ASYMPTOTIC BEHAVIORS OF A FREE BOUNDARY ARISED FROM CORPORATE BOND EVALUATION WITH CREDIT RATING MIGRATION RISKS

by

Wanying Fu

B.S, Fudan University, 2013

M.S, University of Pittsburgh, 2018

Submitted to the Graduate Faculty of

the Dietrich School of Arts and Science in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

University of Pittsburgh

2019

UNIVERSITY OF PITTSBURGH

DIETRICH SCHOOL OF ARTS AND SCIENCES

This dissertation was presented

by

Wanying Fu

It was defended on

May 30th 2019

and approved by

Xinfu Chen, Mathematics Department, University of Pittsburgh.

John M.Chadam, Mathematics Department, University of Pittsburgh.

Song Yao, Mathematics Department, University of Pittsburgh.

Noel J.Walkington, Department of Mathematical Sciences, Carnegie Mellon University.

Dissertation Director: Xinfu Chen, Mathematics Department, University of Pittsburgh.

ASYMPTOTIC BEHAVIORS OF A FREE BOUNDARY ARISED FROM CORPORATE BOND EVALUATION WITH CREDIT RATING MIGRATION RISKS

Wanying Fu, PhD

University of Pittsburgh, 2019

In this thesis, we study asymptotic behaviors of a free boundary arised from the evaluation of a corporate bond subject to changes of the credit rating of the underlying company. The credit rating migration is modeled by a free boundary which separates different credit rating regions in a state space. We first formulate the mathematical problem and then we establish the well-posedness of the problem and the long time-to-expiry behavior of the solution. As a result, we describe the location and asymptotic line of the free boundary. Certain numerical simulations are also provided.

Keywords: credit rating, corporate bond, free boundary, asymptotic behavior.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

PREFACE

This dissertation is for the degree Doctor of Philosophy at the University of Pittsburgh. The research is deduced under the supervision of Professor Xinfu Chen in Mathematics Department in University of Pittsburgh during the time period from August 2015 to April 2019.

I am fortunate enough to have Professor Xinfu Chen to be my PhD advisor, who helps and encourages me to complete my research. He is fantastically erudite with a large stock of knowledge and ideas. He can always solve my doubts and guide me to work well on my research. He emphasizes the importance of details and the logical frame, which influence me to raise a careful and target-oriented research habit. During my PhD life, there are ups and downs and I really appreciate his support and encouragement for me.

I would also like to thank the other members in my defense committee: Professor Chadam John Martin, Professor Song Yao and Professor Noel Walkington. They are always willing to instruct, motivate, sponsor and support me. Even though there are some stressful periods for me, I know I am not alone on the research routine due to the kind help of all committee members.

Moreover, as a part of the teaching assignments and my interests, I step into the field of Actuarial Science. It is a most important experience for me, which broaden my knowledge fields. Thank Dr. Xiong Sheng and Professor Chadam for their help and encouragment for me in this field. Till now, I have passed 4 SOA exams already and I decide to pursue this route as my interest in the future.

Many thanks to Mathematics Department at University of Pittsburgh. Thanks for the support and opportunities of teaching in the department, which are unforgettable for me and motivate me to insist on academia after graduation. The experience also allows me to communicate with diverse students, staffs and faculties, which I really enjoy in.

Thanks to my fellow researchers. I have shared an office with Jing Liu, Jason Pina, Yujie Ding, Ho Chi Low. I enjoy the time we exchange ideas and thoughts. Thanks to my friends and all the people I met at University of Pittsburgh and Carnegie Mellon University during my time in Pittsburgh.

Many thanks to the support from Free Boundary Problems and Interfacial Dynamics 1516344.

At last, I feel grateful for my beloved family who always give me selfless encouragement and support.

1.0 INTRODUCTION

This thesis is concerned with a free boundary problem modeling the evaluation of a corporate bond, where the underlying company undergoes credit rating changes.

The background of the problem is measuring financial credit risks. With the globalization and complexification of financial markets, people pay more and more attention to credit risks. There are two main types of credit risks: default and credit rating migration. Although the credit rating migration is indeed of great importance in financial activities, up until now, most research attention is focused on measuring default risks. In this thesis, we add credit rating change risks into consideration.

The 2008 finance crisis caused global damaging of financial institutions, subsequent collapse of home and stock markets, and prolonged unemployment. Failure of key business and the followed downturn on economic activities played an important role on the European sovereign-debt criss and the Great Recession during 2008-2012. The credit rating migration serves as one crucial reason for these events. As a result, it is important to study credit rating migration.

There are plenty of academic researches about measuring default risks. Two main traditional models are the structural models and the intensity models. In a structural model, a default occurs when the value of a state variable (such as the firm's asset value) falls below a sovereign level. As a pioneer of applying a structural model, Merton [1] assumes that a default of corporate bond of an underlying company may happen on the expiration time. Two years later, Black and Cox [2] develop Merton's model to a first-passage time model by introducing safety covenants that give the bondholders the right to recognize a firm if the company asset value is less than certain level. This model considers that default may happen at anytime. In this direction, similar models have been developed; see Longstarff and Schwartz [3], Leland [4], Leland and Toft [5], Briys and de Varenne [6], Chance [7], and Shimko, Tejima and vanDeventer [8]. In an intensity model, the time of default is described, based on a probabilistic approach, by first arrived times of Possion Processes of variable intensities depending on state variables; see Jarrow and Turnbull [9], Lando [10], Litterman and Iben [11], Duffie and Singleton [12].

To study credit rating migration, one uses the switches of intensity matrices of Markov chains; see Lando [13], Thomas, Allen and Morkel-Kingsbury [14], Hurd and Kuznetsov [15], Jarrow, Lando and Turnbull [16], Das and Tufano [17]. Liang and Zeng [18], Liang, Zhao and Zhang [19], Liang, Wu and Hu [20], for the first time, adapt structure models to price bonds for firms undergoing credit rating migration. In their papers, they set a predetermined migration threshold and separate firm's asset value into high rating region and low rating region under the assumption that firm's value is stochastic. Hu, Liang and Wu [21] modify the model by using the ratio of debt to firm's value as the threshold and transfer the model to a free boundary problem in partial differential equation system. In their papers, they give the well-posedness of the problem and some properties of the free boundary, which separates different credit rating regions in state space. In 2017, the smoothness and boundedness of the free boundary are provided by Liang, Yin, Chen and Wu [22].

In this thesis, we adapt the model developed in Hu, Liang and Wu [21] and study in more detail the solution and the free boundary. The main contributions in our study are the following:

1. We give a proof of the well-posedness in a more general way. In particular, we simplify the uniqueness proof in Hu, Liang and Wu[21].
2. We find two paralleled lines that bounds the free boundary, (i.e. it lies in a stripe of two paralleled lines with explicit expressions).
3. We provide the asymptotic (long-expiration-time) behavior of the free boundary.
4. As expiry time goes to infinity, We show that the company's ratio of debt to asset value (the solution of the parabolic equation) presents a stack of two traveling waves. The first wave connects 0 and γ (i.e. the threshold for the company's ratio of debt to asset value which separates the high and low credit rating), and the second wave connects γ and 1. Both waves travel with different velocities.

5. As a demonstration of our asymptotic results, we provide some numerical simulations. For long time behavior, the scheme is so designed that it can cover an arbitrary long expiry-time period, with any fixed amount of computing time. We find the free boundary and the solution for any expiry-time. The numerical results agree with the theoretical one.

The rest of this thesis is organized as follows: In chapter 2, we set up the model. Chapter 3 shows the main results. In chapter 4, the main results are proven in detail, including the well-posedness establishment and qualitative analysis. Then in chapter 5, we perform numerical simulations. Chapter 6 is a conclusion.

2.0 MATHEMATICAL FORMULATION

In this chapter we first follow Hu, Liang and Wu [21] to derive a mathematical formulation for the price of a corporate bond under credit rating change. Then we perform a dimension reduction and get our main mathematical problem.

2.1 BASICS

A corporate bond is a debt security issued by a corporation and sold to investors. Comparing to government bonds, it provides a higher interest rate to compensate the higher risks. It can be a major source of capital for many businesses. The backing for the bond is usually the payment ability of the company, which is typically money to be earned from future operations. In this thesis, we consider the situation where the company's physical assets are used as collateral for bonds.

We study the price of a corporate bond subject to the change of the underlying company's credit rating, which rates the ability of the underlying company to pay back the borrowing. Here we consider a simple scenario where credit rating is (uniquely) determined by the ratio of the company's debt to its asset value.

We assume that the debt is the market price of the bond and that the asset value of the company undergoes a geometric Brownian motion with volatility depending on credit rating. The relations among the credit rating, debt (bond value), and asset value are depicted in Figure 1.

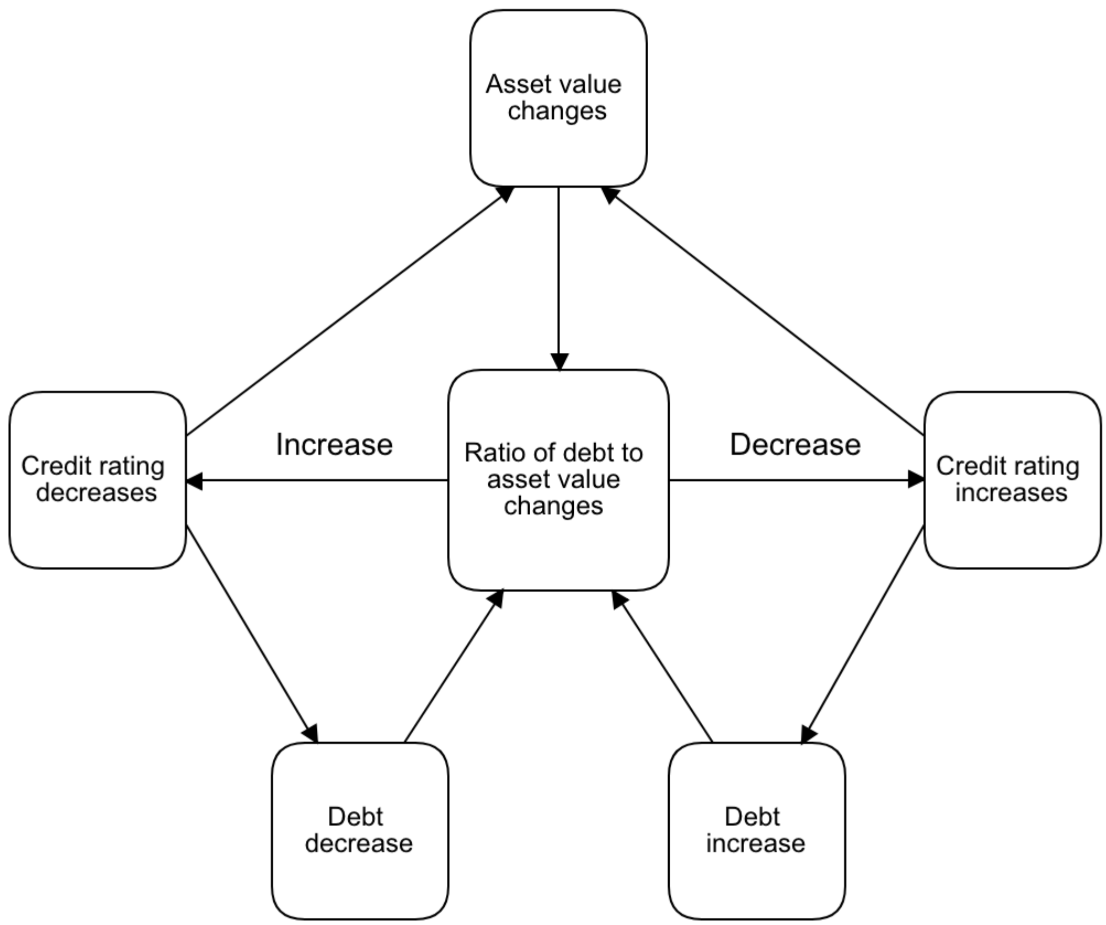

Figure 1: Corporation bond price subject to credit rating change.

First of all, the credit rating is assumed to be determined by the ratio of debt to asset value. Secondly, the debt is assumed to be the market price of the bond.

1. When credit rating increases, then bond price increases which means that debt increases. In the same time, the volatility of the asset value decreases so the asset value becomes more stable.
2. When credit rating decreases, then bond price decreases which means that debt decreases. In the meantime, the volatility of asset value increases, so the asset value changes more drastically.

The changes of the debt and asset value will in turn change their ratio and therefore, change the credit rating. In the study, we shall model the dynamics of these changes in a precise manner.

2.2 MATH FORMULATION

2.2.1 Basic Math Set-up

We consider a company who issued a non-dividend paying bond due at an expiration date at which the bond is paid back either at its face value or at the asset value of the company, whichever is less. The debt of the company is regarded as the market value of the bond, which is (partially) determined by the company's credit rating, which, in turn, is determined by the ratio of the debt to asset value. Following Hu, Liang and Wu [21], we construct a mathematical model as follows.

1. We assume that there is a risk-free bond whose risk free interest rate, $\{r_t\}_{t\geqslant 0}$, is described by a Vasicek process [23]:
$$dr_t = (k - \beta r_t)dt + \sigma dW_t^1,$$
where $\sigma \geqslant 0$, $k > 0$, and $\beta > 0$ are constants and $\{W_t^1\}_{0\leqslant t\leqslant T}$ is a standard Brownian Motion process.

2. Denote by X_t the company's assets value at time t. We assume that $\{X_t\}_{0\leqslant t\leqslant T}$ is a stochastic process under a risk-neutral environment described by:
$$dX_t = r_t X_t dt + \sigma_t X_t dW_t^2,$$
where $\{W_t^2\}_{0\leqslant t\leqslant T}$ is a standard Brownian Motion process under a risk-neutral environment. We assue that r_t and X_t are correlated by assuming the correlation coefficient:
$$\rho = \mathrm{cov}(\mathrm{dW_t^1}, \mathrm{dW_t^2})/\mathrm{dt},$$
where $\rho \in [-1, 1]$ is a constant.

3. Assume that the company debt is the owing of a corporate bond of face value K due at time $T > 0$. The current time is assumed to be zero. Denote the market value of the

bond at time $t \in [0,T]$ by U_t. Based on our assumption, at due date we have that

$$U_T = \min\{X_T, K\}.$$

Note that X_T is the company's assets value at time T.

4. We assume that the credit rating depends only on U_t/X_t, the ratio of the company's debt value to its assets value. We further assume that the volatility of the company's asset value is uniquely determined by the credit rating. For simplicity, we use two rating system. Multiple rating system can be similarly analyzed. Hence, we assume that the volatility process is prescribed by

$$\sigma_t = \Sigma\Big(\frac{U_t}{X_t}\Big),$$

$$\Sigma(s) = \begin{cases} \sigma_H & \text{if } s < \gamma, \\ \sigma_L & \text{if } s \geqslant \gamma; \end{cases}$$

here γ, σ_H and σ_L are positive constants. Because good credit rating corresponds to low volatility, we assume

$$0 < \sigma_H < \sigma_L.$$

Also, we assume that $\gamma \in (0,1)$, since the case $\gamma \geqslant 1$, together with the fact that $U_t \leqslant X_t$, gives the trivial scenario that $\sigma_t = \sigma_H$ for all $t \in [0,T]$.

5. Under the above assumptions and certain routine assumptions needed for Black-Scholes theory, we can derive that U_t, the market value of bond, is a function of X_t, r_t and t, i.e. there exists a function u such that

$$U_t = u(r_t, X_t, t), \quad \forall t \in [0,T],$$

and that $u(\cdot,\cdot,\cdot)$ is the solution of the following Black-Scholes equation with the terminal condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{N}[u] = 0 & \forall r \in \mathbb{R}, x > 0, t \in [0,T); \\ u(r,x,t)|_{t=T} = \min\{K, x\} & \forall r \in \mathbb{R}, x \geqslant 0, \end{cases} \tag{2.1}$$

where $\mathcal{N}$ is a **non-linear** operator given by

$$\mathcal{N}[u] = \frac{1}{2}\Sigma^2\Big(\frac{u}{x}\Big)x^2\frac{\partial^2 u}{\partial x^2} + \rho\sigma\Sigma\Big(\frac{u}{x}\Big)x\frac{\partial^2 u}{\partial x\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial r^2} + xr\frac{\partial u}{\partial x} + (k-\beta r)\frac{\partial u}{\partial r} - ru.$$

In the rest of this thesis, we shall study the non-linear problem (2.1).

2.2.2 Dimension Reduction

Without loss of generality, we assume $K = 1$ in this thesis. In solving problem (2.1), we are dealing with an unknown function $u(x,r,t)$ of three variables. Since the terminal value of u does not depend on r and since we are using the Vasicek model, we shall show below that we can perform a dimension reduction; that is, u is indeed a function of two variables.

For this purpose, we first introduce the price of risk-free bond, $\{P_t\}_{t\geqslant 0}$. By theory of the Vasicek Interest Rate model, we know that

$$P_t = p(r_t, T-t),$$

where $p(r,\tau)$ is the unique solution of

$$\begin{cases} \dfrac{\partial p}{\partial \tau} = \dfrac{\sigma^2}{2}\dfrac{\partial^2 p}{\partial r^2} + (k-\beta r)\dfrac{\partial p}{\partial r} - rp, \quad \forall r\in(0,\infty), \tau>0; \\ p(r,0) = 1; \end{cases}$$

The unique solution[23] is given by:

$$p(r,\tau) = e^{-A(\tau)r - B(\tau)},$$

where $A(\tau)$ and $B(\tau)$ are given by

$$A(\tau) = \frac{1}{\beta}(1-e^{-\beta\tau}),$$

$$B(\tau) = \frac{\sigma_1^2}{4\beta^3}\Big(e^{-2\beta\tau}-1\Big) - \Big(\frac{\sigma_1^2}{\beta^3} - \frac{k}{\beta^2}\Big)\Big(e^{-\beta\tau}-1\Big) + \Big(-\frac{\sigma_1^2}{2\beta^2} + \frac{k}{\beta}\Big)\tau.$$

The detailed calculation is given in Appendix (chapter 7).

