THE STABLE CATEGORY AND INVERTIBLE MODULES FOR INFINITE GROUPS

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ABSTRACT. We construct a well-behaved stable category of modules for a large class of infinite groups. We then consider its Picard group, which is the group of invertible (or endotrivial) modules. We show how this group can be calculated when the group acts on a tree with finite stabilisers.

1. INTRODUCTION

In the modular representation theory of finite groups, the stable module category is of fundamental importance. It is normally constructed by quotienting out all the morphisms that factor through a projective module, but it can also be characterised as the largest quotient category on which the syzygy operator is well defined and invertible. It is a triangulated category when the distinguished triangles are taken to be all the triangles that are isomorphic to the image of a short exact sequence of modules.

In the case of infinite groups we will construct a stable category that has these properties, at least for a fairly large class of groups that we call the groups of type Φ_k (see [35]). This class includes all groups of finite virtual cohomological dimension over k and all groups that admit a finite dimensional classifying space for proper actions, $\underline{E}G$, see Corollary 2.6. However, just quotienting out the projectives is usually not sufficient. We also describe various other possible constructions of this category and show that they all give equivalent results.

Our methods are similar to those of Ikenaga [25], even though he does not mention a stable category and is instead interested in generalising the Tate-Farrell cohomology theory to a larger class of groups than those of finite virtual cohomological dimension. Benson [5] has also described a stable category for infinite groups in the same spirit as ours, but he places restrictions on the modules rather than on the groups. The work of Cornick, Kropholler and coauthors is closely related too, but it is more homological in flavour; see [15] for a recent account and more details of the history of the subject.

This stable category has a well-behaved tensor product over the ground ring k, so is a tensor triangulated category in the language of Balmer. One important invariant of such a category is its Picard group. In this case the elements are the stable isomorphism classes of kG-modules M for which there exists a module N such that $M \otimes_k N$ is in the stable class of the trivial module k. These form a group under tensor product, which we denote by $T_k(G)$. For finite groups this is known as the group of endotrivial modules and has been extensively studied by many authors (see [36] for an excellent survey). Such modules have been classified for p-groups in characteristic p and also for many families of general finite groups [10, 11, 12, 14].

We develop the basic theory of these modules in the infinite case and provide some tools for calculating the Picard group $T_k(G)$. In particular, we have a formula whenever G acts on a tree with finite stabilisers.

One original justification for studying these modules in the finite case, given by Dade when he introduced them [18], was that they form a class of modules that is "small enough to be classified and large enough to be useful". We illustrate how this remains true in the infinite case at the end of the paper, by calculating some examples.

Some notes for a summer school course based on this material will appear in [34].

2. Groups of Type Φ_k

In this paper, k will always be a commutative noetherian ring of finite global dimension; the fundamental case is when k is a field of positive characteristic. All modules will be k-modules; sometimes a fixed k will be understood and we will omit it from the notation. We will consider groups G, usually infinite, and the category of all kG-modules $\mathrm{Mod}(kG)$.

Recall that the projective dimension of a kG-module, $\mathrm{projdim}_{kG} M$, is the shortest possible length of a projective resolution of M (∞ if there is no resolution of finite length). The global dimension of k, $\mathrm{gldim}\, k$, is the supremum of the projective dimensions of all k-modules.

Definition 2.1. A group G is of type Φ_k if it has the property that a kG-module is of finite projective dimension if and only if its restriction to any finite subgroup is of finite projective dimension.

Definition 2.2. The finitistic dimension of kG is
$$\mathrm{findim}\, kG = \sup\{\mathrm{projdim}_{kG} M \,|\, \mathrm{projdim}_{kG} M < \infty\}.$$

Lemma 2.3. *If the group G is finite and the kG-module M has finite projective dimension then* $\mathrm{projdim}_{kG} M \leq \mathrm{gldim}\, k$.

Proof. Let $d = \mathrm{gldim}\, k$ and consider the dth syzygy $\Omega^d M$ of M. This is projective over k, so its projective resolution is split over k. The modules in the resolution are summands of modules induced from the trivial subgroup and induction is equivalent to coinduction for finite groups, so these modules are injective relative to the trivial subgroup. The projective resolution can be taken to be of finite length so, by relative injectivity, it can be split starting from the left, thus $\Omega^d M$ must be projective. $\square$

Lemma 2.4. *If G is of type Φ_k then* $\mathrm{findim}\, kG$ *is finite.*

Proof. If $\mathrm{findim}\, kG$ is not finite, then for each positive integer i we can find a kG-module M_i such that $i \leq \mathrm{projdim}_{kG} M_i < \infty$. Over any finite subgroup F we have $\mathrm{projdim}_{kF} M_i < \infty$, so by Lemma 2.3 we obtain $\mathrm{projdim}_{kF} M_i \leq \mathrm{gldim}\, k$. Let $M = \oplus_i M_i$; then for any finite subgroup F we have $\mathrm{projdim}_{kF} M \leq \mathrm{gldim}\, k$, yet $\mathrm{projdim}_{kG} M = \infty$, so G cannot be of type Φ_k. $\square$

The class of groups of type $\Phi_{\mathbb{Z}}$ was introduced by Talelli [35]. The class of groups of type Φ_k is closed under subgroups and contains many naturally occurring groups; in particular, we have the following result.

Proposition 2.5. *Suppose that there exists a number m and an exact complex of kG-modules*
$$0 \to C_n \to \cdots \to C_0 \to k \to 0,$$
such that each C_i a summand of a sum of modules of the form $k\uparrow_H^G$ with H of type Φ_k and $\mathrm{findim}\, kH \leq m$. *Then G is of type Φ_k and* $\mathrm{findim}\, kG \leq n + m$.

Proof. Let M be a kG-module that is of finite projective dimension on restriction to any finite subgroup of G. The exact complex in the statement of the proposition is split over k, so it remains exact on tensoring with M.

Each $C_i \otimes_k M$ is a summand of a sum of terms of the form $(k\uparrow_H^G) \otimes_k M \cong M\downarrow_H\uparrow^G$. By the hypotheses on H, we know that $M\downarrow_H$ is of finite projective dimension and $\mathrm{projdim}_{kH} M \leq m$, hence $\mathrm{projdim}_{kG} M\downarrow_H\uparrow^G \leq m$.

Setting $D_i = C_i \otimes_k M$ we obtain modules D_i with $\mathrm{projdim}_{kG} D_i \leq m$ and an exact complex

$$0 \to D_n \to \cdots \to D_0 \to M \to 0.$$

An easy induction on n now shows that $\mathrm{projdim}_{kG} M \leq n + m$. $\qquad\square$

Recall that a group G is of finite cohomological dimension over k (finite cd_k) if $\mathrm{projdim}_{kG} k < \infty$ and is of finite virtual cohomological dimension over k (finite vcd_k) if it has a subgroup of finite index that is of finite cohomological dimension over k (see [8, VII.11]).

Corollary 2.6. *A group G is of type Φ_k if either of the following conditions hold:*
  (1) *there is a finite dimensional contractible G-CW-complex with finite stabilisers, or*
  (2) *G is of finite vcd_k.*

Proof. In case (1) we just use the associated G-CW-chain complex in Proposition 2.5. In case (2) the complex is constructed by the method of Swan, following Serre [30, 9.2]. $\qquad\square$

Note that a group G that admits a finite dimensional classifying space for proper actions, $\underline{E}G$, satisfies condition (1) above. It has been conjectured by Talelli that G having a finite dimensional $\underline{E}G$ is actually equivalent to G being of type $\Phi_{\mathbb{Z}}$ [35, Conj. A].

Example 2.7. A free abelian group of infinite rank is not of type Φ_k for any k, so neither is any group that contains it.

3. THE STABLE CATEGORY

Write $\mathrm{Mod}(kG)$ for the category of all kG-modules, possibly infinitely generated, and $\underline{\mathrm{Mod}}(kG)$ the quotient category with the same objects but morphisms

$$\underline{\mathrm{Hom}}_{kG}(M, N) = \mathrm{Hom}_{kG}(M, N)/\mathrm{PHom}_{kG}(M, N),$$

where $\mathrm{PHom}_{kG}(M, N)$ is the submodule of kG-homomorphisms $M \to N$ which factor through a projective module.

The next observation is well known; for a proof see e.g. [24, §2].

Lemma 3.1. *Suppose $M \cong N$ in $\underline{\mathrm{Mod}}(kG)$. Then there exist projective kG-modules P, Q such that $M \oplus P \cong N \oplus Q$ in $\mathrm{Mod}(kG)$.*

If M is finitely generated then Q can be taken to be finitely generated and if N is finitely generated then P can also be taken to be finitely generated.

When G is finite and k is a field then $\underline{\mathrm{Mod}}(kG)$ is taken to be the stable category, but this does not work well in the infinite case, because Ω might not be invertible. We shall introduce various possible definitions of a stable category for infinite groups and show that they all agree for groups of type Φ_k.

Recall that there is a natural map $\Omega : \underline{\mathrm{Hom}}_{kG}(M, N) \to \underline{\mathrm{Hom}}_{kG}(\Omega M, \Omega N)$.

Definition 3.2 ([6]). Let $\mathrm{Stab}(kG)$ be the category with all kG-modules as objects and morphisms given by $\mathrm{Hom}_{\mathrm{Stab}(kG)}(M, N) = \lim_{n \to \infty} \mathrm{Hom}_{\underline{\mathrm{Mod}}(kG)}(\Omega^n M, \Omega^n N)$.

Note that a kG-module has image 0 in $\mathrm{Stab}(kG)$ if and only if it is of finite projective dimension.

This definition can be difficult to work with. In particular, a map in $\mathrm{Hom}_{\mathrm{Stab}(kG)}(M, N)$ might not correspond to any map in $\mathrm{Hom}_{\mathrm{Mod}(kG)}(M, N)$. However, $\mathrm{Stab}(kG)$ is clearly the largest quotient of $\mathrm{Mod}(kG)$ on which Ω is well defined and invertible. It is quite possible that $\mathrm{Stab}(kG) = 0$, for example if G has finite cd_k.

