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Soliton in AdS$_5$×S$_5$ SSSSG Theory Including Grassmann odd Parameters

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"To my family"

Abstract

In this work, starting from the classical Type IIB Superstring in AdS$_5\times$ S$_5$ background theory, written as a 2d-Sigma model on $PSU(2,2|4)$ coset, we show its equivalence to SSSSG theory through the Pohlmeyer Reduction Procedure ([13],[9]). We explicitly construct a classical soliton solution for the generalized semi-symmetric space sine-Gorgon theory (SSSSG) in AdS$_5\times$ S$_5$ background ([11]) including 2 Grassmann odd parameters. Alongside this, we present the purely bosonic case based on S$_5$ symmetric space.

Contents

Introduction

The AdS/CFT correspondence relates the quantum physics of strongly correlated many-body systems to the classical dynamics of gravity in one higher dimension[17]. In its original formulation [13] , the correspondence related a four-dimensional Conformal Field Theory (CFT) to the geometry of an anti-de Sitter(AdS) space in five dimensions. In the study of collective phenomena in condensed matter physics it is quite common that when a system is strongly coupled it reorganizes itself in such a way that new weakly coupled degrees of freedom emerge dynamically and the system can be better described in terms of fields representing the emergent excitations. The holographic duality is a new example of this paradigm. The new feature is that the emergent Fields live in a space with one extra dimension and that the dual theory is a gravity theory.The gauge/gravity duality was discovered in the context of string theory, However, the study of the correspondence has been extended to include very different domains, such as the analysis of the strong coupling dynamics of QCD and the electro-weak theories, the physics of black holes and quantum gravity, relativistic hydrodynamics or different applications in condensed matter physics. Although understanding in detail the interpolation between weak and strong coupling remains an outstanding problem it offers the unique possibility to investigate a four dimensional interacting gauge theory beyond perturbation theory. At strong coupling, string theory becomes tractable in the semi-classical limit where one can study the energies of the corresponding classical string configurations.

Then the concept of integrability is fundamental in both side of correspondence in order to find exact solutions and study its excitation. In this work we will present the dressing method,a particular type of "inverse scattering method" also called Backlünd transformation elaborated to find non trivial solitonic solutions by transforming the vacuum solution of an non linear PDE. So, will be computed the dressing transformation called "dressing factor" which as we will see leads directly to one soliton solutions both of SSSG(Symmetric space sine-Gordon theory) and SSSSG(Semi-Symmetric sine-Gordon theory). A remarkable feature of SSSG and SSSSG theories, is the possibility to establish an equivalence to sting world sheet theories. In this specific case, respectively to world sheet theories on S_5 and $\mathrm{AdS}_5 \times S_5$ background. In this work we will show the equivalence of both 2d sigma model on S_5 symmetric space and 2d Sigma Model on $\mathrm{AdS}_5 \times S_5$ Semi-Symmetric space respectively to generalized sine-Gordon models SSSG and SSSSG using the so called "Pohlmeyer reduction", a method which provide a map between equations of motions of different models, fixing the gauge equivalence arises, in the present case, from the definition of the sigma model as a coset. Then, the soliton solutions are computing through the dressing method for SSSG and SSSSG, which can be mapped to solitons respectively of 2d sigma model on S_5 and 2d sigma model on $\mathrm{AdS}_5 \times \mathrm{S}_5$.

First chapter We present the string action as a 2d sigma model on S_5 Symmetryc space and the 2d Sigma model on $\mathrm{AdS}_5 \times \mathrm{S}_5$ Semi-symmetryc space as a coset.

Second chapter We reduce these two models by Pohlmeyer reduction to SSSG and SSSSG thoeries respectively.

Third chapter We review how to construct the single-soliton solutions in the case of SSSG theory. Finally we construct the soliton solution for SSSSG theory adding 2 fermionic fields.

We start with a brief introduction to AdS/CFT correspondence and to solitons in general.

0.1 Integrability of Classical Fields Theories

In this section we study the concept of integrability of classical field theories and their features:

- Infinite Tower of conserved charges
- Solitonic solutions

We will restrict ourselves to two-dimensional (non-linear) $1 + 1$ field theories $g(x, t)$. The phase space of field theories is infinite-dimensional and, thus, integrability requires infinitely many integrals of motion in involution. A starting point to define an integrable field theory is the concept of "Lax Pair". Suppose that we can find two matrices L, M such that the equations of motion can be written as:

$$\frac{\partial L}{\partial t} - \frac{\partial M}{\partial x} = [M, L] \tag{1}$$

or equivalently, as the " zero curvature" condition

$$[\partial_t - A_t, \partial_x - A_x] = F_{0,1} = 0 \tag{2}$$

where $A_x = L$ and $A_t = M$. Then, we will call such field theories classically integrable, and the pair of matrices $(L, M) = (A_x, A_t)$ is the Lax connection.

The first peculiar fact about integrable field theory is the existence of an infinite tower of conserved charges. The zero-curvature condition allows one to construct an infinite number of conserved quantities as follows. The zero-curvature condition (2) is the compatibility condition of the associated linear problem:

$$[\partial_\mu - A_\mu(\lambda)]\psi = 0 \tag{3}$$

Which determines the "wave function" $\psi = \psi(x,t;\lambda)$ up to a multiplication on the right by a constant matrix, which we can fix by requiring, for instance, $\psi(0,0;\lambda) = 1$. Choosing a path Γ between $(0;0)$ and $(x;t)$, the wave function can be written in terms of the Lax connection as:

$$\psi(x,t;\lambda) = P\exp[\int_\Gamma A_\mu dx^\mu \psi(0,0;\lambda)] \tag{4}$$

where "$P\exp$" denotes the Wilson line. Then the zero-curvature condition $F_{\mu\nu} = 0$ ensures that $\psi(x,r;\lambda)\psi^{-1}(0,0;\lambda)$ does not depend on the choice of Γ. Now in order to define the monodromy matrix, we have to specify the boundary conditions: $A_\mu(x + L,t;\lambda) = A_\mu(x,t;\lambda)$, in other words we are considering a two-dimensional field theory on a cylinder. Then, the monodromy matrix is defined by:

$$T(t;\lambda) = \psi(L,t;\lambda)\psi^{-1}(0,t;\lambda) = P\exp[\int_0^L A_x(x,t;\lambda)dx] \tag{5}$$

which satisfies

$$\partial_t T(t,\lambda) = A_0(L,t;\lambda)T(t;\lambda) - T(t;\lambda)A_0(0,t;\lambda) \tag{6}$$

Since $A_0(L,t;\lambda) = A_0(0,t;\lambda)$, this becomes:

$$\partial_t T(t;\lambda) = [A_0(0,t;\lambda), T(t;\lambda)] \tag{7}$$

which is a Lax equation. Therefore, the trace of $T(t;\lambda)$

$$t(\lambda) = Tr[T(t;\lambda)], \tag{8}$$

called the "transfer matrix", is conserved for all values of the spectral parameter λ. By expanding in λ, one obtains an infinite number of conserved quantities which are the coefficients of the expansion. For instance, if $t(\lambda)$ is analytic around $\lambda = 0$ one gets

$$t(\lambda) = \sum_{n>=0} \lambda^n Q_n, \quad \partial_t Q_n = 0. \tag{9}$$

it is useful to think that wilson's line independence of the path Γ (which is guaranteed by the zero-curvature condition) is due to the geometrical origin of the conserved quantities. For closed paths this implies that:

$$P\exp[\oint_\Gamma A_\mu dx^\mu] = 1 \tag{10}$$

consider the following closed path on the cylinder:

1. circle $t = t_0 \quad (\sigma, t_0), \quad 0 < \sigma < L$
2. vertical line $x - L$: $\quad (L, \tau), \quad t_0 < \tau < t$
3. circle $t = t$: $\quad (L - \sigma, t), \quad 0 < \sigma < L$
4. vertical line $x = 0$: $\quad (0, t_0 + t - \tau), \quad t_0 < \tau < t$

Then, the eq.(10) becomes:

$$T(t;\lambda) = P\exp[\int_0^L A_x(x,t;\lambda)dx] = V(L,t,t_0;\lambda)Pexp[\int_0^L A_x(x,t_0;\lambda)dx]V^{-1}(0,t,t_0;\lambda),$$
$$V(x,t,t_0;\lambda) = Pexp[\int_{t_0}^t A_t(x,\tau;\lambda)d\tau] \tag{11}$$

Assuming periodic boundary conditions, $V(L,t,t_0;\Lambda) = V(0,t,t_0;\lambda)$ this exhibits that the trace of the monodromy matrix is constant. Then we see that the existance of a Lax pair or equivalently a "zero-curvature condition" ,synonymous of integrability of a classical field theory, guarantee the existence of a tower of conserved charges.

0.2 Solitons

Solitons are very special physical phenomena discovered in 1834 by Johnn Scott Russel that observed the motion of some "solitary waves" generated by the motion of a boat. In particular he observed a surprising wave going in the opposite direction of the current of the Union Canal without loss of energy and without changing shape. Then he reproduced the phenomenon in a wave tank and called it the "Wave of Translation". His experiments led to establish qualitatively properties of those waves:

- The waves are stable, and can travel over very large distances despite the normal waves that would tend to either flatten out.
- The speed depends on the size of the wave, and its width on the depth of water
- Unlike normal waves they will never merge so a small wave is overtaken by a large one, rather than the two combining.
- If a wave is too big for the depth of water, it splits into two, one big and one small.

Clearly this is an informal description. His observations could not be reproduced using the current wave theories of Newton and Bernoulli so Russells observations were not welcomed by the whole scientific community. We have to wait until 1989 for a more precise classification of the properties of the solitons by Drazin & Johnson:

- They are of permanent form
- They are localized within a region.
- They can interact with other solitons, and emerge from the collision unchanged, except for a phase shift.

These is the minimum set of properties that a wave should have to be a soliton. But, how can we explain the very existence of these waves?

By the mathematical point of view a soliton is an exact solution of the equations of motion that is localized in space and preserves its shape over time. These field equations are non-linear PDE and, therefore, the properties of this type of solutions rely on a delicate balance between their linear (and thus dispersive) and non-linear terms. The first discovered model which exhibits soliton solutions is the Korteweg-de Vries equation:

$$u_t + u_{xxx} + uu_x = 0 \tag{12}$$

We can see it has two components, one dispersive and another non linear.

- Dispersive term
$$u_t + u_{xxx} = 0 \tag{13}$$
- Non linear term:
$$u_t + uu_x = 0 \tag{14}$$

When both the dispersive and the nonlinear term are present in the equation the two effects can neutralize each other. If the wave has a special shape, the effects are exactly counterbalanced and the wave rolls along undistorted. The soliton shape can be found by direct integration of the KdV equation.

A characteristic feature of many integrable Field theories is that they admit soliton solutions. If the theory is integrable, the existence of an infinite tower of local conserved charges constraints the interactions among solitons so much that they preserve their number and shape even after a collision process. All this suggests that we can interpret this solutions as particles that collide and scatter in a completely elastic way. Moreover, it turns out that the scattering of more than two solitons occurs in a sequence of pairwise interactions, and that it is independent of the sequence in which they scatter. This important feature is called "factorized scattering".

This expectation turns out to be true and, thus, knowledge of the classical soliton solutions provides information about quantum particle states in a systematic semi-classical expansion. Moreover, in general this information is non-perturbative since the corresponding classical solutions are usually themselves non-perturbative (they become singular when the non-linear coupling tends to zero).

The particle-like features of soliton solutions makes us naturally expect that they have something to do with particles in the corresponding quantum field theory.

We can see some properties of solitons considering a scalar field theory in two dimensional Minkowsky. As in ordinary QFT, the classical solutions will be the vacuum for which we quantize our theory. Thus understanding the classical solutions is instrumental in understanding the full quantum theory. To this end we study the Euler-Lagrange equations of motion,

$$\partial_\mu \frac{\partial L}{\partial(\partial^\mu \phi)} = \frac{\partial L}{\partial \phi} \tag{15}$$

We see solutions with finite energy:

$$E = \int dx \epsilon = (\mathbf{x}, t) \tag{16}$$

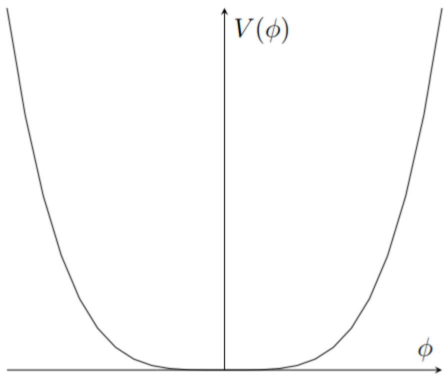
where, $\epsilon = T_0^0$. Furthermore, we want vacuum states which are stable over time, This condition can be written as,

$$\lim_{x \to \infty} \max_x \epsilon(\mathbf{x}, t) \neq 0 \tag{17}$$

That conditions defining the solitons solutions, another formulation involve a requirement that the superposition of two solitons remain a soliton.

We can start the discussion considering the $\lambda\phi^4$theory:

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4}\lambda\phi^4 \tag{18}$$



that leads to the following equation of motion:

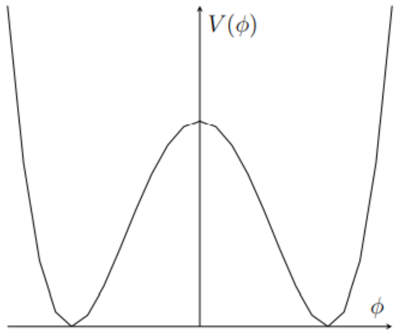
$$\Box\phi + \lambda\phi^3 = 0 \tag{19}$$

the energy of given solution is:

$$E[\phi] = \int_{-\infty}^{+\infty} dx \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + \frac{\lambda}{4}\phi^4 \right) \tag{20}$$

since is definite positive, it is zero only if $\phi = 0$ that is the only configuration at finite energy, then in this case we do not have non trivial solutions because we have only one vacua. In order to find non trivial solutions, we have to consider a theory with,at least two vacua, given a potential of the form:

$$V(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \tag{21}$$



The energy in this case is:

$$E[\phi] = \int_{-\infty}^{+\infty} dx \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 - \frac{1}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \tag{22}$$

the minima occur at $\phi_\pm = \frac{1}{\sqrt{\lambda}}$. The condition to have finite energy solutions explicitly is:

$$\lim_{x\to\pm\infty} \phi \to \phi_\pm \tag{23}$$

now we can see that the behaviour of this model is more rich so much that it is possible to find non trivial solutions. Considering a potential more general:

$$L = \frac{1}{2}(\partial_\mu\phi)^2 - U(\phi) \tag{24}$$

With $U(\phi)$ an arbitrary function. Denoting the minima with ϕ_i, then there are many possible pasitions where the potencial is vanishing. The equation of motion is given by:

$$\ddot{x} - \phi'' = \frac{\partial U}{\partial \phi} \tag{25}$$

The energy density is given by,

$$E(\phi) = \int_{-\infty}^{+\infty} dx \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + U(\phi) \tag{26}$$

we have $E[\phi] = 0$ for $\phi = \phi_i$. Then $E[\phi] < \infty$ implies that,

$$\lim_{x\to+\infty}\phi=\phi_i \qquad \lim_{x\to-\infty}\phi=\phi_j \tag{27}$$

Then, The finite energy solutions **interpolate between two of the zeroin the potential**. Now we can discuss the particle like property of solitons by considering an explicit solution of the equation of motion:

$$\frac{1}{2}\phi'^2=U(\phi) \tag{28}$$

separating variables:

$$x-x_0=\pm\int_{\phi_0}^{\phi(x)}d\widetilde{\phi}\frac{1}{\sqrt{2U(\widetilde{\phi})}} \tag{29}$$

considering the "kink solution" with the potential:

$$U(\phi)=\frac{\lambda}{4}(\phi^2-\frac{m^2}{\lambda})^2 \tag{30}$$

the vacua are at:

$$\phi_0=\pm\sqrt{\frac{m^2}{\lambda}} \tag{31}$$

Plugging in (29) and integrating, we obtain:

$$\phi(x)=\pm\frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m}{\sqrt{2}}(x-x_0)\right] \tag{32}$$

That is the explicit soliton solution, centered in $\phi_0=\phi(0)=0$. Our aim is emphasize the particle-like behaviour of this solutions but we can see that this field configuration it isn't! in fact its only the energy density that needs to be localized, this determines if one can have a particle-like behavior or not. In this case the energy density is,

$$\epsilon(x)=\frac{1}{2}\phi'^2+U(\phi) \tag{33}$$

substituing the value of ϕ':

$$\epsilon(x)=\frac{m^4}{2\lambda\cosh^4\left[\frac{m}{\sqrt{2}}(x-x_0)\right]} \tag{34}$$