Being an affine term structure model, the solution p has the property

$$\frac{\partial \ln p}{\partial r}(r,\tau) = -A(\tau),$$

which does not depend on r. This important property allows us to perform a dimension reduction.

Using the price of risk-free bond as numerie, we introduce

$$\tau = T - t;$$
$$y = \frac{x}{p(r,\tau)};$$
$$v(r,y,\tau) = \frac{u(r,x,t)}{p(r,\tau)} = \frac{u\left(r, yp(r,\tau), T-\tau\right)}{p(r,\tau)}.$$

Simple computation shows that v satisfies the system

$$\begin{cases} \dfrac{\partial v}{\partial \tau} = a\Big(\tau, \dfrac{v}{y}\Big) y^2 \dfrac{\partial^2 v}{\partial y^2} & \forall y > 0, \tau \in [0,T]; \\ v(0,r,y) = \min\{1,y\} & \forall y > 0, \tau = 0. \end{cases} \tag{2.2}$$

Here $a(\tau, s)$ is a function defined by

$$a(\tau,s) = a_H(\tau) + [a_L(\tau) - a_H(\tau)]H(s-\gamma), \tag{2.3}$$

where

$$H(x) = \begin{cases} 1 & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0; \end{cases} \tag{2.4}$$

$$a_i(\tau) = \frac{\sigma^2}{2}\left(\frac{1-e^{-\beta\tau}}{\beta}\right)^2 + \rho\sigma\sigma_i\left(\frac{1-e^{-\beta\tau}}{\beta}\right) + \frac{1}{2}\sigma_i^2, \quad \text{for} \quad i = H, L. \tag{2.5}$$

The solution of (2.2) does not depend on r. Thus, we further introduce

$$z = \ln y;$$
$$w(\tau,z) = \frac{v(\tau,r,y)}{y} = \frac{v(\tau,r,e^z)}{e^z}.$$

An elementary calculation leads to the main PDE problem stated in next subsection we studied in this thesis.

2.2.3 The Main PDE Problem

From the dimension reduction, we conclude that w is the solution of the following non-linear PDE problem

$$\begin{cases} w_\tau = a(\tau, w)(w_z + w_{zz}) & \forall z \in \mathbb{R}, \tau \in (0, \infty), \\ w(0, z) = \min\{1, e^{-z}\} & \forall z \in \mathbb{R}, \tau = 0; \end{cases} \tag{2.6}$$

here w_τ, w_z and w_{zz} stand for the partial derivatives of w with respect to τ and z.

Hence, our mathematical problem is to study (2.6) where $a(\cdot, \cdot)$ is given by (2.3)-(2.5) with $0 < \gamma < 1$.

Note that when $\sigma = 0$, $a(\cdot, \cdot)$ does not depend on τ. Here we write:

$$a(\tau, s) = a^*(s) := \frac{\sigma_H^2}{2} + \Big(\frac{\sigma_L^2}{2} - \frac{\sigma_H^2}{2}\Big)H(s - \gamma). \tag{2.7}$$

3.0 MAIN RESULTS

In this chapter, we shall first define a strong solution for our math problem 2.6. Then we list two theorems as our main results. One is for well-posedness of the system, i.e. the existence and uniqueness of the strong solution in Sobolev sense. The other theorem expresses the asymptotic behaviors of the solution and the free boundary. Especially, the free boundary, which separates different credit rating regions for the underlying company in a state space, is located in a stripe bounded by two straight lines sharing the same slope. The detailed proofs are presented in the next chapter.

3.1 DEFINITION OF STRONG SOLUTION

Since $a(\tau, s)$ in (2.3) is not continuous, problem (2.6) does not possessed a classical solution. For this reason, we introduce a **strong solution** as follows:

Definition 1. *Let $a(\tau, s)$ defined as in (2.3)–(2.5), where $\sigma \geqslant 0, \sigma_H > 0, \sigma_L > 0, \beta > 0, \gamma \in (0,1)$, and $\rho \in [-1,1]$ are constants. A* ***strong solution*** *of (2.6) is a function $w \in C([0,\infty) \times \mathbb{R})$ that satisfies $w_\tau, w_{zz} \in L^2_{loc}([0,\infty) \times \mathbb{R})$ and*

$$\begin{cases} w_\tau = a(\tau, w)(w_z + w_{zz}) \quad a.e. \;\; in \; (0,\infty) \times \mathbb{R}, \\ w(0,z) = w_0(z). \end{cases}$$

The definition is in Sobolev sense, which helps us analyze the well-posedness of the system.

3.2 WELL-POSEDNESS

We state the well-posedness of the solution as in the following theorem. The detailed existence establishment and uniqueness establishment are given in chapter 4.

Theorem 1. *Assume that $\sigma \geqslant 0$, $\sigma_H > 0$, $\sigma_L > 0$, $\beta > 0$, $\rho \in [-1, 1]$, and $\gamma \in (0, 1)$ are constants and $a(\cdot, \cdot)$ is given by (2.3)–(2.5). Then system (2.6) admits a unique strong solution.*

3.3 FREE BOUNDARY'S LOCATION AND SOLUTION'S ASYMPTOTIC BEHAVIORS

In the τ-z space, the right-half plane is divided into two open regions: one is a high credit rating region in which $w(\tau, z) < \gamma$ and the other is a low credit rating region in which $w(\tau, z) > \gamma$.

The two regions are separated by a smooth and non-decreasing curve $z = s(\tau)$ where $w(\tau, z) = \gamma$ on it in the $\tau - z$ plane, i.e. the state space. Technically, this curve is called the free boundary. We denote the free boundary as

$$\Gamma := \{(\tau, z) \mid \tau \geqslant 0, w(\tau, z) = \gamma\} = \{(\tau, z) \mid \tau \geqslant 0, z = s(\tau)\}.$$

Clearly, the problem of finding credit rating migration is equivalent to find the free boundary.

We would like to describe the location of the free boundary. Note that $a(\tau, w)$ does not depend on τ when $\sigma = 0$. (i.e. The function a is given by (2.7).) For simplicity, we consider the case $\sigma = 0$.

We can provide the following information about the location of the free boundary, as well as the asymptotic behaviors of the free boundary and the solution of system 2.6.

Theorem 2. *Assume the conditions of Theorem 1. Also assume that $\sigma = 0$. Let w be the* *strong solution of* (2.6) *and* $\bar{A}$ *and* $\underline{A}$ *be constants defined by*

$$\bar{A} = max\left\{\ln\frac{2}{\gamma}, \ln\frac{1}{\gamma} + \frac{4\sigma_L}{\sigma_L - \sigma_H}\ln\frac{4\sigma_H(1-\gamma)}{\gamma\sigma_L}\right\},$$

$$\underline{A} = min\left\{\ln\frac{1}{\gamma}, \ln\frac{1}{\gamma} - \frac{4\sigma_H}{\sigma_L - \sigma_H}\ln\frac{2\gamma\sigma_L}{\sigma_H(1-\gamma)}\right\}.$$

Then there exists a function $s(\cdot)$ *such that the free boundary* $\Gamma := \{(\tau, z) \mid \tau \geqslant 0, w(\tau, z) = \gamma)\}$ *is given by the curve* $z = s(\tau)$*; more precisely,* $\Gamma = \{(\tau, s(\tau)) \mid \tau \geqslant 0\}$*. In addition,*

$$\underline{A} - \frac{\sigma_H\sigma_L}{2}\tau \leqslant s(\tau) \leqslant \bar{A} - \frac{\sigma_H\sigma_L}{2}\tau, \quad \forall\tau \geqslant 0.$$

Furthermore, w *has the following asymptotic behavior:* $\forall\eta \in \mathbb{R}$*,*

$$\lim_{\tau\to\infty} w\Big(\tau, -\frac{\sigma_H^2}{2}\tau - \eta\sigma_H\sqrt{\tau}\Big) = \gamma N(\eta);$$

$$\lim_{\tau\to\infty} w\Big(\tau, -\frac{\sigma_L^2}{2}\tau - \eta\sigma_L\sqrt{\tau}\Big) = \gamma + (1-\gamma)N(\eta).$$

Here $N(\cdot)$ *is the cumulative density function of the standard normal distribution, i.e.*

$$N(x) = \int_{-\infty}^{x} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}d\theta, \quad \forall x \in \mathbb{R}.$$

Note that for $w(\tau, \cdot)$ resembles a stack of two travelling waves for any fixed $\tau \gg 0$ as following:

1. When $z \geqslant -\frac{\sigma_H\sigma_L}{2}\tau$,
$$w(\tau, z) \approx \gamma N\Big(-\frac{z + \frac{1}{2}\sigma_H^2\tau}{\sigma_H\sqrt{\tau}}\Big).$$
One traveling wave (from $w = 0$ to $w = \gamma$) appears, which nearly centers at $z = -\frac{1}{2}\sigma_H^2\tau$.
2. When $z \leqslant -\frac{\sigma_H\sigma_L}{2}\tau$,
$$w(\tau, z) \approx \gamma + (1-\gamma)N\Big(-\frac{z + \frac{1}{2}\sigma_L^2\tau}{\sigma_L\sqrt{\tau}}\Big).$$
The other traveling wave (from $w = \gamma$ to $w = 1$) appears, which nearly centers at $z = -\frac{1}{2}\sigma_L^2\tau$.

The two travelling waves travel with velocities $O(\sigma_H\sqrt{\tau})$ and $O(\sigma_L\sqrt{\tau})$ respectively.

In short, when time-to-expiry is big, the value of w, i.e. the debt-to-asset ratio, tends toward resemble two travelling waves. One travels from 0 to γ, centering along the straight line $z = -\frac{1}{2}\sigma_H^2\tau$. The other one travels from γ to 1, centering along the straight line $z = -\frac{1}{2}\sigma_L^2\tau$.

In Figure 2, for any fixed $\tau \gg 0$, the curve of $w = w(\tau, \cdot)$ shows the asymptotic behaviors described above.

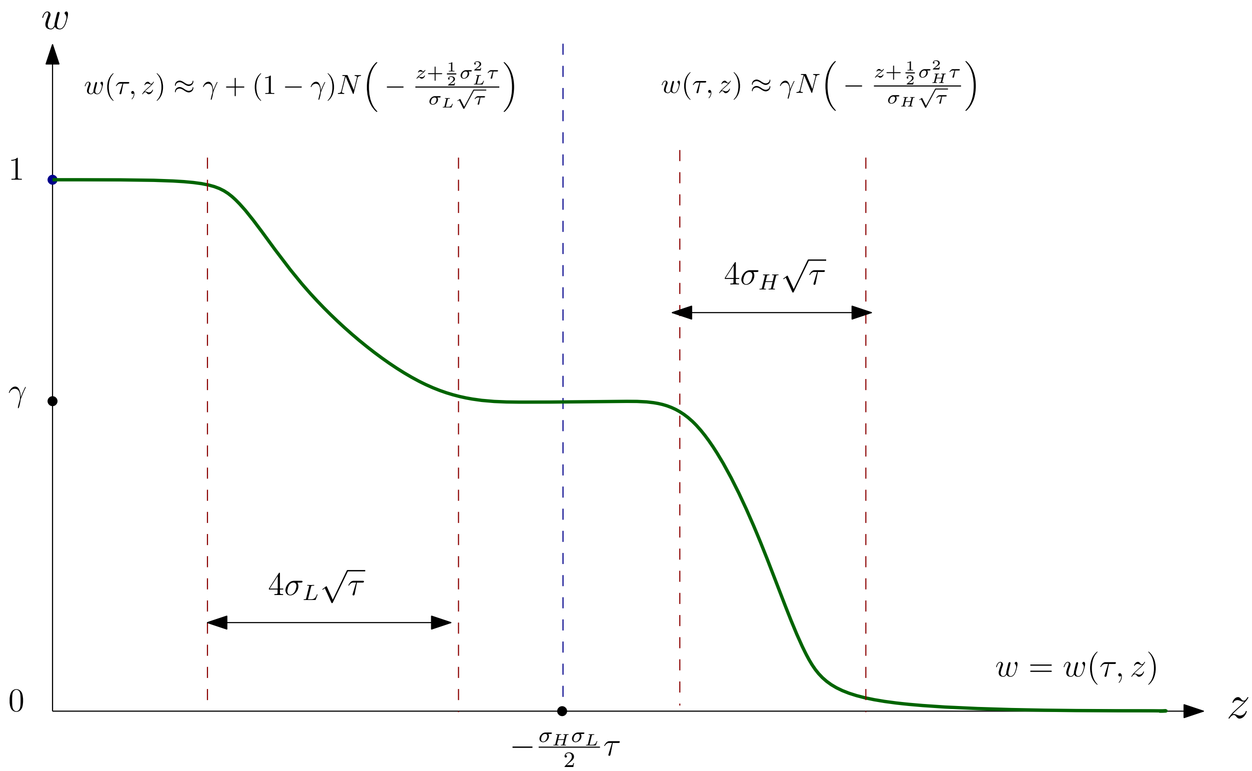

Figure 2: For fixed big τ, the stack of two travelling waves of $w(\tau, \cdot)$.

4.0 PROOFS OF MAIN RESULTS

In chapter 4, we give detailed deductions and proofs of main results. Section 4.1 establishes the well-posedness of system 2.6, including the existence and uniqueness of the defined strong solution. In section 4.2, we do qualitative analysis, which allows us to locate the free boundary and know some asymptotic behaviors of the solution.

4.1 WELL-POSEDNESS ESTABLISHMENT

Section 4.1 contains two subsections. In subsection 4.1.1, we establishes the existence of the strong solution of problem 2.6. In subsection 4.1.2, we prove that the strong solution is unique.

4.1.1 Existence Establishment

In this subsection, we shall establish the existence of a strong solution of (2.6).

First, we regularize the system. For this, let function $\Phi \in C^\infty(\mathbb{R})$ be a standard mollifier, which satisfies $\Phi = 0$ *on* $(-\infty, -1] \cap [1, \infty)$, $\Phi \geqslant 0$ on $\mathbb{R}$ and $\int_{-1}^{1} \Phi(z)dz = 1$. For any positive number ϵ, set $\Phi_\epsilon(z) = \frac{1}{\epsilon}\Phi(\frac{z}{\epsilon})$. Using convolution, we define the mollification of the non-smooth functions H, w_0 and a by

$$H^\epsilon(x) = H * \Phi_\epsilon(x),$$

$$w_0^\epsilon(z) = w_0 * \Phi_\epsilon(z),$$

$$a^\epsilon(\tau, s) = a_H(\tau) + [a_L(\tau) - a_H(\tau)]H^\epsilon(s - \gamma).$$

Based on a standard PDE theory[24], there exists a unique smooth solution w^ϵ to the following system:

$$\begin{cases} w^\epsilon_\tau = a^\epsilon(\tau, w^\epsilon)(w^\epsilon_{zz} + w^\epsilon_z) & \text{in } [0,\infty)\times\mathbb{R}, \\ w^\epsilon(0,z) = w^\epsilon_0(z) & \text{on } \mathbb{R}, \end{cases} \tag{4.1}$$

where $\lim_{|z|\to\infty} w^\epsilon_z = 0$.

Now we establish a priory estimate of w^ϵ and w^ϵ_z.

Lemma 1. *w^ϵ and w^ϵ_z satisfy:*

$$0 \le w^\epsilon \le 1, \quad -1 \le w^\epsilon_z \le 0.$$

Proof. The assertion $w^\epsilon \in [0,1]$ follows by the Maximum Principle. For w^ϵ_z, note that w^ϵ_z satisfies

$$w^\epsilon_z \in L^\infty([0,\infty)\times\mathbb{R}),$$

and

$$\begin{cases} w^\epsilon_{z\tau} = a^\epsilon\Big(w^\epsilon_{zz} + w^\epsilon_{zzz}\Big) + a^\epsilon_s w^\epsilon_z w^\epsilon_{zz} + a^\epsilon_s (w^\epsilon_z)^2, \\ -1 \leqslant w^\epsilon_z(0,z) \leqslant 0. \end{cases}$$

So by the Maximum Principle, we have $w_z \le 0$.

