COBOUNDARY OPERATORS FOR INFINITE FRAMEWORKS

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Dedicated in memory of Richard M. Timoney.

Abstract. We consider, from the point of view of operator theory, a class of infinite matrices in which the matrix entries are determined by an underlying graph structure with accompanying geometric data. This class includes the rigidity matrices of infinite bar-joint frameworks as well as the incidence matrices of infinite directed graphs. We consider the following questions: When do these matrices give rise to bounded operators? Can we compute the operator norm? When are these operators compact? And when are they bounded below?

1. Introduction

Graph rigidity is an interdisciplinary field in which the central aim is to develop theoretical and computational techniques for identifying and characterising forms of rigidity and flexibility in discrete geometric structures. The objects of study can be thought of as an assembly of rigid building blocks, with rotational connecting joints, and are typically categorized by the nature of these blocks and joints; eg. bar-and-joint, body-and-bar and plate-and-hinge frameworks. Constraint systems of these forms are ubiquitous in engineering (eg. trusses, mechanical linkages and deployable structures), in nature (eg. periodic and aperiodic bond-node structures in proteins and materials) and in technology (eg. formation control for autonomous multi-agent systems, sensor network localization, robotics and CAD software) (see for example [2, 3, 9, 10, 18]).

The origins of graph rigidity can be traced back to two seminal results of the 19th century: Cauchy's proof in 1813 that a convex polyhedron in three-dimensional Euclidean space is a (continuously) rigid plate-and-hinge structure [6]; and Maxwell's 1864 observation that the structure graph of a rigid bar-and-joint framework must obey certain counting rules [14]. Dehn's subsequent proof of Cauchy's theorem for simplicial polyhedra developed the now standard analytic and linear methods of *infinitesimal rigidity* [7].

The aim of this article is to understand, from the perspective of operator theory, the *rigidity matrices* which arise from the infinitesimal rigidity theory of infinite bar-joint frameworks. A fundamental issue in this regard is the nature of the null space of the rigidity matrix as a linear transformation on a particular domain of velocity fields. For infinite matrices, this domain can be a normed vector-valued sequence space, such as a Hilbert space of square-summable (finite energy) velocity fields. Moreover, the range of this transformation is a linear space of bar-joint framework

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stresses and so it becomes relevant to determine boundedness properties of the rigidity matrix.

Determining whether an infinite matrix gives rise to a bounded linear operator and, in the affirmative case, computing the operator norm, are in general difficult problems. Schur and Toeplitz are credited with developing the classical theory. For example, Schur's test provides sufficient conditions for ℓ^2-space together with an upper bound for the operator norm, and can be generalised to ℓ^p-space (see for example [11, ch. 5]). Maddox [13] and others have extended some aspects of the classical theory to infinite matrices of operators and we have found these techniques to be particularly useful here. Also of relevance are a series of papers by Mohar and various co-authors in which similar problems are considered for the adjacency matrices of infinite graphs $G = (V, E)$, regarding these infinite matrices as bounded operators on $\ell^2(V)$. See for example the survey article [15]. More recently, Agarwal et. al. [1] have extended these considerations to adjacency matrices on $\ell^p(V)$.

In Section 2, we introduce the notion of a *framework* (G, q) and a *coboundary matrix* $C(G, q)$. These generalise respectively a *directed graph* and an *incidence matrix* commonly used in graph theory and the notion of a *bar-joint framework* and a *rigidity matrix* used in Euclidean space rigidity theory. Also, the k-frame matrices of Whiteley [18] and the rigidity matrices of bar-joint frameworks in non-Euclidean spaces [12] are particular instances of coboundary matrices. We show that, under suitable conditions, coboundary matrices give rise to bounded operators and we either provide a formula for the operator norm, or provide upper and lower bounds for the operator norm. In Section 3, we provide conditions under which the operator norm can be computed by considering subframeworks of the given framework. We also provide necessary and sufficient conditions for coboundary operators to be compact. In Section 4, we provide necessary conditions for a coboundary operator $C(G, q)$ to be bounded below and in particular we show that the graph G cannot be amenable (i.e. have isoperimetric constant 0). On the other hand, coboundary operators with a non-amenable graph structure can be bounded below and, in Section 5, we raise the interesting problem of whether there are natural classes of bar-joint frameworks for which the rigidity matrix gives rise to an operator which is bounded below on various spaces of velocity fields. We also present an example of a bar-joint framework with a non-amenable graph structure for which the associated rigidity matrix determines a bounded operator which is neither compact nor bounded below.

2. Coboundary operators

Throughout this article, $G = (V, E)$ will denote a simple graph (i.e. no loops or multiple edges) with a countably infinite set of vertices. Denote by $E(v)$ the set of edges $e \in E$ which are incident with a vertex $v \in V$ and let $\Delta(G) = \sup_{v\in V} |E(v)|$. The graph G is said to be *locally finite* if $|E(v)| < \infty$ for all $v \in V$ and to have *bounded degree* if $\Delta(G) < \infty$.

Let X and Y be a pair of normed linear spaces over a field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) and denote by $L(X, Y)$ the space of $\mathbb{K}$-linear maps from X to Y. Let $\varphi : V \times V \to L(X, Y)$ be a map with the property that $\varphi(v, w) = -\varphi(w, v)$ whenever $vw \in E$ and $\varphi(v, w) = 0$ whenever $vw \notin E$. We refer to the pair (G, φ) as a *framework*. We will assume throughout the article that X and Y are finite dimensional and we write $\|\varphi\|_\infty = \sup_{vw\in E} \|\varphi(v, w)\|_{op}$ where $\| \cdot \|_{op}$ is the induced operator norm.

Definition 2.1. *A* coboundary matrix *for a framework (G, φ) is a matrix $C(G, \varphi)$ with rows indexed by E, columns indexed by V, and (e, v)-matrix entry,*

$$c_{e,v} = \begin{cases} \varphi(v,w) & \text{if } e = vw, \\ 0 & \text{if } e \notin E(v). \end{cases}$$

Note that a coboundary matrix $C(G, \varphi)$ has the following form,

$$\begin{matrix} \\ \\ e{=}vw \\ \\ \end{matrix} \begin{matrix} & & v & & & & w & & \\ & & \vdots & & & & \vdots & & \\ \cdots & 0 & \varphi(v,w) & 0 & \cdots & 0 & -\varphi(v,w) & 0 & \cdots \\ & & \vdots & & & & \vdots & & \end{matrix}$$

$$\left[\begin{matrix} & & \vdots & & & & \vdots & & \\ \cdots & 0 & \varphi(v,w) & 0 & \cdots & 0 & -\varphi(v,w) & 0 & \cdots \\ & & \vdots & & & & \vdots & & \end{matrix} \right].$$

Moreover, $C(G, \varphi)$ determines in a natural way a linear map $C(G, \varphi) : X^V \to Y^E$, whereby a vector $x = (x_v)_{v \in V} \in X^V$ is mapped to a vector $C(G, \varphi)x \in Y^E$ with e-component,

$$(C(G, \varphi)x)_e = \sum_{\tilde{v} \in V} c_{e,\tilde{v}}(x_{\tilde{v}}) = \varphi(v,w)(x_v - x_w),$$

for each edge $e = vw \in E$.

