Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Maximal left ideals in Banach algebras

M. Cabrera García, H. G. Dales, and Á. Rodríguez Palacios

Abstract

Let A be a Banach algebra. Then frequently each maximal left ideal in A is closed, but there are easy examples that show that a maximal left ideal can be dense and of codimension 1 in A. It has been conjectured that these are the only two possibilities: each maximal left ideal in a Banach algebra A is either closed or of codimension 1 (or both). We shall show that this is the case for many Banach algebras that satisfy some extra condition, but we shall also show that the conjecture is not always true by constructing, for each $n \in \mathbb{N}$, examples of Banach algebras that have a dense maximal left ideal of codimension n. In particular, we shall exhibit a semi-simple Banach algebra with this property. We shall show that the questions concerning maximal left ideals in a Banach algebra A that we are considering are related to automatic continuity questions: When are A-module homomorphisms from A into simple Banach left A-modules automatically continuous?

1. *Introduction*

Let A be an algebra, so that A is a linear algebra over a field $\mathbb{K}$ that is either the real or complex field and A is associative unless stated otherwise. A *left ideal* in A is a linear subspace I of A such that $ax \in I$ whenever $a \in A$ and $x \in I$; a left ideal M is *maximal* if $M \neq A$ and if $I = M$ or $I = A$ whenever I is a left ideal in A with $I \supset M$. In this paper, we shall consider when all maximal left ideals in a Banach algebra are necessarily either closed or of codimension 1, and we shall give some positive results. However, we shall also show that, given $n \in \mathbb{N}$, there are Banach algebras A with a maximal left ideal M such that M is dense and has codimension n in A. We can also arrange that A be primitive, and hence semi-simple, or a Banach $*$-algebra or such that A factors. We do not know whether there exists a Banach algebra A that has a dense maximal left ideal of infinite codimension in A; the existence of such an example is equivalent to that of a Banach algebra A that has a discontinuous left A-module homomorphism into an infinite-dimensional, simple Banach left A-module.

Throughout, we shall concentrate on maximal left ideals in Banach algebras; for us, a *normed algebra* A is an algebra A with a norm $\|\cdot\|$ with respect to which $(A, \|\cdot\|)$ is a normed space and $\|ab\| \leq \|a\|\,\|b\| \ \ (a, b \in A)$, and A is a *Banach algebra* if $(A, \|\cdot\|)$ is complete. For the theory of normed and Banach algebras, see [**1**, **2**, **3**, **4**, **10**]. A *Banach $*$-algebra* is a Banach algebra A with a conjugate-linear involution $*$ such that $(ab)^* = b^*a^* \ \ (a, b \in A)$; for the theory of Banach $*$-algebras, see, in particular, [**10**]. For example, every C^*-algebra is a Banach $*$-algebra.

We shall also make a few remarks about non-associative algebras.

We first recall some standard notation.

For $n \in \mathbb{N}$, set $\mathbb{N}_n = \{1, \ldots, n\}$; the real and complex fields are $\mathbb{R}$ and $\mathbb{C}$, respectively; set $\mathbb{I} = [0, 1]$, the closed unit interval in $\mathbb{R}$, and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc in $\mathbb{C}$. The closed unit ball centred at 0 of a Banach space E is denoted by $E_{[1]}$; the dual space to E is E'.

Let A be an algebra. The *opposite algebra* to A is A^{op}; here the product in A^{op} of a and b in A is ba. The *centre* of A is denoted by $\mathfrak{Z}(A)$, so that $\mathfrak{Z}(A) = \{b \in A : ab = ba \ \ (a \in A)\}$. Take

---

2000 *Mathematics Subject Classification* 46H10, 46H40, 46H70.

two non-empty subsets S and T of A. Then

$$S \cdot T = \{ab : a \in S,\, b \in T\}\,, \quad ST = \operatorname{lin} S \cdot T\,,$$

the linear span of $S \cdot T$. Further, set $A^{[2]} = A \cdot A$ and $A^2 = \operatorname{lin} A^{[2]}$, as in [**4**]. The algebra A *factors* if $A = A^{[2]}$ and A *factors weakly* if $A = A^2$. For results on the factorization of commutative Banach algebras, see [**4**, §2.9] and [**5**].

A *character* on an algebra A over $\mathbb{K}$ is a homomorphism from A onto $\mathbb{K}$; all characters φ on a Banach algebra are continuous, with $\|\varphi\| \leq 1$.

A linear subspace I of an algebra A is a *right ideal* if $IA \subset I$ and an *ideal* if it is both a left and right ideal; an ideal M in A is *maximal* if $M \neq A$ and $I = M$ or $I = A$ whenever I is an ideal in A with $I \supset M$. The quotient algebra of A by an ideal I is A/I; in the case where A is a normed or Banach algebra and I is closed in A, the quotient A/I is a normed or Banach algebra, respectively, with respect to the quotient norm.

Let A be an algebra. Then A is *simple* if $A^2 \neq \{0\}$ and if $\{0\}$ and A are the only ideals in A. For $n \in \mathbb{N}$, we denote by $\mathbb{M}_n$ the algebra of $n \times n$ matrices over $\mathbb{C}$; the algebras $\mathbb{M}_n$ are simple. We also denote by $\mathbb{M}_n(A)$ the algebra of all $n \times n$ matrices with coefficients in A. In the case where A is a Banach algebra, $\mathbb{M}_n(A)$ is also a Banach algebra with respect to the norm given by

$$\|(a_{i,j} : i, j \in \mathbb{N}_n)\| = \sum_{i,j=1}^{n} \|a_{i,j}\| \quad ((a_{i,j} : i, j \in \mathbb{N}_n) \in \mathbb{M}_n(A))\,.$$

Suppose that A is a Banach $*$-algebra. Then $\mathbb{M}_n(A)$ is also a Banach $*$-algebra with respect to the involution given by the transpose map $(a_{i,j}) \mapsto (a^*_{j,i})$.

An element e_A of an algebra A is the *identity* of A if $ae_A = e_A a = a \ (a \in A)$; an algebra is *unital* if it has a non-zero identity; the algebra formed by adding an identity to an algebra A is $A^\sharp$, identified with $\mathbb{K} \times A$, as in [**2**, §1.1.104]. A *left identity* in A is an element $p \in A$ such that $pa = a \ (a \in A)$.

The set of invertible elements in a unital algebra A is denoted by $\operatorname{Inv} A$. More generally, an element a in an algebra A is *quasi-invertible* if there exists $b \in A$ with $a + b - ab = a + b - ba = 0$; the set of quasi-invertible elements in A is denoted by $q - \operatorname{Inv} A$. A unital algebra in which every non-zero element is invertible is a *division algebra*, and a commutative division algebra is a *field*.

A proper left ideal I in an algebra A is *modular* if there exists $u \in A$ with $a - au \in I \ (a \in A)$; in this case, u is a *right modular identity* for I. Let I be a left ideal in an algebra A with a right modular identity u. Then it is immediate from Zorn's lemma that the family of left ideals J in A such that $J \supset I$ and $u \notin J$ (when the family is ordered by inclusion) has a maximal member, say M. Clearly M is a maximal modular left ideal in A, and hence a maximal left ideal in A.

Let A be an algebra, and let F be a subspace of A. The *core* of F in A is the largest ideal of A contained in F. A *primitive ideal* in A is the core of a maximal modular left ideal in A; a non-zero algebra is *primitive* if $\{0\}$ is a primitive ideal. See [**2**, Definition 3.6.12].