Definition 3.3. An acyclic complex of projectives is an unbounded chain complex of projective modules P_* that is exact everywhere. It is totally acyclic if, in addition, $\text{Hom}_{kG}(P_*, Q)$ is acyclic for any projective module Q. The category of totally acyclic complexes of projective kG-modules and chain maps up to homotopy will be denoted by $K_{\text{tac}}(kG)$.

The category $K_{\text{tac}}(kG)$ is naturally triangulated in the usual way for categories based on chain complexes. The triangle shift is given by $[-1]$, where $[n]$ denotes degree shift by n.

Definition 3.4. A Gorenstein projective module is a module that is isomorphic to a kernel in a totally acyclic complex of projectives. The full subcategory of $\text{Mod}(kG)$ on the Gorenstein projective modules will be denoted by $\text{GP}(kG)$ and the full subcategory in $\underline{\text{Mod}}(kG)$ by $\underline{\text{GP}}(kG)$.

In some cases Gorenstein projective modules are easy to recognise.

Lemma 3.5. *Suppose that G is of finite vcd_k. The following conditions on a kG-module M are equivalent.*

    (1) *M is Gorenstein projective,*
    (2) *the restriction of M to some subgroup of finite index that is of finite cd_k is projective,*
    (3) *the restriction of M to any subgroup that is of finite cd_k is projective.*

Proof. Suppose that (1) holds and that H is a subgroup of finite cd_k. Let $n = \text{cd}_k(H)$ and $d = \text{gldim}\,k$; there is a kG-module N such that $M \cong \Omega^{n+d}N \cong (\Omega^n k) \otimes_k (\Omega^d N)$ modulo projectives. Now $\Omega^d N$ is projective over k and, on restriction to H, $\Omega^n k$ is projective, thus M is projective over kH and (3) holds. Trivially, (3) implies (2). If (2) holds then the construction in [8, X.2.1] produces a complete resolution with M as a kernel, so (1) holds. $\square$

In particular, if G is finite then a module is Gorenstein projective if and only if it is projective over k. If k is also a field then all kG-modules are Gorenstein projective.

Remark 3.6. The Gorenstein projective modules correspond to the modules called cofibrant in Benson's treatment [5], at least for groups of type Φ_k (see [2, 35]).

The category $\text{GP}(kG)$ is a Frobenius category, hence $\underline{\text{GP}}(kG)$ is naturally triangulated [19, 2.2]. The shift is Ω^{-1}, which is obtained by taking the kernel in degree one less in the totally acyclic complex used to show that the module is Gorenstein projective.

Definition 3.7. There is a functor $\Omega^0 : K_{\text{tac}}(kG) \to \underline{\text{GP}}(kG)$ obtained by taking the kernel of the boundary map in degree 0. There is also a natural inclusion functor $\text{Inc} : \underline{\text{GP}}(kG) \to \underline{\text{Mod}}(kG)$.

Definition 3.8. A *complete resolution* of a kG-module M consists of a totally acyclic chain complex F_* of projective kG-modules and a projective resolution P_* of M, together with a map $F_* \to P_*$ and an integer n such that the map is an isomorphism in all degrees n and above.

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & F_{n+1} & \longrightarrow & F_n & \longrightarrow & F_{n-1} & \longrightarrow & \ldots & \longrightarrow & F_0 & \longrightarrow & F_{-1} & \longrightarrow & \ldots \\
& & \| & & \| & & \downarrow & & & & \downarrow & & & & \\
\ldots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \ldots & \longrightarrow & P_0 & \overset{\varepsilon}{\longrightarrow} & M & \longrightarrow & 0
\end{array}$$

The index n is called the *coincidence index*.

In the literature this is sometimes called a complete resolution in the strong sense, because of the word totally. We will often abuse the terminology by referring to just F_* as the complete resolution.

It is known that two complete resolutions of the same module must be chain homotopy equivalent (i.e. the F_* are chain homotopy equivalent). A map between two P_* induces a map of the corresponding F_*, unique up to homotopy (cf. [25, Lemma 3, Proposition 13]).

In the same way, we can define a complete resolution of any chain complex M_* with only finitely many non-zero homology groups: we just take P_* to be a projective resolution of M_*, i.e. a quasi-isomorphism $P_* \to M_*$.

In general, a module might not possess a complete resolution, but for groups of type Φ_k they always exist.

Theorem 3.9. *If G is a group of type Φ_k then any complex of kG-modules with only finitely many non-zero homology groups has a complete resolution.*

We defer the proof to the end of this section.

Thus, for groups of type Φ_k, we obtain a functor $\mathrm{CompRes} : D^b(\mathrm{Mod}(kG)) \to K_{\mathrm{tac}}(kG)$, where $D^b(\mathrm{Mod}(kG))$ is the derived category of complexes of kG-modules with only finitely many non-zero homology groups. It is easy to see that $K^b(\mathrm{Proj}(kG))$ is in the kernel, where $K^b(\mathrm{Proj}(kG))$ is the homotopy category of bounded complexes of projective kG-modules, so we have a functor on the Verdier quotient $\mathrm{CompRes} : D^b(\mathrm{Mod}(kG))/K^b(\mathrm{Proj}(kG)) \to K_{\mathrm{tac}}(kG)$.

There is a functor $[0] : \mathrm{Stab}(kG) \to D^b(\mathrm{Mod}(kG))/K^b(\mathrm{Proj}(kG))$. On modules it can be given by regarding a module as the degree 0 term of a complex that is 0 elsewhere. For morphisms it is better to regard this slightly differently. Take a projective resolution P_M of the module M and consider the truncation $\sigma_{\geq n}(P_M)$ for some $n \geq 0$ (see [38, 1.2.7]); it is clearly a projective resolution of $\Omega^n M[-n]$ and it is also equal to P_M in the Verdier localisation. Thus we have $M \cong \Omega^n M[-n]$ in $D^b(\mathrm{Mod}(kG))/K^b(\mathrm{Proj}(kG))$. A morphism $f \in \mathrm{Hom}_{\mathrm{Stab}(kG)}(M, N)$ must correspond to a homomorphism $f' \in \mathrm{Hom}_{\mathrm{Stab}(kG)}(\Omega^n M, \Omega^n N)$ for some n, so $f[0]$ can be defined as $f'[-n] : \Omega^n M[-n] \to \Omega^n N[-n]$.

The next theorem is essentially due to Buchweitz [9], at least for parts (2–4); there are many variants in the literature.

Theorem 3.10. *For a group of type Φ_k, the following categories are equivalent:*
  (1) $\mathrm{Stab}(kG)$,
  (2) $\underline{\mathrm{GP}}(kG)$,
  (3) $K_{\mathrm{tac}}(kG)$,
  (4) $D^b(\mathrm{Mod}(kG))/K^b(\mathrm{Proj}(kG))$.
The equivalences are obtained from the functors $\mathrm{Inc}, \Omega^0, \mathrm{CompRes}$ *and* $[0]$ *introduced in Definition 3.7. These are equivalences of triangulated categories, except for those involving* $\mathrm{Stab}(kG)$, *where a triangulated structure has not yet been defined.*

Proof. The equivalences not involving $\mathrm{Stab}(kG)$ are shown in [4, 4.16] (replacing Inc and $[0]$ by $[0]\,\mathrm{Inc}$). The theorem will follow when we show that the cyclic permutations of the composition $\mathrm{Inc}\,\Omega^0\,\mathrm{CompRes}[0]$ are naturally isomorphic to the identity. But the cases with $[0]$ and Inc adjacent follow from [4, 4.16], so we only have to check $\mathrm{Inc}\,\Omega^0\,\mathrm{CompRes}[0]$ itself. By definition, this sends M to the degree 0 kernel $\tilde{M}$ in its complete resolution. But large enough syzygies of M and $\tilde{M}$ are identical in $\mathrm{Stab}(kG)$. $\qquad \square$

Of course, we can use these equivalences to define a triangulated structure on $\mathrm{Stab}(kG)$. The distinguished triangles are all the triangles isomorphic to a short exact sequence of modules. The shift $\Omega^{-1}M$ is obtained by finding a complete resolution of the module M and then taking the kernel in degree -1. The kernel in degree 0 we denote by $\tilde{M}$; from the definition of a complete resolution, $\tilde{M}$ is Gorenstein projective and it comes with a map of modules $\alpha_M : \tilde{M} \to M$ that is an isomorphism in $\mathrm{Stab}(kG)$.

Definition 3.11. We call the map $\alpha_M : \tilde{M} \to M$ the Gorenstein approximation of M.

We can consider any of these categories to be the stable module category of kG and we will use the notation $\widehat{\mathrm{Mod}}(kG)$ when we do not wish to specify which one. We will use the symbol $\simeq$ to denote isomorphism in $\widehat{\mathrm{Mod}}(kG)$.

A kG-module is called a lattice if it is projective over k. The full subcategory of $\mathrm{Stab}(kG)$ on the lattices is also equivalent to $\widehat{\mathrm{Mod}}(kG)$, because $\mathrm{Stab}(kG)$ also contains the full subcategory $\underline{\mathrm{GP}}(kG)$ (and so $\underline{\mathrm{GP}}(kG) \subseteq \mathrm{Stab}(kG) \subseteq \widehat{\mathrm{Mod}}(kG)$ where $\underline{\mathrm{GP}}(kG)$ and $\widehat{\mathrm{Mod}}(kG)$ are equivalent categories).

Lemma 3.12. *If a kG-module M is Gorenstein projective then for any kG-module N the natural map $\mathrm{Hom}_{kG}(M, N) \to \mathrm{Hom}_{\mathrm{Stab}(kG)}(M, N)$ is surjective.*

Proof. Any $f \in \mathrm{Hom}_{\mathrm{Stab}(kG)}(M, N)$ is the image under Inc of some $\tilde{f} \in \mathrm{Hom}_{\underline{\mathrm{GP}}(kG)}(\tilde{M}, \tilde{N})$, which we can compose with the natural map $\alpha_N \colon \tilde{N} \to N$. Since M itself is Gorenstein projective, we have $\tilde{M} \cong M$ in $\underline{\mathrm{GP}}(kG)$, so we obtain a map $M \to N$ in $\underline{\mathrm{Mod}}(kG)$, which must be the image of one in $\mathrm{Mod}(kG)$. $\qquad\square$

Recall that for $H \leq G$ there are two functors Ind_H^G and Coind_H^G from $\mathrm{Mod}(kH)$ to $\mathrm{Mod}(kG)$, the left and right adjoints of restriction. They satisfy the identity

$$\mathrm{Hom}_k(M, \mathrm{Coind}_H^G N) \cong \mathrm{Coind}_H^G \mathrm{Hom}_k(M, N) \cong \mathrm{Hom}_k(\mathrm{Ind}_H^G, M), \quad M, N \in \mathrm{Mod}(kG),$$

where $\mathrm{Hom}_k(M, N)$ is considered as a kG-module in the usual way and all restrictions are implicit.

