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BLOW-UP OF POSITIVE SOLUTIONS TO WAVE EQUATIONS IN HIGH SPACE DIMENSIONS

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Abstract. This paper is concerned with the Cauchy problem for the semilinear wave equation:

$$u_{tt} - \Delta u = F(u) \quad \text{in} \quad \mathbb{R}^n \times [0, \infty),$$

where the space dimension $n \geq 2$, $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$ with $p > 1$. Here, the Cauchy data are non-zero and non-compactly supported. Our results on the blow-up of positive radial solutions (not necessarily radial in low dimensions $n = 2, 3$) generalize and extend the results of Takamura [19] for zero initial position and Takamura, Uesaka and Wakasa [21] for zero initial velocity. The main technical difficulty in the paper lies in obtaining the lower bounds for the free solution when both initial position and initial velocity are non-identically zero in even space dimensions.

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1. Introduction

We consider the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u = F(u) & \text{in} \quad \mathbb{R}^n \times [0, \infty), \\ u(x,0) = f(x), \ u_t(x,0) = g(x) & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 2$, $u = u(x,t)$ is a scalar unknown function of space-time variables, and $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$, with $p > 1$. The scenario in one space-dimension is fairly simple and one can always find general conditions on the initial data to prove the blow up of classical solution. Thus, we only consider problem (1.1) in high space dimensions, $n \geq 2$.

For the case of compactly supported initial data $\{f, g\}$ and when $F(u) = |u|^p$, we recall Strauss' conjecture. Namely, there exists a critical number $p_0(n)$ such that (1.1) has a global in time solution if the initial data are *sufficiently small* and $p > p_0(n)$; and (1.1) has no global solutions if $1 < p \leq p_0(n)$ and the initial data are positive in some sense. It was conjectured that $p_0(n)$ is the positive root of the equation $(n-1)p^2 - (n+1)p - 2 = 0$. That is,

$$p_0(n) = \frac{1}{2(n-1)} \left[n + 1 + \sqrt{n^2 + 10n - 7} \right].$$

We note here that $p_0(n)$ comes from the integrability of a certain weight function in the iteration argument for (1.1).

The conjecture was first verified by John [7] for $n = 3$, but not for $p = p_0(3)$. Glassey [5, 6] verified the conjecture for $n = 2$, but not for $p = p_0(2)$. The case of the critical exponents $p = p_0(2)$ and $p = p_0(3)$ were proven by Schaeffer [17]. In high space dimensions, $n \geq 4$, the subcritical case $1 < p < p_0(n)$ was handled by Sideris [18], and later by Rammaha [16] who provided a simplified proof. The super critical case $p > p_0(n)$ was proven by Georgiev, Lindblad, and Sogge [4]. See also Tataru [22]. Finally, the critical case $p = p_0(n)$, $n \geq 4$ was handled by Yordanov and Zhang [26], and independently by Zhou [27]. Thus, Strauss' conjecture has been completely resolved and all of the cited results above on Strauss' conjecture are summarized in the following table:

	$1 < p < p_0(n)$	$p = p_0(n)$	$p > p_0(n)$
$n = 2$	[6]	[17]	[5]
$n = 3$	[7]	[17]	[7]
$n \geq 4$	[18]	[26], [27] (independently)	[4]

However, the scenario is somewhat different when the initial data are not compactly supported and decaying slowly at infinity. In fact, problem (1.1) may have no global solution even for the supercritical case ($p > 1$ is arbitrarily large). Indeed, the pioneering results on non-compactly supported initial data by Asakura [2] strongly suggests the validity of the following statement:

$$\boxed{\begin{array}{c}\text{There exists a critical decay exponent } \kappa_0 > 0 \text{ such that (1.1)}\\ \text{has a global solution, provided } \kappa \geq \kappa_0,\ p > p_0(n) \text{ and the}\\ \text{initial data are sufficiently small, yet (1.1) has no global}\\ \text{solutions, provided } 0 < \kappa < \kappa_0,\ p > 1, \text{ and the initial data}\\ \text{are positive in some sense.}\end{array}} \tag{1.2}$$

It is remarkable that (see for instance [2]) the critical decay exponent κ_0 is independent of the space dimension n and it is given by:

$$\kappa_0 = \frac{2}{p-1}, \quad p > 1. \tag{1.3}$$

As shown in [2] and later by Takamura, Uesaka and Wakasa [21] that there exists a constant $L > 0$ such that (1.1) has no global solution if the initial data $\{f,\, g\}$ satisfy:

$$f(x) \equiv 0 \quad \text{and} \quad g(x) \geq \frac{\phi(|x|)}{(1+|x|)^{1+\kappa}}, \tag{1.4}$$

or

$$f(x) > 0,\ \Delta f(x) + F(f(x)) \geq \frac{\phi(|x|)}{(1+|x|)^{2+\kappa}} \quad \text{and} \quad g(x) \equiv 0, \tag{1.5}$$

for all $|x| \geq L$, with

$$0 < \kappa < \kappa_0 \quad \text{and} \quad \phi(x) \equiv \text{positive const.}, \tag{1.6}$$

or

$$\kappa = \kappa_0,\ \phi \text{ is positive, monotonously increasing and } \lim_{|x|\to\infty} \phi(|x|) = \infty. \tag{1.7}$$

