall that there is two Frobenius morphisms. The n-th-*absolute Frobenius morphism* denoted $F^n_X : X \to X$ is the identity on the topological space and is the p^n power on the structural sheaves.

Let $\pi : X \to S$ be a morphism of $\mathbb{F}_p$-schemes. We denote $X^{(p^n)}$ the product $X \times_S S$, where S is seen as a S scheme via F^n_S. The second projection $p_2 : X^{(p^n)} \to S$ defined a structure of S-scheme on $X^{(p^n)}$. Moreover, we denote $\varphi^n_X : X^{(p^n)} \to X$ the first projection. Then, we define the n-th *relative Frobenius morphism* denoted $F^n_{X/S}$ with the universal property of the Cartesian product:



We obtain in a previous article two theorems, one about the Picard group of the form of $\mathbb{A}^1_k$ (Theorem [Ach17, Th. 2.4]), the other concerns the Picard functor of their regular completion (Theorem [Ach17, Th. 4.4]). We translate them in terms of restricted Picard functor. First recall some definitions from [Ach17, §1.1].

Definition 2.7. Let X be a form of $\mathbb{A}^1_k$. We consider the two following invariants:

(i) we denote by $n(X)$ the smallest non negative integer such that $X^{(p^n)} \cong \mathbb{A}^1_k$;

(ii) we denote by $n'(X)$ the smallest non negative integer such that $\kappa\left(X^{(p^n)}\right) \cong k(t)$.

Proposition 2.8. *We consider X a form of $\mathbb{A}^1_k$. Then, $\mathrm{Pic}^+_{X/k}$ is representable by a unipotent k-group of $p^{n(X)}$-torsion whose neutral component is of $p^{n'(X)}$-torsion.*

If X is a nontrivial form of $\mathbb{A}^1_k$, then $\mathrm{Pic}^+_{X/k} \neq \{0\}$.

Proof. See [Ach17, Th. 2.4 and Th. 4.4]. □

Example 2.9. We consider $k = \mathbb{F}_p(a, b)$. B. Totaro proved that if U is the subgroup of $\mathbb{G}^3_{a,k}$, defined by the equation

$$x + ax^p + by^p + z^p = 0,$$

then $\mathrm{Pic}(U_{k_s}) = \{0\}$ [Tot13, Exa. 9.7].

In order to show this, he considers the regular completion

$$X = \left\{[x, y, z, w] \in \mathbb{P}^3_k \mid xw^{p-1} + ax^p + by^p + z^p = 0\right\}$$

of U. Thus U satisfies the hypothesis of Theorem 1.1, so $\mathrm{Pic}^+_{U/k}$ is a k-group such that $\mathrm{Pic}^+_{U/k}(k_s) = \{0\}$. By density of the k_s-rational points, $\mathrm{Pic}^+_{U/k} = \{0\}$.

If $p = 2$, then U is a rational unipotent k-group.

Else $p > 2$, we are going to show that the only unirational subgroup of U is the trivial one. Let $k' = k[b^{1/p}]$, then $U_{k'}$ is k'-isomorphic to $\mathbb{G}_{a,k'} \times_{k'} G$ where G is the subgroup of $\mathbb{G}^2_{a,k'}$ defined by the equation $x + ax^p + y^p = 0$. If U was unirational, then G would also be unirational. But G is a nontrivial form of $\mathbb{G}_{a,k'}$ in characteristics greater than 2, thus G is not a rational curve [KMT74, Th. 6.9.2] (so it is not a unirational curve). Moreover, U does not admit a dimension 1 unirational subgroup as such a subgroup would be a nontrivial form of $\mathbb{G}_{a,k}$.

The example above is a k-wound unipotent group of dimension 2 with a trivial Picard group, such a group does not exist in dimension 1.

2.3 First properties of the restricted Picard functor

In this subsection, we look at some well known properties of the Picard group and we translate them in properties of the restricted Picard functor.

Remark 2.10. If X is a form of $\mathbb{A}^d_k$ and Y is a form of $\mathbb{A}^e_k$ then any k-morphism $f : X \to Y$ induce a natural transformation $f^* : \mathrm{Pic}^+_{Y/k} \to \mathrm{Pic}^+_{X/k}$.

Moreover, if $\mathrm{Pic}^+_{Y/k}$, and $\mathrm{Pic}^+_{X/k}$ are representable, then f induces a morphism of k-algebraic groups also denoted $f^* : \mathrm{Pic}^+_{Y/k} \to \mathrm{Pic}^+_{X/k}$.

Lemma 2.11 (Homotopy invariance)**.**

Let X be a form of $\mathbb{A}_k^d$, we denote $p_1 : X \times_k \mathbb{A}_k^1 \to X$ the first projection. Then, the morphism p_1 induces a natural isomorphism p_1^* between the functors $\mathrm{Pic}^+_{X\times\mathbb{A}^1/k}$ and $\mathrm{Pic}^+_{X/k}$.

Moreover, if X admits a regular completion, then the functors $\mathrm{Pic}^+_{X/k}$ and $\mathrm{Pic}^+_{X\times\mathbb{A}^1/k}$ are both representable, and p_1 induces an isomorphism of k-groups:

$$p_1^* : \mathrm{Pic}^+_{X/k} \to \mathrm{Pic}^+_{X\times\mathbb{A}^1/k}.$$

Proof. Let T be a smooth k-scheme, we denote $p_T : X \times_k \mathbb{A}_k^1 \times_k T \to X \times_k T$ the projection. We have a commutative diagram of abelian group:

$$\begin{array}{ccc} \mathrm{Pic}(X \times_k T) & \xrightarrow{p_T^*} & \mathrm{Pic}(X \times_k \mathbb{A}_k^1 \times_k T) \\ \uparrow & & \uparrow \\ \mathrm{Pic}(T) & \xrightarrow[\sim]{} & \mathrm{Pic}(T). \end{array}$$

Moreover p_T^* is injective, indeed any k-rational point $e \in \mathbb{A}_k^1(k)$ induces a section of p_T^*:

$$e^* : \mathrm{Pic}(X \times_k \mathbb{A}_k^1 \times_k T) \to \mathrm{Pic}(X \times_k T).$$

And p_T^* is surjective [EGAIV4, Cor. 21.4.11], so p_T^* is indeed a bijection. Hence, the morphism $p_1 : X \times_k \mathbb{A}_k^1 \to X$ induces a natural isomorphism $p_1^* : \mathrm{Pic}^+_{X\times\mathbb{A}^1/k} \to \mathrm{Pic}^+_{X/k}$.

Let $\overline{X}$ be a regular completion of X, then $\overline{X} \times_k \mathbb{P}_k^1$ is a regular completion of $X \times_k \mathbb{A}_k^1$. Thus, the functor $\mathrm{Pic}^+_{X/k}$ is representable (Theorem 1.1), and likewise $\mathrm{Pic}^+_{X\times\mathbb{A}^1/k}$. Finally, according to Yoneda Lemma p_1^* is an isomorphism of k-algebraic groups. □

Lemma 2.12 (Product)**.**

We consider a form X of $\mathbb{A}_k^d$ and a form Y of $\mathbb{A}_k^e$. The morphisms $p_1 : X \times_k Y \to X$ and $p_2 : X \times_k Y \to Y$ induce a natural transformation of functor of the category of smooth k-schemes into the category of groups:

$$p_1^* \times p_2^* : \mathrm{Pic}^+_{X/k} \times \mathrm{Pic}^+_{Y/k} \to \mathrm{Pic}^+_{X\times Y/k},$$

with trivial kernel, i.e. for any smooth k-scheme T, the morphism of commutative groups:

$$(p_1^* \times p_2^*)(T) : \mathrm{Pic}^+_{X/k}(T) \times \mathrm{Pic}^+_{Y/k}(T) \to \mathrm{Pic}^+_{X\times Y/k}(T)$$

is injective.

Remark 2.13. Over a perfect field of characteristic $p > 0$, there are non smooth k-algebraic group. Over a non perfect field, there are non smooth unipotent k-algebraic group of positive dimension with trivial smoothification!

Indeed, let G be the subgroup of $\mathbb{G}_{a,k}^2$ defined by the equations $x^p = ay^p$ where $a \in k \setminus k^p$. Then G is reduced but is not geometrically reduced. Moreover, $G(k_s) = \{0\}$, so the smoothification G^+ of G is trivial, but $\dim(G) = 1$.

Thus, we cannot directly deduce from Lemma 2.12 that if $\mathrm{Pic}^+_{X/k}$, and $\mathrm{Pic}^+_{Y/k}$, and $\mathrm{Pic}^+_{X\times Y/k}$ are representable, then $p_1^* \times p_2^*$ is a closed immersion.

Finally, we generalize the fact that we can restrict the study of the Picard group of unipotent groups to the k-wound case. First, we need two preliminary lemmas.

Lemma 2.14. *We consider X a k-scheme, Y an affine k-scheme and $f : X \to Y$ a $\mathbb{G}_{a,k}$-torsor.*

Then $f : X \to Y$ is the trivial torsor, in particular X is isomorphic as k-scheme to $Y \times_k \mathbb{G}_{a,k}$.

Proof. The morphism f admits local sections [Ros56, Th. 10]. So the $\mathbb{G}_{a,k}$-torsor f is locally trivial. The isomorphism classes of the locally trivial $\mathbb{G}_{a,k}$-torsors are classified by $\mathrm{H}^1(Y, \mathcal{O}_Y)$. By hypothesis Y is affine, so $\mathrm{H}^1(Y, \mathcal{O}_Y) = \{0\}$. Hence f is the trivial torsor. □

Lemma 2.15. *Let U be a unipotent k-group, we recall that there is a unique normal k-split subgroup U_{split} of U such that the quotient $U_{\mathrm{wd}} = U/U_{\mathrm{split}}$ is k-wound [CGP15, Th. B.3.4].*

Then, the quotient morphism $f : U \to U_{\mathrm{wd}}$ is a trivial U_{split}-torsor, in particular U is isomorphic as k-scheme to the product $U_{\mathrm{wd}} \times_k U_{\mathrm{split}}$.

Proof. If $U_{\mathrm{split}} = \{0\}$, there is nothing to do. Else, U_{split} is of dimension $n \geqslant 1$. As U_{split} is k-split, there is an exact sequence of k-groups

$$1 \to U_1 \cong \mathbb{G}_{a,k} \to U_{\mathrm{split}} \to \frac{U_{\mathrm{split}}}{U_1} \to 1.$$

Then, $U/U_1 \to U_{\mathrm{wd}}$ is a U_{split}/U_1-torsor, and U_{split}/U_1 is still k-split [Bor12, Th. V.15.4]. By induction the U_{split}/U_1-torsor $U/U_1 \to U_{\mathrm{wd}}$ is trivial. Moreover $U \to U/U_1$ is a $\mathbb{G}_{a,k}$-torsor, hence trivial (Lemma 2.14). Thus, the U_{split}-torsor $U \to U_{\mathrm{wd}}$ is trivial. □

Proposition 2.16 (Restriction to the k-wound case)**.**

Let U be a unipotent k-group then, the morphism $q : U \to U_{\mathrm{wd}}$ (see [CGP15, Th. B.3.4]) induce a natural isomorphism $q^* : \mathrm{Pic}^+_{U_{\mathrm{wd}}/k} \to \mathrm{Pic}^+_{U/k}$.

Proof. According to the Lemma 2.15, the U_{split}-torsor $U \to U_{\mathrm{wd}}$ is trivial. Let us recall that any k-split unipotent k-group of dimension n is isomorphic as k-scheme to $\mathbb{A}^n_k$ [DG70, Th. IV.4.4.1]. Thus U is isomorphic as k-scheme to $\mathbb{A}^n_k \times_k U_{\mathrm{wd}}$, where $n = \dim(U_{\mathrm{split}})$. The conclusion follows immediately from n applications of Lemma 2.11. □

2.4 A *dévissage* of the Picard group of unipotent groups

In this Subsection, we provide a method to obtain some explicit informations on the Picard group of a form of $\mathbb{G}^n_{a,k}$. First, we obtain a *dévissage* proposition.

Lemma 2.17. *Let $f : X \to Y$ be a fpqc morphism where X is a locally factorial (i.e. for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ is factorial) k-scheme and Y is an integral normal k-variety of function field $K = \kappa(Y)$. We assume that the fiber X_y is integral for any $y \in Y$ of codimension 1. Then,*

$$\mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \xrightarrow{g^*} \mathrm{Pic}(Z_K) \to 0$$

is an exact sequence, where $g : Z_K \to X$ is the generic fiber of f.

Proof. The exactness at $\mathrm{Pic}(X)$ is a consequence of Corollary [EGAIV4, Cor. 21.4.13]. Moreover, we can identify the Picard group of X and of Z_K with their divisor class group [EGAIV4, Cor. 21.6.10 (ii)]. And, every divisor of Z_K extends to a divisor of X [EGAI, Cor. 6.10.6], thus g^* is surjective. □

We consider G a k-group, and $f : X \to Y$ a G-torsor. We denote by $\widehat{G}$ the character group of G. Then, there are two natural group morphisms $\chi : \mathcal{O}(X)^* \to \widehat{G}$ and $\gamma : \widehat{G} \to \mathrm{Pic}(Y)$ (see e.g. [Bri15, §2.3]).