Lemma 3.13.   (1) *A kG-module is projective if and only if it is a summand of a module induced from a free module for the trivial subgroup; it is injective if and only if it is a summand of a module coinduced from an injective module for the trivial subgroup.*
   (2) *A kG module induced from the trivial subgroup has finite projective dimension. If G has type Φ_k then a module coinduced from the trivial subgroup has finite projective dimension.*

Proof. The first part is trivial. For the second part, if $M = \mathrm{Ind}_1^G X$, take a projective resolution of X over k. This is of finite length since k has finite global dimension. Because Ind_1^G is exact we can apply it to obtain a projective resolution of finite length for M.

If $N = \mathrm{Coind}_1^G Y$, note that, since G is assumed to be of type Φ_k, we only have to check finite projective dimension on restriction to finite subgroups. If F is a finite subgroup then the Mackey formula shows that the restriction of N to F is of the form $\mathrm{Coind}_1^F Z$. But for finite groups, coinduction is equivalent to induction, so we can use the result for induced modules just proved. $\qquad\square$

Now for the proof of Theorem 3.9. In [23], two invariants are defined:

$$\mathrm{spli}(kG) = \sup \{\mathrm{projdim}_{kG} I \mid I \text{ injective}\} \quad \mathrm{silp}(kG) = \sup\{\mathrm{injdim}_{kG} P \mid P \text{ projective}\}.$$

From Lemma 3.13 we know that $\mathrm{projdim}_{kG} I$ is finite for any injective module I. Thus $\mathrm{projdim}_{kG} I$ is bounded by $\mathrm{findim}\, kG$, which is finite by Lemma 2.4, since G is of type Φ_k. We conclude that $\mathrm{spli}(kG)$ is finite. Now [23, 4.1] constructs the acyclic complex of projectives in the definition of a complete resolution (they call this a complete resolution even though they do not require it to be totally acyclic), see also [25]. In fact, total acyclicity is automatic.

Lemma 3.14. *If G is of type Φ_k then any acyclic complex of modules is totally acyclic.*

Proof. Let X_* be the acyclic complex. We claim that $\mathrm{Hom}_{kG}(X_*, M)$ is acyclic for M of finite injective dimension. This is clearly true for M injective and the general case is proved by an easy induction on $\mathrm{injdim}_{kG} M$. Thus all we need to know to finish the proof is that $\mathrm{silp}(kG)$ is finite and this is proved in [23, 2.4]. $\qquad\square$

Remark 3.15. The proof of [23, 2.4] is the only place in this section where the argument requires k to be noetherian.

Following [5, 25], we can define $\widehat{\mathrm{Ext}}_{kG}^i(M, N) = \widehat{\mathrm{Hom}}_{kG}(\Omega^i M, N)$ and $\widehat{H}^i(G; N) = \widehat{\mathrm{Ext}}_{kG}^i(k, N)$, for all $i \in \mathbb{Z}$. When G has finite vcd_k then this is the same as the Tate-Farrell cohomology in [8].

4. FUNCTORS

From now on we assume that all groups are of type Φ_k and we continue to assume that k is noetherian of finite global dimension.

We want to know when and how a functor $F \colon \mathrm{Mod}(kG) \to \mathrm{Mod}(kH)$ induces a triangulated functor between the stable categories $\widehat{\mathrm{Mod}}$. Most, but not all, of the problems only arise when k is not a field. If F is exact and takes projectives to projectives, then clearly it induces a functor of triangulated categories on $\widehat{\mathrm{Mod}}$; this is easy to see using any version of $\widehat{\mathrm{Mod}}$. In this way we obtain induction Ind_H^G and restriction Res_H^G of stable categories for any subgroup H of G.

In fact, it suffices for F to be exact and to take projective modules to modules of finite projective dimension. For a short exact sequence $\Omega M \to P \to M$ in $\mathrm{Mod}(kG)$ with P projective is taken to a short exact sequence $F(\Omega M) \to F(P) \to F(M)$ in $\mathrm{Mod}(kH)$; but $F(P) = 0$ in $\mathrm{Stab}(kH)$, so $F(\Omega M) = \Omega F(M)$ in $\mathrm{Stab}(kH)$. If $f \in \mathrm{Hom}_{\mathrm{Stab}(kG)}(M, N)$ is represented by $f' \colon \Omega^n M \to \Omega^n N$, then $F(f') \colon \Omega^n F(M) \to \Omega^n F(N)$ determines an element $F(f) \in \mathrm{Hom}_{\mathrm{Stab}(kH)}(F(M), F(N))$.

In this way we obtain an inflation map $\mathrm{Inf}_G^{G/N}$ when N is a normal subgroup of G such that G/N is of type Φ_k and any element of of finite order in N has order a unit in k. For given a projective $k(G/N)$-module P inflate it to G; then for any finite subgroup $F \leq G$, P is projective over $k(F/(F \cap N))$ and $|F \cap N|$ is a unit in k, so P is projective over kF. Thus P is of finite projective dimension over kF so, since G is of type Φ_k, we find that P is of finite projective dimension over kG.

More generally, for any homomorphism $h \colon J \to K$ between groups of type Φ_k such that any element of finite order in the kernel has order a unit in k, there is a pullback map $h^* \colon \widehat{\mathrm{Mod}}(kK) \to \widehat{\mathrm{Mod}}(kJ)$.

Lemma 4.1. *Change of scalars across a homomorphism of rings $f \colon k \to \ell$ in either direction, i.e. restriction f^* or base change f_*, induce functors between $\mathrm{Mod}(kG)$ and $\mathrm{Mod}(\ell G)$ that take projective modules to modules of finite projective dimension (the assumptions at the beginning of this section holding for both k and ℓ). Also f^* is exact.*

Proof. By Lemma 3.13(1), a projective module is a summand of one of the form $\mathrm{Ind}_1^G X$ for X free. Both f^* and f_* commute with Ind_1^G; $f_* X$ is still free so $\mathrm{Ind}_1^G f_* X$ is free and for f^* use Lemma 3.13(2). Exactness of f^* is clear. $\qquad\square$

Lemma 4.2. *For a given kG-module M, the following functors from $\mathrm{Mod}(kG)$ to itself take projective modules to modules of finite projective dimension. If M is a lattice then the first two are exact.*

    (1) $- \otimes_k M$,
    (2) $\mathrm{Hom}_k(M, -)$,
    (3) $\mathrm{Hom}_k(-, M)$.

Proof. Any projective module is a summand of $\mathrm{Ind}_1^G X$ for some free k-module X.

1) It is well known that $(\mathrm{Ind}_1^G X) \otimes_k M \cong \mathrm{Ind}_1^G(X \otimes_k M)$. Now use Lemma 3.13.

2) For any finite subgroup $F \leq G$, $\mathrm{Res}_F^G \mathrm{Hom}_k(M, \mathrm{Ind}_1^G X) \cong \mathrm{Hom}_k(M, \mathrm{Ind}_1^F Y)$ for some k-module Y, by the Mackey formula. Now,

$$\mathrm{Hom}_k(M, \mathrm{Ind}_1^F Y) \cong \mathrm{Hom}_k(M, \mathrm{Coind}_1^F Y) \cong \mathrm{Coind}_1^F \mathrm{Hom}_k(M, Y),$$

so we can use Lemma 3.13 to see that $\mathrm{Res}_F^G \mathrm{Hom}_k(M, \mathrm{Ind}_1^G X)$ has finite projective dimension. Since G is of type Φ_k we can deduce that $\mathrm{Hom}_k(M, \mathrm{Ind}_1^G X)$ is also of finite projective dimension.

3) $\mathrm{Hom}_k(\mathrm{Ind}_1^G X, M) \cong \mathrm{Coind}_1^G \mathrm{Hom}_k(X, M)$, so we can use Lemma 3.13 again.

The exactness statement is trivial. $\qquad\square$

The lemmas above together with the discussion at the beginning of this section show that if M is a lattice then $-\otimes_k M$ and $\operatorname{Hom}_k(M,-)$ induce triangulated functors of the stable category as before; so does f^*. In the other cases, we see that each of the functors in Lemmas 4.1 and 4.2 naturally defines a functor $\underline{\mathrm{GP}}(kG) \to \mathrm{Stab}(kG)$ (with the obvious variation for f_*), because now the triangles are isomorphic to short exact sequences, and all modules are projective over k so short exact sequences are split and exactness is automatic. By Theorem 3.10, we can regard this as a functor from $\widehat{\mathrm{Mod}}(kG)$ to itself. This second approach also makes it transparent what the functor does to morphisms, in particular for $\operatorname{Hom}_k(-,M)$.

By Theorem 3.10, the categories $\widehat{\mathrm{Mod}}(kG)$ and $\mathrm{Stab}(kG)$ are equivalent, so we can regard any of these functors as a functor from $\mathrm{Stab}(kG)$ to itself. However, it might not be given by the usual formula on objects that are not Gorenstein projective, because we need to replace a module M by its Gorenstein approximation $\tilde{M}$ before applying the formula (see Definition 3.11).

Example 4.3. Let C be a cyclic group of order p and $\hat{\mathbb{Z}}_p$ the p-adic integers. Consider the trivial module $\mathbb{F}_p$ for $\hat{\mathbb{Z}}_pC$. An easy calculation (cf. [33, 2.6]) shows that $\tilde{\mathbb{F}}_p \simeq \hat{\mathbb{Z}}_p \oplus \Omega\hat{\mathbb{Z}}_p$, hence $\tilde{\mathbb{F}}_p \otimes_{\hat{\mathbb{Z}}_p} \tilde{\mathbb{F}}_p \simeq (\hat{\mathbb{Z}}_p)^2 \oplus (\Omega\hat{\mathbb{Z}}_p)^2$. By definition, $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} \mathbb{F}_p \simeq \tilde{\mathbb{F}}_p \otimes_{\hat{\mathbb{Z}}_p} \tilde{\mathbb{F}}_p \simeq (\mathbb{F}_p)^2$, which is not what might have been expected.