On the other hand (see for instance the results in [13]), (1.1) has a global solution provided

$$(1+|x|)^{1+\kappa}\Big(\frac{|f(x)|}{1+|x|} + \sum_{0<|\alpha|\leq[n/2]+2} |\nabla_x^\alpha f(x)| + \sum_{|\beta|\leq[n/2]+1} |\nabla_x^\beta g(x)|\Big), \tag{1.8}$$

is sufficiently small, $\kappa \geq \kappa_0$ and $p > p_0(n)$, where $n = 2, 3$. In high odd space dimensions $n = 2m+1$, $m \geq 2$, Kubo's results [9] shows that the radially symmetric version of (1.1) has a global solution, provided

$$\sum_{j=0}^{2} |f^{(j)}(r)|\langle r\rangle^{\kappa+j} + \sum_{j=0}^{1} |g^{(j)}(r)|\langle r\rangle^{1+\kappa+j}, \tag{1.9}$$

is sufficiently small, where $\langle r\rangle = \sqrt{1+r^2}$ and $r = |x|$. A similar result was obtained by Kubo and Kubota [11] in the case of even space dimensions $n = 2m$, $m \geq 2$, but under a more stringent condition than (1.9) near $r = 0$. We note that the similar result for the equation with the potential has obtained by Karageorgis [8].

When $n = 3$, Asakura [2] was the first to prove the nonexistence result under the validity of (1.4) or (1.5). In addition, Asakura [2] resolved the existence part under assumption (1.8). The critical case ($\kappa = \kappa_0$) and $n = 3$ was handled by Kubota [13] with the assumption (1.8), and also independently by Tsutaya [25]. For $n = 2$, the nonexistence part with (1.4), (1.5) was verified by Agemi and Takamura [1], and the existence part was verified by Kubota [13], and both parts by Tsutaya [23, 24]. We note here that all of the existence results in [2], [13], [23] and [24] were proven under the validity of (1.8). Also, see the related work of Kubota and Mochizuki [12] for $n = 2$. In higher space dimensions and for radial solutions, the existence part with (1.9) was handled by Kubo [9], Kubo and Kubota [10, 11], and the nonexistence part with (1.4) and (1.5) was verified by Takamura [19]. Another relevant nonexistence result in high dimensions $n \geq 2$ is due to Kurokawa and Takamura [14]. In view of (1.9), we note that the final form of (1.8) will be

$$\sum_{|\alpha|\leq[n/2]+2}(1+|x|)^{\kappa+|\alpha|}|\nabla_x^\alpha f(x)| + \sum_{|\beta|\leq[n/2]+1}(1+|x|)^{\kappa+1+|\beta|}|\nabla_x^\beta g(x)|. \quad (1.10)$$

All of the cited results on non-compactly supported initial data are summarized in the following tables.

Global existence	$\kappa = \kappa_0$	$\kappa > \kappa_0$
$n = 2$	[13], [23] independently	[13], [24] independently
$n = 3$	[13], [25] independently	[2]
$n \geq 4$	[9]	[10] and [11]

Blow-up	(1.4)	(1.5)
(1.6)	[1], [24] independently for $n = 2$ [2] for $n = 3$ [19] for $n \geq 4$	[20]
(1.7)	[14]	[21]

Thus far, all of the cited nonexistence results above were proven with zero initial position, except for Takamura, Uesaka and Wakasa [20, 21] who proved a nonexistence result under the nonzero initial position with the assumption (1.5) by differentiating (1.1) with respect to time.

In this paper, we prove a blow-up result with sharp decay for $f \not\equiv 0$ and $g \not\equiv 0$. The main goal of this work is to obtain the required point-wise lower bounds for the *free* solution of the wave equations by making full use of the formulas by Rammaha [15, 16] in high dimensions. In low space dimensions, one can obtain such lower bounds solutions as in Caffarelli and Friedman [3]. However, it is highly nontrivial to obtain the mentioned lower bounds for the free solution when both initial data are non-zero, particularly in high even dimensions. We overcome the main technical difficulty in high even dimensions by introducing a special change of variables given in (5.7).

2. Main results

In high space dimensions $n \geq 4$, we restrict our analysis to radial solutions. More precisely, we consider the following radially symmetric version of (1.1):

$$\begin{cases} u_{tt} - \dfrac{n-1}{r}u_r - u_{rr} = F(u), \text{ in } \ (0,\infty)\times[0,\infty), \\ u(r,0) = f(r), \ u_t(r,0) = g(r), \text{ in } \ (0,\infty). \end{cases} \tag{2.1}$$

Henceforth, our assumptions (see Assumption 2.1 below) in high dimensions $n \geq 4$ are in reference of the Cauchy problem (2.1).

In order to state our main results, we begin with the assumptions on the initial data and the parameters.

Assumption 2.1.

- ***The nonlinearity:*** $F \in C^1(\mathbb{R})$ *satisfying*

$$F(s) \geq As^p, \quad \textit{for} \ \ s \geq 0, \tag{2.2}$$

where $p > 1$ *and* $A > 0$.

- ***Low space dimensions,*** $n = 2, 3$***:*** *There exists a constant* $R > 0$ *such that the initial data* $f \in C^3(\mathbb{R}^n)$ *and* $g \in C^2(\mathbb{R}^n)$ *satisfying:*

$$\begin{cases} f(x) > 0 \quad \textit{for} \ \ |x| \geq R, \\ \dfrac{f(x)}{1+|x|} - |\nabla f(x)| + g(x) \geq \dfrac{C_0}{(1+|x|)^{1+\kappa}} \quad \textit{for} \ \ |x| \geq R, \end{cases} \tag{2.3}$$

with some positive constant C_0 *and* κ.