Proposition 2.18. *We consider G a k-group, and $f : X \to Y$ is a G-torsor. We assume that $f : X \to Y$ verifies the hypothesis of Lemma 2.17. Then,*

$$0 \to \mathcal{O}(Y)^* \xrightarrow{f^{\#}} \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{G} \xrightarrow{\gamma} \mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \xrightarrow{g^*} \mathrm{Pic}(Z_K) \to 0$$

is an exact sequence. And the generic fiber $Z_K \to \mathrm{Spec}(K)$ is a G_K-torsor over the field $K = \kappa(Y)$.

Proof. The sequence is exact in $\mathcal{O}(Y)^*$ and, $\mathcal{O}(X)^*$ and, $\widehat{G}$ and, $\mathrm{Pic}(Y)$ [Bri15, Pro. 2.10]. And the rest of the sequence is exact by Lemma 2.17. □

We can apply Proposition 2.18 to any exact sequence of unipotent k-groups. Thus, the exact sequence of unipotent k-groups,

$$1 \to U \to U' \to U'' \to 1,$$

imply the exact sequence of Picard groups:

$$0 \to \mathrm{Pic}(U'') \to \mathrm{Pic}(U') \to \mathrm{Pic}\left(Z_{\kappa(U')}\right) \to 0,$$

where the generic fiber $Z_{\kappa(U'')} \to \mathrm{Spec}(\kappa(U''))$ is a $U_{\kappa(U'')}$-torsor.

We are now going to focus on the particular case of a form U of $\mathbb{G}_{a,k}^n$. Then, U is isomorphic to the closed subgroup of $\mathbb{G}_{a,k}^{n+1} = \mathrm{Spec}\left(k[x_1, \ldots, x_{n+1}]\right)$ defined by a p-polynomial $P(x_1, \ldots, x_{n+1})$ [CGP15, Pro. B.1.13].

Any surjective linear morphism $f : k^{n+1} \to k^{n-1}$ induces a morphism of k-group $F : \mathbb{G}_{a,k}^{n+1} \to \mathbb{G}_{a,k}^{n-1}$. If the kernel of f does not contain the line define by the degree 1 monomial of P (we identify them with a base of k^{n+1}), then the kernel of $F_{|U}$ is a form of $\mathbb{G}_{a,k}$ and $F_{|U} : U \to \mathbb{G}_{a,k}^{n-1}$ is a fppf morphism.

Example 2.19. Let k be the field $\mathbb{F}_p(a, b)$. We consider the form U of $\mathbb{G}_{a,k}^2$ defined as the subgroup of $\mathbb{G}_{a,k}^3$ by the p-polynomial $P(x, y, z) = ax^{p^2} + b^p y^{p^2} + z^{p^2} + x$. Then,

$$0 \to G \to U \xrightarrow{(x,y,z) \mapsto y} \mathbb{G}_{a,k} \to 0$$

is an exact sequence of k-groups, where $G \subset \mathbb{G}_{a,k}^2$ is a form of $\mathbb{G}_{a,k}$ isomorphic to the subgroup of $\mathbb{G}_{a,k}^2$ defined by the equation $ax^{p^2} + z^{p^2} + x = 0$.

As shown by the example above, the situation is quite explicit. With Proposition 2.18, we have reduced the problem of the description of the Picard group of the form of $\mathbb{G}_{a,k}^n$ to the study of the Picard group of the G-torsor over a field for G any nontrivial form of $\mathbb{G}_{a,k}$.

We have an upper-bound on the torsion of the Picard group of a form X of $\mathbb{A}_k^1$, and we know that if $X(k) \neq \emptyset$, then $\mathrm{Pic}(X) \neq \{0\}$ [Ach17, Th. 2.2]. The problem is that if $X \to \mathrm{Spec}(k)$ is a G-torsor, then $X(k) \neq \emptyset$ imply that the torsor is trivial; and if U is a k-wound unipotent k-group, the G-torsor obtained by the method described above is always nontrivial.

In general, knowing if the Picard group of a form X of $\mathbb{A}_k^1$ without a rational point is trivial or not is a complicated question. We denote $\overline{X}$ the regular completion of X, and P_∞ the unique point $P_\infty = \overline{X} \setminus X$ [Rus70, Lem. 1.1]. Then

$$0 \to \mathrm{Pic}^0\left(\overline{X}\right) \to \mathrm{Pic}(X) \xrightarrow{\overline{\deg}} \mathbb{Z}/[\kappa(P_\infty) : k]\mathbb{Z}$$

is an exact sequence [Ach17, Eq. (2.1.3)]. Computing $\mathrm{Pic}^0\left(\overline{X}\right)$ is a hard, it is easier to compute the image of the degree function. Indeed, the computation of $[\kappa(P_\infty) : k]$ is easy. Thus, to obtain that the Picard group of X is nontrivial, we look for a point defined over an extension of degree strictly inferior to the degree of the point at infinity.

Example 2.20. We go back to Example 2.19.

The Picard group of U is isomorphic to the Picard group of the form $Z_{k(t)}$ of $\mathbb{A}_{k(t)}^1$ defined as a closed subscheme of $\mathbb{A}_{k(t)}^2$ by the equation $aX^{p^2} + Z^{p^2} + X = -b^p t^{p^2}$ (Proposition 2.18).

Then, $Z_{k(t)}$ does not have any $k(t)$-rational point. But, the point at infinity of the regular completion of $Z_{k(t)}$ is of degree p^2 and $(X, Z) = (0, -b^{1/p} t)$ is a point of $Z_{k(t)}$ of degree p. Thus $\mathrm{Pic}\left(Z_{k(t)}\right) \neq \{0\}$. Hence, the Picard group of U is nontrivial of p^2-torsion [Ach17, Th. 2.2 a)].

3 Extensions of a unipotent group by the multiplicative group

The main goal of this section is to generalize the following Lemma:

Lemma ([Tot13, Lem. 9.4]).

Let U be a k-wound unipotent k-group of dimension 1 over a separably closed field k. Then $\operatorname{Ext}^1(U, \mathbb{G}_{m,k}) \neq \{0\}$.

The proof of Lemma [Tot13, Lem. 9.4] relies on the fact that the neutral component of the Picard functor of the regular completion C of U is a unipotent k-group. Our main tool to generalize Lemma [Tot13, Lem. 9.4] is the neutral component of the restricted Picard functor $\operatorname{Pic}^{+\,0}_{U/k}$. It coincides with $\operatorname{Pic}^0_{C/k}$ in dimension 1 and generalise it in higher dimension (see Example 2.5).

In Subsection 3.1, we give a summary of some results of [Tot13, §9] and [Ros18]. In Subsection 3.2, we define an action of U on $\operatorname{Pic}^+_{U/k}$. In Subsection 3.3, we consider the functor ${\operatorname{Pic}^+_{U/k}}^U$ of fixed points of $\operatorname{Pic}^+_{U/k}$ under the action of U. Then, we state and prove Theorem 3.6 that generalize [Tot13, Lem. 9.4]. Finally, in Subsection 3.5, we give an *ad hoc* characterisation of the commutative k-pseudo-reductive groups among the linear commutative k-groups.

3.1 Extensions of an algebraic group by the multiplicative group

In this subsection, we gather some results on the extensions of a k-group G by the multiplicative group $\mathbb{G}_{m,k}$ from B. Totaro [Tot13] and Z. Rosengarten [Ros18].

First, we consider $\operatorname{Ext}^1(G, \mathbb{G}_{m,k})$, the set of the exact sequence of k-group scheme:

$$1 \to \mathbb{G}_{m,k} \to E \to G \to 1,$$

modulo the usual equivalence relation. Then, such extension is necessary central, so the Braer sum of two extensions is well defined; the Braer sum induce a commutative group law on $\operatorname{Ext}^1(G, \mathbb{G}_{m,k})$. Moreover, if G is commutative, then any such extension E is also commutative [Ros18, Lem. 5.3].

There is a natural map between $\operatorname{Ext}^1(G, \mathbb{G}_{m,k})$ and $\operatorname{Pic}(G)$. Indeed, we can see any extension on G by $\mathbb{G}_{m,k}$ as a $\mathbb{G}_{m,k}$-torsor over G. Moreover, the equivalence classes of such torsors are classified by the cohomology group $\mathrm{H}^1(G, \mathbb{G}_{m,k})$; and $\mathrm{H}^1(G, \mathbb{G}_{m,k})$ identifies with $\operatorname{Pic}(G)$ [DG70, Pro. III.4.4.4]. Thus, there is a morphism of commutative groups:

$$\varphi : \operatorname{Ext}^1(G, \mathbb{G}_{m,k}) \to \operatorname{Pic}(G).$$

We consider $\mathcal{L}$ an element of $\operatorname{Pic}(G)$, and $a \in G$. Then, there is two natural maps from $G_{\kappa(a)}$ into G: the projection p, and $m \circ (id_G \times \mathcal{I}_a)$ (where $\mathcal{I}_a$ is the canonical morphism $\operatorname{Spec}(\kappa(a)) \to G$ associated to a, and m is the group law of G). Then, we denote the pull-back

$$(m \circ (id_G \times \mathcal{I}_a))^* : \operatorname{Pic}(G) \to \operatorname{Pic}(G_{\kappa(a)})$$

by T_a (translation by a). We say that $\mathcal{L}$ is translation invariant if for any $a \in G$, we have $T_a(\mathcal{L}) = p^*(\mathcal{L})$ in $\operatorname{Pic}\left(G_{\kappa(a)}\right)$. We denote the subgroup of translation invariant elements in $\operatorname{Pic}(G)$ by $\operatorname{Pic}(G)^G$.

Let A be a k-abelian variety, it is well known that φ identify the group $\operatorname{Ext}^1(A, \mathbb{G}_{m,k})$ with the subgroup $\operatorname{Pic}^0(A)$ of $\operatorname{Pic}(A)$ (see [Ser12, VII §3.16 Th. 6] for the algebraically closed case and [Oor66, Th. 18.1] for the general case).

Recently, B. Totaro obtained the following analogous for commutative unipotent k-groups:

Lemma ([Tot13, Lem. 9.2]).

Let U be a commutative unipotent k-group. Then, φ identifies the group $\operatorname{Ext}^1(U, \mathbb{G}_{m,k})$ *with the subgroup of elements $\mathcal{L} \in \operatorname{Pic}(U)$ such that the translation $T_a\mathcal{L}$ is isomorphic to $\mathcal{L}$. In short:* $\operatorname{Ext}^1(U, \mathbb{G}_{m,k}) = \operatorname{Pic}(U)^U$.

The group $\operatorname{Ext}^1(U, \mathbb{G}_{m,k})$ *can also be described as the subgroup of primitive elements in* $\operatorname{Pic}(U)$*, i.e.*

$$\operatorname{Ext}^1(U, \mathbb{G}_{m,k}) = \ker\left(m^* - q_1^* - q_2^* : \operatorname{Pic}(U) \to \operatorname{Pic}(U \times_k U)\right),$$

where m is the group law of U and $q_1, q_2 : U \times_k U \to U$ are the two projections.

Z. Rosengarten has generalized the second affirmation of the above Lemma: for any k-group G, the morphism φ induced an isomorphism between $\mathrm{Ext}^1(G, \mathbb{G}_{m,k})$ and $\ker(m^* - q_1^* - q_2^* : \mathrm{Pic}(G) \to \mathrm{Pic}(G \times_k G))$ [Ros18, Pro. 5.1].

Thus, we have the following generalisation of [Tot13, Lem. 9.2]:

Proposition 3.1. *We consider a k-group G, then the morphism*

$$\varphi : \mathrm{Ext}^1(G, \mathbb{G}_{m,k}) \to \mathrm{Pic}(G)$$

identifies $\mathrm{Ext}^1(G, \mathbb{G}_{m,k})$ *with* $\mathrm{Pic}(G)^G$.

Moreover, the group $\mathrm{Ext}^1(G, \mathbb{G}_{m,k})$ *can also be described as the subgroup of primitive elements in* $\mathrm{Pic}(G)$*, meaning that*

$$\mathrm{Ext}^1(G, \mathbb{G}_{m,k}) = \ker(m^* - q_1^* - q_2^* : \mathrm{Pic}(G) \to \mathrm{Pic}(G \times_k G)),$$

where m is the group law of G and $q_1, q_2 : G \times_k G \to G$ are the two projections.

Proof. The only part left is to show that

$$\ker(m^* - q_1^* - q_2^*) = \mathrm{Pic}(G)^G.$$

As the inclusion $\ker(m^* - q_1^* - q_2^*) \subset \mathrm{Pic}(G)^G$ is obvious, let us show the other inclusion.