For any $H \leq G$, coinduction Coind_H^G also induces a functor of stable categories. For a projective kH-module P and any finite subgroup $F \leq G$, we have

$$\operatorname{Res}_F^G \operatorname{Coind}_H^G P = \prod_{g \in F \backslash G / H} \operatorname{Coind}_{F \cap {}^gH}^F {}^g \operatorname{Res}_{F^g \cap H}^H P.$$

Since coinduction is equivalent to induction for finite groups, the factors in the product are clearly projective, thus so is the product, by Lemma 4.4 below. It follows that $\operatorname{Coind}_H^G P$ must be projective.

Lemma 4.4. *If the kG-modules P_i, $i \in I$, are projective then the product $\prod_{i \in I} P_i$ is of finite projective dimension.*

Proof. Because G is of type Φ_k, it suffices to prove the case when G is finite. For each $i \in I$ we can write P_i as a summand of $\operatorname{Ind}_1^G X_i$ for some free k-module X_i. Thus $\prod_{i \in I} P_i$ is a summand of $\prod_{i \in I} \operatorname{Ind}_1^G X_i$. But since G is finite, $\operatorname{Ind}_1^G X_i$ is isomorphic to $\operatorname{Coind}_1^G X_i$, and Coind_1^G, being a right adjoint, commutes with products. Hence $\prod_{i \in I} \operatorname{Ind}_1^G X_i \cong \operatorname{Ind}_1^G \prod_{i \in I} X_i$, which is projective. $\square$

It is easy to check that Ind_H^G is still the left adjoint and Coind_H^G is still the right adjoint of restriction on the stable categories.

Proposition 4.5. $\widehat{\mathrm{Mod}}(kG)$ *has products and coproducts, i.e.,*

$$\prod_i \widehat{\operatorname{Hom}}(A_i, B) \cong \widehat{\operatorname{Hom}}(\oplus_i A_i, B)$$
$$\prod_j \widehat{\operatorname{Hom}}(A, B_j) \cong \widehat{\operatorname{Hom}}(A, \prod_j B_j),$$

where $\prod$ and $\oplus$ are the usual ones in Mod.

Proof. We may assume that all the modules are Gorenstein projective, so we are working in $\underline{\mathrm{Mod}}(kG)$. The only part that is not entirely routine is to check that if each $f_i \colon A_i \to B$ factors through a projective P_i then the induced map $\oplus_i A_i \to B$ factors through a projective. But it factors through $\oplus_i P_i$, which is projective.

If the $g_j \colon A \to B_j$ each factor through a projective Q_j then the induced map $A \to \prod_j B_j$ factors through $\prod_j Q_j$. This has finite projective dimension, by Lemma 4.4, so the induced map is 0 in $\widehat{\mathrm{Mod}}(kG)$. $\square$

5. DECOMPOSITIONS

The Eckmann-Hilton argument, on sets with two monoid structures (also called the Eckmann-Hilton theorem, see [21]), applies and gives us the following fundamental result.

Proposition 5.1. *Let G be a group of type Φ_k. The ring $\widehat{\mathrm{End}}_{kG}(k)$ and the group $\widehat{\mathrm{Aut}}_{kG}(k)$ are commutative. The product under composition agrees with the product under tensor product.*

Unlike $\mathrm{End}_{kG}(k)$, the k-algebra $\widehat{\mathrm{End}}_{kG}(k)$ can be quite complicated, as we shall see.

The next lemma is very useful.

Lemma 5.2. *Let G be a group of type Φ_k and let $f\colon M \to N$ be a morphism in $\widehat{\mathrm{Mod}}(kG)$ that restricts to a stable isomorphism on any finite subgroup. Then f is a stable isomorphism.*

Proof. Consider the cone of f. It is stably 0 on restriction to any finite subgroup, so by the definition of type Φ_k it is stably 0 over G. $\qquad\square$

Proposition 5.3. *Let G be a finite group and let Δ be a G-CW-complex. Suppose that the reduced homology groups $\tilde{H}_i(\Delta^H; k) = 0$ for $i \geq 0$ and any non-trivial subgroup $H \leq G$.*

(1) *The augmented cellular chain complex $\tilde{C}_*(\Delta)$ is chain homotopy equivalent to a bounded-below complex of projective kG-modules.*
(2) *If $H_i(\Delta; k) = 0$ for large enough i then $\tilde{C}_*(\Delta)$ is chain homotopy equivalent to a bounded complex of projective kG-modules.*

The original version of this result is due to Webb [37] for k a complete p-local ring and Δ finite dimensional. There is another proof of the first part by Bouc [7] for k a field and general Δ.

Proof. The first part is proved in [32]; it is the case $H = 1$ of the statement that q is a homotopy equivalence that appears just before the lemma (there is no assumption on k; see the remark at the end). There is a similar proof in [31, 6.6], but note that in 6.4 and 6.6 there the word bounded should not appear unless Δ is finite dimensional.

For the second part, let $C_* = \tilde{C}_*(\Delta)$ and let P_* be the bounded below complex of projectives. Let $\tau_{<n}C_*$ be a good truncation of C_*, truncated in some degree n greater than the degree of any non-zero homology group [38, 1.2.7]. Then we have a quasi-isomorphism $P_* \to \tau_{<n}C_*$. By [31, 6.5], P_* is a summand of a complex that is split in high degrees, so it is split in high degrees itself. Thus P_* is homotopy equivalent to a good truncation of itself. $\qquad\square$

Theorem 5.4. *Let G be a group of type Φ_k and let Δ be a G-CW-complex such that $H_i(\Delta; k) = 0$ for large enough i. Suppose that $\tilde{H}_i(\Delta^H; k) = 0$ for $i \geq 0$ and any non-trivial finite subgroup $H \leq G$. Then the chain complex $C(\Delta)$, considered as an element of $D^b(\mathrm{Mod}(kG))/K^b(\mathrm{Proj}(kG))$, is equal to k. The same is true if k is p-local and we only require $H_i(\Delta^H; k) = 0$ for non-trivial p-groups H. In these cases k decomposes as a direct sum of non-zero pieces corresponding to those path components of Δ/G for which some cell of Δ above them is fixed by an element of finite order not a unit in k.*

Proof. By the previous proposition, for any finite subgroup F (p-subgroup if k a p-local ring), the restriction of the augmented chain complex is 0 in $D^b(\mathrm{Mod}(kF))/K^b(\mathrm{Proj}(kF))$. By Lemma 5.2, the augmented chain complex is 0 in $D^b(\mathrm{Mod}(kG))/K^b(\mathrm{Proj}(kG))$. Thus the augmentation is an isomorphism from $C(\Delta)$ to k.

Let X be the part of Δ corresponding to a given component of Δ/G. If $X^{\langle g \rangle} \neq \emptyset$ for some g then the augmentation of $C(X)$ is split over $\langle g \rangle$. Since $k \neq 0$ over $k\langle g \rangle$ when $|g|$ is finite but not a unit (because $\hat{H}^0(\langle g \rangle; k) = k/|g|k$), we see that $C(X) \neq 0$. If no such g exists, then the restriction of $C(X)$ to any finite subgroup F is a complex of projectives. By the proof of the previous proposition, it is homotopy equivalent to a bounded complex of projectives, so is 0 in

$D^b(\mathrm{Mod}(kF))/K^b(\mathrm{Proj}(kF))$ for all F, hence 0 in $D^b(\mathrm{Mod}(kG))/K^b(\mathrm{Proj}(kG))$, since G is of type Φ_k. $\qquad\square$

An obvious candidate for Δ when k is p-local is the Quillen complex $\Delta(\mathcal{S}_p(G))$, the simplicial complex constructed from chains of non-trivial p-subgroups, or the Brown complex $\Delta(\mathcal{A}_p(G))$, the simplicial complex constructed from chains of non-trivial elementary abelian p-subgroups. These are known to satisfy the condition on fixed point sets and in fact they are equivariantly homotopy equivalent. The latter is finite dimensional if the p-rank of G is finite, so in this case they both satisfy the condition on their homology groups.

For general k we can always use $\Delta(\mathcal{F}(G))$, constructed from chains of non-trivial finite subgroups.

In the p-local case, this decomposition is the best possible. Let I index the components of $\Delta(\mathcal{S}_p(G))/G$ and let k_{e_i} denote the part of $C(\Delta(\mathcal{S}_p(G)))$ corresponding to component i. Then $k \simeq \oplus_i k_{e_i}$. Let $e_i \in \widehat{\mathrm{End}}_{kG}(k)$ be the idempotent corresponding to projection onto k_{e_i}; then $\widehat{\mathrm{End}}_{kG}(k) = \prod \widehat{\mathrm{End}}_{kG}(k_{e_i}) = \prod e_i \widehat{\mathrm{End}}_{kG}(k)$.

Theorem 5.5. *When k is p-local, the primitive idempotents of $\widehat{\mathrm{End}}_{kG}(k)$ correspond to the path components of $\Delta(\mathcal{S}_p(G))/G$, or equivalently to the equivalence classes of non-trivial p-subgroups of G under the equivalence relation generated by inclusion and conjugation.*

Proof. Let X_i be the part of $\Delta(\mathcal{S}_p(G))$ corresponding to component $i \in I$. If P is a p-subgroup that appears in this part of $\mathcal{S}_p(G)$ then $X_i^P \neq \emptyset$, so the augmentation $C(X_i) \to k$ is split over kP and so $k \downarrow_P | C(X_i) \downarrow_P$. But e_i acts as the identity on $C(X_i)$, so $e \downarrow_P$ acts as the identity on $k \downarrow_P$, i.e. $\mathrm{res}_P^G(e_i) = 1$. The idempotents are orthogonal, so $\mathrm{res}_P^G(e_j) = 0$ for $j \neq i$.