- ***High space dimensions,*** $n \geq 4$***:*** *There exists a constant $R > 0$ such that $f \in C^3(\mathbb{R}^n)$ and $g \in C^2(\mathbb{R}^n)$ satisfying:*

$$\begin{cases} f(r) \geq \dfrac{C_1}{(1+r)^\kappa}, \ g(r) > 0 \\ -C_{1,m}\dfrac{f(r)}{r} + g(r) > 0 \quad \text{for} \quad r \geq R, \end{cases} \tag{2.4}$$

for $n = 2m+1$, or

$$\begin{cases} f(r), g(r) > 0 \\ -C_{1,m}\dfrac{f(r)}{r} + g(r) \geq \dfrac{C_2}{(1+r)^{1+\kappa}} \quad \text{for} \quad r \geq R, \end{cases} \tag{2.5}$$

for $n = 2m+1$, and

$$\begin{cases} f(r), g(r) > 0 \\ -C_{2,m}\dfrac{f(r)}{r} - |f'(r)| + \dfrac{1}{2}g(r) \geq \dfrac{C_3}{(1+r)^{1+\kappa}} \quad \text{for} \quad r \geq R, \end{cases} \tag{2.6}$$

for $n = 2m$, where $m = 2, 3, \cdots$, C_1, C_2 and C_3 are positive constants. The constants $C_{1,m}$ and $C_{2,m}$ are given by

$$C_{1,m} = m(m-1), \quad C_{2,m} = m - \frac{3}{8} + \frac{5\zeta_m(m-1)^2}{3},$$

where $\zeta_m > 0$ is as determined in Lemma 5.1.

- ***Parameters:*** $0 < \kappa < \kappa_0$, *where* $\kappa_0 = \frac{2}{p-1}$.

Our first result is on the finite-time blow up of classical solutions in low dimensions, without imposing radial symmetry.

Theorem 2.2. *Assume the validity of Assumption* 2.1 *with $n = 2$ or $n = 3$, and u is a solution of* (1.1). *Then u cannot exist globally in time.*

Our second result addresses the finite-time blow up of radial solutions to the Cauchy problem (2.1).

Theorem 2.3. *Assume the validity of Assumption* 2.1 *with $n \geq 4$, and u is a solution of* (2.1). *Then u cannot exist globally in time.*

Remark 2.1. Let us note here that our assumption on the initial data in (2.4), (2.5) and (2.6) are fairly reasonable in view of the slowly decaying initial data (see for instance (1.9) or remark 2.1 in [20]). In fact, there is a large family of the slowly decaying initial data that satisfies the general conditions in Assumption 2.1.

The paper is organized as follows. In the next section, we illustrate our iteration schemes, which are sufficient to prove Theorem 2.2 and Theorem 2.3. Section 4 is devoted to the treatment of high odd dimensions. In Section 5, we derive the required lower bound in high even dimensions, which is the more technical part of the paper. Finally, Section 6 gives a brief treatment of the low dimensions $n = 2, 3$.

3. ITERATION SCHEME

In this section, we introduce our iteration scheme that allows us to prove the Theorem 2.2 and Theorem 2.3, following the well-known arguments in [7] or [19]. Throughout the paper, we define δ (which depends of the space dimensions n) by:

$$\delta := \max\left\{\frac{2}{\eta_m},\ \frac{2}{\zeta_m}\right\}, \tag{3.1}$$

where $\eta_m,\ \zeta_m > 0$ are given below in Lemma 4.1 and Lemma 5.1; respectively.

Lemma 3.1. *Let u be a solution of* (2.1) *where $n = 2m+1$ or $n = 2m$, $m = 2, 3, 4\cdots$. Then, with the validity of Assumption* 2.1, *we have*

$$u(r,t) \geq \frac{C_4 t}{(1+r+t)^{1+\kappa}} + \frac{1}{8r^m}\int_0^t d\tau \int_{r-t+\tau}^{r+t-\tau} \lambda^m F(u(\lambda,\tau))d\lambda, \tag{3.2}$$

for all $(r,t) \in \Sigma_1$, where

$$\Sigma_1 := \left\{(r,t) \in (0,\infty)^2\ :\ r - t \geq \max\{R,\ \delta t\} > 0\right\}, \tag{3.3}$$

and C_4 is a positive constant.

Lemma 3.2. *Let u be a solution of* (1.1) *with $n = 2$ or $n = 3$, and Assumption* 2.1 *is valid. Then we have*

$$u(x,t) \geq \frac{C_5 t}{(1+|x|+t)^{1+\kappa}} + \int_0^t R(F(u(\cdot,\tau))|x, t-\tau)d\tau, \tag{3.4}$$

for all $(x,t) \in \Sigma_2$, where C_5 is a positive constant,

$$R(\phi|x,t) := \begin{cases} \dfrac{t}{4\pi}\displaystyle\int_{|\omega|=1} \phi(x+t\omega)dS_\omega & \text{for } n = 3, \\ \dfrac{1}{2\pi}\displaystyle\int_0^t \frac{\rho d\rho}{\sqrt{t^2-\rho^2}}\int_{|\omega|=1} \phi(x+\rho\omega)dS_\omega & \text{for } n = 2, \end{cases} \tag{3.5}$$

and

$$\Sigma_2 := \{(x,t) \in \mathbf{R}^n \times (0,\infty)\ :\ |x| - t \geq \max\{R, t-1\}\}. \tag{3.6}$$

As we mentioned earlier, by appealing to iteration arguments in [7] or [19] along with Lemma 3.1 and Lemma 3.2, one can prove the Theorems 2.2 and Theorems 2.3.

The proofs of the above lemmas are provided below. First, let u^0 denotes the free solution of the wave equation. More precisely, u^0 is the solution of the Cauchy problem:

$$\begin{cases} u^0_{tt} - \Delta u^0 = 0, \text{ in } \mathbb{R}^n \times [0,\infty), \\ u^0(x,0) = f(x), \ u^0_t(x,0) = g(x), \text{ in } \mathbb{R}^n, \end{cases} \tag{3.7}$$

if $n = 2$ or $n = 3$ (no radial symmetry is assumed), and for $n \geq 4$, u^0 is the solution of the following radially symmetric version of (3.7):

$$\begin{cases} u^0_{tt} - \dfrac{n-1}{r} u^0_r - u^0_{rr} = 0, \text{ in } (0,\infty) \times [0,\infty), \\ u^0(r,0) = f(r), \ u^0_t(r,0) = g(r), \text{ in } (0,\infty). \end{cases} \tag{3.8}$$

Then, we have the following results.