This is indeed localized as desired. We can also assign a mass by computation of the total energy:

$$M=E=\int_{-\infty}^{+\infty}dx\epsilon(x)=\frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} \tag{35}$$

note that the mass, have a dependence as $\frac{1}{\lambda}$, a thing that never occur in perturbation theory. Clearly, the last solution is "static" then in order to find a solution for all times, we can boost the field:

$$\phi \to \pm \frac{m}{\sqrt{\lambda}} \tanh\left[\frac{\gamma m}{\sqrt{2}}(x - x_0 - vt)\right] \tag{36}$$

and we obtain an energy density:

$$\epsilon = \frac{\gamma m^4}{2\lambda} \mathrm{sech}^4 \left[\frac{\gamma m}{\sqrt{2}}(x - x_0 - vt)\right] \tag{37}$$

and the total energy is:

$$E = \gamma M \tag{38}$$

As final example we can recall the concepts described above throughout the study of the sine-Gordon model. The sine-Gordon model is integrable and admits soliton solutions. The sine Gordon model has the following lagrangian formulation:

$$L = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{\mu^2}{\beta^2}(\cos(\beta\varphi - 1)) = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{\mu^2}{2}\varphi^2 + \frac{\mu^2\beta^2}{4!}\phi^4 + \dots \tag{39}$$

where μ is a mass scale and β is a coupling costant. In order to verify the integrability of this model we have to find the lax connection. It is convenient to switch to light-cone variables:

$$\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x) \to e^{\pm\theta}\partial_\pm \tag{40}$$

and the sine-Gordon equation becomes:

$$\partial_+\partial_-\varphi + \frac{\mu^2}{4\beta}\sin(\beta\phi) = 0 \tag{41}$$

Now we can write the equation of motion as a zero-curvature condition in terms of the Lax pair. The lax pair is defined as the following:

$$A_+ = \frac{1}{4}\mu\lambda\Lambda - \gamma^{-1}\partial_+\gamma, \quad A_- = \frac{1}{4}\mu\lambda^{-1}\gamma^{-1}\Lambda\gamma \tag{42}$$

with:

$$\gamma = \begin{bmatrix} e^{\frac{i\beta\varphi}{2}} & 0 \\ 0 & \frac{-i\beta\varphi}{2} \end{bmatrix} \in \mathrm{SU}(2), \quad \Lambda = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in su(2) \tag{43}$$

now, after having verified that the sine-Gordon model is integrable we can construct the soliton solutions. It is sufficient to perform a boost on a static solution $\varphi(x,t) = \tilde{\varphi}(x)$.

$$x \to \frac{x - vt}{\sqrt{1 - v^2}} = x\mathrm{cosh}\theta - t\mathrm{senh}\theta = \frac{1}{2}(e^{-\theta}x^+ - e^{\theta}x^-) \tag{44}$$

such that:

$$\tilde{\varphi}(x) \to \tilde{\varphi}\left(\frac{x - vt}{\sqrt{1 - v^2}}\right) = \varphi(x, t) \tag{45}$$

This type of solution will clearly keep its shape at any time. Thus, we are looking for solutions of the kind:

$$\tilde{\varphi}'' = \frac{\mu^2}{\beta} \sin \beta\varphi \tag{46}$$

Subject to the boundary conditions:

$$\beta\varphi(\pm\infty, t) \in 2\pi Z \tag{47}$$

The result is:

$$\tilde{\varphi}_{\pm}(x) = \frac{4}{\beta} \arctan(e^{\pm\mu(x-x_0)}) \tag{48}$$

this is a single soliton solution of the integrable sine-Gordon model. This conclude that brief introduction to theory of solitons.

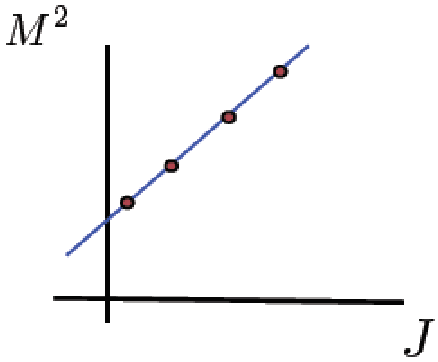
0.3 Introduction to Strings and AdS/CFT correspondence

As with any physical system, it is of great interest to determine any solitonic solutions of the theory, and consider them as the fundamental excitations which one can then use to build the other states of the theory. In the present work we are interested to solitonic solutions in the context of semi/classical analysis of string theory interesting possibly to investigate the AdS/CFT correspondence. The solitonic properties of these solutions were discussed through a reduction of $\mathrm{AdS}_5 \times \mathrm{S}_5$ world sheet theory to the SSSG field theory using the so called Pohlmeyer map.

In a low energy limit, the oscillations of these strings will look like localized excitations - different oscillation modes correspond to the different kinds of particles, including gauge fields. From this perspective, gauge fields appear as non-fundamental objects which are excitations of the fundamental strings. A better understanding of non-perturbative string theory and D-branes has shed light into another interpretation of gauge fields: it has been seen that string theory in certain space-time backgrounds has a dual description as a gauge field theory, thus putting gauge fields and strings as fundamental objects in the respective theories. This gauge/string duality, also known as Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence, was first proposed by Maldacena [13], and identified string theory on an $\mathrm{AdS}_{d+1} \times X_d$ background with a conformal field theory living on the boundary of this $\mathrm{AdS}_d + 1$space (d-dimensional) Gauge/string duality relate two seemingly different quantum physical descriptions, one being a gauge field theory in a number of space-time dimensions, and the other a string theory on a two-dimensional conformal world-sheet. The most studied of these dualities is the AdS_5/CFT_4 correspondence [13], which relates type IIB superstring theory on the $AdS_5\times S_5$. background and $N = 4$ supersymmetric Yang-Mills (SYM) gauge theory in four dimensions. For these reason in this section we present a brief introduction to string theory containing the minimal ingredients to understand the origin of the AdS/CFT duality. to the main concepts of AdS/CFT correspondance. Historically, string theory was introduced in the sixties as an attempt to describe the hadronic resonances of high spin observed in the experiments. Experimentally, the mass square of these particles is linearly related to its spin J:

$$M^2 \sim J \tag{49}$$

It is then said that the hadrons are distributed along Regge trajectoriesString theory was introduced to reproduce this behavior. Actually, it is not difficult to verify qualitatively that the rotational degree of freedom of the relativistic string gives rise to Regge trajectories like that.



Indeed, let us suppose that we have an open string with length L and tension T which is rotating around its center of mass. The mass of this object would be $M \sim TL$, whereas its angular momentum J would be $J \sim PL$, with P being its linear momentum. In a relativistic theory $P \sim M$, which implies that $J \sim PL \sim ML \sim T^{-1}M^2$ or, equivalently, $M^2 \sim TJ$. Thus, we reproduce the Regge behavior in (49) with the slope being proportional to the string tension T.

The basic object of string theory is an object extended along some characteristic distance l_s. Therefore, the theory is non-local. It becomes local in the point-like limit in which the size $l_s \to 0$. The rotation degree of freedom of the string gives rise to Regge trajectories similar to those observed experimentally. In modern language one can regard a meson as a quark-antiquark pair joined by a string. The energy of such a configuration grows linearly with the length, and this constitutes a model of confinement.

Let us consider a relativistic point particle of mass m moving in a at spacetime with Minkowski metric $\eta_{\mu\nu}$. As it moves the particle describes a curve in space-time (the so-called "Worldline"), which can be represented by a function of the type:

$$x^\mu = x^\mu(\tau) \tag{50}$$

where x^μ is the coordinate in the space in which the point particle is moving (the target space) and τ parameterizes the path of the particle.The action of the particle is proportional to the integral of the line element along the trajectory in space-time, with the coeficient being given by the mass m of the particle:

$$S = -m \int ds = -m \int_{\tau_0}^{\tau_1} d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{51}$$

Now we can consider a relativistic string. This once we have a one dimensional object that describe a surface(the world-sheet).Let dA be the area element of the world-sheet hence, the analogue of the action of a relativistic point particle for a string is the so-called Nambu-Goto action:

$$S_{NG} = -T \int dA \tag{52}$$

where T is the tension of the string, given by:

$$T = \frac{1}{2\pi\alpha'} \tag{53}$$

α' is called Regge slope. The string length and mass are defined as:

$$l_s = \sqrt{\alpha'} = \frac{1}{M_s} \tag{54}$$

Then we have that tension, mass and string lenght are related each other:

$$T = \frac{1}{2\pi l_s^2} = \frac{M_s^2}{2\pi} \tag{55}$$

Now we write the Nambu-Goto action in a more explicit manner. We will take two coordinates $\xi^\alpha(\alpha = 0, 1)$ to parameterize the world-sheet $\Sigma[(\xi^0, \xi^1) = (\tau, \sigma)]$ and assuming that the string moves in a target space $\mathcal{M}$ with a metric $G_{\mu\nu}$ we have that embedding of Σ in the space-time $\mathcal{M}$ is characterized by a map $\Sigma \to \mathcal{M}$ with $\xi^\alpha \to X^\mu(\xi^\alpha)$. The induced metric on Σ is:

$$\hat{G}_{\alpha\beta} =_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{56}$$

Thus, the Nambu-Goto action of the string is:

$$S_{NG} = -T \int \sqrt{-det\hat{G}_{\alpha\beta}} d^2\xi \tag{57}$$

The action (57) depends non-linearly on the embedding functions $X^{\mu}(\tau;\sigma)$. The classical equations of motion derived from (57) are partial differential equations which, remarkably, can be solved for a at target space-time with $G_{\mu\nu} = \eta_{\mu\nu}$ Actually, the functions $X^{\mu}(\tau;\sigma)$ can be represented in a Fourier expansion as an infinite superposition of oscillation modes, much as in the string of a violin. The quantization of the string can be carried out simply by canonical quantization, considering that the X^{μ} are operators and by imposing canonical commutation relations between coordinates and momenta. As a result one finds that the different oscillation modes can be interpreted as particles and that the spectrum of the string contains an infinite tower of particles with growing masses and spins that are organized in Regge trajectories, with $\frac{1}{l_s}$ being the mass gap. Quantizing the string we can encounter some peculiar feature:

- **The mass spectrum contain tachions** Quantizing the string we find in the mass spectrum particles with $m^2 < 0$, witch is a signal of instability, in order to avoid this problem one must consider a string which has also fermionic coordinates and require that the system is supersymmetric. In other word it is necessary generalise to superstrings.

- **Number of dimensions fixed to D=10** The quantization process generate constraints. To ensure the consistency of the theory, the number of dimensions D of the space where the strings is moving is fixed. In particular for superstring $D = 10$. This does not mean that the extra dimensions have the same meaning as the ordinary ones of the four-dimensional Minkowski spacetime. Actually, the extra dimensions should be regarded as defining a configuration space. We will see below that this is precisely the interpretation that they have in the context of the AdS/CFT correspondence.

- **The spectrum contain massless particles with spin 2** Analyzing the spectrum of the particles we see that massive particles have a mass multiple of $\frac{1}{l_s}$ then they became undetectable in the low-energy limit $l_s \to 0$. Then, after eliminating the tachyons by using supersymmetry, the massless particles are the low-lying excitations of the spectrum. Now distinguishing the case of open and closed string, we can see that the spectrum of the first contain massless particles of spin one with the couplings needed to have gauge symmetry. These particles can be naturally identified with gauge bosons (photons, gluons,...). Instead seeing the spectrum of opens strings, since it contains a particle of spin two and zero mass which can only be interpreted as the graviton (the quantum of gravity). After this brief introduction to strings we can discuss in witch sense the Ads theory have a correspondence with conformal Quantum field theory.

Although the gauge/gravity duality was discovered in the context of string theory, where it is quite natural to realize (gauge) field theories on hypersurfaces embedded in a higher dimensional space, in a theory containing gravity,however, the study of the correspondence has been extended to include very different domains, such as the analysis of the strong coupling dynamics of QCD and the electroweak theories, the physics of black holes and quantum gravity, relativistic hydrodynamics or different applications in condensed matter physics. We can start by motivating the duality from the Kadano-Wilson renormalization group approach to the analysis of lattice systems. Let us consider a non-gravitational system in a lattice with lattice spacing a and hamiltonian given by:

$$H = \sum_{x,i} J_i(x,a)\mathcal{O}^i(x), \tag{58}$$

where x denotes the different lattice sites and i labels the different operators $\mathcal{O}^i$. The $J_i(x,a)$ are the coupling constants (or sources) of the operators at the point x of the lattice. Notice that we have included a second argument in J_i, to make clear they correspond to a lattice spacing a. In the renormalization group approach we increasing the lattice spacing and replacing multiple sites by a single site with the average value of the lattice variables. In In this process the hamiltonian retains its form (58) but different operators are weighed differently. Accordingly, the couplings $J_i(x;a)$ change in each step. Suppose that we double the lattice spacing in each step. Then, we would have a succession of couplings of the type:

$$J_i(x,a) \to J_i(x,2a) \to J_i(x,4a) \to \dots \tag{59}$$

Therefore, the couplings acquire in this process a dependence on the scale (the lattice spacing) and we can write them as $J_i(x;u)$, where $u = (a;2a;4a;\dots)$ is the length scale at which we probe the system. The evolution of the couplings with the scale is determined by of equations of the form:

$$u\frac{\partial}{\partial u}J_i(x,u) = \beta(J_j(x,u),u), \tag{60}$$

where β_i is the so-called β-function of the i^th coupling constant. At weak coupling the β_i's can be determined in perturbation theory. At strong coupling the AdS/CFT proposal is to consider u as an extra dimension. In this picture the succession of lattices at different values of u are considered as layers of a new higher-dimensional space. Moreover, the sources $J_i(x;u)$ are regarded as fields in a space with one extra dimension and, accordingly we will simply write:

$$J_i(x,u) = \phi_i(x,u). \tag{61}$$

The dynamics of the sources $\phi's$ will be governed by some action. Actually, in the AdS/CFT duality the dynamics of the $\phi's$ is determined by some gravity theory.Therefore, one can consider the holographic duality as a geometrization of the quantum dynamics encoded by the renormalization group. The microscopic couplings of the field theory in the UV can be identified with the values of the bulk fields at the boundary of the extra-dimensional space. Thus, one can say that the field theory lives on the boundary of the higher-dimensional space(immagine). The sources ϕ_i of the dual gravity theory must have the same tensor structure of the corresponding dual operator O_i of field theory, in such a way that the product $\phi_i\mathcal{O}^i$ is a scalar. Therefore, a scalar field will be dual to a scalar operator, a vector field A_μ will be dual to a current J^μ, whereas a spin-two field $g_{\mu\nu}$ will be dual to a symmetric second-order tensor $T_{\mu\nu}$ which can be naturally identified with the energy-momentum tensor $T_{\mu\nu}$ of the field theory. Now in order to find the relation between the coupling constants in both sides of the correspondence, we have to perform a counting of degrees of freedom. Firstly we consider a QFT in a d-dimensional space-time. The number of degrees of freedom of a system is measured by the entropy. On the QFT side the entropy is an extensive quantity. Therefore, if R_{d-1} is $(d-1)$-dimensional spatial region, at constant time, its entropy should be proportional to its volume in $(d-1)$ dimensions:

$$S_{QFT} \propto Vol(R_{d-1}) \tag{62}$$

On the gravity side the theory lives in a $(d+1)$-dimensional space-time. We want explain how it is possible that such higher dimensional theory can contain the same information as its lower dimensional dual. The crucial point to answer this question is the fact that the entropy in quantum gravity is sub-extensive. Indeed, in a gravitational theory the entropy in a volume is bounded by the entropy of a black hole that its inside the volume and, according to the so-called holographic principle, the entropy is proportional to the surface of the black hole horizon. More concretely, the black hole entropy is given by the Bekenstein-Hawking formula:

$$S_{BH} = \frac{1}{4G_n} A_H \tag{63}$$

where A_H is the area of the event horizon and G_N is the Newton constant. In order to apply (63) for our purposes, let R_d be a spatial region in the $(d+1)$-dimension space-time where the gravity theory lives and let us assume that R_d is bounded by a $(d-1)$-dimensional manifold $R_{d-1}(R_{d-1} = \partial R_d)$ Then, according to (63), the gravitational entropy associated to R_d scales as:

$$S_{GR}(R_d) \propto Area(\partial R_d) \propto Vol(R_{d-1}), \tag{64}$$

which agrees with the QFT behavior (62). setting out the geometrical structure of AdS we can find a more precise matching between degrees of freedom of both theories and find the relation between the coupling constants. If we consider a theory at a fixed point of the renormalization group flow, then it has conformal invariance. Let us a QFT in (d)-space-time dimensions.The most general metric in $(d+1)$-dimensions with Poincar invariance in (d)-dimensions is:

$$ds^2 = \Omega^2(z)(-dt^2 + dx^2 + dz^2) \tag{65}$$

z is the coordinate of the extra dimension, $x = (x^1, \dots, x^{d-1})$ and $\Omega(z)$ are to determined. If z rapresents a length scale,since the theory is conformal invariant, then AdS^2 must be invariant under the transformation:

$$(t, x) \to \lambda(t, x), \quad z \to \lambda z \tag{66}$$

Imposing the invariance of the metric under dilatation transformation, $\Omega(z)$ must transform as:

$$\Omega(z) \to \lambda^{-1}\Omega(z) \tag{67}$$

witch fixes $\Omega(z)$ to be:

$$\Omega(z) = \frac{L}{z} \tag{68}$$

where L is a constant. Eventually the metric have the following form:

