Rigidity of frameworks with coordinated constraint relaxations

Hattie Serocold

Supervisor: Dr. Bernd Schulze

Department of Mathematics and Statistics

Lancaster University

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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

This thesis is the result of my own work, except as specified in the text and Acknowledgements. Many of the ideas in this thesis were the product of discussion with my supervisor, Bernd Schulze. This thesis contains research carried out jointly: work that appears in Chapter 4 and Section 7.4 may also be found in the paper co-authored with Bernd Schulze and Louis Theran [SST18]. I declare that I contributed fully to this work.

This dissertation contains fewer than 80,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

Hattie Serocold

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Abstract

This thesis is concerned with the rigidity of coordinated frameworks. These are considered to be bar-joint frameworks for which the requirement that the lengths of bars be kept fixed is relaxed on some collection of bars, with the caveat that all bars within a coordination class must change length by the same amount. We begin by formulating the conditions for a framework to be continuously coordinated rigid, infinitesimally coordinated rigid, and statically coordinated rigid. We prove that static and infinitesimal rigidity are equivalent for coordinated frameworks, and that for regular coordinated frameworks, continuous rigidity and infinitesimal rigidity are equivalent.

We give a characterisation of the rigidity of frameworks in d-dimensional Euclidean space with k coordination classes, based on the rigidity of the structure graph of such a framework. Since minimal infinitesimal rigidity of bar-joint frameworks is characterised in $\mathbb{R}^1$ and $\mathbb{R}^2$, we extend the standard characterisations to a combinatorial characterisation of minimally infinitesimally rigid frameworks with one class of coordinated bars, and with two classes of coordinated bars, in both dimension 1 and dimension 2. We also obtain an inductive characterisation of such minimally infinitesimally rigid frameworks using coordinated analogues to standard inductive constructions. We conclude by considering coordinated frameworks with symmetric realisations, and give some initial results in this area.

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Chapter 1

Introduction

There are many situations in which humans wish to be able to construct rigid structures, which will not deform when a load or force is applied to them. Construction scaffolding is one intuitive example of a framework formed of bars and joints that is required to be rigid, though similar techniques may be applied to much smaller problems such as analysing rigid components of molecules [Hen95].

1.1 History of rigidity

Euler stated the following conjecture on the rigidity of polyhedra with rigid faces, which was an early motivation for the study of rigid structures.

Conjecture 1.1.1 (Euler, 1766 [Eul62])**.** A closed spatial figure allows no changes, as long as it is not ripped apart.

Cauchy [Cau05] gave an initial proof that strictly convex polyhedra with rigid faces are continuously rigid in 1813, though this proof contained some errors. Steinitz and Rademacher [SR34] produced a corrected proof of this result in 1934, and Gluck [Glu75] gave a corresponding result in 1975 that a simply connected closed polyhedron in 3 dimensions is rigid for an open dense subset of the realisations of the vertices.

In contrast, Connelly [Con77] produced a construction for a 3-dimensional closed polyhedron that is flexible, and hence is a counterexample to Euler's 1766 conjecture. The "Bricard octahedra" [Bri97] are used in this construction, and Connelly also verifies that such a flexible convex polyhedron has constant volume. This corresponds to the "Bellows Conjecture", later proved by Connelly, Sabitov and Walz [CSW97], that the volume contained within a flexible triangulated polyhedron is constant.

Polyhedra with rigid faces may be considered as a *2-skeleton*, comprised of vertices, edges and faces. The corresponding *1-skeleton* of a polyhedron consists only of vertices and edges, which may be considered as a bar-joint framework. Alexandrov [Ale05] produced work relating to the rigidity of 1-skeletons of triangulated polyhedra, which Asimow and Roth [AR78] extended to prove that the existence of at least one non-triangular face within the 1-skeleton of a strictly convex 3-dimensional polyhedron implies that the corresponding 3-dimensional bar-joint framework is not rigid.

Another area of classical interest is the concept of *linkages*: 2-dimensional bar-joint frameworks with particular flexes. Kempe's Universality Theorem [Kem76] states that a flexible linkage may be constructed to trace out any given algebraic curve in the plane, though the complete proof is attributed to Kapovich and Millson [KM02]. Chebyshev [Che78] designed flexible frameworks that trace an approximation of a straight line with either a designated vertex (Chebyshev's Lambda Mechanism) or with the centre of a designated bar (the Chebyshev linkage). Hartenberg and Denavit [HD64] include an overview of many types of linkage, including the Peaucellier linkage.

Thus far, we have discussed rigidity in terms of continuous motions. It is a common technique to consider a linearised version known as *infinitesimal rigidity*.

Infinitesimal rigidity is in fact equivalent to *static rigidity*, a perspective on rigidity derived from engineering. Maxwell [Max64b] studied arbitrary d-dimensional bar-joint frameworks, rather than 1-skeletons of polyhedra, and characterised the orthogonal complement to the image of a linear transformation as the *"space of stresses"*. Non-trivial infinitesimal motions of a framework may be seen to be the dual of *equilibrium stresses* within a framework, which we wish to avoid in order to obtain minimally infinitesimally rigid frameworks.

A framework may also satisfy the stronger condition of having a unique realisation within d-dimensional space, which is known as *global rigidity*.

The following key result is the basis for much work relating to static and infinitesimal rigidity [Whi84].