Let $\mathcal{L}$ be a G-invariant element of $\mathrm{Pic}(G)$. We denote

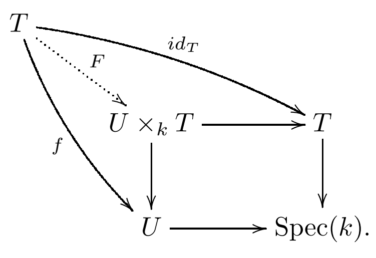
$$\mathcal{M} = m^*(\mathcal{L}) - q_1^*(\mathcal{L}) - q_2^*(\mathcal{L}) \in \mathrm{Pic}(G \times_k G).$$

We consider $g : G_{\kappa(G)} \to G \times_k G$ the generic fiber of the first projection q_1. Then, as $\mathcal{L}$ is invariant by translation by the generic point, $g^*(\mathcal{M}) = 0$ in $\mathrm{Pic}\left(G_{\kappa(G)}\right)$. So $\mathcal{M} = q_1^*(\mathcal{N})$ for some $\mathcal{N} \in \mathrm{Pic}(G)$ (Proposition 3.1). Let $e \in G(k)$ be the identity element of G, then $\mathcal{N} = (id_G \times e)^* \circ q_1^*(\mathcal{N}) = (id_G \times e)^*(\mathcal{M}) = T_e(\mathcal{L}) - \mathcal{L}$ in $\mathrm{Pic}(G)$. As $\mathcal{L}$ is invariant by translation by e, we obtain $\mathcal{N} = 0$. Thus $\mathcal{M} = 0$, and $\mathcal{L} \in \ker(m^* - q_1^* - q_2^*)$. ☐

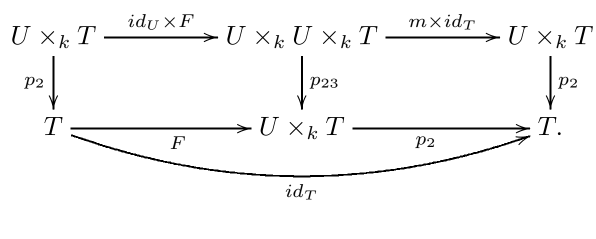
3.2 Action of U on $\mathrm{Pic}^+_{U/k}$

In this subsection, we consider a unipotent k-group U, which admits a regular completion (thus $\mathrm{Pic}^+_{U/k}$ is representable). We are going to defined an action of U on $\mathrm{Pic}^+_{U/k}$. We denote $m : U \times_k U \to U$ the group law of U.

If $f \in U(T)$, then f induces a morphism of k-scheme $F : T \to U \times_k T$ defined by the universal property of Cartesian product:



Thus we have a commutative diagram:



So $(m \times id_T) \circ (id_U \times F)$ induces by pull-back a morphism of abelian group:

$$A_T(f) : \frac{\mathrm{Pic}(U \times_k T)}{p_2^*\mathrm{Pic}(T)} \to \frac{\mathrm{Pic}(U \times_k T)}{p_2^*\mathrm{Pic}(T)}.$$

By definition of $\mathrm{Pic}^+_{U/k}$, we have

$$\frac{\mathrm{Pic}(U \times_k T)}{p_2^*\mathrm{Pic}(T)} = \mathrm{Pic}^+_{U/k}(T).$$

Thus, for any smooth k-scheme T, there is an application

$$A_T : U(T) \times \mathrm{Pic}^+_{U/k}(T) \to \mathrm{Pic}^+_{U/k}(T).$$

Next, we will show that A_T is a group action of $U(T)$ on $\mathrm{Pic}^+_{U/k}(T)$. Let f and G be two morphisms $T \to U$, then we denote $f.g$ the morphism:

$$f.g : T \xrightarrow{(f,g)} U \times_k U \xrightarrow{m} U.$$

We denote $F.G$ the k-scheme morphism $T \to U \times_k T$ induced by $f.g$. Then, we have the commutative diagram (3.1) (see next page). So, $A_T(g) \circ A_T(f) = A_T(g.f)$, and A_T is indeed an action of abstract group.

Moreover, the definition of A_T is functorial in T. Thus, we defined a natural transformation A between the functor of smooth point of the smooth k-scheme $U \times_k \mathrm{Pic}^+_{U/k}$ and $\mathrm{Pic}^+_{U/k}$. As $U \times_k \mathrm{Pic}^+_{U/k}$ and $\mathrm{Pic}^+_{U/k}$ are representable by smooth k-schemes, according to Yoneda Lemma, the morphisms A_T induced a morphism of k-schemes

$$A : U \times_k \mathrm{Pic}^+_{U/k} \to \mathrm{Pic}^+_{U/k}.$$

Moreover, the diagrams,

$$\begin{array}{ccc} U \times_k U \times_k \mathrm{Pic}^+_{U/k} & \xrightarrow{m \times id_+} & U \times_k \mathrm{Pic}^+_{U/k} \\ \downarrow{\scriptstyle id_U \times A} & & \downarrow{\scriptstyle A} \\ U \times_k \mathrm{Pic}^+_{U/k} & \xrightarrow{A} & \mathrm{Pic}^+_{U/k}, \end{array}$$

and,

$$\begin{array}{ccc} \mathrm{Pic}^+_{U/k} & \xrightarrow{e \times id_+} & U \times_k \mathrm{Pic}^+_{U/k} \\ & {\scriptstyle id_+}\searrow & \downarrow{\scriptstyle A} \\ & & \mathrm{Pic}^+_{U/k}, \end{array}$$

are diagram of smooth k-schemes that induce a commutative diagram for any smooth k-scheme T. So according to Yoneda Lemma, there are commutative diagram of k-scheme. Hence A is an action of the k-group U on $\mathrm{Pic}^+_{U/k}$.

3.3 Translation invariant restricted Picard functor

In this subsection, we interpret Lemma [Tot13, Lem. 9.2] in terms of restricted Picard functor (Proposition 3.4) and we generalize Lemma [Tot13, Lem. 9.4] (Theorem 3.6). As in the previous subsection, we consider a unipotent k-group U which admits a regular completion.

We defined two functors of fixed point.

$$
\begin{array}{ccccccccc}
U \times_k T & \xrightarrow{id_U \times F} & U \times_k U \times_k T & \xrightarrow{m \times id_T} & U \times_k T & \xrightarrow{id_U \times G} & U \times_k U \times_k T & \xrightarrow{m \times id_T} & U \times_k T \\
\Vert & & \Vert & & & & \Vert & & \Vert \\
U \times_k T & \xrightarrow{id_U \times F} & U \times_k U \times_k T & \xrightarrow{id_{U \times U} \times G} & U \times_k U \times_k U \times_k T & \xrightarrow{m \times id_{U \times T}} & U \times_k U \times_k T & \xrightarrow{m \times id_T} & U \times_k T \\
\Vert & & \Vert & & \Vert & & & & \Vert \\
U \times_k T & \xrightarrow{id_U \times F} & U \times_k U \times_k T & \xrightarrow{id_{U \times U} \times G} & U \times_k U \times_k U \times_k T & \xrightarrow{id_U \times m \times id_T} & U \times_k U \times_k T & \xrightarrow{m \times id_T} & U \times_k T \\
\Vert & & & & & & \Vert & & \Vert \\
U \times_k T & & & \xrightarrow{id_U \times F.G} & & & U \times_k U \times_k T & \xrightarrow{m \times id_T} & U \times_k T \\
\downarrow{\scriptstyle p_2} & & & & & & & & \downarrow{\scriptstyle p_2} \\
T & & & & \xrightarrow[id_T]{} & & & & T.
\end{array}
\tag{3.1}
$$

Definition 3.2. First, we consider

$$\mathrm{Pic}^{+U}_{U/k} : \quad (\text{Smooth Scheme}/k)^{\circ} \quad \to \quad (\text{Group})$$

the functor of invariants of $\mathrm{Pic}^{+}_{U/k}$ under U: for any smooth k-scheme T, the group $\mathrm{Pic}^{+U}_{U/k}(T)$ is the subgroup formed of the elements $\mathcal{L} \in \mathrm{Pic}^{+}_{U/k}(T)$ such that for all smooth T-schemes S, the element $\mathcal{L}_S$ induced by $\mathcal{L}$ in $\mathrm{Pic}^{+}_{U/k}(S)$ is $U(S)$-invariant.

Next, we consider

$$\mathrm{Pic}^{U}_{U/k} : \quad (\text{Scheme}/k)^{\circ} \quad \to \quad (\text{Group})$$

the "usual" functor of the invariants of the k-schemes $\mathrm{Pic}^{+}_{U/k}$ under the action A of the k-group U (see [DG70, Def. d) II.1.3.4]).

To show that $\mathrm{Pic}^{+U}_{U/k}$ is representable, we are going to use the fact that $\mathrm{Pic}^{U}_{U/k}$ is representable [DG70, Th. d) II.1.3.6].

Lemma 3.3. *For any smooth k-scheme T,*

$$\mathrm{Pic}^{U}_{U/k}(T) = \mathrm{Pic}^{+U}_{U/k}(T).$$

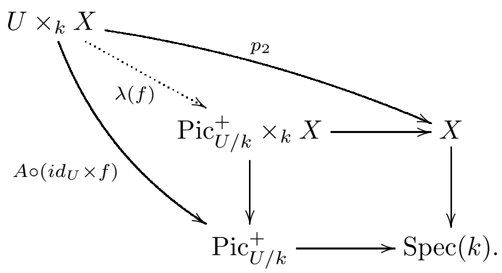
Proof. We consider the functor:

$$\begin{array}{rccc} \mathrm{Hom}_k\left(U, \mathrm{Pic}^{+}_{U/k}\right) : & (\text{Scheme}/k) & \to & (\text{Group}) \\ & T & \mapsto & \mathrm{Hom}_{T-\text{sch}}\left(U \times_k T, \mathrm{Pic}^{+}_{U/k} \times_k T\right). \end{array}$$

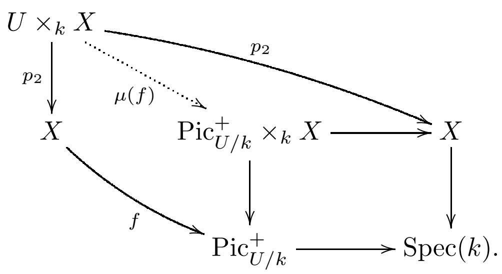
We are going to define two natural transformations,

$$\mu,\ \lambda : \mathrm{Pic}^{+}_{U/k} \to \mathrm{Hom}_k\left(U, \mathrm{Pic}^{+}_{U/k}\right).$$

For any k-scheme X, and $f \in \mathrm{Pic}^{+}_{U/k}(X)$, we defined $\lambda(f)$ by the universal property of Cartesian product:



Likewise for $\mu(f)$:



With the notations above, according to Proposition [DG70, Pro. II.1.3.5], we have the following Cartesian square (of functors of the category of k-schemes into the category of groups):

$$\begin{array}{ccc}
\mathrm{Pic}^{U}_{U/k} & \longrightarrow & \mathrm{Hom}_k\left(U, \mathrm{Pic}^{+}_{U/k}\right) \\
\big\downarrow & & \big\downarrow \scriptstyle\mathrm{diag} \\
\mathrm{Pic}^{+}_{U/k} & \xrightarrow{(\lambda,\mu)} & \mathrm{Hom}_k\left(U, \mathrm{Pic}^{+}_{U/k}\right) \times \mathrm{Hom}_k\left(U, \mathrm{Pic}^{+}_{U/k}\right).
\end{array}$$

Let T be a smooth k-scheme, and let $f \in \mathrm{Pic}^{+U}_{U/k}(T)$. Then $\lambda(f)$ et $\mu(f)$ are two morphisms between two smooth k-scheme: $U \times_k T \to \mathrm{Pic}^{+}_{U/k} \times_k T$. These two morphisms coincide on the S-points for any smooth T-scheme S. We can look at $\lambda(f)$ and $\mu(f)$ as two natural transforms between the functors:

$$\begin{array}{rccc}
U \times_k T: & (\text{Smooth Scheme}/T)^{\circ} & \to & (\text{Set}) \\
 & S & \mapsto & (U \times_k T)(S),
\end{array}$$

and

$$\begin{array}{rccc}
\mathrm{Pic}^{+}_{U/k} \times_k T: & (\text{Smooth Scheme}/T)^{\circ} & \to & (\text{Set}) \\
 & S & \mapsto & \left(\mathrm{Pic}^{+}_{U/k} \times_k T\right)(S).
\end{array}$$

By hypothesis, these two natural transformations are the same, and the k-scheme $U \times_k T$ and $\mathrm{Pic}^{+}_{U/k} \times_k T$ are smooth. Thus, according to Yoneda Lemma, $\lambda(f) = \mu(f)$ hence $f \in \mathrm{Pic}^{U}_{U/k}(T)$. Conversely, by definition the elements of $\mathrm{Pic}^{U}_{U/k}(T)$ are $U(S)$-invariants for any smooth T-scheme S. □

Proposition 3.4. *We consider a unipotent k-group U which admits a regular completion.*

Then, the functor $\mathrm{Pic}^{+U}_{U/k}$ is representable by a smooth subgroup of $\mathrm{Pic}^{+}_{U/k}$. Moreover, for any separable extension L/K,

$$\mathrm{Pic}^{+U}_{U/k}(L) = \mathrm{Ext}^1(U_L, \mathbb{G}_{m,L}). \tag{3.2}$$

Proof. First of all, the functor $\mathrm{Pic}^{U}_{U/k}$ is representable by a closed subgroup of $\mathrm{Pic}^{+}_{U/k}$ [DG70, Th. d) II.1.3.6], we still denote this subgroup as $\mathrm{Pic}^{U}_{U/k}$. Then, $\mathrm{Pic}^{+U}_{U/k}$ is representable by the smoothification of $\mathrm{Pic}^{U}_{U/k}$ (Lemma 3.3, and [CGP15, Lem. C.4.1]).