Suppose that $e = e_i$ decomposes in $\widehat{\mathrm{End}}_{kG}(k)$ as a sum of idempotents, $e = f_1 + f_2$. For any finite p-group P, we have $\widehat{\mathrm{End}}_{kP}(k) = k/|P|k$, which is still local if $P \neq 1$, so the only idempotents are 0 and 1. Thus $\mathrm{res}_P(f_1)$ is 0 or 1. This choice is preserved by restriction and conjugation, so it is constant on all subgroups P in the component i; say $\mathrm{res}_P(f_1) = 1$ and $\mathrm{res}_P(f_2) = 0$. Thus $f_1 \colon k_e \to k_e$ is an isomorphism on restriction to any subgroup in i and also on restriction to any subgroup in any other component because then the restriction of k_e is 0. By Lemma 5.2, f_1 is an an automorphism and so, being an idempotent, $f_1 = e$ and thus $f_2 = 0$. $\quad\square$

A version of this result was obtained for $\mathbf{H}_1\mathfrak{F}$ groups by by Cornick and Leary [16]; this class includes groups of unbounded p-rank. They use the complex that appears in the definition of $\mathbf{H}_1\mathfrak{F}$ as Δ; the condition on the fixed point spaces of finite p-groups is satisfied, by Smith Theory. It is worth noting that, by work of Freyd [22], idempotents in $\widehat{\mathrm{End}}_{kG}(k)$ split k stably; our construction avoids quoting this.

Example 5.6. Let $G = A * B$ be the free product of two groups A and B, both of type Φ_k. We know that G is of type Φ_k and that any non-trivial finite subgroup of G is conjugate to a subgroup of A or B but not both [8, II.A3]. Thus there are at least two components provided both A and B contain elements of order not a unit in k.

Since $\widehat{\mathrm{End}}_{kG}(k) \cong \hat{H}^0(G; k)$, we can use the complete cohomology version of [8, VI.3] to see that $\widehat{\mathrm{End}}_{kG}(k) \cong \widehat{\mathrm{End}}_{kA}(k) \times \widehat{\mathrm{End}}_{kB}(k)$. If both A and B are finite then $\widehat{\mathrm{End}}_{kG}(k) \cong k/|A|k \times k/|B|k$.

This method of calculation can be generalised to any group of type Φ_k that acts on a tree.

6. INVERTIBLE MODULES

Throughout this section, G denotes a group of type Φ_k.

Definition 6.1. A kG-module M is *invertible* if there exists a kG-module N such that $M \otimes N \cong k$ in $\widehat{\mathrm{Mod}}(kG)$. The tensor product $- \otimes - = - \otimes_k -$ of Section 4 equips the set of isomorphism

classes of invertible kG-modules in $\widehat{\mathrm{Mod}}(kG)$ with the structure of an abelian group, which we denote by $T_k(G)$ and call the *group of invertible kG-modules.*

For finite groups, the endotrivial modules are defined in the same way, except that the modules are required to be finitely generated.

If we take M and N to be Gorenstein projective then we have $M \otimes N \cong \tilde{k}$ in $\underline{\mathrm{GP}}(kG)$, with the usual tensor product (recall that the tensor product of two Gorenstein projective modules is Gorenstein projective since $\mathrm{gldim}\, k < \infty$).

First we check that for finite groups the basic theory of endotrivial modules carries over to our context, where k need not be a field and the modules need not be finitely generated. Note that for a finite group a module being Gorenstein projective is equivalent to it being a lattice and if a module is finitely generated then it has a finitely generated Gorenstein projective approximation. This is because a sufficiently high syzygy is finitely generated (since k is noetherian) and Gorenstein projective; it is a kernel in a complete resolution with finitely generated terms by [8, VI.2.6].

The following result is stated for k a field in [3, 2.1]).

Theorem 6.2. *Let G be a finite group and M an invertible kG-module. Then M is stably a summand of a finitely generated kG-module and the natural map $M \to M^{**}$ is a stable isomorphism. If k is a complete local ring and M is a lattice then M decomposes as a kG-module as a direct sum $M' \oplus P$ with M' a finitely generated lattice and P projective.*

Proof. We may assume that M is Gorenstein projective. Let $f : k \to M \otimes N$ be a stable isomorphism; since k is a lattice, f is a genuine homomorphism. Write $f(1) = \sum_i m_i \otimes n_i$ for $m_i \in M$, $n_i \in N$, and let M' be the kG-submodule of M generated by the m_i. We have a stable isomorphism $k \xrightarrow{f} f(k) \le M' \otimes N \subseteq M \otimes N \xrightarrow{f^{-1}} k$, so $k \mid M' \otimes N$ stably. It follows that stably $M \mid M' \otimes N \otimes M \simeq M'$.

Let $L = \tilde{M}'$, so $M \mid L$ stably. L is a finitely generated lattice, so the natural map $L \to L^{**}$ is a stable isomorphism. This property is inherited by stable summands, in particular M.

Somehow we need to deduce the last part. The authors of [3] probably had in mind the method of Rickard [29, 3.2], but this is for k a field; instead we use the Crawley-Jønsson-Warfield Theorem [1, 26.6], which is a version of the Krull-Schmidt Theorem that applies in this case. Since both L and M are Gorenstein projective, we know that M is a summand as a module of some $U = L \oplus (\oplus_{i \in I} kG)$. Both L and kG are finitely generated and k is noetherian, so they can both be expressed as a finite sum of finitely generated indecomposables and thus so can U, say $U = \oplus_{j \in J} X_j$, with only finitely many X_j not projective. By [17, 6.10], since k is a complete local ring, each of these indecomposables has a local endomorphism ring and is clearly countably generated, so the Crawley-Jønsson-Warfield Theorem tells us that M is isomorphic to $\oplus_{j \in K} X_j$ for some $K \subseteq J$. $\qquad\square$

This theorem, together with Lemma 3.1, yields the following result.

Corollary 6.3. *For a finite group G and a complete local (noetherian) ring k we get the same group $T_k(G)$ whether we consider invertible modules in $\widehat{\mathrm{Mod}}(kG)$, as we do in this paper, or the classical group of endotrivial modules in $\underline{\mathrm{mod}}(kG)$.*

The next result is well known when M is finitely generated and k is a field. Note that we do not assume that the trivial module k is stably indecomposable, so we cannot assume that an invertible module is stably indecomposable (a decomposition can occur when $k = \mathbb{Z}$, for example).

Theorem 6.4. *Let G be a finite group and let M and N be kG-modules such that $M \otimes N \simeq k$. Then:*

(1) $N \simeq M^*$,

(2) *the evaluation map* $\mathrm{ev}\colon M \otimes M^* \to k$ *is a stable isomorphism and*

(3) *the natural map* $\theta_M \colon M \otimes M^* \to \mathrm{End}_k(M)$ *is a stable isomorphism.*

Proof. We can assume that M and N are Gorenstein projective and, by the previous theorem, stably isomorphic to their double duals. From the stable isomorphism $f \colon M \otimes N \to k$, realised as a genuine homomorphism, we obtain a map $f' \colon N \to M^*$ such that $f = \mathrm{ev}(\mathrm{Id}_M \otimes f')$. Thus $k \mid M \otimes M^*$ stably and so $N \mid N \otimes M \otimes M^* \simeq M^*$ stably. By symmetry, $M \mid N^*$ stably; say $N \simeq M^* \oplus X$ and $M \simeq N^* \oplus Y$. Eliminating N and using the fact that $M \simeq M^{**}$, we obtain $M \simeq M \oplus X^* \oplus Y$.

By Theorem 6.2, there is a finitely generated lattice L such that $M \mid L$ stably. Applying $\widehat{\mathrm{Hom}}_{kG}(L, -)$ to the equation, we obtain $\widehat{\mathrm{Hom}}_{kG}(L, M) \cong \widehat{\mathrm{Hom}}_{kG}(L, M) \oplus \widehat{\mathrm{Hom}}_{kG}(L, X^*) \oplus \widehat{\mathrm{Hom}}_{kG}(L, Y)$. These are finitely generated k-modules and k is noetherian, so we must have $\widehat{\mathrm{Hom}}_{kG}(L, Y) = 0$. But $Y \mid M$ stably, hence $Y \mid L$ stably, so $\widehat{\mathrm{End}}_{kG}(Y) = 0$; thus $Y \simeq 0$. By symmetry, $X \simeq 0$, yielding $N \simeq M^*$.

The equation $f = \mathrm{ev}(\mathrm{Id}_M \otimes f')$ shows that ev is split, so k is stably a summand of $M \otimes M^*$, say $M \otimes M^* \simeq k \oplus Z$. But we now know that $M \otimes M^* \simeq M \otimes N \simeq k$, so we have $k \simeq k \oplus Z$. As before, applying $\widehat{\mathrm{Hom}}_{kG}(k, -)$ shows that $Z \simeq 0$.

For the final part, L is finitely generated, so we know that the natural map $\theta_L \colon L \otimes L^* \to \mathrm{End}_k(L)$ is a stable isomorphism. But $L \simeq M \oplus U$ for some U, so we can write $L \otimes L^* \simeq (M \otimes M^*) \oplus (M \otimes U^* \oplus U \otimes M^* \oplus U \otimes U^*)$ and $\mathrm{End}_k(L) \simeq (\mathrm{End}_k(M)) \oplus (\mathrm{Hom}_k(U, M) \oplus \mathrm{Hom}_k(M, U) \oplus \mathrm{End}_k(U))$, both considered as a sum of two submodules. By definition, θ_L respects these decompositions and restricts to θ_M on $M \otimes M^*$. In other words, the direct sum of θ_M and another morphism is a stable isomorphism; it follows that θ_M is a stable isomorphism. $\square$

Now for infinite groups of type Φ_k.

Theorem 6.5. *A kG-module M is invertible if and only if $M{\downarrow}^G_F$ is invertible for every finite subgroup F of G. Moreover, if M is invertible, then M^* is an inverse for M.*

Note that if k is p-local then the condition that $M{\downarrow}^G_F$ be invertible for every finite subgroup F of G is satisfied if and only if it is satisfied when F runs through a set of representatives of the conjugacy classes of finite elementary abelian p-subgroups of G ([11]).

Proof. Clearly, if M is invertible, then so is $M{\downarrow}^G_F$. For the converse, consider the map $\mathrm{ev} \colon M \otimes M^* \to k$. The previous theorem shows that for finite groups this is a stable isomorphism if and only if M is invertible. Thus if M is invertible on all finite subgroups then ev is a stable isomorphism on all finite subgroups, hence a stable isomorphism over kG, by Lemma 5.2. $\square$

A similar proof works for the next result.

