Official URL

DOI : https://doi.org/10.1007/978-3-319-91473-2_60

Fuzzy Extensions of Conceptual Structures of Comparison

Didier Dubois[1(✉)], Henri Prade[1,3], and Agnès Rico[2]

[1] IRIT, CNRS & Univ. Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse Cedex 9, France
{dubois,prade}@irit.fr

[2] ERIC & Univ. Claude Bernard Lyon 1, 43 bld du 11 novembre, 69100 Villeurbanne, France
agnes.rico@univ-lyon1.fr

[3] QCIS, University of Technology, Sydney, Australia

Abstract. Comparing two items (objects, images) involves a set of relevant attributes whose values are compared. Such a comparison may be expressed in terms of different modalities such as identity, similarity, difference, opposition, analogy. Recently J.-Y. Béziau has proposed an "analogical hexagon" that organizes the relations linking these modalities. The hexagon structure extends the logical square of opposition invented in Aristotle time (in relation with the theory of syllogisms). The interest of these structures has been recently advocated in logic and in artificial intelligence. When non-Boolean attributes are involved, elementary comparisons may be a matter of degree. Moreover, attributes may not have the same importance. One might only consider *most* attributes rather than all of them, using operators such as ordered weighted min and max. The paper studies in which ways the logical hexagon structure may be preserved in such gradual extensions. As an illustration, we start with the hexagon of equality and inequality due to Blanché and extend it with fuzzy equality and fuzzy inequality.

Keywords: Square of opposition · Hexagon of opposition
Difference · Similarity · Analogy · Ordered weighted min

1 Introduction

In order to compare two objects, two images, etc. (we shall say items, more generally), we use their descriptions. In the sequel, descriptions are understood as plain lists of supposedly relevant attribute values. The two items are assumed to be described by the same set of attributes and the values of these attributes are supposed to be known.

One is then naturally led to state that two items are identical if their respective values for each relevant attribute coincide. Béziau [3] recently pointed out that identity, along with five other modalities pertaining to comparison

https://doi.org/10.1007/978-3-319-91473-2_60

(opposition, similarity, difference, analogy, non-analogy), form a hexagon of opposition. This notion was introduced by Robert Blanché [1,2] as a completion of Aristotle square of opposition [14] (originally introduced in connection with the study of syllogisms). By construction, a hexagon is induced by an abstract three-partition [9] and contains three squares of opposition. Blanché emphasized the point that the hexagonal picture can be found in many conceptual structures, such as arithmetical comparators, or deontic modalities [2].

The logical squares and hexagons are geometrical structures where vertices are traditionally associated to statements that are true or false, possibly involving binary modalities. The use of these structures can be extended to statements for which truth is a matter of degree [5,10]. One may then consider studying gradual comparison operators in the light of the gradual logical hexagon and the framework of fuzzy sets. The study of the compatibility between fuzzy extensions of comparison operations and the logical hexagon is the topic of this paper. In turn, this hexagon-driven approach yields an organized overview of a family of logically related operators.

This paper is organized as follows. Section 2 recalls basics of the logical square and its associated hexagon. Section 3 presents the fuzzy extension of the Blanché hexagon for inequality and equality operators. It is shown that maintaining three squares of opposition inside the hexagon induces strong constraints on aggregation operations involved. We provide two examples of quantitative hexagons for similarity indices based on cardinalities. Section 4 focuses on logical expressions agreeing with Béziau's analogy hexagon, and then on various possible fuzzy extensions. This provides a structure relating gradual indices of opposition, similarity, difference, analogy, and non-analogy. Such gradual extensions take into account approximate equality, attribute importance, and possibly fuzzy quantifiers such as "most".

2 The Square and the Hexagon of Opposition

The traditional square of opposition [14] is built with universally and existentially quantified statements in the following way. Consider a statement (**A**) of the form "all P's are Q's", which is negated by the statement (**O**) "at least one P is not a Q", together with the statement (**E**) "no P is a Q", which is clearly in even stronger opposition to the first statement (**A**). These three statements, together with the negation of the last statement, namely (**I**) "at least one P is a Q" can be displayed on a square whose vertices are traditionally denoted by the letters **A**, **I** (AffIrmative half) and **E**, **O** (nEgative half), as pictured in Fig. 1 (where $\overline{Q}$ stands for "not Q").

As can be checked, noticeable relations hold in the square:

- (i) **A** and **O** (resp. **E** and **I**) are the negation of each other;
- (ii) **A** entails **I**, and **E** entails **O** (it is assumed that there is at least one P for avoiding existential import problems);
- (iii) **A** and **E** cannot be true together, but may be false together;
- (iv) **I** and **O** cannot be false together, but may be true together.