Since $a^\epsilon_s \geqslant 0$, dropping the non-negative term $a^\epsilon_s(w^\epsilon_z)^2$ and using comparison, we find that $w^\epsilon_z \geqslant -1$. This completes the proof of Lemma **1**. □

Lemma 2. *For all fixed $t \geqslant 0$,*

$$\int_{\mathbb{R}} [w^\epsilon_z(t,z)]^2 dz + \int_0^t \int_{\mathbb{R}} \left\{ \frac{[w^\epsilon_\tau(\tau,z)]^2}{a(\tau,w^\epsilon)} + a(\tau,w^\epsilon)[w^\epsilon_{zz}(\tau,z)]^2 \right\} dz d\tau \leqslant e^{\|a\|_\infty t} \int_{\mathbb{R}} [w^\epsilon_{0z}(z)]^2 dz.$$

Proof.

$$\frac{w^\epsilon_\tau}{\sqrt{a}} - \sqrt{a} w^\epsilon_{zz} = \sqrt{a} w^\epsilon_z,$$

square both sides and integrate on $\mathbb{R}$ with respect to z, we get

$$\begin{aligned} \int_{\mathbb{R}} (w^\epsilon_z)^2 a dz &= \int_{\mathbb{R}} \left[\frac{(w^\epsilon_\tau)^2}{a} + (w^\epsilon_{zz})^2 a - 2 w^\epsilon_{zz} w^\epsilon_\tau \right] dz \\ &= \int_{\mathbb{R}} \left[\frac{(w^\epsilon_\tau)^2}{a} + (w^\epsilon_{zz})^2 a \right] dz + \int_{\mathbb{R}} 2 w^\epsilon_{z\tau} w^\epsilon_z dz \\ &= \int_{\mathbb{R}} \left[\frac{(w^\epsilon_\tau)^2}{a} + (w^\epsilon_{zz})^2 a \right] dz + \frac{d}{d\tau} \int_{\mathbb{R}} (w^\epsilon_z)^2 dz. \end{aligned} \tag{4.2}$$

Dropping the term $\int_{\mathbb{R}}\left[\frac{(w_\tau^\epsilon)^2}{a}+(w_{zz}^\epsilon)^2a\right]dz$, we have

$$\frac{d}{d\tau}\int_{\mathbb{R}}(w_z^\epsilon)^2dz \leqslant \int_{\mathbb{R}}(w_z^\epsilon)^2adz \leqslant \|a\|_\infty\int_{\mathbb{R}}(w_z^\epsilon)^2dz.$$

So for any $t\geqslant 0$, w_z^ϵ satisfies

$$\int_{\mathbb{R}}[w_z^\epsilon(t,z)]^2dz \leqslant e^{\|a\|_\infty t}\int_{\mathbb{R}}[w_z^\epsilon(0,z)]^2dz.$$

Integrating (4.2) over $[0,t]$ with respect to τ gives

$$\begin{aligned}\int_{\mathbb{R}}[w_z^\epsilon(t,z)]^2dz+\int_0^t\int_{\mathbb{R}}\left[\frac{(w_\tau^\epsilon)^2}{a}+(w_{zz}^\epsilon)^2a\right]dzd\tau &= \int_{\mathbb{R}}[w_z^\epsilon(0,z)]^2dz+\int_0^t\int_{\mathbb{R}}(w_z^\epsilon)^2adz\\ &\leqslant \int_{\mathbb{R}}[w_z^\epsilon(0,z)]^2dz+\int_0^t\int_{\mathbb{R}}(w_z^\epsilon)^2\|a\|_\infty dz\\ &\leqslant \int_{\mathbb{R}}[w_z^\epsilon(0,z)]^2dz\left[1+\|a\|_\infty\int_0^t e^{\|a\|_\infty s}ds\right]\\ &= e^{\|a\|_\infty t}\int_{\mathbb{R}}[w_z^\epsilon(0,z)]^2dz\\ &= e^{\|a\|_\infty t}\int_{\mathbb{R}}[w_{0z}^\epsilon(z)]^2dz.\end{aligned}$$

This complete the proof of Lemma **2**. □

Theorem 3. *Assume the conditions of Theorem* **1**, *system* (2.6) *admits at least a solution that satisfies*

$$0\le w\le 1,\quad -1\le w_z\le 0\quad \text{on } (0,\infty)\times\mathbb{R}.$$

Proof. By Lemma **1** and Lemma **2**, for arbitrary $T>0$, $\{w_z^\epsilon,w_\tau^\epsilon,w_{zz}^\epsilon\}_{0<\epsilon<1}$ is a bounded family in $L^2([0,T]\times\mathbb{R})$. Hence, by integration $\{w^\epsilon\}_{0<\epsilon<1}$ is a bounded family in $L^\infty([0,T]\times\mathbb{R})$ and $w^\epsilon\in C^{\alpha,\frac{\alpha}{2}}([0,T]\times\mathbb{R})$ for any fixed $\alpha\in(0,\frac{1}{2})$.

Thus, there exists a function $w\in C^{\alpha,\frac{\alpha}{2}}([0,T]\times\mathbb{R})$ with $w_z,w_{zz},w_\tau\in L^2([0,T]\times\mathbb{R})$ such that, along a sequence of $\epsilon\to 0$,

$$\begin{aligned} w^\epsilon &\longrightarrow w && \text{in } C^{\alpha,\frac{\alpha}{2}}([0,T]\times\mathbb{R}),\\ w_z^\epsilon &\longrightarrow w_z && \text{in } L^2([0,T]\times\mathbb{R}),\\ w_{zz}^\epsilon &\longrightarrow w_{zz} && \text{in } L^2([0,T]\times\mathbb{R}) \text{ weakly},\\ w_\tau^\epsilon &\longrightarrow w_\tau && \text{in } L^2([0,T]\times\mathbb{R}) \text{ weakly}.\end{aligned}$$

Define the functions e^ϵ and k^ϵ by:

$$e^\epsilon(\tau,s)=\int_\gamma^s \frac{1}{a^\epsilon(\tau,w)}dz, \quad and \quad k^\epsilon(\tau,s)=-\int_\gamma^s \frac{a^\epsilon_\tau(\tau,w)}{a^\epsilon(\tau,w)^2}dz, \quad \tau\geqslant 0, s\in\mathbb{R}.$$

Since $e^\epsilon, k^\epsilon\in L_\infty((0,\infty)\times\mathbb{R})$, $e^\epsilon_z, k^\epsilon_z\in L^\infty(0,T;L^2)$, and $e^\epsilon_\tau, k^\epsilon_\tau\in L^2([0,T]\times\mathbb{R})$, it is easy to see that

$$\lim_{\epsilon\to 0} e^\epsilon(\tau,s)=e(\tau,s):=\begin{cases}\dfrac{(s-\gamma)}{a(\tau,s)} & \text{if } s\neq\gamma,\\ 0 & \text{if } s=\gamma,\end{cases} \tag{4.3}$$

$$\lim_{\epsilon\to 0} k^\epsilon(\tau,s)=k(\tau,s):=\begin{cases}-(s-\gamma)\dfrac{a_\tau(\tau,s)}{a(\tau,s)^2} & \text{if } s\neq\gamma,\\ 0, & \text{if } s=\gamma.\end{cases} \tag{4.4}$$

Here $k(\tau,w)$ and $e(\tau,w)$ are in $C^{\alpha,\frac{\alpha}{2}}([0,T]\times\mathbb{R})$.

Then the first equation in system (4.1) can be written to be

$$\frac{d}{d\tau}e^\epsilon\Big(w^\epsilon(\tau,z),\tau\Big)=\frac{w^\epsilon_\tau}{a^\epsilon(\tau,w^\epsilon)}+k^\epsilon(\tau,w^\epsilon).$$

Hence, we have

$$\frac{d}{d\tau}e^\epsilon\Big(w^\epsilon(\tau,z),\tau\Big)-k^\epsilon(\tau,w^\epsilon)=w^\epsilon_{zz}+w^\epsilon_z.$$

Sending $\epsilon\to 0$, along the sequence mentioned above, we have

$$\frac{d}{d\tau}e\Big(w(\tau,z),\tau\Big)-k(\tau,w)=w_{zz}+w_z \quad a.e.$$

Because $e(\cdot,\cdot)$ is Lipschitz continuous, we then obtain

$$\frac{d}{d\tau}e\Big(w(\tau,z),\tau\Big)-k(\tau,w)=\frac{w_\tau}{a(\tau,w)} \quad a.e.$$

Thus,we conclude

$$w_\tau=a(\tau,w)(w_z+w_{zz}) \quad a.e.$$

Thus, w is a strong solution of system (2.6). This complete the proof of Theorem **3** $\square$

4.1.2 Uniqueness Establishment

In this subsection, we show the uniqueness of system (2.6). More precisely, we prove the following theorem **4**. Clearly, our main result Theorem **1** follows form Theorem **3** and Theorem **4**.

Theorem 4. *Assume that $\sigma \geqslant 0$, $\sigma_H > 0$, $\sigma_L > 0$, $\beta > 0$, $\rho \in [-1, 1]$, and $\gamma \in (0, 1)$ are constants and $a(\cdot, \cdot)$ is given by (2.3)–(2.5). Then system (2.6) admits at most one strong solution.*

Proof. Let $w_1(\tau, z)$ *and* $w_2(\tau, z) \quad \forall(\tau, z) \in [0, T] \times \mathbb{R}$ be two strong solutions of (2.6). Define $w = w_1 - w_2$.

We want $w \equiv 0$.

Let e and k be defined as in (4.3) and (4.4) with the initial condition:

$$e(\tau, w_*)|_{\tau=0} = e\Big(0, w_0(z)\Big) = e_0(z) \quad \text{on } \{0\} \times \mathbb{R}.$$

Since w_1 *and* w_2 are strong solutions of (2.6), we see that

$$\frac{\partial}{\partial \tau} e(\tau, w_i) = (w_i)_{zz} - (w_i)_z + k(\tau, w_i) \quad (i = 1, 2) \quad a.e.$$

Taking the difference of above equations with $i = 1$ and $i = 2$, we have

$$\frac{\partial}{\partial \tau} [e(\tau, w_1) - e(\tau, w_2)] = w_{zz} - w_z + k(\tau, w_1) - k(\tau, w_2).$$

Fix an arbitrary $t > 0$ and $\tau \in [0, t)$. Multiplying both sides of above equation by $\int_\tau^t w(s, z) ds$, and integrating over $(\tau, z) \in \mathbb{R} \times [0, t]$, we get

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \left\{ \frac{\partial}{\partial \tau} [e(\tau, w_1) - e(\tau, w_2)] \int_\tau^t w(s, z) ds \right\} dz d\tau \\
&= \int_0^t \int_{\mathbb{R}} \left[(w_{zz} + w_z) \int_\tau^t w(s, z) ds \right] dz d\tau + \\
&\quad \int_0^t \int_{\mathbb{R}} \left\{ [k(\tau, w_1) - k(\tau, w_2)] \int_\tau^t w(s, z) ds \right\} dz d\tau.
\end{aligned} \tag{4.5}$$

The left-hand side of (4.5) can be written as, by integration by parts,

$$
\begin{aligned}
&\int_0^t \frac{\partial}{\partial \tau} \int_{\mathbb{R}} \left\{ [e(\tau, w_1) - e(\tau, w_2)] \int_\tau^t w(s,z)ds \right\} dzd\tau \\
&+ \int_0^t \int_{\mathbb{R}} [e(\tau, w_1) - e(\tau, w_2)]\, w(\tau, z) dzd\tau \\
&= \int_0^t \int_{\mathbb{R}} [e(\tau, w_1) - e(\tau, w_2)]\, w(\tau, z) dzd\tau.
\end{aligned}
$$

The first term of the right-hand side of (4.5) is

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \left[w_{zz} \int_\tau^t w(s,z)ds \right] dzd\tau + \int_0^t \int_{\mathbb{R}} \left[w_z \int_\tau^t w(s,z)ds \right] dzd\tau \\
&= -\int_0^t \int_{\mathbb{R}} \left[w_z(\tau, z) \int_\tau^t w_z(s,z)ds \right] dzd\tau + \int_0^t \int_{\mathbb{R}} \left[w_z \int_\tau^t w(s,z)ds \right] dzd\tau \\
&= \int_0^t \int_{\mathbb{R}} \frac{1}{2} \frac{\partial}{\partial \tau} \left[\int_\tau^t w_z(s,z)ds \right]^2 dzd\tau + \int_0^t \int_{\mathbb{R}} \left[w_z(\tau, z) \int_\tau^t w(s,z)ds \right] dzd\tau \\
&= -\frac{1}{2} \int_{\mathbb{R}} \left[\int_0^t w_z(s,z)ds \right]^2 dz + \int_0^t \int_{\mathbb{R}} \left[w_z(\tau, z) \int_\tau^t w(s,z)ds \right] dzd\tau.
\end{aligned}
$$

Applying Mean Value Theorem, the second term of the right-hand side of (4.5) is equal to

$$
\int_0^t \int_{\mathbb{R}} \left[k_s(\tau, \theta) w(\tau, z) \int_\tau^t w(s,z)ds \right] dzd\tau,
$$

where $\theta(\tau, z)$ is between $w_1(\tau, z)$ and $w_2(\tau, z)$ for every $(\tau, z) \in [0, t) \times \mathbb{R}$.

Thus, (4.5) can be written as

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} [e(\tau, w_1) - e(\tau, w_2)]\, w(\tau, z) dzd\tau + \frac{1}{2} \int_{\mathbb{R}} \left[\int_0^t w_z(s,z)ds \right]^2 dz \\
&= \int_0^t \int_{\mathbb{R}} \left[w_z(\tau, z) \int_\tau^t w(s,z)ds \right] dzd\tau + \int_0^t \int_{\mathbb{R}} \left[k_s(\tau, \theta) w(\tau, z) \int_\tau^t w(s,z)ds \right] dzd\tau.
\end{aligned}
\tag{4.6}
$$

From the uniformly boundedness and positiveness of $a(\tau, w)$, we can assert

$$
e(\tau, w_1) - e(\tau, w_2) = \int_{w_2}^{w_1} \frac{2}{\sigma^2(\tau, w)} dw \geqslant \int_{w_2}^{w_1} \frac{1}{sup\{a(\tau, w)\}} dw \geqslant c_0(w_1 - w_2) = c_0 w,
$$

for some positive number c_0.

Equality (4.6) becomes an inequality:

$$
\begin{aligned}
& c_0 \int_0^t \int_{\mathbb{R}} w^2(\tau,z) dz d\tau + \frac{1}{2} \int_{\mathbb{R}} \left[\int_0^t w_z(s,z) ds \right]^2 dz \\
& \leqslant \int_0^t \int_{\mathbb{R}} \left[w_z(\tau,z) \int_\tau^t w(s,z) ds \right] dz d\tau + \int_0^t \int_{\mathbb{R}} \left[k_s(\tau,\theta) w(\tau,z) \int_\tau^t w(s,z) ds \right] dz d\tau \\
& = \int_0^t \int_{\mathbb{R}} \left[w(s,z) \int_0^s w_z(\tau,z) d\tau \right] dz ds + \int_0^t \int_{\mathbb{R}} \left[w(s,z) \int_0^s k_s(\tau,\theta) w(\tau,z) d\tau \right] dz ds \\
& \leqslant \frac{c_0}{4} \int_0^t \int_{\mathbb{R}} w^2(s,z) dz ds + \frac{2}{c_0} \int_0^t \int_{\mathbb{R}} \left[\int_0^s w_z(\tau,z) d\tau \right]^2 dz ds \\
& + \frac{c_0}{4} \int_0^t \int_{\mathbb{R}} w^2(s,z) dz ds + \frac{2}{c_0} \int_0^t \int_{\mathbb{R}} \left[\int_0^s k_s(\tau,\theta) w(\tau,z) d\tau \right]^2 dz ds \\
& \leqslant \frac{c_0}{2} \int_0^t \int_{\mathbb{R}} w^2(s,z) dz ds + \frac{2}{c_0} \int_0^t \int_{\mathbb{R}} \left[\int_0^s w_z(\tau,z) d\tau \right]^2 dz ds \\
& + \frac{2\|k_s\|^2}{c_0} t \int_0^t \int_{\mathbb{R}} \int_0^s w^2(\tau,z) d\tau dz ds.
\end{aligned}
$$

We have

$$
\begin{aligned}
& \frac{c_0}{2} \int_0^t \int_{\mathbb{R}} w^2(\tau,z) dz d\tau + \frac{1}{2} \int_{\mathbb{R}} \left[\int_0^t w_z(s,z) ds \right]^2 dz \\
& \leqslant \frac{2}{c_0} \int_0^t \int_{\mathbb{R}} \left[\int_0^s w_z(\tau,z) d\tau \right]^2 dz ds + \frac{2\|k_s\|^2}{c_0} t \int_0^t \int_{\mathbb{R}} \int_0^s w^2(\tau,z) d\tau dz ds \\
& \leqslant \frac{2}{c_0} (1 + \|k_s\|^2 t) \left\{ \int_0^t \int_{\mathbb{R}} \left[\int_0^s w_z(\tau,z) d\tau \right]^2 dz ds + c_0 \int_0^t \int_{\mathbb{R}} \int_0^s w^2(\tau,z) d\tau dz ds \right\}.
\end{aligned}
$$

Set

$$
G(t) = \frac{c_0}{2} \int_0^t \int_{\mathbb{R}} w^2(\tau,z) dz d\tau + \frac{1}{2} \int_{\mathbb{R}} \left[\int_0^t w_z^2(s,z) ds \right]^2 dz.
$$

We obtain

$$
0 \leqslant G(t) \leqslant \frac{4}{c_0} \left[1 + \|k_s\|^2 t \right] \int_0^t G(s) ds \quad \forall t \geqslant 0.
$$

By Gronwall's inequality, we find that $G(t) = 0 \quad \forall t \geqslant 0$. This means that $w \equiv 0$ and complete the proof of Theorem 4. □

4.2 QUALITATIVE ANALYSIS

In this section, we first introduce the Comparison Principle in subsection 4.2.1, which helps us to construct the sub solutions and super solutions of problem 2.6. In subsection 4.2.2, we construct sub-super solutions to estimate the location of the free boundary and estimate the solution of (2.6). Then in subsection 4.2.3, we study the asymptotic behavior of the solution and the free boundary.

In the study, we consider the case $\sigma = 0$.

We study the following system:

$$\begin{cases} w_\tau = a^*(w)(w_z + w_{zz}) & \forall \tau \in (0, \infty), z \in \mathbb{R}, \\ w(0, z) = min\{1, e^{-z}\} = w_0(z), \end{cases} \tag{4.7}$$

where $a^*(w)$ is defined as in (2.7).

We define the free boundary as

$$\Gamma := \{(\tau, z) \mid \tau \geqslant 0, w(\tau, z) = \gamma\} = \{(\tau, z) \mid \tau \geqslant 0, z = s(\tau)\}.$$

Since $w(\tau, \cdot)$ is strictly decreasing, and

$$\lim_{z\to\infty} w(\tau, z) = 0, \qquad \lim_{z\to-\infty} w(\tau, z) = 1,$$

there exist a unique non-increasing and smooth function $s(\cdot) \in C^\infty[0, \infty)$[22] such that

$$\Gamma := \{(\tau, z) \mid \tau \geqslant 0, w(\tau, z) = \gamma\} = \{(\tau, z) \mid \tau \geqslant 0, z = s(\tau)\}. \tag{4.8}$$

4.2.1 Comparison Principle

First of all, we need a comparison principle to construct the sub-super solutions of (4.7).