Proposition 3.3. *Let u^0 be the solution of (3.8) with $n = 2m+1$ or $n = 2m$, $m = 2, 3, 4, \cdots$. With the validity of Assumption 2.1, then u^0 satisfies:*

$$\begin{aligned} u^0(r,t) \geq & \frac{1}{2r^m}\left\{ f(r+t)(r+t)^m + f(r-t)(r-t)^m \right\} \\ & + \frac{1}{4r^m}\int_{r-t}^{r+t} \lambda^m \left(-C_{1,m}\frac{f(\lambda)}{\lambda} + g(\lambda) \right) d\lambda, \end{aligned} \tag{3.9}$$

for all $(r,t) \in \Sigma_1$, if $n = 2m+1$, and

$$\begin{aligned} u^0(r,t) \geq & \frac{1}{\pi r^{m-1}} \int_0^t \frac{\rho d\eta}{\sqrt{t^2-\rho^2}} \int_{r-\rho}^{r+\rho} \left\{ -2C_{2,m}\frac{f(\lambda)}{\lambda} - 2|f'(\lambda)| + g(\lambda) \right\}, \\ & \times \frac{\lambda^m d\lambda}{\sqrt{\lambda^2 - (r-\rho)^2}\sqrt{(r+\rho)^2 - \lambda^2}} \end{aligned} \tag{3.10}$$

for all $(r,t) \in \Sigma_1$, if $n = 2m$.

Proposition 3.4. *Let $n = 2$ or $n = 3$, u^0 be the solution of (3.7) and Assumption 2.1 is valid. Then, u^0 satisfies:*

$$u^0(x,t) \geq \frac{t}{4\pi}\int_{|\omega|=1} \left\{ \frac{f(x+t\omega)}{1+|x+t\omega|} - |\nabla f(x+t\omega)| + g(x+t\omega) \right\} dS_\omega \tag{3.11}$$

for all $(x,t) \in \Sigma_2$, *if* $n=3$,

$$u^0(x,t) \geq \frac{1}{2\pi}\int_0^t \frac{\rho d\rho}{\sqrt{t^2-\rho^2}} \times \int_{|\omega|=1}\left\{\frac{f(x+\rho\omega)}{1+|x+\rho\omega|} - |\nabla f(x+\rho\omega)| + g(x+\rho\omega)\right\} dS_\omega, \tag{3.12}$$

for all $(x,t) \in \Sigma_2$, *if* $n=2$.

4. High Odd Dimensions: $n = 5, 7, 9, \cdots$

Proof of proposition 3.3 in $n = 2m+1$, $m = 2, 3, 4, \cdots$. According to formula (6a) in [15], we have

$$\begin{aligned} u^0(r,t) = \; & \frac{\partial}{\partial t}\left\{\frac{1}{2r^m}\int_{|r-t|}^{r+t} \lambda^m f(\lambda) P_{m-1}\left(\Theta(\lambda,r,t)\right) d\lambda\right\} \\ & + \frac{1}{2r^m}\int_{|r-t|}^{r+t} \lambda^m g(\lambda) P_{m-1}\left(\Theta(\lambda,r,t)\right) d\lambda, \end{aligned} \tag{4.1}$$

where P_k denotes Legendre polynomials of degree k defined by

$$P_k(z) := \frac{1}{2^k k!}\frac{d^k}{dz^k}(z^2-1)^k, \tag{4.2}$$

and $\Theta = \Theta(\lambda, r, t)$ is given by

$$\Theta(\lambda,r,t) = \frac{\lambda^2 + r^2 - t^2}{2r\lambda}. \tag{4.3}$$

The following auxiliary lemma will be needed in the derivation of the required estimate in this case.

Lemma 4.1. *For* $m = 2, 3, 4, \cdots$, *there exists a positive constant* η_m, *depending only on* m, *such that*

$$P_{m-1}(z) \geq \tfrac{1}{2} \;\; \text{and} \quad 0 < P'_{m-1}(z) \leq \tfrac{1}{2}m(m-1), \;\; \text{for} \; \tfrac{1}{1+\eta_m} \leq z \leq 1. \tag{4.4}$$

Proof. Let us first consider the case of $m=2$. Then, we easily obtain (4.4) by putting $\eta_m = 1$, since $P_1(z) = z$. Now, suppose that $m \geq 3$. Then, by direct computations, we have the following properties of P_k:

$$P'_{m-1}(1) = \tfrac{1}{2}m(m-1) > 0, \tag{4.5}$$

and

$$P''_{m-1}(1) = \frac{1}{4}(m-1)(m-2)\binom{m+1}{m-1} > 0. \tag{4.6}$$

Since $P_1(z) = z$ and $P_m(1) = 1$, then it follows from (4.5), (4.6) and the continuity of $P_{m-1}(z)$, $P'_{m-1}(z)$ and $P''_{m-1}(z)$ that there exists a $\eta_m > 0$ such that (4.4) is valid. □

To use the lemma 4.1 with Θ which is a variable of P_{m-1} or P'_{m-1} in (4.1), we need following lemma.