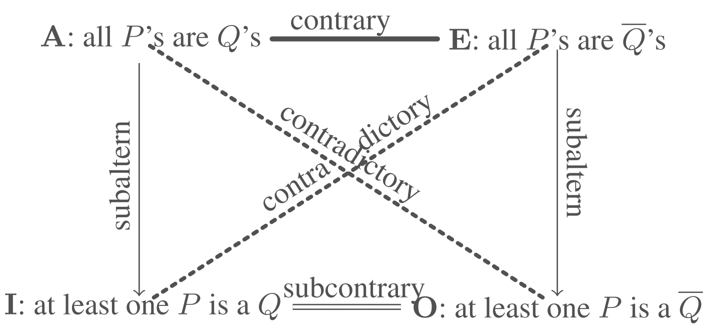


Fig. 1. Square of opposition

Blanché [1,2] noticed that adding two vertices **U** and **Y**, respectively defined as the disjunction of **A** and **E**, and the conjunction of **I** and **O**, to the square, a hexagon **AUEOYI** is obtained that contains 3 squares of opposition, **AEOI**, **YAUO**, and **YEUI**, each obeying the 4 properties above enumerated for the square. Such a hexagon exists each time a three-partition of mutually exclusive situations such as **A**, **E**, and **Y** [9] is considered. Figure 2 represents Blanché's hexagon induced by a complete preorder.

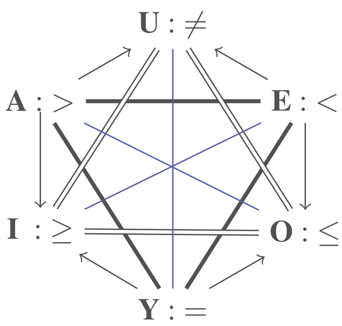


Fig. 2. Blanché's complete preorder hexagon

In the next section, a gradual extension of Blanché's hexagon with fuzzy comparison operators such as *much greater (resp. smaller) than* or *approximately equal to* is proposed.

3 The Fuzzy Blanché Hexagon

We first point out logical constraints bearing on the gradual version of the hexagon, then illustrate them on the case of fuzzy comparison operations.

3.1 Gradual Square and Hexagon

A gradual square of opposition can be defined by attaching variables $\alpha, \epsilon, o, \iota$ valued on a totally ordered set V to vertices A, E, O, I respectively, so as to respect the following constraints [12]:

- α and o (resp. ϵ and ι) are each other's negation, which requires an involutive negation n such that $o = n(\alpha)$ and $\iota = n(\epsilon)$.
- the subaltern relationship between α and ι (resp. ϵ and o) requires a multiple-valued implication operator $I : L \times L \to L$, i.e., decreasing in the first place and increasing in the second place. We must then assume $I(\alpha, \iota) = 1$ and $I(\epsilon, o) = 1$.
- there is mutual exclusion between α and ϵ, i.e., they cannot be simultaneously equal to 1, but can be both 0. We thus need a conjunction operator $C : L \times L \to L$ increasing in both places, and we must enforce $C(\alpha, \epsilon) = 0$.
- ι and o must cover all situations but they can be simultaneously 1. So we need a disjunction operator such that $D(\iota, o) = 1$

A gradual hexagon of opposition [5] is obtained by first assigning variables $\nu = D(\alpha, \epsilon)$ and $\gamma = C(\iota, o)$ to new vertices **U** and **Y**. Then we must require additional conditions to ensure that **YAUO** and **YEUI** are proper squares of opposition playing the same role as **AEOI**. Namely on top of the above conditions for the square, we must have that [5]

- **Y** and **U** are contradictory: $C(\iota, o) = n(D(\alpha, \epsilon))$;
- Subaltern relations:
  $I(\alpha, D(\alpha, \epsilon)) = I(C(\iota, o), o) = I(\epsilon, D(\alpha, \epsilon)) = I(C(\iota, o), \iota) = 1$;
- Contrariety conditions: $C(\alpha, C(\iota, o)) = C(C(\iota, o), \epsilon) = 0$
- Subcontrariety conditions: $D(D(\alpha, \epsilon), o) = D(\iota, D(\alpha, \epsilon)) = 1$
- Conditions for recovering the additional vertices to the two squares **YAUO** and **YEUI** (not mentioned in [5]):
  $\alpha = C(\iota, D(\alpha, \epsilon)), \epsilon = C(D(\alpha, \epsilon), o), \iota = D(\alpha, C(\iota, o)), o = D(C(\iota, o), \epsilon)$.

Note that the conditions in the last item ensure that the three fuzzy sets in the three-partition induced by $(\alpha, \epsilon, \gamma)$ play the same role. If we drop these conditions but preserve the mutual exclusion ones, one may still consider that we have a hexagon of opposition, we call *weak*. However, so-doing we implicitly admit that (α, ϵ) are primitive while γ is derivative, and they cannot be exchanged.

Using a conjunction C, its De-Morgan dual $D(a, b) = n(C(n(a), n(b)))$, and its semi-dual implication $I(a, b) = n(C(a, n(b)))$ then condition $C(\iota, o) = n(D(\alpha, \epsilon))$ is verified, that is γ and ν are contradictories [5]. In the sequel, we denote by (I, C, D) the triplet associated to the hexagon structure with a conjunction C, its semi-dual implication I and its De-Morgan dual D. In that case, the above additional conditions for having a hexagon reduce to $C(\alpha, C(\iota, o)) = C(C(\iota, o), \epsilon) = 0$ and $\iota = D(\alpha, C(\iota, o)), o = D(C(\iota, o), \epsilon)$.

