GELFAND-ZETLIN POLYTOPES AND THE GEOMETRY OF FLAG VARIETIES

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Gelfand-Zetlin polytopes are important in the finite dimensional representation theory of $SL_n(\mathbb{C})$ and the symplectic geometry of coadjoint orbits of the unitary group. We examine the combinatorics of Gelfand-Zetlin polytopes in relation to the geometry of the flag variety of $SL_n(\mathbb{C})$. The two main contributions of the thesis are as follows: (1) we describe virtual Gelfand-Zetlin polytopes associated to non-dominant weights and (2) we identify the cohomology ring of the flag variety with a quotient of the subalgebra of the Chow cohomology ring of the Gelfand-Zetlin toric variety generated in degree one. More precisely, we take the largest quotient of this subalgebra that satisfies Poincarè duality.

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PREFACE

I am grateful to my thesis advisor, Kiumars Kaveh, for countless hours of instruction, discussion, feedback and advice. He patiently answered my numerous questions, usually over breakfast. His friendly encouragement has been immensely helpful throughout this entire process.

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1.0 INTRODUCTION

To each finite dimensional irreducible representation V_λ of $SL_n(\mathbb{C})$ one associates a Gelfand-Zetlin[1] (GZ) polytope $\Delta_\lambda \subset \mathbb{R}^{n(n-1)/2}$. The lattice points in Δ_λ parametrize a natural basis for the irreducible representation V_λ [GZ50]. The geometry of the flag variety $\mathcal{F}\ell_n(\mathbb{C})$ is intimately connected to the representation theory of $SL_n(\mathbb{C})$, and it plays an important role in displaying interactions between representation theory, algebraic geometry, symplectic geometry, and combinatorics. In this thesis we investigate the combinatorics of GZ polytopes in connection to the geometry of $\mathcal{F}\ell_n(\mathbb{C})$. Our three main results can be described as follows. First, we prove that the collection of GZ polytopes of a given dimension have the same normal fan Σ_{GZ} and any polytope normal to this fan is a translation of a GZ polytope. Second, we describe the virtual GZ polytopes in terms of convex chains in the vector space of virtual polytopes following [PK93]. Finally, we identify the cohomology ring of the flag variety as a quotient of the subring of the operational Chow ring of the toric variety of the GZ fan Σ_{GZ} generated in degree one.

We recall that a fan Σ is a finite collection of convex rational polyhedral cones closed under intersection and such that any face of a cone in Σ is also in Σ. The normal fan to a polytope P contains all rays normal to the facets of P , as well as a cone σ_F for each face F which is generated by the rays corresponding to the facets containing F. We consider the normal fan to a GZ polytope. Let $\lambda \in \mathbb{R}^n$ with $\lambda = (\lambda_1 \leq \lambda_2 \ldots \leq \lambda_n)$. The GZ polytope Δ_λ

[1] Note that in the literature, Zetlin is sometimes spelled Cetlin or Tsetlin.

is the set of $(x_{ij}) \in \mathbb{R}^{n(n-1)/2}$ satisfying the array of inequalities (2.1) below.

$$\begin{array}{ccccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & & \dots & & \lambda_n \\ & x_{12} & & x_{23} & & \dots & & x_{(n-1)n} & \\ & & x_{13} & & x_{24} & & \dots & & \\ & & & \ddots & & \iddots & & & \\ & & & & x_{1n} & & & & \end{array}$$

Each small triangle in this array $\begin{array}{ccc} a & & b \\ & c & \end{array}$ corresponds to the inequalities $a \leq c \leq b$. See Section 2.3 for more details.

In Section 3.1, we prove the following propositions about the normal fan of GZ polytopes (for fixed n).

Proposition (3.1.1)**.** *The normal fan Σ_λ for a GZ polytope Δ_λ is independent of λ for $\lambda = (\lambda_1 < \lambda_2 < \ldots < \lambda_n)$ dominant regular.*

This enables us to talk about the GZ fan Σ_{GZ}, rather than the normal fan of a specific Δ_λ, which is important for our later results. The second proposition which we prove in Section 3.1 describes the rest of the polytopes normal to Σ_{GZ}.

Proposition (3.1.2)**.** *Let P be a full dimensional polytope normal to Σ_{GZ}, then $P = c + \Delta_\lambda$ for some dominant regular λ and $c \in \mathbb{R}^N$. Moreover, if P is a lattice polytope then both c and λ are integral.*

With these foundational facts about the GZ polytopes established, we next expand the definition of Δ_λ to arbitrary $\lambda \in \mathbb{R}^n$. Such a Δ_λ is a virtual GZ polytope.

1.1 VIRTUAL GELFAND-ZETLIN POLYTOPES

We can extend the set of convex polytopes to the vector space of virtual polytopes. A virtual polytope is a formal difference $P_1 - P_2$ where P_1 and P_2 are convex polytopes. As we show in 2.3.2, the map $\lambda \mapsto \Delta_\lambda$ is additive, i.e. for $\lambda, \mu \in \mathbb{R}^n$ the polytope $\Delta_{\lambda+\mu}$ is the

Minkowski sum of Δ_λ and Δ_μ. Thus the definition of Δ_λ can be extended to all $\lambda \in \mathbb{R}^n$: for $\lambda = \mu - \gamma$ let $\Delta_\lambda := \Delta_\mu - \Delta_\gamma$. One may naturally ask: *can we describe the virtual GZ polytopes in a similar fashion as the usual GZ polytopes?* We recall the construction from [PK93] of the representation of a virtual polytope as a linear combination of characteristic functions of convex polyhedra. We apply this to the collection of polytopes normal to the GZ fan Σ_{GZ} and describe virtual GZ polytopes in terms of linear combinations of characteristic functions of convex polyhedra. Note that virtual polytopes may consist of multiple, possibly unbounded, convex regions.

To develop an intuition for virtual polytopes, we consider twisted cubes in Section 3.2.4. A twisted cube is a virtual polytope combinatorially equivalent to a hypercube together with a density function. Moreover, we prove the following.

Theorem (3.2.5)**.** *For λ dominant regular, the GZ polytope Δ_λ is a translation of a twisted cube.*

We review the Khovanskii-Pukhlikov theory of convex chains [PK93] in Section 3.2.2. A convex chain is a linear combination of characteristic functions of convex polytopes. The convex chain χ_P associated to a convex polytope P is the characteristic function of the set P. Khovanskii and Pukhlikov show in [PK93] that there is a convolution operation $\star$ on convex chains such that $\chi_P \star \chi_Q = \chi_{P+Q}$ where $P+Q$ is the Minkowski sum of polytopes P and Q. They prove formulas for the convex chain of the interior of a polytope as well as the inverse χ_P^{-1} with respect to $\star$ which we record in Theorem 3.2.1. We describe the convex chain of virtual GZ polytopes. More specifically, we determine the value of the convex chain on each region of a virtual GZ polytope.

The Brianchon-Gram Theorem is required to compute the value of a convex chain. We record this in Theorem 3.3. This describes χ_P in terms of characteristic functions of cones at faces of P. Khovanskii and Pukhlikov extend this to the case of convex chains in [PK93, Section 4 Proposition 2, p. 352].

GZ polytopes are not simple polytopes (except when $n = 1$ or $n = 2$) which complicates the study of corresponding toric varieties. An N-dimensional polytope is called simple if every vertex lies in exactly N facets. The non-simplicity of GZ polytopes can be observed even in the case $n = 3$ where $N = n(n-1)/2 = 3$. See Example 2.3.1. We explore the relations in the normal fan Σ_{GZ} coming from the fact that Δ_λ is not simple in Section 3.2.5. In Section 3.2.6 we carefully consider the virtual GZ polytopes in the case $n = 2$, and in Section 3.2.7 we examine a particular example of a virtual GZ polytope in three dimensions. Finally, in Section 3.2.8 we prove our result about general virtual GZ polytopes.

Theorem (Summary, see Theorem 2.2.1)**.** *A virtual Gelfand-Zetlin polytope corresponds to a convex chain supported on finitely many bounded convex regions. The convex chain takes either the value 1 or -1 on each full-dimensional region. From the GZ array, we determine the inequalities defining each convex region as well as the value of the convex chain on that region.*

We recall that the usual GZ polytope Δ_λ for λ strictly increasing is associated with an irreducible representation V_λ of $SL_n(\mathbb{C})$. We review the relevant representation theory in Section 2.2. In Section 2.4.1 we recall the Borel-Weil-Bott Theorem 2.4.2 which relates the irreducible representation V_λ with the space of sections of line bundle L_λ on the flag variety G/B. We review these definitions in Section 2.4. In Section 2.2.2 we recall the decomposition of V_λ which leads to Theorem 2.2.1 by Gelfand and Zetlin [GZ50] identifying the GZ basis of V_λ. These basis vectors correspond to lattice points in Δ_λ.

1.2 COHOMOLOGY OF G/B AND CHOW COHOMOLOGY OF X_{GZ}

How does the geometry of G/B relate to that of the toric variety X_{GZ} constructed from Δ_λ?

In this section we summarize our results relating the cohomology of G/B with the Chow ring of X_{GZ}.

In Section 2.1 we review the construction of a toric variety from a polytope Δ_λ or equivalently from a fan Σ_{GZ}. Either of these constructions can be used to define the GZ toric variety X_{GZ}. To understand the cohomology of this variety, we first recall the construction of Chow cohomology for smooth toric varieties in Section 3.3. The variety X_{GZ} is not smooth because Δ_λ is not simple, or equivalently because Σ_{GZ} is not simplicial. Thus we need to use operational Chow cohomology instead, which is identified in [FS97] with the ring of Minkowski weights that we describe in Section 3.3.3. We include a very detailed example of the ring of Minkowski weights for X_{GZ} in Section 3.3.3.1.

The flag variety G/B is a smooth projective variety via the Plücker embedding with cellular decomposition given by the Bruhat cells, so Proposition 3.3.6 states that the Chow ring $A^*(G/B) \cong H^*(G/B)$ where the isomorphism doubles degree. The Borel description gives a concrete description of this graded ring as a quotient of the ring of polynomials, see Equation (3.6).

Our main result relating the cohomology of the flag variety and Chow cohomology of X_{GZ} requires the following terminology. The Lefschetz subalgebra of a graded algebra is the subalgebra generated by its degree one piece. The Gorenstein quotient of a graded algebra is the largest quotient of the algebra satisfying Poincaré duality. See Section 3.3.1 for details.

Theorem (3.3.9)**.** *The Chow ring $A^*(G/B)$ can be identified with the Gorenstein quotient of the Lefschetz subalgebra of $A^*(X_{GZ})$.*

To prove Theorem 3.3.9, we first establish two general lemmas about graded algebras, as well as recall an algebra lemma from [Kav11]. Our first lemma characterizes the Gorenstein quotient of a graded ring.

Lemma (3.3.1)**.** *Let $A = \bigoplus_{i=0}^{n} A^i$ with $A^0 \cong \mathbb{Z} \cong A^n$. There exists a homogeneous ideal $I \subset A$ which is minimal with respect to inclusion such that A/I has Poincarè duality. We call this ring A/I the* Gorenstein quotient *$Gor(A)$ of A.*

Our second lemma provides the essential machinery for the proof of Theorem 3.3.9.

Lemma (3.3.3)**.** *Suppose $A = \bigoplus_{i=0}^{n} A^i$ and $B = \bigoplus_{i=0}^{n} B^i$ both have degree zero and degree n pieces isomorphic to $\mathbb{Z}$, are generated in degree one, and ring A has Poincarè duality. Suppose additionally that*

- *there exists isomorphism $\varphi : A^1 \to B^1$ and*
- *for all $a_1, \dots, a_n \in A^1$ we have*

$$a_1 \cdot \ldots \cdot a_n = \varphi(a_1) \cdot \ldots \cdot \varphi(a_n)$$

using fixed isomorphisms $A^n \cong \mathbb{Z} \cong B^n$.

Then φ extends to give an isomorphism of A with the Gorenstein quotient of B, i.e.,

$$\tilde{\varphi} : A \xrightarrow{\cong} Gor(B).$$

Utilizing these results as well as Theorem 3.3.2, we identify both $A^*(G/B)$ and the Gorenstein quotient of the Lefschetz subalgebra of $A^*(X_{GZ})$ with quotients of polynomial rings. Upon inspection, the polynomial rings are isomorphic, and the ideals annihilated in the quotients are isomorphic yielding our result.

The organization of the paper is as follows. Chapter 2 establishes background useful for multiple results organized by section. Chapter 3 is divided into three sections, each developing the more specialized background necessary only for the results in that section. Results about the Gelfand-Zetlin fan are in Section 3.1, virtual GZ polytopes are described in Section 3.2 and the geometry of G/B and X_{GZ} are described and related in Section 3.3.

2.0 PRELIMINARIES

In this chapter we review the necessary background from toric varieties, representation theory, flag varieties, and GZ polytopes.

2.1 TORIC VARIETIES

A *toric variety* is a variety V containing a torus $T \cong (\mathbb{C}^*)^n$ as an open dense subset such that the natural action of T on itself extends to an action of T on V. The existence of such a torus action causes toric varieties to have many combinatorial features, some of which we explain below. More details about toric varieties can be found in [CLS11] or [Ful93].

Recall that a character of T is a group homomorphism $\chi^m : T \to \mathbb{C}^*$ with $\chi^m(t) = t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$ for some $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Hence the group M of characters of T (the *character lattice* of T) can be identified with $\mathbb{Z}^n$.

Dual to this picture, we consider one-parameter subgroups of T which are given by homomorphisms $\mathbb{C}^* \to T$. These are of the form $t \mapsto (t^{u_1}, \ldots, t^{u_n})$ for integers $(u_1, \ldots, u_n) \in \mathbb{Z}^n$. The group N of one-parameter subgroups of T can be identified with $\mathbb{Z}^n$. The lattice N is dual to M as the composition

$$\mathbb{C}^* \to T \to \mathbb{C}^*$$

is given in coordinates by

$$t \mapsto (t^{u_1}, \ldots, t^{u_n}) \mapsto t^{u_1 m_1} \cdots t^{u_n m_n} = t^{u \cdot m}.$$

2.1.1 Constructing Toric Varieties

From a finite subset $\mathcal{A}$ of a lattice M we construct a toric variety in the following manner. Suppose $\mathcal{A} = \{m_1, \dots, m_r\}$ is a finite collection of characters of T and consider the map $T \to \mathbb{C}^r$ given by

$$t \mapsto (t^{m_1}, \dots, t^{m_r}).$$

The variety $Y_{\mathcal{A}}$ is the closure of the image of this map inside $\mathbb{C}^r$. As this space is constructed from characters of the torus, it inherits an action of T. Because each component of the map is given by a monomial, it is algebraic.

Another way to construct a toric variety is to start with an affine semigroup S. An affine semigroup S is a semigroup in $\mathbb{Z}^n$ generated by a finite subset $\mathcal{A} = \{m_1, \dots, m_r\}$, that is,

$$S = \mathbb{N}\mathcal{A} = \left\{ \sum_{i=1}^{r} n_i m_i \ : \ n_i \in \mathbb{N}, m_i \in \mathcal{A} \right\}.$$

To construct an affine toric variety from S, we consider the semigroup algebra

$$\mathbb{C}[S] = \left\{ \sum_{m \in S} c_m \chi^m | \ \text{all but finitely many } c_m \text{ are zero} \right\}$$

where the multiplicative structure of $\mathbb{C}[S]$ is induced by the semigroup structure of S. Since S is finitely generated, say by $\mathcal{A} = \{m_1, \dots, m_r\}$, the semigroup algebra will also be finitely generated by characters $\{\chi^{m_1}, \dots, \chi^{m_r}\}$ of T. Also, $\mathbb{C}[S]$ is an integral domain, and the variety $\mathrm{Spec}(\mathbb{C}[S])$ is toric. Every affine toric variety is of the form $Y_S = \mathrm{Spec}(\mathbb{C}[S])$ for some affine semigroup S.

2.1.2 Cones and Affine Toric Varieties

In addition to the lattices M and N, we will need to consider also the vector spaces $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. For a finite set $S \subset N_{\mathbb{R}}$, the *convex polyhedral cone* generated by S is

$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{s \in S} \lambda_s s \mid \lambda_s \geq 0 \right\}.$$

This cone is convex because it is closed under addition and positive scalar multiplication. It is called *polyhedral* because it is can be realized as the intersection of finitely many half-spaces

$$H_a^+ = \{u \in N_{\mathbb{R}} | \langle u, a\rangle \geq 0\}.$$

We are interested in cones σ which are also *rational*, meaning elements of S are in N rather than just $N_{\mathbb{R}}$, and *strongly convex*, meaning $\{0\}$ is a face of σ, or equivalently, σ does not contain any whole lines.

We have defined the cone σ to be in the vector space $N_{\mathbb{R}}$, while the semigroup S was constructed from elements of $M_{\mathbb{R}}$, the vector space dual to $N_{\mathbb{R}}$. To define a toric variety from the cone σ, we first take its dual

$$\sigma^\vee = \{m \in M_{\mathbb{R}} | \langle m, s\rangle \geq 0 \text{ for all } s \in \sigma\}.$$

If $\sigma^\vee$ is generated by $\{m_1, \ldots, m_r\}$, that is, $\sigma^\vee = \text{Cone}(\{m_1, \ldots, m_r\})$, then the r half-spaces $H_{m_i}^+$ will be exactly the ones which cut out the cone $\sigma \subset N_{\mathbb{R}}$, and for any other $m \in \sigma^\vee$ the cone σ will lie in the half-space H_m^+.

Considering intersecting the cone σ with hyperplane

$$H_m = \{u \in N_{\mathbb{R}} : \langle u, m\rangle = 0\}$$

for $m \in \sigma^\vee$, this will give us a *face* of σ, that is,

$$\tau = \sigma \cap H_m.$$

Of particular interest are *facets* (faces of codimension 1) and *rays* (faces of dimension 1). For a strongly convex rational polyhedral cone σ, there is a particularly nice generating set. For each ray ρ of σ let u_ρ be the smallest nonzero element of the semigroup $\rho \cap N$. Because the cone is rational this intersection $\rho \cap N$ must be nonempty. We call u_ρ the *ray generator* for ray ρ, and then the cone σ is generated by the set $S = \{u_\rho\}_{\rho \text{ ray}}$. The facets τ are important faces of σ because, when our cone is full dimensional, $\tau^\vee$ will be one dimensional and hence lead to a generator for $\sigma^\vee$.

Finally, we construct a toric variety. From a rational convex polyhedral cone σ we obtain a finitely generated affine semigroup $S_\sigma = \sigma^\vee \cap M$ which lies in the proper vector space. We can then define the toric variety $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$.

A face τ of the cone σ is itself a rational convex polyhedral cone, so it is natural to wonder how the two toric varieties U_τ and U_σ relate. Suppose H_m for some $m \in \sigma^\vee$ is the hyperplane whose intersection with σ yields τ, then we have $\sigma^\vee \subset \tau^\vee$ and in particular $m\mathbb{R} \subset \tau^\vee$. It turns out $S_\tau = S_\sigma + m\mathbb{Z}$ and so $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}$. Thus U_τ is an affine open subset of U_σ. This can be used to glue together two affine toric varieties U_σ and $U_{\sigma'}$ along an affine open subset U_τ so long as τ is a face of both cones σ and σ'.