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + dx^2 + dz^2) \tag{69}$$

which is the element of the Ads space in $(d+1)$-dimensions, denoted AdS_{d+1}. L is a constant called "Anti-de Sitter radius". The boundary of AdS is located in $z = 0$. Notice that in $z = 0$ the metric is singular, then if one want define a quantity on the boundary of AdS must introduce a regularization procedure. Now we count explicitly the degrees of freedom, Let us consider first the QFT side. To regulate the theory we put both a UV and IR regulator. We place the system in a spatial box of size R (which serves as an IR cutoff) and we introduce a lattice spacing ϵ that acts as a UV regulator. In (d)-space-time dimensions the system has $\frac{R^{d-1}}{\epsilon^{d-1}}$.Let c_{QFT} be the number of degrees of freedom per lattice site, which we will refer to as the central charge. Then, the total number of degrees of freedom of the QFT is:

$$N_{dof}^{QFT} = \frac{R}{\epsilon}^{d-1} c_{QFT} \tag{70}$$

The central charge is one of the main quantities that characterize a CFT. Let us now compute the number of degrees of freedom of the AdS_{d+1} solution. According to the holographic principle and to the Bekenstein-Hawking formula (63), the number of degrees of freedom contained in a certain region is equal to the maximum entropy, given by

$$N_{dof}^{Ads} = \frac{A_\partial}{4G_N} \tag{71}$$

with A_∂ being the area of the region at boundary $z \to 0$ of AdS$_{d+1}$. Let us evaluate A_∂ by integrating the volume element corresponding to the metricat slice $z = \epsilon \to 0$

$$A_\partial = \int_{R^{d-1}} \quad {}_{z=\epsilon} d^{d-1}x\sqrt{g} = (\frac{L}{\epsilon})^{d-1} \int_{R^{d-1}} d^{d-1}x \tag{72}$$

The last integral is the volume of R^{d-1}, witch is infinite. Then we regulate it by putting the system in a box of size R:

$$\int_{R^{d-1}} d^{d-1}x = R^{d-1} \tag{73}$$

Thus, the area of the A_θ is given by:

$$A_\theta = \left(\frac{RL}{\epsilon}\right)^{d-1} \tag{74}$$

Let us next introduce the Planck length l_P and the Planck mass M_P for a gravity theory in $(d+1)$ dimensions as:

$$G_N = (l_P)^{d-1} = \frac{1}{(M_P)^{d-1}} \tag{75}$$

Then the number of degrees of freedom of the AdS$_{d+1}$ space is:

$$N_{dof}^{Ads} = \frac{1}{4}\left(\frac{R}{\epsilon}\right)^{d-1}\left(\frac{L}{l_p}\right)^{d-1} \tag{76}$$

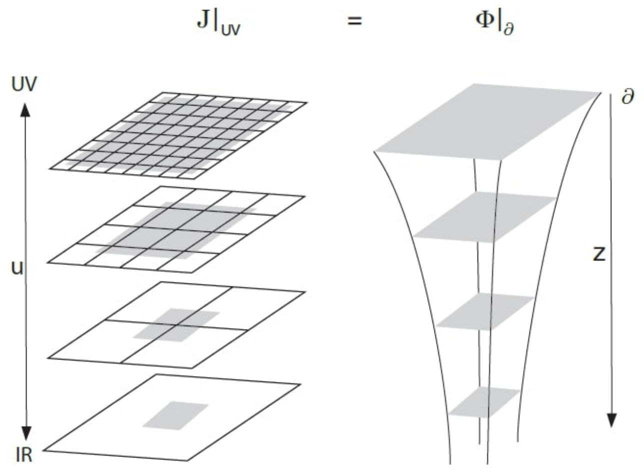
By comparing N_{dof}^{QFT} and N_{dof}^{Ads} we conclude that they scale in the same way with the IR and UV cutoff is R and ϵ and we can identify:

$$\frac{1}{4}\left(\frac{L}{l_P}\right)^{d-1} = c_{QFT} \tag{77}$$

This gives the matching condition between gravity and QFT that we were looking for. Notice that a theory is (semi)classical when the coefficient multiplying its action is large. In this case the path integral is dominated by a saddle point. The action of our gravity theory in the AdS$_{d+1}$ space of radius L contains a factor $\frac{L^{d-1}}{G_N}$. Thus, taking into account the definition of the Planck length in (75), we conclude that the classical gravity theory is reliable if:

$$\text{Classical Gravity in AdS} \rightarrow \left(\frac{L}{l_P}^{d-1}\right) >> 1 \tag{78}$$

which happens when the AdS radius is large in Planck units. Since the scalar curvature goes like $1 = L^2$, the curvature is small in Planck units. Thus, a QFT has a classical gravity dual when c_{QFT} is large, or equivalently if there is a large number of degrees of freedom per unit volume or a large number of species (which corresponds to large N for $SU(N)$ gauge theories).



0.4 AdS Space and Its Conformal Structures

In this section we want describe the Anti-de-Sitter Space and its conformal structure following[16]. So, in order to describe that aspects of AdS space, it is convenient to use the Penrose diagram. In fact, if we want discuss the conformal nature of a given space, we have to perform a compactification through Weyl transformation, the Penrose diagram is a diagram of the latter. Weyl transformations preserves the signature and the angles, then the Penrose diagram describe correctly the casual structure of space-time. Anti-de-Sitter is a solution of Einstein equations with negative cosmological constant arise from the action:

$$S = \frac{1}{16\pi G_N} \int d^{d+1}x\sqrt{g}(R + \gamma) \tag{79}$$

Where $\gamma > 0$. the equations of motions are:

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = \frac{\Gamma}{2}g_{\mu\nu} \quad \rightarrow \quad R = -\frac{d}{d-2}\Gamma \tag{80}$$

We can describe AdS_{d+1} by embedding an hyperboloid into $R^{d,2}$:

$$-X_{-1}^2 - X_0^2 + X_1^2 + \cdots + X_d^2 = -R^2 \quad \text{in} \quad R^{d,2} \tag{81}$$

both the ambient space and the equation have $SO(d,2)$ isometry, thus the resulting space has that isometry too. The metric is induced by:

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + \cdots + dX_d^2 \tag{82}$$

we can solve the equation by:

$$\begin{aligned} X_{-1} &= R\cosh(\rho)\sin(\tau), \\ X_0 &= R\cosh(\rho)\cos(\tau), \\ X_i &= R\sinh(\rho)\Omega_i \\ \text{with} \sum \Omega_i^2 &= 1 \end{aligned} \tag{83}$$

so the induced metric is:

$$ds^2 = R^2[-\cosh^2(\rho)d\tau^2 + d\rho + \sinh^2(\rho)d(\Omega_{d-1}^2)] \tag{84}$$

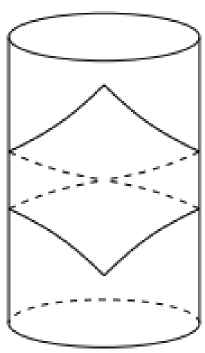
To draw the Penrose $\theta \in [0, \frac{\pi}{2})$:

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2) \tag{85}$$

After removing the denominator, we have the ball $B_d \times R$, this space has a boundary $S^{d-1} \times R$.

Notice that the boundary of conformally compactified AdS_{d-1} is equal to conformal compactification of $R^{d-1,1}$.

A space-time is asymptotically AdS if it has the same boundary structure as AdS after conformal compactification



Another useful parametrization of Ads_{d+1} is the so called "Poincar parametrization":

$$X_{\mu=0....,d-1} = \frac{R}{z}x_\mu, \quad X_{-1} = \frac{R}{2z}(1 + x^2 + z^2), \quad X_d = \frac{R}{2z}(1 - X_\mu{}^2 - z^2) \tag{86}$$

in these coordinaties the metric is:

$$ds^2 = R^2\frac{dz^2 + dx^2}{z^2} \tag{87}$$

where$x = x_0, x_i$. Notice that Poincar coordinates cover only a portion of AdS and that the boundary of the Poincar patch is $R^{d-1,1}$. The isometry of AdS_{d+1} is $SO(2)$ but is not manifest in the metric description. In fact:

- In the Global coordinates the manifest isometry is $SO(d) \times SO(2)$
- In the poincar patch the manifest isometry is $SO(d-1,1) \times SO(1,1)$ where the firs one is the Lorentz group of the boundary, while SO(1,1) is the dilatation realized as $X^\mu \to \lambda x^\mu$, $z \to \lambda z$

Chapter 1

Sigma Model

A sigma-model is a field theory wherein fields take values in a curved manifold M. In other words, a field configuration is a map:

$$X^M(\sigma) : \Sigma \to M \tag{1.1}$$

a vast area of applications of sigma-models is string theory, where they govern string propagation in curved backgrounds. Now, we want introduce string sigma-models relevant for holographic duality whose key feature is complete integrability. For Example, The sigma model on $AdS_5 \times S_5$ is integrable. Given local coordinates X^M on M, the most general two-derivative Lagrangian of a sigma-model is

$$L = \frac{1}{2}\sqrt{\gamma}\gamma^{ab}G_{MN}(X)\partial_a X^M X^M \partial_b X^N \epsilon_{ab} B_{MN}(X)\partial_a X^M \partial_b X^N) \tag{1.2}$$

Transformations that leave the metric and B-field invariant translate into global symmetries of the sigma-model. Symmetries are neither necessary nor sufficient for integrability, but they allow one to build a large class of integrable models.

The sigma-models arising in the holographic duality are precisely of this type. For this reason we concentrate on the cases when the target space M admits an action of a (simple) Lie group (or supergroup) G. Now we need to introduce some geometrical facts aimed to define the sigma models with a certain background before with the definition for Symmetric spaces and after for Semi- symmetric spaces that is the our case of interest.

1.1 Homogeneous spaces

A map

$$T_f(x) : F \times M \to M \tag{1.3}$$

defines left (right) action of group F on manifold M, if:

$$T_f T_h = T_{fg} \quad T_f T_h = T_{gf} \tag{1.4}$$

the stability group of a point $x \in M$ s defined as the set of all elements in F that leave x intact:

$$G_x = \{f \in F | T_f(x) = x\}. \tag{1.5}$$

Definition 1. *M is a left (right) homogeneous space of group F, if the action of F on M is transitive, namely if:*

$$\forall x, y \in M \exists f \in F : T_f(x) = y \tag{1.6}$$

given the definition of homogeneous space, it is possible let the definition of coset space as follow:

Definition 2. *Given a subgroup $G \subset F$, one can define a (right) coset F/G as a set of equivalence classes with respect to right multiplication by G:*

$$F/G = \{f \sim fg | f \in F, g \in G\} \tag{1.7}$$

A set fG, obtained by multiplying all elements of G by f ,constitutes one point in F $\sim G$. One can show that for a closed Lie subgroup $G \subset F$, the coset space $F \sim G$ is a smooth manifold. The left coset G/F is defined in a similar way. the coset F/G is a left homogeneous space of F, with the group action defined by left multiplication:

$$T_k(fG) = kfG \tag{1.8}$$

he converse is also true, in virtue of the following theorem: Homogeneous space M is isomorphic to the coset of its symmetry group F by the stability group G_{x_0} of any point $x_0 \in M : M = F/G_{x_0}$ then we can construct homogeneous spaces as a cosets of groups by they subgroups. A very useful way for our applications in holography.

Example 1. *The case of AdS homogeneous space: The $(d+1)$-dimensional Anti-de-Sitter space has coordinates(x^μ, z) and the line element:*

$$ds^2 = \frac{dx_\mu dx^\mu + dz^2}{z^2} \tag{1.9}$$

where $\mu = 0 \dots d-1$and the indices are contracted with the $- + \cdots +$ Minkowsky metric or Euclidean metric respectively for AdS_{d+1} or $EAdS_{d+1}$

1.1.1 Principal Chiral Field

The principal chiral field is a non-linear field that takes values in a group manifold:

$$f(x) : \Sigma \to F \tag{1.10}$$

We can construct the lagrangian for this field defining a Lie-algebra valued current

$$\mathscr{L} = -\mathrm{Tr}(j_a j^a) = -\mathrm{Tr}(j \wedge *j) \tag{1.11}$$

with:

$$j_a = f^{-1}\partial_a f \in \mathbf{f} \tag{1.12}$$

and "Tr" is an invariant quadratic form. The Lagrangian is invariant under global $F_L \times F_R$ transformations:

$$f(x) \to f_L f(x), \quad f(x) \to f(x) f_R \tag{1.13}$$

The current is left-invariant and transforms in the adjoint under right multiplications: $j_a \to f_R^{-1} j_a g_R$ deriving from the invariance of the quadratic form "Tr".

The equations of motion of the principal chiral field follow from an infinitesimal variation:

$$\delta f = f\xi, \quad \xi \in \mathfrak{f} \tag{1.14}$$

the variation of the current is:

$$\delta j_a = \partial_a \xi + [j_a, \xi] \tag{1.15}$$

The commutator term does not contribute to the variation of the Lagrangian, because of the invariance of the quadratic form under the adjoint action of the Lie algebra. Then we obtain:

$$\delta S = 2 \sum d^2x \mathrm{Tr}(\xi \partial_a j^a) \tag{1.16}$$

The equations of motion are:

$$\partial_a j^a = 0 \tag{1.17}$$

j_a is the Nöether current of the right group multiplication.

1.2 Classical integrability of Sigma model

If we write explicitly j_a in terms of $f(x)$ and writing the boundary conditions:

$$f(x^0, \pm\infty) \tag{1.18}$$

the equations of motion became a well defined Cauchy problem. We would like to treat the current itself like a dynamical variable. To this purpose we have to recognize that the equations (1.12) defines a pure-gauge potential, a flat connection, whose curvature is equal to zero.

$$\partial_a j_b - \partial_b j_a + [j_a, j_b] = 0 \tag{1.19}$$

Once the current is known, $f(x)$ can be reconstructed from (1.18) after imposing the boundary conditions (2.18). the equations of motion now became:

$$\begin{aligned} d * j &= 0 \\ dj + j \wedge j &= 0 \end{aligned} \tag{1.20}$$

Now for to proof the itegrability of this class of models we want to find a Lax representation of our equations of motions. To this and, we have to collect this two equation into one. Let$z$ a complex variable, we can multiply the first equation with it and summing the second one obtain:

$$d(j + z * j) + j \wedge j = 0. \tag{1.21}$$

redefining the current $j + z * j$, using the identities

$$* \, a \wedge b = -a \wedge *b \quad *^2 = 1 \tag{1.22}$$

valid for one forms in two dimensions, up to a rescaling factor we have:

$$(j + z * j) \wedge (j + z * j) = (1 - z^2) j \wedge j. \tag{1.23}$$

if we choose the rescaling factor like:

$$L = \frac{j + z * j}{1 - z^2} \tag{1.24}$$

the current is flat:

$$dL + L \wedge L = 0. \tag{1.25}$$

The current L is the Lax connection that we want to find. explicitly:

$$L_a = \frac{j_a + z\epsilon_{ab} j^b}{1 - z^2} \tag{1.26}$$

then the equations of motion for the principal chiral field are equivalent to the condition that the Lax connection is flat:

$$\partial_a L : b - \partial_b La + [L_a, L_b] = 0 \quad \forall z \tag{1.27}$$

The zero curvature, or Lax representation of the equations of motion is a hallmark of integrability.