Choosing a triangular norm for C, one may wonder if we can obtain a hexagon of opposition. This is the case if $\alpha + \epsilon + \gamma = 1$ (a fuzzy partition in the sense of Ruspini), and we choose the usual involutive negation $n(\cdot) = 1 - (\cdot)$, the Łukasiewicz t-norm $C = \max(0, \cdot + \cdot - 1)$ and the associated co-norm ($D = \min(1, \cdot + \cdot)$). In fact we prove the following, completing a proof in [5]:

Proposition 1. *If $n(a) = 1 - a$, and C is the Łukasiewicz t-norm, then the hexagon obtained from the triplet (I, C, D) is a hexagon of opposition as soon as $\alpha \leq \iota$.*

Proof. It is clear that $\alpha \leq \iota$ is equivalent to $I(\alpha, \iota) = 1$. It is also clear that $\epsilon \leq o$ follows. Moreover, as we use t-norms and co-norms for C and D, we do have that $\max(\alpha, \epsilon) \leq D(\alpha, \epsilon)$ and $C(\iota, o) \leq \min(\iota, o)$. Hence all subaltern relations hold in the hexagon. Now, consider the condition $D(\alpha, C(\iota, o)) = \iota$ at vertex **I**. It expands in $\min(\alpha + \max(\iota + 1 - \alpha - 1, 0), 1) = \iota$ indeed. Moreover $C(\alpha, C(\iota, o)) = \max(\alpha + \iota + 1 - \alpha - 2, 0) = 0$. The three other conditions at vertices **A, E, O** are obtained likewise. □

In contrast with Proposition 1, consider the Kleene-Dienes triplet (I, C, D), that is $(\max(1 - a, b), \min(a, b), \max(a, b))$, it is clear that $I(a, b) = 1$ means: if $a > 0$ then $b = 1$. The hexagon subaltern conditions for vertex **A** read: If $\alpha > 0$, then $\iota = 1$ and $\max(\alpha, \epsilon) = 1$. Assume $\alpha > 0$, then $\epsilon = 1 - \iota = 0$ and $\max(\alpha, \epsilon) = 1$ so that $\alpha = 1$. We get a Boolea hexagon. It shows that there is no weak gradual hexagon of opposition using Kleene-Dienes triplet (I, C, D).

If we relax the Kleene-Dienes triplet, by means of any implication function such that $I(a, b) = 1$ if and only if $a \leq b$, then the subaltern and mutual exclusion conditions ($\alpha \leq \iota$ and $\min(\alpha, \epsilon) = 0$) are compatible with a gradual structure, where $\alpha > 0$ enforces $\epsilon = 0, o = 1 - \alpha, \iota = 1$. However it is only a weak hexagon. Indeed, defining vertex **I** from **A** and **Y** reads $\iota = \max(\alpha, \min(\iota, o)) = \max(\alpha, 1 - \alpha) = 1$. It again enforces a Boolea hexagon. So we can state the following claim

Proposition 2. *Consider a triplet (I, C, D), where $I(a, b) = 1$ if and only if $a \leq b$, C is a triangular norm D its De Morgan dual with respect to an involutive negation n; then if mutual exclusion conditions hold as $C(\alpha, \epsilon) = C(\alpha, C(n(\alpha), n(\epsilon))) = C(C(\iota, n(\alpha)), \epsilon) = 0$, and $\alpha \leq \iota$, we get a weak gradual hexagon of opposition.*

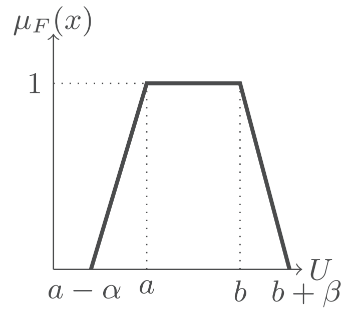
Indeed the subaltern conditions hold in this case. The Łukasiewicz triplet and the relaxed Kleene triplet are examples where this proposition applies. However these conditions do not ensure a full-fledged symmetrical gradual hexagon. Adding condition $C(a, b) = 0$ if and only if $I(a, n(b)) = 1$ seems to be demanding, and the Łukasiewicz triplet is the only known solution then.

3.2 Fuzzy Comparison Operations and Their Hexagon of Opposition

It is possible to construct a gradual hexagon of opposition with fuzzy comparators, in such a way as to extend Blanché's hexagon of Fig. 2. To this end, we define three fuzzy relations that express notions of *approximately equal to, much greater than, much smaller than* as the ones used in the paper [8] for temporal reasoning.

A fuzzy set F on a universe U is a mapping $\mu_F : U \to [0, 1]$ where $\mu_F(x)$ represents the degree of membership of x to F. We denote by $core(F) = \{x | \mu_F(x) = 1\}$ the core of F and $supp(F) = \{x | \mu_F(u) > 0\}$ its support. The complement of F is $\overline{F}$ such that $\mu_{\overline{F}}(x) = 1 - \mu_F(x)$.

Consider for simplicity trapezoidal fuzzy intervals. Such a fuzzy set of the real line pictured on the figure below is parameterized by the 4-tuple of reals (a, b, α, β) where $core(F) = [a, b]$ and $supp(F) =]a - \alpha, b + \beta[$.