Theorem 1.1.2 (Maxwell, 1864 [Max64b])**.** Let $G = (V, E)$ be a framework in d-space. The space of equilibrium stresses of G has dimension at least $|E| - d|V| + \binom{d+1}{2}$, with equality if and only if the framework is infinitesimally rigid.

In more recent years, there has been a key shift in the study of rigidity towards generic rigidity: that is, rigidity as a property of the underlying graph. For generic frameworks in all dimensions, Asimow and Roth prove that finite and infinitesimal rigidity are equivalent [AR78, AR79]. The following result was a key motivator towards the study of generically rigid graphs. It is conventionally referred to as Laman's Theorem [Lam70], although work by Pollaczek-Geiringer [PG27] predates that source.

Theorem 1.1.3 (Laman's Theorem [Lam70])**.** A graph G' is rigid in 2 dimensions if and only if G' contains a spanning subgraph $G = (V, E)$ such that $|E| = 2|V| - 3$ and $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E$.

Laman's Theorem encouraged the study of rigid graphs, rather than particular frameworks, and opened up the study of *combinatorial rigidity*. Graphs that satisfy the conditions of Laman's Theorem themselves are often referred to as *Laman graphs*, or *(2,3)-tight* graphs. An overview of rigidity of frameworks in 1 and 2 dimensions is given by Jackson [Jac07], while parts of Whiteley's 1978 draft book on "structural geometry" also remain available online [Whi78a, Whi78b].

Lovász and Yemini [LY82] give an alternate proof of Laman's Theorem based on matroid theory, which has a natural connection to the sparsity characterisation of rigid graphs in the plane. Laman's Theorem may be viewed as a characterisation of the 2-dimensional generic *rigidity matroid*, which is discussed in detail by Whiteley [Whi92, Whi96]. Jordán [Jor14] gives an overview of the generic rigidity matroid for graphs that are rigid in the plane, and Jackson and Jordán [JJ05] use the rigidity matroid to characterise globally rigid graphs in 2 dimensions.

The abstract rigidity matroid, introduced by Graver [Gra91] as an extension of the generic rigidity matroid, is discussed in more detail by Nguyen [Ngu10]. Graver et al [GSS93] also use the abstract rigidity matroid in their matroidal discussion of rigidity theory, while Servatius and Servatius [SS10] use the rigidity matroid to characterise rigid graphs in the plane.

Crapo [Cra90] gives a characterisation of generically isostatic frameworks in the plane in terms of tree decompositions of graphs, using an adaptation of an algorithm for partitioning matroids. Tay [Tay93] gives an alternative proof of this result, based purely on graph theory. Characterisations of additional classes of graphs also exist, such as planar minimally rigid graphs [HOR$^+$03], 3-connected rigidity circuits [BJ03b] and (k, ℓ)-sparse graphs for $0 \leq \ell \leq 2k - 1$ [LS08].

Henneberg's 1911 work "Die graphische Statik der starren Systeme" [Hen11] summarises much of the existing engineering literature about rigid structures at that time, and uses static rigidity to prove that there are certain recursive constructions that may be applied to a rigid framework to construct a larger rigid framework.

Since Laman's 1970 work, the inductive constructions used by Henneberg have been extended. Tay and Whiteley [TW85] give a generalisation of the 1-extension (often referred to as a Henneberg II move) to replace a larger subset of edges with a vertex of appropriate degree, and Whiteley [Whi90] gives additional vertex splitting moves. A survey of inductive constructions across various types of rigidity is given by Nixon and Ross [NR14], while Jordán and Szabadka [JS09] discuss inductive constructions that preserve global rigidity.

For $d \geq 3$, Connelly [Con05] gives a necessary condition for a d-dimensional bar-joint framework to be globally rigid, while global rigidity of a d-dimensional bar-joint framework is characterised as a property of the underlying graph by Gortler et al [GHT10]. Tanigawa [Tan15] and Hendrickson [Hen92] give additional properties of globally rigid graphs in d dimensions.

Since there is no combinatorial characterisation of generic rigid d-dimensional bar-joint frameworks for $d \geq 3$ for either local or global rigidity, there is interest in special classes of framework in higher dimensions [Tay84, TW84, WW87]. Katoh and Tanigawa [KT11] give a characterisation of generic infinitesimally rigid panel-hinge frameworks, proving Tay and Whiteley's Molecular Conjecture [TW84]. In particular, for $d = 3$ partial characterisations of global rigidity exist for body-bar frameworks [CJW13] and body-hinge frameworks [JKT16].

Along with considering alternative types of framework, many authors have considered the same notions of rigidity within alternative settings to d-dimensional Euclidean space, such as frameworks that are restricted to a particular surface. Necessary conditions for generic global rigidity on some surfaces are given by Jackson et al [JMN14]. Globally rigid frameworks constrained to a family of concentric cylinders are characterised by Jackson and Nixon [JN16], while a characterisation of infinitesimally rigid frameworks constrained to a family of concentric spheres is given by Nixon et al [NOP12]. The minimally rigid graphs for surfaces with exactly two trivial motions, such as the circular cylinder, are characterised as the (2,2)-tight graphs. Nixon et al [NOP12] give a construction for (2,2)-tight graphs, which Nixon and Owen [NO14] extend to a construction of (2,1)-tight graphs.

Another alternative problem is the rigidity of frameworks in non-Euclidean space. Kitson and Power [KP14] state a characterisation for minimally infinitesimally rigid frameworks in 2 dimensions with the ℓ^q-norm with $1 \leq q \leq \infty$, $q \neq 2$, where for $(x_1, x_2) \in \mathbb{R}^2$, $\|(x_1, x_2)\|_q = ((x_1)^q + (x_2)^q)^{\frac{1}{q}}$, while Kitson and Schulze [KS14] give Maxwell-style counts for frameworks in an arbitrary normed space $(X, \|\cdot\|)$.