Example 2.2. *Let G be a directed graph. Define $\varphi : V \times V \to L(\mathbb{R}, \mathbb{R})$ by setting $\varphi(v,w) = 1$ if $vw \in E$ is an edge directed from v to w, $\varphi(v,w) = -1$ if $vw \in E$ is an edge directed from w to v, and $\varphi(v,w) = 0$ if $vw \notin E$. Note that the pair (G, φ) satisfies the conditions of a framework and the associated coboundary matrix $C(G, \varphi)$ is an incidence matrix for G.*

Example 2.3. *A* bar-joint framework *in $\mathbb{R}^d$ is a pair (G, q) where $q : V \to \mathbb{R}^d$ and $q(v) \neq q(w)$ for all edges $vw \in E$. Given such a bar-joint framework, we would like to know if it is* continuously flexible *(i.e. admits a non-trivial continuous motion which preserves the Euclidean distance between pairs of points $q(v)$ and $q(w)$ for all edges $vw \in E$) or* continuously rigid *(i.e. admits only isometric motions).*

A (Euclidean) rigidity matrix for (G, q) is a matrix with rows indexed by E, columns indexed be $V \times \{1, \dots, d\}$ and matrix entries

$$r_{e,(v,k)} = \begin{cases} q(v)_k - q(w)_k & \text{if } e = vw, \\ 0 & \text{if } e \notin E(v), \end{cases}$$

where we write $q(v) = (q(v)_1, \dots, q(v)_d) \in \mathbb{R}^d$ for each $v \in V$. Note that if the columns labelled $(v,1), \dots, (v,d)$ are grouped together for each vertex $v \in V$ then the rigidity matrix $R(G, q)$ has the following form,

$$\begin{matrix} & & (v,1) \cdots (v,d) & & & & (w,1) \cdots (w,d) & & \\ & & \vdots & & & & \vdots & & \\ \cdots & 0 & q(v) - q(w) & 0 & \cdots & 0 & q(w) - q(v) & 0 & \cdots \\ & & \vdots & & & & \vdots & & \end{matrix}$$

$$e{=}vw \left[\begin{matrix} & & \vdots & & & & \vdots & & \\ \cdots & 0 & q(v) - q(w) & 0 & \cdots & 0 & q(w) - q(v) & 0 & \cdots \\ & & \vdots & & & & \vdots & & \end{matrix} \right].$$

If the set of points $\{q(v) : v \in V\}$ affinely spans $\mathbb{R}^d$ then the null space of the linear transformation induced by $R(G, q)$ has dimension at least $\binom{d+1}{2}$. The bar-joint framework (G, q) is said to be infinitesimally rigid *if the dimension of the null space is exactly $\binom{d+1}{2}$. In general, infinitesimal rigidity is a stronger (and more easily verifiable) property than continuous rigidity. (See for example* [3, 9, 18]*).*

Define $\varphi : V \times V \to L(\mathbb{R}^d, \mathbb{R})$ *by setting* $\varphi(v,w)x = (q(v) - q(w)) \cdot x$ *for each* $x \in \mathbb{R}^d$*, if* $vw \in E$*, and* $\varphi(v,w) = 0$ *otherwise. Note that the pair* (G, φ) *satisfies the conditions of a framework and the transformations induced by the associated coboundary matrix* $C(G, \varphi)$ *and the (Euclidean) rigidity matrix* $R(G, q)$ *are isomorphically equivalent.*

Example 2.4. *Let* X *be a finite dimensional real normed linear space. A* bar-joint framework *in* X *is a pair* (G, q) *where* $q : V \to X$ *and* $q(v) \neq q(w)$ *for all edges* $vw \in E$*. As in Example 2.3, we would like to know if* (G, q) *is continuously flexible or continuously rigid with respect to the distance constraints imposed by the norm on* X*. To achieve this we can adopt the following more general approach.*

Recall that a support functional for a point $x \in X \backslash \{0\}$ *is a linear functional* ψ *on* X *which satisfies* $\|\psi\| = \|x\|$ *and* $\|\psi(x)\| = \|x\|^2$*. Also recall that a point in* $X \backslash \{0\}$ *is said to be* smooth *if it has a unique support functional. Suppose* $q(v) - q(w)$ *is a smooth point in* X *for each edge* $vw \in E$ *(note that this assumption is redundant in the Euclidean context). Then we can define* $\varphi : V \times V \to L(X, \mathbb{R})$ *by setting* $\varphi(v,w)$ *to be the unique support functional for* $q(v) - q(w)$ *if* $vw \in E$ *and* $\varphi(v,w) = 0$ *otherwise.*

The pair (G, φ) *satisfies the conditions of a framework. Moreover, the associated coboundary matrix* $C(G, \varphi)$ *plays a similar role to the Euclidean rigidity matrix in Example 2.3. In particular,* $C(G, q)$ *can be used to determine the infinitesimal rigidity and hence rigidity properties of a bar-joint framework in* X*. (See for example* [12] *which considers the special case where* X *is a finite dimensional* ℓ^p*-space).*

2.1. Coboundary operators on ℓ^p-spaces.

In this section, we provide conditions under which a coboundary matrix will determine a bounded linear operator between two normed linear spaces. We also provide formulae, or upper and lower bounds, for the operator norm. The space of bounded linear operators from a normed space Z to a normed space W will be denoted $B(Z, W)$.

For a countable index set I and a normed space X, denote by $\ell^\infty(I; X)$ the space of sequences $x = (x_i)_{i \in I}$ in X with $\sup_{i \in I} \|x_i\| < \infty$ and write $\|x\|_\infty = \sup_{i \in I} \|x_i\|$. Denote by $c_{00}(I; X)$ the vector space of sequences in X with at most finitely many non-zero terms.

Proposition 2.5. *Let* $C(G, \varphi)$ *be a coboundary matrix and let* Z *be a subspace of* $\ell^\infty(V; X)$ *which contains* $c_{00}(V; X)$*. The following statements are equivalent.*

(i) $\varphi : V \times V \to L(X, Y)$ *is a bounded function.*
(ii) $C(G, \varphi) \in B(Z, \ell^\infty(E; Y))$*.*

Moreover, when the above conditions hold, the induced operator norm satisfies,

$$\|C(G, \varphi)\|_{op} = 2\|\varphi\|_\infty.$$

Proof. $(i) \Rightarrow (ii)$. Let $z = (z_v)_{v \in V} \in Z$. Then,

$$\|C(G, \varphi)z\|_\infty = \sup_{vw \in E} \|\varphi(v,w)(z_v - z_w)\| \leq 2\|\varphi\|_\infty \|z\|_\infty.$$

$(ii) \Rightarrow (i)$. Suppose φ is unbounded and let $M > 0$. Then there exists an edge $e = vw \in E$ with $\|\varphi(v,w)\|_{op} > M$. Let $\epsilon > 0$ and choose a unit vector $x \in X$ with $\|\varphi(v,w)x\| > \|\varphi(v,w)\|_{op} - \epsilon$. Consider the unit vector $z = (z_{\tilde{v}})_{\tilde{v} \in V} \in Z$ with,

$$z_{\tilde{v}} = \begin{cases} x & \text{if } \tilde{v} = v, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\|C(G,\varphi)z\|_\infty \geq \|(C(G,\varphi)z)_e\| = \|\varphi(v,w)x\| > M - \epsilon$. Thus $C(G,\varphi)$ is unbounded.

