Mathematical and Algorithmic Methods for Finding Disjoint Rosa-type Sequences

by

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Abstract

A Rosa sequence of order n is a sequence $S = (s_1, s_2, \dots, s_{2n+1})$ of $2n+1$ integers satisfying the conditions: (1) for every $k \in \{1, 2, \dots, n\}$ there are exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$; (2) if $s_i = s_j = k$, $i < j$, then $j - i = k$; and (3) $s_{n+1} = 0$ (s_{n+1} is called the hook). Two Rosa sequences S and S' are disjoint if $s_i = s_j = k = s'_t = s'_u$ implies that $\{i, j\} \neq \{t, u\}$, for all $k = 1, \dots, n$.

In 2014, Linek, Mor, and Shalaby [18] introduced several new constructions for Skolem, hooked Skolem, and Rosa rectangles.

In this thesis, we gave new constructions for four mutually disjoint hooked Rosa sequences and we used them to generate cyclic triple systems $CTS_4(v)$. We also obtained new constructions for two disjoint m-fold Skolem sequences, two disjoint m-fold Rosa sequences, and two disjoint indecomposable 2-fold Rosa sequences of order n. Again, we can use these sequences to construct cyclic 2-fold 3-group divisible design 3-GDD and disjoint cyclically indecomposable $CTS_4(6n+3)$. Finally, we introduced exhaustive search algorithms to find all distinct hooked Rosa sequences, as well as maximal and maximum disjoint subsets of (hooked) Rosa sequences.

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"Do not allow others to be an obstacle to the achievement of dreams; success is available to everyone if we want it" –Fatimah Alruhaymi.

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Chapter 1

Introduction

1.1 History of Skolem-type Sequences

A *Steiner triple system*, denoted as $STS(v)$, is a set of triples formed using v distinct elements, such that each pair of elements occurs in the same triple exactly once. This type of system was found by Plücker [25], in 1839. For example, the sets $\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\},\{5,6,1\},\{6,7,2\},\{7,1,3\}$ form a Steiner triple system with $v = 7$, hence is a $STS(7)$. This Steiner triple system is *cyclic*, since for every triple $\{a,b,c\}, \{a+1,b+1,c+1\}$ is also a triple. In 1847, Kirkman [16] established the existence of a $STS(v)$ for all possible orders; although, he would not be recognized for this contribution for many years [6]. Whether a Steiner triple system exists, was a question posed by Steiner in 1853 [38].

There are two difference problems stated by Heffter [14] in 1897, and the solutions of which are equivalent to the existence of cyclic Steiner triple systems. The first difference problem he asked was if the set $\{1,\ldots,3n\}$ can be partitioned into n ordered triples $\{a_i,b_i,c_i\}$, $i = 1,\ldots,n$, with the condition that for each i, $a_i + b_i \pm c_i \equiv 0$ $(mod\ 6n+1)$. For such a partition, the sets $\{r, a_i + r, b_i + i + r\}$,

$1 \leq i \leq n$, $0 \leq r \leq 6n$ form a cyclic $STS(6n+1)$. For instance, the sets $\{1,3,4\}$ and $\{2,5,6\}$ provide such a partition for $n=2$, so the sets $\{r, 1+r, 4+r\}, \{r, 2+r, 7+r\}, 0 \leq r \leq 12$, form a $STS(13)$. The second difference problem he proposed was a partition of the set $\{1,2,\ldots,2n,2n+2,\ldots,3n+1\}$ into n ordered triples $\{a_i, b_i, c_i\}, i=1,\ldots,n$, such that for each triple, $a_i + b_i \pm c_i \equiv 0 \ (mod\ 6n+3)$. For such a partition, the sets $\{r, a_i+r, a_i+b_i+r\}, \{k, 2n+1+k, 4n+2+k\}$, $1 \leq i \leq n, 0 \leq r \leq 6n+2, 0 \leq k \leq 2n$ form a cyclic $STS(6n+3)$. For example, a solution to Heffter's second difference problem when $n=2$ is $\{1,3,4\}, \{2,6,7\}$, and the sets $\{r,1+r,4+r\}, \{r,2+r,8+r\}, \{k,5+k,10+k\}, 0 \leq r \leq 14, 0 \leq k \leq 4$ form a $STS(15)$. Solutions to Heffter's first difference problem for all $n \geq 1$ and to Heffter's second difference problem for all $n \geq 2$ were found by Peltesohn [24]. The existence of a cyclic $STS(v)$ for all $v \equiv 1, 3 \ (mod\ 6)$, $v \neq 9$ was proved and there is no cyclic $STS(9)$.

While studying Steiner triple systems, Skolem [34] in 1957, considered the following question: "Is it possible to distribute the numbers $1,2,3,\ldots,2n$ in n pairs (a_r, b_r) such that we have $b_r - a_r = r$ for $r = 1,2,\ldots,n$?" and this question led to the study of Skolem sequences. For $n=4$, $\{(1,2),(4,6),(5,8),(3,7)\}$ is an example of a solution to Skolem's problem. The triples (r, a_r+n, b_r+n), $r=1,\ldots,n$ for any solution $\{(a_r,b_r)\}$ to Skolem's problem form a solution to Heffter's first difference problem. Skolem proved that such a distribution is possible if and only if $n \equiv 0,\ 1 \ (mod\ 4)$. In 1958, he proved that the existence of this kind of partition of $\{1,\ldots,2n\}$ implies the existence of a $STS(6n+1)$ [35]. Also Skolem conjectured that a similar partitioning of $\{1,\ldots,2n-1,2n+1\}$ into n pairs (a_r,b_r) with $b_r - a_r = r, r=1,\ldots,n$ would be possible if and only if $n \equiv 2,\ 3 \ (mod\ 4)$ [35]. In 1961, this conjecture was proved true by O'Keefe [23].

Rosa [28] proved in 1966 that a partition of $\{1,\ldots,n,n+2,\ldots,2n+1\}$ into n

pairs (a_r, b_r) with $b_r - a_r = r,\ r = 1, \ldots, n$ is possible if and only if $n \equiv 0,\ 3\ (mod\ 4)$. The triples $(r, a_r + n, b_r + n)$ for such a partition give a solution to Heffter's second difference problem, and thus the existence of such a partition implies the existence of a cyclic $STS(6n + 3)$. He also showed that a similar partition of $\{1, \ldots, n, n + 2, \ldots, 2n, 2n + 2\}$ exists if and only if $n \equiv 1,\ 2\ (mod\ 4)$, where $n \geq 2$ [28].

In 1966, Nickerson [21] was the first to introduce sequential notation for Skolem sequences. He proposed a problem equivalent to Skolem's regarding a sequence which contains each integer $1, 2, \ldots, n$ exactly twice, such that for any integer i, the second appearance of i occurs exactly i positions after the first. The set of ordered pairs of positions $\{(a_i, b_i)\}$, $1 \leq i \leq n$, for such a sequence, forms a solution of Skolem's partitioning problem; for any set of pairs $\{(a_i, b_i)\},\ 1 \leq i \leq n$, which comprises a solution of Skolem's problem, a sequence is formed in which a_i and b_i are ordered pairs of positions i, which is the type described by Nickerson. The sequence $(1, 1, 4, 2, 3, 2, 4, 3)$ is an example which satisfies the conditions proposed by Nickerson and is equivalent to the solutions $\{(1, 2), (4, 6), (5, 8), (3, 7)\}$ to Skolem's partitioning problem. Such sequences became known as Skolem sequences. Solutions to the other types of partitioning problems described above may similarly be written as sequences known as hooked Skolem sequences, Rosa sequences, and hooked Rosa sequences.

Mathematicians have found many different methods to construct Skolem sequences, and have used similar methods in other applications. Some mathematicians, such as Davies [7], Hanani [13], Anderson [1], Hilton [15], and Stanton and Goulden [37] have also constructed sequences or partitions to provide constructions of Steiner triple systems.

In 1993, Baker and Shalaby [4] constructed disjoint (hooked) Skolem sequences. They applied these sequences to the construction of disjoint cyclic Steiner triple systems, Mendelsohn triple systems, and disjoint 1-coverings.

Hill-climbing is a local heuristic search (local improvement algorithm). Tovey (1985) showed that, even though hill-climbing is very quick to reach the local optima, it is unlikely to find the global optima [40]. In 1998, Eldin, Shalaby, and Althukair [9] used a hill-climbing algorithm to generate Skolem sequences for $n = 0,\ 1(mod\ 4)$, for example constructing Skolem sequences of order 84.

A λ-fold triple system of order v, denoted $TS_\lambda(v)$, is a collection B of 3-subsets (called triples or blocks) from a v-set V, such that any given pair of elements in V lies in exactly λ triples. A one-fold triple system is called a *Steiner triple system*, $STS(v)$. A $TS_\lambda(v)$ is simple if it contains no repeated triples. A $TS_\lambda(v)$ is cyclic, denoted $CTS_\lambda(v)$, if its automorphism group contains a v-cycle [26]. In 2005, Grüttmüller, Rees, and Shalaby [12] investigated exhaustively all $CTS_2(v)$ that are constructed by Skolem-type and Rosa-type sequences up to $v \leq 45$ and presented the numbers of distinct Skolem and Rosa sequences of order $n \leq 13$.

In 2014, Linek, Mor, and Shalaby [18] used two types of techniques to construct (hooked) Skolem and Rosa rectangles. These techniques complement each other, as the direct constructions fill the gaps of small orders and the asymptotic constructions provide the only known non-trivial bounds for these rectangles.

Skolem sequences and their generalizations are linked to several combinatorial designs, e.g., Room squares and perfect one-factorization of complete graphs [29]. The known applications of Skolem sequences and their generalizations in the physical world include the interference-free missile guidance code [8] and the construction of binary sequences with controllable complexity [11]. More details may be found in [30]. Wythoff pairs and partitions of sets of numbers were established by Nowakowski [22], which are connections between generalized Skolem and Langford sequences and the golden mean [10]. Furthermore, Rosa made a fine art of applying Skolem-type sequences and their generalizations to combinatorial designs [10].

1.2 Preliminaries

Skolem sequences were first studied for use in constructing cyclic Steiner triple systems. Similar sequences have also been studied and used in the construction of certain combinatorial objects [34]. In this section, we discuss the definitions, the existence, and some examples for (hooked) Skolem sequences, (hooked) Rosa sequences, m-fold Skolem sequences, m-fold Rosa sequences, disjoint sequences, Steiner triple systems STS(v), and group divisible designs GDD.

A *Skolem sequence* of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions: (1) for every $k \in \{1, 2, \ldots, n\}$ there are exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$; and (2) if $s_i = s_j = k,\ i < j$, then $j - i = k$. A *hooked Skolem sequence* of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n+1})$ of $2n + 1$ integers satisfying conditions (1) and (2) as for Skolem sequences, with the added condition that, (3) $s_{2n} = 0$. A sequence $S = (s_1, s_2, \ldots, s_{2n+1})$ of $2n + 1$ integers is a *Rosa sequence* of order n satisfying conditions (1) and (2) as for Skolem sequences, with the added condition that (3) $s_{n+1} = 0$. A *hooked Rosa sequence* of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n+2})$ of $2n + 2$ integers satisfying conditions (1) and (2) as for Skolem sequences, with the added condition that (3) $s_{n+1} = s_{2n+1} = 0$.

A sequence $(4, 2, 3, 2, 4, 3, 1, 1)$ is a Skolem sequence of order 4. The sequence may also be represented as the set of ordered pairs $\{(7, 8),(2, 4),(3, 6),(1, 5)\}$. For each of these pairs (a, b), a and b are the positions of $b - a$ in the sequence. For example, (6, 1, 1, 5, 3, 4, 6, 3, 5, 4, 2, 0, 2) is a hooked Skolem sequence of order 6, $(1, 1, 3, 4, 0, 3, 2, 4, 2)$ is a Rosa sequence of order 4, and (4, 5, 1, 1, 4, 0, 5, 2, 3, 2, 0, 3) is a hooked Rosa sequence of order 5.

The necessary conditions for the existence of (hooked) Skolem, and (hooked) Rosa sequences will be given by the next theorem.

Theorem 1.2.1 (1) *[34] A Skolem sequence of order n can exist only if $n \equiv 0, 1 \pmod 4$.*

(2) *[23] A hooked Skolem sequence of order n can exist only if $n \equiv 2, 3 \pmod 4$.*

(3) *[28] A Rosa sequence of order n can exist only if $n \equiv 0, 3 \pmod 4$.*

(4) *[28] A hooked Rosa sequence of order n can exist only if $n \equiv 1, 2 \pmod 4$.*

Proof: We first consider statement (1):

Consider a set of ordered pairs of positions $\{(a_r, b_r) : r = 1, 2, \ldots, n\}$ for a Skolem sequence of order n. Since $b_r - a_r = r$, then

$$\sum_{r=1}^{n} b_r - \sum_{r=1}^{n} a_r = \sum_{r=1}^{n} (b_r - a_r) = \sum_{r=1}^{n} r = \frac{n(n+1)}{2}. \tag{1.1}$$

However, together these numbers a_r and b_r, $r = 1, 2, \ldots, n$ comprise the set $\{1, 2, \ldots, 2n\}$. Therefore,

$$\sum_{r=1}^{n} b_r + \sum_{r=1}^{n} a_r = \sum_{r=1}^{n} (b_r + a_r) = \sum_{r=1}^{2n} r = \frac{(2n)(2n+1)}{2} = n(2n+1). \tag{1.2}$$

Adding (1.1) and (1.2) gives:

$$2\textstyle\sum_{r=1}^{n} b_r = \frac{n(n+1)}{2} + n(2n+1) = \frac{n(5n+3)}{2}.$$

That is,

$$\textstyle\sum_{r=1}^{n} b_r = \frac{n(5n+3)}{4}.$$

For each $r \in \{1, 2, \ldots, n\}$, b_r is an integer. Therefore, $\sum_{r=1}^{n} b_r$ must also be an integer. Therefore, a Skolem sequence can exist only if $n \equiv 0$ or 1 $(mod\ 4)$.

The verification of results (2), (3), and (4) are similar to the proof of necessary conditions for Skolem sequences. ■

A *Langford sequence* of defect d and length m is a sequence $L = (l_1, l_2, \ldots, l_{2m})$ satisfying the conditions: (1) for every $k \in \{d, d+1, \ldots, d+m-1\}$ there are exactly two elements $l_i, l_j \in L$ such that $l_i = l_j = k$; and (2) if $l_i = l_j = k,\ i < j$, then $j - i = k$. An example of a Langford sequence with defect $d = 4$ and length $m = 7$ is $(10, 8, 6, 4, 9, 7, 5, 4, 6, 8, 10, 5, 7, 9)$. A Langford sequence of defect d and length m exists if and only if $m \geq 2d - 1$ and $m \equiv 0, 1 \pmod 4$ for d odd, or $m \equiv 0, 3 \pmod 4$ for d even [33]. Thus, Skolem sequences are Langford sequences with defect $d = 1$.

A *hooked Langford sequence* of defect d and length m is a sequence $L = (l_1, l_2, \ldots, l_{2m+1})$ satisfying the conditions (1) and (2) as for Langford sequences, with the added condition that (3) $l_{2m} = 0$. For example, (14, 12, 13, 7, 5, 6, 10, 11, 9, 5, 7, 6, 8, 12, 14, 13, 10, 9, 11, 0, 8) is a hooked Langford sequence of defect $d = 5$ and length $m = 10$. A hooked Langford sequence of defect d and length m exists if and only if $m(m + 1 - 2d) + 2 \geq 0$ and $m \equiv 2, 3 \pmod 4$ for d odd, or $m \equiv 1, 2 \pmod 4$ for d even [33]. Thus, hooked Skolem sequences are hooked Langford sequences with defect $d = 1$.

If m and n are integers with $m \leq n$, an *m-near Skolem sequence* of order n and defect m is an integer sequence $S = (s_1, s_2, \ldots, s_{2n-2})$ satisfying the conditions: (1) for every $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$ there are exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$; and (2) if $s_i = s_j = k,\ i < j$, then $j - i = k$. A sequence $(6, 1, 1, 5, 3, 4, 6, 3, 5, 4)$ is a near-Skolem sequence of order 6 and defect 2.

A *hooked m-near Skolem sequence* of order n and defect m is a sequence $S = (s_1, s_2, \ldots, s_{2n-1})$ of $2n - 1$ integers satisfying the conditions (1) and (2) as for near-Skolem sequences, with the added condition that (3) $s_{2n-2} = 0$. An example of a hooked near-Skolem sequence of order 5 and defect 4 is $(5, 3, 1, 1, 3, 5, 2, 0, 2)$.