2.1.3 Projective Toric Varieties and Polytopes

For a finite set $\mathcal{A} \subset M$ we constructed $Y_\mathcal{A}$ as the closure of the image of $t \mapsto (t^{m_1}, \ldots, t^{m_r})$ inside $\mathbb{C}^r$. In a similar way, we can construct a projective toric variety from a finite set $\mathcal{A}$ by composing the above map with the projection $(\mathbb{C}^*)^r \to \mathbb{P}^{r-1}$.

From the same finite set $\mathcal{A}$ we can define a polytope $P = \operatorname{Conv}(\mathcal{A})$. There is an open cover of $X_\mathcal{A}$ parametrized by vertices of P. A vertex $m \in P$ corresponds to the affine open subset $X_\mathcal{A} \cap U_m$ where U_m is the standard affine open subset of $\mathbb{P}^{r-1}$ given by non-vanishing of the coordinate corresponding to m. One shows that $X_\mathcal{A} \cap U_m$ is isomorphic to $\operatorname{Spec}(\mathbb{C}[S_m])$ where S_m is the semigroup $\mathbb{N}(\mathcal{A} - m)$, that is, generated by translates of elements of $\mathcal{A}$ by m. We are interested in the case where P is a *very ample lattice polytope* because each open subset $X_\mathcal{A} \cap U_m$ can be realized as the toric variety of a cone. Recall that a lattice polytope is the convex hull of finitely many lattice points. A polytope P is very ample if for every vertex m, the semigroup $\mathbb{N}(P \cap M - m)$ is *saturated* in M. A semigroup S is saturated in M if for any $k \in \mathbb{N}, m \in M$ we have $km \in S$ implies $m \in S$.

Suppose $P \subset M_\mathbb{R}$ is a very ample lattice polytope of full dimension, then we can associate the projective toric variety $X_\mathcal{A}$ to P, where $\mathcal{A} = P \cap M$ is the finite set used in the construction. This projective variety has a nice affine cover corresponding to vertices of P. The affine variety associated to vertex m_i is $\operatorname{Spec}(\mathbb{C}[\sigma_i^\vee \cap M])$ where $\sigma_i^\vee$ is the cone $\operatorname{Cone}(\mathcal{A} - m_i)$. These cones $\sigma_i^\vee$ fit together into the *normal fan* for the polytope P.

A *fan* Σ is a collection of cones where each face of $\sigma \in \Sigma$ is contained in the collection and if two cones σ and σ' have non-empty intersection, then $\tau = \sigma \cap \sigma'$ is a face of each of the original cones (and hence also contained in Σ). The fan Σ coming from the polytope P is called a normal fan because it can be built from the normal vectors to facets of P. In general, one constructs a toric variety X_Σ from a fan Σ by gluing the affine toric varieties $\{U_\sigma\}_{\sigma\in\Sigma}$ as described in the previous section.

2.2 REPRESENTATION THEORY

We review some fundamentals of the representation theory of SL_n in order to motivate the definition of Gelfand-Zetlin polytopes.

We recall that a representation of a group G is a homomorphism

$$G \mapsto \mathrm{GL}(V)$$

for a vector space V, or equivalently, a representation is a vector space V regarded as a G-module where each $g \in G$ acts linearly. For G-modules V and W, there is a natural representation $V \oplus W$ where $g \cdot (v, w) = (g \cdot v, g \cdot w)$, so the philosophy is to first understand *irreducible representations* which do not contain any proper subrepresentations. In general a representation may be indecomposable (not decompose into a direct sum of two representations) but still contain a proper subrepresentation. This does not happen, however, for the group $G = SL_n(\mathbb{C})$ as it is a reductive group. Every finite dimensional representation of $SL_n(\mathbb{C})$ decomposes into a direct sum of irreducible representations.

2.2.1 Irreducible Representations

In order to study the irreducible representations for SL_n, we first want to recall some information about representations of tori, since representations of a maximal torus $T \subset SL_n$ will help us to understand the irreducible representations of SL_n.

A one-dimensional representation of a torus $T \cong (\mathbb{C}^*)^n$ is an algebraic homomorphism

$$T \to \mathbb{C}^* = GL_1(\mathbb{C})$$

and thus given in coordinates by

$$z = (z_1, \ldots, z_n) \mapsto z_1^{a_1} \cdots z_n^{a_n} = z^a$$

where $a \in \mathbb{Z}^n$. For an arbitrary finite dimensional algebraic representation V of T, we can consider the subspace $V_a \subset V$ defined by

$$V_a = \{v \in V : z \cdot v = z^a v\}$$

and then we will have the decomposition

$$V = \bigoplus_{a \in \mathbb{Z}^n} V_a$$

where only finitely many of the V_a are nontrivial. In other words, any representation for T decomposes completely into components V_a. We call $a \in \mathbb{Z}^n$ the weight of the representation and V_a the weight space. This decomposition of an arbitrary finite dimensional representation of T into one-dimensional weight spaces enables much of what follows.

Since representations of tori decompose completely, it is useful to consider a maximal torus $T \subset SL_n$. In coordinates, T is the set of diagonal matrices with determinant equal to one. There are many maximal tori in SL_n all conjugate to each other. Another important subgroup of SL_n is a Borel subgroup B. We consider the subgroup of upper triangular matrices with determinant one as it contains our maximal torus. There are many Borel subgroups all conjugate to each other, each containing a maximal torus.

To understand an irreducible representation V of $SL_n(\mathbb{C})$, we first consider it as a representation of $T \subset SL_n(\mathbb{C})$. Then V decomposes into a direct sum of V_a for finitely many $a \in \mathbb{Z}^n$. We do this for the adjoint representation.

Recall that in the adjoint representation $G \to GL(\mathfrak{g})$, $g \in G$ acts on the Lie algebra $\mathfrak{g}$ by $g \cdot X = gXg^{-1}$ for $X \in \mathfrak{g}$. For $G = SL_n(\mathbb{C})$, the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is a vector space with basis given by the matrices E_{ij} for $i \neq j$ with zero in every position except for (i, j) where the entry is 1. The Borel subgroup B chosen above has Lie algebra $\mathfrak{b}$ with basis E_{ij} where $i < j$.

We examine the action of T on $\mathfrak{b}$. The eigenvalues of this representation are the positive roots of $\mathfrak{sl}_n$. Simple roots are positive roots of the form $L_i = (0, \cdots, 0, -1, 1, 0, \cdots)$ with the entry -1 occurring in the ith position. The choice of positive roots determines the *positive Weyl chamber*, that is, the set

$$\{x \mid \langle x, \alpha \rangle \geq 0 \text{ for each positive root } \alpha\}.$$

For the simple roots L_i defined above, a vector $x = (x_1, \ldots, x_n)$ is in the positive Weyl chamber exactly when $-x_i + x_{i+1} \geq 0$ for all i, that is, when components of x are increasing $x_1 \leq x_2 \leq \ldots \leq x_n$.

We recall that irreducible representations of SL_n correspond to lattice points in the positive Weyl chamber, so to irreducible representation V we associate the *highest weight vector* $\lambda = (\lambda_1, \ldots, \lambda_n)$ where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. We consider V as a representation of $T \subset SL_n$, then V decomposes into a direct sum of simple representations of T which we know are all one-dimensional, say $V = \bigoplus_\mu V_\mu$. Then λ is the maximal μ which occurs in the sum, where the order is induced by the choice of positive roots. All information about V can be recovered from λ, so we now refer to irreducible representations of $SL_n(\mathbb{C})$ as V_λ.

A weight vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ is *dominant* if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and *regular* if all values λ_i are distinct. We denote the lattice of all weights by Λ.

2.2.2 Decomposition of V_λ

We consider an irreducible representation V_λ for $SL_n(\mathbb{C})$ and determine a basis for this vector space. Of course, counting the size of this basis will tell us the dimension of V_λ. To do this, we consider the action of $SL_{n-1}(\mathbb{C})$ on V_λ. From the branching laws for $SL_n(\mathbb{C})$ (see for example [GW09]), we know how V_λ decomposes as a representation of $SL_{n-1}(\mathbb{C})$. An irreducible representation V_μ for $SL_{n-1}(\mathbb{C})$ occurs in the decomposition of V_λ exactly when the $(n-1)$-dimensional weight vector $\mu = (\mu_1, \ldots, \mu_{n-1})$ interlaces with n-dimensional weight vector λ, meaning,

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \ldots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$

Moreover, the multiplicity of V_μ in V_λ is 1. We continue this process for $SL_{n-2}(\mathbb{C})$, $SL_{n-3}(\mathbb{C})$, etc. until we get to $SL_1(\mathbb{C})$. This last group is a one-dimensional torus, so all of its irreducible representations are one-dimensional. Counting these one-dimensional vector subspaces of V_λ determines $\dim V_\lambda$, and in fact will determine a useful basis for V_λ. Consider the following array of inequalities:

$$\begin{array}{ccccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & & \dots & & \lambda_n \\ & x_{12} & & x_{23} & & \dots & & x_{(n-1)n} & \\ & & x_{13} & & x_{24} & & \dots & & \\ & & & \ddots & & ⋰ & & & \\ & & & & x_{1n} & & & & \end{array} \tag{2.1}$$

where each small triangle $\begin{array}{ccc} a & & b \\ & c & \end{array}$ corresponds to the inequalities $a \le c \le b$. This inductive construction was originally introduced by Gelfand and Zetlin [GZ50], so we refer to Equation (2.1) as a GZ array. The set of solutions (x_{ij}) to the above system of inequalities for $\lambda = (\lambda_1 \le \ldots \le \lambda_n)$ is called a Gelfand-Zetlin (GZ) polytope and denoted Δ_λ. Let $N = n(n-1)/2$; this is $\dim \Delta_\lambda$ when λ is regular. The basis for V_λ described above is parametrized by integer solutions to the system (2.1), hence by integer points inside the polytope Δ_λ; this is called the Gelfand-Zetlin basis for V_λ.

Theorem 2.2.1 (Gelfand-Zetlin)**.** *For a dominant integral weight λ, the number of integer points of Δ_λ is equal to the dimension of the irreducible representation V_λ.*

2.3 GELFAND-ZETLIN POLYTOPES

Gelfand-Zetlin (GZ) polytopes, defined by the GZ array (2.1), were originally defined in [GZ50] and have since been studied by many people including [Kav11] and [Kir09].

Example 2.3.1. Let $\lambda = (-1, 0, 1)$, then the GZ polytope Δ_λ is given by the following inequalities:

$$-1 \le x \le 0, \quad 0 \le y \le 1, \quad x \le z \le y.$$

See Figure 2.1. This polytope $\Delta_{(-1,0,1)}$ has 6 facets, 11 faces of dimension 1, 7 vertices. See

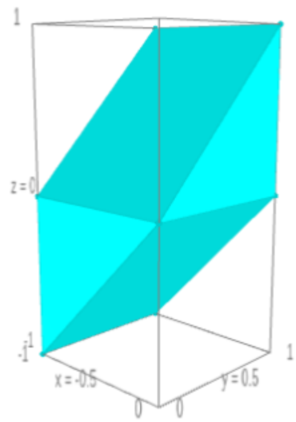


Figure 2.1: GZ Polytope for $\lambda = (-1, 0, 1)$

Figure 2.2.

2.3.1 Minkowski Addition

We recall that for polytopes P and Q, we can define the *Minkowski sum* $P + Q$ to be the polytope

$$P + Q = \{x + y \mid x \in P, y \in Q\}.$$

The collection of GZ polytopes for fixed n behaves well with respect to Minkowski addition, we see in the following proposition.

Proposition 2.3.2 (Additivity). *For $\lambda, \mu \in \mathbb{Z}^n$ both strictly increasing, the assignment $\lambda \mapsto \Delta_\lambda$ is additive. That is,*

$$\Delta_{\lambda+\mu} = \Delta_\lambda + \Delta_\mu$$

where the addition on the right is Minkowski addition of polytopes.

Proof. One inclusion is clear: $\Delta_\lambda + \Delta_\mu \subset \Delta_{\lambda+\mu}$. Suppose $x \in \Delta_\lambda$ and $y \in \Delta_\mu$. Then looking at the top line of inequalities, we have $\lambda_1 \leq x_{11} \leq \lambda_2 \leq x_{12} \leq \ldots \leq \lambda_n$ and similarly for the

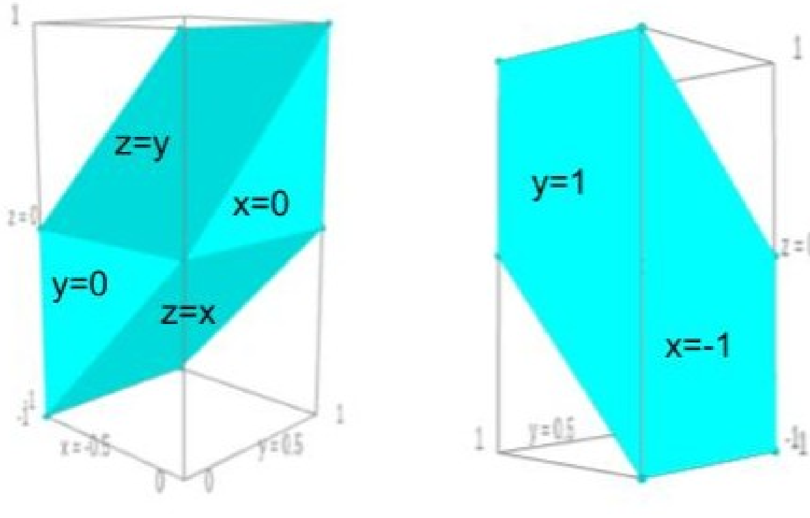


Figure 2.2: Labeled Facets of $\Delta_{(-1,0,1)}$

components of y with μ. Then clearly

$$\lambda_1 + \mu_1 \leq x_{11} + y_{11} \leq \lambda_2 + \mu_2 \leq x_{12} + y_{12} \leq \dots.$$

The lower lines of inequalities follow similarly.

For the other inclusion, let $x \in \Delta_{\lambda+\mu}$ then our goal is to write $x = x' + x''$ with $x' \in \Delta_\lambda$ and $x'' \in \Delta_\mu$. We begin with the top line of inequalities, $\lambda_1 + \mu_1 \leq x_{11} \leq \lambda_2 + \mu_2 \leq \dots$. This can be reduced to a number of inequalities of the form

$$0 \leq y \leq a + b$$

for appropriate y, a, b. We first show that in this situation we can separate $y = y' + y''$ where

$$0 \leq y' \leq a, \quad 0 \leq y'' \leq b.$$

For this, we let

$$y = y\frac{a}{a+b} + y\frac{b}{a+b}, \quad y' = y\frac{a}{a+b}, \quad y'' = y\frac{b}{a+b}.$$

By assumption y, a, b are all positive, so clearly $y', y'' \geq 0$. We just need to show $y' \leq a$ and $y'' \leq b$. We have $y \leq a + b$ so

$$y' = y\frac{a}{a+b} \leq (a+b)\frac{a}{a+b} = a$$

and similarly $y'' \leq b$ as desired.

In this top line of inequalities in our Gelfand-Zetlin array, we convert each $\lambda_i + \mu_i \leq x_{1i} \leq \lambda_{i+1} + \mu_{i+1}$ into $0 \leq y \leq a + b$ by taking

$$y = x_{1i} - \lambda_i - \mu_i, \quad a = \lambda_{i+1} - \lambda_i, \quad b = \mu_{i+1} - \mu_i.$$

These quantities are all positive and satisfy the desired inequality following directly from our assumption. Therefore, we can separate the top line of inequalities involving x into inequalities involving x' and x'', where

$$x'_{1i} = y' + \lambda_i, \quad x''_{1i} = y'' + \mu_i.$$

We then continue inductively on the lower rows of the array.

Thus the assignment $\lambda \mapsto \Delta_\lambda$ is in fact additive for dominant λ. □

Finally, since the definition of a GZ polytope does not require that λ be integral, we can consider $\lambda \in \mathbb{R}^n$ with $\lambda_1 < \lambda_2 \ldots$, in which case the collection of GZ polytopes is closed under multiplication by positive scalars, i.e., $t\Delta_\lambda = \Delta_{t\lambda}$ for $t > 0$. This is apparent from the GZ array (2.1); multiplying all λ_i and x_{ij} by t preserves all inequalities. This gives the set of GZ polytopes the structure of a cone.

GZ polytopes also have strong connections with symplectic geometry. See Guillemin and Sternberg [GS83]. For this, we consider the unitary group $U(n)$, the maximal compact subgroup of $GL_n(\mathbb{C})$. We consider the coadjoint action $U(n) \curvearrowright \mathfrak{u}(n)^*$, which is conjugation of matrices. Each orbit is the collection of matrices with the same eigenvalues. Let the orbit corresponding to integral λ be denoted $\mathcal{O}_\lambda$. This orbit $\mathcal{O}_\lambda$ has a canonical (Kirillov-Kostant) symplectic form. This action is Hamiltonian, and for this case the associated moment map is the inclusion $\mathcal{O}_\lambda \hookrightarrow \mathfrak{u}(n)^*$. The goal is to find $f_1, \ldots, f_N$ that Poisson commute, meaning the associated vector fields X_f commute. Recall that the vector field X_f associated to f

satisfies $\omega(\cdot, X_f) = df$. To do this, Guillemin and Sternberg obtain eigenfunctions $f_1, \dots, f_N$, $N = n(n-1)/2$ from eigenvalues of successively smaller submatrices of $X \in \mathcal{O}_\lambda$. Since X is hermitian, these eigenfunctions are real and by a min-max argument, they interlace just like the GZ array. These eigenfunctions are smooth only on a dense subset corresponding to the interior of the polytope defined by the GZ array.

2.4 FLAG VARIETIES

We begin with the Grassmannian, $Gr(k,n)$, which is the set of k-dimensional linear subspaces of $\mathbb{C}^n$. This is a projective variety embedded in projective space via the Plücker map

$$\varphi : Gr(k,n) \to \mathbb{P}(\Lambda^k \mathbb{C}^n)$$

defined as follows. For a k-dimensional vector space $V \in Gr(k,n)$ with basis $\{v_1, \dots, v_k\}$, we let

$$\varphi(V) = [v_1 \wedge \dots \wedge v_k] \in \mathbb{P}(\Lambda^k \mathbb{C}^n).$$

This map is well defined, since any other basis $\{w_1, \dots, w_k\}$ for V can be obtained from $\{v_1, \dots, v_k\}$ by a change of base matrix B, i.e., $w_i = Bv_i$ for all i, then $w_1 \wedge \dots \wedge w_k = (\det B) v_1 \wedge \dots \wedge v_k$ so the two are linear multiples of each other and hence in the same equivalence class in $\mathbb{P}(\Lambda^k \mathbb{C}^n)$.

Next, we consider a slight generalization of the Grassmannian: nested sequences of subspaces $V_1 \subsetneq V_2 \subsetneq \dots V_k \subsetneq \mathbb{C}^n$. Such a nested collection of subspaces is called a *flag*, and the sequence of dimensions $(\dim V_1, \dim V_2, \dots, \dim V_k)$ is called the *signature* of the flag. If a flag has the signature $(1, 2, 3, \dots, n)$ then it is called a *full flag*. Just as the Grassmannian has the structure of a projective variety, we want to be able to use the tools of algebraic geometry to study these flags. To do this, we define

$$\mathbb{F}(a_1, \dots, a_k) = \{\text{flags with signature } (a_1, \dots, a_k)\}.$$

This set can be embedded in the variety $Gr(a_1, n) \times Gr(a_2, n) \times \dots \times Gr(a_k, n)$ and is called a *flag variety*.