Finally, suppose conditions (i) and (ii) hold. From the proof of $(i) \Rightarrow (ii)$ we have $\|C(G,\varphi)\|_{op} \leq 2\|\varphi\|_\infty$. To show $\|C(G,\varphi)\|_{op} = 2\|\varphi\|_\infty$, let $\epsilon > 0$ and choose an edge $e = vw \in E$ with $\|\varphi(v,w)\|_{op} > \|\varphi\|_\infty - \frac{\epsilon}{4}$. Choose a unit vector $x \in X$ such that $\|\varphi(v,w)x\| > \|\varphi(v,w)\|_{op} - \frac{\epsilon}{4}$. Consider the unit vector $z = (z_{\tilde{v}})_{\tilde{v}\in V} \in Z$ with,

$$z_{\tilde{v}} = \begin{cases} x & \text{if } \tilde{v} = v, \\ -x & \text{if } \tilde{v} = w, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|C(G,\varphi)z\|_\infty \geq \|(C(G,\varphi)z)_e\| = 2\|\varphi(v,w)x\| > 2\|\varphi(v,w)\|_{op} - \frac{\epsilon}{2} > 2\|\varphi\|_\infty - \epsilon$. The result follows. □

Let $p \in [1,\infty)$. For a countable index set I, denote by $\ell^p(I;X)$ the Banach space of sequences $x = (x_i)_{i\in I}$ in X with $\sum_{i\in I}\|x_i\|^p < \infty$ and write $\|x\|_p = \left(\sum_{i\in I}\|x_i\|^p\right)^{\frac{1}{p}}$.

Proposition 2.6. *Let $C(G,\varphi)$ be a coboundary matrix. The following statements are equivalent.*

(i) $\varphi : V \times V \to L(X,Y)$ is a bounded function.
(ii) $C(G,\varphi) \in B(\ell^1(V;X), \ell^\infty(E;Y))$.
(iii) $C(G,\varphi)$ maps $\ell^1(V;X)$ into $\ell^\infty(E;Y)$.

Moreover, when the above conditions hold, the induced operator norm satisfies,

$$\|C(G,\varphi)\|_{op} = \|\varphi\|_\infty.$$

Proof. $(i) \Rightarrow (ii)$ Let $z = (z_v)_{v\in V} \in \ell^1(V;X)$. Then,

$$\|C(G,\varphi)z\|_\infty = \sup_{vw\in E}\|\varphi(v,w)(z_v - z_w)\| \leq \|\varphi\|_\infty\|z\|_1.$$

$(ii) \Rightarrow (iii)$ This is immediate.

$(iii) \Rightarrow (i)$ For each edge $e = vw \in E$, consider the linear map,

$$\psi_e : \ell^1(V;X) \to Y, \quad z \mapsto \varphi(v,w)(z_v - z_w).$$

Note that $\|\varphi(v,w)\|_{op} \leq \|\psi_e\|_{op}$ for each edge $e = vw \in E$. Since $C(G,\varphi)$ maps $\ell^1(V;X)$ into $\ell^\infty(E;Y)$, the set $\{\|\psi_e(z)\| : e \in E\}$ is bounded for each $z \in \ell^1(V;X)$. Thus, by the Banach-Steinhaus theorem, $\sup_{e\in E}\|\psi_e\|_{op} < \infty$ and so the result follows.

Finally, suppose conditions (i)-(iii) hold. To show $\|C(G,\varphi)\|_{op} = \|\varphi\|_\infty$, apply an argument similar to the proof of Proposition 2.5. Note that in this case $\frac{z}{2}$ is a unit vector in $\ell^1(V;X)$ with $\|C(G,\varphi)z\|_\infty > \|\varphi\|_\infty - \epsilon$. □

Proposition 2.7. *Let $C(G,\varphi)$ be a coboundary matrix and let $p \in [1,\infty)$. If the graph G has bounded degree then the following statements are equivalent.*

(i) $\varphi : V \times V \to L(X,Y)$ is a bounded function.
(ii) $C(G,\varphi) \in B(\ell^p(V;X), \ell^p(E;Y))$.
(iii) $C(G,\varphi)$ maps $\ell^p(V;X)$ into $\ell^p(E;Y)$.

Moreover, when the above conditions hold, the induced operator norm satisfies,

$$2^{1-\frac{1}{p}}\|\varphi\|_\infty \leq \|C(G,\varphi)\|_{op} \leq 2^{1-\frac{1}{p}}\|\varphi\|_\infty\Delta(G)^{\frac{1}{p}}.$$

Proof. $(i) \Rightarrow (ii)$ Suppose $p \in (1,\infty)$ and choose p' such that $\frac{1}{p}+\frac{1}{p'}=1$. Let $z=(z_v)_{v\in V}\in \ell^p(V;X)$. By Holder's inequality, for each edge $e\in E$,

$$\begin{aligned}\sum_{v\in V}\|c_{e,v}\|_{op}\|z_v\| &= \sum_{v\in V}\|c_{e,v}\|_{op}^{\frac{1}{p'}}\left(\|c_{e,v}\|_{op}^{\frac{1}{p}}\|z_v\|\right)\\ &\leq \left(\sum_{v\in V}\|c_{e,v}\|_{op}\right)^{\frac{1}{p'}}\left(\sum_{v\in V}\|c_{e,v}\|_{op}\|z_v\|^p\right)^{\frac{1}{p}}\end{aligned}$$

Thus, noting that $\sum_{v\in V}\|c_{e,v}\|_{op}\leq 2\|\varphi\|_\infty$ for each edge $e\in E$,

$$\begin{aligned}\|C(G,\varphi)z\|_p^p &\leq \sum_{e\in E}\left(\sum_{v\in V}\|c_{e,v}\|_{op}\|z_v\|\right)^p\\ &\leq (2\|\varphi\|_\infty)^{\frac{p}{p'}}\sum_{e\in E}\left(\sum_{v\in V}\|c_{e,v}\|_{op}\|z_v\|^p\right)\\ &= (2\|\varphi\|_\infty)^{\frac{p}{p'}}\sum_{v\in V}\left(\sum_{e\in E}\|c_{e,v}\|_{op}\|z_v\|^p\right)\\ &\leq (2\|\varphi\|_\infty)^{\frac{p}{p'}}\|\varphi\|_\infty\Delta(G)\left(\sum_{v\in V}\|z_v\|^p\right)\\ &= 2^{\frac{p}{p'}}\|\varphi\|_\infty^p\Delta(G)\|z\|_p^p\end{aligned}$$

It follows that $C(G,\varphi)$ maps $\ell^p(V;X)$ into $\ell^p(E;Y)$ and that $C(G,\varphi)$ is bounded with $\|C(G,\varphi)\|_{op}\leq 2^{\frac{1}{p'}}\|\varphi\|_\infty\Delta(G)^{\frac{1}{p}}$. For the case $p=1$ a similar (and more direct) argument can be applied.