Theorem 1.2.2 (1) *[30] An m-near Skolem sequence of order n exists if and only if*

$n \equiv 0, 1 \pmod 4$ *and* m *is odd, or* $n \equiv 2, 3 \pmod 4$ *and* m *is even.*

(2) *[30] A hooked* m*-near Skolem sequence of order* n *exists if and only if* $n \equiv 0, 1 \pmod 4$ *and* m *is even, or* $n \equiv 2, 3 \pmod 4$ *and* m *is odd.*

A *near-Rosa sequence* of order n and defect m is a sequence $S = (s_1, s_2, \ldots, s_{2n-1})$ of $2n-1$ integers satisfying the conditions (1) and (2) as for near-Skolem sequences, with the added condition that (3) $s_n = 0$. A *hooked near-Rosa sequence* of order n and defect m is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions (1) and (2) as for near-Skolem sequences, with the added condition that (3) $s_{n+1} = s_{2n-1} = 0$. A sequence $(2, 5, 2, 4, 0, 3, 5, 4, 3)$ is a 1-near-Rosa sequence of order 5.

Theorem 1.2.3 (1) *[32] An* m*-near Rosa sequence of order* n *exists if and only if* $n \equiv 0, 3 \pmod 4$ *and* m *is even, or* $n \equiv 1, 2 \pmod 4$ *and* m *is odd, with the exceptions* $(n, m) = (3, 2), (4, 2)$.

(2) *[32] A hooked* m*-near Rosa sequence of order* n *exists if and only if* $n \equiv 0, 3 \pmod 4$ *and* m *is even, or* $n \equiv 1, 2 \pmod 4$ *and* m *is odd, with the exception* $(n, m) = (2, 1), (3, 3)$.

Two (hooked) Skolem or (hooked) Rosa sequences S and S' of order n are disjoint if $s_i = s_j = k = s'_t = s'_u$ implies that $\{i, j\} \neq \{t, u\}$, for all $k = 1, \ldots, n$. That is, the two occurrences of i are not in the same two positions in both sequences. For example, $(1, 1, 4, 2, 3, 2, 4, 3)$ and $(2, 3, 2, 4, 3, 1, 1, 4)$ are two disjoint Skolem sequences of order 4 even though both have a 3 in position 5. The sequences $(5, 7, 1, 1, 6, 5, 3, 4, 7, 3, 6, 4, 2, 0, 2)$ and $(6, 1, 1, 5, 7, 2, 6, 2, 5, 3, 4, 7, 3, 0, 4)$ are two disjoint hooked Skolem sequences of order 7. However, $(4, 2, 3, 2, 4, 3, 1, 1)$ and $(4, 1, 1, 3, 4, 2, 3, 2)$ are not disjoint since the 4s fall in positions 1 and 5 in both sequences.

A set of m pairwise disjoint hooked Rosa sequences forms a hooked Rosa rectangle of strength m [18]. For example, the following two sequences of order 5 form a hooked Rosa rectangle of strength 2:

$$(3, 1, 1, 3, 5, 0, 2, 4, 2, 5, 0, 4)$$

$$(4, 5, 1, 1, 4, 0, 5, 2, 3, 2, 0, 3).$$

A sequence $mS = (s_1, s_2, \ldots, s_{2mn})$ satisfying the following condition: for every $k \in \{1, 2, \ldots, n\}$ there exist m disjoint pairs $(i, i+k), i, i+k \in \{1, 2, \ldots, 2mn\}$ such that $s_i = s_{i+k} = k$ is an *m-fold Skolem sequence* of order n [27]. For example, (2, 3, 2, 2, 3, 2, 1, 1, 3, 1, 1, 3) is a 2-fold Skolem sequence of order 3. An *m-fold Rosa sequence* of order n is a sequence $mS = (s_1, s_2, \ldots, s_{2mn+2})$ with the following conditions: (1) for every $k \in \{1, 2, \ldots, n\}$ there exist m disjoint pairs $(i, i+k)$, where $i, i+k \in \{1, 2, \ldots, 2mn+2\}$ such that $s_i = s_{i+k} = k$; and (2) $s_{n+1} = s_{3n+2} = 0$. The sequence $(3, 1, 1, 3, 5, 0, 2, 3, 2, 5, 3, 4, 1, 1, 5, 4, 0, 4, 2, 5, 2, 4)$ is a 2-fold Rosa sequence of order 5. It is easy to see that there is no 2-fold Rosa sequence of order 1 [27]. Note that a 1-fold Skolem-type sequence is a Skolem-type sequence [10].

A $(v, k, \lambda; n)$-*difference set* of order $n = k - \lambda$ in an abelian group G of order v is a collection, D_1, D_2, …, D_t, of subsets of G, each of size k, such that each nonzero element of G occurs as a difference of elements in one of the D_i exactly λ times. For example, the set {1, 3, 4, 5, 9} is an $(11, 5, 2; 3)$-*difference set* in the group $\mathbb{Z}_{11}$.

A *Steiner triple system* of order v, STS(v) is a pair (V, B), where V is a set of points, $|V| = v$, and B is a collection of 3-subsets (called triples or blocks) of V such that every pair of points in V occurs in exactly one triple of B. It is well known that a STS(v) exists if and only if $v \equiv 1,\ 3\ (mod\ 6)$.

Theorem 1.2.4 (1) *[35] The existence of a Skolem sequence of order n implies the existence of a cyclic $STS(6n+1)$.*

(2) *[35] The existence of a hooked Skolem sequence of order n implies the existence of a cyclic $STS(6n+1)$.*

(3) *[28] The existence of a Rosa sequence of order n implies the existence of a cyclic $STS(6n+3)$.*

(4) *[28] The existence of a hooked Rosa sequence of order n implies the existence of a cyclic $STS(6n+3)$.*

Proof: (1): Suppose there exists a Skolem sequence of order n. Consider the pairs (a_i, b_i), where a_i and b_i are the set of ordered pairs of positions i with $a_i \leq b_i$, $1 \leq i \leq n$. The sets of differences (mod $6n+1$) of elements in the blocks $\{0, a_i+n, b_i+n\}$, $1 \leq i \leq n$, are :

$$A = \{\pm[(b_i+n) - (a_i+n)]\} = \{\pm(b_i - a_i) : 1 \leq i \leq n\}$$
$$B = \{\pm[(a_i+n) - 0] : 1 \leq i \leq n\} = \{\pm(a_i+n) : 1 \leq i \leq n\}$$
$$C = \{\pm[(b_i+n) - 0] : 1 \leq i \leq n\} = \{\pm(b_i+n) : 1 \leq i \leq n\}$$

The set A consists of all differences of the form $\pm(b_i - a_i)$. However, by the definition of a Skolem sequence, the differences $b_i - a_i$ for $1 \leq i \leq n$ are exactly $1, \ldots, n$. Hence, $A = \{\pm1, \pm2, \ldots, \pm n\}$. Now, together a_i and b_i, $1 \leq i \leq n$ consist of all the elements $1, \ldots, 2n$. Therefore, $a_i + n$ and $b_i + n$ for $1 \leq i \leq n$ are exactly the elements $n+1, n+2, \ldots, 3n$. Thus, $B \cup C = \{\pm(n+1), \pm(n+2), \ldots, \pm(3n)\}$. Therefore, $A \cup B \cup C = \{\pm1, \pm2, \ldots, \pm3n\} = \mathbb{Z}_{6n+1} \setminus \{0\}$, that is, all the non-zero elements of $\mathbb{Z}_{6n+1}$. Further, each of these elements occurs exactly once. Thus the sets $\{0, a_i+n, b_i+n\}$, $1 \leq i \leq n$, form a $(6n+1, 3, 1)$ difference set, which gives rise to a $STS(6n+1)$.

The proof of (2), (3), and (4) are similar to the proof (1) and will be omitted.

■

For $n = 4$, $S = (1,1,3,4,2,3,2,4)$ the pairs $(1,2),(5,7),(3,6)$, and $(4,8)$ form a partition of $\{1,2,3,4,5,6,7,8\}$. This partition gives rise to the base blocks $\{0, r, b_r + n\}$, namely, $\{0,1,6\}$, $\{0,2,11\}$, $\{0,3,10\}$, $\{0,4,12\}$, which when developed modulo 25 yield a cyclic $STS(25)$.

The following definition may be found in [20]. Let $\mathbf{K}$ and $\mathbf{G}$ be sets of positive integers and let λ be a positive integer. A *group divisible design of index* λ *and order* v (($\mathbf{K}$, λ)-GDD) is a triple ($\mathcal{V}$, $\mathcal{G}$, $\mathcal{B}$), where $\mathcal{V}$ is a finite set of cardinality v, $\mathcal{G}$ is a partition of $\mathcal{V}$ into parts (*groups*) whose sizes lie in $\mathbf{G}$, and $\mathcal{B}$ is a family of subsets (*blocks*) of $\mathcal{V}$ which satisfy the properties: (1) if $B \in \mathcal{B}$ then $|B| \in \mathbf{K}$; (2) every pair of distinct elements of $\mathcal{V}$ occurs in exactly λ blocks or one group, but not both; and (3) $|\mathcal{G}|>1$. If $\mathbf{K} = \{k\}$, then the $(\mathbf{K}, \lambda)$-GDD is a (k, λ)-GDD. If $\lambda = 1$, the GDD is a $\mathbf{K}$-GDD. Thus, a $(\{k\}, 1)$-GDD is a k-GDD. If $v = a_1 g_1 + a_2 g_2 + \cdots + a_s g_s$, and if there are a_i groups of size g_i, $i = 1, 2, \ldots, s$, then $(\mathbf{K}, \lambda)$-GDD is of type $g_1^{a_1} g_2^{a_2} \cdots g_s^{a_s}$. This is exponential notation for the group type. For example, a $(3,4)$-GDD of type $1^1 3^3$ or type $[3,3,3,1]$ (the columns below in bold represent the groups, the other columns represent the blocks).

1	**4**	**7**	**10**	1	2	3	1	2	3	1	2	3
2	**5**	**8**		4	5	6	5	6	4	6	4	5
3	**6**	**9**		7	8	9	9	7	8	8	9	7
				10	10	10						

1.3 Outline and Statement of Contribution

The main objective of this investigation is to use Rosa-type sequences to find maximum disjoint Rosa sequences and produce new constructions. The idea of this problem came from Dr.Shalaby as introduced in the paper by Linek, Mor, and Shalaby [18] in 2014. This paper gave several constructions for Skolem, hooked Skolem, and Rosa rectangles for $n \geq 20$.

In Chapter 2 we introduce new constructions for hooked Rosa rectangles when $n \equiv 1,\ 2\ (mod\ 4)$. We introduce four disjoint hooked Rosa sequences of order n, which implies the existence of cyclic triple systems $TS_4(6n+3)$. We show that the existence of two disjoint hooked Rosa sequences of order n implies the existence of a group divisible design $(3,2)$-GDD. This work was completed with the valuable information from Dr.Shalaby.

Chapter 3 originated with work with Dr.Shalaby. We prove that there exist two disjoint m-fold Skolem sequences of order n and that the existence of two disjoint 2-fold Skolem sequences of order n implies the existence of a cyclic $TS_4(6n+1)$. We show the existence of two disjoint m-fold Rosa sequences of order n. Then we use this result, to construct a cyclic 2-fold 3-GDD of type 3^{2n+1}. At the end of this chapter we introduce new constructions for two disjoint indecomposable 2-fold Rosa sequences of order n. This work was completed with the help of a faculty member in the department during the revision of my thesis.

In Chapter 4, we present algorithms for the distinct hooked Rosa sequences algorithm, which finds all hooked Rosa sequences of order n. This algorithm searches exhaustively for all hooked Rosa sequences of order n. The maximal disjoint sequences algorithm is a modification of the distinct hooked Rosa sequences algorithm that finds a maximal disjoint subset containing a given hooked Rosa sequence. The final algorithm modifies the distinct hooked Rosa sequences algorithm to search for a maximum disjoint subset of hooked Rosa sequences. This work has been completed with the help of a faculty member in the department of computer science.

Chapter 5 includes the conclusion and possibilities for future work.

Thanks go to all the faculty and staff members of the Mathematics Department

for their help and support.

Chapter 2

Direct Constructions of Hooked Rosa Rectangles

In this chapter, we introduce (hooked) Skolem and Rosa rectangles and new constructions for mutually disjoint hooked Rosa sequences. Moreover, we use two disjoint hooked Rosa sequences to construct cyclic triple systems $CTS_4(v)$ and a GDD.

2.1 Disjoint Skolem Sequences and Related Disjoint Structures

Linek, Mor, and Shalaby [18] in 2014 introduced several constructions for Skolem, hooked Skolem, and Rosa rectangles for $n \geq 20$, in the following theorems:

Theorem 2.1.1 *[18] For $n \equiv 0, 1 \pmod 4$ and $n \geq 20$ there exist six mutually disjoint Skolem sequences (i.e., a $6 \times n$ Skolem rectangle) of order n.*

Theorem 2.1.2 *[18] For $n \equiv 2, 3 \pmod 4$ and $n \geq 20$ there exist five mutually disjoint hooked Skolem sequences (i.e., a $5 \times n$ hooked Skolem rectangle) of order n.*

Theorem 2.1.3 *[18] For $n \equiv 0, 3 \pmod 4$ and $n \geq 20$ there exist four mutually disjoint Rosa sequences (i.e., a $4 \times n$ Rosa rectangle) of order n.*

In Theorem 2.1.4 below from [4] the authors find the maximum number of mutually disjoint (hooked) Skolem and (hooked) Rosa sequences of order n. Here we will sketch the proof of (1) for illustration.

Theorem 2.1.4 *[4] (1) The maximum number of mutually disjoint Skolem sequences of order n is n.*

(2) The maximum number of mutually disjoint hooked Skolem sequences of order n is $n-1$.

(3) The maximum number of mutually disjoint (hooked) Rosa sequences of order n is $n-1$.

Proof: To prove (1): Assume there are r disjoint Skolem sequences of order n. Let n be in the first position in the first sequence, so n will also be in position $(n+1)$ in the first sequence. Thus, there are $(2n-2)$ positions left to place n in the remaining sequences. Now, let n be in the second position in the second sequence, so n will also be in position $(n+2)$ in the second sequence. Therefore, there are $(2n-4)$ positions left to place n in the remaining sequences. Continue this procedure until n is placed in position n and also in position $2n$. This means that there are $2n-2n=0$ positions left to place n. Clearly, the maximum number of disjoint Skolem sequences of order n is n.

The proof of (2), and (3) are similar to (1). ∎

2.2 Hooked Rosa Rectangles

In 1970, Hilton [15] gave a construction for hooked Rosa sequences when $n \equiv 1, 2 \pmod 4$, and used hooked Rosa sequences of order n to construct Steiner triple systems of order $6n+3$. Here we introduce new direct constructions of four mutually disjoint hooked Rosa sequences for $n \equiv 1, 2 \pmod 4$.

Lemma 2.2.1 *For $n \equiv 1 \pmod 4$ there exist four mutually disjoint hooked Rosa sequences (i.e., a $4 \times n$ hooked Rosa rectangle).*

Proof: Let $n = 4s + 1$. For small order $s = 1$ and $n = 5$ the four mutually disjoint hooked Rosa sequences are:

$$3, 1, 1, 3, 5, 0, 2, 4, 2, 5, 0, 4,$$

$$4, 5, 1, 1, 4, 0, 5, 2, 3, 2, 0, 3,$$

$$1, 1, 5, 3, 4, 0, 3, 5, 4, 2, 0, 2,$$

and

$$2, 3, 2, 4, 3, 0, 5, 4, 1, 1, 0, 5.$$

For $s \geq 2$ the required constructions of the four mutually disjoint hooked Rosa sequences are given in Tables 2.1, 2.2, 2.3, and 2.4.

R_1	i	a_i	b_i	$r \in$
(1)	$4s+1-2r$	r	$4s+1-r$	$[1,2s-1]$
(2)	$4s$	$2s$	$6s$	
(3)	$4s+1$	$2s+1$	$6s+2$	
(4)	$2s$	$4s+1$	$6s+1$	
(5)	$4s-2r$	$4s+2+r$	$8s+2-r$	$[1,s-1]$
(6)	$2s-2r$	$5s+1+r$	$7s+1-r$	$[1,s-2]$
(7)	1	$7s+1$	$7s+2$	
(8)	2	$8s+2$	$8s+4$	

Table 2.1: Construction of R_1 for $n \equiv 1$ $(mod\ 4)$ and $s \geq 2$.