The (Jacobson) *radical* of an algebra A is defined to be the intersection of the maximal modular left ideals of A [**1**, Chapter III], [**2**, Section 3.6], [**4**, §1.5]; it is denoted by $\operatorname{rad} A$, with $\operatorname{rad} A = A$ when A has no maximal modular left ideals. In fact, $\operatorname{rad} A = \operatorname{rad} A^{\mathrm{op}}$ and $\operatorname{rad} A$ is an ideal in A. The algebra A is *semi-simple* when $\operatorname{rad} A = \{0\}$ and *radical* when $\operatorname{rad} A = A$; the quotient algebra $A/\operatorname{rad} A$ is always a semi-simple algebra; a primitive algebra is semi-simple.

An element $a \in A$ is *quasi-nilpotent* if $za \in q - \operatorname{Inv} A \ (z \in \mathbb{K})$; in the case where A is a Banach algebra, $a \in A$ is quasi-nilpotent if and only if $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$. A Banach algebra A is *topologically nilpotent* if

$$\lim_{n\to\infty} \sup\{\|a_1 \cdots a_n\|^{1/n} : a_1, \ldots, a_n \in A_{[1]}\} = 0\,;$$

see [**3**, Subsection 8.4.2 and §8.4.121] and [**10**, §4.8.8].

Let A be an algebra, and let E be a left A-module for the operation

$$(a, x) \mapsto a \cdot x\,, \quad A \times E \to E\,.$$

Then E is *non-trivial* if there exist $a \in A$ and $x \in E$ with $a \cdot x \neq 0$, E is *faithful* if the only element $a \in A$ such that $a \cdot x = 0$ $(x \in E)$ is $a = 0$, and E is *simple* if E is non-trivial and the only left A-modules in E are $\{0\}$ and E. Let E be a simple left A-module, and take $x_0 \in E$ with $x_0 \neq 0$. Then $\{a \cdot x_0 : a \in A\} = E$ and

$$x_0^{\perp} = \{a \in A : a \cdot x_0 = 0\}$$

is a maximal modular left ideal in A. A left A-module is also a left $A^{\sharp}$-module.

Take $n \in \mathbb{N}$. By regarding the elements of E^n as column matrices, the space E^n is naturally a left $\mathbb{M}_n(A)$-module. It is easy to check that E^n is faithful or simple whenever E has the corresponding property. Since an algebra B is primitive if and only if there is a faithful, simple left B-module [**2**, Definition 3.6.35 and Theorem 3.6.38(i)], it follows that $\mathbb{M}_n(A)$ is primitive whenever A is a primitive algebra.

Let A be a Banach algebra. A *Banach left A-module* is a Banach space $(E, \|\cdot\|)$ such that E is a left A-module and

$$\|a \cdot x\| \leq \|a\|\,\|x\| \quad (a \in A,\, x \in E)\,;$$

see [**2**, Subsection 3.6.3] and [**4**, §2.6]. For example, a closed left ideal in A is a Banach left A-module. Similarly, one can define a *Banach right A-module*.

The following theorem, which is originally due to Rickart, is given in [**1**, Lemma 25.2] and [**4**, Theorem 2.6.26].

THEOREM 1.1. *Let A be a Banach algebra, and let E be a simple left A-module. Then there is a norm $\|\cdot\|$ on E such that $(E, \|\cdot\|)$ is a Banach left A-module. In this case, the norm is uniquely so specified up to equivalence of norms.* □

We shall use the following propositions and corollaries; some are already contained in [**3**, pp. 658–659] and [**4**, §1.4] in somewhat different forms.

PROPOSITION 1.2. *Let A be an algebra with $A^2 \subsetneq A$. Then A contains a maximal left ideal that is an ideal in A and that contains A^2. Each maximal left ideal that contains A^2 has codimension 1 in A.*

Proof. Let M be a subspace of codimension 1 in A such that $A^2 \subset M$. Then M is a maximal left ideal and a (maximal) right ideal. Clearly, each maximal left ideal that contains A^2 has codimension 1. □

PROPOSITION 1.3. *Let A be an algebra. Suppose that M is a maximal left ideal in A and that $b \in A$, and set $J_b = \{a \in A : ab \in M\}$. Then either $J_b = A$ or J_b is a maximal modular left ideal in A.*

Proof. Set $E = A/M$, a left A-module. Then either $Ab \subset M$, and hence $J_b = A$, or E is a simple left A-module and $b + M \in E \setminus \{0\}$. In the latter case, $J_b = (b + M)^{\perp}$ is a maximal modular left ideal in A. □

Proposition 1.4. *Let A be an algebra. Then the following are equivalent:*

(a) *A has no maximal left ideal;*

(b) *A is a radical algebra and $A^2 = A$;*

(c) *A has no maximal right ideal.*

Proof. (a) $\Rightarrow$ (b) Since A has no maximal left ideal, it has no maximal modular left ideal, and so A is a radical algebra. By Proposition 1.2, $A^2 = A$.

(b) $\Rightarrow$ (a) Assume that M is a maximal left ideal in A, and take $b \in A$. By Proposition 1.3, $J_b = A$, and so $Ab \subset M$. Hence $A^2 \subset M \subsetneq A$, a contradiction of (b). Thus (a) holds.

(a) $\Leftrightarrow$ (c) Since $\operatorname{rad} A = \operatorname{rad} A^{\mathrm{op}}$, this is immediate. □

An example of a simple, radical algebra is given in [**11**]. Since a simple algebra A is such that $A^2 = A$, it follows from Proposition 1.4 that this algebra has no maximal left or maximal right ideal. However, it does have a maximal ideal, namely $\{0\}$.

Corollary 1.5. *Let R be a radical algebra. Then R has a maximal left ideal if and only if $R^2 \subsetneq R$, and in this case every maximal left ideal contains R^2.* □

We shall also use the following results on primitive algebras.

Lemma 1.6. *Let A be an algebra, and let E be a faithful left A-module. Suppose that, as a left $A^\sharp$-module, E is not faithful. Then A has an identity element.*

Proof. By hypothesis, there is a non-zero element $(\alpha, a) \in A^\sharp$ with $(\alpha, a) \cdot x = 0 \ (x \in E)$. Since E is a faithful left A-module, necessarily $\alpha \neq 0$, and so there exists $e \in A$ such that $(1, -e) \cdot x = 0 \ (x \in E)$. Hence

$$(a - ea) \cdot x = (a - ae) \cdot x = 0 \quad (a \in A, \, x \in E)\,.$$

It follows that $a = ea = ae \ (a \in A)$, and so e is the identity of A. □

Proposition 1.7. *Let A be a primitive algebra, and take $n \in \mathbb{N}$. Suppose that A is non-unital and that $\mathfrak{A}$ is a subalgebra of $\mathbb{M}_n(A^\sharp)$ containing $\mathbb{M}_n(A)$. Then $\mathfrak{A}$ is a primitive algebra.*

Proof. Since A is primitive, there is a faithful, simple left A-module, say E. By Lemma 1.6, E^n is a faithful left $A^\sharp$-module, hence E^n is a faithful $\mathbb{M}_n(A^\sharp)$-module, and hence E^n is a faithful left $\mathfrak{A}$-module. On the other hand, E^n is a simple left $\mathbb{M}_n(A)$-module, and so E^n is a simple left $\mathfrak{A}$-module because $\mathfrak{A} \supset \mathbb{M}_n(A)$. Hence $\mathfrak{A}$ is a primitive algebra. □

Proposition 1.8. *Let A be an algebra containing a maximal left ideal M of codimension 1 such that $A^2 \not\subset M$, and take $n \in \mathbb{N}$. Then the set of matrices $(a_{i,j})$ in $\mathbb{M}_n(A)$ for which $a_{i,1} \in M \ (i \in \mathbb{N}_n)$ is a maximal left ideal in $\mathbb{M}_n(A)$ of codimension n.*

Proof. The matrices that we are considering have the form

$$\mathcal{M} = \begin{pmatrix} M & A & \dots & A \\ M & A & \dots & A \\ \dots & \dots & \dots & \dots \\ M & A & \dots & A \end{pmatrix} .$$

It is clear that $\mathcal{M}$ is a left ideal of codimension n in $\mathbb{M}_n(A)$.