Symmetric frameworks are another area of interest, since symmetry often naturally occurs in applications of rigidity theory. Depending on the geometry of the framework and the symmetry group, additional flexibility may occur within the framework. Forced symmetric frameworks are required to only have motions that preserve symmetry, and for frameworks that are generic with a given symmetry, the existence of a symmetric infinitesimal motion is equivalent to the existence of a symmetry-preserving continuous motion within the framework [Sch10d, SW17b]. Many sources on the rigidity of symmetric frameworks exist [MT14, NSSW14, OP10, Sch10a, Sch10c, ST15, SW11, SW12, The12], and we shall discuss rigidity of symmetric frameworks in further detail in Chapter 8, including the rigidity of frameworks with incidental symmetry.

It is natural that there is much interest in applications of rigidity theory. Hendrickson discusses algorithms for application in biology contexts [Hen95], and fast algorithms for checking Laman-style sparsity characterisations, such as FRODA [FS14] and KINARI [FJLS11], have been implemented in software to analyse rigid components of molecules.

Systems of geometric constraints, such as those discussed by Owen [Owe96], are often used in Computer Aided Design (CAD). Combinatorial algorithms now exist for identifying dependencies within graphs that occur generically, and dependencies that occur due to specific geometric realisations, and these algorithms may be applied to CAD problems [FKS+16]. Rigidity and persistence of directed graphs is also of interest in the context of analysing networks of autonomous agents [HADB07, JJ09].

It is also possible to consider a generalisation of bar-joint frameworks, referred to as *tensegrity frameworks*. Rather than bars having fixed lengths, these frameworks permit bars with minimum or maximum lengths (usually modelled by struts and cables respectively), and are an area of interest within engineering, introduced by Snelson [Sne96]. Williams [Wil03] combines results from both engineering and mathematical literature to give an overview of the kinematic and static rigidity of tensegrity structures. An identical graph may have both rigid and flexible generic realisations, since rigidity is not a generic property of tensegrity frameworks.

Roth and Whiteley [RW81] define various flavours of rigidity for tensegrity frameworks, and prove that static and infinitesimal rigidity are equivalent in the tensegrity context. The authors also characterise infinitesimal rigidity for a tensegrity framework, based on the infinitesimal rigidity of a standard bar-joint framework with the same structure and the space of stresses of this corresponding framework.

1.2 Coordinated rigidity

The main area of study within this thesis is the rigidity of *coordinated frameworks*, in which a subset of bars may change length, but must all change length by the same amount. These frameworks may be considered to be related to tensegrity frameworks, however rigidity of coordinated frameworks is a generic property only of the underlying graph. This allows us to develop a complete combinatorial characterisation for both 1-coordinated and 2-coordinated frameworks in the line and the Euclidean plane.

We model coordinated frameworks using coloured graphs. Motions of the framework are required to preserve the length of any uncoloured bars, while edges with the same colour are permitted to change length, provided that all edges of that colour change length by the same amount. Infinitesimal rigidity of these coordinated frameworks remains a generic property, and we have a Roth-Whiteley style equivalence between infinitesimal rigidity and static rigidity. We have also obtained inductive constructions for 1-coordinated and 2-coordinated frameworks in both $\mathbb{R}^1$ and $\mathbb{R}^2$, along with a characterisation of k-coordinated frameworks based on the graph being redundantly rigid to an appropriate degree.

The initial motivation for the study of coordinated frameworks was the existing work on the rigidity of frameworks on a family of concentric spheres, which may expand [NSTW18]. This type of framework may be viewed as a specific type of coordinated framework, in which all edges from a coordination class have the same length, and are all adjacent to a common vertex.

Another motivation for considering frameworks with coordinated classes of bars is the potential application of a collection of pistons which are all connected to a central pump, and so will extend or contract based on the pressure across the whole system. Discussions with John Owen also identified an alternative potential application, based around frameworks built from multiple different types of materials, which may expand at different rates when the framework is heated.

It seems likely that the concepts of coordinated rigidity will be applicable across many existing types of framework, such as periodic coordinated frameworks, or body-bar frameworks with coordinated bars. There is also the potential for different types of coordination constraint, such as preserving the ratio between lengths of bars from the same coordination class, or preserving the total length of all bars within a coordination class - which could clearly be of use in the construction of systems containing pulleys.

1.3 Overview

We begin by giving an overview of some results for rigidity of standard frameworks in Chapter 2. This includes a brief discussion of continuous rigidity in Section 2.3 and setting up the notions used to consider static rigidity in Section 2.5, along with an introduction to infinitesimal rigidity. The definitions given in Chapter 2 are extended in Chapter 3 to define the analogous concepts within coordinated frameworks, including continuous coordinated rigidity and infinitesimal coordinated rigidity in Section 3.1 and coordinated static rigidity in Section 3.3. We give an Asimow-Roth style result proving the equivalence of these types of rigidity for most frameworks.

We use coordinated static rigidity along with the rigidity matroid to develop a characterisation of infinitesimally rigid k-coordinated frameworks in any dimension in Chapter 4. This characterisation is based on the rigidity of a standard bar-joint framework with the same structure, similar to Roth and Whiteley's characterisation of the rigidity of tensegrities [RW81]. We make use of matroid theory in this chapter, which is introduced in Section 2.6.