1.3 Sigma Model on Symmetric Space

In this section we want to construct the action of world sheet theory on S_5 symmetric space. The action of a Sigma model can be always written in an explicit coordinate system, but for homogeneous space exist a vary convenient construction based on coset representation. A construction that permit to discuss the integrability and the exact solutions in an elegant algebraic way. After we give the definition of Symmetric space:

Let F be a connected Lie group. Then a symmetric space for F is a homogeneous space $\mathcal{M} = F/G$ where G is the isotropy group (or little group) of a typical point$p_0 \in \mathcal{M}$; namely, $G = g \in F : gp_0 = p_0$.the stabilizer G of a typical point is an open subgroup of p_0 set of a $\mathbb{Z}_2$ involution Ω in $\mathrm{Aut}(F)$.

Thus Ω is an automorphism of F with $\Omega^2 = id_F$ and G is an open subgroup of the set

$$F^{\Omega} = f \in F : \Omega(f) = f \tag{1.28}$$

As an automorphism of F, Ω fixes the identity element, and hence, by differentiating at the identity, it induces an automorphism of the Lie algebra $\mathfrak{f}$ of F, also denoted by Ω, whose square is the identity. It follows that the eigenvalues of Ω are ± 1. The $+1$ eigenspace is the Lie algebra $\mathfrak{g}$ of G (since this is the Lie algebra of F^{Ω}), and the 1 eigenspace will be denoted $\mathfrak{p}$. Since Ω is an automorphism of $\mathfrak{f}$, this gives a direct sum decomposition

$$\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p} \tag{1.29}$$

where the element of that algebra composition satisfies the follow commutation relations:

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \tag{1.30}$$
$$[\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p} \tag{1.31}$$
$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g} \tag{1.32}$$

the first one implies that the constraints structure is $p_{\mathfrak{ggp}}$=0, cause the antisymmetry of quadratic form also the $f_{\mathfrak{gpg}}$=0. For $[p, \mathfrak{p}]$ we have that since $p_{\mathfrak{ppg}}$ are related to $p_{\mathfrak{gpp}}$, they are different from zero. the structure constants $p_{\mathfrak{ppp}}$ are not restricted by any principle but an interesting special case arises if they also vanish. Then

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g} \tag{1.33}$$

Now we are ready to introduce the S_5 world sheet theory as a sigma model on a symmetric coset.

We will restrict ourselves to symmetric spaces with F semi-simple. Moreover, we will always consider explicit realizations in terms of matrix representations of F, and we will assume that the corresponding trace form provides a non-degenerate,invariant, bilinear form on $\mathfrak{f}$ such that the decomposition (1.29) is orthogonal. In the case of S_5 we have the following space:

$$\mathfrak{M} = \frac{SO(6)}{SO(5)} \tag{1.34}$$

That is symmetric, compact and $F = SO(6)$ is semi-simple.

Let $f = f(\tau, x)$ be a dimensional field taking values on a faithful matrix representation of F. To formulate the sigma model with target-space $M = F/G$, we introduce a gauge field B_μ on g and define a covariant derivative $D_\mu f = \partial_\mu f - f B_\mu$ with the property that

$$f \to f g^{-1}, \quad B_\mu \to g(B_\mu + \partial_\mu) g^{-1} \quad \Rightarrow D_\mu f \to (D_\mu f) g^{-1} \tag{1.35}$$

for any $g = g(\tau, x)$ taking values on G. It is also useful to introduce the f-valued current

$$J_\mu = f^{-1} D_\mu f = f^{-1} \partial_\mu f - B_\mu \tag{1.36}$$

that is covariant under gauge transformations,

$$J_\mu \to g J_\mu g^{-1} \tag{1.37}$$

Then, if the Lie group F is simple, the nonlinear sigma model is defined by the Lagrangian:

$$\mathscr{L} = -\frac{1}{2k} \mathrm{Tr}(J_\mu J^\mu) \tag{1.38}$$

Where k is an overall normalization constant that plays no role in the classical equations of motion.

That Lagrangian have two symmetry:

- G-Symmetry that is the gauge symmetry defined in (1.35)
- F-Symmetry that is a global symmetry:

$$f \to f_0 f \quad \forall f_0 \in F$$

The equation of motion for this lagrangian are:

- For the field f:

$$D_\mu J^\mu = \partial_\mu J^\mu + [B_\mu, J^\mu] = 0 \tag{1.39}$$

that using the identity $[\partial_\mu + f^{-1}\partial_\mu f, \partial_\mu + f^{-1}\partial_\nu f]$ became:

$$D_\mu J_\nu - D_\nu J_\mu + [J_\mu, J_\nu] + F_{\mu\nu}] = 0 \tag{1.40}$$

where: $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$.

- For the field B_μ:

$$J^\mu = 0 \quad \text{on} \quad \mathfrak{g} \tag{1.41}$$

remembering the commutation relation defining the symmetric space, we have that J_μ must take values in $\mathfrak{p}$.

Using the light-cone variables $x^\pm = \frac{1}{2}(\tau \pm x)$ and $\partial_\pm = \partial_r \pm \partial_x$ the equations (1.40-1.41) splits into:

$$\begin{aligned} D_\pm J_\mp = \partial_\pm J_\mp + [B_\pm, J_\mp] = 0 \quad \text{in} \quad \mathfrak{p} \\ [J_+, J_-] + F_\pm = 0 \quad \text{in} \quad \mathfrak{g} \end{aligned} \tag{1.42}$$

The first equation implies that:

$$\partial_\pm Tr(J_\mp^n) = 0 \tag{1.43}$$

which provides a set of local chiral densities that display the two-dimensional conformal invariance of the sigma model. For $n = 2$ we recover the non-vanishing components of the stress-energy tensor:

$$T_{++} = -\frac{1}{2k}\mathrm{Tr}(J_+^2) \quad \textit{and} \quad T_{--} = -\frac{1}{2k}\mathrm{Tr}(J_-^2) \tag{1.44}$$

1.4 Sigma model on Semi-simmetryc spaces

Semi-symmetric spaces are defined generalizing the cosets symmetric spaces in super-cosets.

That generalization is based on an automorphism $\Omega : \mathfrak{f} \to \mathfrak{f}$ that squares to the fermion parity:

$$\Omega^2 = (-1)^F \tag{1.45}$$

Such automorphism has order four: $\Omega^4 = id$, and the superalgebra has a $\mathbb{Z}_4$ grading:

$$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3 \tag{1.46}$$

where:

$$\Omega(\mathfrak{f}_\mathfrak{n}) = i^4 \mathfrak{f}_\mathfrak{n} \tag{1.47}$$

The $\mathbb{Z}_4$ decomposition is consistent with the Grassmann parity: $\mathfrak{f}_0 \oplus \mathfrak{f}_2$ form the bosonic subalgebra of g and all the supercharges belong to $\mathfrak{f}_1 \oplus \mathfrak{f}_3$ If G_0 is the subgroup of F whose Lie algebra $\mathfrak{f}_0$ is $\mathbb{Z}_4$-invariant, the coset F/G_0 is called semi-symmetric superspace. Then: The current now decomposes into four components according to their Z_4 grading:

$$j = j_0 + j_1 + j_2 + j_3. \tag{1.48}$$

The current components j_0, j_2 expand in bosonic generators of the superalgebra, while the components j_1, j_3 are fermionic they are linear combinations of supercharges with Grassmann-odd coefficients. The action of the $\mathbb{Z}_4$ coset sigma-model is:

$$S = \int d^2x Str(\sqrt{-h}h^{ab} j_{0a} j_{2a} + \epsilon^{ab} j_{1a} j_{3b}) \tag{1.49}$$

The first term is the usual Sigma-model Lagrangian, while the second term descends from the WZ action, which for super-cosets is not topological and can be written in a manifestly 2d form.

Chapter 2

Pohlmeyer Reduction

In this chapter we want reduce both the sigma model on S_5 and the bosonic part of sigma model in $Ads_5 \times S_5$ showing the equivalence to the generalized SSSG and SSSSG theories respectively. In particular we will see that leads to the same action then at the same equations of motion.

To this purpose is needed a procedure called "Pohlmeyer Reduction". The Pohlmeyer reduction was first introduced in 1976 showing the relationship between the $O(3)$ sigma model and sine-Gordon theory Pohlmeyer reduction provides a map between the equations of motion of two-dimensional sigma models and a class of multi-component integrable generalizations of the sine-Gordon equation. Following [9],[7], It relies on the classical conformal invariance of sigma models that can be exploited to choose coordinates such that the components of the stress-energy tensor are constant; namely,

$$T_{++} = T_{--} = \mu^2 \tag{2.1}$$

together with: T_{+-}. From the point of view of the original sigma model degrees of freedom, Pohlmeyer reduction amounts to a non-local transformation of variables that breaks conformal invariance while preserving integrability and two-dimensional Lorentz invariance and gives rise to a mass scale, related to the mass parameter in the sine-Gordon equation.

2.1 Pohlmeyer reduction of classical strings on symmetric coset spaces

In this section we discuss the formulation of the Pohlmeyer reduction for strings on symmetric coset spaces. In particular we are interested to the case of S^5 space. We start by considering strings moving on a target space F/G.Where F is a compact group. We name the algebras associated to the group F and G as $\mathfrak{f} \oplus \mathfrak{g}$. We define therefore $\mathfrak{p}$ as the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{f}$ as:

$$\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p} \quad \mathrm{Tr}(\mathfrak{g}\mathfrak{p}) = 0 \tag{2.2}$$

the condition thatF/G is a symmetric coset space, imposing the following commutation relation:

$$[\mathfrak{g}, \mathfrak{g}] \in \mathfrak{g}, \quad [\mathfrak{g}, \mathfrak{p}] \in \mathfrak{p}, \quad [\mathfrak{p}\mathfrak{p}] \in \mathfrak{g} \tag{2.3}$$

For the purposes of the Pohlmeyer reduction we require the algebra $\mathfrak{f}$ to have additional properties. The maximal abelian subalgebra of $\mathfrak{p}$, which we denote $\mathfrak{a}$, should be one-dimensional. Defining $\mathfrak{h}$ as the centralizer of $\mathfrak{p}$ in $\mathfrak{p}$, that is $[\mathfrak{p}, \mathfrak{p}] = 0$, we further assume the following conditions on the structure of these algebras:

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}, \quad \mathfrak{p} = \mathfrak{a} \oplus \mathfrak{n}, \quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{2.4}$$

$$[\mathfrak{a}, \mathfrak{a}] = 0, \quad [\mathfrak{a}, \mathfrak{h}] = 0 \quad [\mathfrak{a}, \mathfrak{m}] \in \mathfrak{n}, \quad [\mathfrak{a}, \mathfrak{n}] \in \mathfrak{m}, \tag{2.5}$$

$$[\mathfrak{n}, \mathfrak{n}] \in \mathfrak{h}, \quad [\mathfrak{n}, \mathfrak{m}] \in \mathfrak{a} \quad [\mathfrak{m}, \mathfrak{m}] \in \mathfrak{h}, \tag{2.6}$$

$$[\mathfrak{h}, \mathfrak{m}] \in \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{n}] \in \mathfrak{n}, \quad [\mathfrak{h}, \mathfrak{h}] \in \mathfrak{h} \tag{2.7}$$

Where $\mathfrak{h}$ is a superalgebra of $\mathfrak{g}$. $\mathfrak{m}$ and $\mathfrak{n}$ are the orthogonal complements of $\mathfrak{h}/\mathfrak{a}$ in $\mathfrak{g}/\mathfrak{p}$.

2.1.1 Reduction of Sigma model on S_5 background to SSSG theory

In this section we reduce the Sigma model with target space a symmetric space to a generalized sin-Gordon theory, following the scheme proposed in[]We start with general reduction for S^n to SSSG theory. The n-spheres S^n defined by:

$$S^n = (x_1, ..., x_{n+1}) : x^2 + ... + x_{n+1}^2 = 1 = \frac{SO(n+1)}{SO(n)} \tag{2.8}$$

is a compact symmetric space of definite signature. In this case the solutions of the condition:

$$\partial_\pm Tr(J_\mp^n) = 0 \tag{2.9}$$

can be found by using the so-called "polar coordinate decomposition": Let $\mathfrak{a}$ be a maximal abelian subsapce

$\mathfrak{p}$. Then, for any $k \in \mathfrak{p}$ there exists $\mathfrak{g} \in G$ such that $\mathfrak{g}^{-1}k\mathfrak{g} \in \mathfrak{a}$. A more explicit proof specific for $S^n = SO(n+1)/SO(n)$ is given in appendix B of []. The dimension of the maximal abelian subspaces $\mathfrak{a} \subset \mathfrak{p}$ defines the rank of the symmetric space. For symmetric spaces($\frac{F}{G}$) we have the following relation for the ranks:

$$\mathrm{rank}(F) - \mathrm{rank}(G) \leq \mathrm{rank}\left(\frac{F}{G}\right) \leq \mathrm{rank}(F) \tag{2.10}$$

In the case of $S_5 = \frac{SO(6)}{SO(5)}$:

- $\text{rank}(\frac{SO(6)}{SO(5)}) = 1$
- $\text{rank}(SO(6) = 3$
- $\text{rank}\ (SO(5)) = 2$

Now we consider the decomposition $\mathfrak{p} = \mathfrak{a} + \mathfrak{n}$ then we have:

$$[\mathfrak{a}, \mathfrak{a}] = 0 \qquad [\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{n} \tag{2.11}$$

Using the equations of motions for the field B_μ we can write the currents $J_\pm$:

$$J_\pm = \hat{g}_\pm c_\pm \hat{g}_\pm^{-1} \tag{2.12}$$

where $\hat{g}$ and $c_\pm$ are function that take values in G and in $\mathfrak{a}$ respectively. Then now we can rewrite the first equation (1.42):

$$\partial_\pm c_\mp = [\partial_\pm \hat{g}_\mp^{-1} \hat{g}_\mp - \hat{g}_\mp^{-1} B_\pm \hat{g}_\mp, c_\mp] \tag{2.13}$$

which, taking (2.11) in account, imply that c_+ and c_- are chiral,

$$c_+ = c_+(x_+) \quad c_- = c_-(x_-) \tag{2.14}$$

then we can write the chiral densities:

$$\text{Tr}(J_\mp^n) = \text{Tr}(c_\pm^n) \tag{2.15}$$

Which shows that their value is fixed by the value of the components of c_+ and c_-. Then, constraining all the chiral densities $\text{Tr}(J_\pm^n)$ to take constant values is equivalent to constraining the chiral functions c_+ and c_- to be constant. In the present case $\text{rank}(\frac{SO(6)}{SO(5)}) = 1)$ and $\dim(\mathfrak{a}) = 1$. We can write:

$$c_+ = \mu_+(x_+)\Lambda \quad \text{and} \quad c_-(\mu_-(x_-)\Lambda \tag{2.16}$$

where μ_+ and μ_- are real(numeric) functions, Λ a constant generator of $\mathfrak{a}$ and, the components of stress-energy tensor became:

$$T_{++} = -\frac{1}{2k}\mu_+^2(x_+)\text{Tr}(\Lambda^2) \quad \text{and} \quad T_{--} = -\frac{1}{2k}\mu_-^2(x_-)\text{Tr}(\Lambda^2) \tag{2.17}$$

where the value of $k = \pm 1$ is chosen so that T_{++} and T_{--} are always positive. Therefore, the components of the stress-energy tensor are constant if, and only if, μ_+ and $\mu_-$are constant, which is obviously equivalent to the claim that c_+ and c_- are constant elements of $\mathfrak{a}$. a. Since all the maximal abelian subspaces in $\mathfrak{p}$ are conjugated under the adjoint action of G, in this case the reduction procedure gives rise to only one set of SSSG equations, which are indeed equivalent to the equations of motion of the original sigma model up to a (classical) conformal transformation.