Flag varieties are perhaps best understood using the language of algebraic groups. An algebraic group G is both an algebraic variety and a group where the multiplication and inversion operations of the group are algebraic maps. To study flag varieties, we will examine $SL_n(\mathbb{C})$ as an algebraic group. There is a natural action of $SL_n(\mathbb{C})$ on a flag $F: V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq \mathbb{C}^n$ as $SL_n(\mathbb{C})$ acts on each of the vector spaces V_i individually. This action is algebraic, as it is linear, and so we can view the action $SL_n(\mathbb{C}) \times F \to F$ as morphism of varieties. Then $SL_n(\mathbb{C})$ acts (algebraically) on the entire flag variety $\mathbb{F}(a_1, \ldots, a_k)$. This action is transitive. For the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$, we call the flag $E: \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle \subsetneq \langle e_1, e_2, e_3 \rangle \subsetneq \ldots$ the *standard full flag*. We can similarly construct the standard flag of signature $(a_1, \ldots, a_k)$. To see that $SL_n(\mathbb{C})$ will take the standard flag E to an arbitrary $F \in \mathbb{F}(a_1, \ldots, a_k)$, we choose a basis of $\mathbb{C}^n$ subject to the flag F, that is, so that $V_i = \langle v_1, v_2, \ldots, v_{a_i} \rangle$. Consider the matrix where the ith column is the vector v_i. This is a change of base matrix from the standard basis to the basis subject to F. This matrix is an element of $SL_n(\mathbb{C})$ that takes the standard flag E to the flag F.

Since we have established that the action of $SL_n(\mathbb{C})$ on $\mathbb{F}(a_1, \ldots, a_k)$ is transitive, we want to understand the stabilizer of a point. First let us examine the kernel of the map $SL_n(\mathbb{C}) \to \mathbb{F}(1, 2, \ldots, n)$ given by $g \mapsto g \cdot E$. For g to be in the kernel of this map, we need $ge_1 = e_1$, so the first column of g needs to be $[\star, 0, \ldots, 0]^t$. For the next vector space $\langle e_1, e_2 \rangle$ to be preserved, we need $ge_2 \in \langle e_1, e_2 \rangle$ as well (ge_1 is already in this space). This means the second column of g is of the form $[\star, \star, 0, \ldots, 0]$. The other columns follow similarly, and we see that g must be an upper triangular matrix with respect to the standard basis. This tells us that the flag variety $\mathbb{F}(1, 2, \ldots, n) \cong SL_n(\mathbb{C})/B$ where B is a Borel subgroup of $SL_n(\mathbb{C})$, that is, the group of upper triangular matrices. Next, we return our attention to the flag variety $\mathbb{F}(a_1, \ldots, a_k)$. The kernel of the action on the standard flag of this signature will be the set of block upper triangular matrices with blocks of size $a_1, a_2 - a_1, a_3 - a_2, \ldots$. Subgroups of $SL_n(\mathbb{C})$ consisting of block upper triangular matrices are called *parabolic* subgroups. Thus we see that this induces a bijection $\mathbb{F}(a_1, \ldots, a_k) \cong SL_n(\mathbb{C})/P$, and we can study quotients of $SL_n(\mathbb{C})$ by parabolic or Borel subgroups rather than partial or complete flag varieties themselves. Every parabolic subgroup P contains a unique Borel B, so there is a natural inclusion $B \to P$ which induces a surjection $G/B \to G/P$. All quotients by parabolic subgroups, i.e., all partial flag varieties, arise as quotients of the complete flag variety $\mathcal{F}\ell_n(\mathbb{C}) \cong G/B$. Thus, it is sufficient to study the complete flag variety.

2.4.1 Line Bundles on Flag Varieties

To a dominant integral weight λ, we associate a line bundle L_λ on $\mathcal{F}\ell_n = SL_n(\mathbb{C})/B$. We consider the character $\lambda : B \to \mathbb{C}^*$ induced from the character λ of T. Let $\mathbb{C}_\lambda = \mathbb{C}$ be given the structure of a B-module where $b \cdot z = \lambda(b)z$ for $b \in B, z \in \mathbb{C}$. Then we define

$$G \times_B \mathbb{C}_{-\lambda} = (G \times \mathbb{C}_{-\lambda})/\sim$$

where $(g, z) \sim (gb^{-1}, b \cdot z)$. The image of projection onto the first factor $\pi : G \times_B \mathbb{C}_{-\lambda} \to G/B$ is the flag variety, and we show this is a line bundle L_λ. We first show that π is well-defined. For $(gb^{-1}, b \cdot z) \sim (g, z)$ we have $\pi((gb^{-1}, b \cdot z)) = gb^{-1} \in gB$ so the entire equivalence class $[(g, z)]$ maps to $gB \in G/B$. Finally, we verify that $\pi : (G \times_B \mathbb{C}_{-\lambda}) \to G/B$ is a line bundle. With $g \in G$ fixed, we consider

$$\pi^{-1}(gB) = \{[(g, z)] | z \in \mathbb{C}\}$$

which is a line, so we have a line bundle L_λ on the flag variety $\mathcal{F}\ell_n$.

The Borel-Weil theorem relates the space of sections $H^0(\mathcal{F}\ell_n, L_\lambda)$ of L_λ to an irreducible representation of $SL_n(\mathbb{C})$.

Theorem 2.4.1 (Borel-Weil)**.** *As $SL_n(\mathbb{C})$-modules, we have*

$$H^0(\mathcal{F}\ell_n, L_\lambda) \cong (V_\lambda)^*.$$

The space of sections is given by

$$H^0(\mathcal{F}\ell_n, L_\lambda) = \{s : G/B \to G \times_B \mathbb{C}_{-\lambda} \text{ such that } \pi \circ s = Id_{G/B}\}$$

which has the structure of a G-module by $(g \cdot s)(g'B) = gs(g^{-1}(g'B))$. The Borel-Weil-Bott theorem extends the above theorem for arbitrary integral weights $w \cdot \lambda$ for w in the Weyl group W. For $SL_n(\mathbb{C})$, the Weyl group is $W = S_n$, the symmetric group. The action of W on $\lambda \in \Lambda$ in the Borel-Weil-Bott Theorem is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$ where ρ is half of the sum of positive roots.

Theorem 2.4.2 (Borel-Weil-Bott). *As G-modules, we have*

$$H^p(G/B, L_{w\cdot\lambda}) \cong \begin{cases} (V_\lambda)^* & p = \ell(w) \\ 0 & \text{otherwise} \end{cases}$$

where $\ell(w)$ is the length of w in the Weyl group.

We recall that Theorem 2.2.1 equates the number of lattice points of Δ_λ and the dimension of V_λ. By Theorem 2.4.2, the number of lattice points of Δ_λ is also equal to the dimension of the space of sections $H^p(G/B, L_{w\cdot\lambda})$ for the line bundle $L_{w\cdot\lambda}$ on flag variety $\mathcal{F}\ell_n$.

2.5 DIVISORS

Here, we review divisors in general, on flag varieties and on toric varieties. A divisor on the variety X is a codimension one subvariety D. Let $\{D\}$ be the set of prime divisors on X and define a corresponding collection of valuations $\{\nu_D\}$ where $\nu_D(f)$ is the order of vanishing of f along D. The divisor of a rational function f on X is defined by

$$\mathrm{div}(f) = \sum_D \nu_D(f) D.$$

Recall that a *discrete valuation* ν on field K is a group homomorphism $K^* \to \mathbb{Z}$ satisfying $\nu(xy) = \nu(x) + \nu(y)$ and $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$. The abelian group $\mathrm{Div}(X)$ is generated by the collection of prime divisors. An element of this group is called a *Weil divisor*. A Weil divisor is *effective* if all coefficients are non-negative. Clearly, $\mathrm{div}(f)$ is a Weil divisor. A divisor of this form is called *principal* and the set of principal divisors is $\mathrm{Div}_0(X)$.

We are interested in the class group of a variety, that is,

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{Div}_0(X)$$

where divisors $D, E \in \mathrm{Div}(X)$ are equivalent if $D - E = \mathrm{div}(f)$ for some $f \in \mathbb{C}(X)^*$. Such divisors are called *linearly equivalent*. Another useful group of divisors is $\mathrm{CDiv}(X)$, the group of *Cartier divisors*. These are locally principal, meaning there is an open cover $\{U_i\}$

of X such that the restriction of D to U_i is given by $\mathrm{div}(f_i)$ for some $f_i \in \mathbb{C}(U_i)^*$. The group $\mathrm{CDiv}(X)/\mathrm{Div}_0(X) = \mathrm{Pic}(X)$, the Picard group of X.

Next, we recall divisors on projective varieties. This includes the case of flag varieties.

2.5.1 Divisors on Flag Varieties

For a projective variety X of dimension d embedded into $\mathbb{P}^N$,

$$\deg(X) = \#(X \cap H_1 \cap \ldots \cap H_d)$$

where the H_i are generic hyperplanes in $\mathbb{P}^N$. As the Picard group $\mathrm{Pic}(\mathbb{P}^N) \cong \mathbb{Z}$, we can choose a single hyperplane H corresponding to the generator of $\mathrm{Pic}(\mathbb{P}^N)$ and compute the size of the intersection $\#(X \cap H \cap \ldots \cap H)$ in $\mathbb{P}^N$. Alternatively, let H' be the pullback of H to X via the embedding, then H' is a divisor on X and

$$\deg(X) = (H')^d$$

where $(H')^d$ is the self-intersection of the divisor H'.

If the embedding $X \hookrightarrow \mathbb{P}^N$ is given by sections of a very ample line bundle L, that is, $X \hookrightarrow \mathbb{P}(H^0(X,L)^*)$, then Hilbert's Theorem, or the Asymptotic Riemann-Roch Theorem, gives a way to compute the degree of the line bundle.

Theorem 2.5.1 (Asymptotic Riemann-Roch)**.** *For a very ample line bundle L on a projective variety X,*

$$\deg(X) = d! \lim_{m \to \infty} \frac{\dim H^0(X, \mathcal{L}^{\otimes m})}{m^d}.$$

We recall the line bundle L_λ on $\mathcal{F}\ell_n$ defined in Section 2.4.1. This line bundle has the property $\mathcal{L}_\lambda^{\otimes m} = \mathcal{L}_{m\lambda}$, and we recall also from Theorem 2.4.1 that $H^0(\mathcal{F}\ell_n, L_\lambda) \cong V_\lambda^*$. Finally, we recall from Theorem 2.2.1 that $\#(\Delta_\lambda \cap \mathbb{Z}^N) = \dim H^0(G/B, \mathcal{L}_\lambda)$. Combining these results, we compute the degree of this embedding; see for example [Kav11] Remark 2.4.

Proposition 2.5.2. *For flag variety $\mathcal{F}\ell_n$ and λ dominant regular,*

$$\deg(\mathcal{F}\ell_n, L_\lambda) = N!\, Vol_N(\Delta_\lambda),$$

where Δ_λ is the corresponding GZ polytope of dimension $N = n(n-1)/2$.

2.5.2 Divisors on Toric Varieties

We consider divisors on a toric variety X_Σ. The torus action on X_Σ leads to a correspondence between torus orbits in X_Σ and cones in the fan Σ. The *orbit-cone correspondence* relates k-dimensional cones in Σ to codimension k orbits of the torus. Rather than consider all prime divisors on our toric variety X_Σ, it is enough to consider torus-invariant prime divisors. A torus-invariant prime divisor is an irreducible codimension 1 subvariety, and therefore we are interested in codimension 1 orbits which correspond to 1-dimensional cones or rays of our fan. We denote rays in the fan Σ by $\Sigma(1)$, and similarly k-dimensional cones by $\Sigma(k)$.

Let $\rho \in \Sigma(1)$, D_ρ the corresponding torus-invariant divisor, and ν_ρ the associated valuation. Then

$$\nu_\rho(\chi^m) = \langle m, u_\rho \rangle$$

where u_ρ is the ray generator corresponding to ρ. Then the divisor corresponding to the character χ^m is

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \nu_\rho(f) D_\rho = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

We can also construct a polyhedron (not necessarily bounded) from a divisor $D = \sum_\rho a_\rho D_\rho$. This polytope describes which characters χ^m have the property that adding $\operatorname{div}(\chi^m)$ to D yields an effective divisor, that is, for which m the divisor

$$\sum_\rho \langle m, u_\rho \rangle D_\rho + \sum_\rho a_\rho D_\rho = \sum_\rho (\langle m, u_\rho \rangle + a_\rho) D_\rho$$

is effective. We call this polyhedron P_D, and it is defined by

$$P_D = \{x \mid \langle x, u_\rho \rangle \geq -a_\rho\}.$$

When the divisor $D = \sum_\rho a_\rho D_\rho$ is Cartier, the Cartier data $\{m_\sigma\}_{\sigma \in \Sigma}$ satisfies $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for $\rho \in \Sigma(1)$. In this case, P_D is a full-dimensional lattice polytope, so we can construct the toric variety $X_\mathcal{A}$, and also the divisor of the polytope $D_{P_D} = D = \sum_\rho a_\rho D_\rho$, the same divisor with which we started.

It can be useful to describe a Cartier divisor in terms of *support functions*. For a fan $\Sigma \subset N_\mathbb{R}$, the support of Σ, denoted $|\Sigma|$, is $|\Sigma| = \cup_{\sigma \in \Sigma} \sigma \subset N_\mathbb{R}$. A *support function* is a piecewise linear function $\varphi : |\Sigma| \to \mathbb{R}$ which is linear on each cone $\sigma \in \Sigma$. We are interested in the support functions that behave nicely with respect to our lattice N. We say φ is *integral with respect to* N if $\varphi(N \cap |\Sigma|) \subset \mathbb{Z}$, that is, lattice points map to integers under φ. Given a Cartier divisor with Cartier data $\{m_\sigma\}_{\sigma \in \Sigma}$ we can construct a support function integral with respect to N by taking $\varphi(u) = \langle u, m_\sigma \rangle$ whenever $u \in \sigma$. If we start with a support function φ on Σ integral with respect to N then we can determine the coefficients a_ρ of D_ρ for the corresponding divisor by taking $-a_\rho = \varphi(u_\rho)$, then $D = \sum_\rho a_\rho D_\rho$.

Finally, we consider the sheaf $\mathcal{O}_X(D)$ for a Cartier divisor D defined by

$$\mathcal{O}_X(D) = \{f \in \mathbb{C}(X) | \text{div}(f) + D \geq 0\}.$$

This is a sheaf as it is defined locally on open sets (as D is Cartier, and hence locally given by some $\text{div}(f_i)$ on U_i) where the local definition is compatible with restriction to smaller open sets and also with gluing open sets together into a larger open set. When D is Cartier and X is normal, $\mathcal{O}_X(D)$ is the sheaf of sections of a line bundle L_D. In this case, the dimension of $H^0(X, L_D)$ is equal to the number of lattice points in the polytope P_D. If we do this for the divisor D_P obtained from a polytope P, then $P_{D_P} = P$. When a divisor D comes from a polytope, it is ample. Therefore the space of sections of L_{kP} defines an embedding into projective space for $k \in \mathbb{N}$ large enough.

Proposition 2.5.3. *The degree of X_Σ under the embedding given by polytope P is*

$$\begin{aligned} deg(X_\Sigma) &= d! \lim_{m \to \infty} \frac{\dim H^0(X_\Sigma, \mathcal{O}(mP))}{m^d} \\ &= d! \lim_{m \to \infty} \frac{\#(mP \cap \mathbb{Z}^d)}{m^d} \\ &= d!\, Vol_d(P). \end{aligned}$$

3.0 MAIN RESULTS

In this chapter, we establish some facts about the GZ fan in order to define the GZ toric variety X_{GZ}. We then use convex chains to extend the relation between dominant weight λ and GZ polytope Δ_λ to non-dominant weights. These are so-called virtual GZ polytopes. In the final section, we identify the cohomology ring of the flag variety G/B with a quotient of a subalgebra of the Chow cohomology ring of the toric variety X_{GZ}.

3.1 GELFAND-ZETLIN FAN RESULTS

In this section we prove two results about normal fans of GZ polytopes. Let $N = n(n-1)/2$ be the dimension of Δ_λ for λ dominant regular, and consider the normal fan Σ_λ to polytope Δ_λ. Our first result justifies the terminology "Gelfand-Zetlin fan".

Proposition 3.1.1. *The normal fan Σ_λ is independent of λ for λ dominant regular.*

Proof. We recall the coordinates given in the GZ array (2.1). A facet of the polytope Δ_λ is determined by changing a single inequality to an equality in the GZ array, and a lower dimensional face is determined by changing multiple inequalities to equalities. We distinguish between two types of equality: those of the form $x_{1i} = \lambda_j$ and those of the form $x_{ij} = x_{(i-1)k}$. Fix a face of F of Δ_λ, that is, fix a collection of equalities in the array. The second type of inequality is clearly independent of λ, and the first type depends on λ but only changes by translation when λ is varied. Then the cone at that face as λ varies is simply translated based on how many equalities of the first type appear in the array. When we examine the corresponding cone in the fan Σ_λ, we translate the cone at the face F to the origin then take the dual cone. Thus the fan Σ_λ does not actually depend on λ, and we will now refer to the fan normal to any Δ_λ as Σ_{GZ}. □

Next, we show that the only polytopes normal to the fan Σ_{GZ} are GZ polytopes up to shifting.

Proposition 3.1.2. *Let P be a full dimensional polytope normal to Σ_{GZ}, then $P = c + \Delta_\lambda$ for some dominant regular λ and $c \in \mathbb{R}^N$. Moreover, if P is a lattice polytope then both c and λ are integral.*

Proof. Let P be normal to Σ_{GZ}, then the hyperplanes defining P are parallel to those defining any Δ_λ because the fan is independent of λ. Recall that there are two types of equations defining Δ_λ, $x_{1i} = \lambda_j$ and $x_{ij} = x_{(i-1)k}$. We will use variables y_{ij} for P and reserve x_{ij} for a GZ polytope. Now, because the supporting hyperplanes of P are parallel to those for Δ_λ, there are two forms of equation defining P as well: $y_{1i} = a$ and $y_{ij} = y_{(i-1)k}$ for appropriate i, j, k. The polytope P is the set of solutions to the following system of inequalities:

$$a_i \leq y_{1i} \leq b_i \qquad 1 \leq i \leq n-1 \tag{3.1}$$
$$y_{(i-1)j} + a_{ij} \leq y_{ij} \leq y_{(i-1)(j+1)} + b_{ij} \qquad i \in \{2, \ldots, n\}, 1 \leq j \leq n-i+1.$$

If P is a lattice polytope, then all a_i and b_i must be integers.

Our goal is to translate $y = (y_{11}, y_{12}, \ldots, y_{n1})$ to

$$x = y + c$$

such that the inequalities (3.1) fit into a GZ array (2.1). The inequalities $a_i \leq y_{1i} \leq b_i$ will determine λ up to a choice of λ_1. We first shift y so that the first type of inequality for P will interlace as the top two lines of a GZ array.