The inductive constructions that preserve the infinitesimal rigidity of the underlying framework, introduced in Section 2.8, are also extended to coordinated frameworks in Chapter 5. In Chapter 6 we give Henneberg-type constructive characterisations along with Laman-type combinatorial characterisations for generically rigid 1-dimensional frameworks with one and two classes of coordinated edges, along with a Laman-type characterisation in the case of three coordinated classes. Chapter 7 contains both Laman-type combinatorial characterisations, and Henneberg-type inductive constructions, for generically rigid 2-dimensional frameworks with one and two classes of coordinated edges. We also discuss necessary conditions for frameworks in 1 and 2 dimensions with any number of coordination classes, and state conjectures for sufficient conditions.

Chapter 8 discusses existing work on infinitesimal rigidity of symmetric frameworks, and includes some initial work towards extending these results to symmetric coordinated frameworks. Chapter 9 concludes with a discussion of current open problems, and some areas that may yield interesting research in this field.

Chapter 2

Standard Rigidity

In this chapter, we shall begin by defining some different viewpoints on rigidity, and lead into the set up for combinatorial rigidity, which shall be the focus of much of this thesis.

2.1 Flavours of rigidity

Much of the mathematical research into geometric rigidity focuses on the study of bar-joint frameworks. The bars within such a framework are considered to be rigid, with fixed lengths, while the joints allow any motion within the d-dimensional space in which the framework exists. One question that is intuitive to ask is whether or not such a framework has a *non-trivial continuous motion.* A non-trivial continuous motion of a framework will preserve the lengths of the bars, while changing the distances between pairs of joints that are not directly connected. An example of a framework with a clear continuous motion would be the rectangular framework C_4, as illustrated in Figure 2.1a. We may consider the lower pair of joints as being held fixed, along with the bar that connects them, and the upper pair of joints each tracing a circular path around its neighbouring joint. The upper vertices may move in tandem to preserve the lengths of the bars of the framework, while the distances between diagonally opposite pairs of vertices vary. An alternative position of the framework, and the circular paths traced by the upper vertices, are shown in Figure 2.1b. We shall discuss continuous rigidity further in Section 2.3.

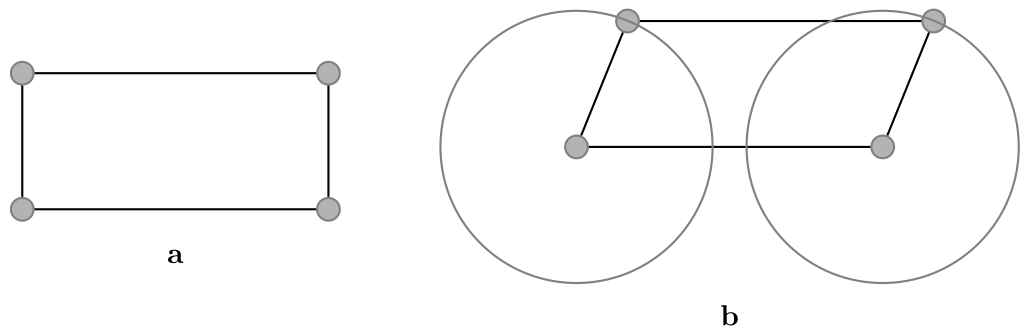
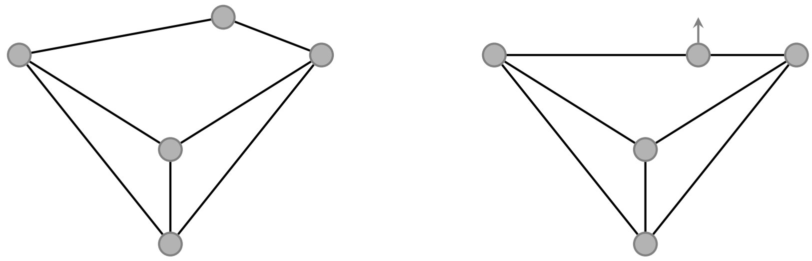


Figure 2.1 A continuous motion links Figure 2.1a and Figure 2.1b, showing that this is clearly not a continuously rigid framework. The motion may be considered as the upper pair of vertices tracing circles centred around the lower pair of vertices.

Another question that it is reasonable to ask is whether a framework, that may not seem to have any continuous motions, still might not be completely rigid, and may instead be noticed to be slightly shaky. We refer to this type of small motion as an *infinitesimal motion*. These motions are considered to be made up of velocity vectors assigned to each joint of the framework, but are still required to preserve the length of each bar of the framework at first order. Some infinitesimal motions may be considered as the very beginnings of a corresponding continuous motion, such as infinitesimal translations and rotations of the framework as a whole, which coincide with the continuous translations and rotations. Additional infinitesimal motions may occur that do not correspond to any continuous motion, such as that illustrated in Figure 2.2b. Section 2.4 defines infinitesimal motions and infinitesimal rigidity in more detail.

An alternative viewpoint on rigidity, which often occurs in an engineering context, is that of *static rigidity*. Instead of being concerned with whether there is a motion of a



a **b**

Figure 2.2 Figure 2.2a and Figure 2.2b share the same graph. The colinearity of the top three vertices in Figure 2.2b creates an infinitesimal motion, denoted by the arrow, which does not exist in the infinitesimally rigid framework of Figure 2.2a.

bar-joint framework, we wish to be sure that any equilibrium load placed on the joints of the framework can be resolved by a stress within the bars within the framework. Stresses indicate whether each bar is in compression or tension, and we wish to avoid stresses that may be resolved without any load being applied to the framework, referred to as *equilibrium stresses*.