To show that $\mathcal{M}$ is a maximal left ideal in $\mathbb{M}_n(A)$, consider a left ideal $\mathcal{J}$ in $\mathbb{M}_n(A)$ such that $\mathcal{J} \supsetneq \mathcal{M}$. Since $A^2 \not\subset M$, there exist $a, b \in A$ with $ab \notin M$, and so $b \notin M$, and this implies that $\mathbb{K}ab + M = \mathbb{K}b + M = A$. Further there are $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, not all zero, such that

$$\begin{pmatrix} \alpha_1 b & 0 & \dots & 0 \\ \alpha_2 b & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \alpha_n b & 0 & \dots & 0 \end{pmatrix} \in \mathcal{J} .$$

Suppose that $\alpha_j \neq 0$, and take $r \in \mathbb{N}_n$. Multiply the above matrix on the left by the matrix that has a in the (r, j)-th position and 0 elsewhere. This gives the matrix that has $\alpha_j ab$ in the $(r, 1)$-th position and 0 elsewhere. Since $\mathbb{K}ab + M = A$, it follows that $\mathcal{J}$ contains each matrix that has any element of A in the $(r, 1)$-th position. Hence $\mathcal{J} = \mathbb{M}_n(A)$, and so $\mathcal{M}$ is a maximal left ideal in $\mathbb{M}_n(A)$. □

The theory of non-associative normed algebras is covered in [**2**, **3**]. Proposition 1.2, the equivalence of clauses (a) and (b) in Proposition 1.4, Corollary 1.5, and Propositions 1.7 and 1.8 also hold if the requirement of associativity in the definition of an 'algebra' be removed. However the equivalence of (a) and (c) in Proposition 1.4 does not necessarily hold in the non-associative case: see [**2**, Corollary 3.6.60].

An algebra A that is also a topological linear space is a *topological algebra* if the product $(a, b) \mapsto ab,\ A \times A \to A$, is continuous. A *Fréchet algebra* is a complete, metrizable topological algebra such that there is a base of neighbourhoods of the origin consisting of sets that are absolutely convex and closed under products. A topological algebra A is a *Q-algebra* if $q - \mathrm{Inv} A$ is open in A; of course, every Banach algebra is a Q-algebra [**2**, Example 3.6.42]. The radical of a Q-algebra is a closed ideal in A. For these definitions and facts, see [**2**, **4**].

Let A be a topological algebra. Then it is obvious that the closure of each left ideal in A is also a left ideal, and so a maximal left ideal in A is either closed or dense in A.

The following theorem is elementary and standard [**4**, Theorem 2.2.28(i)].

THEOREM 1.9. *Every maximal modular left ideal in a Q-algebra is closed.* □

The following corollary is immediate from Proposition 1.3 and Theorem 1.9.

COROLLARY 1.10. *Let A be a Q-algebra, and let M be a maximal left ideal in A. For each non-empty subset S of A, the set*

$$J_S = \{a \in A : aS \subset M\}$$

is a closed left ideal in A. □

On the other hand there are trivial examples of Banach algebras that have maximal left ideals that are not closed.

EXAMPLE 1.11. Let E be an infinite-dimensional Banach space. Then E has a dense subspace F that has codimension 1 in E. The space E is a commutative Banach algebra with respect to the zero product, and F is a maximal (left) ideal in this algebra such that F is not closed. □

We now give a modification of the above example that shows that a Banach algebra that factors may have a dense maximal left ideal of codimension 1.

EXAMPLE 1.12. Let G be any linear space, and set $A = \mathbb{C} \times \mathbb{C} \times G$. Define

$$(\alpha, \zeta, x)(\beta, \eta, y) = \zeta(\beta, \eta, y) \quad (\alpha, \beta, \zeta, \eta \in \mathbb{C},\ x, y \in G)\,.$$

Then A is an associative algebra with respect to this product. Set

$$M = \{(0, \eta, y) : \eta \in \mathbb{C},\ y \in G\}\,.$$

Then M is a left ideal of codimension 1 in the algebra A, and so M is a maximal left ideal. The element $p = (0, 1, 0)$ is a left identity for A, and so the algebra A factors.

Now suppose that E is an infinite-dimensional, complex Banach space. Let $\lambda \in E'$ with $\|\lambda\| = 1$, and choose $e_2 \in E$ with $\lambda(e_2) = 1$. Then $E = \mathbb{C}e_2 \oplus \ker \lambda$, and

$$|\zeta| = |\lambda(\zeta e_2 + y)| \leq \|\zeta e_2 + y\| \quad (\zeta \in \mathbb{C},\ y \in \ker \lambda)\,.$$

Next, choose a dense linear subspace G of $\ker\lambda$ of codimension 1, say $\ker \lambda = \mathbb{C}e_1 \oplus G$, and set $F = \mathbb{C}e_2 + G$, so that F is a dense linear subspace of E of codimension 1. We have

$$|\zeta| \leq \|\alpha e_1 + \zeta e_2 + x\| \quad (\alpha, \zeta \in \mathbb{C},\ x \in G)\,.$$

The linear bijection $\alpha e_1 + \zeta e_2 + x \mapsto (\alpha, \zeta, x),\ E \to A$, identifies F with M and transfers the norm from E to A, so that M is dense in the Banach space $(A, \|\cdot\|)$. For $\alpha, \beta, \zeta, \eta \in \mathbb{C}$ and $x, y \in G$, we have

$$\|(\alpha, \zeta, x)(\beta, \eta, y)\| = |\zeta|\,\|(\beta, \eta, y)\| \leq \|(\alpha, \zeta, x)\|\ \|(\beta, \eta, y)\|\ ,$$

and so $(A, \|\cdot\|)$ is a Banach algebra.

Thus there is a Banach algebra A and $p \in A$ such that $pa = a$ $(a \in A)$, so that $A^{[2]} = A$, and such that A has a dense maximal left ideal of codimension 1.

Clearly every linear subspace of A is a left ideal, so that the maximal left ideals of A are just the subspaces of codimension 1. However $H := \{(\alpha, 0, x) : \alpha \in \mathbb{C},\ x \in G\}$ is clearly the unique maximal modular left ideal (with right modular identity $u = (0, 1, 0)$). Thus $\operatorname{rad} A = H$, and so A is far from being semi-simple. □

EXAMPLE 1.13. A maximal modular left ideal in a (non-commutative) Banach algebra does not necessarily have codimension 1. For example, let E be a complex Banach space, and consider the unital Banach algebra $\mathcal{B}(E)$ of all bounded linear operators on E. Set

$$M_x = \{T \in \mathcal{B}(E) : Tx = 0\}\,,$$

where x is a non-zero element of E. Then M_x is a singly-generated maximal left ideal of $\mathcal{B}(E)$ [**7**, Proposition 2.4] and M_x is closed in $\mathcal{B}(E)$. Take $n \in \mathbb{N}$, and set $E = \mathbb{C}^n$. Then we obtain closed, maximal left ideals in $\mathbb{M}_n$ of codimension n. Now suppose that E is an infinite-dimensional space. Then M_x has infinite codimension in $\mathcal{B}(E)$ for each non-zero $x \in E$. □

These examples suggested the possibility that every maximal left ideal in a Banach algebra is either closed or of codimension equal to 1. We shall show in §4 that this is not the case.