Let $\lambda_1 = a_1$, $\lambda_2 = b_1$ and $x_{11} = y_{11}$, then $\lambda_1 \leq x_{11} \leq \lambda_2$ and it is clear that we need to shift y_{12} to

$$x_{12} = y_{12} + \lambda_2 - a_2$$

so that $\lambda_1 \leq x_{12}$. Similarly, we must have

$$x_{1i} = y_{1i} + \lambda_i - a_i$$

for $2 \le i \le n-1$. Once we have x_{1i}, we determine λ_{i+1} to be

$$\lambda_{i+1} = b_i + \lambda_i - a_i.$$

Thus, we have determined λ and shifted y so that the first kind of inequalities lace together as in a GZ array. If P is a lattice polytope, λ must have integer entries and the components $(\lambda_i - a_i)$ of the shift are integral because all a_i and b_i are integers.

In order to translate the rest of the variables y_{ij} so that they fit in a GZ array, we first need to examine relations occurring in each small diamond $\begin{smallmatrix} & a & \\ b & & c \\ & d & \end{smallmatrix}$ in the GZ array (2.1). When we have equalities $b = a$ and $c = a$, then since $b \le d \le c$ we must have $d = a$. This gives us linear relations among ray generators in the fan Σ which yields relations between the constants a_{ij}, b_{ij} for P.

Suppose for induction that the first $(i-1)$ rows of variables, and the first $(j-1)$ entries of the ith row have been translated to fit in the GZ array. We want to determine the shift $x_{ij} = y_{ij} + c_{ij}$ such that x_{ij} fits into the array. Note that in the argument below, some of the constants a_{ij}, b_{ij} have also been shifted, but only those below the variable in question. The relevant diamond is

$$\begin{matrix} & x_{(i-2)(j+1)} & \\ x_{(i-1)j} & & x_{(i-1)(j+1)} \\ & x_{ij} & \end{matrix}$$

except in the case $i = 2$ where we have λ_{j+1} instead of $x_{(i-2)(j+1)}$. We consider the face of P where $x_{(i-1)j} = x_{(i-2)(j+1)}$ and $x_{(i-1)(j+1)} = x_{(i-2)(j+1)}$. The diamond relation implies that $x_{ij} = x_{(i-1)j} = x_{(i-1)(j+1)}$ as well. In terms of y_{ij}, we have the inequalities

$$x_{(i-1)j} + a_{ij} \le y_{ij} \le x_{(i-1)(j+1)} + b_{ij}, \tag{3.2}$$

which, when we consider the face of P, become equalities

$$x_{(i-2)(j+1)} + a_{ij} = x_{(i-2)(j+1)} + b_{ij} \quad \text{thus} \quad a_{ij} = b_{ij}.$$

Then,

$$x_{ij} = y_{ij} + a_{ij}$$

is the translation required to fill in the next position of the GZ array. Note again that if P is a lattice polytope with a_{ij}, b_{ij} integers then this shift will also be integral. At this point, we also substitute x_{ij} into any remaining inequalities involving y_{ij} and variables not yet shifted. We rename the relevant shifted constants $a_{(i+1)k}, b_{(i+1)k}$.

In this way, we shift all variables for the polytope P so that the defining inequalities fit into a GZ array. Therefore $P = c + \Delta_\lambda$ where c encodes the translations and $\lambda = (\lambda_1, \ldots, \lambda_n)$ is constructed above. As indicated in the proof, when P is a lattice polytope the corresponding c and λ are both integral. □

Remark 3.1.3. Observe that there are $n + n(n-1)/2$ parameters present in $c + \Delta_\lambda$, but a GZ polytope is cut out by $n(n-1)$ facets, one for each ray in $\Sigma(1)$. The dimension of the space of polytopes normal to Σ_{GZ} is therefore much smaller than the number of rays in the fan due to the fact that Δ_λ is not a simple polytope, or equivalently, because the fan Σ_{GZ} is not simplicial.

Remark 3.1.4. We see that the collection of all polytopes normal to Σ_{GZ} is an example of a *linear family*, see [KV18].

3.2 VIRTUAL GELFAND-ZETLIN POLYTOPES

In this section we recall the construction of a vector space of virtual polytopes from Khovanskii and Pukhlikov, [PK93], as well as their use of convex chains to understand virtual polytopes. We examine the extra relations occurring because the GZ fan Σ_{GZ} is not simplicial. Finally, after exploring some small-dimensional examples of virtual GZ polytopes, we prove that virtual GZ polytopes are bounded unions of convex regions on which the value of the associated convex chain is ± 1.

3.2.1 Vector Space of Virtual Polytopes

In order to establish later results on virtual GZ polytopes, we follow the construction of "virtual polytopes" by Khovanskii and Pukhlikov, [PK93]. For fixed n, let $\mathcal{P}(\Sigma_{GZ})$ be the collection of polytopes normal to Σ_{GZ}. Recall that all GZ polytopes have the same normal fan, Prop 3.1.2, and that any polytope normal to Σ_{GZ} is a translation of a GZ polytope. This means the polytopes in $\mathcal{P}(\Sigma_{GZ})$ are all translations of GZ polytopes. Next recall Proposition 2.3.2 where we prove that the assignment $\lambda \mapsto \Delta_\lambda$ is additive, thus the Minkowski sum of two GZ polytopes is again a GZ polytope. In $\mathcal{P}(\Sigma_{GZ})$ we consider all polytopes, not only those with lattice vertices, so $\mathcal{P}(\Sigma_{GZ})$ is closed under positive scalar multiplication as well. This gives $\mathcal{P}(\Sigma_{GZ})$ the structure of a cone.

Because $\mathcal{P}(\Sigma_{GZ})$ is a cone, we can consider the Grothendieck group $\mathcal{P}^*(\Sigma_{GZ})$ obtained by taking formal inverses of elements of $\mathcal{P}(\Sigma_{GZ})$. A typical element of $\mathcal{P}^*(\Sigma_{GZ})$ is of the form

$$\sum \Delta_i - \sum \Delta_j,$$

a formal difference of GZ polytopes. We call such an element a *virtual polytope*, or more specifically, a virtual GZ polytope.

3.2.2 Convex Chains

Next, we recall from Khovanskii and Pukhlikov, [PK93], the notion of convex chains. For a polytope P we consider instead the characteristic function χ_P defined as

$$\chi_P(x) = \begin{cases} 1 & x \in P \\ 0 & x \notin P \end{cases}.$$

A *convex chain* is finite linear combination of characteristic functions of polytopes. In their work [PK93], Khovanskii and Pukhlikov describe a convolution operation $\star$ which has the property that, for polytopes P and Q,

$$\chi_{P+Q} = \chi_P \star \chi_Q.$$

This gives the collection of convex chains the structure of an algebra since it is also clearly closed under addition and scalar multiplication of functions.

This convolution operation is defined by

$$\chi_P \star \chi_Q(x) = \int \chi_P(x-y)\chi_Q(y)d\mu = \mu(\{y : y \in Q \text{ and } x - y \in P\}) = \mu(Q \cap x + (-P)).$$

Here μ is a finitely additive measure coming from the Euler characteristic. To verify $\chi_P \star \chi_Q = \chi_{P+Q}$, where $P+Q$ is the Minkowski sum of the two polytopes, we observe:

$$x \in P+Q \Leftrightarrow Q \cap x + (-P) \neq \varnothing$$
$$\Leftrightarrow \text{ there exists} y \in Q \text{ with } x-y \in P.$$

For a polytope P let $\Gamma(P)$ denote the set of all (proper and improper) faces of P and let P° denote the interior of P with respect to the span of P. Then Khovanskii and Pukhlikov prove the following:

Theorem 3.2.1. *1. The interior of polytope P corresponds to the convex chain*

$$\chi_{P^\circ} = (-1)^{\dim P} \sum_{\Delta \in \Gamma(P)} (-1)^{\dim \Delta} \chi_\Delta.$$

2. The inverse of a characteristic function with respect to the convolution $\star$ can be computed and is given by

$$\chi_P^{-1} = (-1)^{\dim P} \chi_{(-P)^\circ}.$$

This inverse then satisfies $\chi_P^{-1} \star \chi_P = \chi_{\{0\}}$, that is, $\chi_{\{0\}}$ is the identity of this convolution as $\{0\}$ is the identity for Minkowski addition.

The second part of Theorem 3.2.1 implies Ehrhart reciprocity. See [BLD+05].

3.2.3 Brianchon-Gram Theorem

In this section we recall the Brianchon-Gram Theorem as well as its extension to virtual polytopes.

Let $P \subset \mathbb{R}^N$ be a polytope and consider a facet F of P. Recall that each facet lies in a hyperplane which divides $\mathbb{R}^n$ into two regions, one containing P and one disjoint from P. The region containing P together with the hyperplane is called a supporting half-space for P, and is denoted H_F.

Let $\{F_i\}$ denote the collection of all facets of P. Any face F of P is an intersection of some of the facets; suppose $F = \cap_{i=1}^k F_i$. We define the tangent cone at face F to be

$$C_F = \bigcap_{i=1}^k H_{F_i},$$

that is, the cone at face F is the intersection of all of the half-spaces for facets F_i where $F \subset F_i$.

The Brianchon-Gram theorem decomposes the convex chain for P into a sum of characteristic functions of tangent cones at faces of P. See for example [BS15].

Theorem 3.2.2 (Brianchon-Gram)**.** *For polytope P,*

$$\chi_P = \sum_{F \in \Gamma(P)} (-1)^{\dim F} \chi_{C_F}. \tag{3.3}$$

A virtual polytope $P - Q$ corresponds to the convex chain $\chi_P - \chi_Q$. The convolution operation defined in [PK93], where $\chi_{P+Q} = \chi_P \star \chi_Q$, extends to χ_P^{-1} in Theorem 3.2.1 which gives the convex chain corresponding to the virtual polytope $-P$. This shows that the algebra of convex chains is equivalent to the algebra of virtual polytopes. Khovanskii and Pukhlikov examine this algebra for virtual polytopes; we are interested in the special case of virtual GZ polytopes.

Khovanskii and Pukhlikov extend the Brianchon-Gram Theorem, Equation (3.3), to virtual polytopes. See Proposition 2 of [PK93]. Therefore the convex chain of a virtual polytope decomposes into a sum of convex chains of cones at faces of that polytope. As a virtual polytope can be a union of several convex regions, we first fix a convex region then use the faces of that region to determine the value of the convex chain on that fixed region.

3.2.4 Twisted Cubes

In this section we recall the definition of a twisted cube then explore GZ polytopes as a special case.

We recall the definition of a twisted cube from [GK94].

Definition 3.2.3. A *standard twisted cube* for integers $\{a_{ij}\}$ and real numbers $\{\ell_k\}$ is the set of solutions $(x_1, \ldots, x_n)$ to the following inequalities:

$$\begin{aligned} -\ell_n \leq x_n \leq 0 \quad &\text{or} \quad 0 < x_n < -\ell_n \\ -(\ell_{n-1} + a_{n-1,n}x_n) \leq x_{n-1} \leq 0 \quad &\text{or} \quad 0 < x_{n-1} < -(\ell_{n-1} + a_{n-1,n}x_n) \\ &\vdots \\ -(\ell_i + \sum_{n \geq k > i} a_{i,k}x_k) \leq x_i \leq 0 \quad &\text{or} \quad 0 < x_i < -(\ell_i + \sum_{n \geq k > i} a_{i,k}x_k) \end{aligned} \tag{3.4}$$

and a *density function* ρ supported on the twisted cube is defined by $\rho(x) = (-1)^n \prod \operatorname{sgn}(x_i)$ where $\operatorname{sgn}(x_i) = -1$ for $x_i \leq 0$, otherwise $\operatorname{sgn}(x_i) = 1$. Other twisted cubes are obtained by affine isomorphism. A twisted cube is *untwisted* if none of the right hand side inequalities are involved in describing a region.

Example 3.2.4. Consider the twisted cube where $x = x_1, y = x_2,\ \ell_2 = 5, \ell_1 = 2, a_{12} = 1$:

$$\begin{aligned} -5 \leq y \leq 0 \quad &\text{OR} \quad 0 < y < -5 \text{ (does not occur)} \\ -(2 + y) \leq x \leq 0 \quad &\text{OR} \quad 0 < x < -(2 + y). \end{aligned}$$

The lighter region in the fourth quadrant indicates where the associated convex chain associated has value +1. This region is "untwisted". The darker region is where the associated convex chain has the value −1; some of the right hand side inequalities are involved in defining this region.

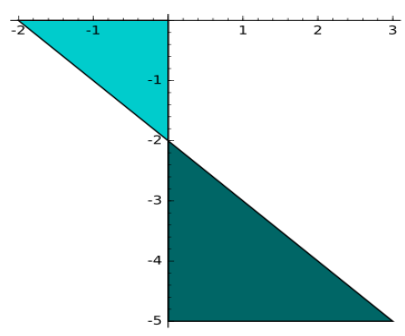


Figure 3.1: Example of a Twisted Cube

Since GZ polytopes are similarly defined by a collection of affine inequalities, and virtual GZ polytopes generalize to the case where the opposite inequalities are also allowed, virtual GZ polytopes together with their convex chains are special cases of twisted cubes. We prove the following.

Theorem 3.2.5. *For λ dominant regular, the GZ polytope Δ_λ is a translation of a twisted cube.*

Proof. First, we translate Δ_λ so that the inequalities defining the polytope are of the form $x_i \geq 0$ or $x_i \leq 0$. We then determine the appropriate affine isomorphism. Finally, we identify all defining constants ℓ_i and a_{ij}.

The polytope Δ_λ is not virtual, so it should correspond to a twisted cube which is not twisted, that is, where the density function is non-negative. Thus we need to translate Δ_λ so that $x_i \leq 0$ for each i. In the GZ array (2.1), we consider variable x_{ij}. We have the string of inequalities from x_{ij} in the upper right direction,

$$x_{ij} \leq x_{(i-1)(j+1)} \leq \ldots \leq x_{1(j+i-1)} \leq \lambda_{i+j},$$

which suggests that each variable x_{ij} should be shifted by λ_{i+j} so that

$$y_{ij} = x_{ij} - \lambda_{i+j} \leq 0.$$

This shifted GZ polytopes is defined by the inequalities

$$\begin{aligned} \lambda_i - \lambda_{i+1} \leq y_{1i} \leq 0 & \quad \text{for } 1 \leq i \leq n-1 \\ y_{(i-1)j} + \lambda_{i+j-1} - \lambda_{i+j} \leq y_{ij} \leq y_{(i-1)(j+1)} & \quad \text{for } i > 1. \end{aligned} \tag{3.5}$$

Note that when $i = n$ this expression involves λ_{n+1}, which is not defined. Hence, let $\lambda_{n+1} = 0$.

The next step is to determine an affine isomorphism which will take the shifted GZ polytope defined in (3.5) to a standard twisted cube. It is clear that the desired shift will take

$$y_{ij} \mapsto y_{ij} - y_{(i-1)(j+1)}$$

(for $i > 1$) so that the defining inequalities will be of the form

$$-(\ell_i + \sum_{n \geq k > i} a_{i,k} x_k) \leq x_i \leq 0.$$

We must relabel our variables y_{ij} to have a single subscript for this. We relabel as follows:

$$\begin{array}{ccccccc} x_N & & x_{N-1} & & \cdots & & \cdots \\ & & & \vdots & & & \\ & x_6 & & x_5 & & x_4 & \\ & & x_3 & & x_2 & & \\ & & & x_1 & & & \end{array}$$

In terms of these new variables, the top row of variables are constrained by inequalities of the form

$$\lambda_{k+1} - \lambda_{k+2} \leq x_{N-k} \leq 0$$

for $k = 0, \ldots, n-2$. The inequalities in the lower rows are more difficult to translate as they depend on both indices i and j, but we will discern the pattern below.

We begin with $x_1 = y_{n1}$ from before, which is involved in the inequalities

$$y_{n-1,1} + \lambda_n - \lambda_{n+1} \leq y_{n1} \leq y_{n-1,2}.$$

Recall that $\lambda_{n+1} = 0$, so in terms of the variables $\{x_i\}$, this becomes:

$$\lambda_n + x_3 \leq x_1 \leq x_2.$$

We apply another transformation, $x_1' = x_1 - x_2$ so that the right hand side is zero. We now have

$$\lambda_n + x_3 - x_2 \leq x_1' \leq 0.$$

We next consider the inequalities involving $x_2 = y_{n-1,2}$, and in the same way we translate the inequalities from (3.5) to our single subscript variables:

$$x_5 + \lambda_n \leq x_2 \leq x_4.$$

Then we transform x_2 to $x'_2 = x_2 - x_4$ so that the above inequality transforms to the desired form. Notice that this will affect the inequality for x_1, but only by adding linear terms to the left hand side.

We continue this process for each variable in order, and at each stage the transformation may affect the left hand side of previous inequalities, but will only be adding linear terms with larger subscripts. Following this pattern, we collect the transformation data into a matrix. The pattern we observe is that each variable is shifted by the variable above right in the array (2.1).

$$A = \begin{bmatrix} 1 & -1 & 0 & \dots & & & & & & \\ 0 & 1 & 0 & -1 & 0 & \dots & & & & \\ \vdots & 0 & 1 & 0 & -1 & 0 & \dots & & & \\ & & 0 & 1 & 0 & 0 & -1 & 0 & \dots & \end{bmatrix}$$

Observe that this matrix is upper triangular with 1's on the diagonal, so it has determinant 1 and is therefore invertible in $SL_N(\mathbb{Z})$. Hence this transformation is an affine isomorphism.

We shifted and then applied an affine isomorphism to our initial GZ polytope, which resulted in inequalities of the form present in the definition of a twisted cube. Note that for x'_i we will have $x'_i \leq 0$ by construction and $x'_i \geq \ell(x'_{i+1}, \dots, x'_N)$ where ℓ is some linear function of variables with greater subscripts. Thus we have proved a GZ polytope is in fact a twisted cube. □

Example 3.2.6. We examine the case of GZ polytopes in $\mathbb{R}^3$. Up to translation, such a polytope is given by $\lambda = (-a, 0, b)$ for some $a, b > 0$. We show more concretely how the above proof identifies $\Delta_{(-a,0,b)}$ with the twisted cube:

$$-a \leq x \leq 0 \quad \text{OR} \quad 0 < x < -a$$
$$-b \leq y \leq 0 \quad \text{OR} \quad 0 < y < -b$$
$$-(b + y - x) \leq z \leq 0 \quad \text{OR} \quad 0 < z < -(b + y - x).$$

First, the GZ array for this polytope is below.

$$\begin{array}{ccccc} -a & & 0 & & b \\ & x & & y & \\ & & z & & \end{array}$$

The inequalities $-a \le x \le 0$ are already in the correct form. We shift the other two variables $y \mapsto y + b, z \mapsto z + b$ and obtain

$$-b \le y \le 0$$
$$x - b \le z \le y.$$

Finally, we apply the affine isomorphism $z \mapsto z + y$ to obtain the twisted cube

$$-a \le x \le 0$$
$$-b \le y \le 0$$
$$-b - y + x \le z \le 0.$$

3.2.5 Faces and Relations for Gelfand-Zetlin Polytopes

With the intuition of virtual polytopes established in Section 3.2.4, we turn our attention to the faces of GZ polytopes. This is necessary in order to compute the value of the associated convex chain on various convex regions of the polytope, but it is complicated because the fan Σ_{GZ} is not simplicial.

Faces of GZ polytopes correspond to choices of equalities rather than inequalities in the GZ array (2.1). This is also true for "faces" of the virtual GZ polytopes. Recall that a virtual polytope is made up of multiple convex regions, so it is not clear what a "face" is for a virtual polytope. To overcome this, we will fix a convex region then discuss faces of that region. To do this, we first introduce notation.