These equilibrium stresses may be seen to be duals of any non-trivial infinitesimal motions using the rigidity matrix, as equilibrium stresses are $\boldsymbol{\omega} \in \mathbb{R}^{|E|}$ such that $\boldsymbol{\omega}^\top R(G,p) = \mathbf{0} \in \mathbb{R}^{d|V|}$, and infinitesimal motions are $\mathbf{u} \in \mathbb{R}^{d|V|}$ such that $R(G,p)\mathbf{u} = \mathbf{0} \in \mathbb{R}^{|E|}$ [Con87b]. Section 2.5 looks more closely at static rigidity, and its relationship with infinitesimal rigidity.

Another type of rigidity of bar-joint frameworks that is often considered is that of *global rigidity*. Two positions of a framework are still required to have the same distances between joints that are connected by a bar, but the requirement for a continuous motion to exist between the two positions within the same space is relaxed, and motions may be considered to move through higher dimensional spaces. Belk and Connelly [BC07] constructed frameworks that are not globally rigid in $\mathbb{R}^d$, but require at least $\mathbb{R}^{2d}$ for a motion between the positions, while Bezdek and Connelly [BC04] proved there is a motion in $\mathbb{R}^{2d}$ between an arbitrary pair of positions of a framework in $\mathbb{R}^d$, so $\mathbb{R}^{2d}$ is sufficient, and also necessary in some cases. Gortler et al [GHT10] proved that if a generic framework in $\mathbb{R}^d$ is not globally rigid, it will have another position that may be attained through a motion in $\mathbb{R}^{d+1}$, however this result does not apply to every alternative position of the framework.

We note that if a framework is not continuously rigid then it is clearly also not globally rigid, as illustrated by the two different positions of the framework in Figure 2.1. The converse does not hold, as may be seen in Figure 2.3. There are many sources on global rigidity, such as [Con05, Tan15, JS09], however we shall not consider global rigidity within this thesis. We shall begin by considering continuous rigidity in Section 2.3, infinitesimal rigidity in Section 2.4 and static rigidity in Section 2.5. Section 2.8 discusses some inductive constructions that may be applied to extend a framework, while preserving the infinitesimal rigidity properties of the underlying framework. We shall discuss analogous inductive constructions for coordinated frameworks in Chapter 5.

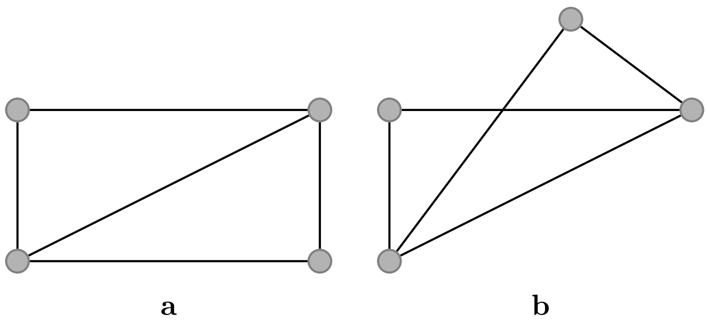


Figure 2.3 Figure 2.3a and 2.3b represent two positions of a framework that is clearly not globally rigid, as the two positions have very different distances between the pair of vertices not connected by an edge. The motion between the two positions is not contained within $\mathbb{R}^2$, and both realisations may be seen to be continuously rigid.

2.2 Definitions

Let $G = (V, E)$ be a graph, where V denotes the vertex set and E is the edge set. We shall frequently consider $|V| = n$, and may often identify V by $\{1, \ldots, n\} = [n]$.

We shall usually represent edges in E by unordered pairs of vertices, $\{i, j\}$. We may sometimes denote $|E|$ by m.

We consider the dimension d to be fixed.

Definition 2.2.1. Given a graph $G = (V, E)$, let $p : V \to \mathbb{R}^d$ be a map taking each vertex $i \in V$ to a position $p(i) \in \mathbb{R}^d$, with $p(i) \neq p(j)$ for any edge $\{i, j\} \in E$. The *d-dimensional framework* (G, p) is the graph G together with the configuration of the vertices p. We may refer to G as the *graph* of the framework (G, p).

When we identify V by $\{1, \ldots, n\}$, we may also consider the map as a point $p := (p(1), \ldots, p(n)) \in \mathbb{R}^{dn}$. We refer to $p \in \mathbb{R}^{dn}$ as a *d-dimensional configuration.*

Remark 2.2.2. In situations where the dimension is clear, we may refer simply to frameworks and configurations instead of d-dimensional frameworks and d-dimensional configurations.

Remark 2.2.3. For convenience of notation, we may sometimes use p_i to denote $p(i)$ for each vertex $i \in V$.

Definition 2.2.4. The *edge-length function* in $\mathbb{R}^d$ for a graph G is $f_G : \mathbb{R}^{dn} \to \mathbb{R}^m$, where $f_G(p)_{\{i,j\}} = \|p(i) - p(j)\|^2$.