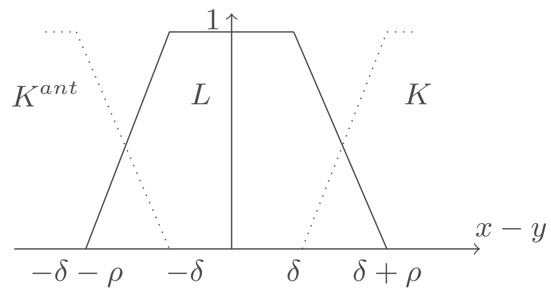


We define a translation operation consisting of adding a constant c to the 4-tuple $F = (a, b, \alpha, \beta)$:

$$F + c = (a + c, b + c, \alpha, \beta).$$

Besides, given F, its antonym is defined by F^{ant} $\mu_{F^{ant}}(x) = \mu_F(-x)$.

Let L be a symmetrical fuzzy interval with respect to the vertical coordinate axis ($L^{ant} = L$). This fuzzy set L is instrumental to define a fuzzy approximate equality relation E as $E(x, y) = L(x - y)$. Likewise, let G be a fuzzy relation representing the concept of *much greater than* of the form $G(x, y) = K(x - y)$, where μ_K is an increasing membership function whose support is in the positive real line. So, the fuzzy relation $P(x, y) = K^{ant}(x - y)$ captures the idea of *much smaller than*. Assume moreover that the three fuzzy sets K^{ant}, L, K form a fuzzy partition, namely $\mu_K(r) + \mu_{K^{ant}}(r) + \mu_L(r) = 1, \forall r \in \mathbb{R}$, as per the figure below for the trapezoidal case. To complete the hexagon, we need the fuzzy counterparts of comparators $\geq$, $\leq$, and $\neq$. To this end we consider the fuzzy set union of K and L, $K \sqcup L$, where $\sqcup$ is defined by Łukasiewicz disjunction, as the fuzzy version of $\geq$ (it is also the convex hull of $K \cup L$ where $\cup$ is modeled by max). It is easy to see that $K \sqcup L = K - 2\delta - \rho$ and $K^{ant} \sqcup L = K^{ant} + 2\delta + \rho$ (which is a fuzzy version of $\leq$). They do correspond to the concept of *approximately greater or equal* and *approximately less or equal* respectively. Finally, the fuzzy version of $\neq$ is $\overline{L} = K \sqcup K^{ant} = K \cup K^{ant}$.



Based on results in [5], and Proposition 1, a gradual hexagon of opposition is obtained (as in Fig. 3) using negation $1 - (\cdot)$, and Łukasiewicz conjunction $(C(a, b) = \max(0, a + b - 1)$. In particular note that $\overline{K} = (K^{ant} \sqcup L)$ and $\overline{L} = K \sqcup K^{ant}$.

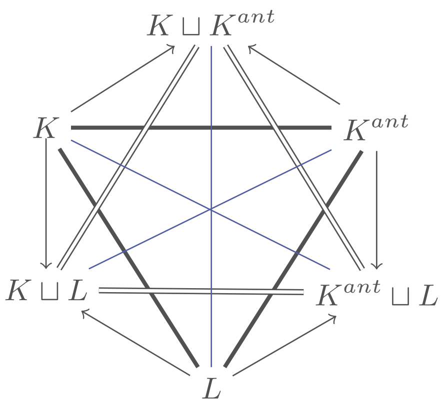


Fig. 3. Fuzzy comparator hexagon

3.3 Hexagons for Quantitative Similarities

Remember that any three-partition gives birth to a hexagon of opposition. Relying on cardinalities of subsets forming a three-partition, it is always possible to obtain a gradual hexagon of opposition. This claim will be illustrated on two cases involving quantitative similarity indices.

Let two items be described by their vectors of Boolean features $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ for a set of attributes $\mathcal{A} = \{1, \cdots, i, \cdots, n\}$.

A first partition of $\mathcal{A}$ is formed by three sets Ag^+, Ag^-, Dif:

- $Ag^+(x, y) = \{i \mid x_i = y_i = 1\}$
- $Ag^-(x, y) = \{i \mid x_i = y_i = 0\}$
- $Dif(x, y) = \{i \mid y_i \neq x_i\}$

The three-partition made of "positive identity" (Ag^+) "negative identity" (Ag^-), "opposition" (Dif) yields the hexagon of Fig. 4. It groups six indices that are all easy to interpret in terms of difference and similarity.

Another hexagon (see Fig. 5) is based on a three-partition of $X \cup Y$, where $X = \{i \mid x_i = 1\}$, $Y = \{i \mid y_i = 1\}$ (hence $X \cap Y = Ag^+(x, y)$, and $\overline{X} \cap \overline{Y} = \overline{X \cup Y} = Ag^-(x, y)$). It consists of the three sets $X \cap Y$, $\overline{X} \cap Y$ and $X \cap \overline{Y}$.

Note that $\frac{|X \cap Y|}{|X \cup Y|} = 1$ if and only if $X = Y$ if and only if $Ag(x, y) = \mathcal{A}$ where $Ag(x, y) = \{i \mid x_i = y_i\} = Ag^+(x, y) \cup Ag^-(x, y)$. Index $\frac{|X \cap Y|}{|X \cup Y|}$ is clearly Jaccard index, i.e., a well-known approximate equality measure, while $\frac{|X \triangle Y|}{|X \cup Y|}$ is a difference index (where $X \triangle Y$ is the symmetric difference). However, $\frac{|Y|}{|X \cup Y|}$ is not really a similarity index as it is not symmetrical; $\frac{|X \cap \overline{Y}|}{|X \cup Y|}$ is an opposition index "inside X", with respect to Y.