R_2	i	a_i	b_i	$r \in$
(1)	$4s-2r$	r	$4s-r$	$[1,s-1]$
(2)	1	s	$s+1$	
(3)	$2s$	$2s$	$4s$	
(4)	$2s-2r$	$s+1+r$	$3s+1-r$	$[1,s-2]$
(5)	$4s-1$	$2s+1$	$6s$	
(6)	$4s$	$2s+2$	$6s+2$	
(7)	$4s+1$	$4s+1$	$8s+2$	
(8)	$4s-1-2r$	$4s+2+r$	$8s+1-r$	$[1,2s-3]$
(9)	2	$6s+1$	$6s+3$	
(10)	3	$8s+1$	$8s+4$	

Table 2.2: Construction of R_2 for $n \equiv 1$ $(mod\ 4)$ and $s \geq 2$.

R_3	i	a_i	b_i	$r \in$
(1)	$4s+2-2r$	r	$4s+2-r$	$[1, s]$
(2)	$4s+1$	$s+1$	$5s+2$	
(3)	$2s+1-2r$	$s+1+r$	$3s+2-r$	$[1, s]$
(4)	$4s+1-2r$	$4s+2+r$	$8s+3-r$	$[1, s-1]$
(5)	$2s+2-2r$	$5s+2+r$	$7s+4-r$	$[1, s]$
(6)	$2s+1$	$6s+3$	$8s+4$	

Table 2.3: Construction of R_3 for $n \equiv 1$ $(mod\ 4)$ and $s \geq 2$.

R_4	i	a_i	b_i	$r \in$
(1)	2	1	3	
(2)	$4s+1$	2	$4s+3$	
(3)	$4s-3-2r$	$4+r$	$4s+1-r$	$[0, 2s-3]$
(4)	$4s-1$	$2s+2$	$6s+1$	
(5)	$4s$	$2s+3$	$6s+3$	
(6)	$4s-2r$	$4s+3+r$	$8s+3-r$	$[1, s-2]$
(7)	$2s+2-2r$	$5s+1+r$	$7s+3-r$	$[1, s-1]$
(8)	$2s+2$	$6s+2$	$8s+4$	
(9)	1	$7s+3$	$7s+4$	

Table 2.4: Construction of R_4 for $n \equiv 1$ $(mod\ 4)$ and $s \geq 2$.

To prove the first construction R_1 in Table 2.1 produces a hooked Rosa sequence, it needs to be shown that each such element of $\{1,\ 2,\ \ldots,\ 2n+2\}$ appears in a pair $(a_i,\ b_i)$ exactly once and that the differences $b_i - a_i$ are exactly the elements $1, 2, \ldots, n$ and the hooks are in positions $n+1$ and $2n+1$. There are $n = 4s+1$ pairs $(a_i,\ b_i)$, and exactly $2n = 8s+2$ elements a_i and b_i. If every element of

$\{1, 2, \ldots, 2n\} = \{1, 2, \ldots, 8s+2\}$ occurs in one of these pairs, each of these elements must occur exactly once. The elements $1, 2, \ldots, 2s-1$ occur in the pairs $(r,\, 4s+1-r)$ for $1 \leq r \leq 2s-1$ in row (1), where $2s$ in row (4) appear in $(4s+1, 6s+1)$. The elements $2s$ and $2s+1$ occur in the pairs $(2s,\, 6s)$ in row (2) and $(2s+1,\, 6s+2)$ in row (3), respectively. While $2s+2, 2s+3, \ldots, 4s$ are given by $(r, 4s+1-r)$ in row (1). Then $4s+1$ appears in row (4). The pairs $(4s+2+r,\, 8s+2-r)$ for $1 \leq r \leq s-1$ in row (5) give the elements $4s+3, 4s+4, \ldots, 5s+1$. For $1 \leq r \leq s-2$ in row (6) the elements $5s+2, 5s+3, \ldots, 6s-1$ occur in the pairs $(5s+1+r, 7s+1-r)$. The elements $6s, 6s+1, 6s+2$ appear in $(2s, 6s)$, $(4s+1, 6s+1)$, and $(2s+1, 6s+2)$ in row (2), (4), and (3). The elements $6s+3, 6s+4, \ldots, 7s$ are present in $(5s+1+r, 7s+1-r)$ for $1 \leq r \leq s-2$ in row (6). Both $7s+1$ and $7s+2$ are given by the pair $(7s+1, 7s+2)$ in row (7). Also, $7s+3, 7s+4, \ldots, 8s+1$ are given by the pairs $(4s+2+r, 8s+2-r)$ for $1 \leq r \leq s-1$ in row (5), $8s+2$ and $8s+4$ are given by the pair $(8s+2, 8s+4)$ in row (8), with $8s+3 = 2n+1$ omitted. Therefore, all elements of $\{1,\, 2,\, \ldots,\, 4s+2,\, \ldots,\, 8s+3,\, 8s+4\}$ occur in the pairs $(a_i,\, b_i)$, and each such elements occurs exactly once as either a_i or b_i for some i and the hooks are in $n+1$ and $2n+1$.

Secondly, it must be verified that the differences $b_i - a_i$ give all values $\{1,\, 2,\, \ldots,\, 4s+1\}$ exactly once. There are $n = 4s+1$ such differences, it must only be shown that each element occurs exactly once. The numbers 1, and 2 are given by $(7s+2) - (7s+1)$ and $(8s+4) - (8s+2)$ from row (7) and (8). The differences $(4s+1-r) - r = 4s+1-2r$ for $1 \leq r \leq 2s-1$ in row (1). The remaining odd element of $\{1,\, 2,\, \ldots,\, 4s+1\}$ occurs as the difference $(6s+2) - (2s+1) = 4s+1$ from row (3). The numbers $2s$ and $4s$ are given by $(6s+1) - (4s+1)$ and $(6s) - 2s$ in row (4) and (2). The differences $(7s+1-r) - (5s+1+r) = 2s-2r$ for $1 \leq r \leq s-2$ from row (6). The remaining even elements $2s+2, 2s+3, \ldots, 4s-2$ are given by $(8s+2-r) - (4s+2+r) = 4s-2r$ for $1 \leq r \leq s-1$ in row (5). Therefore, the sequences that are formed from the construction R_1 are hooked Rosa sequences.

In a similar way, we can prove that R_2, R_3, and R_4 are hooked Rosa sequences.

We now show that construction R_1 and R_2 are disjoint which implies that in both constructions the two occurrences of i are not in the same two positions. The numbers 1 in row (7), and 2 in row (8) for R_1 occur in $(7s+1, 7s+2)$, and $(8s+2, 8s+4)$, respectively, while 1 in row (2), and 2 in row (9) for R_2 occur in $(s, s+1)$, and $(6s+1, 6s+3)$ respectively. The elements $2s$ and $4s$ occur in $(4s+1, 6s+1)$, and $(2s, 6s)$ in row (4), and (2) for R_1, but they occur in $(2s, 4s)$, and $(2s+2, 6s+2)$ for R_2 in row (3), and (6). In row (3) for R_1, the element $4s+1$ appears in $(2s+1, 6s+2)$, and it appears in $(4s+1, 8s+2)$ in row (7) for R_2. For $1 \leq r \leq s-1$, the numbers $4s-2r$ occur in $(4s+2+r, 8s+2-r)$ in row (5) for R_1 and in $(r, 4s-r)$ in row (1) for R_2. The elements $2s-2r$ for $1 \leq r \leq s-2$ occur in $(5s+1+r, 7s+1-r)$ in row (6) for R_1 and they appear in $(s+1+r, 3s+1-r)$ in row (4) for R_2. The odd numbers $4s+1-2r$ in R_1 occur in $(r, 4s+1-r)$ for $1 \leq r \leq 2s-1$ in row (1), while the odd numbers in R_2, $4s-r$ appears in $(2s+1, 6s)$ in row (5), the number 3 appears in $(8s+1, 8s+4)$ in row (10), and the elements $4s-1-2r$ occur in $(4s+2+r, 8s+1-r)$ for $1 \leq r \leq 2s-3$ in row (8). Therefore, the construction R_1 and R_2 are disjoint hooked Rosa sequences.

In a similar way, we can show that each pair of constructions, R_i, R_j, $1 \leq i,j \leq 4$, is disjoint. ■

Lemma 2.2.2 *For $n \equiv 2 \pmod 4$ there exist four mutually disjoint hooked Rosa sequences (i.e., a $4 \times n$ hooked Rosa rectangle).*

Proof: Let $n = 4s+2$. For small order $s = 1$ and $n = 6$ the four mutually disjoint hooked Rosa sequences are:

$$4, 2, 5, 2, 4, 6, 0, 5, 1, 1, 3, 6, 0, 3,$$

$$5, 3, 6, 4, 3, 5, 0, 4, 6, 1, 1, 2, 0, 2,$$

$$3, 1, 1, 3, 4, 5, 0, 6, 4, 2, 5, 2, 0, 6,$$

and

$$2, 4, 2, 6, 3, 4, 0, 3, 5, 6, 1, 1, 0, 5.$$

And for $s = 2$, $n = 10$ the four mutually disjoint hooked Rosa sequences are:

$$9, 7, 5, 3, 10, 6, 3, 5, 7, 9, 0, 6, 4, 8, 10, 2, 4, 2, 1, 1, 0, 8,$$

$$8, 6, 4, 2, 9, 2, 4, 6, 8, 5, 0, 10, 7, 9, 5, 3, 1, 1, 3, 7, 0, 10,$$

$$6, 4, 2, 8, 2, 4, 6, 9, 7, 10, 0, 8, 5, 1, 1, 7, 9, 5, 3, 10, 0, 3,$$

and

$$7, 5, 3, 1, 1, 3, 5, 7, 10, 8, 0, 4, 9, 6, 2, 4, 2, 8, 10, 6, 0, 9.$$

For $s \geq 3$ the required constructions of the four mutually disjoint hooked Rosa sequences are given in Tables 2.5, 2.6, 2.7, and 2.8.

R_1	i	a_i	b_i	$r \in$
(1)	$2s+1$	1	$2s+2$	
(2)	$4s+2-2r$	$r+1$	$4s+3-r$	$[1,s]$
(3)	$4s+1$	$s+2$	$5s+3$	
(4)	$2s+1-2r$	$s+2+r$	$3s+3-r$	$[1,s-1]$
(5)	$4s+2$	$2s+3$	$6s+5$	
(6)	$4s+1-2r$	$4s+3+r$	$8s+4-r$	$[1,s-1]$
(7)	$2s+2-2r$	$5s+3+r$	$7s+5-r$	$[1,s-1]$
(8)	1	$6s+3$	$6s+4$	
(9)	2	$8s+4$	$8s+6$	

Table 2.5: Construction of R_1 for $n \equiv 2$ $(mod\ 4)$ and $s \geq 3$.

R_2	i	a_i	b_i	$r \in$
(1)	1	1	2	
(2)	$4s+1-2r$	$2+r$	$4s+3-r$	$[1,s-1]$
(3)	$4s+2$	$s+2$	$5s+4$	
(4)	$2s+2-2r$	$s+2+r$	$3s+4-r$	$[1,s]$
(5)	$4s+1$	$2s+3$	$6s+4$	
(6)	$4s+2-2r$	$4s+3+r$	$8s+5-r$	$[1,s]$
(7)	$2s+1-2r$	$5s+4+r$	$7s+5-r$	$[1,s-1]$
(8)	$2s+1$	$6s+5$	$8s+6$	

Table 2.6: Construction of R_2 for $n \equiv 2$ $(mod\ 4)$ and $s \geq 3$.

R_3	i	a_i	b_i	$r \in$
(1)	$4s+3-2r$	r	$4s+3-r$	$[1, s+1]$
(2)	$2s-2r$	$s+1+r$	$3s+1-r$	$[1, s-1]$
(3)	$4s$	$2s+1$	$6s+1$	
(4)	$4s+2$	$3s+1$	$7s+3$	
(5)	$4s-2r$	$4s+3+r$	$8s+3-r$	$[1, s-1]$
(6)	$2s+1-2r$	$5s+2+r$	$7s+3-r$	$[1, s-2]$
(7)	1	$6s+2$	$6s+3$	
(8)	$2s$	$6s+4$	$8s+4$	
(9)	3	$8s+3$	$8s+6$	

Table 2.7: Construction of R_3 for $n \equiv 2$ $(mod\ 4)$ and $s \geq 3$.

R_4	i	a_i	b_i	$r \in$
(1)	$4s+2-2r$	r	$4s+2-r$	$[1, s]$
(2)	$2s+1-2r$	$s+r$	$3s+1-r$	$[1, s-1]$
(3)	$4s+1$	$2s$	$6s+1$	
(4)	$4s-1$	$2s+1$	$6s$	
(5)	$2s+1$	$3s+1$	$5s+2$	
(6)	$4s+2$	$4s+2$	$8s+4$	
(7)	$4s-1-2r$	$4s+3+r$	$8s+2-r$	$[1, s-2]$
(8)	$2s-2r$	$5s+2+r$	$7s+2-r$	$[1, s-3]$
(9)	2	$6s+2$	$6s+4$	
(10)	$2s$	$6s+3$	$8s+3$	
(11)	1	$7s+2$	$7s+3$	
(12)	4	$8s+2$	$8s+6$	

Table 2.8: Construction of R_4 for $n \equiv 2$ $(mod\ 4)$ and $s \geq 3$.

To verify the constructions of R_1, R_2, R_3, and R_4 above, one can proceed as shown in the proof of Lemma 2.2.1. Finally, checking that the constructions are disjoint also follows as in the proof of Lemma 2.2.1. ■

From Lemma 2.2.1 and 2.2.2 we then have this theorem.

Theorem 2.2.3 *For* $n \equiv 1, 2$ $(mod\ 4)$ *there exist four mutually disjoint hooked Rosa sequences (i.e., a* $4 \times n$ *hooked Rosa rectangle).*

2.3 Applications for Disjoint Hooked Rosa Sequences

In this section, we prove that two disjoint hooked Rosa sequences give cyclic triple systems CTS_2 for $v \equiv 3$ $(mod\ 6)$ and a $(3, 2)$-GDD.

Theorem 2.3.1 *For* $v \equiv 3$ $(mod\ 6)$ *there exist cyclic triple systems* $CTS_2(v)$.

Proof: We construct a $CTS_2(v)$, $v \equiv 3$ $(mod\ 6)$, as follows. Suppose there exist two disjoint hooked Rosa sequences of order n. For $1 \leq i{,}j \leq n$, let the pairs (a_i, b_i) and (a_j, b_j) be the two disjoint subscripts of i and j in the two disjoint sequences, with $a_i \leq b_i$ and $a_j \leq b_j$. Consider the sets $\{0, a_i + n, b_i + n\}$ from the first sequence, and $\{0, a_j + n, b_j + n\}$ from the second disjoint sequence, with the following sets of differences $(mod\ 6n + 3)$:

$A = \{\pm[(b_i + n) - (a_i + n)]\} = \{\pm(b_i - a_i)\}$,

$B = \{\pm[(a_i + n) - 0]\} = \{\pm(a_i + n)\}$,

$C = \{\pm[(b_i + n) - 0]\} = \{\pm(b_i + n)\}$,

$A' = \{\pm[(b_j + n) - (a_j + n)]\} = \{\pm(b_j - a_j)\}$,

$B' = \{\pm[(a_j + n) - 0]\} = \{\pm(a_j + n)\}$, and $C' = \{\pm[(b_j + n) - 0]\} = \{\pm(b_j + n)\}$.

By the definition of a hooked Rosa sequence, A consists of exactly the numbers ± 1, ± 2, $\ldots$, $\pm n$. Since a_i and b_i make up the numbers $1, 2, \ldots, n, n+2, n+3, \ldots, 2n, 2n+2$, together B and C consist of exactly the numbers $\pm(n+1)$, $\pm(n+2)$, $\ldots$, $\pm 2n$, $\pm(2n+2)$, $\ldots$, $3n$, $\pm(3n+2)$. Similarly, for sets A', B', and C' from the second disjoint sequence. Thus, all elements of $\mathbb{Z}_{6n+3} \setminus \{0\}$ occur as differences exactly twice, except for $2n+1$ and its inverse $4n+2$, so the translates of these blocks give all pairs of elements except for those which differ by $2n+1$.

Adding the short-orbit base block $\{0, 2n+1, 4n+2\}$ with its translates $\{j', (2n)+j', (4n)+j'\}$, where $0 \leq j' \leq 2n$, gives the remaining pairs, which yields a cyclic triple system, $CTS_2(v)$. ■

For example, two disjoint hooked Rosa sequences of order 5:

$$S_5 = (1, 1, 2, 5, 2, 0, 3, 4, 5, 3, 0, 4),$$

and

$$S'_5 = (3, 1, 1, 3, 4, 0, 5, 2, 4, 2, 0, 5)$$

give the cyclic $TS(33)$

$B_1 = \{\{0, 6, 7\}, \{0, 8, 10\}, \{0, 12, 15\}, \{0, 13, 17\}, \{0, 9, 14\}(mod\ 33)\}$, from S_5, and $B'_1 = \{\{0, 7, 8\}, \{0, 13, 15\}, \{0, 6, 9\}, \{0, 10, 14\}, \{0, 12, 17\}(mod\ 33)\}$, from S'_5.