Finally, we prove the equality (3.2). Let L/k be a separable extension, as

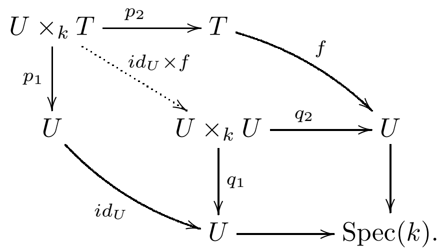
$$\mathrm{Pic}^{+U}_{U/k}(L) = \mathrm{Pic}^{+U_L}_{U_L/L}(L),$$

we can assume that $L = k$. By definition, $\mathrm{Pic}^{+U}_{U/k}(k)$ is the set of the equivalence classes of line bundle $\mathcal{L} \in \mathrm{Pic}(U)$ such that for any smooth k-scheme T, the pull-back $\mathcal{L}_T \in \mathrm{Pic}^{+U}_{U/k}(T)$ is $U(T)$-invariant. Let $f : T \to U$ be a morphism of k-scheme, the diagram

$$\begin{array}{ccccc}
U \times_k T & \xrightarrow{id_U \times F} & U \times_k U \times_k T & \xrightarrow{m \times id_T} & U \times_k T \\
\Big\| & & & & \big\downarrow \scriptstyle p_1 \\
U \times_k T & \xrightarrow[id_U \times f]{} & U \times_k U & \xrightarrow[m]{} & U
\end{array}$$

is commutative. The morphism of the first line $(id_U \times F) \circ (m \times id_T)$ defined by pull-back the action of f on $\mathrm{Pic}(U \times_k T)$. The morphism of the second line is the translation by f denoted T_f. By hypothesis $\mathcal{L}_T = p_1^*\mathcal{L} = A_T(f, \mathcal{L}_T)$ in $\mathrm{Pic}^{+U}_{U/k}(T)$. But, according to the diagram above, $A_T(f, \mathcal{L}_T) = T_f^*\mathcal{L}$. Thus $\mathcal{L}$ is invariant by translation by f. Hence, $\mathcal{L}$ is an element of $\mathrm{Ext}^1(U, \mathbb{G}_{m,k})$ (Proposition 3.1).

Conversely, we consider $\mathcal{L} \in \mathrm{Pic}(U)$ such that $m^*\mathcal{L} = q_1^*\mathcal{L} + q_2^*\mathcal{L}$. Let T be a smooth k-scheme and let $f : T \to U$ be a morphism of k-scheme, we have the following commutative diagram:



Thus
$$(id_U \times f)^* m^*\mathcal{L} = (id_U \times f)^* q_1^*\mathcal{L} + (id_U \times f)^* q_2^*\mathcal{L} = p_1^*\mathcal{L} + p_2^* f^*\mathcal{L}$$
in $\mathrm{Pic}(U \times_k T)$. Hence $A(f, \mathcal{L}) = p_1^*\mathcal{L}$ in $\frac{\mathrm{Pic}(U\times_k T)}{p_2^*\mathrm{Pic}(T)} = \mathrm{Pic}^+_{U/k}(T)$. □

We can finally generalize [Tot13, Lem. 9.4]. The arguments used in the demonstration are inspired by those of the demonstration of [Tot13, Lem. 9.4]: instead of the neutral component of the Picard functor of the regular completion of a form of $\mathbb{G}_{a,k}$, we consider the neutral component of the restricted Picard functor of U. First, we show a preliminary lemma on the action of unipotent group on an other unipotent group.

Lemma 3.5. *Let U and V be two unipotent k-groups. We consider an action of V on U denoted $\alpha : V \times_k U \to U$. If $\dim(U) > 0$, then V acts trivially on a k-subgroup $W \neq \{0\}$ of U.*

Proof. We denote the semi-direct product $V \ltimes U$ (induce by the action α) by G, it is a unipotent k-group. We consider a sequence of unipotent k-group, first $U_0 = U$, and then recursively $U_i = [G, U_{i-1}]$ for $i \geqslant 1$ (see [DG70, Pro. II.5.4.9] for the definition and representability of the commutator subgroup). Then, $(U_i)_i$ is a descending series of normal k-subgroups of U.

Since G is unipotent, it is also a nilpotent group. So U_i is trivial for i large enough. Let j be the last index such that $U_j \neq \{0\}$, then V acts trivially on U_j. □

Theorem 3.6. *We consider a unipotent k-group U which admits a regular completion. Then:*

(i) if $\mathrm{Pic}^+_{U/k}$ is étale, then $\mathrm{Pic}^{+U}_{U/k} = \mathrm{Pic}^+_{U/k}$;

(ii) if $\mathrm{Pic}^{+\,0}_{U/k} \neq \{0\}$, then $\mathrm{Pic}^{+U\,0}_{U/k} \neq \{0\}$;

(iii) if $\mathrm{Pic}(U_{k_s})$ is finite, then $\mathrm{Ext}^1(U, \mathbb{G}_{m,k}) = \mathrm{Pic}(U)$;

(iv) if $\mathrm{Pic}(U_{k_s})$ is infinite, then $\mathrm{Ext}^1(U_{k_s}, \mathbb{G}_{m,k_s})$ is likewise infinite;

(v) if $\mathrm{Pic}(U_{k_s}) \neq \{0\}$, then $\mathrm{Ext}^1(U_{k_s}, \mathbb{G}_{m,k_s}) \neq \{0\}$.

Proof. First, both $\mathrm{Pic}^+_{U/k}$ and $\mathrm{Pic}^{+U}_{U/k}$ are representable (Theorem 1.1 and Proposition 3.4).

Let us show (i), if $\mathrm{Pic}^+_{U/k}$ is an étale k-group then, as U is connected, the action of U on $\mathrm{Pic}^+_{U/k}$ is trivial. Hence $\mathrm{Pic}^{+U}_{U/k} = \mathrm{Pic}^+_{U/k}$.

Then (ii) is a direct consequence of Lemma 3.5 applied to the action of U on $\mathrm{Pic}^{+\,0}_{U/k}$.

If $\mathrm{Pic}(U_{k_s})$ is finite, then $\mathrm{Pic}^{+\,0}_{U/k}$ is trivial so $\mathrm{Pic}^+_{U/k}$ is an étale k-group [DG70, Pro. II.5.1.4]. Thus (iii) is a consequence of (i) and Proposition 3.4.

Likewise, if $\mathrm{Pic}(U_{k_s})$ is infinite, then $\mathrm{Pic}^{+\,0}_{U/k} \neq \{0\}$. So by (ii), $\mathrm{Pic}^{+U\,0}_{U/k} \neq \{0\}$ and $\mathrm{Ext}^1(U_{k_s}, \mathbb{G}_{m,k_s}) = \mathrm{Pic}^{+U}_{U/k}(k_s)$ is infinite.

Finally, (v) is an immediate consequence of (iii) and (iv). □

3.4 Torsion of the extension group of a unipotent group by the multiplicative group

In this subsection, we make an elementary study of the torsion of the $\mathrm{Ext}^1(U, \mathbb{G}_{m,k})$.

Lemma 3.7 (Restriction to the wound case)**.**
Let U be a unipotent k-group. Let us recall that there is a unique k-split normal subgroup U_{split} of U such that $U_{\text{wd}} = U/U_{\text{split}}$ is k-wound [CGP15, Th. B.3.4]. The morphism $q : U \to U_{\text{wd}}$ induces a morphism

$$q^* : \mathrm{Ext}^1(U_{\text{wd}}, \mathbb{G}_{m,k}) \to \mathrm{Ext}^1(U, \mathbb{G}_{m,k}),$$

which is an isomorphism of commutative groups.

Proof. The exact sequence of k-groups:

$$1 \to U_{\text{split}} \to U \to U_{\text{wd}} \to 1,$$

induces an exact sequence of extension group:

$$\mathrm{Hom}(U_{\text{split}}, \mathbb{G}_{m,k}) \to \mathrm{Ext}^1(U_{\text{wd}}, \mathbb{G}_{m,k}) \xrightarrow{q^*} \mathrm{Ext}^1(U, \mathbb{G}_{m,k}) \to \mathrm{Ext}^1(U_{\text{split}}, \mathbb{G}_{m,k}).$$

As $\mathrm{Hom}(U_{\text{split}}, \mathbb{G}_{m,k}) = \{0\}$ and $\mathrm{Ext}^1(U_{\text{split}}, \mathbb{G}_{m,k}) = \{0\}$ [SGAIII2, XVII Th. 6.1.1], the morphism q^* is an isomorphism. □

Let U be a unipotent k-group, let us recall that U is of p^t-torsion for $t \in \mathbb{N}$ big enough.

Definition 3.8. We denote the smallest integer $t \in \mathbb{N}$ such that U is of p^t-torsion by $t(U)$.

Proposition 3.9. *Let U be a unipotent k-group, then $\mathrm{Ext}^1(U, \mathbb{G}_{m,k})$ is of $p^{t(U)}$-torsion.*

Proof. Let

$$1 \to \mathbb{G}_{m,k} \to E \xrightarrow{g} U \to 1$$

be an extension of U by $\mathbb{G}_{m,k}$. Then $p^{t(U)}.[E] = \left[\left(p^{t(U)} Id_U\right)^* E\right]$ where $p^{t(U)} Id_U : U \to U$. But

$$\left(p^{t(U)} Id_U\right)^* E = \ker\left(U \times_k E \xrightarrow{p^{t(U)} Id_U - g} U\right) \cong U \times_k \mathbb{G}_{m,k}.$$

Thus $p^{t(U)}.[E] = 0$ in $\mathrm{Ext}^1(U, \mathbb{G}_{m,k})$. □

Example 3.10. Let U be a nontrivial form of $\mathbb{G}_{a,k}$, then $\mathrm{Pic}_{U/k}^{\overset{U}{+}\,0}$ is representable by a commutative p-torsion unipotent k-group. Thus, $\mathrm{Pic}_{U/k}^{\overset{U}{+}\,0}$ is a form of $\mathbb{G}_{a,k}^d$ (with $d > 0$ if U is not rational) [CGP15, Lem. B.1.10].

As the Picard group of a form of $\mathbb{G}_{a,k}$ is of $p^{n(U)}$-torsion (see Definition 2.7 and [Ach17, Th. 2.2]), the group $\mathrm{Ext}^1(U, \mathbb{G}_{m,k})$ is also of $p^{n(U)}$-torsion. Thus, it is natural to ask if we can compare these two invariants.

Lemma 3.11. *If U is a k-wound commutative k-group, then $t(U) \leqslant n(U)$.*

Proof. Indeed, we denote $n = n(U)$, then $p^n Id_U : U \to U$ factor as

$$\begin{array}{ccc} U & \xrightarrow{\quad p^n Id_U \quad} & U \\ & {\scriptstyle F^n_{U/k}} \searrow \quad \nearrow {\scriptstyle V^n} & \\ & U^{(p^n)} & \end{array}$$

where $F^n_{U/k}$ is the n-th relative Frobenius morphism and V^n is the n-th shift morphism [DG70, IV.3.4.10]. As V^n is a morphism from a k-split unipotent k-group to a k-wound unipotent k-group, it is trivial. Hence $t(U) \leqslant n(U)$. □

3.5 A criteria of pseudo-reductivity

In this subsection, we discuss which commutative unipotent k-groups are quotients of commutative k-pseudo-reductive groups. B. Totaro reduce the problem to the case $k = k_s$ [Tot13, Lem. 9.1], hence throughout this subsection we assume that k is separably closed.

B. Totaro proved that any k-wound unipotent k-group U of dimension 1 is the quotient of a 2 dimensional commutative k-pseudo-reductive group [Tot13, Cor. 9.5]. More precisely, he obtained that $\mathrm{Ext}^1(U, \mathbb{G}_{m,k}) \neq \{0\}$ and that any nontrivial extension of U by $\mathbb{G}_{m,k}$ is a k-pseudo-reductive group.

The objective of this subsection is to obtain an *ad hoc* criteria for the commutative unipotent k-groups that are quotients of a commutative k-pseudo-reductive groups (Proposition 3.13).

Lemma 3.12. *Let U be a commutative unipotent k-group which is the quotient of a commutative k-pseudo-reductive group. We consider a closed immersion $i : V \to U$ where $V \neq \{0\}$, then $\mathrm{Ext}^1(V, \mathbb{G}_{m,k}) \neq \{0\}$ and the pull-back morphism $i^* : \mathrm{Ext}^1(U, \mathbb{G}_{m,k}) \to \mathrm{Ext}^1(V, \mathbb{G}_{m,k})$ is nonzero.*

Proof. By hypothesis, there is an extension

$$0 \to \mathbb{G}_{m,k}^d \to E \to U \to 0,$$

where E is a commutative k-pseudo-reductive group. Let $V \neq \{0\}$ be a closed subgroup of U, we consider the pull-back:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_{m,k}^d & \longrightarrow & i^*E & \longrightarrow & V & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & \mathbb{G}_{m,k}^d & \longrightarrow & E & \longrightarrow & U & \longrightarrow & 0.
\end{array}$$

As E is commutative k-pseudo-reductive, the closed subgroup i^*E of E is also k-pseudo-reductive. Thus, $\mathrm{Ext}^1(V, \mathbb{G}_{m,k}) \neq \{0\}$, and $i^* : \mathrm{Ext}^1(U, \mathbb{G}_{m,k}) \to \mathrm{Ext}^1(V, \mathbb{G}_{m,k})$ is also nonzero. ☐

Proposition 3.13. *Let U be a commutative unipotent k-group such that for any closed subgroup $V \xrightarrow{i} U$ with $V \neq \{0\}$, the pull-back morphism $i^* : \mathrm{Ext}^1(U, \mathbb{G}_{m,k}) \to \mathrm{Ext}^1(V, \mathbb{G}_{m,k})$ is nonzero.*

Then, U is the quotient of a commutative k-pseudo-reductive group.