None of these two hexagons exhibit all the modalities of identity, difference, similarity, opposition and analogy that are supposed to appear in Béziau's intuitive hexagon The latter will be studied in the next section.

It is easy to verify that the two hexagons possess all properties required for being hexagons of opposition. They could be generalized, replacing relative cardinalities by weighted averages, or even Choquet integrals following a suggestion in [6].

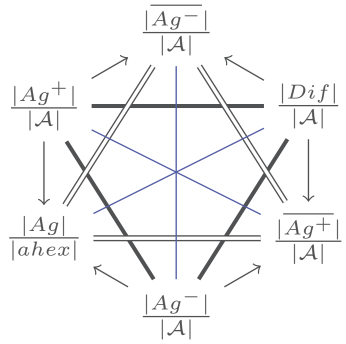


Fig. 4. Hexagon. 3-partition Ag^+, Ag^-, Dif

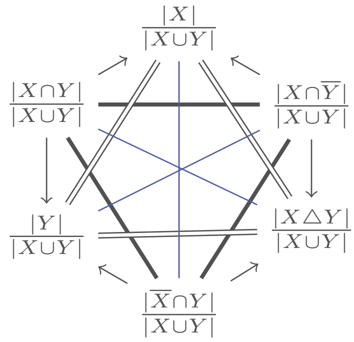


Fig. 5. Jaccard index hexagon

4 Hexagon of Opposition for the Comparison of Items

First, we recall Béziau's informal analogical hexagon [3,4] which organizes the comparison modalities between items supposedly described in terms of attribute values. Then we assume that the equality between attribute values can be approximate, that the attributes do not have the same importance, and that universal or existential quantifiers involved in the comparison modalities can become fuzzy.

4.1 Béziau's Analogical Hexagon

Consider a framework where items x and y are described by their respective attribute values x_i and y_i for attributes $i \in \{1, \cdots, n\}$. At this point attributes are assumed to be Boolean. So the attribute values are 0 or 1. Six comparison modalities between x and y can be defined in this framework:

- Identity: $\forall i,\ x_i = y_i$. Opposition: $\forall i,\ x_i \neq y_i$.
- Difference: $\exists i,\ x_i \neq y_i$. Similarity: $\exists i,\ x_i = y_i$.
- Analogy: $(\exists i, x_i \neq y_i) \wedge (\exists i, x_i = y_i)$. Non-analogy: $(\forall i, x_i = y_i) \vee (\forall i, x_i \neq y_i)$.

This provides a simple reading of the empirical analogical hexagon proposed by Béziau. Note that analogy involves at the same time ideas of similarity and difference, which agrees with the modeling proposed in [15]. It is easy to check that these six modalities are related via logical links requested to realize a hexagon of opposition, as in the analogical hexagon [3]. In particular, difference is the negation of identity and the hexagon makes it clear that analogy is the conjunction between difference and similarity (Fig. 6).

Borrowing from multiple-criteria decision evaluation methods, this hexagon can be extended to gradual modalities using weighted min and max operators on vertices, as we shall see in the sequel.

4.2 Approximate Equality and Weighted Attributes

Suppose from now on that item attributes map to a totally ordered value scale V with least and greatest elements respectively denoted by 0 and 1. The scale

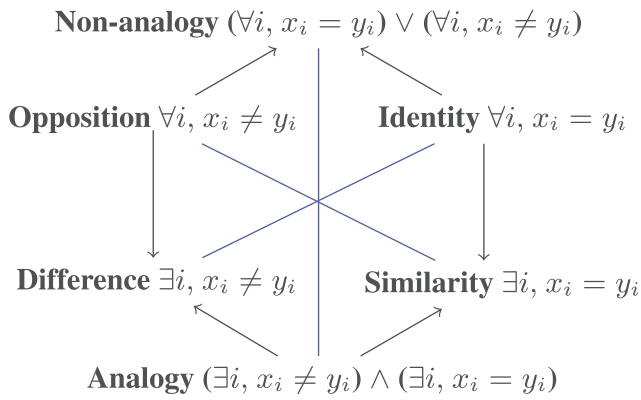


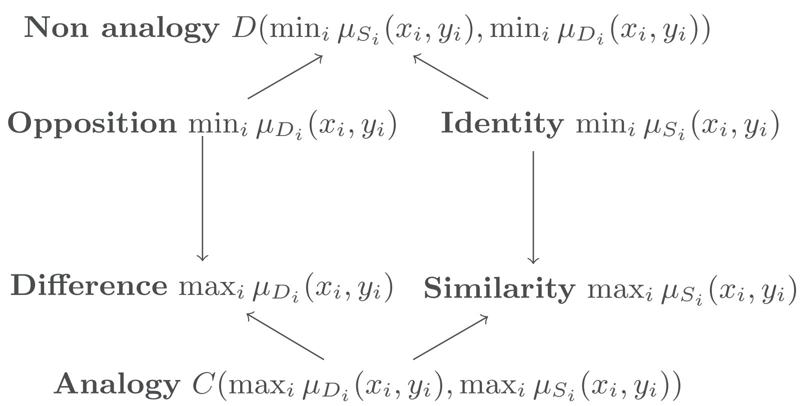
Fig. 6. Analogical hexagon

V is supposed to be equipped with an involutive negation $1-(.)$, i.e. order-reversing on V. On each attribute, equality and difference are evaluated by means of similarity measures $\mu_{S_i} : V \times V \to V$ and dissimilarity measures $\mu_{D_i} : V \times V \to V$. It is natural to assume that $\mu_{S_i} = 1 - \mu_{D_i}$. The vector of similarities between x and y is