We give the comparison principle used in the construction of sub-super solutions as following.

Lemma 3. *Let* $(\tilde{s}, \tilde{w}) \in C^\infty([0, \infty)) \times C([0, \infty) \times \mathbb{R})$ *be a pair of functions. Set* $\tilde{\Gamma} :=$

$\{(\tau, \tilde{s}(\tau)) \mid \tau \geqslant 0\}$. *Assume that* $(\tilde{s}, \tilde{w})$ *satisfies*

$$\begin{cases} \tilde{w}_\tau - a(\tau, \tilde{w})(\tilde{w}_{zz} + \tilde{w}_z) \geqslant 0 & \text{in } (0, \infty) \times \mathbb{R} \setminus \tilde{\Gamma}, \\ \tilde{w} = \gamma \quad \text{on } \tilde{\Gamma}, \\ \tilde{w}(0, z) \geqslant w_0(z) = min\{1, e^{-z}\} & \text{on } \{0\} \times \mathbb{R}, \\ \tilde{w}_z\Big(\tau, \tilde{s}(\tau) + 0\Big) \leqslant \tilde{w}_z\Big(\tau, \tilde{s}(\tau) - 0\Big) & \forall \tau > 0, \\ \tilde{s}(0) > \ln \frac{1}{\gamma}. \end{cases}$$

Then

$$\tilde{s}(\cdot) \geqslant s(\cdot) \text{ on } [0, \infty) \quad \text{and} \quad \tilde{w} \geqslant w \text{ on } [0, \infty) \times \mathbb{R}.$$

Proof. It is assumed that $\tilde{s}(0) > \ln \frac{1}{\gamma} = s(0)$.

Set $\tau^* = \inf\{t > 0 \mid \tilde{s}(t) > s(t)\}$. There are two cases: (1) $\tau^* = \infty$ and (2) $\tau^* < \infty$.

Case (1) $\tau^* = \infty$: We have $\tilde{s}(\tau) > s(\tau)$, $\forall \tau > 0$. Denote $\tilde{Q}^+ = \bigcup_{\tau>0}\{(\tau, z)|z > \tilde{s}(\tau)\}$, $\tilde{Q}^- = \bigcup_{\tau>0}\{(\tau, z)|z < \tilde{s}(\tau)\}$ and $Q^+ = \bigcup_{\tau>0}\{(\tau, z)|z > s(\tau)\}$, $Q^- = \bigcup_{\tau>0}\{(\tau, z)|z < s(\tau)\}$. The parabolic boundary of $\tilde{Q}^+$ is $\tilde{\Gamma} \cup \{\{0\} \times [\tilde{s}(0), \infty)\}$. Since $\tilde{w}(0, z) \geqslant w_0(z)$, $\tilde{s}(\tau) \geqslant s(\tau)$, $w|_{\tilde{\Gamma}} \leqslant \gamma = \tilde{w}|_{\tilde{\Gamma}}$, so $\tilde{w} \geqslant w$ on the parabolic boundary of $\tilde{Q}^+$. Then, by Comparison Principle, $\tilde{w} \geqslant w$ in $\tilde{Q}^+$. Also applying Maximum Principle, we have $w \leqslant \gamma \leqslant \tilde{w}$ in $Q^+ \cap \tilde{Q}^-$. Similarly we can get $\tilde{w} \geqslant w$ in Q^-. Thus $\tilde{w} \geqslant w$ on $[0, \infty) \times \mathbb{R}$.

Case (2) $\tau^* < \infty$: We have $\tilde{s}(\tau^*) = s(\tau^*)$ and $\tilde{s} > s$ in $[0, \tau^*)$. Denote $\tilde{Q}^+_* = \bigcup_{\tau \in (0, \tau_*]}\{(\tau, z)|z > \tilde{s}(\tau)\}$ and $Q^-_* = \bigcup_{\tau \in (0, \tau_*]}\{(\tau, z)|z < s(\tau)\}$. By comparison, we have $\tilde{w} > w$ in $\tilde{Q}^+_* \bigcup Q^-_*$. Since $w - \tilde{w} = 0$ at $(\tau^*, s(\tau^*)]$, we have $(\tilde{w} - w)_z(\tau_*, \tilde{s}(\tau_*) - 0) \leqslant 0$ and $(\tilde{w} - w)_z(\tau_*, \tilde{s}(\tau_*) + 0) > 0$. We know that w_z is continuous everywhere, so we obtain $\tilde{w}_z\Big(\tau_*, \tilde{s}(\tau_*) + 0\Big) - \tilde{w}_z\Big(\tau_*, \tilde{s}(\tau_*) - 0\Big) > 0$, which contradicts with the condition.

Thus, case (2) never happen. That is, we always have $\tau^* = \infty$ and $\tilde{s} > s$ on $[0, \infty)$. Then by comparison we have $\tilde{w} \geqslant w$ on $[0, \infty) \times \mathbb{R}$. This complete the proof. □

We call $(\tilde{s}, \tilde{w})$ a super-solution of system (4.7). Sub-solution can be constructed similarly.

4.2.2 Sub Solutions and Super Solutions

After introducing the comparison principle, we can start to construct a set of sub solutions and a set of super solutions for system (4.7).

During the process, we aim at the special super solutions $\{\bar{w}\}_I$ (where I denotes the index) with $\bar{\Gamma}_I := \{(\tau, z) \mid \tau \geqslant 0, \bar{w}_I(\tau, z) = \gamma\}$ and the special sub solutions $\{\underline{w}\}_J$ (where J denotes the index) with $\underline{\Gamma}_J := \{(\tau, z) \mid \tau \geqslant 0, \underline{w}_J(\tau, z) = \gamma\}$, such that all $\bar{\Gamma}_I$ and $\underline{\Gamma}_J$ are straight lines sharing the same slope. Because we know the free boundary is bounded by $\bar{\Gamma}_i, \forall i \in I$ and $\underline{\Gamma}_j, \forall j \in J$, so in this way we can locate the free boundary in a stripe with two straight line boundaries.

For simplification of the calculation, we consider $\hat{w} = w - \gamma$, where w is the strong solution of (4.7). Then $\hat{w}$ is the solution of following partial differential equation system:

$$\begin{cases} \hat{w}_\tau = \frac{\sigma_H^2}{2}(\hat{w}_z + \hat{w}_{zz}) & \text{for} \quad \hat{w} < 0, \\ \hat{w}_\tau = \frac{\sigma_L^2}{2}(\hat{w}_z + \hat{w}_{zz}) & \text{for} \quad \hat{w} > 0, \\ \hat{w} \; \textit{and} \; \hat{w}_z \; \textit{is continuous in} \; \mathbb{R} \times (0, \infty), \\ \hat{w}(0, z) = \hat{w}_0(z) := min\{1, e^{-z}\} - \gamma. \end{cases} \tag{4.9}$$

Notice the free boundary of (4.9), i.e. $\{(\tau, z) \mid \tau \geqslant 0, \hat{w}(\tau, z) = 0\}$ is exact the free boundary $\Gamma = \{(\tau, z) \mid \tau \geqslant 0, w(\tau, z) = \gamma\} = \{(\tau, z) \mid \tau \geqslant 0, z = s(\tau)\}$ where w is the strong solution of (4.7).

We consider boundary value problem below first,

$$\begin{cases} \hat{w}_\tau = \frac{\hat{\sigma}^2}{2}(\hat{w}_z + \hat{w}_{zz}); \\ \hat{w}(0, z) = g(z), \end{cases}$$

where $\hat{\sigma}$ is a positive constant and $g(\cdot)$ is continuous.

In order to reduce the partial differential equation into the standard heat equation, We introduce $\hat{v}(\tau, \xi)$ by

$$\hat{v}(\tau, \xi) = e^{-\alpha(\xi + A + \kappa\tau) - \lambda\tau} \hat{w}(\tau, \xi + A + \kappa\tau),$$

that is,

$$\xi = z - \kappa\tau - A,$$

$$\hat{w}(\tau, z) = e^{\alpha z + \lambda\tau} \hat{v}(\tau, \xi)|_{\xi = z - A - \kappa\tau}.$$

where α, κ, λ are constants to be determined in the process of constructing the sub-super solutions.

Assume that the following relations hold:

$$\kappa = -\hat{\sigma}^2(\alpha + \frac{1}{2}),$$

$$\lambda = \frac{\hat{\sigma}^2}{2}(\alpha^2 + \alpha).$$

One can easily see

$$\alpha\kappa + \lambda = (\hat{\sigma}\alpha)^2 = -\frac{1}{2}(\frac{\kappa}{\hat{\sigma}} + \frac{\hat{\sigma}}{2})^2.$$

Then (4.9) transforms into a standard heat equation system

$$\begin{cases} \hat{v}_\tau = \frac{\hat{\sigma}^2}{2}\hat{v}_{\xi\xi}, \\ \hat{v}(0,\xi) = e^{-\alpha(\xi+A)}g(\xi+A). \end{cases}$$

From the calculation about $\hat{v}, \hat{w}$, we can get the solution of an system stated in following lemma.

Lemma 4. *The solution of the system*

$$\begin{cases} w_\tau = \frac{\hat{\sigma}^2}{2}(w_z + w_{zz}) & \text{for } z > A + \kappa\tau, \tau > 0, \\ w = \gamma & \text{for } z = A + \kappa\tau, \tau > 0, \\ w(0,z) = g(z) & \text{for } z \geqslant A, \tau = 0, \end{cases} \tag{4.10}$$

is given by, for $z > A + \kappa\tau, \tau > 0$

$$w(\tau,z) = e^{\alpha z + \lambda\tau}\int_0^\infty [g(A+\zeta) - \gamma]e^{-\alpha(A+\zeta)}\left[e^{-\frac{(\xi-\zeta)^2}{2\tau\hat{\sigma}^2}} - e^{-\frac{(\xi+\zeta)^2}{2\tau\hat{\sigma}^2}}\right]\frac{d\zeta}{\sqrt{2\pi\tau\hat{\sigma}^2}} + \gamma, \tag{4.11}$$

where $\xi = z - A - \kappa\tau$.

In addition, w_z near the straight line $z = A + \kappa\tau, \tau > 0$ satisfied

$$\frac{\partial w}{\partial z}(\tau, A+\kappa\tau+0) := \lim_{z\downarrow A+\kappa\tau}\frac{\partial w}{\partial z} = e^{(\alpha\kappa+\lambda)\tau}\int_0^\infty 2\zeta[g(A+\zeta)-\gamma]e^{-\alpha\zeta-\frac{\zeta^2}{2\tau\hat{\sigma}^2}}\frac{d\zeta}{\sqrt{2\pi\tau\hat{\sigma}^2}\tau\hat{\sigma}^2}. \tag{4.12}$$

Proof. System (4.10) is equal to

$$\begin{cases} \hat{w}_\tau = \frac{\hat{\sigma}^2}{2}(\hat{w}_z + \hat{w}_{zz}) & \text{for } z > A + \kappa\tau, \tau > 0, \\ \hat{w} = 0 & \text{for } z = A + \kappa\tau, \tau > 0, \\ \hat{w}(0,z) = g(z) - \gamma & \text{for } z \geqslant A, \tau = 0. \end{cases}$$

That is also equal to:

$$\begin{cases} \hat{v}_\tau = \frac{\hat{\sigma}^2}{2}\hat{v}_{\xi\xi} & \text{for } \xi > 0, \\ \hat{v} = 0 & \text{on } \xi = 0, \\ \hat{v}(0,\xi) = e^{-\alpha(\xi+A)}[g(\xi + A) - \gamma]. \end{cases}$$

By the initial condition expansion below

$$\hat{v}_0(\xi) = \begin{cases} \hat{v}(0,\xi) & \text{for } \xi > 0, \\ 0 & \text{for } \xi = 0, \\ -\hat{v}(0,-\xi) & \text{for } \xi < 0, \end{cases}$$

and the solution of standard heat equation, we derive the solution of the system is

$$\hat{v}(\tau,\xi) = \int_0^\infty \hat{v}_0(\zeta)\left[e^{-\frac{(\xi-\zeta)^2}{2\tau\hat{\sigma}^2}} - e^{-\frac{(\xi+\zeta)^2}{2\tau\hat{\sigma}^2}}\right]\frac{d\zeta}{\sqrt{2\pi\tau\hat{\sigma}^2}}.$$

followed by (4.11).

Then differentiate (4.11), combined with $w(\tau, A + \kappa\tau) = \gamma$ and $\xi \to 0$ as $z \to A + \kappa\tau$, we can get (4.12). This complete the proof. □

Similarly, we can derive the lemma below.

Lemma 5. *The solution of the system*

$$\begin{cases} w_\tau = \frac{\hat{\sigma}^2}{2}(w_z + w_{zz}) & \textit{for } z < A + \kappa\tau, \tau > 0, \\ w = \gamma & \textit{for } z = A + \kappa\tau, \tau > 0, \\ w(0,z) = g(z) & \textit{for } z \leqslant A, \tau = 0, \end{cases}$$

is given by, for $z > A + \kappa\tau, \tau > 0$

$$w(\tau, z) = e^{\alpha z + \lambda\tau} \int_{-\infty}^{0} [g(A+\zeta) - \gamma] e^{-\alpha(A+\zeta)} \left[e^{-\frac{(\xi-\zeta)^2}{2\tau\hat{\sigma}^2}} - e^{-\frac{(\xi+\zeta)^2}{2\tau\hat{\sigma}^2}} \right] \frac{d\zeta}{\sqrt{2\pi\tau\hat{\sigma}^2}} + \gamma,$$

where $\xi = z - A - \kappa\tau$.

In addition, w_z near the straight line $z = A + \kappa\tau, \tau > 0$ satisfied

$$\frac{\partial w}{\partial z}(\tau, A + \kappa\tau - 0) := \lim_{z \uparrow A+\kappa\tau} \frac{\partial w}{\partial z} = e^{(\alpha\kappa+\lambda)\tau} \int_{-\infty}^{0} 2\zeta [g(A+\zeta) - \gamma] e^{-\alpha\zeta - \frac{\zeta^2}{2\tau\hat{\sigma}^2}} \frac{d\zeta}{\sqrt{2\pi\tau\hat{\sigma}^2}\,\tau\hat{\sigma}^2}.$$

Our target is to search for a special sub-solution $\underline{w}$ and super-solution $\bar{w}$ of system (4.9), where $\bar{w}$ satisfies

$$\begin{cases} \bar{w}_\tau = \frac{\sigma_H^2}{2}(\bar{w}_z + \bar{w}_{zz}) & \text{for } z > \bar{A} + \kappa\tau, \tau > 0, \\ \bar{w}_\tau = \frac{\sigma_L^2}{2}(\bar{w}_z + \bar{w}_{zz}) & \text{for } z < \bar{A} + \kappa\tau, \tau > 0, \\ \bar{w} = \gamma & \text{for } z = \bar{A} + \kappa\tau, \tau > 0, \\ \bar{w}(0, z) = \bar{w}_0(z) \geqslant w_0(z) & \text{for } z \in \mathbb{R}, \tau = 0, \\ [\![\frac{\partial \bar{w}}{\partial z}]\!]_{z=\bar{A}+\kappa\tau} := \lim_{h\downarrow 0}\{\frac{\partial \bar{w}}{\partial z}(\tau, \bar{A}+\kappa\tau+h) - \frac{\partial \bar{w}}{\partial z}(\tau, \bar{A}+\kappa\tau-h)\} \leqslant 0, \end{cases} \tag{4.13}$$

and $\underline{w}$ satisfies

$$\begin{cases} \underline{w}_\tau = \frac{\sigma_H^2}{2}(\underline{w}_z + \underline{w}_{zz}) & \text{for } z > \underline{A} + \kappa\tau, \tau > 0, \\ \underline{w}_\tau = \frac{\sigma_L^2}{2}(\underline{w}_z + \underline{w}_{zz}) & \text{for } z < \underline{A} + \kappa\tau, \tau > 0, \\ \underline{w} = \gamma & \text{for } z = \underline{A} + \kappa\tau, \tau > 0, \\ \underline{w}(0, z) = \underline{w}_0(z) \leqslant w_0(z) & \text{for } z \in \mathbb{R}, \tau = 0, \\ [\![\frac{\partial \underline{w}}{\partial z}]\!]_{z=\underline{A}+\kappa\tau} := \lim_{h\downarrow 0}\{\frac{\partial \underline{w}}{\partial z}(\tau, \underline{A}+\kappa\tau+h) - \frac{\partial \underline{w}}{\partial z}(\tau, \underline{A}+\kappa\tau-h)\} \geqslant 0. \end{cases} \tag{4.14}$$

We define following free boundaries:

$$\underline{\Gamma} := \{(\tau, z) | \underline{w}(\tau, z) = \gamma\} = \{(\tau, \underline{s}(\tau)) | \underline{s}(\tau) = \underline{A} + \kappa\tau\};$$

$$\bar{\Gamma} := \{(\tau, z) | \bar{w}(\tau, z) = \gamma\} = \{(\tau, \bar{s}(\tau)) | \bar{s}(\tau) = \bar{A} + \kappa\tau\}.$$

By Comparison Principle we can assert $\underline{w} \leqslant \hat{w} + \gamma \leqslant \bar{w}$ and $\bar{s}(\tau) \geqslant \underline{s}(\tau)$, and we search for a pair of $\underline{w}$ and $\bar{w}$ sharing the same κ and the same $\alpha\kappa + \lambda$. That means $\underline{\Gamma}$ is paralleled to $\bar{\Gamma}$. That is,

$$\frac{\kappa}{\sigma_L} + \frac{\sigma_L}{2} = -[\frac{\kappa}{\sigma_H} + \frac{\sigma_H}{2}].$$

Then, we have

$$\kappa = -\frac{\sigma_L\sigma_H}{2}$$

$$\alpha_H\kappa + \lambda_H = \alpha_L\kappa + \lambda_L = -\frac{(\sigma_L - \sigma_H)^2}{8}$$

$$\alpha_H = -\frac{1}{2} + \frac{\sigma_L}{2\sigma_H} \qquad \alpha_L = -\frac{1}{2} + \frac{\sigma_H}{2\sigma_L}$$

$$\lambda_H = \frac{\sigma_L^2 - \sigma_H^2}{8} \qquad \lambda_L = \frac{\sigma_H^2 - \sigma_L^2}{8}$$

where α_L, λ_L and α_H, λ_H are corresponding designed constants for $z < A+\kappa\tau$ and $z > A+\kappa\tau$ respectively. Here A can be either $\bar{A}$ and $\underline{A}$, because (4.13) and (4.14) share the same α_L, λ_L and α_H, λ_H.