In the following, we will need the centralisers of in G, which are the Lie group:

$$H = g \in G : g^{-1}\Lambda g = \Lambda \tag{2.18}$$

with Lie algebras:

$$\mathfrak{h} = \mathrm{Ker}(Ad_\Lambda \cap \mathfrak{g}) \tag{2.19}$$

The explicit formulation of the reduced model is obtained by imposing a particular gauge-fixing condition to the equations of motion of the sigma model subjected to the constraints(2.16). Now implementing the so-called "partial reduction" gauge condition:

$$J_+ = \mu_+\Lambda \quad \text{and} \quad J_- = \mu_-\gamma^{-1}\Lambda\gamma \tag{2.20}$$

where $\gamma = \hat{g}_-^{-1}\hat{g}_+$ takes values in G. Then, the first two equations in (1.42), $D_\pm J_\mp = 0$ became:

$$[B_-, \Lambda] = 0 \tag{2.21}$$

$$[B_+ - \gamma^{-1}\partial_+\gamma, \gamma^{-1}\Lambda\gamma] = 0 \tag{2.22}$$

The condition (2.20) does not fix the gauge condition (1.35) completely, and the residual gauge transformations correspond to $\gamma \to \gamma\mathfrak{h}^{-1}$ with $\mathfrak{h} \in H$. All these gauge transformations can be summarised as follows:

$$\begin{aligned} \gamma &\to \mathfrak{h}\gamma\mathfrak{h}^{-1} \\ A_-^{(R)} &\to \mathfrak{h}(A_-^{(R)} + \partial_-)\mathfrak{h}_+^{-1}, \\ A_+^{(L)} &\to \mathfrak{h}(A_+^{(L)} + \partial_+)\mathfrak{h}^{-1} \end{aligned} \tag{2.23}$$

the third equation in (1.42) can be written as a 2d-Lorentz invariant zero-curvature condition for γ:

$$[\partial_+ + \gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+^L\gamma + z\mu_+\Lambda, \partial_- + \gamma^{-1}\partial_-\gamma + \gamma^{-1}A_-^R\gamma + z^{-1}\mu_-\Lambda] = 0 \tag{2.24}$$

where z is a spectral parameter. This equation, subjected to the gauge symmetry (2.23), provides the most general form of the SSSG equations specified by $(\frac{F}{G}, \Lambda)$. These euqations derive by the following action:

$$S = S_{gWZW}[\gamma, A_\mu] - \frac{k}{\pi}\int d^2x Tr(\Lambda\gamma^{-1}\Lambda\gamma) \tag{2.25}$$

here, $S_{gWZW}[\gamma, A_\mu]$ is the WZW action for $\frac{G}{H}$:

$$\begin{aligned} S_{gWZW}[\gamma, A_\mu] = -\frac{k}{2\pi}\int d^2x Tr[\gamma^{-1}\partial_+\gamma\gamma^{-1}\partial_-\gamma + 2A_+\partial_-\gamma\gamma^{-1} - 2A_-\gamma^{-1}\partial_+\gamma - 2\gamma^{-1}A_+\gamma A_- + 2A_+A_-] \\ + \frac{k}{12\pi}\int d^3x\epsilon^{abc}Tr[\gamma^{-1}\partial_a\gamma\gamma^{-1}\partial_b\gamma\gamma^{-1}\partial_c\gamma] \end{aligned} \tag{2.26}$$

Actually, it is the integrability condition required to reconstruct the field $f = f(\tau, x)$ corresponding to the currents and the gauge fields .

Namely, using (1.36), f is the solution to the auxiliary linear problem:

$$\begin{aligned}\partial_+ f^{-1} &= -(\gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+^L\gamma + \mu_+\Lambda)f^{-1} \\ \partial_- f^{-1} &= -(\gamma^{-1}\partial_-\gamma + \gamma^{-1}A_+^R\gamma + \mu_-\Lambda)f^{-1}\end{aligned} \tag{2.27}$$

It has a unique solution once the initial condition $f_0 = f(\tau_0, x_0)$ is fixed.

2.2 Polhmeyer Reduction of classical AdS$_5\times$S^5 superstring

In this section we want reduce the AdS$_5\times$ S$_5$ model to SSSSG theory. In the last section we have showed that the sigma model on S^5 symmetric space is equivalent to SSSG thoery fixing the chirality dansities to have a constant value or in other words, by fixing the gauge. In the case of semi-symetryc spaces we have also to fix the so called "k-Symmetry, a symmetry involving the worldsheet fermion of the theory. The resulting reduced system will be still invariant under a residual k-symmetry which can be fixed by an additional gauge condition. That will finally make the number of the fermionic degrees of freedom the same as the number of the physical bosonic degrees of freedom.It will turn out that the resulting system of reduced equations of motion (that originate in particular from the Maurer-Cartan equations and thus are first order in derivatives) will follow from a local Lagrangian containing only first derivatives of the fermionic fields. The bosonic part of the reduced Lagrangian will coincide with the gauged WZW Lagrangian with the same potential as in the bosonic model discussed in the section 1.

In the chapter 1 we have discussed the action of Sigma model on Semi-symmetric space as a coset. In what follows we shall assume the" conformal gauge" condition $\gamma^{ab} = e^{\phi}\eta^{ab}$ so the action became:

$$L_{GS} = Str[J_+^{(2)}J_-^{(2)} + \frac{1}{2}(J_+^{1+}J_-^{3-} - J_-^{1-}J_+^{3+})] \tag{2.28}$$

Having fixed conformal gauge, the Virasoro constraint

$$STr(J_\pm J_\pm) = 0 \tag{2.29}$$

need to be imposed, in addition to the variational equations coming from the action(2.28). The action and the Virasoro constrains are invariant under the G-gauge symmetry:

$$f \to fg \Rightarrow \quad j \to g^{-1}Jg + g^{-1}dg \tag{2.30}$$

Wihich imply:

$$\mathfrak{g}: \qquad J^{(0)} \to g^{-1}J^{(0)}g + g^{-1}dg \tag{2.31}$$
$$\hat{f}_1: \qquad J^{(1)} \to g^{-1}J^{(1)}g \tag{2.32}$$
$$\mathfrak{p}: \qquad J^{(2)} \to g^{-1}J^{(2)}g \tag{2.33}$$
$$\hat{f}_3: \qquad J^{(3)} \to g^{-1}J^{(3)}g \tag{2.34}$$

Varyng $f(x;t)$ in the action we obtain the following equation of motion:

$$\mathcal{D}_+J_-^{(2)} + \mathcal{D}_-J_+^{(2)} + [J_-^{(1)}, J_+^1] + [J_+^{(3)}, J_-^{(3)}] = 0 \tag{2.35}$$
$$[J_+^2, J_-^{(1)}] = 0, [J_-^2 J_+^{(3)}] = 0 \tag{2.36}$$

supplementes by the flatness condition for the Maurer-Cartan one-form J

$$\partial_+J_- - \partial_-J_+ + [J_+, J-] = 0 \tag{2.37}$$

that can be decomposed under the $\mathbb{Z}_4$ grading as:

$$\mathfrak{p}: \quad \mathcal{D}_-J_+^{(2)} - \mathcal{D}_+J_-^{(2)} + [J_-^{(1)}, J_+^{(1)}] + [J_-^{(3)}, J_+^{(3)}] = 0 \tag{2.38}$$
$$\hat{\mathfrak{f}}_1: \quad \mathcal{D}_-J_+^{(1)} + \mathcal{D}_+J_-^{(1)} + [J_-^{(2)}, J_+^{(3)}] - [J_+^{(2)}, J_-^{(3)}] = 0 \tag{2.39}$$
$$\hat{\mathfrak{f}}_3: \quad \mathcal{D}_-, J_+^{(3)} - \mathcal{D}_+J_-^{(3)} + [J_-^{(2)}, J_+^{(1)}] - [J_+^{(2)}, J_-^{(1)}] = 0 \tag{2.40}$$
$$\mathfrak{g}: \quad \mathcal{D}_-, J_+^{(0)} - \mathcal{D}_+J_-^{(0)} + [J_-^{(0)}, J_+^{(0)}] + [J_-^{(1)}, J_+^{(3)}] + [J_-^{(2)}, J_+^{(2)}] + [J_-^{(3)}, J_+^{(1)}] = 0 \tag{2.41}$$

Combining the equation of motion for $J^{(2)}$ and the flatness condition projected on $\hat{f}_2$ we can obtain the eqautions for $J_+^{(2)}$ and $J_-^{(2)}$:

$$\mathcal{D}_+J_-^{(2)} + [J_+^{(3)}, J_-^{(3)}] = 0, \quad \mathcal{D}_-J_+^{(2)} + [J_-^{(1)}, J_+^{(1)}] = 0 \tag{2.42}$$

the first stage of pohlmeyer reduction of the AdS$_5\times$S$_5$ superstring thoery is to fix the gauge on $J_+^{(2)}$:

$$J_+^{(2)} = p_{1+}\Lambda_1 + p_{2+}\Lambda_2 \tag{2.43}$$

where we have choosed the following basis for the maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$:

$$\Lambda_1 = \frac{i}{2}\text{diag}(1, 1, -1, -1, 0, 0, 0, 0) \tag{2.44}$$
$$\Lambda_2 = \frac{i}{2}\text{diag}(0, 0, 0, 0, 1, 1, -1, -1) \tag{2.45}$$

and write $J_-^{(2)}$ as:

$$J_-^{(2)} = p_{1-}\Lambda_1\gamma + p_{2-}\gamma^{-1}\Lambda_2\gamma \tag{2.46}$$

where γ is an element of G and $p_{1\pm}$, $p_{2\pm}$ are functions of the world-sheet coordinates. Now substituing these expressions into the Virasoro constraints implies that $p_{1+} = p_{2+} = p_{1-} = p_{2-}$ thus:

$$J_+^{(2)} = p_+\Lambda, \quad J_-^{(2)} = p_-\gamma^{-1}\Lambda\gamma \tag{2.47}$$

Where Λ is defined as:

$$\Lambda = \Lambda_1 + \Lambda_2 \tag{2.48}$$

The algebra $\mathfrak{h}$ is defined to be the centralizer of Λ in $\mathfrak{g}$ and the corresponding group is denoted H. The matrix element of Λ defines a sub-algebra of $\hat{f}$ denoted $\hat{f}^{\perp}$ which is the centralizer of Λ in $\hat{f}$. The last stage involve the fixing of the so called k-symmetry. This fixing eliminate 16 of 32 fermionis degrees of freedom in the subalgebra $\mathfrak{psu}(2,2|4)$. This is foundamental to obtain just the physical degrees of freedom in the reduced theory. This is achieved by projecting the fermionic courrents onto $\hat{f}_1^{\parallel}$ and $\hat{f}_3^{\parallel}$:

$$J^{(1)} = J^{(1)\parallel} \tag{2.49}$$

$$\gamma J^{(3)}\gamma^{-1} = (\gamma J^{(3)}\gamma^{-1})^{\parallel} \tag{2.50}$$

substituing in the equation (2.40), the resulting equations imply:

$$J_-^{(1)} = J_+^{(3)} = 0 \tag{2.51}$$

then the eqautions (2.45) became:

$$\mathcal{D}_+ J_-^{(2)} = 0 \quad \mathcal{D}_- J_+^{(2)} = 0 \tag{2.52}$$

that equations are solved by taking:

$$J_+^{(0)} = \gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+\gamma \quad J_-^{(0)} = A_-, \quad A_\pm \in \mathfrak{h} \tag{2.53}$$

we make the following redefinitions of the fermionc courrents:

$$\psi_R = \frac{1}{\sqrt{\mu}}(J_+^{(1)})^{\parallel}, \quad \psi_L = \frac{1}{\sqrt{\mu}}(\gamma J_-^{(3)\parallel}\gamma^{-1}) \tag{2.54}$$

Then the equation are solved by the redefinition of the courrent in terms of the new fields $\gamma, A_\pm, \psi_R, \psi_L$. Substituing in the remaining equations we obtain the complete reduced equations of motion for the reduced thoery:

$$\begin{aligned}\partial_-(\gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+\gamma) - \partial_+A_- + [A,\gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+\gamma] - [\psi_+, \gamma^{-1}\psi_-\gamma] - [\Lambda, \gamma^{-1}\Lambda\gamma] = 0 \\ D_\mp\psi_\pm + [\Lambda, \gamma^{\mp 1}\psi_\mp\gamma^{\pm 1} = 0\end{aligned} \tag{2.55}$$

the new fermionic fields transforms down the enhanced $(H_L \times H_R)$-gauge symmetry as:

$$\psi_R \to h_R^{-1}\psi_R h_R, \quad \psi_L \to h_L^{-1}\psi_L h_L \tag{2.56}$$

Where the H_R-gauge symmetry arises as a subgroup of the original G-gauge symmetry in the world-sheet sigma model When The H_L-gauge symmetry arises in defining the reduced-theory field, g. Both of these gauge freedoms exist because $\mathfrak{h}$ is the centralizer of Λ in $\mathfrak{g}$. To writing the right Lagrangian which that generate these equations of motion, that gauge symmetry need to be partialy fixed to an H-symmetry. This is performed by setting:

$$\tau(A_+) = (\gamma^-1\partial_+\gamma + \gamma^{-1}A_+\gamma - \frac{1}{2}[[\Lambda,\psi_R],\psi_R)|_{\mathfrak{h}}] \tag{2.57}$$

$$\tau(A_-) = (\gamma^-1\partial_+\gamma + \gamma^{-1}A_-\gamma - \frac{1}{2}[[\Lambda,\psi_L],\psi_L)|_{\mathfrak{h}}] \tag{2.58}$$

where τ is an automorphism of the algebra $\mathfrak{h}$. Fixed this gauge symmetry the transformations rules for the fermionic field are:

$$\psi_R \to \hat{\tau}(h)^{-1}\psi_R\tau(h), \quad \psi_L \to h^{-1}\psi_L h \tag{2.59}$$

The e.o.m. (2.25) follow from the Lagrangian of the Pohlmeyer reduced AdS$_5\times$ S $_5$ superstring:

$$S = S_gWZW[\gamma, A_\mu] - \frac{k}{\pi}\int d^2xSTr\big(\gamma^{-1}\Lambda\gamma\big) + \frac{k}{2\pi}\int d^2xSTr\big(\psi_+[\Lambda, D_-\psi_+] - \psi_-[\Lambda, D_+\psi_-] - 2\psi_+\gamma^{-1}\psi_-\gamma\big) \tag{2.60}$$

Where S_gWZW is the conventional (bosonic) gauged WZW model for G/H, but involving the supertrace defined as:

$$\mathrm{STr}M = -\mathrm{Tr}m + \mathrm{Tr}n = 0 \tag{2.61}$$

The reduced theory is thus a G/H gauged WZW model with a gauge-invariant integrable potential and fermionic extension. For the our case $G = Sp(2,2) \times Sp(4)$ and $H = [SU(2)]^4$.

The integrability of these models are guaranted by the existence of the following Lax connection:

$$\mathscr{L}_+(z) = \partial_+ + \gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+\gamma + z\psi_+ - z^2\Lambda \tag{2.62}$$

$$\mathscr{L}_- = \partial_- + A_- + z^{-1}\gamma^{-1}\psi_-\gamma - z^{-2}\gamma^{-1}\Lambda\gamma \tag{2.63}$$

where, again, $\gamma \in G$, $\psi_\pm$ are fields taking values in $\mathfrak{f}_{1,3}$, respectively, and hence are fermionic. Lax connection exhibits that the Lorentz transformation:

$$x^\pm \to \lambda^\pm x^\pm \tag{2.64}$$

is equivalent to the rescaling of the spectral parameter:

$$z \to \lambda^2 z \tag{2.65}$$

where $\gamma \in G, \psi_\pm$ are fields taking values in $\mathfrak{f}_{1,3}$,

Chapter 3

The Dressing Method

Integrability provides different methods to generate soliton solutions. The most important one is the Inverse Scattering Method (ISM), which allows one to construct the general solution of integrable field equations making use of an associated linear problem. The ISM can be regarded as a generalisation of the Fourier transform to non-linear problems. Other well known techniques to construct soliton solutions are the Backlünd transformations, the dressing transformations, the Hirota method, etc. In this work we want find soliton solutions by dressing method following [1],[11]. The inverse spectral (or scattering) transform (IST) method serves as the mathematical background for the soliton theory. The modern version of IST is based on the dressing method proposed by Zakharov and Shabat, first in terms of the factorization of integral operators on a line into a product of two Volterra integral operators and then using the RiemannHilbert (RH) problem . The most powerful version of the dressing method incorporates the ∂^- problem formalism. The ∂^- problem was put forward by Beals and Coifman as a generalization of the RH problem and was applied to the study of first-order one-dimensional spectral problems. Generally, the term dressing implies a construction that contains a transformation from a simpler (bare, seed) state of a system to a more advanced, dressed state. In particular cases, dressing transformations, as the purely algebraic construction, are realized in terms of the Backlünd transformations which act in the space of solutions of the nonlinear equation, or the Darboux transformations (DTs) acting in the space of solutions of the associated linear problem. In the present case we want use the "dressing method" to following linear problem:

$$L_\mu(z)\Upsilon(z) = 0 \tag{3.1}$$

where: $\Upsilon(z)$ is an element of the loop group associated to f and so it must satisfy the reality condition lifted to the loop group:

$$H\Upsilon(z^*)^{-1\dagger}H = \Upsilon(z) \tag{3.2}$$

it must have the following behaviour under the automorphism:

$$\mathcal{K}^{-1}\Upsilon(z)^{-1\dagger}\mathcal{K} = \Upsilon(iz) \tag{3.3}$$

and the action of fermionic parity is:

$$\beta\Upsilon(z)^{-1\dagger}\beta = \Upsilon(-z) \tag{3.4}$$

Soliton solutions are special solutions for which $g_+ = g = 1$ in the Riemann-Hilbert problem:

$$\Upsilon_0(z)g_-g_+^{-1}\Upsilon_0^{-1}(z) = \chi(z)^{-1}\gamma\chi(z) \tag{3.5}$$

$$\Upsilon_0(x;z) = exp[(z^2x^+ + z^{-2}x^-)\Lambda] \tag{3.6}$$

the in the solution of the linear problem can be written in two equivalent ways:

$$\Upsilon(x;z) = \chi(x;z)\Upsilon_0(x;z) = \gamma^{-1}\chi(x;z)\Upsilon_0(x;z) \tag{3.7}$$

where $(\gamma = 1, \psi_\pm = 0, A_\mu = 0)$. is the vacuum solution of the linear problem. In the context of solitons, $\chi(z) := \chi(x;z)$ is known as the dressing transformation for the obvious reason that it generates the soliton solutions from the vacuum. The method then proceeds by taking an ansatz for the dressing factor which takes the form of a sum over a finite set of simple poles

$$\chi(z) = 1 + \frac{Q_i}{z-\epsilon_i} \qquad\qquad \chi(z)^{-1} = 1 + \frac{R_i}{z-\mu_i} \tag{3.8}$$