2. *Commutative algebras*

Suppose that M is a maximal modular ideal in a commutative algebra A over $\mathbb{C}$. Then A/M is a field containing $\mathbb{C}$. In the case where A is a Banach algebra, M is necessarily closed, and so A/M is a Banach algebra. By the Gel'fand–Mazur theorem, we have $A/M \cong \mathbb{C}$, and so M is the kernel of a continuous character. In fact, the Gel'fand–Mazur theorem applies to Fréchet algebras (and more general topological algebras) [**4**, Theorem 2.2.42], and so each closed, maximal modular ideal in a commutative Fréchet algebra is the kernel of a continuous character. (It is a formidable open question, called *Michael's problem*, whether all characters on each commutative Fréchet algebra are automatically continuous.)

Now suppose that A is a commutative, unital Fréchet algebra. Then a maximal ideal (which is necessarily modular) in A is not necessarily either closed or of finite codimension, as the following example, which essentially repeats [**4**, Proposition 4.10.27], shows.

EXAMPLE 2.1. Let $O(\mathbb{C})$ denote the space of entire functions on $\mathbb{C}$. This is a commutative, unital algebra for the pointwise algebraic operations, and it is a Fréchet algebra with respect to the topology of uniform convergence on compact subsets of $\mathbb{C}$. It is standard that each maximal ideal M of codimension 1 in $O(\mathbb{C})$ is closed and has the property that there exists $z \in \mathbb{C}$ such that

$$M = M_z := \{f \in O(\mathbb{C}) : f(z) = 0\}\,.$$

Let I be the set of functions $f \in O(\mathbb{C})$ such that $f(n) = 0$ for each sufficiently large $n \in \mathbb{N}$. Clearly I is an ideal in $O(\mathbb{C})$, and it is easy to see that I is dense in $O(\mathbb{C})$. Since $O(\mathbb{C})$ has an identity, I is contained in a maximal modular ideal, say M, of $O(\mathbb{C})$. The ideal M is dense in $O(\mathbb{C})$, but M is not of the form M_z for any $z \in \mathbb{C}$, and so M does not have codimension 1 in $O(\mathbb{C})$. It follows from [**4**, Theorem 1.5.30], that M does not have finite codimension; the quotient A/M is a 'large field'. □

PROPOSITION 2.2. *Let A be a Q-algebra, and let M be a maximal left ideal in A. Suppose that M is also a right ideal in A and that M does not contain A^2. Then M is closed in A.*

Proof. By Corollary 1.10, the set $J_A = \{a \in A : aA \subset M\}$ is a closed left ideal in A. Since M is a right ideal in A, we have $M \subset J_A$. Further, $J_A \neq A$ because $A^2 \not\subset M$, and so $J_A = M$. Hence M is closed in A. □

COROLLARY 2.3. *Let A be a commutative Q-algebra, and suppose that M is a dense maximal ideal in A. Then $A^2 \subset M$.* □

THEOREM 2.4. *Let A be a commutative, normed Q-algebra over a field $\mathbb{K}$, and suppose that M is a maximal ideal in A.*

(i) *Suppose that $\mathbb{K} = \mathbb{C}$. Then M has codimension 1 in A. Further, either $A/M \cong \mathbb{C}$ and M is closed in A or $A^2 \subset M$.*

(ii) *Suppose that $\mathbb{K} = \mathbb{R}$. Then M has codimension 1 or 2 in A. Further, M is closed in A when M has codimension 2.*

Proof. Suppose that $A^2 \subset M$. By Proposition 1.2, M has codimension 1 in A.

Now consider the case where $A^2 \not\subset M$, so that $A^2 + M = A$. By Corollary 2.3, M is closed in A, and then A/M is a normed, commutative, simple algebra, and hence a normed extension field over $\mathbb{K}$, as in [**1**, Lemma 30.2]. It follows from the Gel'fand–Mazur theorem, as in [**2**, Corollary 1.1.43 and Proposition 2.5.40], that A/M is isomorphic to $\mathbb{C}$ when $\mathbb{K} = \mathbb{C}$ and to $\mathbb{R}$ or $\mathbb{C}$ when $\mathbb{K} = \mathbb{R}$. □

The above theorem does not necessarily apply to non-associative algebras. Indeed, take A to be $\mathbb{M}_2$, with the product given by $a \bullet b = (ab + ba)/2$ for $a, b \in A$. Then A is a complete, normed, commutative, non-associative algebra over $\mathbb{C}$. Further, $\{0\}$ is a maximal ideal of A [**2**, Proposition 3.6.11(i)], and this ideal has codimension 4 in A.

We noted in Corollary 1.5 that a commutative, radical Banach algebra R has a maximal ideal if and only if $R^2 \subsetneq R$. There are many examples of commutative, radical Banach algebras R such that $R^2 = R$, and hence such that R has no maximal ideals. For example, it follows from Cohen's factorization theorem (see [**1**, Theorem 11.10] and [**4**, Theorem 2.9.24]) that each Banach algebra A with a bounded approximate identity necessarily factors. In particular, this is the case for the Volterra algebra $\mathcal{V}$, which is the space $L^1(\mathbb{I})$ taken with the (truncated) convolution product $\star$ given by

$$(f \star g)(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s \quad (t \in \mathbb{I})$$

for $f, g \in \mathcal{V}$; this is a commutative, radical Banach algebra with a bounded approximate identity, and so $\mathcal{V}^{[2]} = \mathcal{V}$. There are also examples which are integral domains; see [**4**, §4.7]. An example of a commutative, separable, radical Banach algebra R with $R^{[2]} = R$, but such that R does not have a bounded approximate identity, is given in [**5**].

The following example shows that there are commutative, radical Banach algebras R such that $\overline{R^2} = R$, even $\overline{R^{[2]}} = R$, but such that R does have a (dense) maximal ideal, necessarily of codimension 1.

EXAMPLE 2.5. Define $R = \{f \in C(\mathbb{I}) : f(0) = 0\}$, taken with the uniform norm $|\cdot|_{\mathbb{I}}$ and the above truncated convolution product. Then R is a commutative, radical Banach algebra. By [**3**, Example 8.4.41], R is topologically nilpotent.

For $\alpha > 0$, set

$$I_\alpha = \left\{ f \in R : \lim_{t \to 0+} f(t)/t^\alpha = 0 \right\},$$

so that I_α is a linear subspace of R; in fact, it is immediate that I_α is an ideal in $(R, \star)$. Thus $I := \bigcup\{I_\alpha : \alpha > 0\}$ is a (proper) ideal in R. Take $f, g \in R$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|g(s)| < \varepsilon \ (0 \le s \le \delta)$, and so, for $0 \le t \le \delta$, we have

$$|(f \star g)(t)| \le |f|_{\mathbb{I}} \int_0^t |g(s)|\,\mathrm{d}s < \varepsilon t\,|f|_{\mathbb{I}},$$

whence $f \star g \in I_1 \subset I$. Hence $R^2 \subset I$, and so I is contained in a maximal ideal in R. By Proposition 1.2 and Corollary 1.5, every maximal ideal in R has codimension 1 and contains R^2.