$(ii)\Rightarrow(iii)$ This is immediate.

$(iii)\Rightarrow(i)$ Apply Proposition 2.6.

Finally, suppose conditions (i)-(iii) hold. The proof of $(i)\Rightarrow(ii)$ provides the upper bound for $\|C(G,\varphi)\|_{op}$. To obtain the lower bound, let $\epsilon>0$ and choose $e=vw\in E$ with $\|\varphi(v,w)\|_{op}>\|\varphi\|_\infty-\frac{\epsilon}{4}$. Choose a unit vector $x\in X$ such that $\|\varphi(v,w)x\|>\|\varphi(v,w)\|_{op}-\frac{\epsilon}{4}$. Let $z\in\ell^p(V;X)$ be the vector with components,

$$z_{\tilde{v}}=\begin{cases}x & \text{if } \tilde{v}=v,\\ -x & \text{if } \tilde{v}=w,\\ 0 & \text{otherwise.}\end{cases}$$

Then $\|z\|_p=2^{\frac{1}{p}}\|x\|=2^{\frac{1}{p}}$. Note that $\|(C(G,\varphi)z)_e\|=2\|\varphi(v,w)x\|>2\|\varphi\|_\infty-\epsilon$. Thus, $\|C(G,\varphi)z\|_p>2\|\varphi\|_\infty-\epsilon=2^{1-\frac{1}{p}}\|\varphi\|_\infty\|z\|_p-\epsilon$ and so it follows that $\|C(G,\varphi)\|_{op}\geq 2^{1-\frac{1}{p}}\|\varphi\|_\infty$. □

For a countable index set I, denote by $c_0(I;X)$ the vector space of null sequences $x=(x_i)_{i\in I}$ in X. i.e. $x\in c_0(I;X)$ if, given any $\epsilon>0$, there exists a finite subset $I_0\subset I$ such that $\sup_{i\in I\setminus I_0}\|x_i\|<\epsilon$. Note that $c_0(I;X)$ is a closed subspace of $\ell^\infty(I;X)$.

Proposition 2.8. *Let $C(G,\varphi)$ be a coboundary matrix. If G is locally finite then the following statements are equivalent.*

(i) $\varphi: V\times V\to L(X,Y)$ *is a bounded function.*

(ii) $C(G,\varphi)$ *maps* $c_0(V;X)$ *into* $c_0(E;Y)$.

(iii) $C(G,\varphi) \in B(c_0(V;X), c_0(E;Y))$.

Moreover, when the above conditions hold, the induced operator norm satisfies,

$$\|C(G,\varphi)\|_{op} = 2\|\varphi\|_\infty.$$

Proof. $(i) \Rightarrow (ii)$ Let $x = (x_v)_{v\in V} \in c_0(V;X)$ and let $\epsilon > 0$. Then there exists a finite subset $V_0 \subset V$ such that $\sup_{v\in V\backslash V_0} \|x_v\| < \frac{\epsilon}{2\|\varphi\|_\infty}$. Thus, for each $e \in E$,

$$\|\sum_{v\in V\backslash V_0} c_{e,v}(x_v)\| \leq \sum_{v\in V\backslash V_0} \|c_{e,v}\|_{op}\|x_v\| < \left(\sum_{v\in V\backslash V_0} \|c_{e,v}\|_{op}\right) \frac{\epsilon}{2\|\varphi\|_\infty} < \epsilon.$$

Let $E_0 \subset E$ denote the finite subset of edges which are incident with a vertex in V_0. Then, noting that $c_{e,v} = 0$ whenever $e \in E\backslash E_0$ and $v \in V_0$,

$$\sup_{e\in E\backslash E_0} \|\sum_{v\in V} c_{e,v}(x_v)\| = \sup_{e\in E\backslash E_0} \left(\|\sum_{v\in V\backslash V_0} c_{e,v}(x_v)\|\right) \leq \epsilon.$$

Thus $C(G,\varphi)$ maps $c_0(V;X)$ into $c_0(E;Y)$.

$(ii) \Rightarrow (i)$ Apply Proposition 2.6.

$(ii) \Rightarrow (iii)$ Note that, by the above arguments, φ is bounded. Thus, by Proposition 2.5, $C(G,\varphi) : c_0(V;X) \to c_0(E;Y)$ is bounded.

$(iii) \Rightarrow (ii)$ This is immediate.

The formula for the operator norm follows from Proposition 2.5. □

For ease of reference we note the following corollary.

Corollary 2.9. *Let $C(G,\varphi)$ be a coboundary matrix. If G has bounded degree then the following statements are equivalent.*

(i) $\varphi : V \times V \to L(X,Y)$ *is a bounded function.*
(ii) $C(G,\varphi) \in B(c_0(V;X), c_0(E;Y))$.
(iii) $C(G,\varphi) \in B(\ell^p(V;X), \ell^p(E;Y))$, *for all* $p \in [1,\infty]$.
(iv) $C(G,\varphi) \in B(\ell^p(V;X), \ell^p(E;Y))$, *for some* $p \in [1,\infty]$.

Proof. $(i) \Rightarrow (ii)$ Apply Proposition 2.8.

$(ii) \Rightarrow (i)$ Apply Proposition 2.5.

$(i) \Rightarrow (iii)$ Apply Proposition 2.5 for $p = \infty$ and Proposition 2.7 for $p \in [1,\infty)$.

$(iii) \Rightarrow (iv)$ This is immediate.

$(iv) \Rightarrow (i)$ Apply Proposition 2.5 when $p = \infty$ and Proposition 2.6 when $p \in [1,\infty)$. □

Remark 2.10. *In the case where $X = Y = \mathbb{K}$ and $p = 2$, results on the boundedness and operator norm of coboundary operators $C(G,\varphi) : \ell^2(V;\mathbb{K}) \to \ell^2(E;\mathbb{K})$ can be found in* [19, p. 15] *and in* [4, Section 5.2]. *These results can be applied in particular to incidence matrices for directed graphs, as described in Example 2.2, and to rigidity matrices for one-dimensional bar-joint frameworks, as described in Example 2.3.*

3. Norm approximation and compactness

Let (G,φ) be a framework and let $G_0 = (V_0, E_0)$ be a subgraph of $G = (V,E)$. Denote by (G_0,φ) the *subframework* obtained by restricting φ to $V_0 \times V_0$. Denote by $P_{E_0} : Y^E \to Y^E$ the projection of Y^E onto Y^{E_0} along $Y^{E\backslash E_0}$.

A sequence $(G_k)_{k\in\mathbb{N}}$ of subgraphs of G *converges* to G if G_k is a subgraph of G_{k+1} for each $k \in \mathbb{N}$ and, given any $e \in E$, there exists $N \in \mathbb{N}$ such that $e \in E(G_k)$ for all $k \geq N$. In this case, we write $G_k \to G$ as $k \to \infty$.