Adding the short-orbit base block $\{0, 11, 22\}$ with its translates $\{i, (2n+1)+i, (4n+1)+i\}$, where $0 \leq i \leq 2n$, gives the remaining pairs, which yield cyclic triple systems $CTS_4(33)$.

Theorem 2.3.2 *The existence of two disjoint hooked Rosa sequences of order n implies the existence of a $(3, 2)$-GDD of type 3^{2n+1}.*

Proof: Suppose there exist two disjoint hooked Rosa sequences of order n. For $1 \leq i, j \leq n$, let the pairs (a_i, b_i) and (a_j, b_j) be the two disjoint sets of ordered pairs of positions i and j in the two disjoint sequences, with $a_i \leq b_i$ and $a_j \leq b_j$. Consider the sets $\{0, i, b_i + n\}$ from the first sequence and $\{0, j, b_j + n\}$ from the second disjoint sequence, with the following sets of differences (mod $6n + 3$):

$A = \{\pm[(b_i + n) - (i)]\} = \{\pm(b_i + n - i)\}$,

$B = \{\pm[(b_i + n) - 0]\} = \{\pm(b_i + n)\}$,

$C = \{\pm[(i) - 0]\} = \{\pm(i)\}$,

$A' = \{\pm[(b_j + n) - (j)]\} = \{\pm(b_j + n - j)\}$,

$B' = \{\pm[(b_j + n) - 0]\} = \{\pm(b_j + n)\}$ and $C' = \{\pm(j) - 0]\} = \{\pm(j)\}$.

Thus, all elements of $\mathbb{Z}_{6n+3} \setminus \{0\}$ occur as differences exactly twice except for $2n+1$ and its inverse $4n+2$. Therefore, the translates of these blocks give all pairs of elements except for those which differ by $2n + 1$. Adding the short-orbit base block $\{0, 2n+1, 4n+2\}$ with its translates $\{i', (2n) + i', (4n) + i'\}$, where $0 \leq i' \leq 2n$, gives $2n + 1$ subsets.

$$\underbrace{\begin{array}{ccccc} 0 & 1 & 2 & \dots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \dots & 4n+1 \\ 4n+2 & 4n+3 & 4n+4 & \dots & 6n+2 \end{array}}_{\text{2n+1 } \textit{subsets are the columns called groups}}$$

Each block will intersect with 3 different groups. Adding the short-orbit base block $\{0, 2n + 1, 4n + 2\}$ from the second disjoint sequence with its translates $\{j', (2n) + j, (4n) + j'\}$, where $0 \leq j' \leq 2n$, gives $2n + 1$ subsets.

Similarly, each block will intersect with 3 different groups. Therefore, two disjoint hooked Rosa sequences give $(3, 2)$-GDD. ■

For instance, let $(3, 1, 1, 3, 5, 0, 2, 4, 2, 5, 0, 4)$ and $(4, 5, 1, 1, 4, 0, 5, 2, 3, 2, 0, 3)$ be two disjoint hooked Rosa sequences of order 5. The pairs from the first sequence $(2, 3)$,

$(7,9)$, $(1,4)$, $(8,12)$, $(5,10)$ and $(3,4)$, $(8,10)$, $(9,12)$, $(1,5)$, $(2,7)$ from the second disjoint sequence. Consider the sets $\{0, i, b_i + n\}$ from the first sequence and $\{0, j, b_j + n\}$ from the second disjoint sequence, with the following sets of differences $(mod\ 6n + 3)$:

$\{0,1,8\}, \{0,2,14\}, \{0,3,9\}, \{0,4,17\}, \{0,5,15\}$ and

$\{0,1,9\}, \{0,2,15\}, \{0,3,17\}, \{0,4,10\}, \{0,5,12\}$.

Thus, all elements of $\mathbb{Z}_{6n+3} \setminus \{0\}$ occur as differences exactly twice except for $2n+1$ and its inverse $4n+2$. Therefore, the translates of these blocks give all pairs of elements except for those which differ by $2n+1$. Adding the short-orbit base block $\{0, 2n+1, 4n+2\}$ with its translates $\{i', (2n)+i', (4n)+i'\}$, where $0 \le i' \le 2n$, gives $2n+1$ subsets.

$$\underbrace{\begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \end{array}}_{11\ \textit{subsets are the columns called groups}}$$

Each block will intersect with 3 different groups. Adding the short-orbit base block $\{0, 2n+1, 4n+2\}$ from the second disjoint sequence with its translates $\{j', (2n) + j', (4n) + j'\}$, where $0 \le j' \le 2n$, gives $2n+1$ subsets.

Similarly, each block will intersect with 3 different groups. Therefore, two disjoint hooked Rosa sequences give $(3,2)$-GDD.

Chapter 3

Building Disjoint m-fold Sequences

In this chapter, we give necessary conditions for the existence of m-fold Skolem and Rosa sequences. We then introduce new constructions for two disjoint m-fold Skolem sequences and two disjoint m-fold Rosa sequences of order n. We present the concept of indecomposable Skolem and Rosa sequences and construct two disjoint indecomposable 2-fold Rosa sequences of order n. In addition, we show that the existence of two disjoint 2-fold Rosa sequences of order n implies the existence of a cyclic 2-fold 3-GDD.

3.1 Disjoint m-fold Skolem and Rosa Sequences

We indicated in Chapter 1 that a sequence $mS = (s_1, s_2, \ldots, s_{2mn})$ is an m-fold Skolem sequence of order n with the following condition: (1) for every $k \in \{1, 2, \ldots, n\}$ there exist m disjoint pairs $(i, i+k)$, with $i, i+k \in \{1, 2, \ldots, 2mn\}$ such that $s_i = s_{i+k} = k$. An m-fold extended Skolem sequence of order n is a sequence $mS = (s_1, s_2, \ldots, s_{2mn+1})$ with the same condition (1), and (2) there exists exactly one $s_i = 0$, $1 \leq i \leq 2mn+1$. If $s_{2mn} = 0$, the extended sequence is called an m-fold hooked Skolem sequence. The necessary conditions are sufficient for the existence of m-fold (hooked extended) Skolem sequences and it was shown in [2,3].

Theorem 3.1.1 *[3] An m-fold Skolem sequence of order n exists if and only if*

(1) $n \equiv 0, 1(mod\ 4)$, or

(2) $n \equiv 2, 3(mod\ 4)$ and m even,

and a hooked m-fold Skolem sequence of order n exists if and only if $n \equiv 2, 3(mod\ 4)$ and m is odd.

Theorem 3.1.2 *[2] Let m, n, k be positive integers. There exists an extended m-fold Skolem sequence of order n with $s_k = 0$ if and only if one of the following conditions:*

(1) $n \equiv 0, 1(mod\ 4)$, and k is odd,

(2) $n \equiv 2, 3(mod\ 4)$, m is even and k is odd,

(3) $n \equiv 2, 3(mod\ 4)$, m is odd and k is even.

For instance, $3, 1, 1, 3, 1, 1, 2, 3, 2, 2, 2, 3, 2$ is a 2-fold Skolem sequence of order 3 and $1, 1, 1, 1, 1, 1, 2, 0, 2, 2, 2, 2, 2$ is a 3-fold extended Skolem sequence of order 2. Also, it was shown in [27] that there exists a 2-fold Rosa sequence of order n for every $n \geq 2$.

Theorem 3.1.3 *[27] There exists a 2-fold Rosa sequence of order n if and only if $n \geq 2$.*

As shown in [2,3], the necessary conditions are sufficient for the existence of m-fold (hooked extended) Skolem sequences. We can show the necessary conditions in the following results:

Lemma 3.1.4 (1) *An m-fold Skolem sequence of order n exists only if:*

$n \equiv 0, 1(mod\ 4)$, or

$n \equiv 2, 3(mod\ 4)$, *m is even.*

(2) *An m-fold Rosa sequence of order n exists only if:*

$n \equiv 0, 3(mod\ 4)$, *or*

$n \equiv 1, 2(mod\ 4)$, *m is even.*

Proof: (1) Suppose that $(s_1, s_2, \ldots, s_{2mn})$ is an m-fold Skolem sequence of order n. Consider the set of subscripts $\{(a_i, b_i) : i = 1, 2, \ldots, n\}$, since $b_i - a_i = i$. Then

$$\sum_{i=1}^{n}(b_i - a_i) = m\sum_{i=1}^{n} i = \frac{mn(n+1)}{2} \tag{3.1}$$

However, together these numbers $a_i,\ b_i, i = 1, 2, \ldots, n$ comprise the set $\{1, 2, \ldots, 2mn\}$. Therefore,

$$\sum_{i=1}^{n}(a_i + b_i) = \frac{2mn(2mn+1)}{2} \tag{3.2}$$

Adding (3.1) and (3.2) gives:

$$2\sum_{i=1}^{n} b_i = \frac{mn(n+1)}{2} + \frac{2mn(2mn+1)}{2}$$

That is,

$$\sum_{i=1}^{n} b_i = \frac{mn(4mn+n+3)}{4}$$

For each $i \in \{1, 2, \ldots, n\}$, b_i is an integer. Therefore, either $n \equiv 0, 1(mod\ 4)$ or m must be even.

If $n \equiv 0(mod\ 4)$, let $n = 4s$. Then $\frac{mn(4mn+n+3)}{4} = ms(16ms + 4s + 3)$. If $n \equiv 1(mod\ 4)$, let $n = 4s+1$. Then $\frac{mn(4mn+n+3)}{4} = m(16ms^2+4s^2+8ms+5s+m+1)$. If $n \equiv 2(mod\ 4)$, let $n = 4s + 2$. Then $\frac{mn(4mn+n+3)}{4} = \frac{m(32ms^2+8s^2+32ms+14s+8m+5)}{2}$.

If $n \equiv 3(mod\ 4)$, let $n = 4s + 3$. Then $\frac{mn(4mn+n+3)}{4} = \frac{m(32ms^2+8s^2+48ms+18s+18m+9)}{2}$. Thus, if $n \equiv 2,\ 3(mod\ 4)$ and m is even, then $\sum_{i=1}^{n} b_i$ is an integer.

The proof of (2) is similar to the proof of (1). ■

The following theorems will introduce new constructions to find two disjoint m-fold Skolem and Rosa sequences of order n.

Theorem 3.1.5 *There exist two disjoint m-fold Skolem sequences of order n whenever:*

(1) $n \equiv 0, 1(mod\ 4)$, *or*

(2) $n \equiv 2, 3(mod\ 4)$ *and m even,*

Proof: Case 1: $n \equiv 0, 1(mod\ 4)$.

By Theorem 2.1.1 there exist six mutually disjoint Skolem sequences of order n. We can take any two sequences from these constructions $S = (s_1, s_2, \ldots, s_{2n})$ and $S' = (s'_1, s'_2, \ldots, s'_{2n})$, each with pairs that are not in the same two positions in both sequences. Thus, we can construct a 2-fold Skolem sequence by using two copies of S and S' which are

$$2S = (s_1, s_2, \ldots, s_{2n}, s_1, s_2, \ldots, s_{2n})$$

and

$$2S' = (s'_1, s'_2, \ldots, s'_{2n}, s'_1, s'_2, \ldots, s'_{2n}).$$

Therefore, $2S$ and $2S'$ are disjoint. Appending more copies of S and S' will give two disjoint m-fold Skolem sequences. Then, there exist two disjoint m-fold Skolem sequences of order $n \equiv 0, 1(mod\ 4)$.

Case 2: $n \equiv 2, 3(mod\ 4)$ and m is even.

By Theorem 2.1.2 there exist five mutually disjoint hooked Skolem sequences of order

n. We can take any two sequences from these constructions $S = (s_1, s_2, \ldots, s_{2n+1})$ and $S' = (s'_1, s'_2, \ldots, s'_{2n+1})$. Thus, we can construct two disjoint 2-fold Skolem sequences by using S and S' and their reverse and starting the reverse from the hook position. Then, $2S = (s_1, s_2, \ldots, s_{2n-1}, s_{2n+1}, s_{2n+1}, \ldots, s_2, s_1)$ is a 2-fold Skolem sequence of order n. Also, $2S' = (s'_1, s'_2, \ldots, s'_{2n-1}, s'_{2n+1}, s'_{2n+1}, \ldots, s'_2, s'_1)$ is a 2-fold Skolem sequence of order n. Therefore, $2S$ and $2S'$ are disjoint. Appending $(m-2)$ more copies of S and S' will give two disjoint m-fold Skolem sequences and in this case m always be even. Then, there exist two disjoint m-fold Skolem sequences of order $n \equiv 2, 3(mod\ 4)$. ■

For example,

$$2S = (4, 2, 3, 2, 4, 3, 1, 1, 4, 2, 3, 2, 4, 3, 1, 1)$$

and

$$2S' = (1, 1, 4, 2, 3, 2, 4, 3, 1, 1, 4, 2, 3, 2, 4, 3)$$

are two disjoint 2-fold Skolem sequences of order 4. And,

$$2S = (3, 1, 1, 3, 6, 4, 2, 5, 2, 4, 6, 5, 5, 6, 4, 2, 5, 2, 4, 6, 3, 1, 1, 3)$$

and

$$2S' = (6, 4, 5, 1, 1, 4, 6, 5, 2, 3, 2, 3, 3, 2, 3, 2, 5, 6, 4, 1, 1, 5, 4, 6)$$

are two disjoint 2-fold Skolem sequences of order 6.

Now we will construct two disjoint m-fold Rosa sequences.

Theorem 3.1.6 *There exist two disjoint m-fold Rosa sequences of order n whenever:*
(1) $n \equiv 0, 3(mod\ 4)$ *or,*
(2) $n \equiv 1, 2(mod\ 4)$ *and m is even.*

Proof: Case 1: $n \equiv 0, 3(mod\ 4)$.

By Theorem 2.1.3 there exist four mutually disjoint Rosa sequences of order n. So we can take any two sequences from these constructions $S = (s_1, s_2, \ldots, s_{2n+1})$ and $S' = (s'_1, s'_2, \ldots, s'_{2n+1})$. Thus, we can construct a 2-fold Rosa sequence by using two copies of S and S' which are

$$2S = (s_1, s_2, \ldots, s_{2n+1}, s_1, s_2, \ldots, s_{2n+1})$$

and

$$2S' = (s'_1, s'_2, \ldots, s'_{2n+1}, s'_1, s'_2, \ldots, s'_{2n+1}).$$

Therefore, $2S$ and $2S'$ are disjoint. Appending $(m-2)$ more copies of S and S' will give two disjoint m-fold Rosa sequences. Then, there exist two disjoint m-fold Rosa sequences of order $n \equiv 0, 3(mod\ 4)$.

Case 2: $n \equiv 1, 2(mod\ 4)$ and m is even.

By Theorem 2.2.3 there exist four mutually disjoint hooked Rosa sequences of order n. So we can take any two sequences from these constructions $S = (s_1, s_2, \ldots, s_{2n+2})$ and $S' = (s'_1, s'_2, \ldots, s'_{2n+2})$. Thus, we can construct two disjoint 2-fold Rosa sequences by using S and S' and their reverse and starting the reverse from the second hook position. Then, $2S = (s_1, s_2, \ldots, s_{2n+2}, s_{2n+2}, \ldots, s_2, s_1)$ is a 2-fold Rosa sequence of order n. Also, $2S' = (s'_1, s'_2, \ldots, s'_{2n+2}, s'_{2n+2}, \ldots, s'_2, s'_1)$ is a 2-fold Rosa sequence of order n. Therefore, $2S$ and $2S'$ are disjoint. Appending $(m-2)$ more copies of $2S$ and $2S'$ will give two disjoint m-fold Rosa sequences and in this case m always be even. Then, there exist two disjoint m-fold Rosa sequences of order $n \equiv 1, 2(mod\ 4)$.

■

For example,

$$2S = (2, 4, 2, 3, 0, 4, 3, 1, 1, 2, 4, 2, 3, 0, 4, 3, 1, 1)$$

and

$$2S' = (1,1,3,4,0,3,2,4,2,1,1,3,4,0,3,2,4,2)$$

are two disjoint 2-fold Rosa sequences of order 4.

As we indicated in Chapter 1 a λ-fold triple system of order v, denoted $TS_\lambda(v)$, is a collection B of 3-subsets (called triples or blocks) from a v-set V, such that any given pair of elements in V lies in exactly λ triples [26].