For $n \in \mathbb{N}$, take $e_n \in R$ such that $\operatorname{supp} e_n \subset [0, 1/n]$, such that $e_n(t) \ge 0 \ (t \in \mathbb{I})$, and such that $\int_0^1 e_n(t)\,\mathrm{d}t = 1$. Then (e_n) is a sequence in R. Take $f \in R$. For $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $|f(t-s) - f(t)| < \varepsilon$ for $t \in \mathbb{I}$ and $s \in [0, 1/n_0] \cap [0, t]$ and such that $|f(r)| < \varepsilon$ for $r \in [0, 1/n_0]$. Take $n \ge n_0$. Then, for $t \in [1/n, 1]$, we have

$$|(f \star e_n)(t) - f(t)| \le \int_0^{1/n} |f(t-s) - f(t)|\,e_n(s)\,\mathrm{d}s < \varepsilon,$$

and, for $t \in [0, 1/n]$, we have $|(f \star e_n)(t) - f(t)| \leq 2\varepsilon$. Hence $|f \star e_n - f|_{\mathbb{I}} \leq 2\varepsilon$, and so $\lim_{n\to\infty} f \star e_n = f$ in R. This shows that (e_n) is an approximate identity in R; in particular, $\overline{R^{[2]}} = R$. Thus every maximal ideal in R is dense in R. □

3. *Non-commutative algebras*

We now consider some conditions on a Banach algebra A that imply that every (or at least some) maximal left ideal in A is either closed or of codimension 1. The first theorem of the section follows immediately from Propositions 1.2 and 2.2.

THEOREM 3.1. *Let A be a Q-algebra. Suppose that M is a maximal left ideal and a right ideal in A. Then either M is closed in A or M has codimension 1 in A.* □

PROPOSITION 3.2. *Let A be a Q-algebra. Suppose that M is a maximal left ideal in A such that $A\mathfrak{Z}(A) \not\subset M$. Then M is closed in A.*

Proof. Set $Z = \mathfrak{Z}(A)$. By Corollary 1.10, the set $J_Z = \{a \in A : aZ \subset M\}$ is a closed left ideal in A, and $M \subset J_Z$. Further, $J_Z \neq A$ because $AZ \not\subset M$. Thus $M = J_Z$ is closed in A. □

PROPOSITION 3.3. *Let A be a topologically nilpotent Banach algebra. Then A has maximal left ideals, and every maximal left ideal contains A^2 and has codimension 1.*

Proof. By [**10**, Theorem 4.8.9] (see also [**2**, Proposition 4.4.59(i)] and [**3**, Proposition 8.4.56 and Remark 8.4.67]), A is radical and $A^2 \subsetneq A$. By Corollary 1.5, A contains maximal left ideals, and every maximal left ideal in A contains A^2. By Proposition 1.2, each maximal left ideal in A has codimension 1. □

The definition of a topologically nilpotent Banach algebra can be suitably extended to the non-associative setting in such a way that Proposition 3.3 still holds; see [**3**, Definition 8.4.10 and p. 620].

PROPOSITION 3.4. *Let A be a Banach algebra over $\mathbb{C}$ that is separable as a Banach space, and suppose that A^2 has countable codimension in A and that $A^2 \subsetneq A$. Then A contains a maximal left ideal that is closed and of codimension 1 in A.*

Proof. By a theorem of Loy given as [**4**, Theorem 2.2.16], A^2 is closed and of finite codimension in A. By Proposition 1.2, there is a maximal left ideal M in A that has codimension 1 in A and contains A^2. Clearly M is closed in A. □

Let $(E, \|\cdot\|)$ be a Banach space. Then a *null sequence in E* is a sequence (x_n) in E such that $\lim_{n\to\infty} \|x_n\| = 0$; the space of null sequences in E is denoted by $c_0(E)$, and $c_0(E)$ is itself a Banach space for the norm defined by

$$\|(x_n)\| = \sup\{\|x_n\| : n \in \mathbb{N}\} \quad ((x_n) \in c_0(E)).$$

Let A be a Banach algebra. Then $c_0(A)$ is a Banach right A-module for the action defined by $(a_n) \cdot a = (a_n a)$, and *null sequences factor* (on the right) in A if, for each (a_n) in $c_0(A)$, there exist $a \in A$ and (b_n) in $c_0(A)$ with $a_n = b_n a$ $(n \in \mathbb{N})$. It follows from Cohen's factorization theorem [**4**, Corollary 2.9.29] that null sequences factor for each Banach algebra that has a bounded right approximate identity (but the converse does not necessarily hold [**5**]).

The following result is [**4**, Proposition 2.6.13].

PROPOSITION 3.5. *Let A be a Banach algebra for which null sequences factor. Then every maximal left ideal in A is closed.*

Proof. Let M be a maximal left ideal in A.

Take $a \in A$ and (a_n) in M such that $\lim_{n\to\infty} a_n = a$. By hypothesis, there exist $b, b_0 \in A$ and $(b_n) \in c_0(A)$ with $a = b_0 b$ and $a - a_n = b_n b$ $(n \in \mathbb{N})$. Set $J = \{x \in A : xb \in M\}$. By Proposition 1.3, either $J = A$ or J is a maximal modular left ideal in A; in either case, J is closed in A. It follows that $(b_0 - b_n)b = a_n \in M$ $(n \in \mathbb{N})$, and so $b_0 = \lim_{n\to\infty}(b_0 - b_n) \in J$. Thus $a \in M$, and so M is closed. □

COROLLARY 3.6. *Let A be a C^*-algebra. Then every maximal left ideal in A is closed.*

Proof. Every C^*-algebra has a bounded approximate identity; for example, see [**2**, Proposition 3.5.23] or [**4**, Lemma 3.2.20]. □

Thus, to find a maximal left ideal M in a Banach algebra A that is neither closed nor of codimension 1, one must at least construct an example A without a bounded right approximate identity and with a maximal left ideal M such that $A^2 \not\subset M$, such that $A\mathfrak{Z}(A) \subset M$, and such that M is not also a right ideal.

We shall now see that the question that we are considering is related to an 'automatic continuity' question. (See [**4**].)

THEOREM 3.7. *Let A be a Banach algebra. Then the following conditions are equivalent:*

(a) *each maximal left ideal M in A such that $A^2 \not\subset M$ is closed in A;*

(b) *each A-module homomorphism from A into a simple Banach left A-module is automatically continuous.*

Proof. (a) ⇒ (b) Let E be a simple Banach left A-module, and let $\theta : A \to E$ be an A-module homomorphism. We may suppose that $\theta \neq 0$. Thus θ is surjective and $M := \ker\theta$ is a maximal left ideal in A. Clearly there exist $a, b \in A$ with $\theta(ab) = a \cdot \theta(b) \neq 0$, and so $A^2 \not\subset M$. By (a), M is closed in A. Thus E is a Banach left A-module with respect to the quotient norm on E, and so, by Theorem 1.1, this norm is equivalent to the given norm on E. Hence θ is continuous.

(b) ⇒ (a) Let M be a maximal left ideal in A with $A^2 \not\subset M$. Then $E := A/M$ is a simple left A-module. By Theorem 1.1, E is a Banach left A-module with respect to a norm that is equivalent to the quotient norm on E. The quotient map $q : A \to A/M$ is an A-module homomorphism from A onto E, and so $M = \ker q$ is closed in A. □

4. *Examples*

We shall now construct Banach algebras A each having a maximal left ideal M such that M is neither closed nor of codimension 1 in A. Indeed, given $n \in \mathbb{N}$, there are such examples such that M has codimension n in A.