Theorem 3.1. *Let $C(G, \varphi)$ be a coboundary matrix where G has bounded degree and $\varphi : V \times V \to L(X, Y)$ is a bounded function.*

(i) If $G_k \to G$ as $k \to \infty$ then,

$$\|C(G, \varphi)\|_{op} = \lim_{k\to\infty} \|C(G_k, \varphi)\|_{op}.$$

(ii) If $\mathcal{S}$ and $\mathcal{S}'$ denote respectively the set of all subgraphs of G and the set of all finite subgraphs of G then,

$$\|C(G, \varphi)\|_{op} = \sup_{G_0\in\mathcal{S}} \|C(G_0, \varphi)\|_{op} = \sup_{G_0\in\mathcal{S}'} \|C(G_0, \varphi)\|_{op}.$$

The operator norms in (i) and (ii) refer to the following two cases:

(a) $C(G, \varphi) \in B(c_0(V; X), c_0(E; Y))$ and $C(G_k, \varphi) \in B(c_0(V_k; X), c_0(E_k; Y))$.

(b) $C(G, \varphi) \in B(\ell^p(V; X), \ell^p(E; Y))$ and $C(G_k, \varphi) \in B(\ell^p(V_k; X), \ell^p(E_k; Y))$, where $p \in [1, \infty)$.

Proof. We will prove the statements for case (a) (similar arguments apply for case (b)).

(i) Let $\epsilon > 0$. Note that the sequence $(\|C(G_k, \varphi)\|_{op})_{k\in\mathbb{N}}$ is bounded above by $\|C(G, \varphi)\|_{op}$ and that $\|C(G_k, \varphi)\|_{op} = \|P_{E_k} \circ C(G, \varphi)\|_{op}$ for each $k \in \mathbb{N}$. Also note that $P_{E_k} \in B(c_0(E; Y), c_0(E; Y))$ for each $k \in \mathbb{N}$ and, since $G_k \to G$ as $k \to \infty$, the sequence $(P_{E_k})_{k\in\mathbb{N}}$ converges strongly to the identity map in $B(c_0(E; Y), c_0(E; Y))$. Thus $P_{E_k} \circ C(G, \varphi) \to C(G, \varphi)$ strongly as $k \to \infty$. Since the operator norm is strongly lower semi-continuous, there exists $N \in \mathbb{N}$ such that,

$$|\|C(G, \varphi)\|_{op} - \|C(G_k, \varphi)\|_{op}| = \|C(G, \varphi)\|_{op} - \|P_{E_k} \circ C(G, \varphi)\|_{op} < \epsilon,$$

for all $k \geq N$. This proves (i).

(ii) Note that $\|C(G, \varphi)\|_{op}$ is an upper bound for $\{\|C(G_0, \varphi)\|_{op} : G_0 \in \mathcal{S}\}$ and so

$$\|C(G, \varphi)\|_{op} \geq \sup_{G_0\in\mathcal{S}} \|C(G_0, \varphi)\|_{op} \geq \sup_{G_0\in\mathcal{S}'} \|C(G_0, \varphi)\|_{op}.$$

For the reverse inequalities, let $\epsilon > 0$. Choose a sequence $(G_k)_{k\in\mathbb{N}}$ of finite subgraphs of G such that $G_k \to G$ as $k \to \infty$. By (i), there exists $k \in \mathbb{N}$ such that $\|C(G_k, \varphi)\|_{op} > \|C(G, \varphi)\|_{op} - \epsilon$. The result follows. □

The set of compact operators $T : Z \to W$ between two Banach spaces Z and W is denoted $K(Z, W)$. Recall that the essential operator norm for an operator $T \in B(Z, W)$ is given by,

$$\|T\|_e = \inf_{K\in K(Z,W)} \|T - K\|_{op}.$$

Given a framework (G, φ) we define,

$$k(G, \varphi) = \inf_{E_0 \text{ finite}} \left(\sup_{vw\in E\setminus E_0} \|\varphi(v, w)\|_{op} \right),$$

where the infimum is taken over all finite subsets $E_0 \subseteq E$.

Proposition 3.2. *Let $C(G, \varphi)$ be a coboundary matrix and suppose $\varphi : V \times V \to L(X, Y)$ is a bounded function.*

(i) *If Z is a closed subspace of $\ell^\infty(V;X)$ which contains $c_{00}(V;X)$ then the bounded operator $C(G,\varphi): Z \to \ell^\infty(E;Y)$ satisfies,*

$$\|C(G,\varphi)\|_e \leq 2k(G,\varphi).$$

(ii) *If G has bounded degree and $p \in [1,\infty)$ then the bounded operator $C(G,\varphi): \ell^p(V;X) \to \ell^p(E;Y)$ satisfies,*

$$\|C(G,\varphi)\|_e \leq 2^{1-\frac{1}{p}} k(G,\varphi)\Delta(G)^{\frac{1}{p}}.$$

Proof. (i) Let E_0 be a finite subset of E. Define $K: Z \to \ell^\infty(E;Y)$ by setting $(K(z))_e = C(G,\varphi)z$ if $e \in E_0$ and $(K(z))_e = 0$ otherwise. Since E_0 is finite, and X and Y are finite dimensional, K is a finite rank operator. Note that $C(G,\varphi) - K$ is similar to a column operator,

$$\begin{bmatrix} 0 \\ C(G_0,\varphi) \end{bmatrix} : Z \to \ell^\infty(E_0;Y) \oplus \ell^\infty(E\backslash E_0;Y),$$

where G_0 is the subgraph of G with vertex set V and edge set $E\backslash E_0$. (Note that G_0 may contain vertices of degree zero). Since φ is bounded, by Proposition 2.5,

$$\|C(G,\varphi)\|_e \leq \|C(G,\varphi) - K\|_{op} = \|C(G_0,\varphi)\|_{op} = 2 \sup_{vw \in E\backslash E_0} \|\varphi(v,w)\|_{op}.$$

The proof of (ii) is similar and uses Proposition 2.7. □

The function $\varphi: V \times V \to L(X,Y)$ is said to *vanish at infinity* if given any $\epsilon > 0$, there exists a finite subset $E_0 \subset E$ such that $\sup_{vw \in E\backslash E_0} \|\varphi(v,w)\|_{op} < \epsilon$.

Corollary 3.3. *Let $C(G,\varphi)$ be a coboundary matrix and suppose $\varphi: V \times V \to L(X,Y)$ vanishes at infinity.*

(i) *If Z is a closed subspace of $\ell^\infty(V;X)$ which contains $c_{00}(V;X)$ then $C(G,\varphi) \in K(Z, \ell^\infty(E;Y))$.*
(ii) *If G has bounded degree and $p \in [1,\infty)$ then $C(G,\varphi) \in K(\ell^p(V;X), \ell^p(E;Y))$.*

Proof. (i) Since $\varphi: V \times V \to L(X,Y)$ vanishes at infinity, $k(G,\varphi) = 0$. Thus, by Proposition 3.2(i), $\|C(G,\varphi)\|_e = 0$ and so the result follows. A similar argument applies for (ii). □

Proposition 3.4. *Let $C(G,\varphi)$ be a coboundary matrix and suppose one of the following conditions holds.*

(i) *$C(G,\varphi) \in K(c_0(V;X), c_0(E;Y))$.*
(ii) *$C(G,\varphi) \in K(\ell^p(V;X), \ell^p(E;Y))$, where $p \in [1,\infty)$.*

Then the function $\varphi: V \times V \to L(X,Y)$ vanishes at infinity.