Just as a polytope is the intersection of its supporting half-spaces, a virtual polytope normal to a fan is also defined by the hyperplanes normal to the rays in the fan. One important difference is that a virtual polytope may consist of convex regions located on both sides of a given hyperplane. We let a solid line denote an inequality agreeing with the standard GZ polytope, and a dashed line denote an opposite inequality. In Figure 3.2 we see the four possibilities for inequalities which agree or disagree with the standard GZ array for a single variable.

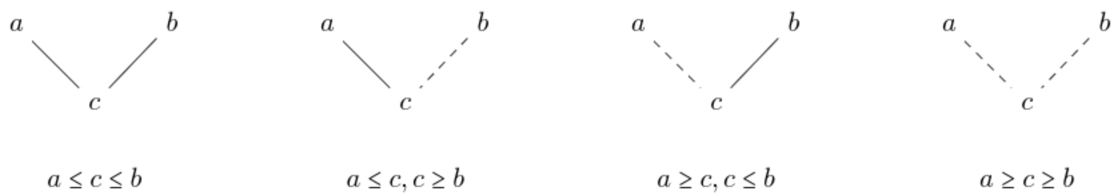


Figure 3.2: Types of Inequality in GZ Array

A GZ array decorated with a solid or dashed line between every pair of diagonally adjacent variables indicates a convex region of Δ_λ by intersecting corresponding half-spaces. For solid lines, we intersect the usual side of the half-space, and for a dashed line we use the opposite side. The philosophy is to consider every possible combination of solid and dashed lines and compute the values of the corresponding convex chains. Then, the virtual polytope consists of any regions where the convex chain is non-zero.

Now that we have notation to designate a particular convex region as an intersection of half-spaces, we can discuss the issues coming from the fact that Σ_{GZ} is not simplicial. To designate a face of a convex region, we include equalities in the GZ array in addition to the solid and dashed lines. Figure 3.3 illustrates the *diamond relation*.

We temporarily use x_i, x_k, etc for ease of notation rather than unnecessarily using $x_{i,j}$ and $x_{k,j+1}$ while we discuss faces. Regardless of whether $\mu = \lambda_i$ or $\mu = x_\ell$, such a diamond gives a relation. Consider the face where $x_i = \mu = x_{i+1}$ in Figure 3.4.

Since $x_i \leq x_{k+1} \leq x_{i+1}$ we must also have $x_{k+1} = \mu$. Thus two equalities decrease the dimension of this face by 3, so it seems we cannot simply count the number of equalities defining a face in order to know its dimension. We need to know the dimension of the face for the coefficient $(-1)^{\dim F}$ appearing in the Brianchon-Gram Theorem, Theorem 3.3.

The other complication from this diamond relation is that the same face is represented by multiple diagrams. The two diagrams in Figure 3.5 both represent the same face as the

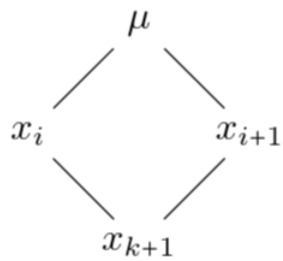


Figure 3.3: Diamonds in GZ Array Yield Relations

diagram in Figure 3.4. We will later show that naively counting faces with a coefficient of $(-1)^{\#\text{ equalities}}$ yields the correct number after cancellation. Notice also that each variable x_{k+1} has at most one equality above it, and we think of each equality as being associated with the variable in the lower line. This will help us to count faces and equalities later.

3.2.6 Virtual Gelfand-Zetlin Polytopes in One Dimension

Before stating and proving our main result about virtual GZ polytopes, we explore examples for $n = 2$ and $n = 3$ dimensions to develop intuition. We begin with the case $n = 2$, where GZ polytopes are of the form $\lambda_1 < x < \lambda_2$. In this section we represent such a polytope by the interval $[\lambda_1, \lambda_2]$. The interior of this polytope is (λ_1, λ_2), and we use the formula in Theorem 3.2.1 to compute the convex chain of (λ_1, λ_2).

$$
\begin{aligned}
\chi_{(\lambda_1,\lambda_2)} &= (-1)^{\dim[\lambda_1,\lambda_2]} \sum_{\Delta\in\Gamma([\lambda_1,\lambda_2])} (-1)^{\dim\Delta}\chi_\Delta \\
&= (-1)\left[(-1)^{\dim[\lambda_1,\lambda_2]}\chi_{[\lambda_1,\lambda_2]} + (-1)^{\dim\{\lambda_1\}}\chi_{\{\lambda_1\}} + (-1)^{\dim\{\lambda_2\}}\chi_{\{\lambda_2\}}\right] \\
&= (-1)\left[(-1)\chi_{[\lambda_1,\lambda_2]} + \chi_{\{\lambda_1\}} + \chi_{\{\lambda_2\}}\right] \\
&= \chi_{[\lambda_1,\lambda_2]} - \chi_{\{\lambda_1\}} - \chi_{\{\lambda_2\}}.
\end{aligned}
$$

This shows the value of the convex chain is compatible with the usual notation for intervals, that is, the convex chain of the interior of an interval is the characteristic function of that open interval.

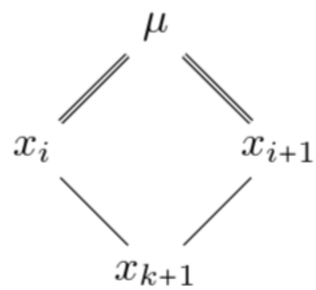


Figure 3.4: Diamond Relation

We next compute $\chi^{-1}_{[\lambda_1,\lambda_2]}$. We combine the formula in Theorem 3.2.1 with the above computation of $\chi_{(\lambda_1,\lambda_2)}$.

$$\begin{aligned}
\chi^{-1}_{[\lambda_1,\lambda_2]} &= (-1)^{\dim[\lambda_1,\lambda_2]}\chi_{(-[\lambda_1,\lambda_2])^\circ} \\
&= (-1)\chi_{(-\lambda_2,-\lambda_1)} \\
&= (-1)\left[\chi_{[-\lambda_2,-\lambda_1]} - \chi_{\{-\lambda_2\}} - \chi_{\{-\lambda_1\}}\right] \\
&= -\chi_{[-\lambda_2,-\lambda_1]} + \chi_{\{-\lambda_2\}} + \chi_{\{-\lambda_1\}}
\end{aligned}$$

Thus we see that the inverse to $\chi_{[\lambda_1,\lambda_2]}$ is the convex chain with value -1 on the open interval $(-\lambda_2,-\lambda_1)$.

We next compute $\chi_{[\lambda_1,\lambda_2]} \star \chi^{-1}_{[\lambda_1,\lambda_2]}$ to illustrate this convolution operation as well as to verify that the inverse convex chain computed above is correct. We have:

$$\begin{aligned}
\chi_{[\lambda_1,\lambda_2]} \star \chi^{-1}_{[\lambda_1,\lambda_2]} &= \chi_{[\lambda_1,\lambda_2]} \star \left(-\chi_{[-\lambda_2,-\lambda_1]} + \chi_{\{-\lambda_2\}} + \chi_{\{-\lambda_1\}}\right) \\
&= -\chi_{[\lambda_1,\lambda_2]+[-\lambda_2,-\lambda_1]} + \chi_{[\lambda_1,\lambda_2]+\{-\lambda_2\}} + \chi_{[\lambda_1,\lambda_2]+\{-\lambda_1\}} \\
&= -\chi_{[\lambda_1-\lambda_2,\lambda_2-\lambda_1]} + \chi_{[\lambda_1-\lambda_2,0]} + \chi_{[0,\lambda_2-\lambda_1]} \\
&= \chi_{\{0\}}.
\end{aligned}$$

Note that we are able to compute this convolution of convex chains merely using the correspondence between $\star$ and the Minkowski sum of polytopes. For intervals, we have $[a,b]+[c,d] = [a+c,b+d]$, as well as $[a,b]+\{c\} = [a+c,b+c]$.

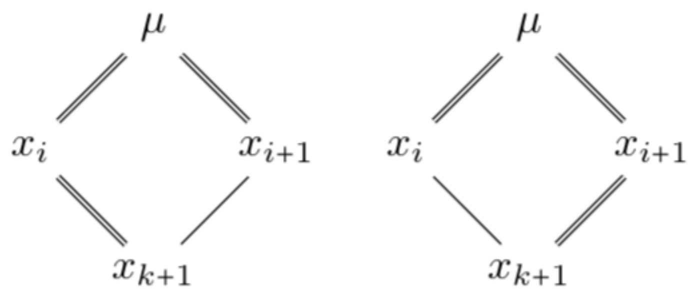


Figure 3.5: Equivalent Faces

In the case $n = 2$ there are only two Weyl chambers. In the positive chamber $\lambda_1 < \lambda_2$, and in the opposite chamber $\lambda_1 > \lambda_2$. On the boundary between the Weyl chambers, we have $\lambda_1 = \lambda_2$, which gives the inequality

$$\lambda_1 \leq x \leq \lambda_1,$$

hence the polytope is just the point $\{\lambda_1\} \in \mathbb{R}$.

We examine the convex chain associated to a virtual GZ polytope for λ in the opposite Weyl chamber. Let $\lambda_1 > \lambda_2$. The half-spaces cut out three regions of $\mathbb{R}$ on which we need to examine the values of the convex chain. The polytope $[\lambda_1, \lambda_2]$ has three faces: the entire polytope, and the endpoints $\{\lambda_1\}$ and $\{\lambda_2\}$. From the extended Brianchon-Gram theorem, we need to understand the characteristic functions of cones at these faces. For the interval, we have $\chi_{\mathbb{R}}$, and for the endpoints we have $\chi_{[\lambda_1,\infty)}$ and $\chi_{(-\infty,\lambda_2]}$ respectively. We thus obtain:

$$\begin{aligned} \chi_{[\lambda_1,\lambda_2]} &= (-1)\chi_{\mathbb{R}} + \chi_{[\lambda_1,\infty)} + \chi_{(-\infty,\lambda_2]} \\ &= -\chi_{(\lambda_2,\lambda_2)}. \end{aligned}$$

We recall Theorem 3.2.1 which shows $\chi^{-1}_{[-\lambda_1,-\lambda_2]}$ is also supported on the interval (λ_2, λ_1) with value -1, hence this is the convex chain associated to this virtual polytope.

Remark 3.2.7. We observe that convex chains of virtual polytopes in the case $n = 2$ take either the value 1 or -1. We will later use this to prove our main result.

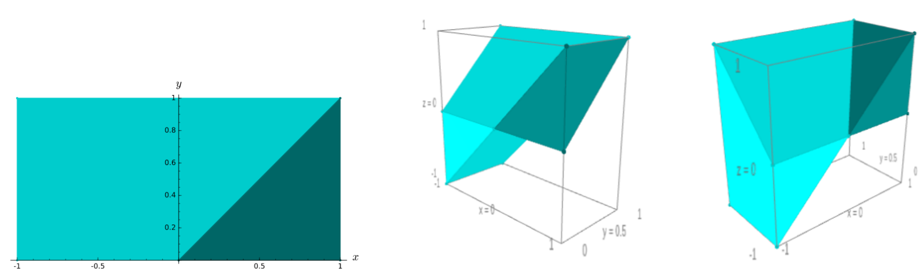
3.2.7 Virtual Gelfand-Zetlin Polytopes in Three Dimensions

The next simplest case to examine is $n = 3$. We will study the particular virtual GZ polytope $\Delta_{(-1,1,0)}$, then generalize to the rest of the virtual GZ polytopes in this dimension. Recall that the virtual GZ polytopes correspond to λ not dominant (not increasing), but we will restrict our attention to those which are regular.

Example 3.2.8. Recall that in Example 2.3.1 we examined the GZ polytope $\Delta_{(-1,0,1)}$. We permute the entries of that dominant weight, and examine $\Delta_{(-1,1,0)}$. This virtual polytope has x-coordinate ranging between -1 and 1, y-coordinate ranging between 1 and 0, and z-coordinate ranging between x and y. The inequalities may be the opposite of those found in the standard GZ array, but we will just change them as needed.

We first observe that the x and y coordinates form a rectangle (and similarly, when $n > 3$ the top line of variables will define a region which is a product of intervals).

Since the inequalities involving y are opposite of the corresponding inequalities in the GZ array, there are two different convex regions for Δ_λ: one region where $x > z > y$ and one where $x < z < y$. These two regions meet at the line $y = x$. Because of this, we can understand this 3-dimensional object by projecting to the xy-plane. See Figure 3.6. The



2D view of $\Delta_{(-1,1,0)}$ View 1 of $\Delta_{(-1,1,0)}$ View 2 of $\Delta_{(-1,1,0)}$

Figure 3.6: $\Delta_{(-1,1,0)}$ from Various Angles

lighter region is where $x < y$ and hence $x < z < y$ for the 3-dimensional images whereas the darker region is where $y > x$, and thus $y > z > x$. The line $y = x$ is where the two polytopes meet.

For both regions, x satisfies the standard inequalities and y satisfies the opposite inequalities. In the lighter region, z also satisfies the typical inequalities so the value of the convex chain is -1. In the darker region the value is $(-1)(-1) = 1$ since the inequalities for z are opposite.

Using the notation defined above with a solid line corresponding to an inequality agreeing with the standard GZ array and a dashed line corresponding to an inequality opposite of the usual, these two regions could be represented by the diagrams in Figure 3.7.

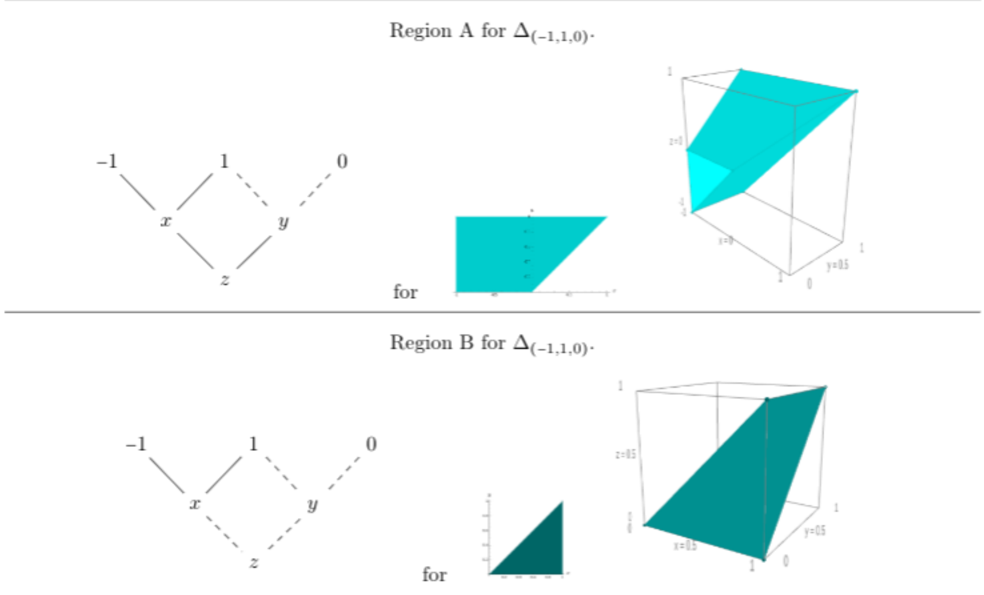


Figure 3.7: Two Convex Regions of $\Delta_{(-1,1,0)}$

Recall the construction of the cone C_F at a face F of a convex polytope. Suppose $F = \bigcap_{i=1}^{k} F_i$ for facets $\{F_i\}$ of the polytope, then C_F is the intersection of the half-spaces corresponding to the facets F_i which contain F. In the convex case, each half-space contains the polytope. For virtual polytopes we first fix a convex region, then do the analogous intersection of half-spaces. The half-space for a facet F_i is the hyperplane spanned by F_i together with the the side which would contain a standard GZ polytope.

Consider the facet $F : x = z$, see View 2 in Figure 3.6. This is the bottom face of the lighter region and the top face of the darker region, and is one of two hyperplanes separating the two convex regions of $\Delta_{(-1,1,0)}$. The inequality relating x and z in the standard GZ array is $x \le z$, meaning the cone C_F contains the lighter region and does not contain the darker region.

To relate this face cone back to the solid and dashed line diagrams, consider Figure 3.7. The cone C_F contains Region A and the diagram for Region A has a solid line between x and z, whereas C_F does not contain Region B, and the diagram for Region B has a dashed line between x and z. This suggests that the tangent cone to a face will contain a convex region exactly when the lines in the diagram which define the face are all solid.

In the Brianchon-Gram theorem, we add up χ_{C_F} so if a cone C_F does not contain the convex region of interest, the value of this function is zero. We add the values corresponding to all possible faces which can be made from only solid lines in the diagram.

We summarize the virtual GZ polytopes for the rest of the orbit of $\lambda = (-1, 0, 1)$ under the action of the Weyl group. See Figure 3.8.

3.2.8 Virtual Gelfand-Zetlin Polytopes as Convex Chains

To state our main result about virtual GZ polytopes, we first fix a regular $\lambda \in \mathbb{Z}^n$ as well as a convex region of Δ_λ, that is, a choice of solid or dashed line for each inequality in the GZ array. We consider an inductive sequence Δ_λ^k of polytopes. Let ε_k denote the value of the convex chain associated to Δ_λ^k. The result below describes ε_{k+1} in terms of ε_k. More specifically, let Δ_λ^k be the polytope defined using only the first k variables in the fixed GZ array. For this we must relabel the variables in the array so that they have a single subscript. Let $x_1 = x_{11}$ and let the indices increase across the row then continue in the leftmost position of the following row. Clearly, Δ_λ^{k+1} is obtained from Δ_λ^k by taking the product $\Delta_\lambda^k \times \mathbb{R}$, where x_{k+1} is the coordinate on $\mathbb{R}$, and intersecting this with the two half-spaces defined by the two inequalities involving x_{k+1}. Note that for the given choice of solid and dashed lines it is possible for the resulting polytope to be empty. This inductive sequence of polytopes will enable us to count the number of faces of Δ_λ^{k+1} of a given dimension in terms of faces of Δ_λ^k.