Lemma 6. *Let $\kappa = -\frac{\sigma_L\sigma_H}{2}$, then the solution of the system*

$$\begin{cases} w_\tau = \frac{\sigma_H^2}{2}(w_z + w_{zz}) & \text{for } z > A+\kappa\tau, \\ w_\tau = \frac{\sigma_L^2}{2}(w_z + w_{zz}) & \text{for } z < A+\kappa\tau, \\ w = \gamma & \text{for } z = A+\kappa\tau, \\ w(0,z) = g(z) & \text{for } z \in \mathbb{R}, \tau = 0, \end{cases} \tag{4.15}$$

satisfies

$$[\![\frac{\partial w}{\partial z}]\!]_{z=A+\kappa\tau} := \\ \sqrt{2}e^{-\frac{(\sigma_H-\sigma_L)^2}{8}\tau}\int_0^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}}\left[\frac{g(A+\sigma_H\theta)-\gamma}{\sigma_H} + \frac{g(A-\sigma_L\theta)-\gamma}{\sigma_L}\right]\frac{d\theta}{\sqrt{\pi\tau^3}}.$$

Proof. By Lemma (4) and Lemma (5), the solution of system (4.15) satisfies:

$$w(\tau,z) - \gamma = \\ \begin{cases} e^{\alpha_H z + \lambda_H\tau}\int_0^\infty \frac{g(A+\zeta)-\gamma}{e^{\alpha_H(A+\zeta)}}\left[e^{-\frac{(\xi-\zeta)^2}{2\tau\sigma_H^2}} - e^{-\frac{(\xi+\zeta)^2}{2\tau\sigma_H^2}}\right]\frac{d\zeta}{\sqrt{2\pi\tau\sigma_H^2}} & \text{if } z > A+\kappa\tau, \tau > 0; \\ 0 & \text{if } z = A+\kappa\tau, \tau > 0; \\ e^{\alpha_L z + \lambda_L\tau}\int_{-\infty}^0 \frac{g(A+\zeta)-\gamma}{e^{\alpha_L(A+\zeta)}}\left[e^{-\frac{(\xi-\zeta)^2}{2\tau\sigma_L^2}} - e^{-\frac{(\xi+\zeta)^2}{2\tau\sigma_L^2}}\right]\frac{d\zeta}{\sqrt{2\pi\tau\sigma_L^2}} & \text{if } z < A+\kappa\tau, \tau > 0, \end{cases}$$

where $\xi = z - A - \kappa\tau$.

We use $\hat{g}(\cdot)$ to stand for $g(\cdot) - \gamma$ and the jump of $\frac{\partial w}{\partial z}$ over $z = A + \kappa\tau$ is:

$$
\begin{aligned}
&[\![\frac{\partial w}{\partial z}]\!]_{z=A+\kappa\tau} \\
&= \frac{\sqrt{2}e^{-\frac{(\sigma_H-\sigma_L)^2}{8}\tau}}{\sqrt{\pi\tau^3}} \int_0^\infty \zeta\hat{g}(A+\zeta)\exp\left(\frac{1}{2}(1-\frac{\sigma_L}{\sigma_H})\zeta - \frac{\zeta^2}{2\tau\sigma_H^2}\right)\frac{d\zeta}{\sigma_H^3} \\
&\quad - \frac{\sqrt{2}e^{-\frac{(\sigma_H-\sigma_L)^2}{8}\tau}}{\sqrt{\pi\tau^3}} \int_{-\infty}^0 \zeta\hat{g}(A+\zeta)\exp\left(\frac{1}{2}(1-\frac{\sigma_H}{\sigma_L})\zeta - \frac{\zeta^2}{2\tau\sigma_L^2}\right)\frac{d\zeta}{\sigma_L^3} \\
&= \frac{\sqrt{2}e^{-\frac{(\sigma_H-\sigma_L)^2}{8}\tau}}{\sqrt{\pi\tau^3}} \int_0^\infty \theta\hat{g}(A+\sigma_H\theta)\exp\left(\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}\right)\frac{d\theta}{\sigma_H} \\
&\quad + \frac{\sqrt{2}e^{-\frac{(\sigma_H-\sigma_L)^2}{8}\tau}}{\sqrt{\pi\tau^3}} \int_0^\infty \theta\hat{g}(A-\sigma_L\theta)\exp\left(\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}\right)\frac{d\theta}{\sigma_L} \\
&= \frac{\sqrt{2}e^{-\frac{(\sigma_H-\sigma_L)^2}{8}\tau}}{\sqrt{\pi\tau^3}} \int_0^\infty \theta\exp\left(\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}\right)\left[\frac{\hat{g}(A+\sigma_H\theta)}{\sigma_H} + \frac{\hat{g}(A-\sigma_L\theta)}{\sigma_L}\right]d\theta \\
&= \sqrt{2}e^{-\frac{(\sigma_H-\sigma_L)^2}{8}\tau}\int_0^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta-\frac{\theta^2}{2\tau}}\left[\frac{g(A+\sigma_H\theta)-\gamma}{\sigma_H} + \frac{g(A-\sigma_L\theta)-\gamma}{\sigma_L}\right]\frac{d\theta}{\sqrt{\pi\tau^3}}.
\end{aligned}
$$

In this equation, we did the variable substitution that $\zeta = \sigma_H\theta,\ for\ \zeta > 0$ and $\zeta = -\sigma_L\theta,\ for\ \zeta < 0$. This complete the proof. $\square$

Based on above analysis, we can derive the conditions to construct the sub-super solutions of system (4.7), illustrated by following Lemma **7** and **8**.

Lemma 7. *Assume that $(\bar{w}_0, \bar{A})$ is a function-constant pair satisfying the following.*

1. *The function $\bar{w}_0(\cdot) \geqslant w_0(\cdot)$ on $\mathbb{R}$;*
2. *The pair satisfies*

$$
\int_0^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta-\frac{\theta^2}{2\tau}}\left[\frac{\bar{w}_0(\bar{A}+\sigma_H\theta)-\gamma}{\sigma_H} + \frac{\bar{w}_0(\bar{A}-\sigma_L\theta)-\gamma}{\sigma_L}\right]d\theta \leqslant 0, \quad \forall\tau > 0;
$$

Then the solution $\bar{w}$ given by Lemma (6) with $A = \bar{A}$ and initial condition $g(z) = \bar{w}_0(z)$ is a super-solution of system (4.7).

Consequently,

$$
s(\tau) \leqslant \bar{A} - \frac{\sigma_H\sigma_L}{2}\tau.
$$

Similarly we derive the conditions for sub-solutions of (4.7) in the lemma below.

Lemma 8. *Assume that* $(\underline{w}_0, \underline{A})$ *is a function-constant pair satisfying the following:*

1. *The function* $\underline{w}_0(\cdot) \leqslant w_0(\cdot)$ *on* $\mathbb{R}$*;*
2. *The pair satisfies*

$$\int_0^{\infty} \theta e^{\frac{1}{2}(\sigma_H - \sigma_L)\theta - \frac{\theta^2}{2\tau}} \left[\frac{\underline{w}_0(\underline{A} + \sigma_H \theta) - \gamma}{\sigma_H} + \frac{\underline{w}_0(\underline{A} - \sigma_L \theta) - \gamma}{\sigma_L} \right] d\theta \geqslant 0, \quad \forall \tau > 0;$$

Then the solution $\underline{w}$ *given by Lemma* (6) *with* $A = \underline{A}$ *and initial condition* $g(z) = \underline{w}_0(z)$ *is a sub-solution of system* (4.7)*.*

Consequently,

$$s(\tau) \geqslant \underline{A} - \frac{\sigma_H \sigma_L}{2} \tau.$$

As a result, we manage to find special sub-super solutions satisfying our requirements, given by following Lemma **9** and Lemma **10**. In Lemma **9** and Lemma **10**, we also state the asymptotic behaviors of the sub-super solutions.

Lemma 9. *Let* $(\bar{w}_0, \bar{A})$ *satisfies:*

$$\bar{w}_0(z) = \begin{cases} 1 & \text{for } z \leqslant \ln \frac{1}{\gamma}, \\ \gamma & \text{for } \ln \frac{1}{\gamma} < z < \bar{A}, \\ e^{-z} & \text{for } z \geqslant \bar{A}, \end{cases} \tag{4.16}$$

$$\frac{2(1-\gamma)}{\sigma_L} (\gamma e^{\bar{A}})^{\frac{\sigma_H - \sigma_L}{4\sigma_L}} \leqslant \frac{\gamma - e^{-\bar{A}}}{\sigma_H}. \tag{4.17}$$

Then the solution $\bar{w}$ *given by Lemma* (6) *with* $(\bar{w}_0, \bar{A})$ *above is a super-solution of system* (4.7)*.*

Particularly, we can pick

$$\bar{A} = max \left\{ \ln \frac{2}{\gamma}, \ln \frac{1}{\gamma} + \frac{4\sigma_L}{\sigma_L - \sigma_H} \ln \frac{4\sigma_H(1-\gamma)}{\gamma \sigma_L} \right\}, \tag{4.18}$$

to construct a super-solution of (4.7) *that satisfies:*

For any fixed $\eta_1 \in \mathbb{R}$*, on every curve* $z = -\frac{\sigma_H^2}{2}\tau + \eta_1 \sigma_H \sqrt{\tau} + \bar{A}$*,*

$$\lim_{\tau \to \infty} \bar{w}(\tau, z) = \gamma - \gamma N(\eta_1);$$

For any fixed $\eta_2 \in \mathbb{R}$, on every curve $z = -\frac{\sigma_L^2}{2}\tau - \eta_2\sigma_L\sqrt{\tau} + \ln\frac{1}{\gamma}$,

$$\lim_{\tau\to\infty} \bar{w}(\tau, z) = \gamma + (1-\gamma)N(\eta_2)$$

Here $N(\cdot)$ is the standard normal probability function, i.e.

$$N(x) = \int_{-\infty}^{x} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} d\theta, \quad \forall x \in \mathbb{R}.$$

Proof. By Lemma (7), a super-solution need to satisfies

$$\int_0^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}} \left[\frac{\bar{w}_0(\bar{A}+\sigma_H\theta)-\gamma}{\sigma_H} + \frac{\bar{w}_0(\bar{A}-\sigma_L\theta)-\gamma}{\sigma_L}\right] d\theta \leqslant 0, \quad \forall \tau > 0 \tag{4.19}$$

The first term of (4.19) is

$$\begin{aligned}
&\int_0^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}} \left[\frac{\bar{w}_0(\bar{A}+\sigma_H\theta)-\gamma}{\sigma_H}\right] d\theta \\
&= \int_0^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}} \left[\frac{1}{\sigma_H}(e^{-\bar{A}-\sigma_H\theta}-\gamma)\right] d\theta \\
&\leqslant \int_0^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}} \left[\frac{1}{\sigma_H}(e^{-\bar{A}}-\gamma)\right] d\theta \\
&= \frac{e^{-\bar{A}}-\gamma}{\sigma_H} \int_0^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}} d\theta
\end{aligned}$$

The second term of (4.19) is

$$\begin{aligned}
&\int_0^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}} \left[\frac{\bar{w}_0(\bar{A}-\sigma_L\theta)-\gamma}{\sigma_L}\right] d\theta \\
&= \int_{\frac{\bar{A}-\ln\frac{1}{\gamma}}{\sigma_L}}^\infty \frac{1-\gamma}{\sigma_L} \cdot \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}} d\theta \\
&= \int_{\frac{\bar{A}-\ln\frac{1}{\gamma}}{2\sigma_L}}^\infty \frac{1-\gamma}{\sigma_L} \cdot (\theta + \frac{\bar{A}-\ln\frac{1}{\gamma}}{2\sigma_L}) exp\left[\frac{1}{2}(\sigma_H-\sigma_L)\Big(\theta + \frac{\bar{A}-\ln\frac{1}{\gamma}}{2\sigma_L}\Big) - \frac{1}{2\tau}\Big(\theta + \frac{\bar{A}-\ln\frac{1}{\gamma}}{2\sigma_L}\Big)^2\right] d\theta \\
&\leqslant \int_{\frac{\bar{A}-\ln\frac{1}{\gamma}}{2\sigma_L}}^\infty \frac{1-\gamma}{\sigma_L} \cdot 2\theta \cdot e^{\frac{1}{4\sigma_L}(\sigma_H-\sigma_L)(\bar{A}-\ln\frac{1}{\gamma})} exp\left[\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{1}{2\tau}\Big(\theta + \frac{\bar{A}-\ln\frac{1}{\gamma}}{2\sigma_L}\Big)^2\right] d\theta \\
&\leqslant \frac{2(1-\gamma)}{\sigma_L} \exp\left(\frac{1}{4\sigma_L}(\sigma_H-\sigma_L)(\bar{A}-\ln\frac{1}{\gamma})\right) \int_{\frac{\bar{A}-\ln\frac{1}{\gamma}}{2\sigma_L}}^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}} d\theta \\
&\leqslant \frac{2(1-\gamma)}{\sigma_L} \exp\left(\frac{1}{4\sigma_L}(\sigma_H-\sigma_L)(\bar{A}-\ln\frac{1}{\gamma})\right) \int_0^\infty \theta e^{\frac{1}{2}(\sigma_H-\sigma_L)\theta - \frac{\theta^2}{2\tau}} d\theta.
\end{aligned}$$

Then (4.19) provided that:

$$\frac{e^{-\bar{A}}-\gamma}{\sigma_H}+\frac{2(1-\gamma)}{\sigma_L}e^{\frac{1}{4\sigma_L}(\sigma_H-\sigma_L)(\bar{A}-\ln\frac{1}{\gamma})}\leqslant 0,$$

which yields to (4.18).

So we can pick

$$\bar{A}=max\left\{\ln\frac{2}{\gamma},\ln\frac{1}{\gamma}+\frac{4\sigma_L}{\sigma_L-\sigma_H}\ln\frac{4\sigma_H(1-\gamma)}{\gamma\sigma_L}\right\}.$$

Let $\xi=z-\bar{A}-\kappa\tau$,

$$\bar{w}(\tau,z)-\gamma=$$
$$\begin{cases}\mathrm{I}=e^{\alpha_H z+\lambda_H\tau}\displaystyle\int_0^\infty\frac{\bar{w}_0(\bar{A}+\zeta)-\gamma}{e^{\alpha_H(\bar{A}+\zeta)}}\left[e^{-\frac{(\xi-\zeta)^2}{2\tau\sigma_H^2}}-e^{-\frac{(\xi+\zeta)^2}{2\tau\sigma_H^2}}\right]\frac{d\zeta}{\sqrt{2\pi\tau\sigma_H^2}} & \text{if } z>\bar{A}+\kappa\tau,\tau>0,\\ 0 & \text{if } z=\bar{A}+\kappa\tau,\tau>0,\\ \mathrm{II}=e^{\alpha_L z+\lambda_L\tau}\displaystyle\int_{-\infty}^0\frac{\bar{w}_0(\bar{A}+\zeta)-\gamma}{e^{\alpha_L(\bar{A}+\zeta)}}\left[e^{-\frac{(\xi-\zeta)^2}{2\tau\sigma_L^2}}-e^{-\frac{(\xi+\zeta)^2}{2\tau\sigma_L^2}}\right]\frac{d\zeta}{\sqrt{2\pi\tau\sigma_L^2}} & \text{if } z<\bar{A}+\kappa\tau,\tau>0.\end{cases}$$

$$\begin{aligned}\mathrm{I}=&\exp\left(\alpha_H z+\lambda_H\tau-\alpha_H\bar{A}\right)\int_0^\infty[e^{-(\bar{A}+\zeta)}-\gamma]\exp\left(-\alpha_H\zeta-\frac{(\xi-\zeta)^2}{2\sigma_H^2\tau}\right)\frac{d\tau}{\sqrt{2\pi\sigma_H^2\tau}}\\ &-\exp\left(\alpha_H z+\lambda_H\tau-\alpha_H\bar{A}\right)\int_0^\infty[e^{-(\bar{A}+\zeta)}-\gamma]\exp\left(-\alpha_H\zeta-\frac{(\xi+\zeta)^2}{2\sigma_H^2\tau}\right)\frac{d\tau}{\sqrt{2\pi\sigma_H^2\tau}}\\ =&\exp\left(\alpha_H z+\lambda_H\tau-\alpha_H\bar{A}-\bar{A}\right)\int_0^\infty\exp\left(-(1+\alpha_H)\zeta-\frac{(\xi-\zeta)^2}{2\sigma_H^2\tau}\right)\frac{d\tau}{\sqrt{2\pi\sigma_H^2\tau}}\\ &-\exp\left(\alpha_H z+\lambda_H\tau-\alpha_H\bar{A}-\bar{A}\right)\int_0^\infty\exp\left(-(1+\alpha_H)\zeta-\frac{(\xi+\zeta)^2}{2\sigma_H^2\tau}\right)\frac{d\tau}{\sqrt{2\pi\sigma_H^2\tau}}\\ &-\gamma\exp\left(\alpha_H z+\lambda_H\tau-\alpha_H\bar{A}\right)\int_0^\infty\exp\left(-\alpha_H\zeta-\frac{(\xi-\zeta)^2}{2\sigma_H^2\tau}\right)\frac{d\tau}{\sqrt{2\pi\sigma_H^2\tau}}\\ &-\gamma\exp\left(\alpha_H z+\lambda_H\tau-\alpha_H\bar{A}\right)\int_0^\infty\exp\left(-\alpha_H\zeta-\frac{(\xi+\zeta)^2}{2\sigma_H^2\tau}\right)\frac{d\tau}{\sqrt{2\pi\sigma_H^2\tau}}\\ =&\,\mathrm{III}-\mathrm{IV},\end{aligned}$$

where among the four terms, the first two are denoted as III and the last two terms are denoted as IV.