Then the our linear problem in the gauge $A_\pm = 0$ gives rise to the two equations:

$$\begin{aligned} \partial_+\chi(z)\chi(z)^{-1} + z^2\chi(z)\Lambda\chi(z)^{-1} &= -\gamma^{-1}\partial_+\gamma - z\psi_+ + z^2\Lambda \\ \partial_-\chi(z)\chi(z)^{-1} + z^2\chi(z)\Lambda\chi(z)^{-1} &= -z^{-1}\gamma^{-1}\psi_- + z^{-2}\gamma^{-1}\Lambda\gamma \end{aligned} \tag{3.9}$$

To obtain the fields,it is sufficient expand $\chi(z)$ around $z = 0$ and $z = \infty$:

$$\chi(z) = 1 + z^{-1}W_{-1} + z^2(W_{-2} + \frac{1}{2}W_{-1}^2) + O(z^{-3} \tag{3.10}$$

$$= \gamma^{-1}(1 + zW_1 + z^2(W_2 + \frac{1}{2}W_1^2) + O(z^3) \tag{3.11}$$

Then we obtain:

$$\gamma = \chi(0)^{-1} \tag{3.12}$$

Now, since the dependence on z of the right-hand-side of is explicit, the residues of the left-hand-side at $z = \epsilon_i$ and μ_i must vanish, giving

$$\begin{aligned} (\xi_i^{\mp 2}\partial_\pm Q_i + Q_i\Lambda)\left(1 + \frac{R_j}{\epsilon_i - \mu_j}\right) = 0 \\ \left(1 + \frac{Q_j}{\mu_i - \xi_j}\right)(\mu_i^{\mp 2}\partial_\pm R_i + R_i\Lambda) = 0 \end{aligned} \tag{3.13}$$

The key to solving them is to propose that the residues have rank one

$$Q_i = \mathbf{X}_i\mathbf{F}_i^\dagger \quad R_i = \mathbf{H}_i\mathbf{K}_i^\dagger \tag{3.14}$$

where 8-vectors are written in boldface. The point is that in order to preserve the fermionic grading, the vectors must have the structure

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} \tag{3.15}$$

where either of the 4-vectors $\mathbf{v}_1$, or $\mathbf{v}_2$, must be Grassmann odd. We shall fix the choice by realizing that there are a known consistent bosonic solitons solutions for the S_5= SU(4)/Sp(4). This solutions would be obtained by taking a dressing ansatz where all the vectors have $\mathbf{v}_1 = 0$. In other words, the sub 4-vector $\mathbf{v}_1$ must be Grassmann odd and the sub 4-vector $\mathbf{v}_2$ must be Grassmann even. The solution of (3.13) is

$$F_i = (\Upsilon_0(\xi_i)^\dagger)^{-1}\varpi_i \qquad\qquad H_i = \Upsilon_0(\mu_i)\pi_i \tag{3.16}$$

for constant complex graded 8-vectors ϖ_i and pi_i along with

$$\mathbf{X}_i\Gamma_{ij} = \mathbf{H}_j \quad \mathbf{K}_i(\Gamma^\dagger)_{ij} = -\mathbf{F}_j \tag{3.17}$$

where the matrix

$$\Gamma_{ij} = \frac{\mathbf{F}_i^* \cdot \mathbf{H}_j}{\xi_i - \mu_j} \tag{3.18}$$

This is a solution very general, we need to implement some conditions. First of all, the reality condition:

$$\frac{H(\mathbf{H}_i\mathbf{K}_i^\dagger)H}{z - \mu_i^*} = \frac{\mathbf{X}_j\mathbf{F}_j}{z - \xi_j} \tag{3.19}$$

which is solved by taking

$$\mu_i = \xi_i^*, \quad \mathbf{K}_i = H\mathbf{X}_i \quad \mathbf{H}_i = H\mathbf{F}_i \tag{3.20}$$

and so:

$$\Gamma_{ij} = \frac{\mathbf{F}_i^* \cdot H\mathbf{F}_j}{\xi_i - \xi_j^*} = -\Gamma_{ij}^* \tag{3.21}$$

Now the condition (3.3)

$$\frac{\mathcal{K}^{-1}(\mathbf{H}_i\mathbf{K}_i^\dagger)^{\mathrm{st}}\mathcal{K}}{z-\xi_i^*} = \frac{\mathbf{X}_j\mathbf{F}_j^\dagger}{iz-\xi_j} \tag{3.22}$$

which means that as a set $\xi_i^* = -i\xi_i$.Consequently, we define the operator η with:

$$\xi_i^* = -i\xi_{\eta(i)}, \quad \mathbf{F}_i = \epsilon_i H\mathcal{K}\mathbf{F}_{\eta(i)}^* \quad \mathbf{X}_i = i\epsilon_i\beta H\mathcal{K}X_{\eta(i)}^* \tag{3.23}$$

where $\epsilon_i = \pm 1$. We can notice that $\eta^2(i) = i$:

$$\xi_{\eta^2(i)} = i\xi_{\eta(i)}^* = i(i\xi_i^*)^* = \xi_i \tag{3.24}$$

where we have used that $\mathcal{K}^2 = -1$. Therefore, we have the constraint

$$\epsilon_i\epsilon_{\eta(i)} = -1 \tag{3.25}$$

and we shall choose

$$\epsilon_1 = \epsilon_3 = -1, \quad \epsilon_2 = \epsilon_4 = 1 \tag{3.26}$$

Finally, the condition of the action of fermionic parity gives:

$$\frac{\beta(\mathbf{X}_i\mathbf{F}_i^\dagger)\beta}{z-\xi_i} = -\frac{\mathbf{X}_j\mathbf{F}_j^\dagger}{z+\xi_j} \tag{3.27}$$

therefore, as a set $\xi_i = -\xi_i$, and

$$\xi_i = -\xi_{\rho(i)} \to \mathbf{X}_i = -\beta\mathbf{X}_{\rho(i)}, \quad \mathbf{F}_i = \beta\mathbf{F}_{\rho(i)} \tag{3.28}$$

with $\rho^2(i) = i$. Taken together, these conditions require the dressing data to have four poles. Choosing the ordering

$$\xi_i = \{\xi, i\xi^*, -\xi, -i\xi^*\} \tag{3.29}$$

we have $\eta(1,2,3,4)$=(2,1,4,3) and $\rho(1,2,3,4) = (3,4,1,2)$ and

$$\mathbf{F}_i = \{\mathbf{F}, \mathcal{K}H\mathbf{F}^*, \beta\mathbf{F}, \mathcal{K}H\mathbf{F}^*\} \tag{3.30}$$

which means that the constant (4|4) vectors are:

$$\varpi_i = \{\varpi, \mathcal{K}\mathbf{H}, \varpi^*, \beta\varpi, \mathcal{K}\mathbf{H}\varpi^*\} \tag{3.31}$$

Fineally the " dressing factor" is:

$$\chi(x;z) = 1 + \frac{H\mathbf{F}\Gamma_{ij}^{-1}\mathbf{F}}{z-\sigma(\xi)} \tag{3.32}$$

Where σ are operators $\sigma_i, i = 1,2,3,4$, such that $\sigma_i(\xi) = \xi_i$ and $\sigma(\varpi) = \varpi_i$, with $\sigma_1(\xi) := \xi$ and $\sigma(\varpi) = \varpi$.

3.1 Soliton of SSSG on S_5

In this section we can see how work the dressing method finding explicitly the dressing factor for the pure bosonic Sigma Model as coset. As we have seen in the last section, the collective coordinaties of the soliton consist in a (4|4) constant vector. The first 4 components of the vector are Grassmann odd while the second 4 components are ordinary c-numbers.This means that the soliton solution for the group fields and wave function has the structural form:

$$\gamma = \begin{pmatrix} fermionic^2 & fermionic \\ fermionic & bosonic \end{pmatrix} \tag{3.33}$$

So the part of the solution in the AdS part of the geometry is a bosonic quantity that is a composite at least quadraticof the Grassmann collective coordinates. In this section we are interested to purely bosonic solutions. They lie entirely in the subgroup $SU(4) \subset PSU(2,2|4)$ associated to the S^5 part of the geometry. In SU(4) subspace, the solitons have collective coordinate on the form of a constant 4-vectorϖ. Using global symmetries we can bring ϖ into the form,

$$\varpi = (1,0,1,0) \tag{3.34}$$

We then define the following $\mathbb{Z}_2$ action on the soliton data:

$$\xi_i = \{\xi, -\xi^*\} \quad \varpi_i = (\varpi, \hat{\mathcal{K}}\varpi*) \tag{3.35}$$

where:

$$\hat{\mathcal{K}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.36}$$

Then we define $F_i = \Upsilon_0(\sqrt{\xi_i^*})\varpi$ where the latter is the vacuum solution in the $SU(4)$ subspace:

$$\mathbf{F}_i = \psi(\sqrt{\xi_i^*})\varpi = \exp[(\xi^2 x^+ + \xi^{-2} x^-)\Lambda]\varpi \tag{3.37}$$

for S_5 case the expression of Λ is explicitly:

$$\Lambda = \frac{i}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.38}$$

explicitly for the $\mathbf{F}_i$ we have:

$$\mathbf{F}_1 = \begin{pmatrix} e^{-X*} \\ 0 \\ e^{X*} \\ 0 \end{pmatrix} \quad \mathbf{F}_1^* = \begin{pmatrix} e^{-X} \\ 0 \\ e^{X} \\ 0 \end{pmatrix}$$
$$\mathbf{F}_2 = \begin{pmatrix} 0 \\ e^{-X} \\ 0 \\ e^{X} \end{pmatrix} \quad \mathbf{F}_2^* = \begin{pmatrix} 0 \\ e^{-X*} \\ 0 \\ e^{X*} \end{pmatrix} \tag{3.39}$$

where we have defined:

$$X := \frac{1}{2}(\xi x^+ + \xi^{-1}x^-) \tag{3.40}$$

now we have to compute Γ_{ij}^{-1}:

The tensorial products results are:

$$\mathbf{F}_1 \otimes \mathbf{F}_1^* = \begin{pmatrix} e^{-X^- X*} & 0 & e^{X-X^*} & \\ 0 & 0 & 0 & 0 \\ e^{-X+X^*} & 0 & e^{X+X^*} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.41}$$

$$\mathbf{F}_1 \otimes \mathbf{F}_2^* = \begin{pmatrix} 0 & e^{-2X*} & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & e^{2X^*} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.42}$$

$$\mathbf{F}_2 \otimes \mathbf{F}_1^* = \begin{pmatrix} 0 & e^{-2X} & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & e^{2X} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.43}$$

$$\mathbf{F}_2 \otimes \mathbf{F}_2^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{-X^- X*} & 0 & e^{-X+X^*} & 0 \\ 0 & 0 & 0 & 0 \\ e^{X-X^*} & 0 & e^{X+X^*} & 0 \end{pmatrix} \tag{3.44}$$

The inverse of Γ_{ij} is simply:

$$\begin{pmatrix} \frac{\xi-\xi^*}{e^{(-X-X*)}+e^{(X+X*)}} & 0 \\ 0 & \frac{-\xi+\xi^*}{e^{(-X-X*)}+e^{(X+X*)}} \end{pmatrix} \tag{3.45}$$

Finally we have the expression for the dressing factor:

$$\Theta(x;z) = 1 + \frac{\xi - \xi^*}{e^{(-X-X*)} + e^{(X+X*)}} \begin{bmatrix} \frac{e^{-X-X^*}}{z^2-\xi} & 0 & \frac{e^{X-X^*}}{z^2-\xi} & 0 \\ 0 & \frac{e^{-X-X^*}}{z^2+\xi^*} & 0 & \frac{e^{-X+X^*}}{z^2+\xi^*} \\ \frac{e^{-X+X^*}}{z^2-\xi} & 0 & \frac{e^{X+X^*}}{z^2-\xi} & 0 \\ 0 & \frac{e^{X-X^*}}{z^2+\xi^*} & 0 & \frac{e^{X+X^*}}{z^2+\xi^*} \end{bmatrix} \tag{3.46}$$

giving the dressing factor, we can simply compute the expressions of the group valued field in for SSSG theory by the following relations:

$$\gamma(x) = \Theta(x;0)^{-1} \tag{3.47}$$

For the SSSG field we have:

$$\gamma = \begin{bmatrix} 1 - \frac{\xi^{*2}-\xi^2}{\xi^{*2}\left(e^{2(X+X^*)}+1\right)} & 0 & -\frac{\left(\xi^{*2}-\xi^2\right)e^{X-X^*}\operatorname{sech}(X+X^*)}{2\xi^{*2}} & 0 \\ 0 & 1 - \frac{\xi^2-\xi^{*2}}{\xi^2\left(e^{2(X+X^*)}+1\right)} & 0 & \frac{\left(\xi^{*2}-\xi^2\right)e^{X^*-X}\operatorname{sech}(X+X^*)}{2\xi^2} \\ -\frac{\left(\xi^{*2}-\xi^2\right)e^{X-X^*}\operatorname{sech}(X+X^*)}{2\xi^{*2}} & 0 & 1 - \frac{\left(\xi^{*2}-\xi^2\right)(\tanh(X+X^*)+1)}{2\xi^{*2}} & 0 \\ 0 & \frac{\left(\xi^{*2}-\xi^2\right)e^{X-X^*}\operatorname{sech}(X+X^*)}{2\xi^2} & 0 & \frac{\left(\xi^{*2}-\xi^2\right)(\tanh(X+X^*)+1)}{2\xi^2} + 1 \end{bmatrix} \tag{3.48}$$

Notice that, giving the dressing factor it is possible obtain also the soliton for the 2d sigma model:

$$f(x) = \exp(2t\Lambda)\Theta(x,1)^{-1} \tag{3.49}$$

3.2 Soliton Solution for SSSSG on $AdS_5 \times S_5$ Including Fermionic Variables

Now we want to introduce two fermionic variabiles into the collective coordinates vector . Starting from:

$$\varpi = \begin{pmatrix} 0, & 0, & 0, & 0, & 0, & 0, & 1, & 1 \end{pmatrix} \tag{3.50}$$

Using the global symmetry we can transform ϖ:

$$\varpi \to U\varpi \tag{3.51}$$

where:

$$U = e^f, \qquad\qquad f \in PSU(2,2|4) \tag{3.52}$$

With:

$$\text{f}= \begin{bmatrix} \mathbf{0} & Grass_1 \\ Grass_2 & \mathbf{0} \end{bmatrix}$$

$$Grass_1 = \begin{bmatrix} 0 & 0 & \alpha_1 + i\alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_1 - i\alpha_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Grass_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\alpha_1 + \alpha_2 & 0 & 0 & 0 \\ 0 & i\alpha_1 - \alpha_2 & 0 & 0 \end{bmatrix} \tag{3.53}$$