Proposition 6.6. *If M is an invertible kG-module then $\theta_M \colon M \otimes M^* \to \mathrm{End}_k(M)$ is a stable isomorphism, as is the natural map $M \to M^{**}$.*

Here is another result from the theory of endotrivial modules for finite groups which extends to invertible modules for groups of type Φ_k.

Corollary 6.7. *Let M be a kG-module. For any $n \in \mathbb{Z}$, if $M \otimes N \simeq k$ then $\Omega^n M \otimes \Omega^{-n} N \simeq k$.*

Proof. The assertion holds for all finite subgroups of G (see [11, Section 2]) and therefore for G too. $\square$

Lemma 6.8. *Let H be a subgroup of G and let N be a normal subgroup such that any element of finite order has order a unit in k and the quotient group G/N is of type Φ_k. Restriction and inflation induce group homomorphisms*

$$\mathrm{Res}^G_H \colon T(G) \to T(H) \quad and \quad \mathrm{Inf}^{G/N}_G \colon T(G/N) \to T(G),$$

which commute with Ω. *More generally, there is a pullback map* $h^* : T(K) \to T(J)$ *for any homomorphism* $h \colon J \to K$ *such that any element of finite order in the kernel has order a unit in* k. *There is also a base change map* $f_* \colon T_k(G) \to T_\ell(G)$ *for any* $f \colon k \to \ell$ *for suitable* ℓ.

Proof. This follows immediately from the fact that the functors commute with tensor product in $\widehat{\mathrm{Mod}}$. However, given the way we have defined functors on $\widehat{\mathrm{Mod}}$ in Section 4, the latter is not quite obvious. In the case of Inf, for example, we need to check that $\mathrm{Inf}_G^{G/N} M \otimes \mathrm{Inf}_G^{G/N} N \simeq \mathrm{Inf}_G^{G/N}(M \otimes N)$. By taking Gorenstein approximations (Definition 3.11), we can assume that M and N are Gorenstein projective, hence lattices; the same will be true of their inflations. By the discussion at the beginning of Section 4, both $\mathrm{Inf}_G^{G/N}(-)$ and $- \otimes_k -$ applied to these modules have their usual meaning, and therefore commute. $\qquad\square$

Lemma 6.9. *If G and N are as above and the quotient map $\pi \colon G \to G/N$ is split then $\mathrm{Inf}_G^{G/N}$ is split.*

Proof. Let σ be a splitting homomorphism, so $\pi\sigma = \mathrm{Id}$. Then $\sigma^* \mathrm{Inf}_G^{G/N} = \sigma^*\pi^* = \mathrm{Id}$. $\qquad\square$

Another source of invertible modules comes from the observation that any kG-module that is a free k-module of rank 1 is invertible.

Proposition 6.10. *Suppose that G is finite and k is a complete discrete valuation ring with residue class field $\overline{k}$. Then there is a short exact sequence*

$$0 \to \mathrm{Hom}(G, \mathrm{tor}_p(k^\times)) \to T_k(G) \to T_{\overline{k}}(G) \to 0,$$

which is split. Here tor_p *denotes the p-torsion subgroup and we are identifying* $\mathrm{Hom}(G, k^\times)$ *with the rank 1 lattices.*

Proof. By Theorem 6.2, we may take the modules to be finite rank lattices. Surjectivity at $T_{\overline{k}}(G)$ is proved in [26, 1.3] in the case when k has characteristic 0. When it has characteristic p then $\overline{k}$ lifts to a subring of k (Cohen's Structure Theorem, see e.g. [27, 28.3]), so modules lift too.

If M is a kG-lattice such that $\overline{k} \otimes_k M \cong \overline{k} \oplus (\mathrm{proj})$ as $\overline{k}G$-modules, then the projective summand lifts to a projective summand of M. The complement is of rank 1 and thus corresponds to an element of $\mathrm{Hom}(G, \mathrm{tor}_p(k^\times))$; this proves exactness in the middle. For injectivity on the left, note that if two rank 1 lattices are isomorphic modulo projectives then they are isomorphic, by the Krull-Schmidt theorem, unless p does not divide $|G|$, in which case the proposition is trivial.

For the splitting, first note that if k has characteristic p then $\mathrm{tor}_p(k^\times) = 1$ and there is nothing to prove, so we may assume that k has characteristic 0. Let $\ell \leq k$ be a coefficient ring (Cohen's Structure Theorem again, [27, 29.3]), so ℓ is a complete local ring with maximal ideal generated by p, and also $\overline{\ell} \cong \overline{k}$. Thus ℓ contains no pth roots of unity and the first part of the proposition tells us that $T_{\overline{k}}(G) \cong T_{\overline{\ell}}(G) \cong T_\ell(G)$. The base change map $T_\ell(G) \to T_k(G)$ of Lemma 6.8 now completes the splitting. $\qquad\square$

7. Groups Acting on Trees

In this section, we describe $T_k(G)$ for groups G of type Φ_k which act on trees, in other words for the fundamental group of a graph of groups. We follow [20] for the notation and conventions. As special cases, we will obtain $T_k(G)$ for amalgamated free products and HNN extensions of groups of type Φ_k.

Suppose that a group G acts on a tree T with vertex set VT and edge set ET. If all the vertex stabilisers are of type Φ_k (hence the edge stabilisers too) and there exists a common bound on their finistic dimensions, then G is of type Φ_k by Proposition 2.5. In particular, this applies when the stabilisers are finite.

We continue to impose our assumptions on k from the beginning of Section 2 , and all groups will be of type Φ_k except where indicated (and of course, this does not apply to groups such as $T_k(G)$). We will omit the subscript k on T and Φ and write $\widehat{\mathrm{Aut}}_G(k)$ for $\widehat{\mathrm{Aut}}_{kG}(k)$.

The edges of T are oriented, so each edge $e \in ET$ has an initial vertex $\iota(e)$ and a terminal vertex $\tau(e)$. We pick a fundamental G-transversal Y of T and a maximal subtree Y_0 of Y. (So Y is a fundamental domain for the action of G on T and $VY = VY_0$; an edge in Y_0 has both vertices in Y_0, but an edge in $Y \smallsetminus Y_0$ has only its initial vertex in Y_0.) For each vertex $v \in VT$, let $\bar{v}$ be the unique vertex of Y_0 in the same G-orbit as v and choose $t_v \in G$ such that $t_v \bar{v} = v$, with $t_v = 1$ if $\bar{v} = v$.

Let G_v and G_e denote the stabilisers in G of $v \in VT$ and $e \in ET$, so $G_e = G_{i(e)} \cap G_{t(e)}$. For each edge $v \in VT$ there is an isomorphism $f_v : G_v \to G_{\bar{v}}$, given by $g \mapsto t_v^{-1} g t_v$.

Suppose that for each $v \in VY$, we are given a Gorenstein projective kG_v-module M_v, and for each edge $e \in EY$ a stable kG_e-isomorphism (which we can assume to be a genuine morphism, by Lemma 3.12)

$$\varphi_e : M_{\iota(e)}\downarrow_{G_e}^{G_{\iota(e)}} \longrightarrow \left(t_{\tau(e)} \otimes M_{\overline{\tau(e)}}\right)\downarrow_{G_e}^{G_{\tau(e)}} ,$$

where $t_{\tau(e)} \otimes M_{\overline{\tau(e)}}$ is considered as a $kG_{\tau(e)}$-module in the same way as it would be in the restriction of $kG \otimes_{kG_{\overline{\tau(e)}}} M_{kG_{\overline{\tau(e)}}}$, i.e. $g(t_{\tau(e)} \otimes m) = t_{\tau(e)} \otimes (t_{\tau(e)}^{-1} g t_{\tau(e)})m$ for $g \in G_{\tau(e)}$. Thus $t_{\tau(e)} \otimes M_{\overline{\tau(e)}} \simeq f_{\tau(e)}^* M_{\overline{\tau(e)}}$. We refer to this data as $(\underline{M}, \underline{\varphi})$.

Now, for $v \in VT$, set $M_v = t_v \otimes M_{\bar{v}}$, a kG_v-module, and for $e \in ET$, $M_e = M_{\iota(e)}\downarrow_{G_e}^{G_{\iota(e)}}$. Define

$$\varphi_e : M_e \longrightarrow M_{\tau(e)}\downarrow_{G_e}^{G_{\tau(e)}} \qquad \text{by} \qquad \varphi_e(t_{\iota(e)} \otimes m) = t_{\tau(e)} t_{\overline{\tau(e)}}^{-1} \varphi_e(m).$$

Set

$$C_0 = \bigoplus_{v \in VT} M_v \quad \text{and} \quad C_1 = \bigoplus_{e \in ET} M_e,$$

equipped with the canonical G-action, so that

$$C_0 \cong \bigoplus_{v \in VY} M_v\uparrow_{G_v}^{G} \quad \text{and} \quad C_1 \cong \bigoplus_{e \in EY} M_e\uparrow_{G_e}^{G} .$$

Define a kG-homomorphism $d : C_1 \to C_0$ as follows: for $e \in ET$ and $m \in M_e = M_{i(e)}\downarrow_{G_e}^{G_{i(e)}}$, let dm be the difference of $m \in M_{\iota(e)}$ and $\varphi_e(m)$. Let $C(\underline{M}, \underline{\varphi})$ be the two-term complex $C_1 \xrightarrow{d} C_0$ and define $\overline{C}(\underline{M}, \underline{\varphi})$ to be its cone.

Note that this description of d requires us to work with modules and genuine homomorphisms. A construction entirely in the stable category would proceed by defining maps $M_e \to C_0\downarrow_{G_e}^{G}$ for $e \in EY$ and then inducing.

This construction gives canonical maps $\alpha_v : M_v \longrightarrow \overline{C}(\underline{M}, \underline{\varphi})\downarrow_{G_v}^{G}$.