Proof. By hypothesis, for any $d \geqslant 1$, the group $\mathrm{Ext}^1(U, \mathbb{G}_{m,k}^d)$ is nontrivial. Let E be an extension of U by $\mathbb{G}_{m,k}^d$, then the unipotent radical $\mathcal{R}_{u,k}(E)$ is a subgroup of U of dimension $n \geqslant 0$. We consider an extension E such that n is minimal among all the extensions of U by $\mathbb{G}_{m,k}^d$ for any $d \geqslant 1$. If $n = 0$, then E is k-pseudo-reductive.

Let us assume that $n \geqslant 1$. Then, we denote $V = \mathcal{R}_{u,k}(E)$ and we have the commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V & = & V & & \\
 & & & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & \mathbb{G}_{m,k}^d & \longrightarrow & E & \xrightarrow{q} & U & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{G}_{m,k}^d & \longrightarrow & E/V & \longrightarrow & U/V & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

It induces a commutative diagram of Ext group:

$$\begin{array}{ccc} \mathrm{Ext}^1(U,\mathbb{G}_{m,k}) & \xrightarrow{\quad q^* \quad} & \mathrm{Ext}^1(E,\mathbb{G}_{m,k}) \\ & {\scriptstyle i^*}\searrow \qquad \swarrow{\scriptstyle j^*} & \\ & \mathrm{Ext}^1(V,\mathbb{G}_{m,k}). & \end{array}$$

As $n \geqslant 1$, in particular $V \neq \{0\}$. Thus, by hypothesis, the morphism $i^* : \mathrm{Ext}^1(U,\mathbb{G}_{m,k}) \to \mathrm{Ext}^1(V,\mathbb{G}_{m,k})$ is nonzero, and $j^* : \mathrm{Ext}^1(E,\mathbb{G}_{m,k}) \to \mathrm{Ext}^1(V,\mathbb{G}_{m,k})$ is also nonzero.

Let us consider $[F] \in \mathrm{Ext}^1(E,\mathbb{G}_{m,k})$ such that $j^*[F] \neq 0$ in $\mathrm{Ext}^1(V,\mathbb{G}_{m,k})$. As $\mathcal{R}_{u,k}(F) \cap \mathbb{G}_{m,k} = \{0\}$, we identify $\mathcal{R}_{u,k}(F)$ with a closed subgroup of V. Then j^*F is a nontrivial extension of V by $\mathbb{G}_{m,k}$, thus $\mathcal{R}_{u,k}(j^*F)$ is a strict subgroup of V. Moreover, the natural closed immersion $\mathcal{R}_{u,k}(j^*F) \to \mathcal{R}_{u,k}(F)$ is an isomorphism.

Hence, F is an extension of U by $\mathbb{G}_{m,k}^{d+1}$ with a unipotent radical of dimension strictly lower than the dimension of V. This contradicts the minimality hypothesis on the dimension of V. □

Checking directly the criteria of Proposition 3.13 is not reasonable. But, the proof of the Proposition 3.13 contains an algorithm: we do not need to check if i^* is nonzero for any subgroup of U, we only need to do it for at most $\dim(U)$ subgroups.

Lemma 3.14. *We consider*

$$0 \to V \xrightarrow{i} U \xrightarrow{\varphi} W \to 0,$$

an exact sequence of commutative unipotent k-groups.

If the pull-back morphism $i^* : \mathrm{Ext}^1(U,\mathbb{G}_{m,k}) \to \mathrm{Ext}^1(V,\mathbb{G}_{m,k})$ is nonzero, then the pull-back morphism $\varphi^* : \mathrm{Pic}(W) \to \mathrm{Piv}(U)$ is not an isomorphism (or equivalently the Picard group of the generic fiber of $U \to V$ is nontrivial).

Proof. The diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Ext}^1(W,\mathbb{G}_{m,k}) & \longrightarrow & \mathrm{Pic}(W) & \xrightarrow{\alpha} & \mathrm{Pic}(W \times_k W) \\ & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle (\varphi\times\varphi)^*} \\ 0 \longrightarrow & \mathrm{Ext}^1(U,\mathbb{G}_{m,k}) & \longrightarrow & \mathrm{Pic}(U) & \xrightarrow{\beta} & \mathrm{Pic}(U \times_k U), \end{array}$$

is a commutative diagram of commutative (abstract) group where α and β are the morphisms of Lemma [Tot13, Lem. 9.2]; Thus the lines are exact.

If $\mathrm{Pic}(W) \xrightarrow{\varphi^*} \mathrm{Pic}(U)$ is surjective, then any element $\mathcal{L} \in \ker(\beta)$ is the image of some $\mathcal{L}' \in \mathrm{Pic}(W)$. As the diagram is commutative $(\varphi \times \varphi)^* \circ \alpha(\mathcal{L}') = 0$. Moreover, $(\varphi \times \varphi)^*$ is injective (see Proposition 2.18), thus $\mathcal{L}' \in \ker(\alpha)$. So $\mathrm{Ext}^1(W,\mathbb{G}_{m,k}) \xrightarrow{\varphi^*} \mathrm{Ext}^1(U,\mathbb{G}_{m,k})$ is surjective, hence $\mathrm{Ext}^1(U,\mathbb{G}_{m,k}) \xrightarrow{i^*} \mathrm{Ext}^1(V,\mathbb{G}_{m,k})$ is the zero morphism.

Moreover, $\mathrm{Pic}(W) \xrightarrow{\varphi^*} \mathrm{Pic}(U)$ is surjective if and only if the Picard group of the generic fiber of $U \to V$ is trivial (Proposition 2.18). □

We are now going to simplify the criteria of Proposition 3.13 in a special case.

Remark 3.15 (Extension of unipotent groups of finite Picard groups).

Let us consider V and W two commutative unipotent k-groups such that $\mathrm{Pic}(V)$ and $\mathrm{Pic}(W)$ are both finite (as we will see in the next section, the unirational unipotent k-groups are an important class of unipotent groups that satisfy this hypothesis).

Let

$$0 \to V \xrightarrow{i} U \xrightarrow{\varphi} W \to 0$$

be an exact sequence of commutative unipotent k-groups. We denote by $Z_{\kappa(W)}$ the generic fiber of $U \to W$. As $\mathrm{Pic}(W)$ is finite, $\mathrm{Pic}\left(Z_{\kappa(W)}\right)$ is also finite, thus $\mathrm{Pic}(U)$ is finite

(Proposition 2.18). And, as $\mathrm{Pic}(U)$ is finite, we have $\mathrm{Ext}^1(U, \mathbb{G}_{m,k}) = \mathrm{Pic}(U)$ (and likewise for V and W). Thus, we can write the exact sequence of Ext-groups as:

$$0 \to \mathrm{Pic}(W) \xrightarrow{\varphi^*} \mathrm{Pic}(U) \xrightarrow{i^*} \mathrm{Pic}(V).$$

Finally i^* is nonzero if and only if φ^* is not surjective. And φ^* is not surjective if and only if $\mathrm{Pic}\left(Z_{\kappa(W)}\right) \neq \{0\}$ (Proposition 2.18).

Question 3.16. *Does the remark above generalize to any commutative extension of unipotent k-groups (i.e. is the converse of Lemma 3.14 true)?*

Finally, we apply Proposition 3.13 and Remark 3.15 to an explicit example.

Example 3.17. Let k be a non perfect field of characteristic 2, we consider $a, b \in k \setminus k^p$. Let V (resp. W) be the unipotent k-group defined as the subgroup of $\mathbb{G}_{a,k}^2$ by the equation $y^p = x + ax^p$ (resp. $z^p = t + bt^p$).

We consider a commutative extension of k-groups:

$$0 \to V \xrightarrow{i} U \xrightarrow{\varphi} W \to 0.$$

We denote by $Z_{\kappa(W)}$ the generic fiber of $U \to W$.

As $\mathrm{Pic}(V) = \mathrm{Pic}(W) = \frac{\mathbb{Z}}{2\mathbb{Z}}$, we are in the situation of Remark 3.15. Thus, if U is the quotient of some k-pseudo-reductive group, then $\mathrm{Pic}\left(Z_{\kappa(W)}\right) \neq \{0\}$. Conversely, W admits an extension G by $\mathbb{G}_{m,k}$ that is pseudo-reductive [Tot13, Cor. 9.5], thus φ^*G is an extension of U by $\mathbb{G}_{m,k}$ with k-unipotent radical V. As $\mathrm{Pic}\left(Z_{\kappa(W)}\right) \neq \{0\}$, there is an extension G' of φ^*G by $\mathbb{G}_{m,k}$ such that $\mathcal{R}_{u,k}(G')$ is a strict subgroup of V, so $\mathcal{R}_{u,k}(G') = \{0\}$. Hence, U is the quotient of a k-pseudo-reductive group if and only if $\mathrm{Pic}\left(Z_{\kappa(W)}\right) \neq \{0\}$.

There are two cases, either $Z_{\kappa(W)}$ is isomorphic to $V_{\kappa(W)}$ (i.e. $Z_{\kappa(W)} \to \mathrm{Spec}(\kappa(W))$ is the trivial $V_{\kappa(W)}$-torsor). Then, there is an extension of U by $\mathbb{G}_{m,k}^2$ that is a commutative k-pseudo-reductive group.

The other possibility is that $Z_{\kappa(W)}$ is a nontrivial $V_{\kappa(W)}$-torsor, then U is a nontrivial extension of W by V, and $Z_{\kappa(W)}(\kappa(W)) = \emptyset$. Moreover $Z_{\kappa(W)}(\kappa(W)) = \emptyset$ imply that $Z_{\kappa(W)}$ has no rational point on any extension of odd degree (see Corollary [EKM08, Cor. 18.5] by T. A. Springer). And, $\mathrm{Pic}\left(Z_{\kappa(W)}\right) \cong \mathrm{Im}(\deg)/2\mathbb{Z}$ where $\deg : \mathrm{Pic}\left(Z_{\kappa(W)}\right) \to \mathbb{Z}$ is the degree function (see the exact sequence [Ach17, Eq. (2.1.3)]). Thus $\mathrm{Pic}\left(Z_{\kappa(W)}\right) = \{0\}$, and U is not the quotient of some commutative k-pseudo-reductive group.

Finally, we give an example of nontrivial extension of W by V with $Z_{\kappa(W)}(\kappa(W)) = \emptyset$. We fix $k = \mathbb{F}_2(a, b)$, we consider the subgroup U of $\mathbb{G}_{a,k}^4 = \mathrm{Spec}\left(k[x, y, z, t]\right)$ defined by the equations:

$$\begin{cases} z^2 = t + bt^2 \\ y^2 = x + ax^2 + t. \end{cases}$$

Then U is a form of $\mathbb{G}_{a,k}^2$, and

$$\begin{array}{ccccccccc} & & (x,y) \mapsto (x,y,0,0) & & & & & & \\ 0 & \longrightarrow & V & \longrightarrow & U & \longrightarrow & W & \longrightarrow & 0 \\ & & & & (x,y,z,t) \mapsto (z,t), & & & & \end{array}$$

is an exact sequence of commutative k-groups.

The generic fiber $Z_{\kappa(W)}$ of $U \to W$ is isomorphic to the closed subscheme of $\mathbb{A}^2_{\kappa(W)}$ defined by the equation $y^2 = x + ax^2 + t$ where $t \in \kappa(W)$.

And $\kappa(W) = \mathrm{Frac}\left(\dfrac{k[z,t]}{< z^2 - t - bt^2 >}\right) = k(w)$ where $w = t/z$. As,

$$w^2 = \frac{t^2}{t + bt^2} \Leftrightarrow t = \frac{w^2}{1 - bw^2},$$

we have to show that the equation $y^2 = x + ax^2 + \frac{w^2}{1-bw^2}$ with x and y in $\mathbb{F}_2(a,b,w)$ have no solution. Let us show that the only solution of

$$(1-bw^2)A^2 = (1-bw^2)BC + a(1-bw^2)B^2 + w^2C^2,$$

where $A, B, C \in \mathbb{F}_2[a,b,w]$ is $A = B = C = 0$.

Let us look at the total degree of the polynomial in the above equation:

$$2\deg(A) + 3 = \deg((1-bw^2)BC + a(1-bw^2)B^2 + w^2C^2).$$

If $\deg(B) \geqslant \deg(C)$, then $\deg((1-bw^2)BC + a(1-bw^2)B^2 + w^2C^2) = 4 + 2\deg(B)$. Thus $2\deg(A) + 3 = 4 + 2\deg(B)$, this is false. If $\deg(B) < \deg(C) - 1$, then $2\deg(A) + 3 = 2 + 2\deg(C)$, this is impossible.

Hence $\deg(B) = \deg(C) - 1$, and the monomials of highest degree of bw^2BC, abw^2B^2 and w^2C^2 cancel each others. We denote the monomial of highest degree of B as $\widetilde{B}$, and likewise $\widetilde{C}$ for the monomial of highest degree of C.