$$\mu_S(x, y) = (\mu_{S_1}(x_1, y_1), \cdots, \mu_{S_n}(x_n, y_n))$$

while for dissimilarity it is $\mu_D(x, y) = (\mu_{D_1}(x_1, y_1), \cdots, \mu_{D_n}(x_n, y_n))$. For any two items x and y, it is also supposed that separability holds, namely: $\mu_{S_i}(x_i, y_i) = 1$ (resp. $\mu_{D_i}(x_i, y_i) = 1$) if and only if x_i and y_i are perfectly similar (resp. dissimilar).

Proposition 2 gives conditions under which the following picture is a *weak* hexagon of opposition, in particular the conditions of mutual exclusion: $C(Opposition, Identity) = C(Opposition, Analogy) = C(Identity, Analogy) = 0$.



Under this representation, two items are perfectly *opposite* (resp. *identical*) if they are perfectly dissimilar (resp. similar) on each attribute. They are perfectly *different* (resp. *similar*) if there is at least one attribute for which they are perfectly dissimilar (resp. similar). For *analogy* and *non-analogy* between two items, the result depends on the conjunction C and the disjunction D involved.

Example 1. *Consider Łukasiewicz triplet* (I, C, D). *From Proposition 1, a hexagon of opposition is obtained since* $\min_i \mu_{D_i}(x_i, y_i) \leq \max_i \mu_{D_i}(x_i, y_i)$ *In this case two items are perfectly* **analogical** *if they are perfectly similar and perfectly different. They are* **non-analogical** *if* $\min_i \mu_{D_i}(x_i, y_i) + \min_i \mu_{S_i}(x_i, y_i) \geq 1$, *i.e.,* $\min_i \mu_{D_i}(x_i, y_i) \geq \max_i \mu_{D_i}(x_i, y_i)$. *In this case, similarity and dissimilarity measures are constant and equal to 1 or 0. Then, the two items are either perfectly identical or perfectly opposite.*

Operations min and max are qualitative elementary operators that can be extended by means of importance weights or priorities π_i assigned to attributes. The closer π_i to 1, the more important the attribute. Such importance weights may alter local evaluations in various ways [11], leading to operators of the form

$$MIN_\pi^{\rightarrow}(x) = \min_{i=1}^{n} \pi_i \rightarrow x_i, \quad MAX_\pi^{\otimes}(x) = \max_{i=1}^{n} \pi_i \otimes x_i,$$

where $(\rightarrow, \otimes)$ is a pair of semi-dual implication and conjunction in $\{(\rightarrow_{KD}, \otimes_{KD}), (\rightarrow_G, \otimes_G), (\rightarrow_{GC}, \otimes_{GC})\}$. $a \otimes_{KD} b = a \wedge b$ is Kleene-Dienes conjunction, and its semi-dual implication is $a \rightarrow_{KD} b = (1 - a) \vee b$. Gödel implication and conjunction are respectively defined by: $a \rightarrow_G b = 1$ if $a \leq b$ and b otherwise, and $a \otimes_G b = 0$ if $a \leq 1 - b$ and b otherwise. The contrapositive Gödel implication and conjunction are defined by: $a \rightarrow_{GC} b = 1$ if $a \leq b$ and $1 - a$ otherwise and $a \otimes_{GC} b = 0$ if $a \leq 1 - b$ and a otherwise.

We build the following hexagon where we use shorthand $Id.$, $Op.$, $Dif.$, $Sim.$, $An.$ and $NonAn.$ for the vertices of the analogical hexagon.

Proposition 3. *If* $n(a) = 1 - a$, *and* C *is the Łukasiewicz t-norm, then the fuzzy weighted analogical hexagon of Fig. 7, obtained from the Łukasiewicz triplet* (I, C, D), *is a hexagon of opposition as soon as there is an attribute such that* $\pi_i = 1$, *and implication* $I = \rightarrow$ *is such that* $(1 - a) \rightarrow 0 \leq 1 - (a \rightarrow 0)$.