Lemma 7.1. *The following hold.*

(1) *For each $v \in VT$, the map α_v is a stable kG_v-isomorphism.*

(2) *For each $e \in ET$, we have $\left(\alpha_{\tau(e)}\downarrow_{G_e}^{G_{\tau(e)}}\right)^{-1}\left(\alpha_{\iota(e)}\downarrow_{G_e}^{G_{\iota(e)}}\right) = \varphi_e$ in $\widehat{\mathrm{Mod}}(kG_e)$.*

Proof. For (1) it suffices to show that the composite map $\bar{d} : C_1 \xrightarrow{d} C_0 \longrightarrow C_0/M_w$ is a kG_w-isomorphism for all $w \in VY$. We shall construct an inverse s. Write $(C_0/M_w)\downarrow_{G_w}^{G} = \bigoplus_{i=1}^{\infty} C_0^i$, where C_0^i is the sum of all the M_v for v at a distance i from w. We shall define s inductively on the C_0^i as follows. Suppose that we have already defined s on $\bigoplus_{i=1}^{j} C_0^i$ for some $j \geq 0$. Given a

vertex v at a distance $j+1$ from w, there is a unique edge e with one vertex v and the other at a distance j from w. Let $m \in M_v$: if $v = \iota(e)$ then define $s(m)$ to be the sum of $m \in M_e = M_v{\downarrow}_{G_e}^{G_v}$ and $s\varphi_e(m)$; if $v = \tau(e)$ define $s(m)$ to be the sum of $-\varphi_e^{-1}(m) \in M_e = M_{\iota(e)}{\downarrow}_{G_e}^{G_{\iota(e)}}$ and $s(\varphi_e^{-1}(m))$. Here φ_e^{-1} denotes a genuine homomorphism that is a stable inverse to φ_e.

It is straightforward to check that s is a kG_w-homomorphism that is a stable inverse to $\bar{d}$.

For (2), note that, by construction, we have $\alpha_{i(e)}(m) - \alpha_{t(e)}(\varphi_e(m)) \in \operatorname{im}(d)$ for all $m \in M_v$ and all $v \in VT$. The equality follows. $\qquad\square$

Suppose that each M_v is stably trivial. Let $\tilde{k}$ be the Gorenstein projective approximation of the trivial kG-module k and choose stable isomorphisms $\theta_v : \tilde{k}{\downarrow}_{G_v}^{G} \longrightarrow M_v$ for all $v \in VY$. We extend the definition of these maps to all of VT by setting $\theta_v = t_v \theta_{\bar{v}} t_v^{-1}$ for $v \in VT$.

Lemma 7.2. *The assignment $\varphi_e \mapsto \varphi'_e = \theta_{\tau(e)}^{-1} \varphi_e \theta_{\iota(e)}$ identifies the stable classes of isomophisms $M_{\iota(e)}{\downarrow}_{G_e}^{G_{\iota(e)}} \to M_{\tau(e)}{\downarrow}_{G_e}^{G_{\tau(e)}}$ with the elements of $\widehat{\operatorname{Aut}}_{G_e}(k)$.*

Proof. This is immediate from the definition of the maps θ_v and φ_e. $\qquad\square$

As a direct consequence of Lemmas 7.1 and 7.2, we obtain the following.

Corollary 7.3. *For every edge $e \in ET$, $\varphi'_e = \theta_{t(e)}^{-1} \alpha_{t(e)}^{-1} \alpha_{i(e)} \theta_{i(e)}$ in $\widehat{\operatorname{Mod}}(kG_e)$.*

Now take $M_v = \tilde{k}{\downarrow}_{G_v}^{G}$ and θ_v to be the identity map. Define

$$(1) \qquad \partial \colon \prod_{e \in EY} \widehat{\operatorname{Aut}}_{G_e}(k) \longrightarrow T(G) \quad \text{by} \quad \partial\big((\varphi'_e)_{e \in EY}\big) = \overline{C}(\underline{k}, \underline{\varphi}),$$

where $\overline{C}(\underline{k}, \underline{\varphi})$ is the cone defined above. This yields an invertible module by Lemma 7.1 and Theorem 6.5, since every finite subgroup must fix some vertex, but we do not yet know that ∂ is a group homomorphism.

Theorem 7.4. *Let G be a group acting on a tree with vertex set VT and edge set ET, and assume that all the vertex stabilisers are groups of type Φ_k and that there exists a common bound on their finistic dimensions. Then G is of type Φ_k and there is an exact sequence of abelian groups*

$$\widehat{\operatorname{Aut}}_G(k) \xrightarrow{\operatorname{Res}} \prod_{v \in VY} \widehat{\operatorname{Aut}}_{G_v}(k) \xrightarrow{\operatorname{Res} - \operatorname{Res}_f} \prod_{e \in EY} \widehat{\operatorname{Aut}}_{G_e}(k) \xrightarrow{\partial} \dots$$

$$\dots \xrightarrow{\partial} T(G) \xrightarrow{\operatorname{Res}} \prod_{v \in VY} T(G_v) \xrightarrow{\operatorname{Res} - \operatorname{Res}_f} \prod_{e \in EY} T(G_e).$$

Here, for $M \in T(G)$, the $T(G_v)$-coordinate of $\operatorname{Res}(M)$ is $M{\downarrow}_{G_v}^{G}$; for $M_v \in T(G_v)$,

$$\text{the } T(G_e)\text{-coordinate of } \operatorname{Res}(M_v) \text{ is } \begin{cases} M_v{\downarrow}_{G_e}^{G_v} & \text{if } v = \iota(e) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and the } T(G_e)\text{-coordinate of } \operatorname{Res}_f(M_v) \text{ is } \begin{cases} f_e^*(M_v){\downarrow}_{G_e}^{G_v} & \text{if } v = \overline{\tau(e)} \\ 0 & \text{otherwise.} \end{cases}$$

We take the product of the maps on the coordinates, which is well defined since each edge has only two vertex maps associate to it. The maps for the groups of stable automorphisms are defined analogously and ∂ is the map defined in Equation (1).

Proof. First we check that ∂ is constant on the cosets of the image of $\operatorname{Res} - \operatorname{Res}_f$. Given $\psi_v \in \widehat{\operatorname{Aut}}_{G_v}(k)$ for $v \in VY$, we extend it to $v \in VT$ by $\psi_v = t_v^{-1} \psi_{\bar{v}} t_v$. Then

$$(\operatorname{Res} - \operatorname{Res}_f)(\psi_v) = \big(\psi_{\iota(e)}{\downarrow}_{G_e}^{G_{\iota(e)}}\big)\big(\psi_{\tau(e)}{\downarrow}_{G_e}^{G_{\tau(e)}}\big)^{-1}.$$

The maps ψ_v combine in the obvious way to yield a map of complexes

$$C(\tilde{k}{\downarrow}_{G_v}^G, \varphi_e) \longrightarrow C\big(\tilde{k}{\downarrow}_{G_v}^G, \varphi_e(\psi_{\iota(e)}{\downarrow}_{G_e}^{G_{\iota(e)}})(\psi_{\tau(e)}{\downarrow}_{G_e}^{G_{\tau(e)}})^{-1}\big),$$

in the sense that the diagram commutes in the stable module category. It is also a stable isomorphism on the modules, and so the third objects in the triangles are isomorphic, as required.

The image of ∂ is contained in $\ker(\mathrm{Res})$ by Lemma 7.1, so we obtain a function

$$\overline{\partial} : \mathrm{coker}(\mathrm{Res} - \mathrm{Res}_f) \longrightarrow \ker(\mathrm{Res}).$$

We shall construct an inverse s as follows. Given $M \in T(G)$ such that $M{\downarrow}_{G_v}^G \simeq k$ for $v \in VY$, set $M_v = M{\downarrow}_{G_v}^G$ and choose isomorphisms $\theta_v : \tilde{k}{\downarrow}_{G_v}^G \longrightarrow M_v$; extend these to all the vertices VT by setting $\theta_v = s_v \otimes \theta_{\bar{v}}$. Put $\varphi_e' = \big(\theta_{\tau(e)}{\downarrow}_{G_e}^{G_{\tau(e)}}\big)^{-1}(\theta_{\iota(e)}{\downarrow}_{G_e}^{G_{\iota(e)}})$ for $e \in EY$ and define $s(M) = (\varphi_e')_{e \in EY}$. As element of $\mathrm{coker}(\mathrm{Res} - \mathrm{Res}_f)$, $s(M)$ does not depend on the choice of the maps θ_v. This function s is actually a group homomorphism, because given $\theta_j \in \widehat{\mathrm{Hom}}_{G_e}(\tilde{k}, M_j)$ for $j = 1, 2$, we can combine them to obtain a homomorphism

$$k \longrightarrow k \otimes k \xrightarrow{\theta_1 \otimes \theta_2} M_1 \otimes M_2 \ .$$

Moreover, $\sigma\overline{\partial}$ is the identity by Corollary 7.3, since we can choose $\theta_v = \alpha_v^{-1}$ (the maps θ_v in the statement of Corollary 7.3 are now the identity maps).

Given $M \in \ker(\mathrm{Res})$ and θ_v as above, notice that $C(\tilde{\underline{k}}, \underline{\mathrm{Id}}) = k$, since this is the simplicial chain complex on the tree T and so M is the third module in the triangle for $C(\tilde{\underline{k}}, \underline{\mathrm{Id}}) \otimes M$. The maps θ_v provide an isomorphism of complexes

$$C(\tilde{\underline{k}}, \underline{\mathrm{Id}}) \otimes M \longrightarrow C\big(\tilde{\underline{k}}, \underline{(\theta_{\tau(e)}{\downarrow}_{G_e}^{G_{\tau(e)}})^{-1}(\theta_{\iota(e)}{\downarrow}_{G_e}^{G_{\iota(e)}})}\big),$$

so $M = \overline{\partial}\sigma(M)$ as required. It follows that ∂ is a group homomorphism and that the complex in the statement of the theorem is exact at $\prod_{e \in EY} \widehat{\mathrm{Aut}}_{G_e}(k)$ and $T(G)$.

To check the exactness at $\prod_{v \in VY} T(G_v)$, observe that if $(M_v)_{v \in VY} \in \ker(\mathrm{Res} - \mathrm{Res}_f)$, then for each edge $e \in EY$ we have $M_{\iota(e)}{\downarrow}_{G_e}^{G_{\iota(e)}} \simeq f_e^* M_{\overline{\tau(e)}}$; let $\varphi_e : M_{\iota(e)}{\downarrow}_{G_e}^{G_{\iota(e)}} \longrightarrow M_{\tau(e)}{\downarrow}_{G_e}^{G_{\tau(e)}}$ be a stable isomorphism. Then $(M_v)_{v \in VY} = \mathrm{Res}\big(\overline{C}(\underline{M}, \underline{\varphi})\big)$ by Lemma 7.1.