For the two terms in III, we do the variable substitution respectively by $\theta=\frac{\zeta+(\alpha_H+1)\sigma_H^2\tau-\xi}{\sqrt{\sigma_H^2\tau}}$

and $\theta = \frac{\zeta+(\alpha_H+1)\sigma_H^2\tau+\xi}{\sqrt{\sigma_H^2\tau}}$.

We denote the standard normal probability function as $N(\cdot)$. Then we get:

$$
\begin{aligned}
\mathrm{III} = e^{\alpha_H(z-\bar{A})+\lambda_H\tau-\bar{A}} & \left[\int_{\frac{\sigma_H^2(\alpha_H+1)\tau-\xi}{\sqrt{\sigma_H^2\tau}}}^{\infty} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} d\theta \cdot e^{\frac{\sigma_H^2(\alpha_H+1)^2\tau}{2}-(\alpha_H+1)\xi} \right. \\
& \left. - \int_{\frac{\sigma_H^2(\alpha_H+1)\tau+\xi}{\sqrt{\sigma_H^2\tau}}}^{\infty} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} d\theta \cdot e^{\frac{\sigma_H^2(\alpha_H+1)^2\tau}{2}+(\alpha_H+1)\xi} \right] \\
=& exp\left[\alpha_H(z-\bar{A})+\lambda_H\tau-\bar{A}+\frac{\sigma_H^2(\alpha_H+1)^2\tau}{2}-(\alpha_H+1)\xi\right] \int_{\frac{\sigma_H^2(\alpha_H+1)\tau-\xi}{\sqrt{\sigma_H^2\tau}}}^{\infty} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} d\theta \\
& - exp\left[\alpha_H(z-\bar{A})+\lambda_H\tau-\bar{A}+\frac{\sigma_H^2(\alpha_H+1)^2\tau}{2}+(\alpha_H+1)\xi\right] \int_{\frac{\sigma_H^2(\alpha_H+1)\tau+\xi}{\sqrt{\sigma_H^2\tau}}}^{\infty} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} d\theta.
\end{aligned}
$$

From

$$
\begin{gathered}
\kappa = -\frac{\sigma_L\sigma_H}{2}, \\
\alpha_H = -\frac{1}{2}(1-\frac{\sigma_L}{\sigma_H}), \quad \alpha_L = -\frac{1}{2}(1-\frac{\sigma_H}{\sigma_L}), \\
\lambda_H = \frac{\sigma_L^2-\sigma_H^2}{8}, \quad \lambda_L = \frac{\sigma_H^2-\sigma_L^2}{8},
\end{gathered}
$$

we get

$$
\alpha_i(z-\bar{A})+\lambda_i\tau+\frac{\sigma_i^2\alpha_i^2\tau}{2}-\alpha_i\xi = 0, \quad i \in \{H, L\}.
$$

Then

$$
\begin{aligned}
& \alpha_H(z-\bar{A})+\lambda_H\tau-\bar{A}+\frac{\sigma_H^2(\alpha_H+1)^2\tau}{2}-(\alpha_H+1)\xi \\
=& -\bar{A}+\frac{\tau\sigma_H^2(2\alpha_H+1)}{2}-\xi \\
=& -\bar{A}+\frac{\sigma_L\sigma_H}{2}\tau-\xi.
\end{aligned}
$$

For any fixed $\eta_1 \in \mathbb{R}$, on every curve $\eta_1 = \frac{\xi-\sigma_H^2\alpha_H\tau}{\sqrt{\sigma_H^2\tau}}$, i.e. $z = -\frac{\sigma_H^2}{2}\tau+\eta_1\sigma_H\sqrt{\tau}+\bar{A}$, we have $\xi = \eta_1\sigma_H\sqrt{\tau}-\frac{\sigma_H^2}{2}\tau-\kappa\tau$ on the curve.

So

$$
-\bar{A}+\frac{\sigma_L\sigma_H}{2}\tau-\xi = -\bar{A}-\eta_1\sigma_H\sqrt{\tau}+\frac{\sigma_H^2}{2}\tau.
$$

Similarly,

$$\alpha_H(z-\bar{A})+\lambda_H\tau-\bar{A}+\frac{\sigma_H^2(\alpha_H+1)^2\tau}{2}+(\alpha_H+1)\xi=-\bar{A}+\eta_1\sigma_L\sqrt{\tau}+\frac{\sigma_L^2}{2}\tau$$

Additionally, by the tail bounds of the standard normal distribution, we have

$$\begin{aligned}
\text{III}&=e^{-\bar{A}-\eta_1\sigma_H\sqrt{\tau}+\frac{\sigma_H^2}{2}\tau}\int_{\frac{\sigma_H^2(\alpha_H+1)\tau-\xi}{\sqrt{\sigma_H^2\tau}}}^{\infty}\frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}d\theta-e^{-\bar{A}+\eta_1\sigma_L\sqrt{\tau}+\frac{\sigma_L^2}{2}\tau}\int_{\frac{\sigma_H^2(\alpha_H+1)\tau+\xi}{\sqrt{\sigma_H^2\tau}}}^{\infty}\frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}}d\theta\\
&\leqslant\frac{\sqrt{\sigma_H^2\tau}}{\sqrt{2\pi}[\sigma_H^2(\alpha_H+1)\tau-\xi]}exp\left[-\bar{A}-\eta_1\sigma_H\sqrt{\tau}+\frac{\sigma_H^2}{2}\tau-\frac{(\xi-\sigma_H^2(\alpha_H+1)\tau)^2}{2\sigma_H^2\tau}\right]\\
&\quad-\frac{\sqrt{\sigma_H^2\tau}}{\sqrt{2\pi}[\sigma_H^2(\alpha_H+1)\tau+\xi]}exp\left[-\bar{A}+\eta_1\sigma_L\sqrt{\tau}+\frac{\sigma_L^2}{2}\tau-\frac{(\xi+\sigma_H^2(\alpha_H+1)\tau)^2}{2\sigma_H^2\tau}\right]\\
&=\frac{\sqrt{\sigma_H^2\tau}}{\sqrt{2\pi}[\sigma_H^2(\alpha_H+1)\tau-\xi]}exp\left[-\bar{A}-\eta_1\sigma_H\sqrt{\tau}+\frac{\sigma_H^2}{2}\tau-\frac{(\eta_1-\sigma_H\sqrt{\tau})^2}{2}\right]\\
&\quad-\frac{\sqrt{\sigma_H^2\tau}}{\sqrt{2\pi}(\sigma_H^2[\alpha_H+1)\tau+\xi]}exp\left[-\bar{A}+\eta_1\sigma_L\sqrt{\tau}+\frac{\sigma_L^2}{2}\tau-\frac{(\eta_1+\sigma_H\sqrt{\tau})^2}{2}\right]\\
&=\frac{\sqrt{\sigma_H^2\tau}}{\sqrt{2\pi}[\sigma_H^2(\alpha_H+1)\tau-\xi]}e^{-\bar{A}-\frac{\eta_1^2}{2}}-\frac{\sqrt{\sigma_H^2\tau}}{\sqrt{2\pi}[\sigma_H^2(\alpha_H+1)\tau+\xi]}e^{-\bar{A}-\frac{\eta_1^2}{2}}\\
&\to 0,\ as\ \tau\to\infty.
\end{aligned}$$

After variable substitution in IV and the fact

$$\alpha_i(z-\bar{A})+\lambda_i\tau+\frac{\sigma_i^2\alpha_i^2\tau}{2}-\alpha_i\xi=0, i\in\{H,L\},$$

we have

$$\begin{aligned}
\text{IV}&=\gamma\left[N\Big(\frac{\xi-\alpha_H\sigma_H^2\tau}{\sqrt{\sigma_H^2\tau}}\Big)-e^{2\alpha_H\xi}N\Big(-\frac{\xi+\alpha_H\sigma_H^2\tau}{\sqrt{\sigma_H^2\tau}}\Big)\right]\\
&=\gamma\left[N(\eta_1)-e^{2\alpha_H\xi}N\Big(-\frac{\xi+\alpha_H\sigma_H^2\tau}{\sqrt{\sigma_H^2\tau}}\Big)\right]\\
&\leqslant\gamma\left\{N(\eta_1)-\frac{\sqrt{\sigma_H^2\tau}}{\sqrt{2\pi}(\sigma_H^2\alpha_H\tau+\xi)}exp\left[2\alpha_H\xi-\frac{(\xi+\alpha_H\sigma_H^2\tau)^2}{2\sigma_H^2\tau}\right]\right\}\\
&=\gamma\left\{N(\eta_1)-\frac{\sqrt{\sigma_H^2\tau}}{\sqrt{2\pi}(\sigma_H^2\alpha_H\tau+\xi)}exp\left[2\alpha_H\xi-\frac{\Big(\frac{\xi}{\sqrt{\sigma_H^2\tau}}+\alpha_H\sqrt{\sigma_H^2\tau}\Big)^2}{2}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \gamma \left\{ N(\eta_1) - \frac{\sqrt{\sigma_H^2 \tau}}{\sqrt{2\pi}(\sigma_H^2 \alpha_H \tau + \xi)} exp\left[-\frac{\left(\frac{\xi}{\sqrt{\sigma_H^2 \tau}} - \alpha_H \sqrt{\sigma_H^2 \tau}\right)^2}{2} \right] \right\} \\
&= \gamma \left[N(\eta_1) - \frac{\sqrt{\sigma_H^2 \tau}}{\sqrt{2\pi}(\sigma_H^2 \alpha_H \tau + \xi)} e^{-\frac{\eta_1^2}{2}} \right] \\
&\to \gamma N(\eta_1),\ \ as\ \tau \to \infty.
\end{aligned}
$$

In result, when $z = \bar{A} + \kappa\tau, \tau > 0$

$$
\begin{aligned}
\bar{w}(\tau, z) = \gamma &+ \exp\left(-\bar{A} - \eta_1 \sigma_H \sqrt{\tau} + \frac{\sigma_H^2}{2}\tau \right) N\Big(- \frac{\sigma_H^2(\alpha_H + 1)\tau - \xi}{\sqrt{\sigma_H^2 \tau}} \Big) \\
&- \exp\left(-\bar{A} + \eta_1 \sigma_L \sqrt{\tau} + \frac{\sigma_L^2}{2}\tau \right) N\Big(- \frac{\sigma_H^2(\alpha_H + 1)\tau + \xi}{\sqrt{\sigma_H^2 \tau}} \Big) \\
&- \gamma \left[N\Big(\frac{\xi - \alpha_H \sigma_H^2 \tau}{\sqrt{\sigma_H^2 \tau}} \Big) - e^{2\alpha_H \xi} N\Big(- \frac{\xi + \alpha_H \sigma_H^2 \tau}{\sqrt{\sigma_H^2 \tau}} \Big) \right]
\end{aligned}
$$

and

$$
\lim_{\tau \to \infty} \mathrm{I} = \gamma - \gamma N(\eta_1).
$$

In other words, for any fixed $\eta_1 \in \mathbb{R}$, on every curve $z = -\frac{\sigma_H^2}{2}\tau + \eta_1 \sigma_H \sqrt{\tau} + \bar{A}$,

$$
\lim_{\tau \to \infty} \bar{w}(\tau, z) = \gamma - \gamma N(\eta_1).
$$

By similar process,

$$
\begin{aligned}
\mathrm{II} &= e^{\alpha_L z + \lambda_L \tau} \int_{-\infty}^{\ln \frac{1}{\gamma} - \bar{A}} (1-\gamma) e^{-\alpha_L(\bar{A} + \zeta)} \left[e^{-\frac{(\xi - \zeta)^2}{2\tau\sigma_L^2}} - e^{-\frac{(\xi + \zeta)^2}{2\tau\sigma_L^2}} \right] \frac{d\zeta}{\sqrt{2\pi\tau\sigma_L^2}} \\
&= (1-\gamma) \left[N\Big(\frac{\ln \frac{1}{\gamma} - \bar{A} + \alpha_L \sigma_L^2 \tau - \xi}{\sqrt{\sigma_L^2 \tau}} \Big) - e^{2\alpha_L \xi} N\Big(\frac{\ln \frac{1}{\gamma} - \bar{A} + \alpha_L \sigma_L^2 \tau + \xi}{\sqrt{\sigma_L^2 \tau}} \Big) \right]
\end{aligned}
$$

which is the expression of $\bar{w}(\tau, z)$ during $z < \bar{A} + \kappa\tau, \tau > 0$.

For any fixed $\eta_2 \in \mathbb{R}$, on every curve $\eta_2 = \frac{\ln \frac{1}{\gamma} - \bar{A} + \alpha_L \sigma_L^2 \tau - \xi}{\sqrt{\sigma_L^2 \tau}}$, i.e. $z = -\frac{\sigma_L^2}{2}\tau - \eta_2 \sigma_L \sqrt{\tau} + \ln \frac{1}{\gamma}$,

$$
\lim_{\tau \to \infty} \bar{w}(\tau, z) = \lim_{\tau \to \infty} \mathrm{II} = \gamma + (1-\gamma) N(\eta_2)
$$

This completes the proof. $\square$

Lemma 10. *Let* $(\underline{w}_0, \underline{A})$ *satisfies:*

$$\underline{w}_0(z) = \begin{cases} 1 & \textit{for } z \leqslant \underline{A}, \\ \gamma & \textit{for } \underline{A} < z < \ln \frac{1}{\gamma}, \\ 0 & \textit{for } z \geqslant \ln \frac{1}{\gamma}, \end{cases} \tag{4.20}$$

$$\underline{A} \leqslant min \left\{ \ln \frac{1}{\gamma}, \ln \frac{1}{\gamma} - \frac{4\sigma_H}{\sigma_L - \sigma_H} \ln \frac{2\gamma\sigma_L}{\sigma_H(1-\gamma)} \right\}. \tag{4.21}$$

Then the solution $\underline{w}$ *given by Lemma* (6) *with* $(\underline{w}_0, \underline{A})$ *above is a sub-solution of system* (4.7).

Particularly, we can pick

$$\underline{A} = min \left\{ \ln \frac{1}{\gamma}, \ln \frac{1}{\gamma} - \frac{4\sigma_H}{\sigma_L - \sigma_H} \ln \frac{2\gamma\sigma_L}{\sigma_H(1-\gamma)} \right\},$$

to construct a sub-solution of (4.7) *that that satisfies:*

For any fixed $\eta_1 \in \mathbb{R}$*, on every curve* $z = -\frac{\sigma_H^2}{2}\tau + \eta_1\sigma_H\sqrt{\tau} + \ln\frac{1}{\gamma}$,

$$\lim_{\tau\to\infty} \underline{w}(\tau, z) = \gamma - \gamma N(\eta_1);$$

For any fixed $\eta_2 \in \mathbb{R}$*, On every curve* $z = -\frac{\sigma_L^2}{2}\tau - \eta_2\sigma_L\sqrt{\tau} + \underline{A}$,

$$\lim_{\tau\to\infty} \underline{w}(\tau, z) = \gamma + (1-\gamma)N(\eta_2).$$

Here $N(\cdot)$ *is the standard normal probability function, i.e.*

$$N(x) = \int_{-\infty}^{x} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} d\theta, \forall x \in \mathbb{R}.$$

Proof. Similar to the proof of Lemma (9). □

4.2.3 Asymptotic Behaviors of the Solution and the Free Boundary

Based on the sub-super solutions of system (4.7) mentioned in Lemma **9** and Lemma **10**, we can derive the location of the free boundary Γ and the asymptotic behaviors of the strong solution w and the free boundary for system (4.7), as stated in the following Theorem **5**.

Theorem 5. *Assume $\bar{A}, \underline{A}$ are constants satisfied the conditions in Lemma (9) and Lemma (10). Then the free boundary $\Gamma = \{(\tau, s(\tau)) | w(\tau, s(\tau)) = \gamma\}$ of w, the strong solution of (4.7), satisfies:*

$$\underline{A} - \frac{\sigma_H \sigma_L}{2}\tau \leqslant s(\tau) \leqslant \bar{A} - \frac{\sigma_H \sigma_L}{2}\tau,$$

In addition, the long-time expiration behavior of w satisfies: for any fixed $\eta_1 \in \mathbb{R}$ and any fixed constant $A_1 \in [\ln \frac{1}{\gamma}, \bar{A}]$,

$$\lim_{\tau \to \infty} w\Big(\tau, -\frac{\sigma_H^2}{2}\tau - \eta_1 \sigma_H \sqrt{\tau} + A_1\Big) = \gamma N(\eta_1);$$

For any fixed $\eta_2 \in \mathbb{R}$ and any fixed constant $A_2 \in [\underline{A}, \ln \frac{1}{\gamma}]$,

$$\lim_{\tau \to \infty} w\Big(\tau, -\frac{\sigma_L^2}{2}\tau - \eta_2 \sigma_L \sqrt{\tau} + A_2\Big) = \gamma + (1-\gamma) N(\eta_2).$$

Here $N(\cdot)$ is the cumulative density function of the standard normal distribution, i.e.

$$N(x) = \int_{-\infty}^{x} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} d\theta \quad \forall x \in \mathbb{R}.$$

Proof. From Lemma (7) and Lemma (8) we get $\underline{A} - \frac{\sigma_H \sigma_L}{2}\tau \leqslant s(\tau) \leqslant \bar{A} - \frac{\sigma_H \sigma_L}{2}\tau$.