THEOREM 4.1. *Let $n \in \mathbb{N}$. Then there is a Banach algebra $\mathcal{A}$ with a left identity, so that $\mathcal{A}$ factors, and such that $\mathcal{A}$ has a dense maximal left ideal of codimension n.*

Proof. Let A be the Banach algebra described in Example 1.12, and set $\mathcal{A} = \mathbb{M}_n(A)$, so that $\mathcal{A}$ is a Banach algebra. Take $\mathcal{M}$ to be as specified in Proposition 1.8, so that $\mathcal{M}$ is a maximal left ideal of codimension n in $\mathcal{A}$; clearly $\mathcal{M}$ is dense in $\mathcal{A}$. Take P to be the matrix in $\mathcal{A}$ with the element p in each diagonal position and with 0 in all other positions. Then P is a left identity for $\mathcal{A}$. In particular, A factors. □

However, the algebra $\mathcal{A}$ of the above theorem is not semi-simple. We now seek examples of Banach algebras $\mathcal{A}$ with dense maximal left ideals of codimension n such that $\mathcal{A}$ is semi-simple and has some other properties.

DEFINITION 4.2. Let A be an algebra with a character φ. Then M_φ is the kernel of φ and

$$J_\varphi = \mathrm{lin}\,\{ab - \varphi(a)b : a, b \in A\}\,.$$

Certainly J_φ is a right ideal in A and $M_\varphi A \subset J_\varphi \subset M_\varphi$.

LEMMA 4.3. *Let A be an algebra with a character φ. Suppose that there exists $u \in A \setminus M_\varphi$ with $u^2 = u$. Then*

$$J_\varphi = M_\varphi^2 + M_\varphi u + (1-u)M_\varphi\,. \tag{4.1}$$

Proof. Clearly $M_\varphi^2 + M_\varphi u + (1-u)M_\varphi \subset J_\varphi$.

Now take $a, b \in A$, say $a = \alpha u + x$ and $b = \beta u + y$, where $\alpha, \beta \in \mathbb{K}$ and $x, y \in M_\varphi$. Then

$$ab - \varphi(a)b = xy + \beta xu - \alpha(1-u)y \in M_\varphi^2 + M_\varphi u + (1-u)M_\varphi\,,$$

and so $J_\varphi \subset M_\varphi^2 + M_\varphi u + (1-u)M_\varphi$. This gives the result. □

LEMMA 4.4. *Let A be an algebra with a character φ. Suppose that λ is a non-zero linear functional on A such that $\lambda \mid J_\varphi = 0$. Then $M := \ker\lambda$ is a maximal left ideal in A such that $A^2 \not\subset M$. Further, suppose that $\lambda \mid M_\varphi \neq 0$. Then M is not a modular left ideal.*

Proof. Since $\lambda \mid J_\varphi = 0$, it follows that $\lambda(ab) = \varphi(a)\lambda(b)\ \ (a, b \in A)$, and so M is a proper left ideal in A of codimension 1 such that $A^2 \not\subset M$. Hence M is a maximal left ideal in A.

Suppose that $\lambda \mid M_\varphi \neq 0$, and assume that u is a right modular identity for M. Then

$$\lambda(a) - \varphi(a)\lambda(u) = \lambda(a - au) = 0 \quad (a \in A)\,,$$

and so $\lambda \mid M_\varphi = 0$, a contradiction. So M is not a modular left ideal. □

We continue to use the above notation in the next theorem.

THEOREM 4.5. *Let A be a topological algebra with a character φ, and suppose that J_φ is not closed in A. Then A contains a maximal left ideal M such that $A^2 \not\subset M$, such that M has codimension 1 in A, and such that M is dense in A.*

Proof. Since J_φ is not closed in A, there is a (discontinuous) linear functional λ on A such that $\lambda \mid J_\varphi = 0$ and $\lambda \mid \overline{J_\varphi} \neq 0$. By Lemma 4.4, $M := \ker \lambda$ is a maximal left ideal in A such that $A^2 \not\subset M$, and M has codimension 1 in A. Clearly M is not closed in A, and so M is dense in A. □

Note that M is not a modular left ideal and that, by Proposition 2.2, M cannot be a right ideal in A in the case where A is a Q-algebra.

We now give a construction of a Banach algebra from a certain 'starting point', as follows.

Starting point: We suppose that we have a Banach algebra $(I, \|\cdot\|_I)$ over a field $\mathbb{K}$ such that $I^2 \subsetneq \overline{I^2} = I$, and we take $B = I^\sharp$ to be the unitization of I, so that B is a unital Banach algebra, with identity e_B, say, and I is a maximal ideal in B.

Several examples that show that we can reach the starting point (with algebras I with various additional properties) will be given in Examples 4.7, below. We shall note that, for some of these examples, the starting ideal I is a Banach $*$-algebra and a primitive algebra.

Construction: From our starting point, we consider the Banach algebra $\mathfrak{B} = \mathbb{M}_2(B)$, so that $\mathfrak{B}$ is also a unital Banach algebra. Set $\mathfrak{I} = \mathbb{M}_2(I)$. Then $\mathfrak{I}$ is a closed ideal in $\mathfrak{B}$ (of codimension 4).

Consider the elements

$$P = \begin{pmatrix} e_B & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & e_B \end{pmatrix}$$

in $\mathfrak{B}$. Then $P^2 = P$, $Q^2 = Q$, $PQ = QP = 0$, and $P + Q$ is the identity of $\mathfrak{B}$.

Next, consider the subset $\mathfrak{A} = \mathfrak{I} + \mathbb{C}P$ in $\mathfrak{B}$. Symbolically, $\mathfrak{A}$ has the form

$$\mathfrak{A} = \begin{pmatrix} B & I \\ I & I \end{pmatrix}.$$

Then $\mathfrak{A}$ is a closed subalgebra of $\mathfrak{B}$, and $\mathfrak{I}$ is a maximal ideal in $\mathfrak{A}$ of codimenison 1; the quotient map $\varphi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} = \mathbb{K}(P + \mathfrak{I})$ is a character on $\mathfrak{A}$.

We define M_φ and J_φ (in relation to $\mathfrak{A}$ and the character φ) as in Definition 4.2. Then we see that $\mathfrak{I} = M_\varphi$ and that, by Lemma 4.3,

$$J_\varphi = \mathfrak{I}^2 + \mathfrak{I}P + Q\mathfrak{I} \subset P\mathfrak{I}^2 Q + P\mathfrak{I}P + Q\mathfrak{I} \subset \mathfrak{I}\,, \tag{4.2}$$

and so $\mathfrak{I}^2 \subset J_\varphi \subset \mathfrak{I} = M_\varphi$.

We *claim* that $\mathfrak{I}^2$ is dense in M_φ. Indeed, given $\varepsilon > 0$ and $x \in I$, there exist $n \in \mathbb{N}$ and $u_1, \dots, u_n, v_1, \dots, v_n \in I$ such that $\|x - \sum_{i=1}^n u_i v_i\|_I < \varepsilon$ because $\overline{I^2} = I$. It follows that

$$\left\| \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} - \sum_{i=1}^n \begin{pmatrix} u_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_i & 0 \\ 0 & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} x - \sum_{i=1}^n u_i v_i & 0 \\ 0 & 0 \end{pmatrix} \right\| < \varepsilon\,,$$

with similar calculations in the other three positions. The claim follows. Hence $\overline{J_\varphi} = M_\varphi$.