One can conclude that the existence of two disjoint 2-fold Rosa sequences of order n implies the existence of a cyclic 2-fold 3-GDD and $TS_4(6n+3)$. Let $2S = (s_1, s_2, \ldots, s_{4n+2})$, and $2S' = (s'_1, s'_2, \ldots, s'_{4n+2})$ be two disjoint 2-fold Rosa sequences of order n. For each $i, j \in \{1, 2, \ldots, n\}$ the pairs of positions i and j in $2S$ are $(a_i, b_i), (a_j, b_j)$, and $(a'_i, b'_i), (a'_j, b'_j)$ are the pairs of positions i and j in $2S'$. In particular, $s_{n+1} = s_{3n+2} = 0$. The set of triples $\{\{0, i, b_i + n\}, \{0, j, b_j + n\}, \{0, i', b'_i + n\}, \{0, j', b'_j + n\}\}$ form the base blocks for disjoint cyclic 2-fold 3-GDD of type 3^{2n+1} (the groups of which are given by $\{0, 2n+1, 4n+2\}(mod\ 6n+3)$) which in turn gives rise to a $TS_4(6n+3)$.

3.2 Disjoint Indecomposable 2-fold Rosa sequences

A t-indecomposable m-fold (hooked) Skolem sequence of order n is an m-fold (hooked) Skolem sequence of order n such that for all subscripts $i, j, 1 \leq i<j \leq 2mn$ $(i, j, 1 \leq i<j \leq 2mn+1)$, the proper subsequence $(s_i, s_{i+1}, \ldots, s_j)$ is not a t-fold (hooked) Skolem sequence of order r where $1<r \leq n$. If an m-fold (hooked) Skolem sequence of order n is t-indecomposable for all $t<m$, then it is called indecomposable [27]. For example, $1,1,2,2,2,2,1,1$ is an indecomposable 2-fold Skolem sequence of order 2. For more details on indecompoasable (hooked) Skolem sequences see [26, 27]. A $TS_\lambda(v)$ is called indecomposable if its block set B cannot be partitioned into sets B_1,

B_2 of blocks to form $TS_{\lambda_1}(v)$ and $TS_{\lambda_2}(v)$, where $\lambda_1 + \lambda_2 = \lambda$.

A $CTS_\lambda(v)$ is called cyclically indecomposable if its block set B cannot be partitioned into sets B_1, B_2 of blocks to form $CTS_{\lambda_1}(v)$ and $CTS_{\lambda_2}(v)$, where $\lambda_1+\lambda_2 = \lambda$, $\lambda_1, \lambda_2 \geq 1$. We focus our attention on two disjoint indecomposable 2-fold Rosa sequences of order n and construct disjoint indecomposable 2-fold triple systems for all admissible orders. There is a totally indecomposable 2-fold Rosa sequence of order n, which is a 2-fold Rosa sequence that does not contain a proper Rosa subsequence when $n \equiv 0, 3(mod\ 4)$ or a proper hooked Rosa subsequence when $n \equiv 1, 2(mod\ 4)$.

Theorem 3.2.1 *[27] For every $n \geq 2$ there is a totally indecomposable 2-fold Rosa sequence of order n.*

Now, the new results can be shown.

Lemma 3.2.2 *There exist two disjoint indecomposable 2-fold Rosa sequences of order $n \equiv 0\ (mod\ 4)$.*

Proof: Let $n = 4s$. For small order when $s = 1$, $n = 4$ the two disjoint indecomposable 2-fold Rosa sequences are

$$3, 1, 1, 3, 0, 4, 2, 4, 2, 4, 2, 4, 2, 0, 3, 1, 1, 3$$

and

$$2, 4, 2, 3, 0, 4, 3, 3, 1, 1, 3, 4, 2, 0, 2, 4, 1, 1.$$

For $n > 4$ the required constructions of two disjoint indecomposable 2-fold Rosa sequences are given in Tables 3.1, 3.2, 3.3, and 3.4.

R_{1i}	i	a_i	b_i	$r \in$
(1)	$4s-2r$	r	$4s-r$	$[1, 2s-1]$
(2)	$4s$	$2s$	$6s$	
(3)	$4s-1-2r$	$4s+1+r$	$8s-r$	$[1, 2s-2]$
(4)	1	$9s+2$	$9s+3$	
(5)	$4s-1$	$10s+2$	$14s+1$	

Table 3.1: Construction of R_{1i} for $n \equiv 0$ $(mod\ 4)$ and $s \geq 2$.

R_{1j}	j	a_j	b_j	$r \in$
(1)	$4s$	$4s$	$8s$	
(2)	$2s+1$	$6s+1$	$8s+2$	
(3)	2	$8s+1$	$8s+3$	
(4)	$4s-1-2r$	$8s+3+r$	$12s+2-r$	$[1, s-2]$
(5)	1	$11s+2$	$11s+3$	
(6)	$4s-1$	$10s+3$	$14s+2$	
(7)	$2s-1-2r$	$9s+3+r$	$11s+2-r$	$[1, s-2]$
(8)	$4s-2r$	$12s+2+r$	$16s+2-r$	$[1, 2s-2]$
(9)	$2s-1$	$14s+3$	$16s+2$	

Table 3.2: Construction of R_{1j} for $n \equiv 0$ $(mod\ 4)$ and $s \geq 2$.

R_{2i}	i	a_i	b_i	$r \in$
(1)	$4s+1-2r$	r	$4s+1-r$	$[1, 2s]$
(2)	$4s+2-2r$	$4s+1+r$	$8s+3-r$	$[1, 2s]$

Table 3.3: Construction of R_{2i} for $n \equiv 0$ $(mod\ 4)$ and $s \geq 2$.

R_{2j}	j	a_j	b_j	$r \in$
(1)	$4s$	$6s+2$	$10s+2$	
(2)	$4s-2r$	$8s+2+r$	$12s+2-r$	$[1, 2s-1]$
(3)	$4s+1-2r$	$12s+2+r$	$16s+3-r$	$[1, 2s]$

Table 3.4: Construction of R_{2j} for $n \equiv 0 \ (mod\ 4)$ and $s \geq 2$.

To prove these constructions are disjoint it is similar to the proof of disjoint hooked Rosa sequences as shown in Chapter 2. Our constructions here are indecomposable as there is no Rosa subsequence or hooked Rosa subsequence as stated below.

For instance, from constructions R_{1i}, and R_{1j} for $n = 8$, we obtain the following sequence

$$6, 4, 2, 8, 2, 4, 6, 8, 0, 5, 3, 8, 5, 3, 5, 8, 2, 5, 2, 1, 1, 7, 7, 1, 1, 0, 6, 4, 7, 7, 3, 4, 6, 3.$$

In this case $1, 1$ is not in the first and second positions:

$$0, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 2, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0.$$

It is easy to see that there is no Rosa subsequence or hooked Rosa subsequence. Hence, constructions R_{1i}, and R_{1j} for $s \geq 2$ give indecomposable 2-fold Rosa sequences of order $n \equiv 0 \ (mod\ 4)$.

Also, from constructions R_{2i} and R_{2j} for $n = 8$, we obtain the following sequence

$$7, 5, 3, 1, 1, 3, 5, 7, 0, 8, 6, 4, 2, 8, 2, 4, 6, 8, 6, 4, 2, 8, 2, 4, 6, 0, 7, 5, 3, 1, 1, 3, 5, 7.$$

In this case $1, 1$ is not in the first and second positions:

$$0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 2, 0, 2, 0, 0, 0, 0, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0.$$

Again, it is easy to see that there is no Rosa subsequence or hooked Rosa subsequence.

Thus, constructions R_{2i} and R_{2j} for $s \geq 2$ give indecomposable 2-fold Rosa sequences of order $n \equiv 0$ $(mod\ 4)$.

Therefore,

$$6, 4, 2, 8, 2, 4, 6, 8, 0, 5, 3, 8, 5, 3, 5, 8, 2, 5, 2, 1, 1, 7, 7, 1, 1, 0, 6, 4, 7, 7, 3, 4, 6, 3$$

and

$$7, 5, 3, 1, 1, 3, 5, 7, 0, 8, 6, 4, 2, 8, 2, 4, 6, 8, 6, 4, 2, 8, 2, 4, 6, 0, 7, 5, 3, 1, 1, 3, 5, 7$$

are two disjoint indecomposable 2-fold Rosa sequences of order 8. Hence, there exist two disjoint indecomposable 2-fold Rosa sequences of order $n \equiv 0$ $(mod\ 4)$. ■

Lemma 3.2.3 *There exist two disjoint indecomposable* 2*-fold Rosa sequences of order* $n \equiv 1$ $(mod\ 4)$.

Proof: Let $n = 4s + 1$. For small order when $s = 1$, $n = 5$ the two disjoint indecomposable 2-fold Rosa sequences are

$$4, 2, 5, 2, 4, 0, 5, 5, 4, 1, 1, 5, 4, 2, 3, 2, 0, 3, 3, 1, 1, 3$$

and

$$3, 1, 1, 3, 3, 0, 2, 3, 2, 4, 5, 1, 1, 4, 5, 5, 0, 4, 2, 5, 2, 4.$$

For $n > 5$ the required constructions of two disjoint indecomposable 2-fold Rosa sequences are given in Tables 3.5, 3.6, 3.7, and 3.8.

R_{1i}	i	a_i	b_i	$r \in$
(1)	$4s + 2 - 2r$	r	$4s + 2 - r$	$[1, 2s]$
(2)	$4s + 1$	$2s + 1$	$6s + 2$	
(3)	$4s + 1 - 2r$	$4s + 3 + r$	$8s + 4 - r$	$[1, 2s - 2]$
(4)	1	$12s$	$12s + 1$	
(5)	3	$12s + 3$	$12s + 6$	

Table 3.5: Construction of R_{1i} for $n \equiv 1$ $(mod\ 4)$ and $s \geq 2$.

R_{1j}	j	a_j	b_j	$r \in$
(1)	$4s+1$	$4s+3$	$8s+4$	
(2)	$4s+2-2r$	$6s+2+r$	$10s+4-r$	$[1,3]$
(3)	$4s-4-2r$	$8s+4+r$	$12s-r$	$[1,2s-4]$
(4)	2	$12s+2$	$12s+4$	
(5)	$4s+1-2r$	$12s+6+r$	$16s+7-r$	$[1,2s]$

Table 3.6: Construction of R_{1j} for $n \equiv 1$ $(mod\ 4)$ and $s \geq 2$.

R_{2i}	i	a_i	b_i	$r \in$
(1)	$4s+1-2r$	r	$4s+1-r$	$[1,2s]$
(2)	2	$4s+3$	$4s+5$	
(3)	$4s+2-2r$	$6s+3+r$	$10s+5-r$	$[1,3]$
(4)	$4s-4-2r$	$4s+7+r$	$8s+3-r$	$[1,2s-4]$
(5)	$4s+1$	$10s+5$	$14s+6$	

Table 3.7: Construction of R_{2i} for $n \equiv 1$ $(mod\ 4)$ and $s \geq 2$.

R_{2j}	j	a_j	b_j	$r \in$
(1)	3	$4s+1$	$4s+4$	
(2)	1	$4s+6$	$4s+7$	
(3)	$4s+3-2r$	$8s+2+r$	$12s+5-r$	$[1,2s-1]$
(4)	$4s+2-2r$	$12s+5+r$	$16s+7-r$	$[1,2s]$

Table 3.8: Construction of R_{2j} for $n \equiv 1$ $(mod\ 4)$ and $s \geq 2$.

In a similar way, we can prove that R_{1i}, R_{1j}, R_{2i}, and R_{2j} are two disjoint indecomposable 2-fold Rosa sequences of order $n \equiv 1 \ (mod\ 4)$ as in the proof of Lemma 3.2.2. ■

Lemma 3.2.4 *There exist two disjoint indecomposable 2-fold Rosa sequences of order $n \equiv 2 \ (mod\ 4)$.*

Proof: Let $n = 4s + 2$. For small order when $s = 1$, $n = 6$ the two disjoint indecomposable 2-fold Rosa sequences are

$$5, 3, 6, 4, 3, 5, 0, 4, 6, 1, 1, 3, 1, 1, 3, 5, 2, 6, 2, 0, 5, 4, 2, 6, 2, 4$$

and

$$2, 6, 2, 2, 5, 2, 0, 6, 4, 5, 6, 4, 4, 5, 3, 4, 6, 3, 5, 0, 3, 1, 1, 3, 1, 1.$$

For $n > 6$ the required constructions of two disjoint indecomposable 2-fold Rosa sequences are given in Tables 3.9, 3.10, 3.11, and 3.12.

R_{1i}	i	a_i	b_i	$r \in$
(1)	$4s+3-2r$	r	$4s+3-r$	$[1, 2s]$
(2)	$4s+2$	$2s+1$	$6s+3$	
(3)	$4s$	$2s+2$	$6s+2$	
(4)	1	$6s+4$	$6s+5$	
(5)	$4s-2r$	$8s+8+r$	$12s+8-r$	$[1, 2s-1]$

Table 3.9: Construction of R_{1i} for $n \equiv 2 \ (mod\ 4)$ and $s \geq 2$.

R_{1j}	j	a_j	b_j	$r \in$
(1)	3	$8s+4$	$8s+7$	
(2)	1	$8s+5$	$8s+6$	
(3)	$4s+1-2r$	$4s+3+r$	$8s+4-r$	$[1, 2s-2]$
(4)	$4s+1$	$8s+8$	$12s+9$	
(5)	$4s+2$	$10s+8$	$14s+10$	
(6)	$4s+2-2r$	$12s+9+r$	$16s+11-r$	$[1, 2s]$

Table 3.10: Construction of R_{1j} for $n \equiv 2 \ (mod\ 4)$ and $s \geq 2$.

R_{2i}	i	a_i	b_i	$r \in$
(1)	$4s-2r$	$3+r$	$4s+3-r$	$[1, 2s-1]$
(2)	$4s+2$	$6s+5$	$10s+7$	
(3)	$4s$	$6s+6$	$10s+6$	
(4)	$4s+3-2r$	$8s+5+r$	$12s+8-r$	$[1, 2s]$
(5)	1	$16s+9$	$16s+10$	

Table 3.11: Construction of R_{2i} for $n \equiv 2 \ (mod\ 4)$ and $s \geq 2$.

R_{2j}	j	a_j	b_j	$r \in$
(1)	2	1	3	
(2)	$4s+2$	2	$4s+4$	
(3)	$4s+1$	$2s+3$	$6s+4$	
(4)	$4s+2-2r$	$4s+4+r$	$8s+6-r$	$[1, 2s-1]$
(5)	$4s+1-2r$	$12s+8+r$	$16s+9-r$	$[1, 2s]$

Table 3.12: Construction of R_{2j} for $n \equiv 2 \ (mod\ 4)$ and $s \geq 2$.

■

Lemma 3.2.5 *There exist two disjoint indecomposable 2-fold Rosa sequences of order $n \equiv 3 \ (mod\ 4)$.*

Proof: Let $n = 4s + 3$. The required constructions for two disjoint indecomposable 2-fold Rosa sequences are given in Tables 3.13, 3.14, 3.15, and 3.16.

R_{1i}	i	a_i	b_i	$r \in$
(1)	$4s+4-2r$	r	$4s+4-r$	$[1, 2s+1]$
(2)	$4s+3$	$2s+2$	$6s+5$	
(3)	$4s+3-2r$	$4s+4+r$	$8s+7-r$	$[1, 2s]$
(4)	1	$12s+8$	$12s+9$	

Table 3.13: Construction of R_{1i} for $n \equiv 3 \ (mod\ 4)$.

R_{1j}	j	a_j	b_j	$r \in$
(1)	$4s+1$	$6s+6$	$10s+7$	
(2)	$4s+2-2r$	$8s+6+r$	$12s+8-r$	$[1, 2s]$
(3)	$4s+3$	$12s+10$	$16s+13$	
(4)	$4s+2$	$12s+12$	$16s+14$	
(5)	$4s+1-2r$	$12s+12+r$	$16s+13-r$	$[1, 2s]$

Table 3.14: Construction of R_{1j} for $n \equiv 3 \ (mod\ 4)$.

R_{2i}	i	a_i	b_i	$r \in$
(1)	2	1	3	
(2)	$4s+3$	2	$4s+5$	
(3)	$4s+1-2r$	$3+r$	$4s+4-r$	$[1, 2s]$
(4)	$4s+2$	$6s+7$	$10s+9$	
(5)	$4s+2-2r$	$8s+9+r$	$12s+11-r$	$[1, 2s-1]$
(6)	$4s+1$	$10s+11$	$14s+12$	

Table 3.15: Construction of R_{2i} for $n \equiv 3$ $(mod\ 4)$.