Expanding in series the exponential matrix:

$$e^f = 1 + f + \frac{f^2}{2} \tag{3.54}$$

we can see that is truncated at the second order because all the terms quadratic in α_1 and α_2 get 0. Finally the transformed ϖ vector is:

$$\varpi' = \begin{pmatrix} 0, & \alpha_1 + i\alpha_2, & 0, & 0, & 1 - i\alpha_1\alpha_2, & 0, & 1, & 0 \end{pmatrix} \tag{3.55}$$

the dressing factor have the form:

$$\chi = 1 + \sum_{ij} \frac{HF_i\Gamma_{ij}^{-1}F_j^\dagger}{z - \xi_j}, \quad \Gamma_{ij} = \frac{F_i^* F_j}{\xi_i - \xi_j} \tag{3.56}$$

the poles are:

$$\xi_i = \{\xi, i\xi^*, -\xi, -i\xi^*\} \tag{3.57}$$

And the F_i:

$$\mathbf{F}_i = \{\mathbf{F}, \mathbf{F}H\mathbf{F}^*, \beta\mathbf{F}, KH\beta\mathbf{F}^*\} \tag{3.58}$$

where:

$$H = \begin{bmatrix} 1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & 1_2 \end{bmatrix} \quad \beta = \begin{bmatrix} -1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & 1_2 \end{bmatrix} \tag{3.59}$$

when $\mathbf{F}$ is:

$$\mathbf{F} = \exp it\cos(q) + x\sin(q)\mathbf{v}_+ + \exp(it\cos(q) - x\sin(q))\mathbf{v}_- \tag{3.60}$$

but we can define $X := it\cos(q) + x\sin(q)$ so that we can write $\mathbf{F}$ in a more compact way.

$$\mathbf{F} = \begin{pmatrix} 0, & (\alpha_1 + i\alpha_2)e^{-X^*}, & 0, & 0, & (1 - i\alpha_1\alpha_2)e^{-X^*}, & 0 & e^{X^*}, & 0 \end{pmatrix} \tag{3.61}$$

Now, in order to compute the projector Γ_{ij} we have to compute all the 8 vectors $\mathbf{F}_i$. The result is:

$$
\begin{aligned}
\mathbf{F}_1 &= \big[0,\ \ (\alpha_1 + i\alpha_2)e^{-X^*},\ \ 0,\ \ 0,\ \ (1 - i\alpha_1\alpha_2)e^{-X^*},\ \ 0,\ \ e^{X^*},\ \ 0\big] && (3.62)\\
\mathbf{F}_1^* &= \big[0,\ \ (\alpha_1 - i\alpha_2)e^{-X},\ \ 0,\ \ 0,\ \ (1 - i\alpha_1\alpha_2)e^{-X},\ \ 0,\ \ e^{X},\ \ 0\big] && (3.63)\\
\mathbf{F}_2 &= \big[(-\alpha_1 + i\alpha_2)e^{-X},\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ (1 - i\alpha_1\alpha_2)e^{-X},\ \ 0,\ \ e^{X}\big] && (3.64)\\
\mathbf{F}_2^* &= \big[(-\alpha_1 - i\alpha_2)e^{-X^*},\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ (1 - i\alpha_1\alpha_2)e^{-X^*},\ \ 0,\ \ e^{X^*}\big] && (3.65)\\
\mathbf{F}_3 &= \big[0,\ \ (-\alpha_1 - \imath\alpha_2)e^{-X^*},\ \ 0,\ \ 0,\ \ (1 - i\alpha_1\alpha_2)e^{-X^*},\ \ 0,\ \ e^{X^*},\ \ 0\big] && (3.66)\\
\mathbf{F}_3^* &= \big[0,\ \ (-\alpha_1 + i\alpha_2)e^{-X},\ \ 0,\ \ 0,\ \ (1 - i\alpha_1\alpha_2)e^{-X},\ \ 0,\ \ e^{X},\ \ 0\big] && (3.67)\\
\mathbf{F}_4 &= \big[(\alpha_1 - i\alpha_2)e^{-X},\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ (1 - i\alpha_1\alpha_2)e^{-X},\ \ 0,\ \ e^{X}\big] && (3.68)\\
\mathbf{F}_4^* &= \big[(\alpha_1 + i\alpha_2)e^{-X^*},\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ (1 - i\alpha_1\alpha_2)e^{-X^*},\ \ 0,\ \ e^{X^*}\big] && (3.69)
\end{aligned}
$$

then we can obtain The value of gamma:

$$
\Gamma_{ij} = \begin{bmatrix}
\frac{e^{-X-X^*}+e^{X+X^*}}{\xi-\xi^*} & 0 & \frac{(1-4i\alpha_1\alpha_2)e^{-X-X^*}+e^{X+X^*}}{(\xi+\xi^*)} & 0 \\
0 & \frac{(1-4i\alpha_1\alpha_2)e^{-X-X^*}+e^{X+X^*}}{i\xi+i\xi^*} & 0 & \frac{e^{-X-X^*}+e^{X+X^*}}{-i\xi+i\xi^*} \\
\frac{(1-4i\alpha_1\alpha_2)e^{-X-X^*}+e^{X+X^*}}{-\xi-\xi^*} & 0 & \frac{(1-4i\alpha_1\alpha_2)e^{-X-X^*}+e^{xX+xXc}}{-\xi+\xi^*} & 0 \\
0 & \frac{e^{-X-X^*}+e^{X+X^*}}{i\xi-i\xi^*} & 0 & \frac{e^{-X-X^*}+e^{X+X^*}}{-i\xi-i\xi^*}
\end{bmatrix} \tag{3.70}
$$

To compute Γ_{ij}^{-1} it is useful decompose it as a polynomial in α_1, α_2:

$$
\Gamma = \Gamma_0 + \alpha_1\Gamma_1 + \alpha_2\Gamma + \alpha_1 + \alpha_2\Gamma \tag{3.71}
$$

Using this decomposition, the expression for the inverse can be computed all in terms of Γ_0^{-1}, it reads:

$$
\Gamma^{-1} = \Gamma_0^{-1}(\Gamma_0 - \alpha_1\Gamma_1 - \alpha_2\Gamma_2 - \alpha_1\alpha_2(\Gamma_3 + \Gamma_2\Gamma_0^{-1}\Gamma_1 - \Gamma_1\Gamma_0^{-1}\Gamma_2))\Gamma_0^{-1} \tag{3.72}
$$

Insertig all the data in the expression for the dressing factor() we obtain an 8×8 matrix that we present as 16 (2×2)-blocks:

$$
\begin{bmatrix}
a_1 & a_2 & b_1 & b_2 \\
a_3 & a_4 & a_3 & b_4 \\
c_1 & c_2 & d_1 & d_2 \\
c_3 & c_4 & d_3 & d_4
\end{bmatrix} \tag{3.73}
$$

where:

$$
a_1 = \begin{bmatrix}
-\frac{2i\alpha_1\alpha_2\xi^*(\xi^2-\xi^{2*})}{\xi(\xi^{*2}+z^2)(e^{2(X+X^*)}+1)} & 0 \\
0 & \frac{2i\alpha_1\alpha_2\xi(\xi^2-\xi^{2*})}{\xi^*(\xi^2-z^2)(e^{2(X+X^*)}+1)}
\end{bmatrix} \tag{3.74}
$$

a_2, a_3, a_4 are null matrices.

$$b_1 = \begin{bmatrix} 0 & -\frac{(\alpha_2+i\alpha_1)z(\xi^2-\xi^{2*})}{\xi(\xi^{*2}+z^2)(e^{2(X+X^*)}+1)} \\ -\frac{(\alpha_1+i\alpha_2)z(\xi^2-\xi^{2*})}{\xi^*(\xi^2-z^2)(e^{2(X+X^*)}+1)} & 0 \end{bmatrix} \tag{3.75}$$

$$b_2 = \begin{bmatrix} 0 & -\frac{(\alpha_2+i\alpha_1)z(\xi^2-\xi^{2*})e^{X^*-X}(X+X^*)}{2\xi(\xi^{*2}+z^2)} \\ -\frac{(\alpha_1+i\alpha_2)z(\xi^2-\xi^{2*})e^{X-X^*}(X+X^*)}{2\xi^*(\xi^2-z^2)} & 0 \end{bmatrix} \tag{3.76}$$

b_3 and b_4 are null matrices.

$$c_1 = \begin{bmatrix} 0 & -\frac{(\alpha_1+i\alpha_2)z(\xi^2-\xi^{2*})}{\xi(\xi^2-z^2)(e^{2(X+X^*)}+1)} \\ \frac{(i\alpha_1+\alpha_2+)z(\xi^2-\xi^{2*})}{\xi^*(\xi^{*2}+z^2)(e^{2(X+X^*)}+1)} & 0 \end{bmatrix} \tag{3.77}$$

$$c_3 = \begin{bmatrix} 0 & \frac{(-\alpha_1+i\alpha_2)z(\xi^2-\xi^{2*})e^{X^*-X}(X+X^*)}{2\xi(\xi^2-z^2)} \\ \frac{(\alpha_2+i\alpha_1)z(\xi^2-\xi^{2*})e^{X-X^*}(X+X^*)}{2\xi^*(\xi^{*2}+z^2)} & 0 \end{bmatrix} \tag{3.78}$$

c_2 and c_4 are null matrices.The last four terms are:

$$d_1 = \begin{bmatrix} \frac{\xi^{*2}-\xi^2}{(\xi^2-z^2)(e^{2(X+X^*)}+1)} + \frac{2i\alpha_1\alpha_2(\xi^2-\xi^{*2})(\xi e^{2(X+X^*)}+\xi^*)}{\xi(\xi^2-z^2)(e^{2(X+X^*)}+1)^2} & 0 \\ 0 & \frac{\xi^2-\xi^{*2}}{(\xi^{*2}+z^2)(e^{2(X+X^*)}+1)} - \frac{2i\alpha_1\alpha_2(\xi^2-\xi^{*2})(\xi+\xi^* e^{2(X+X^*)})}{\xi^*(\xi^{*2}+z^2)(e^{2(X+X^*)}+1)^2} \end{bmatrix}$$

$$d_2 = \begin{bmatrix} -\frac{(\xi^2-\xi^{*2})e^{X-X^*}\operatorname{sech}(X+X^*)}{2(\xi^2-z^2)} + \frac{i\alpha_1\alpha_2(\xi^2-\xi^{*2})e^{X-X^*}\operatorname{sech}(X+X^*)[(\xi-\xi^*)\tanh(X+X^*)+\xi^*]}{2\xi(\xi^2-z^2)} & 0 \\ 0 & \frac{(\xi^2-\xi^{*2})e^{X-X^*}\operatorname{sech}(X+X^*)}{2(\xi^{*2}+z^2)} + \frac{i\alpha_1\alpha_2(\xi^2-\xi^{*2})e^{X-X^*}\operatorname{sech}(X+X^*)[(\xi-\xi^*\tanh(X+X^*)-\xi]}{2\xi^*(\xi^{*2}+z^2)} \end{bmatrix}$$

$$d_3 = \begin{bmatrix} -\frac{(\xi^2-\xi^{*2})e^{X-X^*}\operatorname{sech}(X+X^*)}{2(\xi^2-z^2)} + \frac{i\alpha_1\alpha_2(\xi^2-\xi^{*2})e^{X-X^*}\operatorname{sech}(X+X^*)[(\xi-\xi^*)\tanh(X+X^*)+\xi^*]}{2\xi(\xi^2-z^2)} & 0 \\ 0 & \frac{(\xi^2-\xi^{*2})e^{X-X^*}\operatorname{sech}(X+X^*)}{2(\xi^{*2}+z^2)} + \frac{i\alpha_1\alpha_2(\xi^2-\xi^{*2})e^{X-X^*}\operatorname{sech}(X+X^*)[(\xi-\xi^*)\tanh(X+X^*)-\xi]}{2\xi^*(\xi^{*2}+z^2)} \quad 0 \end{bmatrix}$$

$$d_4 = \begin{bmatrix} -\frac{(\xi^2-\xi^{*2})e^{X+X^*}\operatorname{sech}(X+X^*)}{2(\xi^2-z^2)} + \frac{i\alpha_1\alpha_2(\xi-\xi^*)^2(\xi+\xi^*)e^{X+X^*}[\tanh(X+X^*)-1]\operatorname{sech}(X+X^*)}{2\xi(\xi^2-z^2)} & 0 \\ 0 & \frac{(\xi^2-\xi^{*2})e^{X+X^*}\operatorname{sech}(X+X^*)}{2(\xi^{*2}+z^2)} + \frac{i\alpha_1\alpha_2(\xi+\xi^*)(\xi-\xi^*)^2e^{X+X^*}[\tanh(X+X^*)-1]\operatorname{sech}(X+X^*)}{2\xi^*(\xi^{*2}+z^2)} \end{bmatrix} \tag{3.79}$$

Giving the expression for the dressing factor we can compute explicitly the field valued expression for the SSSSG theory:

$$\gamma = \chi(0)^{-1} \tag{3.80}$$

To obtain the inverse of the dressing factor one can consider the following reality condition:

$$\chi(z)^{-1} = H\chi(z^*)^{\dagger}H \tag{3.81}$$

as for the dressing factor,we present the result as 16 (2×2)-blocks matrices, the only non zero blocks are:

$$a_1 = \begin{bmatrix} -\frac{2i\alpha_1\alpha_2(\xi+\xi^*)(\xi^{*2}-\xi^2)}{\xi\xi^*(e^{2(X+X^*)}+1)} & 0 \\ 0 & \frac{2i\alpha_1\alpha_2(\xi^{*2}-\xi^2)}{\xi\xi^*(e^{2(X+X^*)}+1)} \end{bmatrix} \tag{3.82}$$

$$
\begin{aligned}
d_1 &= \begin{bmatrix} \frac{\xi^2-\xi^{*2}}{\xi^{*2}(e^{2(X+X^*)}+1)} + \frac{2i\alpha_1\alpha_2(\xi^{*2}-\xi^2)(\xi+\xi^* e^{2(X+X^*)})}{\xi^{*3}(e^{2(X+X^*)}+1)^2} & 0 \\ 0 & \frac{\xi^{*2}-\xi^2}{\xi^2(e^{2(X+X^*)}+1)} - \frac{2i\alpha_1\alpha_2(\xi^{*2}-\xi^2)(\xi e^{2(X+X^*)}+\xi^*)}{\xi^3(e^{2(X+X^*)}+1)^2} \end{bmatrix} \\
d_2 &= \begin{bmatrix} -\frac{(\xi^{*2}-\xi^2)e^{X-X^*}\operatorname{sech}(X+X^*)}{2\xi^{*2}} + \frac{i\alpha_1\alpha_2(\xi^{*2}-\xi^2)e^{X-X^*}\operatorname{sech}(X+X^*)[(\xi^*-\xi)\tanh(X+X^*)+\xi]}{2\xi^{*3}} & 0 \\ 0 & \frac{(\xi^{*2}-\xi^2)e^{X-X^*}\operatorname{sech}(X+X^*)}{2\xi^2} + \frac{i\alpha_1\alpha_2(\xi^{*2}-\xi^2)e^{X-X^*}\operatorname{sech}(X+X^*)[(\xi^*-\xi)\tanh(X+X^*)-\xi^*]}{2\xi^3} \end{bmatrix} \\
d_3 &= \begin{bmatrix} -\frac{(\xi^{*2}-\xi^2)e^{X-X^*}\operatorname{sech}(X+X^*)}{2\xi^{*2}} + \frac{i\alpha_1\alpha_2(\xi^{*2}-\xi^2)e^{X-X^*}\operatorname{sech}(X+X^*)[(\xi^*-\xi)\tanh(X+X^*)+\xi]}{2\xi^{*3}} & 0 \\ 0 & \frac{(\xi^{*2}-\xi^2)e^{X-X^*}\operatorname{sech}(X+X^*)}{2\xi^2} + \frac{i\alpha_1\alpha_2(\xi^{*2}-\xi^2)e^{(X-X^*)}\operatorname{sech}(X+X^*)[(\xi^*-\xi)\tanh(X+X^*)-\xi^*]}{2\xi^3} \end{bmatrix} \\
d_4 &= \begin{bmatrix} -\frac{(\xi^{*2}-\xi^2)e^{X+X^*}\operatorname{sech}(X+X^*)}{2\xi^{*2}} + \frac{i\alpha_1\alpha_2(\xi^*-\xi)^2(\xi+\xi^*)e^{X+X^*}[\tanh(X+X^*)-1]\operatorname{sech}(X+X^*)}{2\xi^{*3}} & 0 \\ 0 & \frac{(\xi^{*2}-\xi^2)e^{X+X^*}\operatorname{sech}(X+X^*)}{2\xi^2} + \frac{i\alpha_1\alpha_2(\xi+\xi^*-\xi^2)(\xi^*-\xi)^2e^{X+X^*}[\tanh(X+X^*)-1]\operatorname{sech}(X+X^*)}{2\xi^3} \end{bmatrix}
\end{aligned}
\tag{3.83}
$$

notice that if we turn to 0 the grassmann variables we obtain the purely bosonic soliton solution showed in the case of SSSG theory. Giving the dressing factor it's also possible to compute the solution for the 2d-sigma model:

$$f = \Upsilon_0(1)^{-1}\chi(1)^{-1} \tag{3.84}$$

Chapter 4

Conclusions

This work moves from the AdS/CFT correspondence, a duality which enstablish the equivalance between a string theory and a conformal field theory. This conjecture received much attention due to its potential application to the study of strong-coupled gauge theories relative to the standard model. The most important and well understood case of correspondence is the one which connects the Type IIB superstring theory on $\mathrm{Ads}_5 \times \mathrm{S}_5$ to $N = 4$ super Yang-Mills theory in four dimensions. Among the many aspects of this duality, a very interesting one concerns the non-perturbative scheme in the AdS side. The classical integrability of Type IIB superstring theory on $\mathrm{Ads}_5 \times \mathrm{S}_5$ plays a prominent role in studying non-perturbative objects as solitons: parcticle-like exact solutions of non-linear PDEs. This work really focused on computing the explicit soliton solution for generalized sine-Gordon theories connected to superstring theory through the so called "Pohlmeyer reduction" procedure. In the first chapter, we presented the string world sheet theory as a 2d sigma model on S_5 symmetric space which is a coset based on $SO(6)/SO(5)$ and its generalization to semi-simmetryc space $\mathrm{AdS}_5 \times \mathrm{S}_5$ based on $PSU(2, 2|4)$ super-group (as we have previously seen, the last one is a generalization which involves fermionic fields). In the second chapter, we described how to apply the Pholmeyer reduction for both cases of interest. The procedure leads to equations of motion written in terms of new currents which involve only physical degrees of freedom, in particular, we understood that the 2d sigma model on S_5 and $\mathrm{Ads}_5 \times \mathrm{S}_5$ is equivalent to a sine-gordon like theory perturbed by a potential term, respectively in a symmetric and semi-symmetric space background. In the last chapter we explicitly constructed the soliton solutions using the dressing method. We retraced the steps of this tecnique for SSSG on S_5 background, then, exploiting the invariance of the dressing data we could choose an appropriate one to explicitly construct a soliton solution for the case of $\mathrm{Ads}_5 \times \mathrm{S}_5$[11] with two Grassmann odd variables.The result is a generalization of the purely bosonic case, that exhibits the right structure and well verifies all the properties provided by the algebra structure and by the dressing method.