Definition 2.2.5. Let $p, q \in \mathbb{R}^{dn}$ be two distinct d-dimensional configurations. The frameworks (G, p) and (G, q) are *equivalent* if

$$\|p(i) - p(j)\|^2 = \|q(i) - q(j)\|^2 \text{ for all } \{i, j\} \in E. \tag{2.1}$$

The frameworks (G, p) and (G, q) are *congruent* if

$$\|p(i) - p(j)\|^2 = \|q(i) - q(j)\|^2 \text{ for all } i, j \in V. \tag{2.2}$$

Remark 2.2.6. It is straightforward to see that a pair of d-dimensional frameworks (G, p) and (G, q) being equivalent corresponds to the configurations $p, q \in \mathbb{R}^{dn}$ having the same edge-length function, $f_G(p) = f_G(q)$.

Figure 2.1 shows two equivalent configurations of the rectangle C_4.

2.3 Finite rigidity

Definition 2.3.1. A framework (G, p) is *(locally) rigid* if there is a neighbourhood U of $p \in \mathbb{R}^{dn}$ such that, if $q \in U$ and the frameworks (G, p) and (G, q) are equivalent, then the frameworks (G, p) and (G, q) are congruent.

If a framework is not locally rigid, we may refer to it as *flexible*.

Definition 2.3.2 ([Gra01])**.** A *continuous motion* of a framework (G, p) is a family of continous functions $P_i : [0, 1] \to \mathbb{R}^d$, indexed by the vertex set V, such that:

1. $P_i(0) = p(i)$ for all $i \in V$;

2. $P_i(t)$ is differentiable on the interval $[0, 1]$ for all $i \in V$;

3. $\|P_i(t) - P_j(t)\|^2 = \|p(i) - p(j)\|^2$ for all $t \in [0, 1]$, for all edges $\{i, j\} \in E$.

A continuous motion is *non-trivial* if there is some $t_0 \in (0, 1]$ such that setting $q(i) = P_i(t_0)$ gives a framework (G, q) that is equivalent, but not congruent, to the framework (G, p).

A *trivial continuous motion* of a framework (G, p) preserves the distance between every pair of vertices within the framework.

Remark 2.3.3. The trivial continuous motions of any d-dimensional framework (G, p) may be viewed as being isometries of the space $\mathbb{R}^d$ applied at the vertices of the framework. In Euclidean space, these shall be translations, rotations, or combinations of these. We may also refer to the trivial motions of a framework as *rigid body motions*.

Remark 2.3.4. If a pair of d-dimensional frameworks, (G, p) and (G, q), are related through a trivial motion, then the framework (G, p) is clearly congruent to the framework (G, q).

Asimow and Roth [AR78] prove that a framework being flexible is equivalent to that framework having a non-trivial motion.

Proposition 2.3.5 (Proposition 1 [AR78])**.** Let G be a graph on n vertices, K be the complete graph on n vertices, and $p \in \mathbb{R}^{dn}$. The following are equivalent:

1. (G, p) is not rigid in $\mathbb{R}^d$;

2. (G, p) has a non-trivial continuous motion;

3. There exists a continuous path y in $f_G^{-1}(f_G(p))$ with $y(0) = p$ and $y(t) \notin f_K^{-1}(f_K(p))$ for some $t \in (0, 1]$.

The notation of the third condition looks a bit unusual in our context, but occurs quite naturally as Asimow and Roth [AR78] define a framework (G, p) as *not rigid* if there is no neighbourhood $U \subset \mathbb{R}^{dn}$ of p, such that the real algebraic varieties $f_G^{-1}(f_G(p))$ and $f_K^{-1}(f_K(p))$ coincide on U. The style of proof used in [AR78] is therefore very different to the other proofs within this thesis, so we shall give a restatement using slightly more intuitive notation, and using our Definition 2.3.1 in the first condition.

The third condition may in fact be seen to be equivalent to the existence of a continuous path $y : [0, 1] \to \mathbb{R}^{dn}$, where $y(0) = p$ and $(G, y(t))$ is not congruent to (G, p) for some $t \in (0, 1]$. It is straightforward to see the equivalence between this path y, and a non-trivial continuous motion of the framework (G, p).

If a framework (G, p) is not locally rigid (as in Definition 2.3.1), any neighbourhood U of the configuration p that contains an equivalent configuration, will contain an equivalent configuration that is not congruent to p. We may choose arbitrarily small neighbourhoods, and use these non-congruent configurations to construct a non-trivial motion of the framework (G, p).

Conversely, if there is a non-trivial motion $\{Q_i\}_{i \in V}$ of the framework (G, p), let $t_0 \in [0, 1)$ be the last point in the motion where the configuration given by $q_{t_0}(i) = Q_i(t_0)$ gives a framework (G, q_{t_0}) that is congruent to (G, p).

There is an isometry T of $\mathbb{R}^{dn}$ such that $T(q_{t_0}) = p$. If there is a neighbourhood U of q_{t_0} with $q_t \in U$ for some $t \in (t_0, 1]$, then there is a neighbourhood $T(U)$ of p that contains $T(q_t)$. The frameworks (G, q_t) and $(G, T(q_t))$ are congruent, since T is an isometry, but (G, q_t) is not congruent to (G, p). Hence the neighbourhood $T(U)$ of p contains a configuration $T(q_t)$ such that $(G, T(q_t))$ is equivalent, but not congruent, to the framework (G, p). As each motion in $\{Q_i\}_{i \in V}$ is continuous, any neighbourhood of q_{t_0} will contain at least one q_t for $t \in (t_0, 1]$, and so there will be a configuration which gives an equivalent but not congruent framework in every neighbourhood of p, and so (G, p) is not locally rigid.