Proof. Based on results from Proposition 17 in [12], the two conditions $\pi_i = 1$ for some i, and $(1-a) \rightarrow 0 \leq 1-(a \rightarrow 0)$ are sufficient to get $MIN_\pi^{\rightarrow}(\mu_D(x, y)) \leq MAX_\pi^{\otimes} \mu_D(x, y)$, that is $\alpha \leq \iota$ in the hexagon. The rest follows by Proposition 1. □

We give three examples of semi-dual pairs $(\rightarrow, \otimes)$ where this proposition applies:

1. **Kleene-Dienes.** Then: $MIN_\pi^{\rightarrow_{KD}}(x) = \min_{i=1}^{n} \max(1 - \pi_i, x_i) \leq MAX_\pi^{\otimes_{KD}}(x) = \max_{i=1}^{n} \min(\pi_i, x_i)$ is well-known. When computing the external minimum (resp. maximum), the partial evaluation of attributes of low importance is increased (resp. decreased). Such attributes will thus have limited influence

on the global rating. Then two items will be perfectly *opposite* (resp. *identical*) if for each attribute either its importance is zero, or dissimilarity (resp. similarity) between items is perfect. They will be perfectly *different* (resp. *similar*) if there exists at least one attribute with importance 1 for which there is perfect dissimilarity (resp. similarity) between items.

2. **Gödel implication:** Since $(1-a) \to_G 0 = 0$, it follows that $MIN_\pi^{\to_G}(\mu_D(x,y)) \leq MAX_\pi^{\otimes_G} \mu_D(x,y)$ if $\pi_i = 1$ for some i. The weights π_i only play the role of thresholds. Then two items will be perfectly *opposite* (resp. *identical*) if all local dissimilarities (resp. similarities) are above their thresholds. They will be perfectly *different* (resp. *similar*) if there exists at least one attribute with non-zero importance π_i and perfect dissimilarity (resp. similarity) between items.
3. **Contrapositive Gödel implication:** We can check that $(1-a) \to_{GC} 0 = 1 - a = 1 - (a \to_{GC} 0)$. It follows that $MIN_\pi^{\to_{GC}}(\mu_D(x,y)) \leq MAX_\pi^{\otimes_{GC}} \mu_D(x,y)$, i.e., $\min_{i|\pi_i > \mu_{D_i}(x_i,y_i)} 1 - \pi_i \leq \max_{i|\mu_{D_i}(x_i,y_i) > 1-\pi_i} \pi_i$. Then two items will be perfectly *opposite* (resp. *identical*) if all local dissimilarities (resp. similarities) are above their thresholds. They will be *different* (resp. *similar*) if there exists at least one attribute with $\pi_i = 1$ and non-zero dissimilarity (resp. similarity) between items.

$$\begin{tikzcd}
 & \textbf{NonAn. } D(MIN_\pi^{\to}(\mu_S(x,y)), MIN_\pi^{\to}(\mu_D(x,y))) & \\
\textbf{Op. } MIN_\pi^{\to}(\mu_D(x,y)) \arrow[ur] \arrow[d] & & \textbf{Id. } MIN_\pi^{\to}(\mu_S(x,y)) \arrow[ul] \arrow[d] \\
\textbf{Dif. } MAX_\pi^{\otimes}(\mu_D(x,y)), & & \textbf{Sim. } MAX_\pi^{\otimes}(\mu_S(x,y)) \\
 & \textbf{An. } C(MAX_\pi^{\otimes}(\mu_D(x,y)), MAX_\pi^{\otimes}(\mu_S(x,y))) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Fig. 7. Fuzzy weighted analogical hexagon

4.3 OWmin and OWmax

With operations $MIN_\pi^{\to}$ and $MAX_\pi^{\otimes}$ the result depends on all local evaluations for all attributes. One interesting issue is whether the hexagon structure can survive when the quantifiers involved are weakened using ordered weighted min and max (shorthand $OWmin$ and $OWmax$) [7]. In such operations, the quantifier *for all* i is replaced by *for the* k *best*, where the selection of the best ones is based on local evaluations. For a given vector $x \in V^n$, let σ be the permutation of attributes such that $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$.

Define $\mu_k \colon \{1, \cdots, n\} \to V$ by $\mu_k(i) = 1$ if $1 \leq i \leq k$ and 0 otherwise. Then,

$$OWmin_{\mu_k}(x) = \min_{i=1}^{n} \max(1 - \mu_k(i), x_{\sigma(i)})$$

$$\begin{tikzcd}
 & \textbf{NonAn. } D(OWmin_{\mu_k}(\mu_S(x,y)), OWmin_{\mu_k}(\mu_D(x,y))) & \\
\textbf{Op. } OWmin_{\mu_k}(\mu_D(x,y)) \arrow[ur] \arrow[d] & & \textbf{Id. } OWmin_{\overrightarrow{\mu_k}}(\mu_S(x,y)) \arrow[ul] \arrow[d] \\
\textbf{Dif. } OWmax_{\mu_k}(\mu_D(x,y)) & & \textbf{Sim. } OWmax_{\mu_k}(\mu_S(x,y)) \\
 & \textbf{An. } C(OWmax_{\mu_k}(\mu_D(x,y)), OWmax_{\mu_k}(\mu_S(x,y))) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Fig. 8. OWmin-OWmax hexagon

only uses the worst among the k best attributes with respect to their partial evaluations. Likewise, from De Morgan duality,

$$OWmax_{\mu_k}(x) = 1 - OWmin_{\mu_k}(1-x) = \max_{i=1}^{n} \min(\mu_k(i), x_{\sigma(i)})$$

(where σ is now such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$) only uses the best among the k worst attributes with respect to their partial evaluations. μ_k thus represents the quantifier *for at least the k best attributes*. Such a quantifier can vary from *for all attributes* to *for at least the best attribute.*