R_{2j}	j	a_j	b_j	$r \in$
(1)	$4s+5-2r$	$4s+5+r$	$8s+10-r$	$[1, 2s+1]$
(2)	$4s+2$	$6s+8$	$10s+10$	
(3)	$4s+2-2r$	$12s+11+r$	$16s+13-r$	$[1, 2s]$
(4)	1	$16s+13$	$16s+14$	

Table 3.16: Construction of R_{2j} for $n \equiv 3$ $(mod\ 4)$.

■

From Lemma 3.2.2 - 3.2.5 we have the following theorem.

Theorem 3.2.6 *There exist two disjoint indecomposable 2-fold Rosa sequences of order n.*

We will use the following results and our constructions to produce disjoint indecomposable $CTS_4(6n+3)$:

Construction (Rees, Shalaby, Sharary, [27]) Let $2T = (t_1, t_2, \ldots, t_{4n+2})$ be a 2-fold Rosa sequence of order n. In particular, $t_{n+1} = t_{3n+2} = 0$. The set of triples

$\{\{0, r, b_r + n\}, \{0, r, d_r + n\}, : r = 1, 2, \ldots, n\}$ form the base blocks for a cyclic 2-fold 3-GDD of type 3^{2n+1} (the groups of which are given by $\{0, 2n+1, 4n+2\}(mod\ 6n+3)$) which in turn gives rise to a $CTS_2(6n + 3)$.

Lemma 3.2.7 *(Rees, Shalaby, [26]) If* $2T = (t_1, t_2, \ldots, t_{4n+2})$ *is a 2-fold Rosa sequence of order* n *and the pairs* (a_r, b_r)*,* (c_r, d_r) *contain among them a pair* (x_r, y_r) *where* $x_r + y_r = 4n + 3$ *then the corresponding* $CTS_2(6n + 3)$ *(arising out of* ***Construction*** *[27]) is indecomposable.*

By our constructions for two disjoint indecomposable 2-fold Rosa sequences we can produce disjoint cyclically indecomposable $CTS_4(6n + 3)$. For example,

$$6, 4, 2, 7, 2, 4, 6, 0, 5, 3, 7, 5, 3, 5, 4, 2, 5, 2, 4, 1, 1, 7, 0, 6, 3, 1, 1, 3, 7, 6$$

and

$$2, 7, 2, 3, 1, 1, 3, 0, 7, 7, 5, 3, 6, 6, 3, 5, 7, 4, 6, 6, 5, 4, 0, 4, 2, 5, 2, 4, 1, 1$$

are two disjoint indecomposable 2-fold Rosa sequences of order 7.

The pairs from the first sequence are (a_i, b_i): $(20, 21)$, $(3, 5)$, $(10, 13)$, $(2, 6)$, $(9, 14)$, $(1, 7)$, $(4, 11)$ and (a_j, b_j) where $i = j = 1, 2, 3, 4, 5, 6, 7$: $(26, 27)$, $(16, 18)$, $(25, 28)$, $(15, 19)$, $(12, 17)$, $(24, 30)$, $(22, 29)$. The set of triples $\{0, i, b_i + n\}$: $\{0, 1, 28\}$, $\{0, 2, 12\}$, $\{0, 3, 20\}$, $\{0, 4, 13\}$, $\{0, 5, 21\}$, $\{0, 6, 14\}$, $\{0, 7, 18\}$ and $\{0, j, b_j + n\}$: $\{0, 1, 34\}$, $\{0, 2, 25\}$, $\{0, 3, 35\}$, $\{0, 4, 26\}$, $\{0, 5, 24\}$, $\{0, 6, 37\}$, $\{0, 7, 36\}$, and adding the short-orbit base block $\{0, 15, 30\}$, yield a cyclically indecomposable 2-fold cyclic triple system $CTS_2(6n + 3)$.

The pairs from the second sequence are (a'_i, b'_i): $(5, 6)$, $(1, 3)$, $(4, 7)$, $(18, 22)$, $(21, 26)$, $(13, 19)$, $(2, 9)$ and (a'_j, b'_j) where $i = j = 1, 2, 3, 4, 5, 6, 7$: $(29, 30)$, $(25, 27)$, $(12, 15)$, $(24, 28)$, $(11, 16)$, $(14, 20)$, $(10, 17)$. The set of triples $\{0, i, b'_i + n\}$: $\{0, 1, 13\}$, $\{0, 2, 10\}$, $\{0, 3, 14\}$, $\{0, 4, 29\}$, $\{0, 5, 33\}$, $\{0, 6, 26\}$, $\{0, 7, 16\}$ and $\{0, j, b'_j + n\}$:

$\{0,1,37\}$, $\{0,2,34\}$, $\{0,3,22\}$, $\{0,4,35\}$, $\{0,5,23\}$, $\{0,6,27\}$, $\{0,7,24\}$, and adding the short-orbit base block $\{0,15,30\}$, yield a cyclically indecomposable 2-fold cyclic triple system $CTS_2(6n+3)$.

Hence, the two disjoint indecomposable 2-fold Rosa sequences of order 7 give disjoint cyclically indecomposable $CTS_4(6n+3)$.

Chapter 4

Algorithms and Computational Results

In this chapter, we find the number of distinct Skolem, Rosa, (2-fold) Skolem and Rosa sequences of small orders. We present algorithms to find the number of distinct hooked Rosa sequences of order n <11, and a maximal disjoint set of (hooked) Rosa sequences containing a given (hooked) Rosa sequence as well as maximum disjoint (hooked) Rosa sequences for small orders $2 \leq n \leq 21$.

We can find all or most of the possible solutions to a computational problem by backtracking algorithms. In 2005, Grüttmüller, Rees, and Shalaby [12] investigated exhaustively and constructed cyclically indecomposable 2-fold triple systems $CTS_2(v)$ for all admissible orders by using Skolem-type and Rosa-type sequences. They obtained the results shown in Table 4.1 and Table 4.2.

Order	1	2	3	4	5	6	7	8	9	10	11	12	13
Number of Skolem sequences	1	0	0	6	10	0	0	504	2656	0	0	455936	3040560
Number of Rosa sequences	0	0	2	2	0	0	44	260	0	0	33104	203712	0

Table 4.1: Number of Skolem and Rosa sequences of order $n \leq 13$

Order	1	2	3	4	5	6	7	8	9
Number of 2-fold Skolem sequences	1	3	12	186	3212	79238	2770026	127860956	>5000000000
Number of 2-fold Rosa sequences	0	1	8	50	912	22286	782374	36649766	

Table 4.2: Number of 2-fold Skolem and 2-fold Rosa sequences of order $n \leq 9$

In 2009, Larsen [17] used the inclusion-exclusion principle to count the number of Skolem sequences where the principle of inclusion-exclusion comes from set theory. They produce the results in Tables 4.3 and 4.4 [17].

Order	15	16	17	18
Number of hooked Skolem sequences	168870048	0	0	113071735648

Table 4.3: Number of hooked Skolem sequences of order $15 \leq n \leq 18$

Order	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Number of Rosa sequences	0	0	2	2	0	0	44	260	0	0	33104	203712	0	0	75499696	621309008	0	0

Table 4.4: The number of distinct Rosa sequences of order $n \leq 18$

Skolem sequences are used to construct other kinds of systems, but the algorithm from [17] does not generate Skolem sequences, and does not help in those areas. There is no other way than checking all $2n!$ permutations to actually generate all the possible Skolem sequences of a given order [17].

In 2013, Burrill and Yen [5] used a different strategy which is arc diagrams, or arc annotated. These sequences are structures that encode a variety of combinatorial classes. This algorithm exhaustively constructs all Skolem sequences of order n by building a generating tree to generate all of the sequences up to size 7, and some of size 8. The importance of this method was to reduce the search space significantly [5].

As we indicated in Chapter 2, the maximum number of mutually disjoint (hooked) Rosa sequences of order n is $n-1$. We will consider the following problems:

1. For $n \equiv 1, 2(mod\ 4)$, find all distinct hooked Rosa Sequences of order n.

2. Given a (hooked) Rosa sequence R of order n, find a maximal set of disjoint (hooked) Rosa sequences containing R.

3. For $n \equiv 1, 2(mod\ 4)$, find a set of mutually disjoint (hooked) Rosa sequences of order n of maximum cardinality.

We give algorithms to solve these problems. These algorithms perform exhaustive searches. An exhaustive search is an algorithm that systematically examines all values in the search space. We chose to use an exhaustive search because such algorithms are easy to formulate and are guaranteed to provide a solution. Algorithm 1 enumerates all distinct hooked Rosa sequences of a given order n. This algorithm is effective for n<11. Algorithm 2 adapts Algorithm 1 to find a maximal disjoint subset of hooked Rosa sequences. We have run this algorithm successfully for $n \leq 14$. Algorithm 3 also adapts Algorithm 1, and searches for a maximum disjoint subset of hooked Rosa sequences. We have run Algorithm 3 successfully for $n \leq 21$.

The following sections describe the algorithms to solve these problems.

4.1 Distinct Hooked Rosa Sequences

```
input  : n /* order of the hooked Rosa Sequences n ≡ 1, 2(mod 4)          */
output: list of valid hooked Rosa sequences; total number of sequences
Function place(i, S):
    feasiblePos = {(k, k+i) | S[k] = S[k+i] = 0, 1 ≤ k, k+i ≤ length (S)}
    if feasiblePos = {} then
        return
    end
    if i = n then
        /* feasiblePos has one element, (k, k+i)                        */
        S[k] = S[k+i] = i
        numSeqs = numSeqs + 1
    else
        foreach (k, k+i) in feasiblePos do
            S[k] = S[k+i] = i
            Call place(i+1, S)
            S[k] = S[k+i] = 0
        end
    end
Initialization begin
    R = [0, ..., 0] /* an array of 2n+2 zeros                            */
    R[n+1] = R[2n+1] = "*"
    numSeqs = 0
end
Program begin
    Call place(1, R)
    Output numSeqs
end
```

Algorithm 1: Number of Hooked Rosa Sequences of order n <11

Now we will explain how this algorithm works. Create an array sequence of length $2n+2$ with all elements other than the hooks set to blank. Place $1, 1$ in the far left of the empty sequence. Then, starting from the left, find the first two blank spots in the sequence that are two elements apart. Place a 2 in each spot. Then place a 3 in the first two blank spots that are three elements apart, and so on. If you reach a number i that does not fit into either of the two blank spots in the sequence that are i elements apart, backtrack by removing the pair of $i-1$'s from the sequence and place $i-1$ in the next two blank spots in the sequences that are $i-1$ spaces apart. Then try to place i in the sequence and proceed. If $i-1$ has been tried in all the spots that can fit it, remove $i-2$ from the sequence, and so on. Following this procedure you will occasionally manage to place all n numbers into the sequence. In this case you have discovered a hooked Rosa sequence. Add the sequence to the list of hooked Rosa sequences for order n, backtrack by removing $n-1$ from the sequence and continue the above process. Eventually you will have tried $1, 1$ in every possible position (and the rest of the numbers in all feasible positions). At this point the program ends and outputs the total number of hooked Rosa sequences discovered.

How do we know that the above algorithm is correct? Firstly, it will not falsely report a non-hooked Rosa sequence as a hooked Rosa sequence. Considering the three conditions listed in the definition of hooked Rosa sequence condition (3) is satisfied during the creation of the empty sequence. The algorithm starts with $i = 1$ and moves toward $i = n$, incrementing i by 1, each time the previous pair has been placed successfully. Thus, it does not place any pair more than once. Algorithm 1 also does not skip any numbers: if the pair $i \dots i$ cannot be placed successfully, the algorithm decrements i by 1, and tries new spots for $i-1$. As a result, each pair $i \dots i$ is placed exactly once for each $i \in \{1, \dots, n\}$, satisfying condition (1). Finally, condition (2) is honoured because each pair $i \dots i$ is placed i spaces apart.

Next, we need to show that Algorithm 1 always terminates. The search described above is conducted by recursively calling a procedure **place**. This procedure takes as input a partially completed hooked Rosa sequence and an integer i. It then generates a list of all pairs of blank positions in the sequence that are i spaces apart. It places the pair $i \ldots i$ in the first of these positions and calls **place** with the updated sequence and the integer $i+1$. When this sub-call returns, the procedure removes the pair $i \ldots i$ from the first available position and places it in the next available position, calling **place** again, and so on, through each position in the list. When it is finished, the procedure returns control to the procedure that called it initially. When **place** is called with $i = n$ and there are two blank spaces left n places apart in the partially completed sequence, the procedure **place** returns. From here we can see by induction that **place** eventually returns for any integer $1 \leq i \leq n$.

To prove that the algorithm finds all hooked Rosa sequences of order n, let R be an arbitrary hooked Rosa sequence of order n. The pair 1 1 occurs somewhere in R. Since the algorithm considers placing the pair 1 1 at all possible feasible positions in turn, and is known to terminate, it must at some point consider the sequence fragment with all spaces blank except for the pair 1 1 found in the position in which it occurs in R. At this point it starts to consider placing pairs of twos, until it places them in the position in which they are found in R. It proceeds through the integers up to n, at which point it has found R. We have now shown that the algorithm finds all hooked Rosa sequences of order n, and does not falsely report non-hooked Rosa sequences.

What is the computational complexity of this algorithm? When searching for hooked Rosa sequences, there are initially $2n$ empty places in the sequence. The first one in the pair can be placed in $2n - 3$ of these places, but since we are searching for an upper bound, we can assume that all $2n$ places are valid. For each potential spot to place a pair of ones, the algorithm tries to place a pair of twos into empty spaces. Since the pair of ones takes up two spaces, there are at least two fewer spaces in which to place a pair of twos; *i.e.* no more than $2n-2$ possible places. For each possible place for a pair of twos, there are at most $2n-4$ possible spaces for a pair of threes. Continuing this way, we see that the algorithm is completed in less than $2n \cdot (2n-2) \cdot (2n-4) \cdot \ldots \cdot 4 \cdot 2 = 2^n n!$ steps.

Here we present the result for the number of distinct hooked Rosa sequences is found by our algorithm of order n <11.

Order	1	2	3	4	5	6	7	8	9	10
Number of hooked Rosa sequences	0	1	0	0	8	18	0	0	1208	6332

Table 4.5: Number of Hooked Rosa Sequences of order n<11

4.2 Maximal Disjoint Hooked Rosa Sequences

input : A Rosa sequence *givenR*

output: A maximal disjoint subset containing *givenR*

Initialization **begin**

  if *givenR is a hooked Rosa sequence* **then**

    $R = [0, \ldots, 0]$ `/* an array of 2n+2 zeros */`

    $R[n+1] = R[2n+1] = “*''$

  else

    $R = [0, \ldots, 0]$ `/* an array of 2n+1 zeros */`

    $R[n+1] = “*''$

  end

  $infeasiblePos = \{\}$

  $currentSet = \{givenR\}$

end

Program **begin**

  Call $AddInfeasiblePositions(givenR)$

  Call $Place(1, R)$

end

Function `AddInfeasiblePosition(`S`):`

  `/* This function add all pair positions in` S `to infeasiblePos */`

  begin

    $infeasiblePos = infeasiblePos \cup \{(i, i+k)|S[i] = S[i+k] = i\}$

  end

```
Function place(i, S):
    /* Whether a sequence was found and added to currentSet */
    begin
        feasiblePos = {(k, k+i) | (k, k+i) ∉ infeasiblePos, S[k] = S[k+i] = 0,
        1 ≤ k, k+i ≤ length (S)}
        if feasiblePos = {} then
            return False
        end
        if i = n then
            /* feasiblePos has one element, (k, k+i) */
            S[k] = S[k+i] = i
            currentSet = currentSet ∪ {S}
            Call AddInfeasiblePositions(S)
            Output S
            return True
        else
            foreach (k, k+i) in feasiblePos do
                S[k] = S[k+i] = i
                FoundSequence? = Place(i+1, S)
                S[k] = S[k+i] = 0
            end
            if FoundSequence? = True and i > 1 then
                Return True
            end
        end
        return False
    end
```

Algorithm 2: Maximal Disjoint Hooked Rosa Sequences

How we can describe this algorithm? Initialize a set D of disjoint sequences, initially containing just S. Run the Distinct Hooked Rosa Sequences (Algorithm 1), with one change. When considering feasible positions for the number i, exclude any pair of positions for that number that is used by a sequence already in D. At the conclusion of the algorithm, D contains a maximal subset of disjoint hooked Rosa sequences of order n. (i.e. any hooked Rosa sequence not in D is not disjoint with at least one hooked Rosa sequence in D). As we have shown that the previous algorithm terminates, we can make the same assertion about this algorithm, since it follows the same procedure, but considers fewer positions to place pairs of numbers. The algorithm discovers a subset of the set of all hooked Rosa sequences of order n such that all excluded sequences contain pairs in positions already used by a hooked Rosa sequence in D. Thus all hooked Rosa sequences that could potentially be placed in D have been considered. As a result, we can state that D is a maximal subset of disjoint hooked Rosa sequences. Since it considers fewer positions than the above algorithm, this algorithm also completes in $O(2^n n!)$ steps.