Proof. Suppose (i) holds and suppose $\varphi: V \times V \to L(X,Y)$ does not vanish at infinity. Then there exists $\epsilon > 0$ and a countably infinite subset $E' \subset E$ such that $\inf_{e=vw \in E'} \|\varphi(v,w)\|_{op} \geq \epsilon$. For each edge $e = vw \in E'$, choose a unit vector $x_e \in X$ with $\|\varphi(v,w)(x_e)\| > \|\varphi(v,w)\|_{op} - \frac{\epsilon}{2}$. Then let $z^e \in c_{00}(V;X)$ be the vector with components,

$$z^e_{\tilde{v}} = \begin{cases} x_e & \text{if } \tilde{v} = v, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\|z^e\|_\infty = \|x_e\| = 1$ for all $e \in E'$ and so the set $\{z^e : e \in E'\}$ is bounded in $c_0(V;X)$. Since $C(G,\varphi) \in K(c_0(V;X), c_0(E;Y))$, the image $\{C(G,\varphi)z^e : e \in E'\}$ is precompact in $c_0(E;Y)$ and hence contains a sequence $(C(G,\varphi)z^{e_k})_{k\in\mathbb{N}}$ which converges in $c_0(E;Y)$. Let $y=(y_e)_{e\in E}\in c_0(E;Y)$ be the limit of this sequence. Note that,

$$\|(C(G,\varphi)z^{e_k})_{e_k}\| = \|\varphi(v_k,w_k)x_{e_k}\| \geq \|\varphi(v_k,w_k)\|_{op} - \frac{\epsilon}{2} \geq \frac{\epsilon}{2},$$

for each $e_k = v_k w_k$. On the other hand,

$$\|(C(G,\varphi)z^{e_k})_{e_k}\| \leq \|(C(G,\varphi)z^{e_k})_{e_k} - y_{e_k}\| + \|y_{e_k}\| \leq \|C(G,\varphi)z^{e_k} - y\|_\infty + \|y_{e_k}\|,$$

where the right-hand side tends to 0 as $k\to\infty$. This is a contradiction and so (i) is proved. A similar argument applies when (ii) holds. □

Corollary 3.5. *Let $C(G,\varphi)$ be a coboundary matrix. If G has bounded degree then the following statements are equivalent.*

(i) The function $\varphi: V\times V\to L(X,Y)$ vanishes at infinity.
(ii) $C(G,\varphi)\in K(c_0(V;X),c_0(E;Y))$.
(iii) $C(G,\varphi)\in K(\ell^p(V;X),\ell^p(E;Y))$, where $p\in[1,\infty)$.

Proof. For the implications $(i)\Rightarrow(ii)$ and $(i)\Rightarrow(iii)$, apply Corollary 3.3. For the implications $(ii)\Rightarrow(i)$ and $(iii)\Rightarrow(i)$, apply Proposition 3.4. □

Proposition 3.6. *Let $C(G,\varphi)$ be a coboundary matrix and suppose $\varphi\in\ell^1(V\times V,L(X,Y))$.*

(i) $C(G,\varphi)\in B(\ell^\infty(V;X),\ell^1(E;Y))$ and the induced operator norm satisfies,

$$\|C(G,\varphi)\|_{op} \leq 2\sum_{vw\in E}\|\varphi(v,w)\|_{op}.$$

(ii) $C(G,\varphi)\in K(c_0(V;X),\ell^1(E;Y))$.

Proof. (i) Let $L=\sum_{vw\in E}\|\varphi(v,w)\|_{op}$. If $x=(x_v)_{v\in V}\in\ell^\infty(V;X)$ then,

$$\|C(G,\varphi)x\|_1 = \sum_{e\in E}\|\sum_{v\in V}\varphi(v,w)x_v\| \leq \sum_{e\in E}\left(\sum_{v\in V}\|\varphi(v,w)\|_{op}\|x_v\|\right) \leq 2L\|x\|_\infty.$$

Thus, $C(G,\varphi)x$ lies in $\ell^1(E;Y)$ and $\|C(G,\varphi)\|_{op}\leq 2L$.

(ii) By (i), $C(G,\varphi)\in B(c_0(V;X),\ell^1(E;Y))$. Now apply Pitt's theorem [8, proposition 6.25]. □

4. The bounded below property

Let $G=(V,E)$ be a simple graph. If V_0 is a finite subset of V then denote by ∂V_0 the set of edges of G with exactly one vertex in V_0. The *isoperimetric constant* for G is the value,

$$i(G) = \inf_{V_0 \text{ finite}} \frac{|\partial V_0|}{|V_0|},$$

where the infimum is taken over all finite subsets V_0 of V. (See [5] eg.)

Denote by $\chi(V;X)$ the set of finitely supported vectors in X^V with constant non-zero entries. i.e. $z\in\chi(V;X)$ if $z\in X^V$ and there exists a finite subset $V_0\subset V$ and a non-zero vector $x\in X$ such that $z_v=x$ for all $v\in V_0$ and $z_v=0$ otherwise.

Proposition 4.1. *Let $C(G,\varphi)$ be a coboundary matrix where $G=(V,E)$ is locally finite, $\varphi: V\times V\to L(X,Y)$ is a bounded function and $p\in[1,\infty)$.*

(i)

$$\inf\{\|C(G,\varphi)z\|_p : z\in\chi(V;X),\ \|z\|_p=1\} \leq i(G)^{\frac{1}{p}}\,\|\varphi\|_\infty.$$

(ii) If $\dim X = \dim Y = 1$ *then,*

$$i(G)^{\frac{1}{p}}\left(\inf_{vw\in E}\|\varphi(v,w)\|_{op}\right) \leq \{\|C(G,\varphi)z\|_p : z\in\chi(V;\mathbb{K}),\ \|z\|_p = 1\}.$$

In particular, if $\|\varphi(v,w)\|_{op}$ *is constant on* $\{(v,w): vw\in E\}$ *then,*

$$\inf\{\|C(G,\varphi)z\|_p : z\in\chi(V;\mathbb{K}),\ \|z\|_p = 1\} = i(G)^{\frac{1}{p}}\|\varphi\|_\infty.$$

Proof. (i) Let V_0 be a finite subset of V. Choose a unit vector $x\in X$ and define a unit vector $z\in\ell^p(V;X)$ by setting,

$$z_v = \begin{cases} \left(\frac{1}{|V_0|}\right)^{\frac{1}{p}} x & \text{if } v\in V_0,\\ 0 & \text{otherwise.}\end{cases}$$

Note that $z\in\chi(V;X)$. Moreover,

$$\|C(G,\varphi)z\|_p^p = \frac{1}{|V_0|}\sum_{vw\in\partial V_0}\|\varphi(v,w)x\|^p \leq \frac{|\partial V_0|}{|V_0|}\|\varphi\|_\infty^p.$$

The result follows.