Let us now look at the partial degree in a of $bw^2\widetilde{B}\widetilde{C}$, $abw^2\widetilde{B}^2$ and $w^2\widetilde{C}^2$. Then, as $\deg_a\left(abw^2\widetilde{B}^2\right) = 1 + 2\deg_a\left(\widetilde{B}\right) \neq 2\deg_a\left(\widetilde{C}\right) = \deg_a\left(w^2\widetilde{C}^2\right)$, we have

$$\deg_a\left(bw^2\widetilde{B}\widetilde{C}\right) = \deg_a\left(w^2\widetilde{C}^2\right) \quad \text{or} \quad \deg_a\left(bw^2\widetilde{B}\widetilde{C}\right) = \deg_a\left(abw^2\widetilde{B}^2\right).$$

If we have the first equality, then $\deg_a\left(\widetilde{C}\right) = \deg_a\left(\widetilde{B}\right)$, so

$$\deg_a\left(abw^2\widetilde{B}^2\right) > \deg_a\left(bw^2\widetilde{B}\widetilde{C}\right)$$

and these polynomial can not cancel each others. We have the same kind of contradiction for the second inequality.

Thus $Z_{\kappa(W)}$ does not have any $\kappa(W)$-rational point, and the extension is nontrivial.

This example show that, even in dimension 2, the situation is unintuitive. A lot more work is needed to obtain a complete classification of the unipotent groups that are quotient of a commutative k-pseudo-reductive groups.

4 Unirational forms of the affine space

If K is a perfect field, then any linear K-group is unirational [Bor12, V Th. 18.2] and any unipotent K-group is rational.

Over a non perfect field, the situation is way more complicated. The k-split unipotent k-groups are rational. The reductive k-groups are still unirational [Bor12, V Th. 18.2]. And, if G is a perfect (i.e. $G = [G : G]$), then G is unirational [CGP15, Pro. A.2.11]. But neither the k-pseudo-reductive groups, nor the k-wound unipotent k-groups are in general unirational.

In this section, we study the subtle relationship between the notion of unirationality and the unipotent k-groups. When possible we try to obtain results about unirational forms of the affine n-space.

4.1 An example of unirational k-wound unipotent k-group

We consider a purely inseparable extension k' of k, we denote p^n the degree of the extension k'/k. We are going to study the quotient

$$U = R_{k'/k}(\mathbb{G}_{m,k'})/\mathbb{G}_{m,k},$$

where $R_{k'/k}$ denote the Weil restriction (see [BLR90, §7.6] for definition and properties of the Weil restrictions). J. Oesterlé proved that U is a k-wound unirational unipotent k-group [Oes84, Lem. VI.5.1]. We are going to compute the Picard group of U and the group $\mathrm{Ext}^1(U, \mathbb{G}_{m,k})$ of the extension of U by $\mathbb{G}_{m,k}$.

First, $R_{k'/k}(\mathbb{G}_{m,k'})$ is isomorphic as a k-scheme to the principal open subscheme of $\mathbb{A}_k^{p^n}$ defined by the equation $N \neq 0$ where N is the norm of k'/k thus

$$\mathrm{Pic}(R_{k'/k}(\mathbb{G}_{m,k'})) = \{0\}.$$

Moreover, the exact sequence of k-group:

$$0 \to \mathbb{G}_{m,k} \to R_{k'/k}(\mathbb{G}_{m,k'}) \to U \to 0, \tag{4.1}$$

induces an exact sequence of commutative group:

$$0 \to \mathcal{O}(U)^* \to \mathcal{O}(R_{k'/k}(\mathbb{G}_{m,k'}))^* \to \widehat{\mathbb{G}_{m,k}} \to \mathrm{Pic}(U) \to \mathrm{Pic}(R_{k'/k}(\mathbb{G}_{m,k'})) = \{0\}$$

where $\widehat{\mathbb{G}_{m,k}}$ is the character group of $\mathbb{G}_{m,k}$ (Proposition 2.18). Moreover, the quotient $\mathcal{O}(R_{k'/k}(\mathbb{G}_{m,k'}))^*/k^*$ is the free $\mathbb{Z}$-module generated by the class $[N]$ induced by the norm N, and $\widehat{\mathbb{G}_{m,k}} \cong \mathbb{Z}$. In addition, the image of $[N]$ in $\widehat{\mathbb{G}_{m,k}}$ is p^n. Thus $\mathrm{Pic}(U) = \mathbb{Z}/p^n\mathbb{Z}$.

Let us recall that $\mathrm{Ext}^1(U, \mathbb{G}_{m,k})$ identify with the subgroup $\mathrm{Pic}(U)^U$ of $\mathrm{Pic}(U)$ consisting of the translation invariant classes [Tot13, Lem. 9.2]. In this case, we have shown that the group $\mathrm{Pic}(U)$ is generated by $[N]$ and that $[N]$ induces the extension of U by $\mathbb{G}_{m,k}$ given by the exact sequence (4.1).

Thus, we have obtain the following results:

Proposition 4.1. *With the notations and the hypothesis above:*

$$\mathrm{Ext}^1(U, \mathbb{G}_{m,k}) = \mathrm{Pic}(U)^U = \mathrm{Pic}(U) = \frac{\mathbb{Z}}{p^n\mathbb{Z}}.$$

Moreover, the restricted Picard functor $\mathrm{Pic}^+_{U/k}$ *is representable and:*

$$\mathrm{Pic}^+_{U/k} = \left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right)_k,$$

where $\left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right)_k$ *is the constant* k*-group associated to* $\frac{\mathbb{Z}}{p^n\mathbb{Z}}$.

Proof. The only point left is the last one. We just have to remark that U admits a smooth completion, indeed $U = \mathbb{P}^r_k \setminus V(N)$ where $r = p^n - 1$, and $V(N)$ is the closed subscheme defined by the norm N of k'/k. Hence, $\mathrm{Pic}^+_{U/k}$ is representable (Theorem 1.1). □

This example is quite interesting because J. Oesterlé proved that if K is the function field of a curve over a finite field, then the K-wound unipotent K-group of dimension strictly inferior to $p-1$ have a finite number of K-rational points [Oes84, Th. VI.3.1]. In particular they are not unirational. If $[K' : K] = p$ and T is a K-torus, then $R_{K'/K}(T_{K'})/T$ is K-wound unirational and of dimension $p-1$. This suggests that these groups play an important role among unirational unipotent K-groups. Hence, J. Oesterlé asks the following question:

Question 4.2. *[Oes84, p. 80]*

Does any unirational commutative unipotent k*-group admit a subgroup isomorphic to* $R_{k'/k}(T_{k'})/T$ *(where* T *is a* k*-torus and* k'/k *is a finite purely inseparable extension)?*

Remark 4.3. (*i*) We can generalize Proposition 4.1. Indeed, we consider k'/k a purely inseparable extension of degree p^n and T a k-torus of dimension d. Then $U = R_{k'/k}(T_{k'})/T$ is a k-wound unipotent k-group, and $\mathrm{Pic}(U) = \mathrm{Ext}^1(U, \mathbb{G}_{m,k})$ is a subgroup of $(\mathbb{Z}/p^n\mathbb{Z})^d$.

(*ii*) Our arguments also apply to any finite field extension E/k:

$$\mathrm{Pic}\left(\frac{R_{E/k}(\mathbb{G}_{m,E})}{\mathbb{G}_{m,k}}\right) = \frac{\mathbb{Z}}{[E:k]\mathbb{Z}}.$$

But $R_{E/k}(\mathbb{G}_{m,E})/\mathbb{G}_{m,k}$ is not necessary unipotent. For example, if E/k is separable, then $R_{E/k}(\mathbb{G}_{m,E})$ and $R_{E/k}(\mathbb{G}_{m,E})/\mathbb{G}_{m,k}$ are k-tori.

(*iii*) In the particular case where $[k^{1/p} : k] = p$, J. Oesterlé has obtained an explicit description of the unipotent k-group $R_{k^{1/p}/k}\left(\mathbb{G}_{m,k^{1/p}}\right)/\mathbb{G}_{m,k}$ [Oes84, Pro. VI.5.3].

(*iv*) Let U be a connected unipotent k-group, the Picard group of U can be trivial even if U is k-wound [Tot13, Exa. 9.7]. But if U has a quotient that is a nontrivial form G of $\mathbb{G}_{a,k}$, then the Picard group of U is nontrivial. Indeed, the Picard group of a nontrivial form of $\mathbb{G}_{a,k}$ is nontrivial [Ach17, Th. 2.2] and the group morphism $\mathrm{Pic}(G) \to \mathrm{Pic}(U)$ is an injection (Proposition 2.18). The converse is false, at least if $p > 2$. Indeed, if k'/k is a nontrivial purely inseparable extension, then the Picard group of $R_{k'/k}(\mathbb{G}_{m,k'})/\mathbb{G}_{m,k}$ is nontrivial. But U does not admit a quotient G which is a nontrivial form of $\mathbb{G}_{a,k}$. Because, if G is such a quotient, then G is a unirational (so rational) form of $\mathbb{G}_{a,k}$. But, if $p > 2$, the only rational form of $\mathbb{G}_{a,k}$ is the trivial one [KMT74, Th. 6.9.2].

4.2 Unirationality and structure of commutative k-group

Proposition 4.4. *Let G a commutative k-group, then there is a unique maximal unirational k-subgroup of G denoted by G_{ur}.*

Moreover, if X is a geometrically reduced unirational k-variety, then any morphism $X \to G$ whose image contains the identity element of G factors via G_{ur}.

Proof. We consider X a geometrically reduced unirational k-variety, and $X \to G$ a morphism whose image contains the identity element of G. Then X generates a smooth k-algebraic subgroup of G [SGAIII1, VI.B Pro. 7.1] that is unirational [SGAIII1, VI.B Pro. 7.6] and connected [SGAIII1, VI.B Pro. 7.2.1].

Thus, two unirational k-subgroups of G generate a third k-subgroup that is also unirational. Hence, there is a unique largest unirational k-subgroup G_{ur} of G, and G_{ur} contains the image of any morphism from a geometrically reduced unirational k-variety to G whose image contains the identity element of G. □

Example 4.5. (*i*) If G is a k-abelian variety, then $G_{ur} = \{0\}$.

(*ii*) If T is a k-torus, then $T_{ur} = T$.

(*iii*) If U is a commutative k-split unipotent k-group, then $U_{ur} = U$.

(*iv*) If U is a nontrivial form of $\mathbb{G}_{a,k}$. Then, either $\mathrm{char}(k) = 2$ and U is isomorphic to the subgroup of $\mathbb{G}_{a,k}^2$ defined by the equation $y^2 = x + ax^2$ where $a \notin k^2$; and then U is rational, thus $U_{ur} = U$ [KMT74, Th. 6.9.2]. Or, U is not rational, and $U_{ur} = \{0\}$.

(*v*) Let U be a nontrivial form of $\mathbb{G}_{a,k}$. We consider G an extension of U by $\mathbb{G}_{m,k}$. Then, either U is rational, and $G_{ur} = G$. Or U is not rational, and $G_{ur} = \mathbb{G}_{m,k}$.

(*vi*) Let U be the group of Example 2.9. If $\mathrm{char}(k) = 2$, then $U_{ur} = U$. Else $\mathrm{char}(k) > 2$ and $U_{ur} = \{0\}$.

(*vii*) If T is a k-torus, and k'/k is a finite purely inseparable field extension. Then, the unipotent k-group

$$U = R_{k'/k}(T_{k'})/T$$

is unirational, hence $U_{ur} = U$.

(*viii*) We consider the function field K of an algebraic curve over a finite group. Let U be a commutative unipotent K-group of dimension $< p - 1$, then $U_{ur} = \{0\}$ [Oes84, Th. VI.3.1].

As a commutative unipotent k-group U such that $U_{ur} = \{0\}$ is k-wound, we use the following terminology:

Definition 4.6. A commutative k-group G such that $G_{ur} = \{0\}$ is called k-*strongly wound*.

Proposition 4.7. *Let X be a form of $\mathbb{A}_k^n$ which admits a regular completion, then $\mathrm{Pic}^{+\,0}_{X/k}$ is k-strongly wound.*

Proof. Indeed, if $\mathrm{Pic}^{+\,0}_{X/k}$ as not k-strongly wound, then there would be a nonconstant morphism $V \to \mathrm{Pic}^{+\,0}_{X/k}$ for some open V of an $\mathbb{A}_k^n$. This would contradict Lemma 1.3. □

As a commutative k-group is in general neither unirational nor k-strongly wound; it is natural to look for a *dévissage* results.

Proposition 4.8. *Let G be a commutative k-group. Then, G admits a unique quotient G_{sw} such that G_{sw} is k-strongly wound and any morphism $G \to H$ to a k-strongly wound k-group H factors in a unique way into $G \to G_{sw}$ follow by a morphism $G_{sw} \to H$. Moreover, the kernel of $G \to G_{sw}$ contains G_{ur}.*

Proof. If $G_{ur} = \{0\}$, then $G = G_{sw}$. Else, we consider $G' = G/G_{ur}$, by induction on the dimension of G, there is a unique quotient $G' \to G'_{sw}$ that is k-strongly wound and satisfies the mapping properties. Finally, we remark that $G_{sw} := G'_{sw}$ suits. ☐

Question 4.9. *Are the commutative extensions of a unirational commutative k-group by an other unirational commutative k-group still unirational?*

If the answer to the question above is affirmative, then the natural morphism $G/G_{ur} \to G_{sw}$ would be an isomorphism.