By comparison principle, for the super-solution $\bar{w}$ and the sub-solution $\underline{A}$ defined with $\bar{A}$ and $\underline{A}$ as showed in Lemma (9) and Lemma (10). we can know $\underline{w} \leqslant w \leqslant \bar{w}$. The asymptotic behaviors of $\bar{w}$ and $\underline{w}$ showed in Lemma (9) and Lemma (10), combined with the fact $w_z \leqslant 0$ and the Squeeze Theorem, we can get the asymptotic behaviors of w as asserted in this theorem. This completes the proof. □

Theorem 2 tells us, when τ big, $-\frac{\sigma_H^2}{2}\tau$ and $-\frac{\sigma_L^2}{2}\tau$ are dominated terms, thus the z-τ plane is nearly separated into three parts by the two lines: $z = -\frac{\sigma_L^2}{2}\tau + O(1)$ and $z = -\frac{\sigma_H^2}{2}\tau + O(1)$. And $w(\tau, z)$ approximately equals to 1 in $\{z < -\frac{\sigma_L^2}{2}\tau + O(1)\}$, γ in $\{-\frac{\sigma_H^2}{2}\tau + O(1) > z > -\frac{\sigma_L^2}{2}\tau + O(1)\}$, and 0 in $\{z > -\frac{\sigma_H^2}{2}\tau + O(1)\}$. We'll verify the results by numerical simulations later.

Clearly, our main result Theorem **2** is derived from Lemma **9**, Lemma **10** and Theorem **5**.

5.0 NUMERICAL SIMULATION

In this chapter, we perform numerical simulations to verify our analytic results. We pay attention to the asymptotic behaviors of w and the free boundary $\Gamma = \{(z,\tau)|w(z,\tau) = \gamma\}$.

The chapter includes two sections. Section 5.1 describes our simulation strategy, which allows us to simulate the large time behaviors of the strong solution and the free boundary. In section 5.2, the simulation results are presented, which exactly match the theoretical main results in chapter 3.

5.1 SIMULATION SCHEME

In this section, we aim at a numerical strategy that can simulate the system 2.6 for time-to-expiry of arbitrary length. For this reason, we do variable substitutions and deduce the PDE to be simulated in the subsection 5.1.1. Then, we list all the chosen parameters in the subsection 5.1.2.

5.1.1 PDE Simulated

To simulate the large time behavior of w in the neighborhood of the lines $z = -\frac{\sigma_H^2}{2}\tau + O(1)$ and $z = -\frac{\sigma_L^2}{2}\tau + O(1)$, we make a change of independent variables from (z,τ) to (ϑ, s) and dependent variable $w(z,\tau)$ to $\phi(\vartheta, s)$ via

$$\begin{cases} \vartheta = \Big(z - \mu\tau - \ln\dfrac{1}{\gamma}\Big)\dfrac{1}{\tau + M}, \\ s = \ln\dfrac{\tau + M}{M}, \\ \phi(\vartheta, s) = w\Big(\vartheta e^s M + \mu\tau + \ln\dfrac{1}{\gamma}, (e^s - 1)M\Big), \quad \forall\vartheta \in \mathbb{R}, s \in [0, \infty). \end{cases} \tag{5.1}$$

where M, μ are constants chosen at our convenience. We simulate $\phi(\vartheta, s)$ first then we can simulate $w(z, \tau)$ later.

The differential equation (4.7) thus leading to:

$$\begin{cases} \phi_s = \dfrac{a^*(\phi)}{e^s M}\phi_{\vartheta\vartheta} + (a^*(\phi) + \mu + \vartheta)\phi_\vartheta & \forall\vartheta \in \mathbb{R}, s \in [0, \infty), \\ \phi(\vartheta, 0) = \min\left\{1, \gamma e^{-M\vartheta}\right\} & \forall\vartheta \in \mathbb{R}. \end{cases} \tag{5.2}$$

Now we describe our numerical algorithm for problem (5.2). We use uniform mesh size h for s variable and l for ϑ variable. Here we set:

$$s_i = ih, \quad i = 0, 1, 2, ..., I;$$

$$\vartheta_j = jl, \quad j = -J, -(J-1), ..., J-1, J;$$

$$\phi^i_j = \phi(\vartheta_j, s_i);$$

$$a^i_j = a^*(\phi^i_j).$$

Using the central difference scheme for ϕ_ϑ, we can discretize the term $(a^*(\phi) + \mu + \vartheta)\phi_\vartheta$ at (ϑ_j, s_i) by:

$$\begin{aligned} (a^i_j + \mu + \vartheta_j)\phi_y(\vartheta_j, s_i) &= \begin{cases} (a^i_j + \mu + \vartheta_j)\frac{\phi^i_{j+1} - \phi^i_j}{l} & \text{if } a^i_j + \mu + \vartheta_j \geqslant 0 \\ -(a^i_j + \mu + \vartheta_j)\frac{\phi^i_{j-1} - \phi^i_j}{l} & \text{if } a^i_j + \mu + \vartheta_j < 0 \end{cases} \\ &= \frac{1}{l}H(a^i_j + \mu + \vartheta_j)|a^i_j + \mu + \vartheta_j|(\phi^i_{j+1} - \phi^i_j) \\ &\quad + \frac{1}{l}H(-a^i_j - \mu - \vartheta_j)|a^i_j + \mu + \vartheta_j|(\phi^i_{j-1} - \phi^i_j). \end{aligned}$$

Hence, we obtain an explicit scheme for (49):

$$\begin{aligned}\phi_j^{i+1} =&\phi_{j+1}^i\Big[\frac{h * a_j^i}{l^2 e^{s_i} M} + \frac{h}{l} H(a_j^i + \mu + \vartheta_j)|a_j^i + \mu + \vartheta_j|\Big] \\ &+ \phi_j^i\Big[1 - \frac{2a_j^i * h}{l^2 e^{s_i} M} - \frac{h}{l}|a_j^i + \mu + \vartheta_j|\Big] \\ &+ \phi_{j-1}^i\Big[\frac{h * a_j^i}{l^2 e^{s_i} M} + \frac{h}{l} H(-a_j^i - \mu - \vartheta_j)|a_j^i + \mu + \vartheta_j|\Big]\end{aligned}$$

We use the boundary condition:

$$\phi_J^i = 0,\ \phi_{-J}^i = 1 \quad for\ i \geqslant 0.$$

5.1.2 Parameters Chosen

In our numerical simulation, we take the parameters as below:

Table 1: Constants Setting Up

Parameters Setting Up	
Parameters	value
σ_L	0.4
σ_H	0.2
γ	0.5
M	200
$\mu = -(\sigma_H^2 + \sigma_L^2)/4$	-0.05
h	0.0005
l	0.001
I	30000
J	750

5.2 NUMERICAL SIMULATION RESULTS

In this section, we show our numerical results of the strong solution w and the free boundary Γ, which match the theoretical analysis stated in section 3.3.

In subsection 5.2.1, we recover the numerical results of w from the simulated PDE solution. The results show a stack of two travelling waves as time-to-expiry is big.

In subsection 5.2.2, the free boundary is simulated and we find its asymptotic line.

5.2.1 The Simulated Strong Solution and Two Travelling Waves

The functions $\phi(\cdot, s)$ at s =0, 2.5, 5, 7.5, 10, 12.5, 15 are shown in Figure 3. In period $s \in [0, 3]$, the corresponding $w(\cdot, \tau)$ for τ =0, 56.8, 129.74, 223.4, 343.66, 498.07, 696.34, 950.92, 1277.8, 1697.5, 2236.5 are shown in Figure 4.

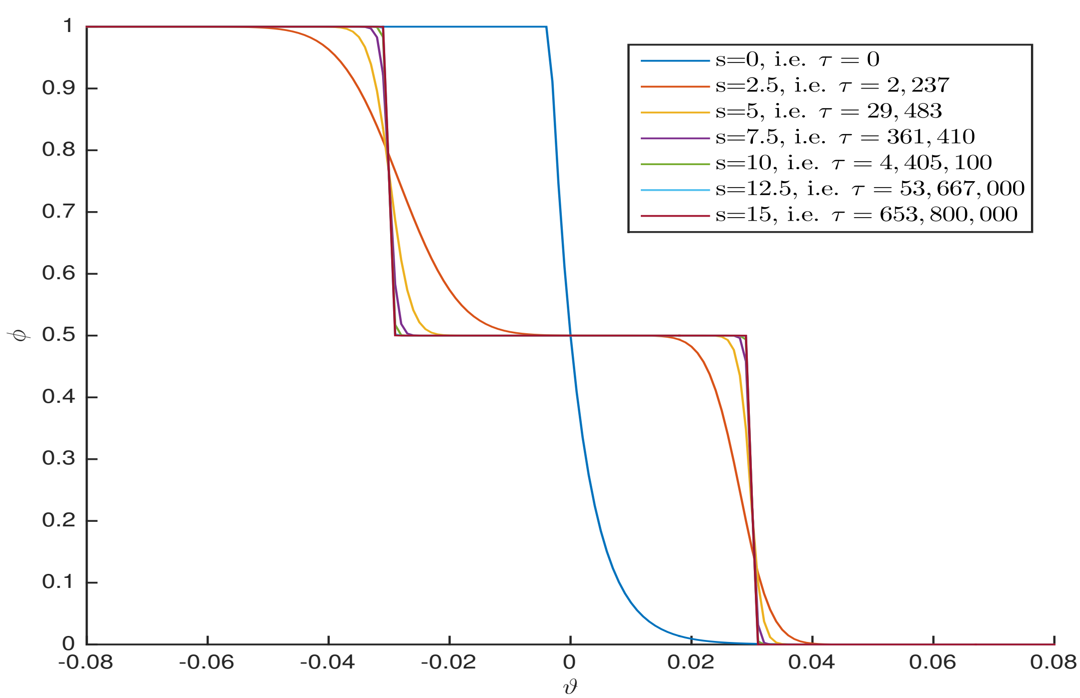

Figure 3: $\vartheta - \phi$ plots at different s from 0 to 15.

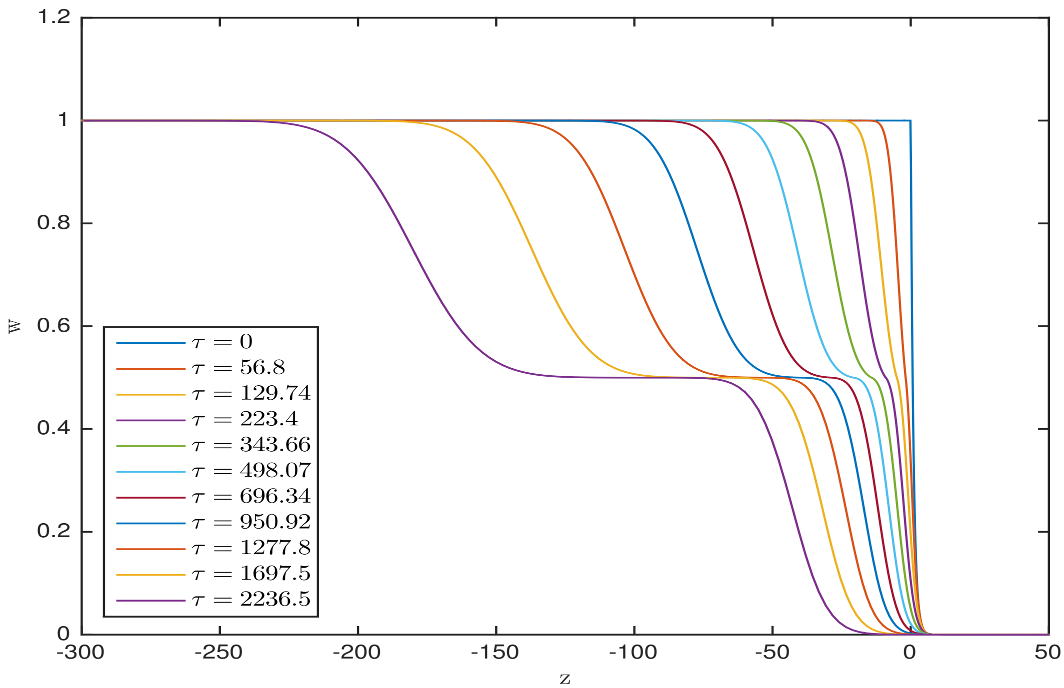

Figure 4: $z - w$ plots at different time τ from 0 to 2,237.

In Figure 3, we can see that

$$\lim_{s\to\infty} \phi(\vartheta, s) = \gamma H(\vartheta_1 - \vartheta) + (1-\gamma)H(\vartheta_2 - \vartheta),$$

where $\gamma = 0.5, \vartheta_1 = 0.03, \vartheta_2 = -0.03$.

By recovering the expression in variable (z, τ) from ϑ_1, ϑ_2, we find the result exactly matches the two lines $z = -\frac{\sigma_L^2}{2}\tau + O(1)$ and $z = -\frac{\sigma_H^2}{2}\tau + O(1)$ that separate the $z - \tau$ plane into three parts where w approximately equals to 0,γ and 1 correspondingly, as mentioned in our analytic assertion.

In Figure 4, the curve of $w(\cdot, \tau)$ for big fixed τ shows a stack of two travelling waves depicted in Figure 4, which are nearly centered along the two straight lines, $z = -\frac{\sigma_L^2}{2}\tau + O(1)$ and $z = -\frac{\sigma_H^2}{2}\tau + O(1)$ respectively.

As the time τ goes by, w tends to take on three main values, i.e. 0, γ and 1. The time for each travelling wave is in order $O(\sqrt{\tau})$.

5.2.2 The Simulated Free Boundary and Asymptotic Lines

We are very interested in the three curves where $w(z,\tau) = \gamma/2, \gamma, (\gamma+1)/2$ respectively. So we give the plots of the three important boundaries in Figure 5 in $z-\tau$ plane till $\tau = 2,237$.

In our theoretical results, we assert that the three curves should be close to the three straight lines

$$z = -\frac{\sigma_H^2}{2}\tau + O(1)$$
$$z = -\frac{\sigma_H\sigma_L}{2}\tau + O(1)$$
$$z = -\frac{\sigma_L^2}{2}\tau + O(1)$$

correspondingly and Figure 5 matches this assertion.

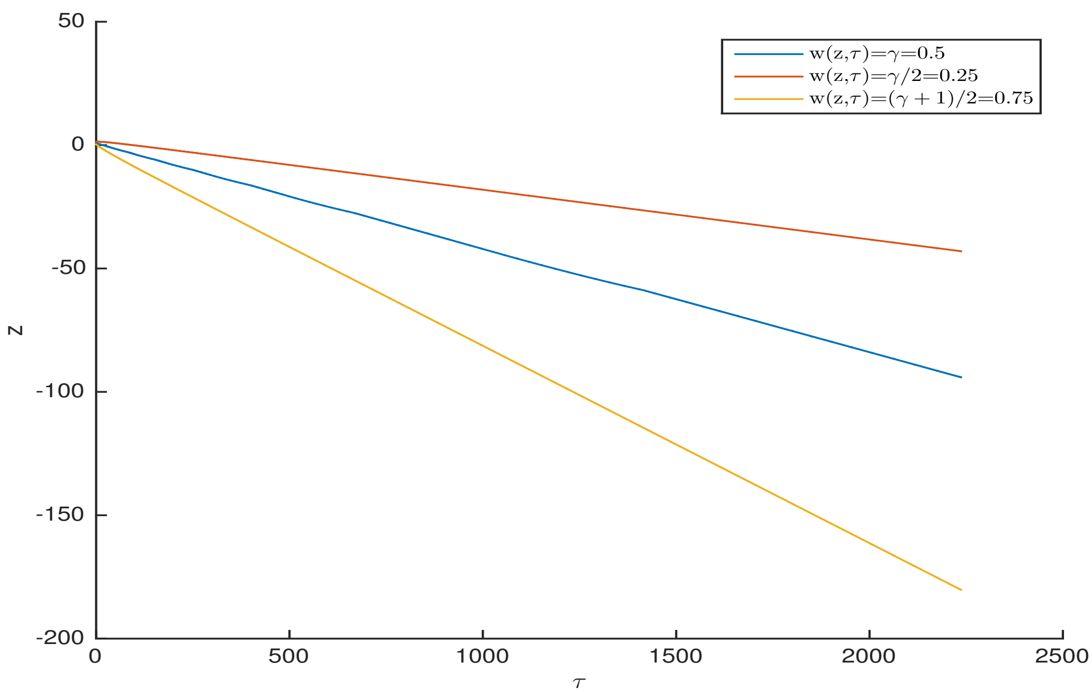

Figure 5: The 3 curves where $w = \gamma$,i.e the free boundary and $w = \gamma/2, (\gamma+1)/2$.

The data of time period $\tau \in [0, 6.5\times 10^8]$ shows that $\phi(0.01, 15) = 0.5 - 2.7756 \times 10^{-15}$, $\phi(-0.029, 15) = 0.5003$, $\phi(-0.031, 15) = 0.9997$, $\phi(0.029, 15) = 0.4999$, $\phi(0.031, 15) = 0.0001$, where the step size for ϑ is 0.001. Combining with the known slope of the asymptotic lines, we can have $\phi(0.01, 15) = 0.5$, $\phi(-0.03, 15) = 0.75$, $\phi(0.03, 15) = 0.25$. Then by (5.1) we can get the estimated asymptotic lines for the three free boundaries in $\tau - z$ plane:

$$z = -0.04\tau + 2.6931, \quad \textit{the asymptotic line for } w(z,\tau) = 0.5;$$

$$z = -0.08\tau - 5.3069, \quad \textit{the asymptotic line for } w(z,\tau) = 0.75;$$

$$z = -0.02\tau + 6.6931, \quad \textit{the asymptotic line for } w(z,\tau) = 0.25.$$

The simulated asymptotic line for the free boundary is $z = -0.04\tau + 2.6931$, which indeed locates in the region asserted in Theorem (2):

$$-0.04\tau - 4.8520 \leqslant s(\tau) \leqslant -0.04\tau + 6.2383.$$

For small time-to-expiry, we also simulate the three free boundaries where $w = \gamma$, $\gamma/2$, $(\gamma + 1)/2$ till time $\tau = 60$, 12, 4.5, 2 in Figure 6-9.