(ii) Let $z\in\chi(V;\mathbb{K})$ with $\|z\|_p = 1$ and let V_0 be the support of z. Then $z_v = x$ for all $v\in V_0$, for some non-zero $x\in\mathbb{K}$, and $z_v = 0$ otherwise. Note that $|x| = \left(\frac{1}{|V_0|}\right)^{\frac{1}{p}}$. Thus,

$$\|C(G,\varphi)z\|_p^p = \sum_{vw\in\partial V_0}\|\varphi(v,w)\|_{op}^p|x|^p \quad\geq\quad \left(\inf_{vw\in E}\|\varphi(v,w)\|_{op}^p\right)\frac{|\partial V_0|}{|V_0|}.$$

The inequality now follows. For the final statement, combine this inequality with (i). □

Remark 4.2. *Note that Proposition 4.1(ii) applies in the particular case where* $C(G,\varphi)$ *is an incidence matrix for an infinite directed graph* G*. In this case,* $X = Y = \mathbb{R}$ *and* $\|\varphi(v,w)\|_{op} = 1$ *for all edges* $vw\in E$*.*

Given a framework (G,φ) and a unit vector $x\in X$ we define,

$$i(G,\varphi;x) = \inf_{V_0\text{ finite}}\left(\sup_{vw\in\partial V_0}\|\varphi(v,w)x\|\right),$$

where the infimum is taken over all finite subsets V_0 of V.

Proposition 4.3. *Let* $C(G,\varphi)$ *be a coboundary matrix, let* $p\in[1,\infty)$ *and let* $x\in X$ *be a unit vector. If* G *has bounded degree then,*

$$\inf\{\|C(G,\varphi)z\|_p : z\in\chi(V;\mathbb{K}^d), \|z\|_p = 1\} \leq i(G,\varphi;x)\Delta(G)^{\frac{1}{p}}.$$

Proof. Let V_0 be a finite subset of V and construct a unit vector $z\in\ell^p(V;X)$ as in the proof of Proposition 4.1(i). Then,

$$\|C(G,\varphi)z\|_p^p = \frac{1}{|V_0|}\sum_{vw\in\partial V_0}\|\varphi(v,w)x\|^p \leq \Delta(G)\left(\sup_{vw\in\partial V_0}\|\varphi(v,w)x\|^p\right).$$

The result follows. □

Recall that a linear operator $T: Z\to W$ between normed spaces Z and W is said to be *bounded below* if $\inf_{\|z\|=1}\|Tz\| > 0$.

Corollary 4.4. *Let $C(G,\varphi)$ be a coboundary matrix, where $\varphi : V\times V \to L(X,Y)$ is a bounded function, and let $p\in[1,\infty)$. Suppose at least one of the following conditions holds.*

(a) G is locally finite and $i(G)=0$.
(b) G has bounded degree and $i(G,\varphi;x)=0$ for some unit vector $x\in X$.

Then the operator $C(G,\varphi):\ell^p(V;X)\to\ell^p(E;Y)$ is not bounded below.

Proof. Apply Proposition 4.1 in case (a) and Proposition 4.3 in case (b). □

Remark 4.5. *Note that if a graph G is periodic then $i(G)=0$. (Recall that a graph G is periodic if the automorphism group of G contains a free abelian subgroup Γ which acts on G freely and the set $\{\Gamma v : v\in V\}$ of vertex orbits is finite). Thus, for any framework (G,φ) with φ bounded, condition (a) of Corollary 4.4 is satisfied and so $C(G,\varphi):\ell^p(V;X)\to\ell^p(E;Y)$ is not bounded below for all $p\in[1,\infty)$. This applies, in particular, to the rigidity matrices of periodic bar-joint structures ([10]).*

Proposition 4.6. *Let $C(G,\varphi)$ be a coboundary matrix and let $x\in X$ be a unit vector.*

(i)
$$\inf\{\|C(G,\varphi)z\|_\infty : z\in\chi(V;X),\ \|z\|_\infty=1\}\le i(G,\varphi;x).$$

(ii) If $\dim X=\dim Y=1$ then,
$$\inf\{\|C(G,\varphi)z\|_\infty : z\in\chi(V;\mathbb{K}),\ \|z\|_\infty=1\}= i(G,\varphi;x).$$

Proof. (i) Let V_0 be a finite subset of V and define a unit vector $z\in\ell^\infty(V;X)$ by setting,
$$z_v=\begin{cases} x & \text{if } v\in V_0,\\ 0 & \text{otherwise.}\end{cases}$$
For each edge $e=vw\in E(G)$,
$$\|(C(G,\varphi)z)_e\|=\begin{cases} \|\varphi(v,w)x\| & \text{if } vw\in\partial V_0,\\ 0 & \text{otherwise.}\end{cases}$$
Thus,
$$\|C(G,\varphi)z\|_\infty\le \sup_{vw\in\partial V_0}\|\varphi(v,w)x\|.$$
The result follows since $z\in\chi(V;X)$ and $\|z\|_\infty=1$.

(ii) Let $z\in\chi(V;\mathbb{K})$ with $\|z\|_\infty=1$ and let V_0 be the support of z. Then $z_v=x$ for all $v\in V_0$ where $|x|=1$. Thus,
$$\|C(G,\varphi)z\|_\infty=\sup_{vw\in\partial V_0}\|\varphi(v,w)\|_{op}|x|\ge i(G,\varphi;x).$$
For the reverse inequality apply (i). □

Corollary 4.7. *Let $C(G,\varphi)$ be a coboundary matrix, where $\varphi:V\times V\to L(X,Y)$ is a bounded function, and let Z be a subspace of $\ell^\infty(V;X)$ which contains $c_{00}(V;X)$. If $i(G,\varphi;x)=0$ for some unit vector $x\in X$ then the operator $C(G,\varphi):Z\to\ell^\infty(E;Y)$ is not bounded below.*

Proof. Note that Z contains $\chi(V;X)$ and so the result follows from Proposition 4.6. □

Let (G, φ) be a framework and, for each $v \in V$, define

$$l(v) = \sup_{vw \in E(v)} \|\varphi(v,w)\|_{op},$$

where the supremum is taken over all edges incident with v. The function $\varphi : V \times V \to L(X, Y)$ is said to be *weakly bounded away from zero* if $\inf_{v\in V} l(v) > 0$.

Corollary 4.8. *Let $C(G, \varphi)$ be a coboundary matrix and suppose one of the following conditions holds.*

(a) $C(G, \varphi) \in B(Z, \ell^\infty(E; Y))$, where Z is a subspace of $\ell^\infty(V; X)$ which contains $c_{00}(V; X)$, and the operator $C(G, \varphi) : Z \to \ell^\infty(E; Y)$ is bounded below.

(b) G has bounded degree, $C(G, \varphi) \in B(\ell^p(V; X), \ell^p(E; Y))$ and the operator $C(G, \varphi) : \ell^p(V; X) \to \ell^p(E; Y)$ is bounded below, where $p \in [1, \infty)$.

Then the function $\varphi : V \times V \to L(X, Y)$ is weakly bounded away from zero.