4.3 Picard group of unirational forms of the affine space

We denote by A and B two functors from the category of smooth k-schemes to the category of sets, we assume that there is two natural transformations $e : \mathrm{Spec}(k) \to A$ and $f : \mathrm{Spec}(k) \to B$. Then, we denote by $\mathrm{Nat}(A, B)$ the set of natural transformations from A to B, and by $\mathrm{Nat}_{pt.}(A, B)$ set of natural transformations η from A to B such that $\eta \circ f = e$.

Lemma 4.10. *We consider X a form of $\mathbb{A}^n_k$ such that $X(k) \neq \emptyset$. Then, for all smooth k-schemes W such that $W(k) \neq \emptyset$,*

$$\mathrm{Nat}_{pt.}\left(W, \mathrm{Pic}^+_{X/k}\right) \cong \frac{\mathrm{Pic}\,(X \times_k W)}{p_1^*\mathrm{Pic}(X) \times p_2^*\mathrm{Pic}\,(W)}.$$

In particular, if Y is a form of $\mathbb{A}^c_k$ such that $Y(k) \neq \emptyset$, then

$$\mathrm{Nat}_{pt.}\left(Y, \mathrm{Pic}^+_{X/k}\right) \cong \mathrm{Nat}_{pt.}\left(X, \mathrm{Pic}^+_{Y/k}\right).$$

Proof. First of all, according to Yoneda Lemma, $\mathrm{Nat}\left(W, \mathrm{Pic}^+_{X/k}\right) = \mathrm{Pic}^+_{X/k}(W)$, and as W is smooth,

$$\mathrm{Nat}\left(W, \mathrm{Pic}^+_{X/k}\right) = \frac{\mathrm{Pic}(X \times_k W)}{p_2^*\mathrm{Pic}(W)}.$$

Let e be a k-rational point of W, then e induces a morphism $e^* : \mathrm{Pic}^+_{X/k}(W) \to \mathrm{Pic}(X)$. Likewise for f, a k-rational point of X. And, $\mathrm{Nat}_{pt.}\left(W, \mathrm{Pic}^+_{X/k}\right)$ is the kernel of e^*. Moreover, $e^* \times f^* : \mathrm{Pic}(X \times_k W) \to \mathrm{Pic}(X) \times \mathrm{Pic}(W)$ is a retraction of

$$p_1^* \times p_2^* : \mathrm{Pic}(X) \times \mathrm{Pic}(W) \to \mathrm{Pic}(X \times_k W).$$

Thus, $\mathrm{Nat}_{pt.}\left(W, \mathrm{Pic}^+_{X/k}\right)$ is isomorphic to

$$\frac{\mathrm{Pic}\,(X \times_k W)}{p_1^*\mathrm{Pic}(X) \times p_2^*\mathrm{Pic}\,(W)}.$$

☐

The idea of Lemma 4.10 comes from a well known property of abelian variety. If A and B are two k-abelian varieties, then $\mathrm{Pic}^0_{A/k}$ and $\mathrm{Pic}^0_{B/k}$ are two abelian varieties and with the exact same arguments as in the proof of Lemma 4.10, we can prove that

$$\mathrm{Hom}_{pt.}\left(A, \mathrm{Pic}^0_{B/k}\right) \cong \mathrm{Hom}_{pt.}\left(B, \mathrm{Pic}^0_{A/k}\right).$$

We are going to apply Lemma 4.10 to the study of the restricted Picard functor of unirational forms of $\mathbb{A}^n_k$.

Theorem 4.11. *Let X be a unirational form of $\mathbb{A}^n_k$ which admits a regular completion. Then:*

(i) The unipotent k-algebraic group $\mathrm{Pic}^+_{X/k}$ is étale;

(ii) The groups $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X_{k_s})$ are finite.

Proof. First, (ii) is an immediate consequence of (i). To prove that $\mathrm{Pic}^+_{X/k}$ is an *étale* k-algebraic group, we only need to prove that $\mathrm{Pic}^{+\,0}_{X/k}$ is trivial [DG70, Pro. II.5.1.4]. We can assume that $k = k_s$, then X admits a k-rational point.

The functor $\mathrm{Pic}^+_{X/k}$ is representable by a smooth unipotent k-algebraic group (Theorem 1.1), thus $\mathrm{Pic}^{+\,0}_{X/k}$ is a form of $\mathbb{A}^e_k$. According to Lemma 4.10:

$$\mathrm{Nat}_{pt.}\left(X, \mathrm{Pic}^+_{\mathrm{Pic}^{+\,0}_{X/k}/k}\right) \cong \mathrm{Nat}_{pt.}\left(\mathrm{Pic}^{+\,0}_{X/k}, \mathrm{Pic}^+_{X/k}\right).$$

Let W be an open of $\mathbb{A}^n_k$ ($n \in \mathbb{N}^*$), the only natural transformations from W into $\mathrm{Pic}^+_{\mathrm{Pic}^{+\,0}_{X/k}/k}$ are the constant one. Indeed, according to Lemma 4.10:

$$\mathrm{Nat}_{pt.}\left(W, \mathrm{Pic}^+_{\mathrm{Pic}^{+\,0}_{X/k}/k}\right) \cong \frac{\mathrm{Pic}\left(\mathrm{Pic}^{+\,0}_{X/k} \times_k W\right)}{p_1^*\mathrm{Pic}\left(\mathrm{Pic}^{+\,0}_{X/k}\right) \times p_2^*\mathrm{Pic}(W)}.$$

Moreover $\mathrm{Pic}(W) = \{0\}$, and

$$p_1^* : \mathrm{Pic}\left(\mathrm{Pic}^{+\,0}_{X/k}\right) \to \mathrm{Pic}\left(\mathrm{Pic}^{+\,0}_{X/k} \times_k W\right)$$

is an isomorphism. Indeed, the group morphism p_1^* is injective because p_1 admits a section induced by a rational point of W, and p_1^* is surjective [EGAIV4, Cor. 21.4.11].

Thus $\mathrm{Nat}_{pt.}\left(\mathrm{Pic}^{+\,0}_{X/k}, \mathrm{Pic}^+_{X/k}\right)$ is trivial, hence $\mathrm{Hom}_{pt.}\left(\mathrm{Pic}^{+\,0}_{X/k}, \mathrm{Pic}^{+\,0}_{X/k}\right)$ is also trivial. If $\mathrm{Pic}^{+\,0}_{X/k} \neq \{0\}$, then the identity is a nonconstant pointed endomorphism of $\mathrm{Pic}^{+\,0}_{X/k}$. Thus $\mathrm{Pic}^{+\,0}_{X/k} = \{0\}$, and we have prove (i). □

There is an other situation where the restricted Picard functor is *étale*:

Lemma 4.12. *Let X a form of $\mathbb{A}^n_k$ which admits a smooth completion. Then, $\mathrm{Pic}^+_{X/k}$ is an étale unipotent k-algebraic group.*

Proof. Let $\overline{X}$ be a smooth completion of X, then $\mathrm{Pic}^{+\,0}_{X/k} \cong \mathrm{Pic}^{+\,0}_{\overline{X}/k}$. Moreover, for some field extension K/k, we have $X_K \cong \mathbb{A}^n_K$. And $\overline{X}_K$ is a smooth completion of X_K, so

$$\{0\} = \mathrm{Pic}^{+\,0}_{X_K/K} \cong \mathrm{Pic}^{+\,0}_{\overline{X}_K/K} = \left[\left(\mathrm{Pic}_{\overline{X}/k}\right)_K\right]^{+\,0}.$$

Finally, by universal property of the smoothification, $\left(\mathrm{Pic}^{+\,0}_{\overline{X}/k}\right)_K$ is a subgroup of $\left[\left(\mathrm{Pic}_{\overline{X}/k}\right)_K\right]^{+\,0}$. Thus it is trivial. □

Corollary 4.13. *Let X be a form of $\mathbb{A}^n_k$, we assume that $\mathrm{Pic}^+_{X/k}$ is representable by an étale unipotent k-algebraic group (e.g. X is unirational and admits a regular completion).*

(i) Let W be a smooth k-scheme with a k-rational point. Then,

$$p_1^* \times p_2^* : \mathrm{Pic}(X) \times \mathrm{Pic}(W) \to \mathrm{Pic}(X \times_k W)$$

is an isomorphism of commutative groups.

(ii) We consider Y a form of $\mathbb{A}^c_k$, and we assume that $\mathrm{Pic}^+_{Y/k}$ is representable. Then, the restricted Picard functor $\mathrm{Pic}^+_{X\times Y/k}$ is representable and

$$p_1^* \times p_2^* : \mathrm{Pic}^+_{X/k} \times_k \mathrm{Pic}^+_{Y/k} \to \mathrm{Pic}^+_{X\times_k Y/k}$$

is an isomorphism of k-algebraic groups.

Proof. The affirmation (i) is a consequence of the triviality of $\mathrm{Pic}^{+\,0}_{X/k}$, and of Lemma 4.10. For (ii), we can assume that $k = k_s$; then the proof followed from (i) and Yoneda Lemma. $\square$

Corollary 4.13 is not unexpected, as the result is well known if we assume that X is rational. Hence, as far as the Picard group is concerned, unirational forms of $\mathbb{A}^n_k$ behave like the rational ones.

Corollary 4.14. *Let U be a unipotent k-group. We assume one of the following hypothesis:*

(a) U admits a regular completion, and U is unirational,

(b) U admits a smooth completion.

Then:

(i) The unipotent k-algebraic group $\mathrm{Pic}^{+}_{U/k}$ is étale.

(ii) $\mathrm{Pic}^{+}_{U/k} = \mathrm{Pic}^{+^U}_{U/k}$.

(iii) $\mathrm{Pic}(U) = \mathrm{Ext}^1(U, \mathbb{G}_{m,k})$ and this is a finite group of $p^{t(U)}$-torsion.

Proof. First (i) is a particular case of Theorem 4.11 or Lemma 4.12.

Then, (ii) is a consequence of the fact that any action of a connected k-algebraic group on an étale k-scheme is trivial, and of the definition of $\mathrm{Pic}^{+^U}_{U/k}$.

Finally, (iii) is a consequence of (i), (ii), Proposition 3.4, and Proposition 3.9. $\square$

Finally, let us remark that the methods we use in Lemma 4.10 and Theorem 4.11 generalize with the "usual" Picard functor.

Proposition 4.15. *If X is a geometrically integral, geometrically normal, projective and unirational k-algebraic variety, then $\left(\mathrm{Pic}^0_{X/k}\right)_{red}$ is trivial. Where $\left(\mathrm{Pic}^0_{X/k}\right)_{red}$ is the unique reduced closed subscheme of $\mathrm{Pic}^0_{X/k}$ having the same underlying topological space as $\mathrm{Pic}^0_{X/k}$ (see e.g. [Liu06, Pro. 2.4.2]).*

Moreover, if k is a field of characteristic 0, then $\mathrm{Pic}^0_{X/k}$ and $\mathrm{H}^1(X, \mathcal{O}_X)$ are trivial.

Proof. First, under our assumptions $\mathrm{Pic}^0_{X/k}$ is representable by a projective k-variety [Kle05, Th. 5.4]. Thus $\left(\mathrm{Pic}^0_{X/k}\right)_{red}$ is an Abelian variety whose formation commute with field extensions [Bri17, Lem. 3.3.7]. We can assume that $k = \overline{k}$, then $X(k) \neq \emptyset$. With the same arguments as in Lemma 4.10, we have:

$$\begin{aligned}
\mathrm{Hom}_{pt.}\left(\left(\mathrm{Pic}^0_{X/k}\right)_{red}, \left(\mathrm{Pic}^0_{X/k}\right)_{red}\right) &= \mathrm{Hom}_{pt.}\left(\left(\mathrm{Pic}^0_{X/k}\right)_{red}, \left(\mathrm{Pic}_{X/k}\right)_{red}\right) \\
&= \mathrm{Hom}_{pt.}\left(\left(\mathrm{Pic}^0_{X/k}\right)_{red}, \mathrm{Pic}_{X/k}\right) \\
&\cong \frac{\mathrm{Pic}\left(X \times_k \left(\mathrm{Pic}^0_{X/k}\right)_{red}\right)}{p_1^*\mathrm{Pic}(X) \times p_2^*\mathrm{Pic}\left(\left(\mathrm{Pic}^0_{X/k}\right)_{red}\right)} \\
&\cong \mathrm{Hom}_{pt.}\left(X, \mathrm{Pic}_{\left(\mathrm{Pic}^0_{X/k}\right)_{red}/k}\right).
\end{aligned}$$

As $\left(\mathrm{Pic}^0_{X/k}\right)_{red}$ is a smooth projective variety, any morphism from an open U of an affine space to $\mathrm{Pic}_{\left(\mathrm{Pic}^0_{X/k}\right)_{red}/k}$ is constant (Lemma 1.3). Thus $\left(\mathrm{Pic}^0_{X/k}\right)_{red}$ is trivial.

The second point is a consequence of the first one and of [BLR90, 8.4 Th. 1 (b)]. $\square$

4.4 Torsion of the restricted Picard functor

The Picard group of a form X of $\mathbb{A}^d_k$ is of p^n-torsion for some integer n larg enough [Ach17, Pro. 2.6]. In this subsection, we try to find the minimal n such that $\mathrm{Pic}(X)$ is of p^n-torsion.