We also *claim* that $J_\varphi \neq M_\varphi$. Assume towards a contradiction that $J_\varphi = M_\varphi$. Then it follows from (4.2) that

$$\mathfrak{I} = P\mathfrak{I}P + P\mathfrak{I}Q + Q\mathfrak{I} = P\mathfrak{I}^2 Q + P\mathfrak{I}P + Q\mathfrak{I}\,.$$

Since $\mathfrak{I} = P\mathfrak{I}P \oplus P\mathfrak{I}Q \oplus Q\mathfrak{I}$, this implies that $P\mathfrak{I}Q = P\mathfrak{I}^2Q$. However, take $x \in I \setminus I^2$, and consider the element

$$\mathbf{x} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \mathfrak{I}\,.$$

Since $P\mathbf{x}Q = \mathbf{x}$, we see that $\mathbf{x} \in P\mathfrak{I}Q$. But every element of $P\mathfrak{I}^2Q$ has the form

$$\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix},$$

where $u \in I^2$, and so $\mathbf{x} \notin P\mathfrak{I}^2Q$, the required contradiction. Thus the claim holds.

We conclude at this stage that

$$\mathfrak{I}^2 \subset J_\varphi \subsetneq \overline{J_\varphi} = \mathfrak{I} = M_\varphi\,.$$

Suppose that the starting ideal I is a primitive algebra. Then the corresponding algebra $\mathfrak{A}$ is a primitive algebra by Proposition 1.7. In the case where I is a Banach $*$-algebra, the corresponding algebra $\mathfrak{A}$ is also a Banach $*$-algebra.

We note that we could have defined $\mathfrak{B}$ and $\mathfrak{I}$ as the spaces of upper-triangular matrices in $\mathbb{M}_2(B)$ and $\mathbb{M}_2(I)$, respectively; the same arguments would lead to the same conclusion, save that $\mathfrak{I}$ would now have codimension 3 in $\mathfrak{B}$ and the corresponding algebra $\mathfrak{A}$ would not necessarily be primitive or a Banach $*$-algebra when I has these properties.

The following theorem now follows from Theorems 3.7 and 4.5.

THEOREM 4.6. *The Banach algebra $\mathfrak{A}$ contains a maximal left ideal $\mathfrak{M}$ such that $\mathfrak{A}^2 \not\subset \mathfrak{M}$, such that $\mathfrak{M}$ has codimension 1 in $\mathfrak{A}$, and such that $\mathfrak{M}$ is dense in $\mathfrak{A}$. There is a discontinuous $\mathfrak{A}$-module homomorphism from $\mathfrak{A}$ into a simple Banach left $\mathfrak{A}$-module.* □

We now give various examples that show that we can reach our starting point. Recall that we require Banach algebras I such that I^2 is dense in I and $I^2 \subsetneq I$.

EXAMPLES 4.7. (i) Let $I = (\ell^p, \|\cdot\|_p)$, where $1 \le p < \infty$, taken with the coordinatewise product, as in [**4**, Example 4.1.42], so that I is a commutative, semi-simple Banach algebra. Clearly $I^{[2]}$ is dense in I and $I^2 = \ell^{p/2} \subsetneq I$. Further, I is a Banach $*$-algebra for the involution $(\alpha_n) \mapsto (\overline{\alpha}_n)$.

(ii) Take K to be a non-empty, compact, metric space without isolated points, and take $\alpha \in (0,1)$. Let B be the Lipschitz algebra $\mathrm{lip}_\alpha K$, as in [**4**, §4.4], so that B is a commutative, unital, semi-simple Banach algebra, and take I to be any maximal ideal of B. Then, by [**4**, Theorem 4.4.30, (i) and (iv)], I has an approximate identity, so that $I^{[2]}$ is dense in I, and I^2 has infinite codimension in I. Again, I is a Banach $*$-algebra for the involution $f \mapsto \overline{f}$.

(iii) Take I to be the commutative, radical Banach algebra R of Example 2.5. Then I has the required properties. Again, I is a Banach $*$-algebra for the involution $f \mapsto \overline{f}$.

(iv) Let X be a compact plane set, and let $R(X)$ be the usual uniform algebra on X.

For example, consider the 'road-runner' set, defined as follows [**8**, p. 52]. For $x > 0$ and r with $0 < r < x$, set $D(x,r) = \{z \in \mathbb{C} : |z - x| < r\}$. Let X be the compact set in $\mathbb{C}$ obtained by deleting from $\overline{\mathbb{D}}$ a sequence $(D_n = D(x_n, r_n))$ of open discs, where we ensure that the closed discs $\overline{D_n}$ are contained in $\overline{\mathbb{D}}$, are pairwise-disjoint, and that (x_n) decreases to 0. Consider the maximal ideal

$$I = M_0 = \{f \in R(X) : f(0) = 0\}\,.$$

It follows from a result of Hallstrom [**9**, p. 156] that M_0^2 is dense in M_0 if and only if $\sum_{i=1}^\infty r_i/x_i^2 = \infty$, and it follows from Melnikov's Criterion [**8**, Theorem VIII.4.5] that $M_0^2 = M_0$

if and only if $\sum_{i=1}^{\infty} r_i/x_i = \infty$. Thus there is a choice of (x_n) and (r_n) such that M_0 is a uniform algebra satisfying the required conditions on I.

(v) Let H be an infinite-dimensional Hilbert space, and take I to be the non-commutative Banach algebra of all Hilbert–Schmidt operators on H, with the standard norm on I. Then $I^2 = I^{[2]}$ is the space of trace-class operators. Here I is a primitive algebra ([**2**, Example 3.6.40], [**4**, Theorem 2.5.8(i)]) and a Banach $*$-algebra, so that the corresponding algebra $\mathfrak{A}$ has the same properties. For details and definitions for this example, see [**12**].

(vi) Let E be an infinite-dimensional Banach space, and let $I = \mathcal{N}(E)$, the nuclear operators on E, so that I is a non-commutative Banach algebra with respect to the nuclear norm [**4**, §2.5]. Then $I^{[2]}$ is dense in I and I^2 has infinite codimension in I [**6**], as required. Again, I is a primitive algebra, and so the corresponding algebra $\mathfrak{A}$ is also a primitive algebra. □

We can combine the above results to exhibit our main example.

THEOREM 4.8. *Let $n \in \mathbb{N}$. Then there is a Banach algebra $\mathcal{A}$ with a maximal left ideal $\mathcal{M}$ such that $\mathcal{M}$ is dense in $\mathcal{A}$ and has codimension n in $\mathcal{A}$. In the case where the starting algebra I is primitive, $\mathcal{A}$ is also primitive, and, in the case where the starting algebra I is a Banach $*$-algebra, $\mathcal{A}$ is also a Banach $*$-algebra.*

Proof. By Theorem 4.6, there is a Banach algebra $\mathfrak{A}$ with a maximal left ideal $\mathfrak{M}$ such that $\mathfrak{A}^2 \not\subset \mathfrak{M}$, such that $\mathfrak{M}$ has codimension 1 in $\mathfrak{A}$, and such that $\mathfrak{M}$ is dense in $\mathfrak{A}$. Set $\mathcal{A} = \mathbb{M}_n(\mathfrak{A})$, and take $\mathcal{M}$ to be the corresponding maximal left ideal in $\mathcal{A}$ specified in Proposition 1.8. Then $\mathcal{M}$ has codimension n in $\mathcal{A}$, and it is clear that $\mathcal{M}$ is dense in $\mathcal{A}$.