Appendix A

Conformal Trnasformations in D=2

This chapter provide some basics concept on 2d CFT field theory. Compared to ordinary quantum field theories in four dimensions, conformal field theories in two dimensions can be defined in a rather abstract way via operator algebras and their representation theory. In fact, there are many examples of CFTs where the usual description in terms of a Lagrangian action with resulting perturbative expansion is not even known. Instead, following a so-called boot-strap approach, one can define these theories without making reference to an action and sometimes one can even solve them exactly. Such a procedure is possible because the algebra of infinitesimal conformal transformations in two dimensions is infinite dimensional and therefore highly constraining. The main feature of a conformal field theory is the invariance under conformal transformations. In other word these are transformations leaving angles invariant and a particular example is the scaling $x \to ax$ of a point x by some constant a. A field theory exhibiting such a symmetry has no preferred scale and one can only expect a physical system to have this property, if there are no dimensionful scales involved. Although the mathematical property of CFT in lower dimension, this type of theories posses a vast area of applications like the description of critical fenomena and second order phase transitions. Here, the CFT arises as a two-dimensional field theory living on the world-volume of a string which moves in some background spacetime.The dynamics of this string is governed by a so-called non-linear sigma model. Then, physical systems with a conformal symmetry are thus more common than one would have naively expected. We can start by instroducing conformal transformations and determing the condition for conformal invariance. Let us consider differantiable maps ϕ: U $\to V$, where U$\subset M$ and $V \subset M'$ are open subsets. A map ϕ is called *Conformal Transformation* if the metric tensor satisfies $\phi * g' = \Lambda g$. Denoting $x' = \phi(x)$ with $x \in U$ we can express this condition in the following way:

$$g'_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{x'\sigma}{x^{\nu}} = \Lambda(x)g_{\mu\nu}(x) \tag{A.1}$$

where the positive function $\Lambda(x)$ is called "scale factor". In the case of interest, $g = g'$ and the metric is of the form $\eta_{\mu\nu} = \mathrm{diag}(-1, ... + 1, ...)$.In this case the conformal transformation became:

$$\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\eta_{\mu\nu} \tag{A.2}$$

For $= 0$ we recover simply the poincar group. Now consider an infinitesimal coordinate transformation:

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + O(\epsilon^2) \tag{A.3}$$

we want find a restriction so that this transofmations it's conformal. Transforming the first member of (A.2) we have:

$$\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \eta_{\mu\nu} + \frac{\partial \epsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial \epsilon_{\nu}}{\partial x^{\mu}} + O(\epsilon^2) \tag{A.4}$$

Then, the condition is:

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = K(x)\eta_{\mu\nu} \tag{A.5}$$

where $K(x)$ is a function which we can determine tracing the eqaution above with $\eta_{\mu\nu}$:

$$\eta^{\mu\nu}(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}) = K(x)\eta^{\mu\nu}\eta_{\mu\nu}2\partial^{\mu}\epsilon_{\mu} = K(x)d \tag{A.6}$$

Solving for $K(x)$d we have the following restriction of (A.3) to be conformal:

$$\partial_{\mu\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial\dot{\epsilon})\eta_{\mu\nu} \tag{A.7}$$

Obteined that restriction to the infinitesimal transformation of coordinate, we can define the conformal group and conformal algebra in the case of $D = 2$. The condition A.6 for invariance under infinitesimal conformal transformations in two dimensions is:

$$_0\epsilon_0 = +\partial_1\epsilon_1 \quad \partial_0\epsilon_1 = -\partial_1\epsilon_0 \tag{A.8}$$

we see that this equations are the familiar Cauchy-Riemann equations which specifies, in an open set, how they should be the real and imaginary part of a complex function to do holomorphic. Turning to complex variables in the following way:

$$z = x^0 + ix^1 \quad \epsilon = \epsilon^0 + i\epsilon^1 \quad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1), \overline{z} = x^0 - ix^1 \quad \epsilon = \epsilon^0 - i\epsilon^1 \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_0 + i\partial_1), \tag{A.9}$$

Since $\epsilon(z)$ is holopmorphic, so is $f(z) = z + \epsilon(z)$ from which we conclude that:

A holomorphic function $f(z) = z + \epsilon(z)$ gives rise to an infinitesimal two-dimensional conformal transformation $z \to f(z)$

remembering that $\epsilon(z)$ has to be holomorphic, we can expand it in laurent series:

$$z' = z + \epsilon(z) = z + \sum_{n \in Z} \epsilon_n(-z^{n+1}), \overline{z}' = \overline{z} + \overline{\epsilon}(\overline{z}) = \overline{z} + \sum_{n \in Z} \overline{\epsilon}_n(-\overline{z}^{(n+1)}), \tag{A.10}$$

The generators are:

$$l_n = -z^{n+1}\partial_z \quad \bar{l}_n = -\overline{z}^{n+1}\partial_z \tag{A.11}$$

So, the number of indipendento infinitesimal transformations is *infinite*. Compute the commutator between that generators we have:

$$[l_m, l_n] = (m-n)l_{m+n} \tag{A.12}$$

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \tag{A.13}$$

$$[l_m, \bar{l}_n] = 0 \tag{A.14}$$

then we can say that:

The algebra of infinitesimal conformal transformations in an Euclidean two-dimensional space is infinite dimensional

This is a very important property of two dimension conformal transformations, because is crucial for proving the integrability of conformal field theory. Note that, considering the algebra generated by l_n, the generators are not defined everywhere in $\mathbb{R}^2 = C$ In particular we have the $z = 0$ point and the point inf which are ambiguous. To resolve this ambiguity we have to consider not $\mathbb{R}^2$ but his conformal compactification S^2. Analyzing the expression of the generators in $z = 0, z = \infty$:

- z=0:

  $$l_n = -z^{n+1}\partial_z, \tag{A.15}$$

  non singular at z=0 only for $n \geq -1$

- $z = \infty$, performing the change of variables$z = -\frac{1}{w}$ and study for $w \to 0$

  $$l_n = -(-\frac{1}{w})^{n-1}\partial_w \tag{A.16}$$

  non singular at $w = 0$ only for $n \leq +1$

So we can conclude that:

Globally defined conformal transformations on the Riemann sphere $S^2 = C \cup \infty$ are generated by $\{l_{-1}, l_0, l_{+1}\}$.

After having determined the operators generating global conformal transformations we wll now determine the conformal group. We have:

- l_{-1} generates the translations $z \to z + b$
- l_0 generates transformations of the type:$z \to az$ with $a \in C$.
- The operator l_{+1} corresponds to Special Conformal Transformations.

In summary, we have that the operators$\{l_{-1}, l_0, l_{+1}\}$ generate trasformations of the form:

$$z \to \frac{az + b}{cz + d} \quad a, b, c, d \in C \tag{A.17}$$

The conformal group of the Riemann sphere $S^2 = C \cup \infty$ is the mobius group $SL(2, C)/Z_2$

Usually, a Field Theory is defined in terms of a Lagrangian action from which one can derive various objects and properties of the theory.Since the algebra of infinitesimal conformal transformations in two dimensions is infinite dimensional, there are strong constraints on a conformal field theory. In particular, it turns out to be possible to study such a theory without knowing the explicit form of the action. The only information needed is the behaviour under conformal transformations which is encoded in the energymomentum tensor. Then we concluding this brefly review with the following remarkable property of conformal field theories:

In a conformal field theory, the energymomentum tensor $T_{\mu\nu}$ is traceless, that is, T^{μ}_{μ}

This result leads directly from the study of energy-momentum tensor under conformal transformations. In two dimensional theories, in particular have:

The two non-vanishing components of the energymomentum tensor are a chiral and an anti-chiral field

$$T_{zz}(z, \overline{z}) =: T(z), \quad T_{\overline{zz}}(z, \overline{z}) =: \overline{T}(\overline{z}) \tag{A.18}$$

Appendix B

PSU$(2,2|4)$ realization

In this appendix we provide a particular matrix rapresentation of $\mathfrak{psu}(2,2|4)$ used explicitly in the chapter 3.

The generic element of the algebra $\mathfrak{psu}(2,2|4)$ can be written as follows:

$$\hat{f} = \begin{bmatrix} m & \theta \\ \eta & n \end{bmatrix}, \quad \mathrm{Tr}(m) = \mathrm{Tr}(n) = 0 \tag{B.1}$$

The non compact real form $psu(2,2|4)$ is picked out Imposing the reality condition:

$$\hat{f} = -H\hat{f}^{\dagger}H, \tag{B.2}$$

where:

$$H = \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_4 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix} \tag{B.3}$$

m, n are 4×4 matrices whose elements are Grassmann even and θ, η are Grassmann odd. The algebra $psu(2,2|4)$ admits a $\mathbb{Z}_4$ automorphism defined:

$$\hat{f} \to \sigma_-(\hat{f}) = \mathcal{K}M^{\mathrm{st}}\mathcal{K}^{-1} \tag{B.4}$$

where:

$$\mathcal{K} = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix} \qquad J = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{B.5}$$

And the st denotes the super-transposition defined as:

$$\hat{f}^{\mathrm{st}} = \begin{bmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{bmatrix} \tag{B.6}$$

We can decompose $\hat{f}$ throught a $\mathbb{Z}_4$ automorphism as follows:

$$\hat{f} = \hat{f}_0 \oplus \hat{f}_1 \oplus \hat{f}_2 \oplus \hat{f}_3 \qquad\qquad \sigma_-(f_{(j)}) = i^j f_{(j)} \tag{B.7}$$

the bosonic subspaces $\mathfrak{f}_0, \mathfrak{f}_2$ are rapresented by matrices of the form:

$$f = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \quad \Sigma m^\dagger \Sigma = -m, \quad n^\dagger = -n, \tag{B.8}$$

in particular, the subspace $\mathfrak{f}_0$ is formed by matrices satisfying also:

$$-\Sigma m_0^t \Sigma = m_0, \quad \Sigma n_0^t \Sigma = n_0, \tag{B.9}$$

Furthermore we have these conditions on $\mathfrak{f}_2$ subspace

$$\mathcal{K} m_{(2)}^t \mathcal{K} = m_{(2)}^t \qquad\qquad \mathcal{K} n_{(2)}^t \mathcal{K} = n_{(2)}^t \tag{B.10}$$

Now we want explicit the fermionic sector $\mathfrak{f}_1 \oplus \mathfrak{f}_3$. The form of matrices in $\mathfrak{f}_1$ is:

$$\begin{pmatrix} 0 & \theta \\ \eta & 0 \end{pmatrix} \quad \mathcal{K}\eta^t\mathcal{K} = i\theta, \qquad\qquad i\Sigma\eta^\dagger = \theta \tag{B.11}$$

Since $\Sigma = \mathcal{K}$ gives $\eta^\dagger + = -\eta^t\mathcal{K}$.

The reality condition for the subspace $\mathfrak{f}_3$ is:

$$i\theta_{(3)} = \mathcal{K}\eta_{(3)}^t\mathcal{K} \tag{B.12}$$

The subspaces of this decomposition fulfil the following commutation relation:

$$[\hat{f}_m, \hat{f}_n] \subset \hat{f}_{m+n}^{\mathrm{mod}(4)} \tag{B.13}$$

identifyng $\hat{f}_0 = \mathfrak{g}$ and $\hat{f}_2 = \mathfrak{p}$ then $\mathfrak{g}$ forms a subalgebra. It is possible to perform a $\mathbb{Z}_2$ decomposition allowing to define the group H in the G/H gauged WZW model. If we identify $\mathfrak{h} = \hat{f}_0^\perp$,$\mathfrak{m} = \hat{f}_0^{||}$,$\mathfrak{a} = \hat{f}_2^\perp$, $\mathfrak{n} = \hat{f}_2^{||}$. $\mathfrak{h}$ is thus a subalgebra and the corresponding subgroup is then identified as the group H in the G/H gauged WZW model. It is possible to show that $\mathfrak{h}$ has the following form:

$$\begin{bmatrix} \mathfrak{su}(2) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathfrak{su}(2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathfrak{su}(2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathfrak{su}(2) \end{bmatrix} \tag{B.14}$$

The physical fields of the Pohlmeyer-reduced theory, X, ψ_R and ψ_L , take values in $\hat{f}_0^{||}$,$\hat{f}_1^{||}$ and $\hat{f}_3^{||}$ respectively. Here we explcitly write out the basis of these subspaces. An arbitrary element of bosonic subspace $\hat{f}_0^{\|}$ is:

$$\begin{bmatrix} 0 & 0 & x_1 + ix_2 & -x_3 - ix_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_3 + ix_4 & -x_1 + ix_2 & 0 & 0 & 0 & 0 \\ x_1 - ix_2 & -x_3 - ix_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_3 + ix_4 & -x_1 - ix_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_5 + ix_6 & x_7 + ix_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_7 + ix_8 & x_5 - ix_6 \\ 0 & 0 & 0 & 0 & -x_5 + ix_6 & x_7 + ix_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_7 + ix_8 & -x_5 - ix_6 & 0 & 0 \end{bmatrix} \tag{B.15}$$

where x_i are commuting variables. An arbitrary element of fermionic subspace $\hat{f}_1^{\|}$ is:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 + i\alpha_2 & \alpha_3 + i\alpha_4 & \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 & \\ 0 & 0 & 0 & 0 & \alpha_5 - i\alpha_6 & -\alpha_7 - i\alpha_8 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \alpha_7 + i\alpha_8 & \alpha_5 - i\alpha_6 & 0 & 0 & \\ 0 & 0 & i\alpha_5 + \alpha_6 & i\alpha_7\alpha_8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\alpha_7 + \alpha_8 & i\alpha_5 + \alpha_6 & 0 & 0 & 0 & 0 & \\ i\alpha_1 + \alpha_2 & i\alpha_3 + \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & \\ i\alpha_3 + \alpha_4 & i\alpha_1 - \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & \end{bmatrix} \tag{B.16}$$

where α_i are anticommuting variables. The arbitrary element of the fermionic subspace $\hat{f}_3^{\|}$ can be written in terms of $f_1^{\|}(\alpha_i)$ as :

$$f_3^{\|}(\alpha_i) = 2\Lambda f_1^{\|}(\alpha_i) \tag{B.17}$$

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