We saw in Example 3.2.8 that the top line of inequalities were determined by λ but that the lower line, z, could be given solid lines or dashed lines. We will show that for a region where variable x_k has one solid line and one dashed line above it the value of the associated convex chain is zero, so for each variable x_k we only need to consider the cases where both

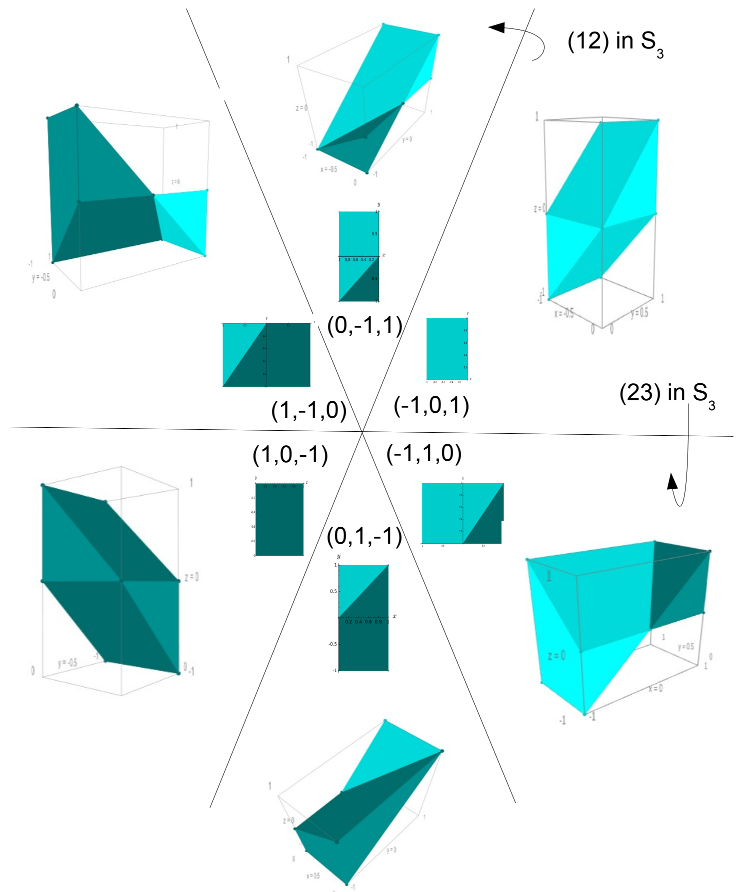


Figure 3.8: Virtual GZ Polytopes for Orbit of $\lambda = (-1, 0, 1)$ under Weyl Group

inequalities agree or both inequalities disagree. This simplifies the process of counting faces required to compute the value of the convex chain.

Theorem 3.2.9.

1. *If the inequalities for variable* x_{k+1} *both agree with the standard case then* $\varepsilon_{k+1} = \varepsilon_k$. *This case corresponds to a diagram where both lines above* x_{k+1} *are solid.*
2. *If the inequalities for variable* x_{k+1} *both disagree with the standard case then* $\varepsilon_{k+1} = -\varepsilon_k$. *This case corresponds to a diagram where both lines above* x_{k+1} *are dashed.*
3. *If exactly one of the inequalities for variable* x_{k+1} *agrees with the standard case then* $\varepsilon_{k+1} = 0$. *This case corresponds to a diagram where one line above* x_{k+1} *is solid and the other is dashed.*

Proof. Our strategy is to use the inductively defined sequence Δ_λ^k to determine the number of j-dimensional faces of Δ_λ^{k+1} with tangent cone containing the fixed convex region in terms of the numbers of relevant faces of Δ_λ^k. These are exactly the cones with the fixed region in the support of the corresponding characteristic functions in the Brianchon-Gram theorem.

Let c_j denote the number of j-dimensional faces of Δ_λ^k with tangent cone containing the fixed region. Then the value of the convex chain for Δ_λ^k is given by

$$\varepsilon_k = \sum_{j=0}^{k} (-1)^j c_j.$$

In each case below, we determine an expression for the number of relevant j-dimensional faces of Δ_λ^{k+1} in terms of $\{c_j\}$ and examine the corresponding sum to determine the value of ε_{k+1}. The base case was established in Section 3.2.6 where we showed $\varepsilon_1 = 1$ when both inequalities agree, $\varepsilon_1 = -1$ when both inequalities disagree, and $\varepsilon = 0$ when exactly one inequality agrees.

We next consider separately the three cases.

Case 1: Suppose the inequalities involving x_{k+1} in the diagram agree with the standard case (are both solid lines). We will show that the diamond constraint does not pose a problem to naively counting the codimension of each face as the number of equalities defining it. Consider a face with GZ array diagram containing one of the three equivalent diamonds in Figure 3.9, as well as the other two representations of that face. If we naively compute the

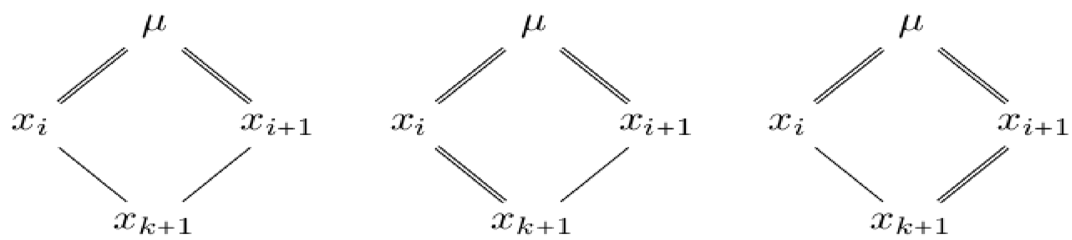


Figure 3.9: Equivalent Diagrams for a Face

dimension based on the number of constrained variables, that is, with the codimension of the face equal to the number of constraints, we will be wrong for the first diamond but correct for the second and third. This will cause the first copy of our face, from the first diamond, to be counted with the wrong sign as the two constraints lower the dimension by three rather than two. However, the second and third figures will cause this face to be double counted, so the first and second copies of this face effectively cancel out and we end up counting only the third copy. This happens for every diamond, so though it is difficult to count the number of faces of dimension j for a (virtual) GZ polytope, we need the count only for the Brianchon-Gram theorem, and hence the naive count is sufficient. We now count the number of relevant j-dimensional faces of Δ_λ^{k+1}. This is the number ways to choose $(k+1)-j$ solid lines from our fixed partial diagram.

Since the new variable x_{k+1} comes with two solid lines in this case, there are $2c_j$ faces of dimension j coming from j-dimensional faces of Δ_λ^k. We must also add the number of j-dimensional faces of Δ_λ^{k+1} in which the variable x_{k+1} is unconstrained. There are c_{j-1} of these. Thus the number of relevant j-dimensional faces of Δ_λ^{k+1} is:

$$2c_j + c_{j-1}.$$

For ease of notation let $c_{k+1} = 0 = c_{-1}$, then the value of the convex chain for Δ_λ^{k+1} on the

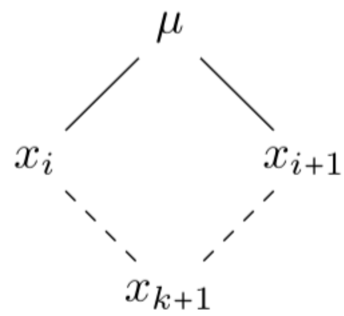


Figure 3.10: Inequalities for x_{k+1} Disagree with Standard Case

fixed convex region satisfies

$$\begin{aligned}
\varepsilon_{k+1} &= \sum_{j=0}^{k+1}(-1)^j\left(2c_j + c_{j-1}\right) \\
&= 2\sum_{j=0}^{k+1}(-1)^j c_j + \sum_{j=0}^{k+1}(-1)^j c_{j-1} \\
&= 2\sum_{j=0}^{k}(-1)^j c_j - \sum_{j=0}^{k}(-1)^j c_j \\
&= \sum_{j=0}^{k}(-1)^j c_j \\
&= \varepsilon_k
\end{aligned}$$

as desired.

Case 2: Suppose the inequalities involving x_{k+1} in the diagram both disagree with the standard case (are both dashed lines). We do not need to worry about our diamond constraint in this case because the diagram in Figure 3.10 encodes inequalities

$$x_i \le \mu \le x_{i+1} \quad \text{and} \quad x_i \ge x_{k+1} \ge x_{i+1}$$

that are inconsistent because $x_{k+1} \le x_i \le \mu$ but also $x_{k+1} \ge x_{i+1} \ge \mu$. We have already excluded the case where $x_{k+1} = x_i = \mu$ and $x_{k+1} = x_{i+1} = \mu$ by considering only full-dimensional polytopes.

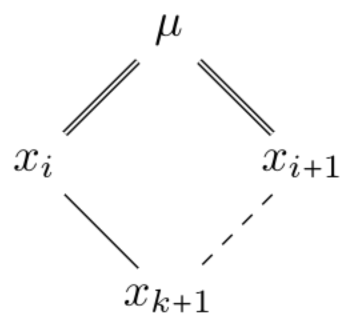


Figure 3.11: Exactly One Inequality Agrees

We count the number of relevant faces of Δ_λ^{k+1}. Since x_{k+1} is added with two dashed lines, neither line will be counted when we count subsets of solid lines. Hence the relevant j-dimensional faces all come from $j-1$ dimensional faces of Δ_λ^k, so the value of the convex chain on the indicated region of Δ_λ^{k+1} is

$$\begin{aligned}\varepsilon_{k+1} &= \sum_{j=0}^{k+1}(-1)^j c_{j-1} \\ &= -\sum_{j=0}^{k}(-1)^j c_j \\ &= -\varepsilon_k\end{aligned}$$

as desired.

Case 3: Suppose exactly one of the inequalities involving x_{k+1} in the diagram agrees with the standard case (one solid line, one dashed). We do not need to worry about the diamond constraint, see Figure 3.11, because if we have $x_i = \mu = x_{i+1}$ then $x_{k+1} \geq \mu$. Hence a diamond no longer implies constraint when there is one dashed line and one solid line. We see that the fixed convex region is unbounded, but we will show that the value of ε_{k+1} is zero.

The relevant j-dimensional faces for Δ_λ^{k+1} will come from j-dimensional faces of Δ_λ^k with variable x_{k+1} constrained, or from $j-1$-dimensional faces with x_{k+1} free. In this case there is only one solid line, and hence one way to constrain x_{k+1}, so the number of j-dimensional faces coming from j-dimensional faces of Δ_λ^k is equal to c_j. There are c_{j-1} faces with x_{k+1} free, so the total number of relevant j-dimensional faces of Δ_λ^{k+1} is

$$c_{j-1} + c_j.$$

Again we let $c_{-1} = 0 = c_{k+1}$, then we see that

$$\begin{aligned}
\varepsilon_{k+1} &= \sum_{j=0}^{k+1} (-1)^j \left(c_{j-1} + c_j\right) \\
&= \sum_{j=0}^{k} (-1)^j c_j - \sum_{j=0}^{k} (-1)^j c_j \\
&= 0.
\end{aligned}$$

The above arguments imply that the only values of a convex chain corresponding to a virtual GZ polytope Δ_λ are $0, 1$, or -1, and in particular, the convex chain is supported only on finitely many bounded regions. □

3.3 CHOW RING OF X_{GZ} AND COHOMOLOGY OF G/B

We begin this section with some algebra lemmas. We use these to relate the cohomology ring of G/B with the operational Chow ring of X_{GZ}. We recall the intersection theory of the flag variety as well as the operational Chow cohomology for (singular) toric varieties. This Chow ring is isomorphic to the ring of Minkowski weights, which is a more computationally friendly combinatorial object. Finally, we identify the cohomology ring of G/B with a quotient of the subring of the Chow cohomology ring of X_{GZ} generated in degree one.

3.3.1 Algebra Results

Let $A = \bigoplus_{i=0}^n A^i$ be a graded ring with $A^0 \cong \mathbb{Z} \cong A^n$. Then following [HW17], the *Lefschetz subalgebra* L_A is the graded subalgebra of A generated by A^1. We recall that A has *Poincarè duality* if the multiplication maps

$$A^i \times A^{n-i} \to A^n \cong \mathbb{Z}$$

are non-degenerate for all i. Since our goal is to compare $A^*(G/B) \cong H^*(G/B)$, which has Poincarè duality, with the ring $A^*(X_{GZ})$, which may not, we are interested in how an abstract graded ring might be related to one with Poincarè duality.

Lemma 3.3.1. *Let $A = \bigoplus_{i=0}^n A^i$ with $A^0 \cong \mathbb{Z} \cong A^n$. There exists a homogeneous ideal $I \subset A$ which is minimal with respect to inclusion such that A/I has Poincarè duality. We call this ring A/I the* Gorenstein quotient *$Gor(A)$ of A.*

Proof. Consider the ideal I generated by all homogeneous elements $x \in A$ such that

$$x \cdot A^{n-\deg(x)} = 0.$$

We first show that A/I has Poincarè duality. Suppose for contradiction that A/I does not have Poincarè duality. Then, there is some $x \in A$ which we can take to be homogeneous, say $x \in A^i$, such that for all $y \in A^{n-i}$ we have $xy \in I$. As the degrees of x and y are complementary, $\deg xy = n$, so these products lie in the nth graded piece of the ideal I. Observe, however, that for any $z \in I$ with $\deg z = n$, we must have $z = \sum_i c_i x_i$ where $\{x_i\}$ are generators of I with $\deg x_i = d_i$. It is sufficient to consider only terms with degree n, so assume $\deg c_i = n - d_i$. Since the x_i generate I, by assumption they satisfy $x_i A^{n-d_i} = 0$, so $c_i x_i = 0$ for all i and hence $z = 0$. We have shown that the degree n part of I is trivial, and hence $xy \in I$ with $\deg xy = n$ implies that $xy = 0$. This implies that $x \in I$, contradicting our assumption that it is not. Thus A/I does have Poincarè duality.

We next show that I is the minimal such homogeneous ideal. Suppose not, then there exists homogeneous ideal J such that A/J has Poincarè duality. Let $x \in (I \setminus J)$. As both ideals are homogeneous, we can take x to be homogeneous say of degree i. Since $x \in I$ we must have $x \cdot A^{n-i} = 0$. Since $x \notin J$, it corresponds to a non-trivial coset $\bar{x}$ in A/J. However, this $\bar{x}$ satisfies $\bar{x} \cdot (A/J)^{n-i} = 0$, contradicting Poincarè duality for A/J. □

We next recall an algebra result required to prove our main result. (See [Kav11, Theorem 1.1] and [Eis95, Exercise 21.7].)

Theorem 3.3.2. *Let $A = \bigoplus_{i=0}^{n} A^i$ be a finite dimensional graded algebra which is generated by A^1, satisfies $A^0 \cong \mathbb{Z} \cong A^n$, and has Poincarè duality.*

Fix a basis $\{a_1, \ldots, a_r\}$ for A^1, and consider the polynomial $P : \mathbb{Z}^r \to \mathbb{Z}$ defined by

$$P(x_1, \ldots, x_r) = (x_1 a_1 + \cdots + x_r a_r)^n \in A^n \cong \mathbb{Z}.$$

Then we obtain an isomorphism of graded algebras

$$A \cong \mathbb{Z}[\partial_1, \ldots, \partial_r]/I$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and I is the ideal of polynomials in the operators $\partial_1, \ldots, \partial_r$ which annihilate P.

Proof. We follow the sketch outlined in [Kav11]. Consider the evaluation homomorphism

$$\Phi : \mathbb{Z}[t_1, \ldots, t_r] \to A$$

under which $t_i \mapsto a_i$. Since A is generated by A^1, this map is clearly surjective. We aim to show that $\ker \Phi = I$, so that we will have $A \cong \mathbb{Z}[t_1, \ldots, t_r]/I$. Since Φ respects the degree one grading, and both rings are generated in degree one it is a graded morphism, and hence $\ker \Phi$ is a graded ideal, i.e., is generated by homogeneous elements.

We now consider $f \in \mathbb{Z}[t_1, \ldots, t_r]$ homogeneous of degree n, say

$$f(t_1, \ldots, t_r) = \sum_{\beta_1 + \ldots + \beta_r = n} c_{\beta_1, \ldots, \beta_r} t_1^{\beta_1} \cdots t_r^{\beta_r}.$$

Then

$$\begin{aligned}
f(\partial_1,\ldots,\partial_r)\cdot P &= \left(\sum_{\beta_1+\ldots+\beta_r=n} c_{\beta_1,\ldots,\beta_r}\partial_1^{\beta_1}\cdots\partial_r^{\beta_r}\right)\cdot(x_1a_1+\ldots+x_ra_r)^n \\
&= \left(\sum_{\beta_1+\ldots+\beta_r=n} c_{\beta_1,\ldots,\beta_r}\partial_1^{\beta_1}\cdots\partial_r^{\beta_r}\right)\cdot\left(\sum_{\alpha_1+\ldots+\alpha_r=n}\binom{n}{\alpha_1,\ldots,\alpha_r}a_1^{\alpha_1}\cdots a_r^{\alpha_r}x_1^{\alpha_1}\cdots x_r^{\alpha_r}\right) \\
&= \sum_{\beta_1+\ldots+\beta_r=n}\sum_{\alpha_1+\ldots+\alpha_r=n} c_{\beta_1,\ldots,\beta_r}a_1^{\alpha_1}\cdots a_r^{\alpha_r}\binom{n}{\alpha_1,\ldots,\alpha_r}\partial_1^{\beta_1}\cdots\partial_r^{\beta_r}\cdot(x_1^{\alpha_1}\cdots x_r^{\alpha_r}) \\
&= \sum_{\beta_1+\ldots+\beta_r=n} c_{\beta_1,\ldots,\beta_r}a_1^{\beta_1}\cdots a_r^{\beta_r}\binom{n}{\beta_1,\ldots,\beta_r}\partial_1^{\beta_1}\cdots\partial_r^{\beta_r}\cdot\left(x_1^{\beta_1}\cdots x_r^{\beta_r}\right) \\
&= \sum_{\beta_1+\ldots+\beta_r=n} c_{\beta_1,\ldots,\beta_r}a_1^{\beta_1}\cdots a_r^{\beta_r}\frac{n!}{\beta_1!\cdots\beta_r!}\beta_1!\cdots\beta_r! \\
&= n!f(a_1,\ldots,a_r).
\end{aligned}$$

From this we see that $f(a_1,\ldots,a_r)=0$, i.e. $f\in\ker\Phi$, if and only if f annihilates P so $f\in I$.

It remains to show that the same holds for f homogeneous of degree $m<n$. Let

$$f(t_1,\ldots,t_r)=\sum_{\beta_1+\ldots+\beta_r=m} c_{\beta_1,\ldots,\beta_r}t_1^{\beta_1}\cdots t_r^{\beta_r}.$$

Suppose first that f is not in $\ker\Phi$, then $f(a_1,\ldots,a_r)\neq 0$. Since A has Poincarè duality and $f(a_1,\ldots,a_r)\in A^m$ there must be some $a'\in A^{n-m}$ such that $a'f(a_1,\ldots,a_r)\neq 0$. As A is generated in degree one, there is a homogeneous polynomial g of degree $n-m$ which gives this element a'. Then gf is a nonzero homogeneous polynomial of degree n, and the above computation shows that $(gf)(\partial_1,\ldots,\partial_r)\cdot P$ must not be zero. Then $f(\partial_1,\ldots,\partial_r)\cdot P$ cannot be zero, so f is not in I. Thus we have showed that f in I implies f in $\ker\Phi$.