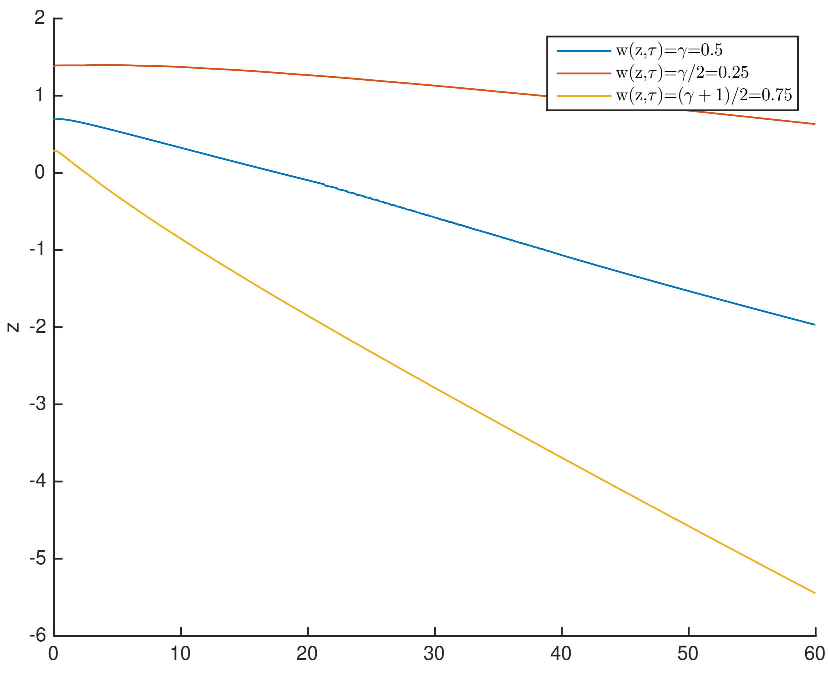

Figure 6: Three free boundaries where $w = \gamma, \gamma/2, (\gamma + 1)/2$ till time $\tau = 60$.

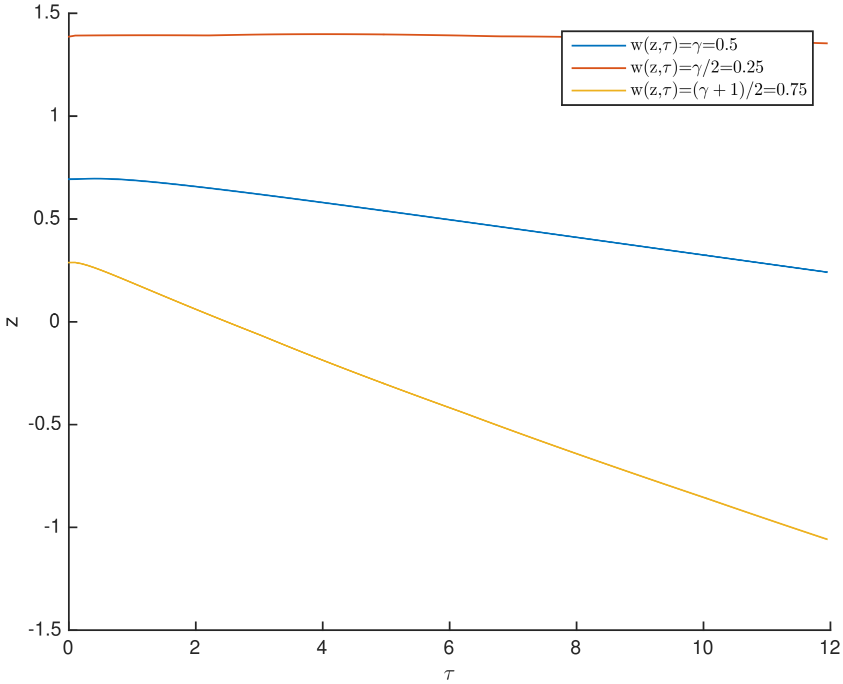

Figure 7: Three free boundaries where $w = \gamma, \gamma/2, (\gamma+1)/2$ till time $\tau = 12$.

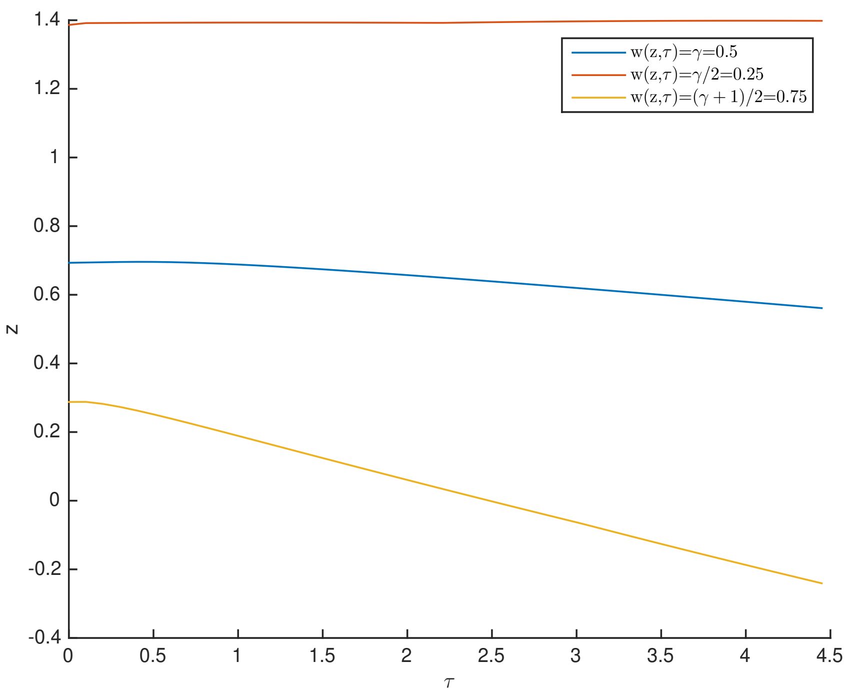

Figure 8: Three free boundaries where $w = \gamma, \gamma/2, (\gamma+1)/2$ till time $\tau = 4.5$.

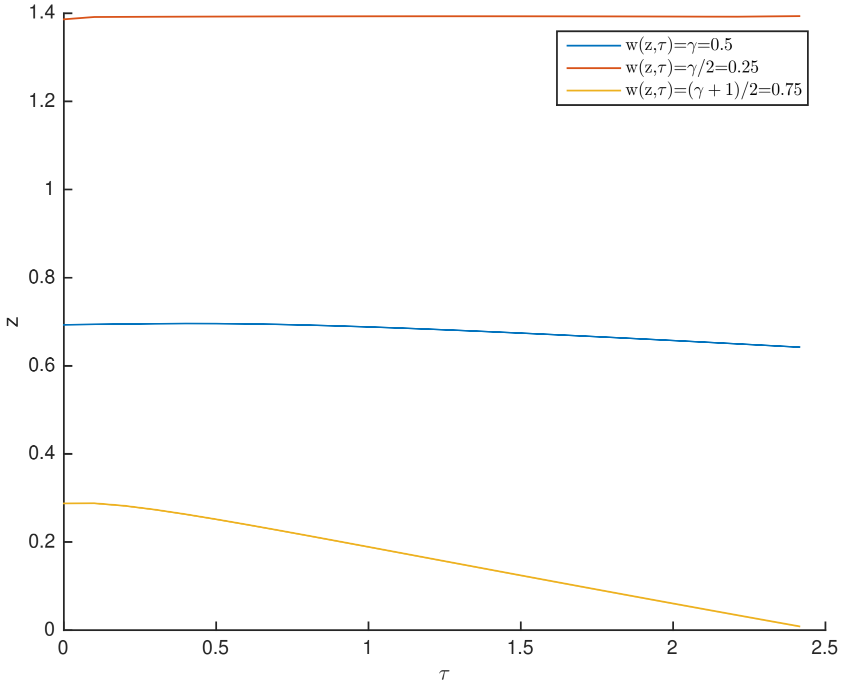

Figure 9: Three free boundaries where $w = \gamma, \gamma/2, (\gamma+1)/2$ till time $\tau = 2$.

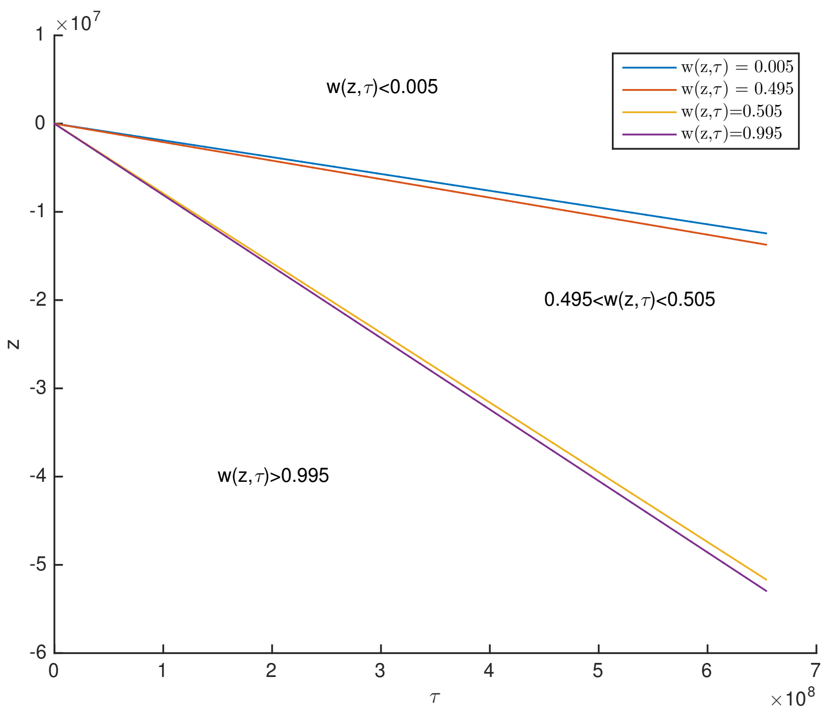

Figure 10: 4 boundaries where $w = 0.005, 0.495, 0.505, 0.995$ in $z - \tau$ plane.

At last, in Figure 10 we plot the four curves where $w(z,\tau) = 0.005, 0.495, 0.505, 0.995$ in Figure 10. One can see the $z-\tau$ plane is approximately separated into three parts where $w(z,\tau)$ is almost 0, γ, 1 respectively, which means the underlying firm's ratio of debt to asset value mainly takes one of the three values: 0, 1 and γ, i.e. the threshold separating different rating regions, in the corresponding zoom in the state plane.

6.0 CONCLUSIONS

In this thesis, we analyze the price of corporate bond of the underlying firm by connecting its debt-to-asset ratio and credit rating change. We as well study the free boundary, across which the underlying company's credit rating will change. We first manage to do dimension reduction and establish a general proof of the well-posedness, including existence and uniqueness of the solution. Secondly, we prove that the free boundary located in a stripe bounded by two straight line sharing the same slope. At the meantime, the asymptotic behaviors of the debt-to-asset ratio and the free boundary are given. Especially, the debt-to-asset ratio has a tendency of presenting a stack of two stage traveling waves as time passes by. At last, numerical simulations are given, which match the theoretical assertions.

7.0 APPENDIX

7.1 THE RISK FREE BOND PRICE

In the study we consider the risk free bond price under the Vasicek Interest Rate Model:

$$dr_t = (k - \beta r_t)dt + \sigma_1 dW_t^1$$

Based on the assumptions and analysis in chapter 3, the risk free bond price function $P_t = p(r_t, t, T)$ is the solution of the Black-Scholes equation with the terminal condition:

$$\begin{cases} \frac{\partial p}{\partial t} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 p}{\partial r^2} + (k - \beta r)\frac{\partial p}{\partial r} - rp = 0 \\ p|_{t=T} = 1 \end{cases}$$

Set $\tau = T - t$ and we guess the solution p has the form $p(\tau, r) = e^{-A(\tau)r - B(\tau)}$, where $A(\tau)$ and $B(\tau)$ is undetermined functions of τ.

By the terminal condition at time T, we have $A(0)r + B(0) = 0, \ \forall r \geqslant 0$. That says

$$A(0) = 0, \ B(0) = 0$$

And the partial differential equation gives

$$p\{r[A'(\tau) + \beta A(\tau) - 1] + [B'(\tau) + \frac{1}{2}\sigma_1^2 A(\tau)^2 - kA(\tau)]\} = 0, \ \forall r \geqslant 0$$

It leads to

$$\begin{cases} A'(\tau) = -\beta A(\tau) + 1 \\ B'(\tau) = -\frac{1}{2}\sigma_1^2 A(\tau)^2 + kA(\tau) \end{cases}$$

Solving the equations combined with the initial conditions, we get

$$A(\tau) = \frac{1}{\beta}(1 - e^{-\beta\tau})$$

$$B(\tau) = \frac{\sigma_1^2}{4\beta^3}e^{-2\beta\tau} - (\frac{\sigma_1^2}{\beta^3} - \frac{k}{\beta^2})e^{-\beta\tau} + (-\frac{\sigma_1^2}{2\beta^2} + \frac{k}{\beta})\tau - \frac{\sigma_1^2}{4\beta^3} + \frac{\sigma_1^2}{\beta^3} - \frac{k}{\beta^2}$$

Thus

$$p(\tau, r) = e^{-A(\tau)r - B(\tau)}$$

where $A(\tau)$ and $B(\tau)$ showed above. And

$$\frac{p_r}{p} = -A(\tau) = -\frac{1}{\beta}(1 - e^{-\beta\tau})$$

8.0 BIBLIOGRAPHY

[1] Robert C. Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, 1974.

[2] Fischer Black and John C Cox. Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, 31(2):351–67, 1976.

[3] Francis A. Longstaff and Eduardo S. Schwartz. A simple approach to valuing risky fixed and floating rate debt. *The Journal of Finance*, 50(3):789–819, 1995.

[4] Hayne E. Leland. Corporate debt value, bond covenants, and optimal capital structure. *The Journal of Finance*, 49(4):1213–1252, 1994.

[5] Hayne E. Leland and Klaus Bjerre Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The Journal of Finance*, 51(3):987–1019, 1996.

[6] Eric Briys and Franois de Varenne. Valuing risky fixed rate debt: An extension. *The Journal of Financial and Quantitative Analysis*, 32(2):239–248, 1997.

[7] Don M. Chance. Default risk and the duration of zero coupon bonds. *The Journal of Finance*, 45(1):265–274, 1990.

[8] David C. Shimko, Naohiko Tejima, and Donald R. Van Deventer. The pricing of risky debt when interest rates are stochastic. *The Journal of Fixed Income*, 3(2):58–65, 1993.

[9] Robert A. Jarrow and Stuart M. Turnbull. Pricing derivatives on financial securities subject to credit risk. *The Journal of Finance*, 50(1):53–85.

[10] David Lando. On cox processes and credit risky securities. *Review of Derivatives research*, 2(2-3):99–120, 1998.

[11] Robert B Litterman and Thomas Iben. Corporate bond valuation and the term structure of credit spreads. *The journal of portfolio management*, 17(3):52–64, 1991.

[12] Darrell Duffie and Kenneth J Singleton. Modeling term structures of defaultable bonds. *The review of financial studies*, 12(4):687–720, 1999.

[13] David Lando. Some elements of rating-based credit risk modeling. *Advanced Fixed-Income Valuation Tools*, pages 193–215, 2000.

[14] Lyn C Thomas, David E Allen, and Nigel Morkel-Kingsbury. A hidden markov chain model for the term structure of bond credit risk spreads. *International Review of Financial Analysis*, 11(3):311–329, 2002.

[15] Tom Hurd and Alexey Kuznetsov. Affine markov chain models of multifirm credit migration. *Journal of Credit Risk*, 3(1):3–29, 2007.

[16] Robert A Jarrow, David Lando, and Stuart M Turnbull. A markov model for the term structure of credit risk spreads. *The review of financial studies*, 10(2):481–523, 1997.

[17] Sanjiv R Das and Peter Tufano. Pricing credit sensitive debt when interest rates, credit ratings and credit spreads are stochastic. 1995.

[18] J Liang and ZK Zeng. Pricing on corporate bonds with credit rating migration under structure framework. *Appl. Math. J. Chinese Univ. Ser. A*, 30(1):61–70, 2015.

[19] Jin Liang, Yuejuan Zhao, and Xudan Zhang. Utility indifference valuation of corporate bond with credit rating migration by structure approach. *Economic Modelling*, 54:339–346, 2016.

[20] Jin Liang, Yuan Wu, and Bei Hu. Asymptotic traveling wave solution for a credit rating migration problem. *Journal of Differential Equations*, 261(2):1017 – 1045, 2016.

[21] Bei Hu, Jin Liang, and Yuan Wu. A free boundary problem for corporate bond with credit rating migration. *Journal of Mathematical Analysis and Applications*, 428(2):896–909, 2015.

[22] Jin Liang, Hong-Ming Yin, Xinfu Chen, and Yuan Wu. On a corporate bond pricing model with credit rating migration risks and stochastic interest rate. *Quantitative Finance and Economics*, 1(QFE-01-00300):300–319, 10 2017.

[23] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177 – 188, 1977.

[24] A. Friedman. Partial differential equations of parabolic type. *Englewood Cliffs, N.J.*, 1964.