Unfortunately, the inequality $OWmin_{\mu_k}(\mu_D(x,y)) \leq OWmax_{\mu_k}(\mu_D(x,y))$ generally does not hold. So, the hexagon that could be constructed from these operations may not always be one of opposition. Recall that $OWmin_{\mu_k}$ and $OWmax_{\mu_k}$ are special cases of a Sugeno integral that can be written in two ways:

$$S_\gamma(x) = \max_{E \subseteq \{1,\ldots,n\}} \min(\gamma(E), \min_{i \in E} x_i) = \min_{T \subseteq \{1,\ldots,n\}} \max(1 - \gamma^c(T), \max_{i \in T} x_i).$$

where γ is a capacity and $\gamma^c(T) = 1 - \gamma(\overline{T})$ its conjugate. In the special case considered here, capacities are of the form $\gamma_k(E) = 1$ if $|E| \geq k$ and 0 otherwise. Then $OWmax_{\mu_k} = S_{\gamma_k}$. It is easy to check that the conjugate of γ_k is γ_{n-k+1}. From the above equality, it follows easily that $OWmin_{\mu_k} = OWmax_{\mu_{n-k+1}}$. Hence, the inequality $OWmin_{\mu_k} \leq OWmax_{\mu_k}$ also reads $OWmin_{\mu_k} \leq OWmin_{\mu_{n-k+1}}$ and will hold only when $k \geq (n+1)/2$. In other words, operator $OWmin_{\mu_k}$ must be sufficiently demanding to match with a necessity-like modality. Under such conditions the hexagon in Fig. 8 is a hexagon of opposition. It would be worthwhile comparing this approach with the one in [13].

5 Conclusion

We have shown that the geometrical structure called hexagon of opposition, which organizes the relationship between various comparison modalities can survive under various gradual extensions of such modalities. A study of conditions needed for the gradual hexagon of opposition (as we did for the cube of opposition in [12]) in a more general algebraic setting has been carried out, introducing weak and full-fledged graded versions of the hexagon. Further investigation is still needed to characterize proper algebraic settings that support the gradual hexagon. One may also consider the possibility of introducing an inner negation for defining from the similarity degree $\mu_S(x, y)$ a "remoteness" degree defined by $\mu_S(x, 1-y)$ (with the requirement that $\mu_S(x, 1-y) = \mu_S(1-x, y)$) for $x, y \in [0, 1]$. This might lead to a cube-like structure of opposition.

References

1. Blanché, R.: Sur l'opposition des concepts. Theoria **19**, 89–130 (1953)
2. Blanché, R.: Structures Intellectuelles. Essai sur l'Organisation Systématique des Concepts, Vrin (1966)
3. Béziau, J.-Y.: The power of the hexagon. Log. Univers. **6**(1–2), 1–43 (2012)
4. Béziau, J.-Y.: An analogical hexagon. Int. J. Approx. Reason. **94**, 1–17 (2018)
5. Ciucci, D., Dubois, D., Prade, H.: Structures of opposition induced by relations - the Boolean and the gradual cases. Ann. Math. Artif. Intell. **76**(3–4), 351–373 (2016)
6. Coletti, G., Petturiti, D., Vantaggi, B.: Fuzzy weighted attribute combinations based similarity measures. In: Antonucci, A., Cholvy, L., Papini, O. (eds.) ECSQARU 2017. LNCS (LNAI), vol. 10369, pp. 364–374. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-61581-3_33
7. Dubois, D., Fargier, H., Prade, H.: Beyond min aggregation in multicriteria decision: (ordered) weighted min, discri-min, leximin. In: Yager, R.R., Kacprzyk, J. (eds.) The Ordered Weighted Averaging Operators: Theory and Applications, pp. 181–191. Kluwer Academic Publishers, Dordrecht (1997)
8. Dubois, D., HadjAli, A., Prade, H.: Fuzziness and uncertainty in temporal reasoning. J. Univers. Comput. Sci. **9**, 1168–1194 (2003)
9. Dubois, D., Prade, H.: From Blanché's hexagonal organization of concepts to formal concept analysis and possibility theory. Log. Univers. **6**, 149–169 (2012)
10. Dubois, D., Prade, H.: Gradual structures of oppositions. In: Magdalena, L., Verdegay, J.L., Esteva, F. (eds.) Enric Trillas: A Passion for Fuzzy Sets. SFSC, vol. 322, pp. 79–91. Springer, Cham (2015). https://doi.org/10.1007/978-3-319-16235-5_7
11. Dubois, D., Prade, H., Rico, A.: Residuated variants of Sugeno integrals: towards new weighting schemes for qualitative aggregation methods. Inf. Sci. **329**, 765–781 (2016)
12. Dubois, D., Prade, H., Rico, A.: Graded cubes of opposition and possibility theory with fuzzy events. Int. J. Approx. Reason. **84**, 168–185 (2017)
13. Murinová, P., Novák, V.: Analysis of generalized square of opposition with intermediate quantifiers. Fuzzy Sets Syst. **242**, 89–113 (2014)
14. Parsons, T.: The traditional square of opposition. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy (2008)
15. Prade, H., Richard, G.: From analogical proportion to logical proportions. Log. Univers. **7**, 441–505 (2013)