Suppose now that $f(a_1,\ldots,a_r)=0$, so f is in $\ker\Phi$. Then

$$\begin{aligned}
&f(\partial_1,\ldots,\partial_r)\cdot P \\
&\quad= \sum_{\beta_1+\ldots+\beta_r=m}\sum_{\alpha_1+\ldots+\alpha_r=n} c_{\beta_1,\ldots,\beta_r}a_1^{\alpha_1}\cdots a_r^{\alpha_r}\binom{n}{\alpha_1,\ldots,\alpha_r}\partial_1^{\beta_1}\cdots\partial_r^{\beta_r}\cdot(x_1^{\alpha_1}\cdots x_r^{\alpha_r}) \\
&\quad= \sum_{\substack{\beta_1+\ldots+\beta_r=m\\ \alpha_1+\ldots+\alpha_r=n\\ \beta_i\leq\alpha_i \text{ for } i=1,\ldots,r}} c_{\beta_1,\ldots,\beta_r}a_1^{\alpha_1}\cdots a_r^{\alpha_r}\binom{n}{\alpha_1,\ldots,\alpha_r}\frac{\alpha_1!}{(\alpha_1-\beta_1)!}\cdots\frac{\alpha_r!}{(\alpha_r-\beta_r)!}\left(x_1^{\alpha_1-\beta_1}\cdots x_r^{\alpha_r-\beta_r}\right)
\end{aligned}$$

Substituting $\gamma_i = \alpha_i - \beta_i$, notice that $\sum \gamma_i = \sum \alpha_i - \sum \beta_i = n - m$ and so we obtain:

$$\begin{aligned}
&= \sum_{\gamma_1+\ldots+\gamma_r=n-m} \sum_{\beta_1+\ldots+\beta_r=m} c_{\beta_1,\ldots,\beta_r} a_1^{\beta_1}\cdots a_r^{\beta_r} a_1^{\gamma_1}\cdots a_r^{\gamma_r} \binom{n}{\gamma_1,\ldots,\gamma_r} (x_1^{\gamma_1}\cdots x_r^{\gamma_r}) \\
&= \left(\sum_{\beta_1+\ldots+\beta_r=m} c_{\beta_1,\ldots,\beta_r} a_1^{\beta_1}\cdots a_r^{\beta_r} \right) \left(\sum_{\gamma_1+\ldots+\gamma_r=n-m} a_1^{\gamma_1}\cdots a_r^{\gamma_r} \binom{n}{\gamma_1,\ldots,\gamma_r} (x_1^{\gamma_1}\cdots x_r^{\gamma_r}) \right) \\
&= f(a_1,\ldots,a_r) \left(\sum_{\gamma_1+\ldots+\gamma_r=n-m} a_1^{\gamma_1}\cdots a_r^{\gamma_r} \binom{n}{\gamma_1,\ldots,\gamma_r} (x_1^{\gamma_1}\cdots x_r^{\gamma_r}) \right) \\
&= 0
\end{aligned}$$

thus f is in the ideal I. □

We now use Theorem 3.3.2 to prove the following main lemma required for our result.

Lemma 3.3.3. *Suppose $A = \bigoplus_{i=0}^n A^i$ and $B = \bigoplus_{i=0}^n B^i$ both have degree zero and degree n pieces isomorphic to $\mathbb{Z}$, are generated in degree one, and ring A has Poincarè duality. Suppose additionally that*

- *there exists isomorphism $\varphi : A^1 \to B^1$ and*
- *for all $a_1, \ldots, a_n \in A^1$ we have*

$$a_1 \cdot \ldots \cdot a_n = \varphi(a_1) \cdot \ldots \cdot \varphi(a_n)$$

using fixed isomorphisms $A^n \cong \mathbb{Z} \cong B^n$.

Then φ extends to give an isomorphism of A with the Gorenstein quotient of B, i.e.,

$$\tilde{\varphi} : A \xrightarrow{\cong} Gor(B).$$

Proof. We apply Theorem 3.3.2 to A and to the Gorenstein quotient $\mathrm{Gor}(B)$. It is clear that A already satisfies the conditions of Theorem 3.3.2, so $A \cong \mathbb{Z}[\partial_1, \ldots, \partial_r]/I$ where r is the rank of A^1 and ideal I is the annihilator of the power map P described in Theorem 3.3.2.

We need to show that $\mathrm{Gor}(B)$ also satisfies these conditions. First note that $B^0 \cong \mathbb{Z} \cong B^n$ so the multiplication $B^0 \times B^n \to B^n \cong \mathbb{Z}$ is already non-degenerate, therefore $\mathrm{Gor}(B)^0 \cong \mathbb{Z} \cong \mathrm{Gor}(B)^n$. Also, by construction, $\mathrm{Gor}(B)$ has Poincarè duality. Finally, we consider the map on degree one pieces:

$$A^1 \xrightarrow{\varphi} B^1 \xrightarrow{q} \mathrm{Gor}(B)^1,$$

where q is the map in the construction of the Gorenstein quotient. Call this composition $\tilde{\varphi} : A^1 \to \mathrm{Gor}(B)^1$. We claim this composition is an isomorphism. Since φ is an isomorphism and q is surjective, $\tilde{\varphi}$ is surjective and we only need to verify injectivity. Suppose for contradiction that some nonzero $a \in A^1$ has image $\tilde{\varphi}(a) = q(\varphi(a)) = 0$ in $\mathrm{Gor}(B)^1$. Since φ is an isomorphism, $\varphi(a) = b$ for some nonzero $b \in B^1$. Then b is in the ideal in Lemma 3.3.1, so it is a linear combination of x_i satisfying $x_i \cdot B^{n-\deg(x_i)} = 0$. Since $b \in B^1$, x_i's generating it must be in degree zero or one. We argued above that B^0 is not annihilated in this construction, so we can only have $x_i \in B^1$. Any linear combination of elements annihilating B^{n-1} must also annihilate B^{n-1} so we must have $b \cdot B^{n-1} = 0$. This gives a contradiction because A has Poincarè duality, but the element a pairs with A^{n-1} to give 0 via the fact $a_1 \cdots a_n = \varphi(a_1) \cdots \varphi(a_n)$ with $a_1 = a$ and $b = \varphi(a_1)$.

Thus $\mathrm{Gor}(B)$ satisfies the conditions in Theorem 3.3.2, and hence $\mathrm{Gor}(B) \cong \mathbb{Z}[\partial_1, \ldots, \partial_r]/I$. We already know that A is isomorphic to this operator algebra, thus $A \cong \mathrm{Gor}(B)$. □

3.3.2 Intersection Theory

We now recall the definitions of Chow groups and, for the case of smooth toric varieties, Chow cohomology.

Definition 3.3.4. For a toric variety X_Σ, the Chow group $A_k(X_\Sigma)$ is generated by orbit closures $V(\sigma)$ for $\sigma \in \Sigma$ of codimension k.

When the variety X_Σ is smooth, we define $A^k(X_\Sigma) = A_{n-k}(X_\Sigma)$. There is an intersection product on $A^*(X_\Sigma)$ which respects the grading. When X_Σ is smooth and projective, we have the following description of the Chow ring; see [Ful93].

Proposition 3.3.5. *For X_Σ a smooth projective toric variety, $A^*(X_\Sigma) \cong H^*(X_\Sigma) \cong \mathbb{Z}[D_1, \ldots, D_d]/I$ where the D_i are T-invariant divisors on X_Σ corresponding to ray generators v_i and I is the ideal generated by the following types of relation:*

- *$D_{i_1} \cdots D_{i_k}$ for $v_{i_1}, \ldots, v_{i_k}$ not contained in any cone of Σ and*
- *$\sum_{i=1}^{d} \langle u, v_i \rangle D_i$ for $u \in M$.*

Unfortunately, neither of the two varieties we are interested in, X_{GZ} and G/B, are smooth projective toric varieties.

More generally, one can define Chow groups for an arbitrary variety; see [Ful13]. Let $A_k(X)$ be the group of algebraic k-cycles, formal sums of irreducible subvarieties of X of dimension k modulo rational equivalence. These rational equivalences are generated as divisors of rational functions on $k+1$-dimensional subvarieties of X. When X is a smooth variety, we let $A^k(X) = A_{n-k}(X)$, then the product defined using transverse intersection gives $A^*(X)$ the structure of a graded algebra ([Ful13, Proposition 8.3]). Again, an irreducible subvariety $V \subset X$ of dimension k generates a class $[V] \subset A^{n-k}(X)$ just as in the case of toric varieties, though this time we do not have the correspondence between closed T-invariant subvarieties and cones in the fan. For certain important classes of varieties, there are relations between the Chow ring $A^*(X)$ and the cohomology ring $H^*(X)$. The following proposition can be found in [Ful13, Example 19.1.11].

Proposition 3.3.6. *If X has a cellular decomposition such as is the case for $X = G/B$ with the Bruhat decomposition, then there is an isomorphism*

$$H^*(X) \cong A^*(X).$$

This can be combined with the well-known Borel description of the cohomology ring of G/B,

$$H^*(G/B) \cong \mathbb{Z}[\Lambda_{\mathbb{R}}]/I_W, \tag{3.6}$$

where $\Lambda_{\mathbb{R}} \cong \mathbb{R}^{n-1}$ is the weight lattice tensored with $\mathbb{R}$, and I_W is the ideal generated by non-constant Weyl group invariant polynomials. We recall that the map $\Lambda \to H^2(G/B)$ given by $\lambda \mapsto c_1(\mathcal{L}_\lambda)$ is additive as $\mathcal{L}_\lambda \otimes \mathcal{L}_\mu = \mathcal{L}_{\lambda+\mu}$, and extends to give an isomorphism of graded rings as both are generated by the above graded pieces. We have the following isomorphism:

$$\Lambda \cong H^2(G/B) \cong \mathrm{Pic}(G/B). \tag{3.7}$$

Alternatively, $H^*(G/B)$ can be viewed as the polytope algebra of the GZ family, see [Kav11, Corollary 5.3]. There it is shown that

$$H^*(G/B) \cong \mathrm{Sym}(\Lambda_{\mathbb{R}})/I$$

where I is the ideal of polynomials which, when viewed as differential operators, annihilate the volume polynomial of the GZ polytopes.

Next, we recall the Chow cohomology of a non-smooth toric variety X_Σ. Chow groups and rings have been defined for singular toric varieties, the so-called operational Chow ring, by Fulton and MacPherson in [FM81]. As in the smooth case, the Chow group $A_k(X)$ is generated by orbit closures $\overline{V(\sigma)}$ for $\sigma \in \Sigma(n-k)$, however these groups may have torsion. In the case that the fan Σ is complete (as it is for Σ_{GZ}), we define $A^k(X) = \mathrm{Hom}(A_k(X), \mathbb{Z})$ (see [FS97]). The main goal of Fulton and Sturmfels in [FS97] is to construct an isomorphism between $A^*(X_\Sigma)$ and another graded ring more combinatorial in nature. We discuss this in the following section.

3.3.3 Minkowski Weights

In this section we recall the description of the Chow cohomology ring of a toric variety in terms of Minkowski weights (see [FS97]). Let $\Sigma(k)$ be the set of cones of dimension k in a fan Σ.

Definition 3.3.7. A function $c : \Sigma(n-k) \to \mathbb{Z}$ is a *Minkowski weight* if it satisfies a *balancing condition*

$$\sum_{\sigma \in \Sigma(n-k), \sigma \supset \tau} \langle u, n_{\sigma,\tau} \rangle c(\sigma) = 0 \tag{3.8}$$

where $n_{\sigma,\tau}$ is a lattice point in σ which generates N_σ / N_τ, the quotient of the lattices spanned by σ and τ. The above equation must be satisfied for all $u \in M(\tau)$, the lattice perpendicular to the span of τ.

Let MW^k denote the set of Minkowski weights on cones of codimension k. For two Minkowski weights, $c \in MW^p$ and $\tilde{c} \in MW^q$, the product $c \cup \tilde{c} \in MW^{p+q}$ is given by

$$(c \cup \tilde{c})(\gamma) = \sum_{(\sigma,\tau) \in \Sigma(n-p) \times \Sigma(n-q)} m^{\gamma}_{\sigma,\tau} c(\sigma) \tilde{c}(\tau)$$

where γ is a cone of codimension $p+q$, and $m^{\gamma}_{\sigma,\tau} = [N : N_\sigma + N_\tau]$ and the sum is over all pairs of cones (σ, τ) which both contain γ and such that σ meets $\tau + v$ for fixed generic vector v. This is the content of [FS97] Theorem 4.2.

The goal of [FS97] is to give an isomorphism between the ring of Minkowski weights and the operational Chow ring of a complete toric variety X_Σ. First, they show $MW^k \cong A^k(X_\Sigma)$ in [FS97, Theorem 3.1]. As a consequence,

$$\mathrm{Pic}(X_\Sigma) \cong A^1(X_\Sigma) \tag{3.9}$$

in this situation. It is proven in [KP08, Corollary 4.6] that for any toric variety there is an isomorphism $\mathrm{Pic}(X) \cong A^1(X)$. The multiplication of Minkowski weights described above gives $MW^*(X_\Sigma)$ the structure of a graded ring isomorphic to the operational Chow ring.

Example 3.3.8 (Hypersimplex)**.** The following is an example of a variety where the ring MW^* is not generated by MW^1. See Example 3.5 of [FS97] or, equivalently, Example 4.2 of [KP08]. We consider the fan Σ_H over the cube in $\mathbb{R}^3$ with vertices $(\pm1, \pm1, \pm1)$ then examine the ring of Minkowski weights for the toric variety X_{Σ_H}. The rays in the fan Σ_H will be notated as follows:

$$\begin{aligned} \rho_1 &= \langle 1,1,1\rangle & \rho_5 &= -\rho_1 \\ \rho_2 &= \langle 1,1,-1\rangle & \rho_6 &= -\rho_2 \\ \rho_3 &= \langle 1,-1,1\rangle & \rho_7 &= -\rho_3 \\ \rho_4 &= \langle -1,1,1\rangle & \rho_8 &= -\rho_4. \end{aligned}$$

The 2-dimensional cone spanned by ρ_i and ρ_j will be denoted σ_{ij}, and similarly the 3-dimensional cone spanned by ρ_i, ρ_j and ρ_k will be denoted σ_{ijk}.

We first show that $MW^1 \cong \mathbb{Z}$. We recall that a weight $c \in MW^1$ is a map on cones of codimension 1, which in this example will have dimension 2. Let

$$c(\sigma_{ij}) = c_{ij}$$

for each cone σ_{ij}. We will have a relation as in Equation (3.8) for each ray ρ_k. As Σ_H is the fan over a cube, without loss of generality, we can consider the equation for the ray ρ_1 and by symmetry draw conclusions about the relations corresponding to other rays. Each ray is contained in exactly three cones of dimension 2. For ρ_1, the balancing condition will involve the cones σ_{12}, σ_{13} and σ_{14}. The other ingredients we require are the vectors n_{σ_{1i},ρ_1} for

$i = 2, \ldots, 4$. Again, appealing to symmetry, it will be enough to understand n_{12}. Recall n_{12} is the lattice point in σ_{12} which generates the lattice $N_{\sigma_{12}}/N_{\rho_1}$, so since the rays are orthogonal, we can take $n_{12} = \rho_2$ and similarly $n_{1i} = \rho_i$. It is enough to consider the balancing equations for $u \in \{\langle 1, 0, -1\rangle, \langle 0, 1, -1\rangle\}$ as these vectors form a basis for the lattice $M(\rho_1)$ orthogonal to ρ_1. We obtain:

$$0 = \sum_{i=2}^{4} \langle \langle 1, 0, -1\rangle, \rho_i \rangle c_{1i} = 2c_{12} - 2c_{14}$$
$$0 = \sum_{i=2}^{4} \langle \langle 0, 1, -1\rangle, \rho_i \rangle c_{1i} = 2c_{12} - 2c_{13}.$$

Thus the balancing equations associated with ρ_1 imply that the value of c on all 2-dimensional cones is the same. The symmetry of our fan implies that this same computation can be done for any other ray, and hence the value of c on all 2-dimensional cones is the same, so therefore $MW^1 \cong \mathbb{Z}$.

To show that MW^* is not generated by MW^1, it is enough to show that rank $MW^2 > 1$. To prove this, let $c \in MW^2$. Recall that codimension 2 cones in Σ_H are rays. Let $c(\rho_i) = c_i$, then the balancing condition

$$\sum_{i=1}^{8} c_i \rho_i = 0$$

must be satisfied. As this equation is a 3-dimensional vector equation, our 8 values $\{c_1, \ldots, c_8\}$ must satisfy at most 3 additional equations, hence rank $MW^2 \geq 5$. It can be shown that these equations are independent and that $MW^2 \cong \mathbb{Z}^5$ and thus cannot be generated by products of elements of MW^1.

3.3.3.1 GZ Example, $n = 3$ We next compute the Chow ring of X_{GZ} for $n = 3$ using Minkowski weights. We consider the variety constructed from the weight $\lambda = (-1, 0, 1)$ for ease of computation. The polytope Δ_λ is defined by the following array of inequalities

$$\begin{array}{ccccc} -1 & & 0 & & 1 \\ & x & & y & \\ & & z & & \end{array}$$

and has normal fan Σ_{GZ} as in Figure 3.12. We enumerate the rays as follows:

$$\rho_1 = (1,0,0) \quad \rho_3 = (0,1,0) \quad \rho_5 = (1,0,-1)$$
$$\rho_2 = (-1,0,0) \quad \rho_4 = (0,-1,0) \quad \rho_6 = (0,-1,1).$$

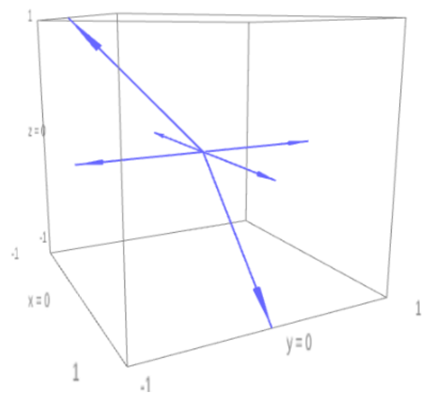


Figure 3.12: Rays of Σ_{GZ} for $n = 3$

Likewise, we let σ_{ij} denote the 2-dimensional cone spanned by rays ρ_i and ρ_j.

$$\sigma_{13} \quad \sigma_{23} \quad \sigma_{24}$$
$$\sigma_{15} \quad \sigma_{25} \quad \sigma_{35} \quad \sigma_{45}$$
$$\sigma_{16} \quad \sigma_{26} \quad \sigma_{36} \quad \sigma_{46}$$

Similarly, the collection of 3-dimensional cones are:

$$\gamma_{135} \quad \gamma_{235} \quad \gamma_{245} \quad \gamma_{1456}$$
$$\gamma_{136} \quad \gamma_{236} \quad \gamma_{246}$$

We now determine MW^k for each value $k = 0, \ldots, 3$, as these are the only codimensions in the fan Σ. We first compute MW^3. There is a single cone of codimension 3, namely, the origin. Then a Minkowski weight on $\Sigma(3)$ is a map $0 \to \mathbb{Z}$, and there are no cones $\tau \subset 0$, thus no relations to satisfy. Hence

$$MW^3 \cong \mathbb{Z}. \tag{3.10}$$

We next determine MW^2. A weight $c \in MW^2$ is a function on cones of codimension 2, i.e., on rays ρ_i. Let $c(\rho_i) = c_i$, then the single relation coming from the cone $\tau = 0$ which is a subcone of all ρ_i is given by

$$\sum_{i=1}^{6} c_i \rho_i \tag{3.11}$$

as the positive generator of the lattice N_{ρ_i}/N_0 is just the ray ρ_i, and the lattice orthogonal to 0 is the entire lattice. Expanding this equation in terms of our basis, we get three relations:

$$\begin{aligned} c_1 - c_2 + c_5 &= 0 \\ c_3 - c_4 - c_6 &= 0 \\ -c_5 + c_6 &= 0. \end{aligned}$$

We see from this that any weight $c \in MW^2$ is determined by its value on three rays, suppose, $c(\rho_2) = a$, $c(\rho_4) = b$ and $c(\rho_6) = c$, then

$$\begin{array}{ccc} c(\rho_1) = a - c & c(\rho_3) = b + c & c(\rho_5) = c \\ c(\rho_2) = a & c(\rho_4) = b & c(\rho_6) = c. \end{array} \tag{3.12}$$

Thus $MW^2 \cong \mathbb{Z}^3$.