Lemma 4.2. *Let Θ be the function defined by* (4.3). *Then, Θ satisfies*

$$\Theta(\lambda, r, t) \geq \frac{\delta}{\delta + 2} \quad \textit{for} \quad r - t \leq \lambda \leq r + t \tag{4.7}$$

provided $(r, t) \in \Sigma_1$.

Proof. Its easy to see that

$$\Theta(\lambda, r, t) \geq \frac{(r-t)^2 + r^2 - t^2}{2r(r+t)} = \frac{r-t}{r+t} \geq \frac{\delta}{\delta + 2}$$

for $r - t \leq \lambda \leq r + t$ and $(r, t) \in \Sigma_1$. □

Let us first note that (4.1) yields:

$$\begin{aligned} u^0(r,t) = \quad & \frac{1}{2r^m}\{f(r+t)(r+t)^m + f(r-t)(r-t)^m\} \\ & + \frac{1}{2r^m}\int_{r-t}^{r+t} \lambda^m f(\lambda) P'_{m-1}(\Theta(\lambda, r, t)) \left(-\frac{t}{r\lambda}\right) d\lambda \\ & + \frac{1}{2r^m}\int_{r-t}^{r+t} \lambda^m g(\lambda) P_{m-1}(\Theta(\lambda, r, t))\, d\lambda. \end{aligned} \tag{4.8}$$

Thanks to Lemma 4.1, Lemma 4.2, and the assumption (2.4) or (2.5), then (3.9) holds, for all $(r, t) \in \Sigma_1$. Hence, the proof of the proposition 3.3 in odd space dimension $n = 2m + 1,\ m = 2, 3, 4, \cdots$ is complete. □

Completion of the Proof of Lemma 3.1. Let us note here that the first term in (3.2) of Lemma 3.1 is obtained as follows. Since we may assume that $\eta_m \leq 1$ for all $m = 2, 3, 4, \cdots$, we have $\delta \geq 2$ so that $r \geq 3t$ holds for $(r, t) \in \Sigma_1$. Thanks to (2.4), then (3.9) yields

$$u^0(r,t) \geq \frac{C_1}{2(1+r+t)^\kappa}\left\{1 + \left(\frac{r-t}{r}\right)^m\right\} \tag{4.9}$$

$$\geq \frac{C_1}{2}\left\{1 + (\tfrac{2}{3})^m\right\}\frac{t}{(1+r+t)^{1+\kappa}} \tag{4.10}$$

for all $(r, t) \in \Sigma_1$. Hence, the first term in (3.2) valid for $C_4 = C_1 2^{-1}\{1 + (\frac{3}{2})^m\}$. Next, we shall show by using the assumption (2.5). Then, (3.9) yields

$$u^0(r,t) \geq \frac{C_2}{4r^m} \int_{r-t}^{r+t} \lambda^m (1+\lambda)^{-\kappa-1} d\lambda \geq \frac{C_2 t}{4(1+r+t)^{\kappa+1}}. \tag{4.11}$$

for all $(r,t) \in \Sigma_1$. Thus, the first term in (3.2) valid for $C_4 = C_2/4$. □

5. HIGH EVEN DIMENSIONS: $n = 4, 6, 8, \ldots$

Proof of proposition 3.3 in $n = 2m$, $m = 2, 3, 4, \cdots$. According to formula (6b) in [15], we have

$$u^0(r,t) = \frac{\partial}{\partial t} \frac{2}{\pi r^{m-1}} I(r,t,u^0(\cdot,0)) + \frac{2}{\pi r^{m-1}} I(r,t,u_t^0(\cdot,0)), \tag{5.1}$$

where,

$$I(r,t,\psi(\cdot)) = \int_0^t \frac{\rho d\rho}{\sqrt{t^2-\rho^2}} \int_{|r-\rho|}^{r+\rho} \frac{\lambda^m \psi(\lambda) T_{m-1}\left(\Theta(\lambda,r,\rho)\right) d\lambda}{\sqrt{\lambda^2-(r-\rho)^2}\sqrt{(r+\rho)^2-\lambda^2}}, \tag{5.2}$$

and as usual, in (5.2) T_k denotes Tschebyscheff polynomials of degree k defined by

$$T_k(z) := \frac{(-1)^k}{(2k-1)!!}(1-z^2)^{1/2} \frac{d^k}{dz^k}(1-z^2)^{k-(1/2)}. \tag{5.3}$$

The following auxiliary lemma will be needed.

Lemma 5.1. *For $m = 2, 3, 4, \cdots$, there exists a positive constant ζ_m, depending only on m, such that*

$$\frac{1}{2} \leq T_{m-1}(z) \leq 1, \quad 0 < T'_{m-1}(z) \leq (m-1)^2, \tag{5.4}$$

for all $\dfrac{1}{1+\zeta_m} \leq z \leq 1$.

Proof. Let us first consider the case of $m = 2$. Then, we easily obtain (5.4) since $T_1(z) = z$ and we may take $\zeta_m = 1$. Now, let $m \geq 3$. Since $T_{m-1}(1) = 1$, then the first assertion is trivial as long as $\zeta_m > 0$ is sufficiently small. For $m \geq 2$, we recall that the Tchebysheff polynomial $T_{m-1}(z)$ satisfies the ODE:

$$(1-z^2)T''_{m-1}(z) - zT'_{m-1}(z) + (m-1)^2 T_{m-1}(z) = 0 \quad \text{for} \quad |z| \leq 1. \tag{5.5}$$

Thus, (5.5) yields

$$\begin{cases} T'_{m-1}(1) = (m-1)^2, \\ T''_{m-1}(1) = \frac{1}{3}m(m-2)(m-1)^2, \end{cases} \tag{5.6}$$

for $m \geq 3$. Hence, the second assertion of the lemma follows from continuity $T'_{m-1}(z)$ and $T''_{m-1}(z)$. $\square$