We list the number of maximal disjoint hooked Rosa sequences for a given hooked Rosa sequence. Note that there may be a larger maximal set of disjoint Rosa sequences that includes R:

R	$n = 5, (2, 3, 2, 4, 3, 0, 5, 4, 1, 1, 0, 5)$
$N_{maximal\ R}$	3
R	$n = 6, (1, 1, 5, 2, 6, 2, 0, 5, 3, 4, 6, 3, 0, 4)$
$N_{maximal\ R}$	4
R	$n = 9, (3, 7, 8, 3, 2, 9, 2, 4, 7, 0, 8, 4, 5, 6, 9, 1, 1, 5, 0, 6)$
$N_{maximal\ R}$	6
R	$n = 10, (5, 8, 6, 9, 3, 5, 10, 3, 6, 8, 0, 7, 9, 4, 1, 1, 10, 4, 7, 2, 0, 2)$
$N_{maximal\ R}$	7
R	$n = 13, (11, 9, 7, 5, 3, 12, 13, 3, 5, 7, 9, 11, 6, 0, 10, 8, 4, 12, 6, 13, 4, 1, 1, 8, 10, 2, 0, 2)$
$N_{maximal\ R}$	11
R	$n = 14, (7, 12, 10, 8, 13, 5, 3, 7, 14, 3, 5, 8, 10, 12, 0, 11, 9, 13, 6, 4, 1, 1, 14, 4, 6, 9, 11, 2, 0, 2)$
$N_{maximal\ R}$	12

Table 4.6: Maximal Disjoint Hooked Rosa Sequences

4.3 Maximum Disjoint (Hooked) Rosa Sequences

```
input : n/* Order of the Rosa Sequences */
output: A maximum disjoint subset of Rosa Sequences of order n
Function initialSet():
    begin
        if n = 1 or n = 2(mod4) then
            R = [0, ..., 0] /* an array of 2n+2 zeros */
            R[n + 1] = R[2n + 1] = "*"
        else
            R = [0, ..., 0] /* an array of 2n+1 zeros */
            R[n + 1] = "*"
        end
    end
    return R
Initialization begin
    R = initialSet()
    currentSet = {}
end
Program begin
    Call Place({}, n, R, {})
end
```

Function place(*infeasiblePos, i, S, triedSeqs*):

/* infeasiblePos: Set of pair positions occupied by pairs in sequences in currentSet */

/* S: sequence under construction */

/* i: value of pair to be placed in S */

/* triedSeqs: set of sequences tried and rejected by parent invocation of place */

/* Return value: Set of sequences found and rejected by place */

begin

$feasiblePos = \{(k, k+i) | (k, k+i) \notin infeasiblePos, S[k] = S[k+i] = 0,\ 1 \leq k,\ k+i \leq \text{length}(S)\}$

if $feasiblePos = \{\}$ **then**

  return $\{\}$

end

if $i = 1$ **then**

  /* feasiblePos has one element, $(k, k+i)$ */

  $S[k] = S[k+i] = i$

  if $S \in triedSeqs$ **then**

    return $\{\}$

  end

  $currentSet = currentSet \cup \{S\}$

  if $length(currentSet) = n - 1$ **then**

    Output *currentSet*

    End Program

  else

    Call place($infeasiblePos \cup \{(i, i+k) | S[i] = S[i+k] = i\}$, n, initialSet(), *triedSeqs*)

  end

  $currentSet = currentSet / \{S\}$

  return $\{S\}$

end

else

  foreach $(k, k+i)$ *in* $feasiblePos$ **do**

    $S[k] = S[k+i] = i$

    $triedSeqs = triedSeqs \cup place(infeasiblePos, i-1, S, triedSeqs)$

    $S[k] = S[k+i] = 0$

  end

end

return *triedSeqs*

end

Algorithm 3: Maximum Disjoint (Hooked) Rosa Sequences of orders $2 \leq n \leq 21$

The details for this algorithm are given below.

Initialize a set D of disjoint sequences to empty. Run the Distinct Sequences algorithm (Algorithm 1), with the following change: when considering feasible positions for a number, exclude any pair of positions for that number that is used by a sequence already in D. When a hooked Rosa sequence is found, place it in D. The algorithm will reach a point where no further hooked Rosa sequences can be found. At this point, D contains a maximal subset of disjoint hooked Rosa sequences of order n. If this subset is a maximum disjoint subset, output it and end. Otherwise, remove the last sequence placed in D, and continue executing the modified Algorithm 1 described above to find new hooked Rosa sequences. Continue in this way: whenever a new hooked Rosa sequence is found, add it to D; whenever the algorithm cannot find a new hooked Rosa sequence disjoint from all sequences in D, remove the last sequence placed in D. When D contains a maximum disjoint subset, output it and end the program.

By backtracking whenever it cannot find a hooked Rosa sequence, Algorithm 3 will eventually enumerate all possible maximum subsets of mutually disjoint hooked Rosa sequences. However, as presented, it will create some subsets more than once. Suppose, for instance, that S_1, S_2, and S_3 are the first three mutually disjoint hooked Rosa sequences created by Algorithm 3. Algorithm 3 first places S_1 in D, then S_2, then S_3. After searching for a maximum disjoint subset containing these three sequences, Algorithm 3 backtracks by removing S_3 and searching for maximum disjoint subsets containing S_1 and S_2 without S_3, then backtracks by removing S_2 from D, at which point D contains only S_1. As it continues to search for maximum subsets containing S_1, at some point Algorithm 3 will place S_3 into D and search for maximum disjoint subsets containing S_1 and S_3. As S_2 is disjoint from both S_1 and S_3, it will find S_2 in the course of its search, place S_2 in D and start searching anew for a maximum disjoint subset containing S_1, S_2, and S_3.

Each time we add a hooked Rosa sequence S to D, the algorithm's procedure is, in effect, to find all hooked Rosa sequences that are mutually disjoint with all the sequences in D, performing a recursive subsearch with each one as it is found. To avoid duplicate subsearches, we maintain a set of sequences that we have already performed subsearches with, and skip these sequences when they are discovered in future subsearches. When backtracking by removing S from D, we discard this set of sequences, because one of them may be able to form a maximum disjoint subset in combination with a different set of sequences in D.

To express the time complexity of this algorithm, we must make some worst-case assumptions. In our analysis of Algorithm 1, we showed that there are at most $R = 2^n n!$ Rosa sequences of order n, and concluded that Algorithm 1 terminates in $O(R)$ time. Since Algorithm 3 uses Algorithm 1 to search for hooked Rosa sequences, we assume that Algorithm 3 requires R steps to find a hooked Rosa sequence. We also assume that there exist R distinct hooked Rosa sequences of order n. We know that there exists a maximum disjoint subset of cardinality $n-1$ for $n \geq 5$. Assuming that Algorithm 3 examines all $\binom{R}{n-2}$ possible subsets of $n-2$ hooked Rosa sequences before finding a maximum disjoint subset, and that it takes R steps to find each hooked Rosa sequence each time, it takes $O[\binom{R}{n-2}R(n-2)]$ steps to find a maximum disjoint subset. We have used this algorithm effectively to find maximum disjoint subsets of (hooked) Rosa sequences for small orders $2 \leq n \leq 21$.

The results from this algorithm are found in the Appendix (6.1).

Chapter 5

Conclusion

This thesis uses Rosa-type sequences to find maximum disjoint Rosa sequences and produce new constructions. We also use Rosa-type sequences to construct a cyclic 2-fold 3-GDD of type 3^{2n+1} as well as constructions cyclically indecomposable triple systems.

Chapter 2 introduces a direct method for finding $4 \times n$ hooked Rosa rectangles of order $n = 1, 2(mod\ 4)$ and we construct cyclic triple systems $CTS_4(v)$ and a GDD by using two disjoint hooked Rosa sequences. In 2014, Linek, Mor, and Shalaby [18] show asymptotic constructions for Skolem sequences, hooked Skolem sequences, and Rosa sequences that provide the only known non-trivial bounds; we are interested in the future to find asymptotic constructions for hooked Rosa sequences, and in whether or not our direct constructions for hooked Rosa rectangles can be generalized to produce all disjoint hooked Rosa sequences of order $n = 1,\ 2(mod\ 4)$.

Chapter 3 establishes new constructions for two disjoint m-fold Skolem, m-fold Rosa, and indecomposable 2-fold Rosa sequences of order n. We may find more disjoint m-fold Skolem and Rosa sequences and use these to construct λ-fold triple system of order v in the future. We use two disjoint indecomposable 2-fold Rosa sequences of order n to construct disjoint cyclically indecomposable $CTS_4(6n+3)$.

Chapter 4 gives exhaustive algorithms to find all distinct hooked Rosa sequences of order n, as well as finding a maximum disjoint subset of hooked Rosa sequences for small orders $n \leq 21$. In 1998, Eldin, Shalaby, and Althukair [9] used a hill-climbing algorithm to generate Skolem sequences, for example constructing Skolem sequences of order 84. In future work we are interested in using a hill-climbing algorithm to find a maximum disjoint subset of (hooked) Skolem and Rosa sequences for all admissible orders.

Chapter 6

Appendix

6.1 List of Maximum Disjoint (Hooked) Rosa Sequences

Maximum disjoint (hooked) Rosa sequences for small orders $2 \leq n \leq 21$

$n = 5$

$1, 1, 5, 3, 4, 0, 3, 5, 4, 2, 0, 2$

$2, 3, 2, 4, 3, 0, 5, 4, 1, 1, 0, 5$

$3, 1, 1, 3, 5, 0, 2, 4, 2, 5, 0, 4$

$4, 5, 1, 1, 4, 0, 5, 2, 3, 2, 0, 3$

$n = 7$

$1, 1, 6, 3, 7, 5, 3, 0, 6, 4, 5, 7, 2, 4, 2$

$2, 4, 2, 7, 5, 4, 6, 0, 3, 5, 7, 3, 6, 1, 1$

$3, 1, 1, 3, 4, 6, 7, 0, 4, 5, 2, 6, 2, 7, 5$

$4, 5, 1, 1, 4, 7, 5, 0, 6, 2, 3, 2, 7, 3, 6$

$5, 7, 2, 6, 2, 5, 4, 0, 7, 6, 4, 3, 1, 1, 3$

$6, 3, 7, 2, 3, 2, 6, 0, 5, 7, 4, 1, 1, 5, 4$

$n = 8$

$1, 1, 5, 7, 2, 8, 2, 5, 0, 6, 7, 3, 4, 8, 3, 6, 4$

$2, 4, 2, 6, 8, 4, 5, 7, 0, 6, 3, 5, 8, 3, 7, 1, 1$

$3, 5, 2, 3, 2, 6, 5, 8, 0, 7, 4, 6, 1, 1, 4, 8, 7$

$4, 1, 1, 8, 4, 7, 3, 6, 0, 3, 5, 8, 7, 6, 2, 5, 2$

$5, 3, 7, 4, 3, 5, 8, 4, 0, 7, 6, 1, 1, 2, 8, 2, 6$

$6, 2, 8, 2, 7, 4, 6, 5, 0, 4, 8, 7, 5, 3, 1, 1, 3$

$7, 8, 4, 2, 6, 2, 4, 7, 0, 8, 6, 5, 3, 1, 1, 3, 5$

$n = 9$

$1, 1, 2, 3, 2, 7, 3, 8, 9, 0, 4, 5, 7, 6, 4, 8, 5, 9, 0, 6$

$2, 4, 2, 5, 9, 4, 8, 3, 5, 0, 3, 6, 7, 9, 8, 1, 1, 6, 0, 7$

$3, 1, 1, 3, 6, 9, 7, 5, 8, 0, 6, 4, 5, 7, 9, 4, 8, 2, 0, 2$

$4, 5, 3, 8, 4, 3, 5, 6, 7, 0, 9, 8, 2, 6, 2, 7, 1, 1, 0, 9$

$5, 3, 6, 4, 3, 5, 9, 4, 6, 0, 7, 8, 1, 1, 2, 9, 2, 7, 0, 8$

$6, 7, 8, 9, 1, 1, 6, 4, 7, 0, 8, 4, 9, 3, 5, 2, 3, 2, 0, 5$

$7, 2, 9, 2, 8, 1, 1, 7, 5, 0, 6, 9, 8, 5, 3, 4, 6, 3, 0, 4$

$8, 9, 5, 2, 7, 2, 6, 5, 8, 0, 9, 7, 6, 4, 1, 1, 3, 4, 0, 3$

$n = 10$

$1, 1, 2, 4, 2, 9, 10, 4, 3, 8, 0, 3, 7, 5, 9, 6, 10, 8, 5, 7, 0, 6$

$2, 5, 2, 8, 1, 1, 5, 6, 9, 10, 0, 8, 4, 6, 7, 3, 4, 9, 3, 10, 0, 7$

$3, 1, 1, 3, 4, 10, 6, 9, 4, 5, 0, 7, 6, 8, 5, 10, 9, 2, 7, 2, 0, 8$

$4, 6, 1, 1, 4, 7, 8, 6, 5, 9, 0, 10, 7, 5, 8, 2, 3, 2, 9, 3, 0, 10$

$5, 3, 4, 6, 3, 5, 4, 10, 7, 6, 0, 8, 9, 1, 1, 7, 2, 10, 2, 8, 0, 9$

$6, 7, 3, 9, 10, 3, 6, 8, 7, 2, 0, 2, 9, 4, 10, 8, 5, 4, 1, 1, 0, 5$

$7, 8, 9, 10, 2, 3, 2, 7, 3, 8, 0, 9, 6, 10, 5, 1, 1, 4, 6, 5, 0, 4$

$8, 4, 10, 3, 9, 4, 3, 7, 8, 6, 0, 5, 10, 9, 7, 6, 5, 1, 1, 2, 0, 2$

$9, 10, 5, 2, 8, 2, 7, 5, 6, 9, 0, 10, 8, 7, 6, 4, 1, 1, 3, 4, 0, 3$

$n = 11$

$1, 1, 3, 6, 8, 3, 10, 7, 9, 6, 11, 0, 8, 5, 7, 4, 10, 9, 5, 4, 2, 11, 2$

$2, 5, 2, 7, 1, 1, 5, 9, 10, 11, 7, 0, 3, 6, 8, 3, 9, 4, 10, 6, 11, 4, 8$

$3, 1, 1, 3, 4, 10, 11, 5, 4, 7, 8, 0, 5, 9, 6, 10, 7, 11, 8, 2, 6, 2, 9$

$4, 2, 6, 2, 4, 8, 3, 11, 6, 3, 9, 0, 10, 8, 7, 5, 1, 1, 11, 9, 5, 7, 10$

$5, 6, 1, 1, 2, 5, 2, 6, 11, 9, 10, 0, 4, 8, 3, 7, 4, 3, 9, 11, 10, 8, 7$

$6, 3, 8, 9, 3, 11, 6, 10, 7, 4, 8, 0, 9, 4, 5, 7, 11, 10, 2, 5, 2, 1, 1$

$7, 8, 4, 10, 11, 9, 4, 7, 2, 8, 2, 0, 5, 10, 9, 11, 6, 5, 3, 1, 1, 3, 6$

$8, 9, 10, 11, 3, 1, 1, 3, 8, 6, 9, 0, 10, 7, 11, 6, 5, 2, 4, 2, 7, 5, 4$

$9, 7, 11, 2, 10, 2, 4, 8, 7, 9, 4, 0, 6, 11, 10, 8, 3, 5, 6, 3, 1, 1, 5$

$10, 11, 7, 5, 9, 2, 8, 2, 5, 7, 10, 0, 11, 9, 8, 6, 4, 1, 1, 3, 4, 6, 3$

$n = 12$

$1, 1, 3, 4, 2, 3, 2, 4, 9, 10, 12, 7, 0, 11, 6, 8, 5, 9, 7, 10, 6, 5, 12, 8, 11$

$2, 4, 2, 6, 7, 4, 11, 12, 10, 6, 3, 7, 0, 3, 9, 5, 8, 11, 10, 12, 5, 1, 1, 9, 8$

$3, 6, 2, 3, 2, 8, 10, 6, 12, 9, 11, 4, 0, 8, 5, 4, 10, 7, 9, 5, 12, 11, 1, 1, 7$

$4, 1, 1, 8, 4, 11, 12, 10, 5, 6, 9, 8, 0, 5, 7, 6, 11, 10, 12, 9, 3, 7, 2, 3, 2$

$5, 7, 4, 1, 1, 5, 4, 8, 7, 11, 6, 12, 0, 9, 10, 8, 6, 2, 3, 2, 11, 3, 9, 12, 10$