2.4 Infinitesimal rigidity

The study of continuous motions is a logical place to begin when considering whether a given framework is rigid or flexible, however in general this problem is coNP-hard when $d \geq 2$ [Abb08]. Rather than consider quadratic algebra to find a motion for a 2-dimensional framework, it is useful to linearise the problem into something rather more tractable. Instead of our requirement that a continuous motion preserve the length of each bar, we now require that the projection along the bar of the motion on the vertices at each end of the bar be equal.

We have the following definition:

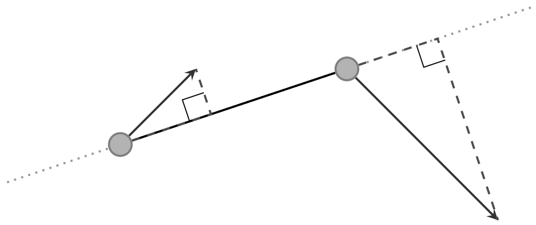


Figure 2.4 An infinitesimal motion of two joints and a single bar.

Definition 2.4.1. An *infinitesimal motion* of the framework (G, p) is a velocity field $p' \in \mathbb{R}^{dn}$ supported on p, such that

$$[p(i) - p(j)] \cdot [p'(i) - p'(j)] = 0 \text{ for all } \{i, j\} \in E. \tag{2.3}$$

Definition 2.4.2. The infinitesimal motion p' is a *trivial infinitesimal motion* of the framework (G, p) if it corresponds to a trivial motion in the continuous context, as defined in Definition 2.3.2, by differentiating and evaluating at $t = 0$.

A trivial infinitesimal motion p' of a framework (G, p) will satisfy the following equation:

$$[p(i) - p(j)] \cdot [p'(i) - p'(j)] = 0 \text{ for all } i, j \in V. \tag{2.4}$$

Remark 2.4.3. If a framework (G, p) affinely spans $\mathbb{R}^d$, the space of trivial infinitesimal motions of (G, p) has dimension $\binom{d+1}{2} = d + \binom{d}{2}$. This space is generated by d infinitesimal translations, and $\binom{d}{2}$ infinitesimal rotations, which correspond to the continuous translations and rotations.

If the framework (G, p) is contained within some strictly smaller subspace $\mathbb{R}^f \subsetneq \mathbb{R}^d$, there may be non-trivial infinitesimal motions that satisfy Equation (2.4). Example 2.4.4 describes one such framework, which is illustrated in Figure 2.5.

Example 2.4.4. Figure 2.5 shows the triangle K_3 with the configuration p given by $p(a) = (-2, 0)$, $p(b) = (0, 0)$, $p(c) = (2, 0)$, which has the following non-trivial infinitesimal motion p':

$$p'(a) = (0, 0) \qquad p'(b) = \left(0, \frac{1}{2}\right) \qquad p'(c) = (0, 0)$$

Note that the edge $\{a, c\}$ is illustrated as being curved for clarity, but would be considered to have length $\|p(a) - p(c)\| = 4$.

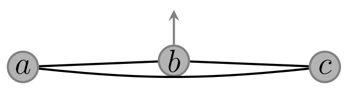


Figure 2.5 Example 2.4.4: a co-linear copy of K_3 with a non-trivial infinitesimal motion.

It is straightforward to check that the following all hold:

$$\begin{aligned}
[p(a) - p(b)] \cdot [p'(a) - p'(b)] &= [(-2, 0) - (0, 0)] \cdot \left[(0, 0) - \left(0, \frac{1}{2}\right)\right] \\
&= (-2, 0) \cdot \left(0, -\frac{1}{2}\right) = 0; \\
[p(b) - p(c)] \cdot [p'(b) - p'(c)] &= [(0, 0) - (2, 0)] \cdot \left[\left(0, \frac{1}{2}\right) - (0, 0)\right] \\
&= (-2, 0) \cdot \left(0, \frac{1}{2}\right) = 0; \\
[p(a) - p(c)] \cdot [p'(a) - p'(c)] &= [(-2, 0) - (2, 0)] \cdot [(0, 0) - (0, 0)] \\
&= (-4, 0) \cdot (0, 0) = 0.
\end{aligned}$$

Hence p' satisfies Equation (2.4), though it does not correspond to any trivial motion of the whole framework (G, p). This is due to the fact that $p' \notin \mathrm{span}(p)$.

Remark 2.4.5. We note that Example 2.4.4 gives a situation in which an infinitesimal motion of a framework satisfies Equation (2.4) while not being a trivial infinitesimal motion, as defined in Definition 2.4.2. This suggests that we shall want to avoid configurations that create degenerate frameworks or subframeworks, which in turn motivates some of the later definitions of this section, such as Definition 2.4.14.

Definition 2.4.6. The framework (G, p) is *infinitesimally rigid* if the only infinitesimal motions of the framework are the trivial infinitesimal motions.

Since the edge-length function f_G, defined in Definition 2.2.4, is clearly continuously differentiable, we make the following definition.

Definition 2.4.7. The *rigidity matrix* $R(G, p)$ for a d-dimensional framework (G, p) is an m by dn matrix derived from the edge-length function (Definition 2.2.4), with $df_G(p) = 2R(G, p)$.