Proof. Note that in general, $i(G, \varphi; x) \leq \inf_{v\in V} l(v)$ for each unit vector $x \in X$. Thus the results follow from Corollary 4.4 and Corollary 4.7. □

Remark 4.9. *By* [17, Theorem 4.27]*, the incidence matrix of a directed graph G with bounded degree is bounded below if and only if $i(G) > 0$. In the next section, we present an example which suggests there is no simple variant of this result for general coboundary operators. In particular, we construct a bar-joint framework (G, q) in the Euclidean plane with the property that the rigidity matrix $R(G, q) : \ell^2(V; \mathbb{R}^2) \to \ell^2(E; \mathbb{R})$ is not bounded below and $i(G) > 0$.*

5. An application to rigidity theory

In this section, we apply the results of the preceding sections to a countably infinite bar-joint framework (G, q) in the Euclidean plane. The associated rigidity matrix is denoted $R(G, q)$. (See Example 2.3 for the relevant definitions.)

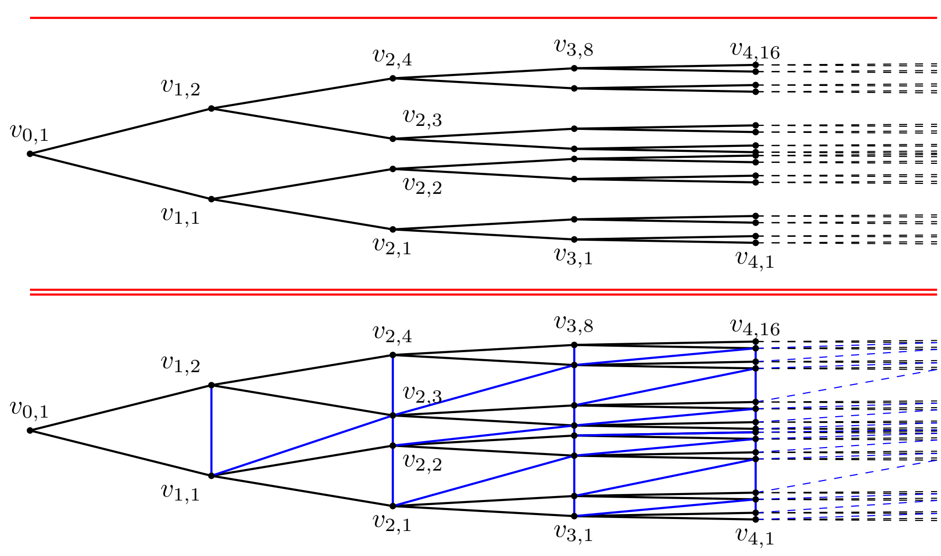

Figure 1. Placement q of the full binary tree G (top) and the augmented bar-joint framework (G', q) in Example 5.1 (bottom).

Example 5.1 (Binary tree framework)**.** *Define inductively the following unbounded placement q of the full binary tree G in $\mathbb{R}^2$. Place the root node $v_{0,1}$ at $(0, \frac{1}{2})$. Given now a joint $q(v_{n,k}) = (x, y)$ that corresponds on a vertex lying on the nth-depth of G, we assign its two children at the points $q(v_{n+1,2k-1}) = (x + 1, y - \frac{1}{3^{n+1}})$ and $q(v_{n+1,2k}) = (x + 1, y + \frac{1}{3^{n+1}})$ (see top of Figure 1).*

Note that the vertices of the same depth of the tree share the same first coordinate. Moreover, each joint of the framework satisfies $q(v_{n,k}) = (n, y)$ for some $y \in (0, 1)$.

We insert extra bars on the flexible framework (G, q) to obtain an infinitesimally rigid framework on the plane. Namely, we add the set of vertical bars

$$\{q(v_{n,k})q(v_{n,k+1}) : 1 \leq k \leq 2^n - 1,\, n \in \mathbb{N}\},$$

and the sets of diagonal bars

$$\{q(v_{n,k})q(v_{n+1,2k+1}) : 1 \leq k \leq 2^n - 1,\, n \in \mathbb{N}\}.$$

Let (G', q) be the resulting framework (see bottom of Figure 1). It is evident that G is a spanning tree of G'. Moreover, (G', q) is infinitesimally rigid, since it is sequentially infinitesimally rigid (see [12]). Indeed, let G_n be the subgraph of G' that is determined by the restriction of (G', q) on the half-plane $H_n := \{(x, y) : x \leq n\}$. Observe that $\{(G_n, q\big|_{H_n})\}_n$ is a vertex-complete tower of infinitesimally rigid bar-joint frameworks in (G', q).

Note that by the full binary tree theorem we have $i(G') = 1$. We claim that the rigidity operator $R(G', q) : c_0(V; \mathbb{R}^2) \to c_0(E; \mathbb{R})$ is not bounded below and that the rigidity operator $R(G', q) : \ell^p(V; \mathbb{R}^2) \to \ell^p(E; \mathbb{R})$ is not bounded below for each $p \in [1, \infty)$. By Corollary 4.4 and Corollary 4.7, it suffices to show that given any $n \in \mathbb{N}$ there exists a finite set V_0 and some direction specified by a unit vector x in $\mathbb{R}^2$, such that

$$\sup_{vw \in \partial V_0} |\varphi(v, w)x| = \sup_{vw \in \partial V_0} |(q(v) - q(w)) \cdot x| \leq \frac{1}{n}.$$

Let $n \in \mathbb{N}$. Choose $V_0 = \{v_{n,1}, v_{n+1,1}\}$ and let x be the unit vector $(0, 1)$. Check that for every edge vw in ∂V_0 we have $\sup_{vw \in \partial V_0} |(q(v) - q(w)) \cdot x| = \frac{2}{3^n} \leq \frac{1}{n}$, so the proof of our claim is complete.

In light of Example 5.1 and the preceding results, we make the following conjecture.

Conjecture 5.2. *Let (G, q) be a bar-joint framework in $\mathbb{R}^2$ with the following properties:*

(i) G has bounded degree,
(ii) $q : V \to \mathbb{R}^2$ is a planar embedding of G (i.e. no edge-crossings are allowed),
(iii) (G, q) is bounded in $\mathbb{R}^2$ (i.e. $\sup_{v \in V} \|q(v)\| < \infty$).

Then the rigidity operator $R(G, q) : c_0(V; \mathbb{R}^2) \to c_0(E; \mathbb{R})$ is not bounded below and, for each $p \in [1, \infty)$, the rigidity operator $R(G, q) : \ell^p(V; \mathbb{R}^2) \to \ell^p(E; \mathbb{R})$ is not bounded below.

Finally, we pose a general problem. Note that if a rigidity operator $R(G, q)$ is bounded below in either of the cases stated below then the bar-joint framework (G, q) may be viewed as satisfying a robust form of rigidity for the given space of vanishing velocity fields.

Problem 5.3. *Let (G, q) be a bar-joint framework in $\mathbb{R}^d$ where G has bounded degree and $\sup_{vw\in E} \|q(v) - q(w)\|_2 < \infty$.*

(a) Find necessary and sufficient conditions on G and q for the rigidity operator $R(G, q) : c_0(V; \mathbb{R}^d) \to c_0(E; \mathbb{R})$ to be bounded below.

(b) Given $p \in [1, \infty)$, find necessary and sufficient conditions on G and q for the rigidity operator $R(G, q) : \ell^p(V; \mathbb{R}^d) \to \ell^p(E; \mathbb{R})$ to be bounded below.

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