In Subsection 2.2, we have defined two invariants $n(X)$ and $n'(X)$ of the forms of $\mathbb{A}^1_k$ (Definition 2.7). We can naturally generalize these definitions:

Definition 4.16. We consider a form X of $\mathbb{A}^d_k$.

(*i*) We denote by $n(X)$ the smallest non negative integer n such that $(X_{k_s})^{(p^n)} \cong \mathbb{A}^d_{k_s}$.

(*ii*) We denote by $n'(X)$ the smallest non negative integer n such that $(X_{k_s})^{(p^n)}$ is unirational.

We are now going to generalize results obtain in a previous article about the torsion of the Picard group of a form of $\mathbb{A}^1_k$ [Ach17, Th. 2.4 and Th. 4.4].

Proposition 4.17. *We consider a form X of $\mathbb{A}^d_k$. Then:*

(i) $\mathrm{Pic}(X)$ *is of* $p^{n(X)}$*-torsion;*

(ii) if X admits a regular completion, then $\mathrm{Pic}^+_{X/k}$ *is of* $p^{n(X)}$*-torsion;*

(iii) if X and $X^{\left(p^{n'(X)}\right)}$ admit regular completions, then $\mathrm{Pic}^{+\,0}_{X/k}$ *is of* $p^{n'(X)}$*-torsion.*

Proof. Without loss of generality, we can assume that k is separably closed (see Subsection 2.1, or the term of low degree of the Hochschild-Serre spectral sequence [SGAIV2, VIII Cor. 8.5]).

In Subsection 2.2, we recall the definition of the absolute and relative Frobenius morphism. In particular, there is a commutative diagram of morphisms of schemes (but not of morphisms of k-schemes!):

$$\begin{array}{ccc} X & \xrightarrow{\quad F^n_X \quad} & X \\ & {\scriptstyle F^n_{X/k}} \searrow \quad \nearrow {\scriptstyle \varphi^n_X} & \\ & X^{(p^n)}. & \end{array}$$

Moreover, if $\mathcal{L}$ is an invertible sheaf on X, then $(F^n_X)^*(\mathcal{L}) \cong \mathcal{L}^{\otimes p^n}$; thus, $(F^n_X)^*(\mathcal{L}) = p^n.\mathcal{L}$ in $\mathrm{Pic}(X)$. In particular, if $n = n(X)$, then $(F^n_X)^*(\mathcal{L}) = \left(F^n_{X/k}\right)^*(\mathcal{M})$ where $\mathcal{M} = (\varphi^n_X)^*(\mathcal{L}) = 0$ in $\mathrm{Pic}\left(X^{(p^n)}\right) = \{0\}$. Hence, $p^n.\mathcal{L} = 0$ in $\mathrm{Pic}(X)$, this is (*i*). Then, (*ii*) is an immediate consequence of (*i*) and Theorem 1.1.

Finally, we show (*iii*). As neither F^n_X, nor φ^n_X are morphisms of k-schemes we need to be a little bit careful when we define the pull-back on the restricted Picard functor. For any smooth k-scheme T, there is a commutative diagram:

$$\begin{array}{ccc} X \times_k T & \xrightarrow{F^n_{X\times T}} & X \times_k T \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ T & \xrightarrow[F^n_T]{} & T, \end{array}$$

that is functorial in T Hence, the absolute Frobenius morphism induces a natural transformation of group functor

$$(F^n_X)^* : \mathrm{Pic}^+_{X/k} \to \mathrm{Pic}^+_{X/k}.$$

Likewise, there is a commutative diagram

$$\begin{array}{ccc} X^{(p^n)} \times_k T & \xrightarrow{\varphi^n_X \times F^n_T} & X \times_k T \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ T & \xrightarrow[F^n_T]{} & T, \end{array}$$

that is functorial in T. Hence, φ_X^n induces a natural transformation

$$(\varphi_X^n)^* : \mathrm{Pic}^+_{X/k} \to \mathrm{Pic}^+_{X^{(p^n)}/k}.$$

Moreover, $F^n_{X/k}$ is a morphism of k-scheme, thus it induces a natural transformation $\left(F^n_{X/k}\right)^*$, with $(F^n_X)^* = (\varphi_X^n)^* \circ \left(F^n_{X/k}\right)^*$. We apply it with $n = n'(X)$, then $\mathrm{Pic}^+_{X^{(p^n)}/k}$ is *étale* (Theorem 4.11). Hence, $(\varphi_X^n)^* : \mathrm{Pic}^{+\,0}_{X/k} \to \mathrm{Pic}^+_{X^{(p^n)}/k}$ is the zero-morphism. And finally, $\mathrm{Pic}^{+\,0}_{X/k}$ is of $p^{n'(X)}$-torsion. □

4.5 Reduction to the k-strongly wound case

We consider an unipotent k-group U of dimension d such that $\mathrm{Pic}^+_{U/k}$ is representable by an *étale* unipotent k-algebraic group (e.g. U is unirational with a regular completion). Let $f : X \to Y$ be a U-torsor, where Y is a form of $\mathbb{A}^n_k$ (then X is a form of $\mathbb{A}^{n+d}_k$). We denote by $\alpha : U \times_k X \to X$ the action of U on X, and by $p_2 : U \times_k X \to X$ the second projection. Then, for any smooth k-scheme T, the sequence of abstract groups

$$0 \to \mathrm{Pic}^+_{Y/k}(T) \xrightarrow{f_T^*} \mathrm{Pic}^+_{X/k}(T) \xrightarrow{\alpha_T^* - p_{2T}^*} \mathrm{Pic}^+_{U\times X/k}(T),$$

is exact (see e.g. [Bri15, Pro. 2.10]). We will just say that

$$0 \to \mathrm{Pic}^+_{Y/k} \xrightarrow{f^*} \mathrm{Pic}^+_{X/k} \xrightarrow{\alpha^* - p_2^*} \mathrm{Pic}^+_{U\times X/k},$$

is an exact sequence of group functors.

Moreover, we have a natural transformation

$$\pi_1 : \mathrm{Pic}^+_{U\times X/k} \to \mathrm{Pic}^+_{U/k}$$

induced by the the isomorphism of functor of Corollary 4.13 (*ii*). We denote

$$\varphi = \pi_1 \circ (\alpha^* - p_2^*).$$

Then,

$$0 \to \mathrm{Pic}^+_{Y/k} \xrightarrow{f^*} \mathrm{Pic}^+_{X/k} \xrightarrow{\varphi} \mathrm{Pic}^+_{U/k}$$

is an exact sequence of group functors.

Moreover, if $\mathrm{Pic}^+_{X/k}$ is representable, then $\mathrm{Pic}^+_{Y/k}$ is representable by the kernel of φ. And $\varphi : \mathrm{Pic}^{+\,0}_{X/k} \to \mathrm{Pic}^+_{U/k}$ is trivial. Thus, $\mathrm{Pic}^{+\,0}_{Y/k} \xrightarrow{f^*} \mathrm{Pic}^{+\,0}_{X/k}$ is an isomorphism of k-groups.

Hence, we have obtained the following result:

Proposition 4.18. *We consider a unipotent k-group U, a form of the affine n-space Y and a U-torsor $f : X \to Y$. We assume that the restricted Picard functor of U is representable by an étale unipotent k-algebraic group.*

Then, the sequence of group functors:

$$0 \to \mathrm{Pic}^+_{Y/k} \xrightarrow{f^*} \mathrm{Pic}^+_{X/k} \xrightarrow{\varphi} \mathrm{Pic}^+_{U/k}$$

is exact.

Moreover, if $\mathrm{Pic}^+_{X/k}$ is representable, then $\mathrm{Pic}^+_{Y/k}$ is also representable and

$$\mathrm{Pic}^{+\,0}_{Y/k} \xrightarrow{f^*} \mathrm{Pic}^{+\,0}_{X/k}$$

is an isomorphism of k-groups.

If U is a commutative unipotent k-group of dimension d, then there is an fppf morphism $f : U \to U_{sw}$ where U_{sw} is k-strongly wound (Proposition 4.8). Let us assume resolution of singularities in dimension inferior or equal to d. As U_{sw} is obtained after a finite number of quotients by some unirational k-group, we can apply Proposition 4.18 for each quotient. And thus,

$$\mathrm{Pic}^{+\ 0}_{U_{sw}/k} \xrightarrow{f^*} \mathrm{Pic}^{+\ 0}_{U/k}$$

is an isomorphism of k-groups.

Hence, we have reduced (up to resolution of singularities) the study of the neutral component of the restricted Picard functor of a commutative unipotent k-group to the case of commutative k-strongly wound unipotent k-group.

4.6 Mapping property of the restricted Picard functor of a form of the affine line

In this subsection, we obtain a mapping property for the restricted Picard functor of a form of $\mathbb{A}^1_k$ with a k-rational point. This universal property is a particular case of the universal properties of the "generalized Jacobian varieties". Generalized Jacobian varieties have been studied by M. Rosenlicht [Ros54, Ros57] and J.-P. Serre [Ser12, Chap. V]. In this subsection, we use the more modern presentation of S. Bosch, W. Lütkebohmert and M. Raynaud [BLR90, §10.3].

Let us consider X a form of $\mathbb{A}^1_k$ with a k-rational point P_0. We consider the morphism of k-scheme $\mathcal{I}_{P_0} : X \to \mathrm{Pic}^{+\ 0}_{X/k}$ defined by the divisor $\Delta - [P_0] \times X$ on $X \times_k X$, where Δ is the diagonal divisor [KMT74, Lem. 6.7.4]. Then, if the arithmetic genus of X is positive, $\mathcal{I}_{P_0}$ is a closed immersion [KMT74, Th. 6.7.9].

Proposition 4.19. *We consider a form of $\mathbb{A}^1_k$ with a k-rational point P_0 denoted X, a commutative k-strongly wound unipotent k-group U and a morphism of k-schemes $f : X \to U$ such that the image of P_0 is the identity element of U.*

Then, there is a unique morphism of k-algebraic group $F : \mathrm{Pic}^{+\ 0}_{X/k} \to U$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathcal{I}_{P_0}} & \mathrm{Pic}^{+\ 0}_{X/k} \\ & \searrow^{f} & \downarrow F \\ & & U \end{array}$$

is commutative.

Proof. We denote by $\overline{X}$ the (canonical) regular completion of X, then $\mathrm{Pic}^{+\ 0}_{X/k} = \mathrm{Pic}^0_{\overline{X}/k}$ is k-strongly wound. Thus, the existence and uniqueness of F is a particular case of Corollary [BLR90, 10.3 Cor. 3]. □

We use the mapping property above and the results of Subsection 4.3 to compute some particular case of Picard group of the product of two forms of the affine space.

First, we will recall a well known formula for the Picard group of the product of curves: if C and C' are two proper geometrically connected smooth k-curves, then

$$0 \to \mathrm{Pic}(C) \times \mathrm{Pic}(C') \to \mathrm{Pic}(C \times_k C') \to \mathrm{Hom}_{grp.}\left(\mathrm{Pic}^0_{C/k}, \mathrm{Pic}^0_{C'/k}\right) \to 0$$

is an exact sequence of commutative groups.

Proposition 4.20. *Let X be a form of $\mathbb{A}^1_k$ with a k-rational point, and Y be a form of $\mathbb{A}^n_k$ with a regular completion. Then,*

$$0 \to \mathrm{Pic}(X) \times \mathrm{Pic}(Y) \to \mathrm{Pic}(X \times_k Y) \to \mathrm{Hom}_{grp.}\left(\mathrm{Pic}^{+\ 0}_{X/k}, \mathrm{Pic}^{+\ 0}_{Y/k}\right) \to 0,$$

is an exact sequence commutative groups.

Proof. This is a consequence of Lemma 4.10, and Proposition 4.19. □

Example 4.21. Let k be a non perfect field of characteristic $p = 2$, and $a, b \in k$ such that $a \notin k^2$ or $b \notin k^2$. We consider G the form of $\mathbb{G}_{a,k}$ defined by the equation $y^4 = x + bx^2 + a^2x^4$. Then the arithmetic genus of the regular completion of G is one, and $\mathrm{Pic}^{+\,0}_{G/k}$ is isomorphic to G.

Moreover, $\mathrm{Hom}_{grp}(G, G) = \mathbb{Z}/2\mathbb{Z}$ [Rus70, Th. 3.1]. Thus, we have an exact sequence:

$$0 \to \mathrm{Pic}(G) \times \mathrm{Pic}(G) \to \mathrm{Pic}(G \times_k G) \to \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0.$$

Example 4.22. Let k be a non perfect field of characteristic $p = 3$, and $a \in k$ such that $a \notin k^3$. We consider G the form of $\mathbb{G}_{a,k}$ defined by the equation $y^3 = x + ax^3$.

Then likewise, $\mathrm{Pic}^{+\,0}_{G/k}$ is isomorphic to G, and $\mathrm{Hom}_{grp}(G, G) = \mathbb{Z}/3\mathbb{Z}$ [Rus70, Th. 3.1]. Thus, we have an exact sequence:

$$0 \to \mathrm{Pic}(G) \times \mathrm{Pic}(G) \to \mathrm{Pic}(G \times_k G) \to \frac{\mathbb{Z}}{3\mathbb{Z}} \to 0.$$

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