Exactness at $\prod_{v \in VY} \widehat{\mathrm{Aut}}_{G_v}(k)$ is proved by an argument analogous to that used for cohomology [8, VII, §9], and is left to the reader.

$\square$

As special cases of Theorem 7.4, we obtain descriptions of $T(G)$ for amalgamated free products and HNN extensions of groups of type Φ.

Corollary 7.5. *Let $G = A *_C B$ be an amalgamated free product with A and B of type Φ_k. Then G is of type Φ_k and there is an exact sequence of abelian groups*

$$\widehat{\mathrm{Aut}}_G k \xrightarrow{\mathrm{Res}} \widehat{\mathrm{Aut}}_A k \oplus \widehat{\mathrm{Aut}}_B k \xrightarrow{\mathrm{Res} - \mathrm{Res}_f} \widehat{\mathrm{Aut}}_C(k) \xrightarrow{\partial} T(G) \xrightarrow{\mathrm{Res}} T(A) \oplus T(B) \xrightarrow{\mathrm{Res} - \mathrm{Res}_f} T(C),$$

where the maps are those defined in Theorem 7.4.

If C is finite and k is a field or a complete discrete valuation ring then $\partial = 0$.

Proof. The main part is a special case of Theorem 7.4. If C is finite then $\widehat{\mathrm{End}}_C(k) \cong \hat{H}^0(C; k) \cong k/|C|k$, so $\widehat{\mathrm{Aut}}_C(k) \cong (k/|C|k)^\times$. Given the restrictions on k, these units lift to units in k, so they are certainly in the image of restriction from A, thus restriction is onto and $\partial = 0$. $\square$

Example 7.6. Let $G = \mathrm{SL}(2, \mathbb{Z}) \cong C_6 *_{C_2} C_4$. Then for k a field or a complete discrete valuation ring the sequence

$$0 \longrightarrow T(G) \longrightarrow T(C_6) \oplus T(C_4) \longrightarrow T(C_2) \longrightarrow 0$$

is exact. This follows directly from Corollary 7.5, except for exactness at $T(C_2)$, where it follows from the fact that $T(C_2)$ is generated by Ωk, which is in the image of restriction from $T(C_4)$.

(1) Suppose that k is a field of characteristic 2. Then $T(C_2) \cong 0$, $T(C_4) \cong \mathbb{Z}/2$ and $T(C_6) \cong \mathrm{Hom}(C_6, k^\times)$. It follows that $T(G) \cong \mathbb{Z}/6$ if k contains a cube root of unity and $T(G) \cong \mathbb{Z}/2$ otherwise.

(2) Suppose that k is a field of characteristic 3. Then $T(C_4) \cong 0 \cong T(C_2)$ and $T(C_6) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (using [28] for $T(C_6)$). Thus $T(G) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

Here is an example of inflation, which shows that inflation can be neither surjective nor injective.

Example 7.7. Let $G = X *_Z Y \cong C_4 *_{C_2} C_4$ where X and Y are generated by x and y respectively. Let N be the normal subgroup generated by $xyxy^{-1}$, let $\bar{G} = G/N \cong Q_8$, and denote the quotient map by π. The subgroup N has no torsion, since any torsion subgroup of G must be conjugate to a subgroup of either X or Y. Thus we do have an inflation map $\mathrm{Inf}_G^{\bar{G}}$.

Let k be a field of characteristic 2. Then $T(C_4) \cong \mathbb{Z}/2$ and $T(C_2) \cong 0$, so by Corollary 7.5 $T(G) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. We claim that the image of $\mathrm{Inf}_G^{\bar{G}}$ is the diagonal subgroup of $T(G)$. The diagonal subgroup is generated by Ωk, which is the image under inflation of the module of the same name in $T(\bar{G})$. The elements of $T(G)$ are detected by restriction to the cyclic subgroups X and Y, and $\mathrm{Res}_X^G \mathrm{Inf}_G^{\bar{G}} = (\pi \restriction_X)^* \mathrm{Res}_{\bar{X}}^{\bar{G}}$ and similarly for Y. But $\mathrm{Res}_{\bar{X}}^{\bar{G}}$ and $\mathrm{Res}_{\bar{Y}}^{\bar{G}}$ take the same value, 0 or 1: this can be seen by an easy calculation, since $T(Q_8)$ is generated by Ωk together with an explicit module of dimension 3 if k contains a cube root of unity [13, 6.3]. Both $\pi \restriction_X$ and $\pi \restriction_Y$ are isomorphisms, so $\mathrm{Res}_X^G \mathrm{Inf}_G^{\bar{G}} = \mathrm{Res}_Y^G \mathrm{Inf}_G^{\bar{G}}$.

The group $T(Q_8)$ has order at least 4 (it depends on k), so the inflation map is not injective.

We now turn to the case of HNN extensions. An HNN extension $G = H*_f$, for $A \leq H$ and $f : A \hookrightarrow H$, is the fundamental group of a graph of groups with one vertex H and one edge A, where A is included into H at the initial vertex and is mapped by f at the terminal vertex. In terms of generators and relations, $G = \langle H, t \mid tat^{-1} = f(a) \, \forall \, a \in A \rangle$.

Corollary 7.8. *Let $G = H*_f$ be a HNN extension with H of type Φ_k. Then G is of type Φ_k and there is an exact sequence of abelian groups*

$$\widehat{\mathrm{Aut}}_G k \xrightarrow{\mathrm{Res}} \widehat{\mathrm{Aut}}_H k \xrightarrow{\mathrm{Res} - \mathrm{Res}_f} \widehat{\mathrm{Aut}}_A(k) \xrightarrow{\partial} T(G) \xrightarrow{\mathrm{Res}} T(H) \xrightarrow{\mathrm{Res} - \mathrm{Res}_f} T(A),$$

where the maps are those defined in Theorem 7.4. If H is finite and k is a field or a complete discrete valuation ring then ∂ is injective.

Proof. The main part is a special case of Theorem 7.4. If H is finite then $\mathrm{Res} : \widehat{\mathrm{Aut}}_G(k) \to \widehat{\mathrm{Aut}}_H(k)$ is onto by the same argument as in the previous proof. Thus $\mathrm{Res} - \mathrm{Res}_f = 0$ and ∂ is injective. $\square$

Note that if f is the identity map on H then $H*_f \cong \mathbb{Z} \times H$ and $\mathrm{Res} - \mathrm{Res}_f = 0 = 0$

In the next three examples, k is a field of characteristic p or a complete discrete valuation ring with residue field of characteristic p and A is a finite group of order divisible by p.

Example 7.9. Consider $G = \mathbb{Z} \times A$ as an HNN extension and apply Corollary 7.8. Since f is the identity, we have $\mathrm{Res} - \mathrm{Res}_f = 0$, so there is a short exact sequence $0 \to (k/|A|k)^\times \xrightarrow{\partial} T(G) \xrightarrow{\mathrm{Res}} T(A) \to 0$; by Lemma 6.9 it is split by inflation. Thus $T(G)$ is generated by modules inflated from A and the rank 1 lattices.

This shows that $T(G)$ can be very big: we cannot bound its cardinality independently of k.

We want to calculate $\widehat{\mathrm{Aut}}_G(k)$ for use in the next example. This can be done by noting that $\widehat{\mathrm{End}}_G(k) \cong \hat{H}^0(G; k)$ (the cup product agrees with the composition product [8, VI.6, X.ex.6]) and using the spectral sequence in Tate-Farrell cohomology $H^p(\mathbb{Z}; \hat{H}^q(A; k)) \Rightarrow \hat{H}^{p+q}(G; k)$ from [8, X.4 ex.5]. The result is that $\widehat{\mathrm{End}}_G(k) \cong k[x]/(x^2)$ as a k-algebra if $p = 0$ in k, and $\widehat{\mathrm{End}}_G(k) \cong k/|A|k$ otherwise. Thus $\widehat{\mathrm{Aut}}_G(k) \cong k^\times \oplus k$ as a group if $p = 0$ in k and $\widehat{\mathrm{Aut}}_G(k) \cong (k/|A|k)^\times$ otherwise. We can write this in a uniform fashion as $\widehat{\mathrm{Aut}}_G(k) \cong (k/|A|k)^\times \oplus \mathrm{tor}_p(k)$.

Example 7.10. Consider $F = \mathbb{Z} \times G = \mathbb{Z} \times \mathbb{Z} \times A$. A similar calculation to the previous one shows that $T(F) \cong \widehat{\mathrm{Aut}}_G(k) \oplus T(G)$. Combining this with our previous calculations we obtain $T(F) \cong \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, (k/|A|k)^\times) \oplus \mathrm{tor}_p(k) \oplus T(A)$.

The rank 1 lattices still appear, as $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, (k/|A|k)^\times)$ and part of $T(A)$, but with identifications. The image of inflation from A accounts for the $T(A)$ summand. The modules corresponding to the $\mathrm{tor}_p(k)$ summand are more mysterious. They do not appear to be stably isomorphic to a lattice of finite rank. They also only occur when $p = 0$ in k, so they cannot lift to characteristic 0, in contrast to the case for finite groups in Proposition 6.10.

Example 7.11. Consider $E = \mathbb{Z} \times (A * A)$. From Corollary 7.5 we know that $T(A * A) \cong T(A) \oplus T(A)$. By Example 5.6 we have $\widehat{\mathrm{Aut}}_{A*A}(k) \cong \widehat{\mathrm{Aut}}_A(k) \oplus \widehat{\mathrm{Aut}}_A(k) \cong (k/|A|k)^\times \oplus (k/|A|k)^\times$.

We obtain $T(G) \cong (k/|A|k)^\times \oplus (k/|A|k)^\times \oplus T(A) \oplus T(A)$. Only one copy of $(k/|A|k)^\times$ can be explained by the rank 1 lattices; in some sense this is because we should think instead of homomorphisms from $\mathbb{Z}$ to $\widehat{\mathrm{Aut}}_{A*A}(k)$,

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