In order to obtain the desired lower bound in high even dimensions, we shall use the following change variables in (5.2): For $(r,t) \in \Sigma_1$, we introduce:

$$\begin{cases} \xi = \dfrac{r+\rho-\lambda}{2\rho} & \text{in the } \lambda\text{-integral}, \\ \rho = t\eta & \text{in the } \rho\text{-integral}. \end{cases} \tag{5.7}$$

Then, with this change of variables then (5.2) reduces to:

$$\begin{aligned} & I(r,t,\psi(\cdot)) \\ & = \frac{t}{2}\int_0^1 \frac{\eta d\eta}{\sqrt{1-\eta^2}} \int_0^1 \frac{K(r,t,\eta,\xi)\psi(r+t\eta-2t\eta\xi)}{\sqrt{\xi}\sqrt{1-\xi}} T_{m-1}\left(\Xi(r,t,\eta,\xi)\right) d\xi. \end{aligned} \tag{5.8}$$

In addition,

$$\begin{aligned} & \frac{\partial}{\partial t} I(r,t,\psi(\cdot)) \\ & = \frac{1}{2}\int_0^1 \frac{\eta d\eta}{\sqrt{1-\eta^2}} \int_0^1 \Big\{ K(r,t,\eta,\xi)\psi(r+t\eta-2t\eta\xi)T_{m-1}(\Xi(r,t,\eta,\xi)) \\ & + t\frac{\partial}{\partial t}\left\{K(r,t,\eta,\xi)\psi(r+t\eta-2t\eta\xi)T_{m-1}(\Xi(r,t,\eta,\xi))\right\}\Big\} \frac{d\xi}{\sqrt{\xi}\sqrt{1-\xi}}, \end{aligned} \tag{5.9}$$

where

$$K(r,t,\eta,\xi) = \frac{(r+t\eta-2t\eta\xi)^m}{\sqrt{r+t\eta-t\eta\xi}\sqrt{r-\xi t\eta}}, \tag{5.10}$$

and

$$\Xi(r,t,\eta,\xi) := \Theta\left(r+t\eta-2t\eta\xi, r, t\eta\right) = \frac{(r+t\eta-2t\eta\xi)^2+r^2-t^2\eta^2}{2r(r+t\eta-2t\eta\xi)}. \tag{5.11}$$

The following proposition is crucial to the rest of the proof.

Proposition 5.2. *Let $m = 2,3,4\cdots$. Assume that $w \in C^1((0,\infty))$ and $w(y) > 0$ for $y \geq R$, where R is as given in (2.6). Then, for $0 \leq \xi, \eta \leq 1$ and $(r,t) \in \Sigma_1$, we have*

$$\begin{aligned} & \frac{\partial}{\partial t}\left\{K(r,t,\eta,\xi)w(r+t\eta-2t\eta\xi)T_{m-1}(\Xi(r,t,\eta,\xi))\right\} \\ & \quad \geq -\left\{E_m \frac{w(r+t\eta-2t\eta\xi)}{r+t\eta-2t\eta\xi} + |w'(r+t\eta-2t\eta\xi)|\right\} K(r,t,\eta,\xi), \end{aligned} \tag{5.12}$$

where E_m is defined by

$$E_m = m + \frac{1}{8} + \frac{5\zeta_m(m-1)^2}{3}.$$

Proof. By direct computation, we obtain:

$$\begin{aligned}&\frac{\partial}{\partial t}\Big\{K(r,t,\eta,\xi)w(r+t\eta-2t\eta\xi)T_{m-1}(\Xi(r,t,\eta,\xi))\Big\}\\ &= K(r,t,\eta,\xi)\Big\{\eta\left\{I_1+I_2+I_3+I_4\right\}T_{m-1}(\Xi(r,t,\eta,\xi))+I_5\Big\},\end{aligned}$$

where,

$$I_1 = m(1-2\xi)\frac{w(r+t\eta-2t\eta\xi)}{r+t\eta-2t\eta\xi}, \quad I_2 = (1-2\xi)w'(r+t\eta-2t\eta\xi), \tag{5.13}$$

$$I_3 = -\frac{1}{2}(1-\xi)\frac{w(r+t\eta-2t\eta\xi)}{r+t\eta-t\eta\xi}, \quad I_4 = \frac{\xi}{2}\frac{w(r+t\eta-2t\eta\xi)}{r-\xi t\eta},$$

$$I_5 = w(r+t\eta-2t\eta\xi)T'_{m-1}(\Xi(r,t,\eta,\xi))\frac{\partial}{\partial t}\Xi(r,t,\eta,\xi).$$

Estimates for terms involving $\mathbf{I_1}, \cdots, \mathbf{I_4}$. It follows from Lemma 5.1 and Lemma 4.2 with $\lambda = r+t\eta-2t\eta\xi$ and $t = t\eta$ for $0 \le \eta, \xi \le 1$ and $(r,t) \in \Sigma_1$ that

$$I_1 \cdot T_{m-1}(\Xi) \ge \frac{m(1-2\xi)}{2}\frac{w(r+t\eta-2t\eta\xi)}{r+t\eta-2t\eta\xi} \quad \text{for } 0 \le \xi \le \frac{1}{2} \text{ and } 0 \le \eta \le 1,$$

$$I_1 \cdot T_{m-1}(\Xi) \ge m(1-2\xi)\frac{w(r+t\eta-2t\eta\xi)}{r+t\eta-2t\eta\xi} \quad \text{for } \frac{1}{2} \le \xi \le 1 \text{ and } 0 \le \eta \le 1,$$

$$I_3 \cdot T_{m-1}(\Xi) \ge -\frac{(1-\xi)}{2}\frac{w(r+t\eta-2t\eta\xi)}{r+t\eta-2t\eta\xi} \quad \text{for } 0 \le \xi \le 1 \text{ and } 0 \le \eta \le 1,$$

$$I_4 \cdot T_{m-1}(\Xi) \ge \frac{\xi}{4}\frac{w(r+t\eta-2t\eta\xi)}{r+t\eta-2t\eta\xi} \quad \text{for } 0 \le \xi \le 1 \text{ and } 0 \le \eta \le 1.$$