$6, 3, 9, 10, 3, 12, 6, 11, 1, 1, 5, 9, 0, 10, 8, 5, 7, 12, 11, 2, 4, 2, 8, 7, 4$

$7, 2, 6, 2, 1, 1, 8, 7, 6, 12, 10, 11, 0, 4, 8, 9, 3, 4, 5, 3, 10, 12, 11, 5, 9$

$8, 9, 7, 11, 12, 10, 5, 6, 8, 7, 9, 5, 0, 6, 11, 10, 12, 4, 2, 3, 2, 4, 3, 1, 1$

$9, 10, 11, 12, 4, 1, 1, 7, 4, 9, 8, 10, 0, 11, 7, 12, 6, 3, 8, 5, 3, 2, 6, 2, 5$

$10, 8, 12, 2, 11, 2, 7, 9, 3, 8, 10, 3, 0, 7, 12, 11, 9, 5, 6, 4, 1, 1, 5, 4, 6$

11, 12, 8, 5, 10, 2, 9, 2, 5, 7, 8, 11, 0, 12, 10, 9, 7, 6, 4, 1, 1, 3, 4, 6, 3

$n = 13$

1, 1, 10, 11, 6, 13, 5, 12, 9, 7, 6, 5, 10, 0, 11, 8, 7, 9, 13, 12, 2, 4, 2, 8, 3, 4, 0, 3

2, 3, 2, 8, 3, 10, 1, 1, 12, 5, 13, 8, 9, 0, 5, 10, 11, 7, 4, 6, 12, 9, 4, 13, 7, 6, 0, 11

3, 8, 2, 3, 2, 12, 13, 11, 4, 8, 5, 10, 4, 0, 9, 5, 6, 12, 11, 13, 7, 10, 6, 9, 1, 1, 0, 7

4, 5, 6, 2, 4, 2, 5, 9, 6, 11, 7, 8, 13, 0, 10, 12, 9, 7, 3, 8, 11, 3, 1, 1, 10, 13, 0, 12

5, 7, 1, 1, 10, 5, 12, 13, 7, 3, 11, 4, 3, 0, 10, 4, 9, 6, 12, 8, 13, 11, 2, 6, 2, 9, 0, 8

6, 4, 5, 1, 1, 4, 6, 5, 7, 13, 9, 12, 8, 0, 11, 7, 2, 10, 2, 9, 8, 3, 13, 12, 3, 11, 0, 10

7, 2, 3, 2, 5, 3, 11, 7, 13, 5, 12, 1, 1, 0, 6, 10, 8, 11, 9, 4, 6, 13, 12, 4, 8, 10, 0, 9

8, 1, 1, 4, 2, 6, 2, 4, 8, 12, 10, 6, 11, 0, 13, 9, 3, 5, 7, 3, 10, 12, 5, 11, 9, 7, 0, 13

9, 10, 8, 12, 13, 11, 2, 7, 2, 9, 8, 10, 6, 0, 7, 12, 11, 13, 6, 5, 1, 1, 3, 4, 5, 3, 0, 4

10, 11, 12, 13, 4, 2, 9, 2, 4, 8, 10, 7, 11, 0, 12, 9, 13, 8, 7, 3, 5, 6, 3, 1, 1, 5, 0, 6

11, 9, 13, 6, 12, 3, 8, 10, 3, 6, 9, 11, 7, 0, 8, 13, 12, 10, 5, 7, 4, 1, 1, 5, 4, 2, 0, 2

12, 13, 9, 7, 11, 4, 10, 1, 1, 4, 7, 9, 12, 0, 13, 11, 10, 8, 6, 2, 3, 2, 5, 3, 6, 8, 0, 5

$n = 14$

1, 1, 6, 2, 3, 2, 5, 3, 6, 12, 10, 5, 13, 11, 0, 14, 7, 9, 4, 8, 10, 12, 4, 7, 11, 13, 9, 8, 0, 14

2, 6, 2, 9, 7, 11, 12, 6, 14, 4, 13, 7, 9, 4, 0, 10, 11, 3, 12, 5, 3, 8, 14, 13, 5, 10, 1, 1, 0, 8

3, 7, 5, 3, 1, 1, 4, 5, 7, 10, 4, 13, 11, 14, 0, 6, 9, 12, 8, 10, 2, 6, 2, 11, 13, 9, 8, 14, 0, 12

4, 1, 1, 3, 4, 10, 3, 5, 13, 14, 12, 7, 5, 6, 0, 10, 11, 8, 7, 6, 9, 13, 12, 14, 2, 8, 2, 11, 0, 9

5, 2, 8, 2, 9, 5, 13, 14, 12, 7, 8, 4, 10, 9, 0, 4, 7, 6, 11, 13, 12, 14, 10, 6, 3, 1, 1, 3, 0, 11

6, 3, 1, 1, 3, 2, 6, 2, 8, 13, 14, 12, 5, 7, 0, 11, 8, 5, 9, 10, 7, 4, 13, 12, 14, 4, 11, 9, 0, 10

7, 8, 3, 5, 2, 3, 2, 7, 5, 8, 11, 9, 14, 12, 0, 4, 13, 10, 6, 4, 9, 11, 1, 1, 6, 12, 14, 10, 0, 13

8, 9, 7, 10, 11, 13, 14, 12, 8, 7, 9, 6, 4, 10, 0, 11, 4, 6, 13, 12, 14, 5, 3, 1, 1, 3, 5, 2, 0, 2

9, 5, 11, 12, 6, 14, 5, 13, 10, 9, 6, 1, 1, 11, 0, 12, 8, 4, 10, 14, 13, 4, 7, 3, 8, 2, 3, 2, 0, 7

10, 11, 9, 13, 14, 12, 3, 8, 5, 3, 10, 9, 11, 5, 0, 8, 13, 12, 14, 2, 7, 2, 4, 6, 1, 1, 4, 7, 0, 6

11, 12, 13, 14, 8, 3, 10, 2, 3, 2, 9, 11, 8, 12, 0, 13, 10, 14, 7, 9, 6, 1, 1, 4, 5, 7, 6, 4, 0, 5

$12, 10, 14, 7, 13, 8, 9, 11, 1, 1, 7, 10, 12, 8, 0, 9, 14, 13, 11, 6, 4, 2, 5, 2, 4, 6, 3, 5, 0, 3$

$13, 14, 10, 8, 12, 5, 11, 1, 1, 9, 5, 8, 10, 13, 0, 14, 12, 11, 9, 7, 3, 6, 2, 3, 2, 4, 7, 6, 0, 4$

$n = 15$

$1, 1, 7, 4, 8, 11, 13, 4, 15, 7, 14, 3, 8, 12, 3, 0, 11, 5, 10, 13, 6, 9, 5, 15, 14, 12, 6, 2, 10, 2, 9$

$2, 9, 2, 6, 10, 12, 14, 15, 13, 6, 9, 5, 7, 11, 10, 0, 5, 12, 8, 7, 14, 13, 15, 4, 11, 3, 8, 4, 3, 1, 1$

$3, 7, 5, 3, 1, 1, 4, 5, 7, 13, 4, 15, 6, 14, 10, 0, 9, 12, 6, 11, 8, 2, 13, 2, 10, 9, 15, 14, 8, 12, 11$

$4, 5, 1, 1, 4, 6, 5, 7, 12, 8, 11, 6, 13, 15, 7, 0, 14, 8, 9, 10, 12, 11, 2, 3, 2, 13, 3, 9, 15, 10, 14$

$5, 1, 1, 3, 7, 5, 3, 11, 14, 15, 13, 7, 4, 8, 12, 0, 4, 10, 11, 6, 9, 8, 14, 13, 15, 6, 12, 10, 2, 9, 2$

$6, 2, 9, 2, 3, 8, 6, 3, 10, 11, 12, 9, 14, 8, 15, 0, 7, 13, 10, 5, 11, 4, 12, 7, 5, 4, 14, 1, 1, 15, 13$

$7, 4, 6, 10, 5, 4, 8, 7, 6, 5, 9, 14, 15, 10, 8, 0, 13, 11, 12, 9, 3, 1, 1, 3, 2, 14, 2, 15, 11, 13, 12$

$8, 6, 3, 9, 2, 3, 2, 6, 8, 14, 15, 13, 9, 1, 1, 0, 12, 4, 11, 7, 10, 4, 5, 14, 13, 15, 7, 5, 12, 11, 10$

$9, 10, 8, 11, 12, 14, 15, 13, 4, 9, 8, 10, 4, 5, 11, 0, 12, 6, 5, 14, 13, 15, 7, 6, 1, 1, 2, 3, 2, 7, 3$

$10, 3, 12, 13, 3, 15, 6, 14, 11, 9, 10, 2, 6, 2, 12, 0, 13, 7, 9, 11, 15, 14, 8, 5, 7, 4, 1, 1, 5, 4, 8$

$11, 12, 10, 14, 15, 13, 2, 9, 2, 1, 1, 11, 10, 12, 8, 0, 9, 14, 13, 15, 5, 6, 8, 7, 4, 5, 3, 6, 4, 3, 7$

$12, 13, 14, 15, 9, 2, 11, 2, 1, 1, 10, 8, 12, 9, 13, 0, 14, 11, 15, 8, 10, 7, 4, 6, 3, 5, 4, 3, 7, 6, 5$

$13, 11, 15, 8, 14, 4, 10, 12, 2, 4, 2, 8, 11, 13, 9, 0, 10, 15, 14, 12, 7, 5, 3, 9, 6, 3, 5, 7, 1, 1, 6$

$14, 15, 11, 7, 13, 3, 12, 5, 3, 10, 7, 9, 5, 11, 14, 0, 15, 13, 12, 10, 9, 8, 6, 1, 1, 2, 4, 2, 6, 8, 4$

$n = 16$

$1, 1, 8, 3, 4, 9, 3, 11, 4, 6, 8, 15, 16, 14, 9, 6, 0, 13, 11, 5, 12, 10, 7, 2, 5, 2, 15, 14, 16, 7, 13, 10, 12$

$2, 3, 2, 4, 3, 8, 6, 4, 9, 15, 16, 14, 6, 8, 13, 7, 0, 9, 5, 12, 10, 11, 7, 5, 15, 14, 16, 13, 1, 1, 10, 12, 11$

$3, 6, 10, 3, 7, 12, 14, 6, 16, 13, 15, 7, 10, 8, 9, 4, 0, 12, 11, 4, 14, 8, 13, 9, 16, 15, 5, 1, 1, 11, 2, 5, 2$

$4, 10, 7, 11, 4, 13, 15, 16, 14, 7, 3, 10, 12, 3, 11, 5, 0, 8, 13, 9, 5, 15, 14, 16, 12, 8, 6, 2, 9, 2, 1, 1, 6$

$5, 2, 9, 2, 11, 5, 8, 12, 15, 16, 14, 9, 6, 13, 8, 11, 0, 3, 6, 12, 3, 7, 10, 15, 14, 16, 13, 4, 7, 1, 1, 4, 10$

$6, 4, 5, 1, 1, 4, 6, 5, 13, 14, 12, 16, 8, 15, 10, 3, 0, 9, 3, 11, 8, 13, 12, 14, 10, 7, 9, 16, 15, 2, 11, 2, 7$

$7, 8, 4, 5, 1, 1, 4, 7, 5, 8, 9, 13, 10, 12, 15, 16, 0, 11, 14, 9, 6, 2, 10, 2, 13, 12, 6, 3, 11, 15, 3, 16, 14$

$8, 5, 1, 1, 6, 2, 5, 2, 8, 12, 6, 11, 14, 7, 16, 9, 0, 15, 13, 10, 7, 12, 11, 4, 9, 3, 14, 4, 3, 10, 16, 13, 15$

9, 7, 2, 8, 2, 5, 1, 1, 7, 9, 5, 8, 15, 16, 14, 10, 0, 6, 12, 13, 11, 3, 4, 6, 3, 10, 4, 15, 14, 16, 12, 11, 13

10, 11, 6, 12, 13, 15, 16, 14, 6, 9, 10, 8, 11, 1, 1, 12, 0, 13, 9, 8, 15, 14, 16, 5, 7, 2, 4, 2, 5, 3, 4, 7, 3

11, 9, 13, 14, 8, 16, 7, 15, 12, 10, 9, 11, 8, 7, 4, 13, 0, 14, 4, 10, 12, 16, 15, 6, 3, 5, 2, 3, 2, 6, 5, 1, 1

12, 13, 11, 15, 16, 14, 4, 10, 7, 2, 4, 2, 12, 11, 13, 7, 0, 10, 15, 14, 16, 6, 8, 9, 5, 1, 1, 6, 3, 5, 8, 3, 9

13, 14, 15, 16, 10, 6, 12, 5, 1, 1, 11, 6, 5, 13, 10, 14, 0, 15, 12, 16, 7, 11, 9, 3, 8, 4, 3, 7, 2, 4, 2, 9, 8

14, 12, 16, 9, 15, 10, 11, 13, 3, 1, 1, 3, 9, 12, 14, 10, 0, 11, 16, 15, 13, 8, 5, 7, 2, 6, 2, 5, 4, 8, 7, 6, 4

15, 16, 12, 10, 14, 7, 13, 4, 2, 11, 2, 4, 7, 10, 12, 15, 0, 16, 14, 13, 11, 9, 6, 8, 1, 1, 3, 5, 6, 3, 9, 8, 5

$n = 17$

1, 1, 3, 4, 12, 3, 9, 4, 5, 16, 17, 15, 7, 5, 14, 9, 12, 0, 6, 7, 11, 8, 13, 10, 6, 16, 15, 17, 14, 8, 2, 11, 2, 10, 0, 13

2, 6, 2, 5, 1, 1, 8, 6, 5, 7, 14, 16, 17, 15, 8, 4, 7, 0, 13, 4, 12, 9, 11, 3, 14, 10, 3, 16, 15, 17, 9, 13, 12, 11, 0, 10

3, 9, 6, 3, 8, 2, 7, 2, 6, 13, 9, 14, 8, 7, 17, 11, 16, 0, 10, 5, 15, 12, 13, 4, 5, 14, 11, 4, 10, 1, 1, 17, 16, 12, 0, 15

4, 1, 1, 3, 4, 10, 3, 13, 14, 15, 11, 17, 6, 16, 2, 10, 2, 0, 6, 12, 13, 11, 14, 7, 15, 8, 9, 5, 17, 16, 7, 12, 5, 8, 0, 9

5, 7, 4, 6, 10, 5, 4, 12, 7, 6, 8, 11, 16, 17, 10, 9, 15, 0, 8, 12, 13, 14, 11, 2, 9, 2, 1, 1, 16, 3, 17, 15, 3, 13, 0, 14

6, 4, 2, 8, 2, 4, 6, 5, 16, 17, 15, 8, 5, 14, 7, 13, 3, 0, 11, 3, 10, 7, 9, 12, 16, 15, 17, 14, 13, 11, 10, 9, 1, 1, 0, 12

7, 3, 9, 10, 3, 13, 15, 7, 17, 14, 16, 9, 8, 10, 5, 12, 6, 0, 13, 5, 8, 15, 6, 14, 11, 17, 16, 12, 2, 4, 2, 1, 1, 4, 0, 11

8, 2, 5, 2, 6, 4, 10, 5, 8, 4, 6, 12, 15, 13, 16, 7, 10, 0, 17, 14, 9, 11, 7, 12, 1, 1, 13, 15, 3, 9, 16, 3, 11, 14, 0, 17

9, 11, 8, 2, 4, 2, 1, 1, 4, 9, 8, 13, 11, 7, 12, 17, 14, 0, 15, 16, 7, 10, 5, 6, 13, 3, 12, 5, 3, 6, 14, 10, 17, 15, 0, 16

10, 8, 11, 12, 7, 14, 16, 17, 15, 8, 10, 7, 13, 11, 6, 12, 5, 0, 9, 14, 6, 5, 16, 15, 17, 13, 4, 9, 1, 1, 4, 2, 3, 2, 0, 3

11, 12, 10, 13, 14, 16, 17, 15, 7, 5, 9, 11, 10, 12, 5, 7, 13, 0, 14, 9, 8, 16, 15, 17, 3, 4, 6, 3, 8, 4, 1, 1, 6, 2, 0, 2

12, 10, 14, 15, 9, 17, 4, 16, 13, 11, 4, 10, 12, 9, 1, 1, 14, 0, 15, 6, 11, 13, 17, 16, 8, 6, 5, 2, 7, 2, 3, 5, 8, 3, 0, 7

13, 14, 12, 16, 17, 15, 3, 11, 4, 3, 1, 1, 4, 13, 12, 14, 9, 0, 11, 16, 15, 17, 10, 5, 6, 9, 7, 8, 5, 2, 6, 2, 10, 7, 0, 8

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