Suppose that the starting algebra I is primitive or a Banach $*$-algebra. Then we have noted that $\mathfrak{A}$ and $\mathcal{A}$ both have the corresponding properties. □

COROLLARY 4.9. *Let $n \in \mathbb{N}$. Then there is a primitive Banach $*$-algebra $\mathcal{A}$ with a maximal left ideal $\mathcal{M}$ such that $\mathcal{M}$ is dense in $\mathcal{A}$ and has codimension n in $\mathcal{A}$.* □

In particular, the algebra $\mathcal{A}$ is semi-simple.

As we said, we do not know the answer to the following question:

Question 1 Is there a Banach algebra that has a dense maximal left ideal of infinite codimension?

As in Theorem 3.7, the existence of such an example is equivalent to the existence of a Banach algebra A that has a discontinuous left A-module homomorphism into an infinite-dimensional, simple Banach left A-module.

We shall now show that, given $n \in \mathbb{N}$, we can modify the above example to obtain a semi-simple Banach algebra $\mathcal{A}$ and a dense maximal left ideal of codimension n and, additionally, such that $\mathcal{A}$ factors weakly.

Take I, B, $\mathfrak{B} = \mathbb{M}_2(B)$, and elements P and Q in $\mathfrak{B}$ as before, but now set

$$\mathfrak{A} = \begin{pmatrix} B & I \\ B & I \end{pmatrix} \quad \text{and} \quad \mathfrak{I} = \begin{pmatrix} I & I \\ B & I \end{pmatrix}.$$

We see that $\mathfrak{A}$ is again a closed subalgebra of $\mathfrak{B}$ and that $\mathfrak{I}$ is a closed maximal ideal in $\mathfrak{A}$ of codimension 1. Further, $\varphi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} = \mathbb{K}(P + \mathfrak{I})$ is still a character on $\mathfrak{A}$. We define M_φ and J_φ (in relation to the new algebra $\mathfrak{A}$ and the character φ) as before. Certainly, equation

(4.2) still holds, and now, as before, $\mathfrak{I}^2$ is dense in $\mathbb{M}_2(I)$. We *claim* that this implies that J_φ is dense in $\mathfrak{I}$. Indeed, choose

$$\mathbf{x} = \begin{pmatrix} 0 & 0 \\ e_B & 0 \end{pmatrix} \in \mathfrak{B}\,,$$

and note that $\mathbf{x} = \mathbf{x}P = \mathbf{x}P - \varphi(\mathbf{x})P \in J_\varphi$. It follows that

$$\mathfrak{I} = \mathbb{M}_2(I) + \mathbb{K}\mathbf{x} = \overline{\mathfrak{I}^2} + \mathbb{K}\mathbf{x} \subset \overline{J_\varphi}\,.$$

Since $J_\varphi \subset \mathfrak{I}$ and $\mathfrak{I}$ is closed in $\mathfrak{A}$, it follows that $\overline{J_\varphi} = \mathfrak{I}$, as claimed. As before, $J_\varphi \neq M_\varphi$, and so we again have a Banach algebra $\mathfrak{A}$ with a character φ such that J_φ is not closed in $\mathfrak{A}$.

We also *claim* that $\mathfrak{A}$ factors weakly. Indeed, take

$$\mathbf{x} = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \in \mathfrak{A}\,,$$

where $x_{1,1}, x_{2,1} \in B$ and $x_{1,2}, x_{2,2} \in I$. Then

$$\mathbf{x} = P\mathbf{x} + \begin{pmatrix} 0 & 0 \\ e_B & 0 \end{pmatrix} \begin{pmatrix} x_{2,1} & x_{2,2} \\ 0 & 0 \end{pmatrix} \in \mathfrak{A}^2\,,$$

as required for the claim.

Now take $n \in \mathbb{N}$, and set $\mathcal{A} = \mathbb{M}_n(\mathfrak{A})$, as before. Then there is a maximal left ideal $\mathcal{M}$ in $\mathcal{A}$ such that $\mathcal{M}$ is dense and has codimension n in $\mathcal{A}$. Again it follows from Proposition 1.7 that we can arrange that $\mathcal{A}$ be primitive and, in particular, semi-simple. The extra point is that now $\mathcal{A}^2 = \mathcal{A}$, and so we have proved the following theorem.

THEOREM 4.10. *Let $n \in \mathbb{N}$. Then there is a semi-simple Banach algebra $\mathcal{A}$ that factors weakly and that has a dense maximal left ideal of codimension n in $\mathcal{A}$.* □

This suggests the following question.

Question 2 Given $n \in \mathbb{N}$, is there a semi-simple Banach algebra A that factors and has a dense maximal left ideal of codimension n in A?

Acknowledgements. The authors were partially supported by the Spanish government grant MTMT2016-76327-C3-2-P; the first and third authors were also partially supported by the Junta de Andalucia grant FQM199; the second author is grateful for generous hospitality in Granada.

References

1. F. F. Bonsall and J. Duncan, *Complete normed algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 80 (Springer–Verlag, New York, 1973).
2. M. Cabrera García and Á. Rodríguez Palacios, *Non-associative normed algebras, Volume 1, The Vidav–Palmer and Gelfand–Naimark theorems*, Encyclopedia of Mathematics and its Applications, 154 (Cambridge University Press, Cambridge, 2014).
3. M. Cabrera García and Á. Rodríguez Palacios, *Non-associative normed algebras, Volume 2, Representation theory and the Zel'manov approach*, Encyclopedia of Mathematics and its Applications, 167 (Cambridge University Press, Cambridge, 2018).
4. H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs, Volume 24 (Clarendon Press, Oxford, 2000).
5. H. G. Dales, J. F. Feinstein, and H. L. Pham, 'Factorization in commutative Banach algebras', submitted.
6. H. G. Dales and H. Jarchow, 'Continuity of homomorphisms and derivations from algebras of approximable and nuclear operators', *Math. Proc. Cambridge Philos. Soc.*, 116 (1994), 465–473.
7. H. G. Dales, T. Kania, T. Kochanek, P. Koszmider, and N. J. Laustsen, 'Maximal left ideals of the Banach algebra of bounded operators on a Banach space', *Studia Math.*, 212 (2013), 245–286.

8. T. W. Gamelin, *Uniform algebras* (Prentice-Hall, Englewood Cliffs, New Jersey, 1969).
9. A. P. Hallstrom, 'On bounded point derivations and analytic capacity', *J. Functional Analysis*, 4 (1969), 153–165.
10. T. W. Palmer, *Banach algebras and the general theory of ∗-algebras, Volume I, Algebras and Banach algebras*, Encyclopedia of Mathematics and its Applications, 49 (Cambridge University Press, Cambridge, 1994).
11. E. Sasiada and P. M. Cohn, 'An example of a simple radical ring', *J. Algebra*, 5 (1967), 373–377.
12. R. Schatten, *Norm ideals of completely continuous operators*, Second printing, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 27 (Springer–Verlag, Berlin–New York, 1970).

M. Cabrera García and Á. Rodríguez Palacios
Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada
Spain

cabrera@ugr.es, apalacio@ugr.es

H. G. Dales
Department of Mathematics and Statistics
University of Lancaster
Lancaster LA1 4YF
United Kingdom

g.dales@lancaster.ac.uk