Then, we have

$$\begin{aligned}&\eta(I_1+I_2+I_3+I_4)T_{m-1}(\Xi) \\ &\ge \eta\left\{\frac{m-1}{2}+\frac{(3-4m)\xi}{4}\right\}\frac{w(r+t\eta-2t\eta\xi)}{r+t\eta-2t\eta\xi} - |w'(r+t\eta-2t\eta\xi)| \\ &\ge -\frac{1}{8}\frac{w(r+t\eta-2t\eta\xi)}{r+t\eta-2t\eta\xi} - |w'(r+t\eta-2t\eta\xi)|,\end{aligned} \tag{5.14}$$

for $0 \le \xi \le \frac{1}{2}$ and $0 \le \eta \le 1$.

Similarly, for $\frac{1}{2} \le \xi \le 1$, $0 \le \eta \le 1$ and $(r,t) \in \Sigma_1$, we have

$$\eta(I_1+I_2+I_3+I_4)T_{m-1}(\Xi) \tag{5.15}$$

$$\geq \eta \left\{(1-2\xi)m + \tfrac{3}{4}\xi - \tfrac{1}{2}\right\} \frac{w(r+t\eta-2t\eta\xi)}{r+\eta-2t\eta\xi} - |w'(r+t\eta-2\eta\xi)|$$
$$\geq -\left(m+\tfrac{1}{8}\right) \frac{w(r+t\eta-2t\eta\xi)}{r+\eta-2t\eta\xi} - |w'(r+t\eta-2\eta\xi)|.$$

Estimates for the term involving $\mathbf{I_5}$. In order to obtain the proper estimate for this term, we first aim to prove the following property:

For $0 \leq \xi, \eta \leq 1$, $(r,t) \in \Sigma_1$ and $m \geq 2$, we have

$$-\frac{5\zeta_m}{3(r+t\eta-2t\eta\xi)} \leq \frac{\partial}{\partial t}\Xi(r,t,\eta,\xi) \leq 0. \tag{5.16}$$

Indeed, direct computation shows

$$\frac{\partial}{\partial t}\Xi(r,t,\eta,\xi) = \frac{\eta N(r,t,\eta,\xi)}{2r(r+t\eta-2t\eta\xi)^2}, \tag{5.17}$$

where

$$\begin{aligned} N(r,t,\eta,\xi) = &\left\{2(r+t\eta-2t\eta\xi)(1-2\xi)-2t\eta\right\}(r+t\eta-2t\eta\xi) \\ &- \left\{(r+t\eta-2t\eta\xi)^2+r^2-t^2\eta^2\right\}(1-2\xi). \end{aligned}$$

However, a straightforward computation yields

$$\begin{aligned} N(r,t,\eta,\xi) &= -8t^2\eta^2\xi^3 + (12t^2\eta^2+8rt\eta)\xi^2 - (8rt\eta+4t^2\eta^2)\xi \\ &= -4t\eta\xi(\xi-1)(2t\eta\xi-(2r+t\eta)). \end{aligned} \tag{5.18}$$

Since $\frac{2r+t\eta}{2t\eta} > 1$, for $(r,t) \in \Sigma_1$, then for each fixed η it follows from (5.17) and (5.18) that $\frac{\partial}{\partial t}\Xi(r,t,\eta,\xi) \leq 0$.

In order to prove the lower bound for (5.16), we compute the minimum value of $N(r,t,\eta,\xi)$ as a function of $\xi \in [0,1]$; but for fixed η. Indeed,

$$\frac{\partial}{\partial \xi}N(r,t,\eta,\xi) = -24t^2\eta^2\xi^2 + 4t\eta(6t\eta+4r)\xi - 4t\eta(2r+t\eta) = 0,$$

if and only if,

$$\xi = \frac{(3t\eta+2r) \pm \sqrt{3t^2\eta^2+4r^2}}{6t\eta}.$$

Put

$$\xi_+ = \frac{(3t\eta+2r) + \sqrt{3t^2\eta^2+4r^2}}{6t\eta}, \quad \xi_- = \frac{(3t\eta+2r) - \sqrt{3t^2\eta^2+4r^2}}{6t\eta}.$$

Obviously, we have $\xi_+ > 1$ and $0 < \xi_- < 1$, for all $(r,t) \in \Sigma_1$ and fixed η. Therefore,

$$N(r,t,\eta,\xi_-) = \frac{1}{27t\eta}(3t\eta + 2r - \sqrt{3t^2\eta^2 + 4r^2})(\sqrt{3t^2\eta^2+4r^2} + 3t\eta - 2r) \times(-4r - \sqrt{3t^2\eta^2+4r^2}).$$

Here, it is important to note that the following inequalities hold:

$$\frac{3t\eta + 2r - \sqrt{3t^2\eta^2+4r^2}}{t\eta} \leq 3 + \frac{2r}{t\eta} - \frac{2r}{t\eta} = 3,$$

$$\sqrt{3t^2\eta^2+4r^2} + 3t\eta - 2r \leq \sqrt{4t^2\eta^2 + 8rt\eta + 4r^2} + 3t\eta - 2r \leq 5t\eta,$$

$$-4r - \sqrt{3\tau^2\eta^2+4r^2} \geq -4r - 2(r+t\eta) \geq -6(r+t\eta).$$

for $t, \eta \geq 0$. Thus,

$$\frac{\partial}{\partial t}\Xi(r,t,\eta,\xi) \geq -\frac{5}{3r}\cdot\frac{t\eta(r+t\eta)}{(r+t\eta-2t\eta\xi)^2}.$$

for all $0 \leq \xi, \eta \leq 1$ and $(r,t) \in \Sigma_1$. Finally, we note that

$$r + t\eta \leq 2r \quad \text{and} \quad r + t\eta - 2t\eta\xi \geq r - t\eta \geq r - t \geq \frac{2}{\zeta_m}t,$$

hold for $0 \leq \xi, \eta \leq 1$ and $(r,t) \in \Sigma_1$. Hence, the lower bound of (5.16) follows.