Next, we examine MW^1. These are functions on codimension 1 cones σ_{ij}. Let $c \in MW^1$ and suppose the value on cone σ_{ij} is $c(\sigma_{ij}) = c_{ij}$. Then, a weight of codimension 1 is given by the data

$$\begin{array}{cccc} c_{13} & c_{23} & c_{24} & \\ c_{15} & c_{25} & c_{35} & c_{45} \\ c_{16} & c_{26} & c_{36} & c_{46} \end{array}$$

subject to relations coming from the rays $\{\rho_i\}$.

First, the relation for $\tau = \rho_1$ involves the cones σ_{13}, σ_{15} and σ_{16}. For each of these, we need to compute $n_{\sigma\tau}$, the lattice point in σ which generates the one-dimensional lattice N_σ/N_τ. The relation will be a vector equation in the vector space perpendicular to $\rho_1 = (1,0,0)$. We compute:

$$n_{13} = (0,1,0), \quad n_{15} = (0,0,-1), \quad \text{and} \quad n_{16} = (0,-1,1)$$

where all vectors are considered modulo ρ_1. The relation equation becomes

$$c_{13} \cdot (0,1,0) + c_{15} \cdot (0,0,-1) + c_{16} \cdot (0,1,-1) = 0$$

which implies

$$c_{13} = c_{15} = c_{16}.$$

Similar computations for the other rays yield the following results:

$$c_{13} = c_{15} = c_{16} = c_{25} = c_{26}$$
$$c_{24} = c_{35} = c_{36} = c_{45} = c_{46}$$
$$c_{23} = c_{13} + c_{24}$$

For later computations, we will let a and b be the generators of $MW^1 \cong \mathbb{Z}^2$, that is,

$$a = c_{13} = c_{15} = c_{16} = c_{25} = c_{26}$$
$$b = c_{24} = c_{35} = c_{36} = c_{45} = c_{46}$$
$$c_{23} = a + b.$$

We now examine MW^0. A weight $c \in MW^0$ is a function on top-dimensional cones subject to relations coming from each 2-dimensional cone. Each 2 dimensional cone σ_{ij} separates two top-dimensional cones, and the corresponding relation gives equality between the values of c on each pair of top-dimensional cones. Hence $MW^0 \cong \mathbb{Z}$ as the value of c on each 3-dimensional cone must be the same. In summary, we have the following:

$$MW^0 \cong \mathbb{Z}$$
$$MW^1 \cong \mathbb{Z}^2$$
$$MW^2 \cong \mathbb{Z}^3$$
$$MW^3 \cong \mathbb{Z}.$$

Before understanding the product structure on MW^*, it is already clear that the ring cannot have Poincarè duality as the rank of MW^2 is greater than MW^1.

Our next goal is to understand the product structure on $MW^*(X_{GZ})$. For weights $c \in MW^p$ and $\tilde{c} \in MW^q$, their product is a function on cones of codimension $p+q$, and its value on a cone $\gamma \in MW^{p+q}$ is given by

$$(c \cup \tilde{c})(\gamma) = \sum_{\sigma,\tau \in \Sigma(n-p)\times\Sigma(n-q)} m^{\gamma}_{\sigma\tau} \cdot c(\sigma) \cdot \tilde{c}(\tau), \tag{3.13}$$

where $m^{\gamma}_{\sigma\tau}$ is $[N : N_\sigma + N_\tau]$ as long as

(a) $\sigma, \tau \supset \gamma$

(b) σ meets $\tau + v$ for a generic fixed $v \in N$

otherwise $m^{\gamma}_{\sigma\tau} = 0$. Recall also that $\Sigma(n-p)$ is the set of cones in Σ of dimension $n-p$.

Our goal is to compute products of Minkowski weights in our example to determine whether $MW^*(X_{GZ})$ is generated in degree 1. To this end, let $c, \tilde{c} \in MW^1(X_{GZ})$, such that

$$\begin{aligned}
c : \{\sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{25}, \sigma_{26}\} &\mapsto a \\
c : \{\sigma_{24}, \sigma_{35}, \sigma_{36}, \sigma_{45}, \sigma_{46}\} &\mapsto b \\
c : \{\sigma_{23}\} &\mapsto a + b \\
\tilde{c} : \{\sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{25}, \sigma_{26}\} &\mapsto \tilde{a} \\
\tilde{c} : \{\sigma_{24}, \sigma_{35}, \sigma_{36}, \sigma_{45}, \sigma_{46}\} &\mapsto \tilde{b} \\
\tilde{c} : \{\sigma_{23}\} &\mapsto \tilde{a} + \tilde{b}.
\end{aligned}$$

Then $c \cup \tilde{c} \in MW^2$ will be evaluated on cones of codimension 2, i.e., rays. It is enough to determine the value of this weight on the rays ρ_2, ρ_4 and ρ_5; see Equation (3.12).

We begin by examining $(c \cup \tilde{c})(\rho_2)$ via Equation (3.13). Recall that this involves looking at all pairs $(\sigma, \tau) \in \Sigma(2) \times \Sigma(2)$ where σ and τ both contain ρ_2 and σ meets $\tau + v$ for a generic fixed $v \in N$. The cones in $\Sigma(2)$ which contain ρ_2 are $\{\sigma_{23}, \sigma_{24}, \sigma_{25}, \sigma_{26}\}$, so σ, τ will come from this collection. Since all these cones involve $\rho_2 = (-1, 0, 0)$, we can sketch the relevant cones in the yz-plane where, for example, σ_{23} can be viewed as $\rho_3 = (1, 0)$. In Figure 3.13, we see the cones for c in blue, and for $\tilde{c}$ in green using a shift of $v = (.1, .1, .1)$. Then there are two pairs (σ, τ) which meet for this vector v, either $(\sigma, \tau) = (\sigma_{23}, \sigma_{25})$ or $(\sigma, \tau) = (\sigma_{26}, \sigma_{24})$.

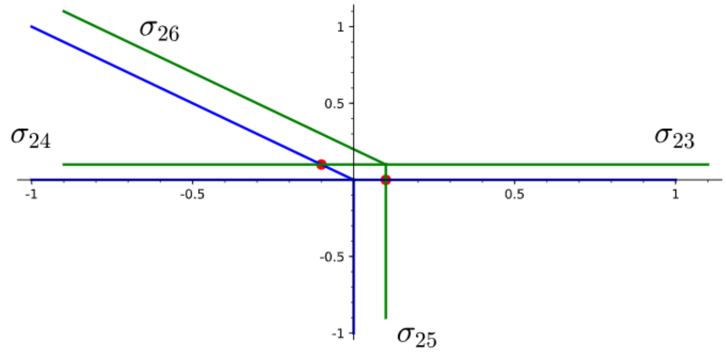


Figure 3.13: Intersection of σ and $\tau + v$

The last ingredient required to compute this product are the coefficients $m^{\rho_2}_{\sigma\tau}$ for the sum. Recall $m^{\gamma}_{\sigma\tau}$ is $[N : N_\sigma + N_\tau]$. In both cases, $N_\sigma + N_\tau = N$ so $m^{\rho_2}_{\sigma\tau} = 1$. Thus we have

$$\begin{aligned}(c \cup \tilde{c})(\rho_2) &= c(\sigma_{23})\tilde{c}(\sigma_{25}) + c(\sigma_{26})\tilde{c}(\sigma_{24})\\ &= (a+b)\tilde{a} + a(\tilde{b})\\ &= a\tilde{a} + b\tilde{a} + a\tilde{b}.\end{aligned}$$

Similar computations for $(c \cup \tilde{c})(\rho_4)$ and $(c \cup \tilde{c})(\rho_5)$ yield:

$$\begin{aligned}(c \cup \tilde{c})(\rho_4) &= b\tilde{b}\\ (c \cup \tilde{c})(\rho_5) &= b\tilde{a} + a\tilde{b}.\end{aligned}$$

Thus we see that products $c \cup \tilde{c}$ in fact generate the entire 3-dimensional space MW^2, and hence MW^* for Σ_{GZ} is generated in degree 1 for the case $n = 3$.

3.3.3.2 GZ Example, $n = 4$

For computations of $A^*(X_{GZ})$ in the case $n = 4$ we utilize SageMath [The19]. See Appendix 4 for the code. We fix an order on the set of cones of codimension k and represent a weight $c \in MW^k$ as a vector in $\mathbb{Z}^{|\Sigma(n-k)|}$. We then use linear algebra to determine the rank of MW^k as well as a basis. We also implement the product structure of MW^*. We obtain the following results for the case $n = 4$:

$$\begin{aligned} \text{rank } MW^0 = \text{rank } MW^6 = 1 \qquad & \text{rank } MW^3 = 11 \\ \text{rank } MW^1 = 3 \qquad & \text{rank } MW^4 = 12 \\ \text{rank } MW^2 = 6 \qquad & \text{rank } MW^5 = 6. \end{aligned}$$

We choose a basis for MW^1 and compute products of these elements in order to determine the Lefschetz subalgebra generated by MW^1. We compute that the rank of the degree three graded piece of L_{MW^*} is 10, so there must be a generator of MW^* in degree three. In particular, we have an example where the Chow ring $A^*(X_{GZ})$ is not generated in degree one.

3.3.4 Main Theorem

We now state and prove our main theorem relating the Chow ring of the toric variety X_{GZ} constructed from the fan of the Gelfand-Zetlin polytope to the cohomology ring of the flag variety G/B. Recall from Proposition 3.3.6 that $A^*(G/B) \cong H^*(G/B)$ where the isomorphism doubles degree.

Theorem 3.3.9. *For X_{GZ} the toric variety associated to GZ fan $\Sigma \subset \mathbb{R}^N$ and the flag variety G/B for $G = SL_n(\mathbb{C})$, the Chow ring $A^*(G/B)$ can be identified with the Gorenstein quotient of the Lefschetz subalgebra of $A^*(X_{GZ})$.*

Proof. We first show that there is an isomorphism of groups $A^1(G/B) \cong A^1(X_{GZ})$. In Equation (3.7) we recalled that

$$A^1(G/B) \cong \text{Pic}(G/B) \cong \Lambda.$$

In Equation (3.9) we recalled that since Σ_{GZ} is a complete toric variety,

$$A^1(X_{GZ}) \cong \text{Pic}(X_{GZ}).$$

We next use the correspondence between Cartier divisors and piecewise linear functions to establish

$$\mathrm{Pic}(X_{GZ}) \cong PL(\Sigma_{GZ}). \tag{3.14}$$

Recall that for $[D] \in \mathrm{Pic}(X_{GZ})$, where $D = \sum a_\rho D_\rho$ is Cartier, the corresponding piecewise linear function f satisfies $f(v_\rho) = a_\rho$ where $\rho \in \Sigma(1)$, D_ρ is the corresponding T-invariant divisor, and v_ρ is the ray generator. In $PL(\Sigma_{GZ})$ the functions are defined up to shifting. Such a piecewise linear function corresponds to a virtual polytope normal to Σ_{GZ} with support numbers $\{a_\rho\}$.

$$PL(\Sigma_{GZ}) \cong \mathcal{P}^*(\Sigma_{GZ}) \tag{3.15}$$

Finally, we can identify $\mathcal{P}^*(\Sigma_{GZ})$ with Λ via the correspondence between Δ and λ established in Proposition 3.1.2. We identify $\Delta \in \mathcal{P}^*(\Sigma_{GZ})$ as a difference $P - Q$ of convex polytopes normal to Σ_{GZ}, apply Proposition 3.1.2 to each, then simplify the resulting difference of GZ polytopes to obtain $\Delta = c + \Delta_\lambda$ where Δ_λ may be virtual. This map $\mathcal{P}^*(\Sigma_{GZ}) \to \Lambda$ is clearly surjective. We briefly justify injectivity. Suppose $\Delta = c + \Delta_\lambda = c' + \Delta_{\lambda'}$ is a shift of two GZ polytopes, then λ is a shift of λ', and hence the two are identified in Λ. It is a homomorphism because of additivity of GZ polytopes, see Proposition 2.3.2. Combining these facts, we have established that

$$A^1(X_{GZ}) \cong \Lambda \cong A^1(G/B).$$

The next step is to show that self-intersection numbers on $A^1(X_{GZ}) \cong A^1(G/B)$ match. Since both groups are isomorphic to Picard groups, it makes sense to consider the degree of the line bundle associated to $\lambda \in \Lambda$ for each variety. It will be enough to show that degrees match for λ dominant. Let λ be a dominant weight, then $\mathcal{L}_\lambda$ and L_{Δ_λ} are the associated line bundles on G/B and X_{GZ} respectively. We recall that by Proposition 2.5.2 and Proposition 2.5.3 we have

$$\deg(G/B, \mathcal{L}_\lambda) = N!\mathrm{Vol}_N(\Delta_\lambda)$$

and

$$\deg(X_{GZ}, L_{\Delta_\lambda)} = N!\mathrm{Vol}_N(\Delta_\lambda).$$

Finally, we show that this isomorphism between $A^1(G/B)$ and $A^1(X_{GZ})$ extends using Lemma 3.3.3 to give our desired result. We apply this lemma with $A = A^*(G/B)$ and $B = L_{A^*(X_{GZ})}$ the Lefschetz subalgebra of $A^*(X_{GZ})$. Since A is the cohomology ring of the flag variety, we have $A^0 \cong A^N \cong \mathbb{Z}$, and we have $B^0 \cong \mathbb{Z}$. Note that in the Lefschetz subalgebra we will have $B^N \cong \mathbb{Z}$ as $\deg(X_{GZ}, L_{\Delta_\lambda}) \neq 0$. Both A and B are generated in degree one, and A has Poincarè duality. Finally, degrees of line bundles corresponding to $\lambda \in \Lambda$ match. Consequently, we obtain an isomorphism

$$A^*(G/B) \cong \mathrm{Gor}(L_{A^* X_{GZ}})$$

as desired. □

4.0 APPENDIX

```
from __future__ import print_function
from sage.matrix.constructor import Matrix
from sage.misc.functional import rank
from sage.misc.prandom import random
from sage.geometry.cone import Cone
from sage.combinat.integer_vector_weighted import
    WeightedIntegerVectors


class MW:
    def __init__(self,fan):
        """The graded ring of Minkowski weights on fan."""
        self._fan = fan
        self._cones = fan.cones
        self._dim = fan.dim()
        self._rk, self._A = self._set_ranks_matrices()
        self._A_RR = [M.echelon_form() for M in self._A]
        self._bases = [M.T.kernel().basis() for M in self._A]
        self._N = fan.lattice()
        self._subalgebra_generated = False

    def basis(self,k):
        """Returns a basis for MW^k."""
        return self._bases[k]

    def ranks(self):
        """Returns list of ranks of MW graded by codimension."""
        return self._rk

    def product(self,w1,cd1,w2,cd2):
        """Computes product of two weights."""
        cd = cd1+cd2
        d = self._dim - cd
        d1 = self._dim - cd1
        d2 = self._dim - cd2
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        assert self._check_weight(w1,cd1),"Weight w1 is not a valid
            weight of codimension cd1"
        assert self._check_weight(w2,cd2),"Weight w2 is not a valid
            weight of codimension cd2"
        if d < 0:
            return []
        else: #compute product and evaluate on all cones of dim d
            soln = []
            cones = self._fan(d)
            Sigmas = self._fan(d1)
            Taus = self._fan(d2)
            #determine generic v
            v = self._generic(self._fan)
            new_weight = []
            for c in cones:
                # only need m_{sig,tau} when both sig, tau contain c
                sig_c = self._cones_containing(c,Sigmas)
                tau_c = self._cones_containing(c,Taus)
                relevant = self._check_generic(sig_c,tau_c,v)
                c_sum = 0
                #for each pair, compute m_{sig,tau} = [N:N_sig + N_tau]
                for (sig,tau) in relevant:
                    N_sig = sig.sublattice()
                    N_tau = tau.sublattice()
                    N_sum = self._N.submodule(N_sig.basis()+N_tau.basis
                        ())
                    #N_sum.index_in(N) is m_{sig,tau}
                    c_sum += N_sum.index_in(self._N)*w1[Sigmas.index(
                        sig)]*w2[Taus.index(tau)]
                new_weight.append(c_sum)

            assert self._check_weight(new_weight,cd),"New weight fails"
            return new_weight

    def _balancing(self, tau):
        """Returns relations associated with cone tau."""
        relns = []
        d = tau.dim()
        l = len(self._cones(d+1)) # dimension of relations
        relevant = tau.facet_of()
        basis = tau.orthogonal_sublattice().basis()
        for b in basis: #each b gives relation
            v = [] # vector holding relations
            for c in self._cones(d+1):
                if c in tau.facet_of():
                    Q = c.relative_quotient(tau)
                    n = Q.gens()[0]
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                v.append(b*n)
            else:
                v.append(0)
        relns.append(v)
    return relns

def _set_rank(self,cd):
    """Returns the rank of degree `cd` as well as the matrix of
       relations."""
    d = self._dim - cd
    ConeList = self._cones(d)
    n = len(ConeList)
    #generate balancing conditions
    ConeRelns = self._cones(d-1)
    relations = []
    for c in ConeRelns:
        balance = self._balancing(c)
        for b in balance:
            relations.append(b)
    #relations may be redundant, determine rank
    A = Matrix(relations)
    r = rank(A)
    #basis of kernel(A.T) is basis for MW^cd
    return n-r, A

def _set_ranks_matrices(self):
    """Iterates through all codimensions and initializes lists of
       ranks and relation matrices"""
    rnks = []
    mtrx = []
    for cd in range(self._dim):
        r,M = self._set_rank(cd)
        rnks.append(r)
        mtrx.append(M)
    rnks.append(1)
    mtrx.append(Matrix(1))
    return rnks, mtrx

def _check_weight(self,w,cd):
    """Determines whether `w` is a balanced weight of codimension `
       cd`."""
    #first check that length of w is compatible with codimension cd
    d = self._dim-cd
    if len(w)!=len(self._fan(d)):
        return False
    #lengths compatible, multiply matrices
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        res = self._A[cd]*Matrix(w).T ##res should be zero vector of
            length = num relations
        return res.is_zero()

    def _generic(self,fan):
        """Returns a vector `v` which is generic with respect to the
            given fan."""
        d = fan.dim()
        #random candidate for genric vector
        v = [random() for r in range(d)]
        #check whether generic, i.e., v NOT in any cones of cd 1
        needToCheck = True
        while needToCheck:
            coneFlag = False #change to true if v in a cone of codim 1
            for c in fan.cones(d-1):
                if v in c:
                    coneFlag = True
            if coneFlag:
                #generate new random vector and try again
                v = [random() for r in range(d)]
            else:
                #vector v is generic
                needToCheck = False
        return v

    def _check_generic(self,cones1,cones2,v):
        """Returns pairs of cones which intersect with respect to `v`.
            """
        good_pairs = []
        for c1 in cones1:
            for c2 in cones2:
                # check if v is in c1-c2 (mink sum c1, -c2)
                C = Cone(rays = [r for r in c1.rays()]+[-1*r for r in
                    c2.rays()])
                if C.contains(v):
                    good_pairs.append((c1,c2))
        return good_pairs

    def _cones_containing(self,cone,conelist):
        """Returns sublist of conelist whose cones have cone as a face.
            """
        toreturn = []
        for c in conelist:
            if cone.is_face_of(c):
                toreturn.append(c)
        return toreturn
```

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