By combining the estimates, (5.14), (5.15), (5.16) and (5.4), then (5.12) follows, completing the proof of proposition 5.2. □

Since we may assume that $\zeta_m \leq 1$ for all $m = 2,3,4,\cdots$, we have $\delta \geq 2$ so that $r \geq 2t$ holds for $(r,t) \in \Sigma_1$. We also note that

$$t \leq r - t\eta \leq r + (1-2\xi)t\eta \quad \text{for} \quad (r,t) \in \Sigma_1.$$

By using proposition 5.2, (5.4) and (2.6), then (5.1) implies

$$u^0(r,t) \geq \frac{t}{\pi r^{m-1}}\int_0^1 \frac{\eta d\eta}{\sqrt{1-\eta^2}}\int_0^1 \left\{\frac{f(r+t\eta-2t\eta\xi)}{2t} - E_m\frac{f(r+t\eta-2t\eta\xi)}{r+t\eta-2t\eta\xi} - |f'(r+t\eta-2t\eta\xi)| + \frac{g(r+t\eta-2t\eta\xi)}{2}\right\}\frac{K(r,t,\eta,\xi)d\xi}{\sqrt{\xi}\sqrt{1-\xi}}, \quad (5.19)$$

in Σ_1. Thus, by returning to the original variables (5.7), then (3.10) follows. The proof of the proposition 3.3 for $n = 2m,\ m = 2,3,4,\cdots$ is now complete. □

Completion of the Proof of Lemma 3.1. Finally, we shall derive the first term in (3.2). It follows from (2.6) and (3.10) that

$$\begin{aligned} u^0(r,t) &\geq \frac{t}{\pi r^{m-1}}\int_0^1 \frac{\eta d\eta}{\sqrt{1-\eta^2}}\int_0^1 \frac{K(r,t,\eta,\xi)}{(1+r+t\eta-2t\eta\xi)^{\kappa+1}}\frac{d\xi}{\sqrt{\xi}\sqrt{1-\xi}} \\ &\geq \frac{C_3 t}{\sqrt{2}\pi(1+r+t)^{\kappa+1}}\int_0^1 \frac{\eta d\eta}{\sqrt{1-\eta^2}}\int_0^{1/2}\frac{d\xi}{\sqrt{\xi}\sqrt{1-\xi}} \\ &\geq \frac{C_3 t}{2\pi(1+r+t)^{\kappa+1}}, \end{aligned} \tag{5.20}$$

in Σ_1. □

6. Low dimensions: $n = 2, 3$

Proofs of Proposition 3.4 and Lemma 3.2. Let u^0 be the solution of (3.7). Then, u^0 is given by:

$$u^0(x,t) = \partial_t R(f|x,t) + R(g|x,t), \tag{6.1}$$

where R is as defined in (3.5).

First, we consider the case $n = 3$. By using (2.3), it follows from (6.1) that

$$\begin{aligned} u^0(x,t) &= \frac{1}{4\pi}\int_{|\omega|=1}\{f + t\omega\cdot\nabla f + tg)\}(x+t\omega)dS_\omega \\ &\geq \frac{t}{4\pi}\int_{|\omega|=1}\left\{\frac{f}{t} - |\nabla f| + g)\right\}(x+t\omega)dS_\omega, \end{aligned} \tag{6.2}$$

for all $(x,t) \in \Sigma_2$. Then, (3.11) follows by noting $t \leq 1 + |x + t\omega|$, for all $(x,t) \in \Sigma_2$. Furthermore, we easily obtain the first term in (3.4) by substituting (2.3) into (3.11).

Next, we consider the case of $n = 2$. Here, we make the change variables: $\rho = t\xi$ in the ρ-integral of (3.5). Thus,

$$R(\phi|x,t) = \frac{t}{2\pi}\int_0^1 \frac{\xi d\xi}{\sqrt{1-\xi^2}}\int_{|\omega|=1}\phi(x+t\xi\omega)dS_\omega.$$

As in (6.2), we obtain

$$\begin{aligned} u^0(x,t) &= \frac{1}{2\pi}\int_0^1 \frac{\xi d\xi}{\sqrt{1-\xi^2}}\int_{|\omega|=1}\{f + t\xi\omega\cdot\nabla f + tg\}(x+t\xi\omega)dS_\omega \\ &\geq \frac{t}{2\pi}\int_0^1 \frac{\xi d\xi}{\sqrt{1-\xi^2}}\int_{|\omega|=1}\left\{\frac{f}{t} - |\nabla f| + g\right\}(x+t\xi\omega)dS_\omega, \end{aligned}$$

in Σ_2. Since $t \leq 1 + |x + t\xi\omega|$ for all $(x, t) \in \Sigma_2$, then (3.12) follows, after going back to the original variables. Furthermore, we easily obtain the first term in (3.4) as in the case of $n = 3$. Therefore, the proof of lemma 3.2 is complete. □

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