The entries of $R(G, p)$ are functions of the coordinates of $p \in \mathbb{R}^{dn}$. These may be written in terms of the vectors $p(i)$ for $i \in V$. Each vertex $i \in V$ has d columns corresponding to it, with one row corresponding to each edge $\{i, j\} \in E$. The rigidity matrix is illustrated in Equation (2.5).

$$\begin{array}{rcl} & \begin{array}{ccccccc} \phantom{\mathbf{0}} & & i & & j & & \phantom{\mathbf{0}} \end{array} & \\ R(G,p) = & \begin{bmatrix} \vdots & \ddots & \vdots & \ldots & \vdots & \ddots & \vdots \\ \mathbf{0} & \ldots & \big(p(i) - p(j)\big) & \ldots & \big(p(j) - p(i)\big) & \ldots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ldots & \vdots & \ddots & \vdots \end{bmatrix} & \{i, j\} \end{array} \tag{2.5}$$

Remark 2.4.8. In situations where the graph of the framework (G, p) is clear, we may sometimes abbreviate $R(G, p)$ to simply $R(p)$.

Let (G, p) be a framework, and consider $p' \in \mathbb{R}^{dn}$ as an arbitrary vector. The set of equations defined by Equation (2.3) correspond exactly to the matrix equation $R(G, p)p' = 0$ when p' is an infinitesimal motion of the framework (G, p).

In fact, $R(G,p)p'$ may be considered for any pair of vectors $p, p' \in \mathbb{R}^{dn}$. If p is not a valid configuration for the graph G and has $p(i) = p(j)$ for an edge $\{i,j\} \in E$, there will be a zero row within $R(G,p)$.

In Remark 2.4.3 we saw that the space of trivial infinitesimal motions of a framework spanning $\mathbb{R}^d$ has dimension $\binom{d+1}{2}$. This leads straightforwardly to the conclusion that, as a framework (G,p) will be infinitesimally rigid if and only if the rigidity matrix $R(G,p)$ has maximal possible rank, (G,p) will be infinitesimally rigid if and only if the kernel of $R(G,p)$ has dimension $\binom{d+1}{2}$. We therefore have the following result.

Theorem 2.4.9 ([AR78])**.** A framework (G,p) that affinely spans $\mathbb{R}^d$ is infinitesimally rigid if and only if $\operatorname{rank} R(G,p) = dn - \binom{d+1}{2}$.

Definition 2.4.10. If the rows of $R(G,p)$ are independent, we refer to the framework (G,p) as being *independent*.

The framework (G,p) is *isostatic* if it is infinitesimally rigid and independent.

An independent framework will have $\operatorname{rank} R(G,p) = |E| = m$. As an isostatic framework simultaneously requires $\operatorname{rank} R(G,p) = d|V| - \binom{d+1}{2}$ and $\operatorname{rank} R(G,p) = |E|$, we reach the following standard results, which may be found in [SW17a] and other sources.

Theorem 2.4.11. Let (G,p) be a framework in $\mathbb{R}^d$ with $|V| \geq d$. The following are equivalent:

1. (G,p) is isostatic;

2. (G,p) is independent, and $|E| = d|V| - \binom{d+1}{2}$;

3. (G,p) is infinitesimally rigid, and $|E| = d|V| - \binom{d+1}{2}$;

4. (G,p) is infinitesimally rigid and $(G \setminus \{e\}, p)$ is infinitesimally flexible for any $e \in E$.

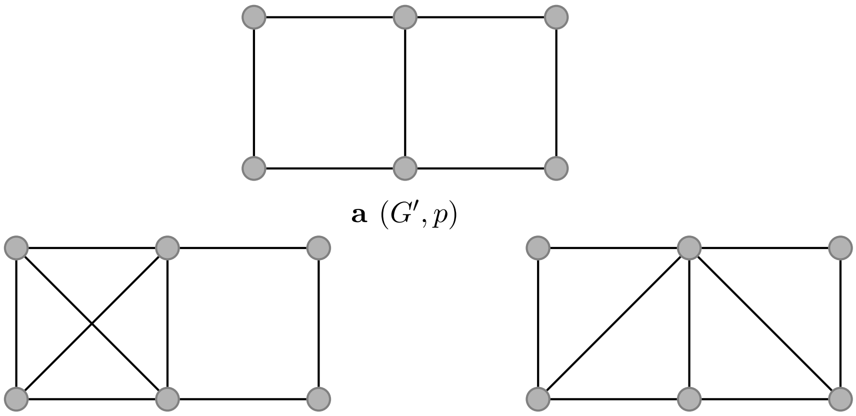
Corollary 2.4.12. A framework (G,p) in $\mathbb{R}^d$ with $n \leq d$ is isostatic if and only if $\operatorname{rank} R(G,p) = \binom{n}{2}$. Equivalently, (G,p) in $\mathbb{R}^d$ with $n \leq d$ is isostatic if and only if G is the complete graph on n vertices and the vertices of (G,p) do not lie in an affine space of dimension $n-2$.

Example 2.4.13. Figure 2.6 shows three frameworks, where (G_1,p) and (G_2,p), illustrated in Figure 2.6b and 2.6c respectively, are clearly both obtained by adding two edges to the framework (G',p) shown in Figure 2.6a.

We note that from Theorem 2.4.11, the framework (G',p) clearly cannot be isostatic, as $|E'| = 7 < 9 = 2|V'| - \binom{3}{2}$, though it is straightforward to see that the rows of $R(G',p)$ will be independent. This suggests that adding two edges will be sufficient to obtain an isostatic framework from (G',p), provided that adding these edges preserves independence.

It is useful to note that the rows of $R(K_4,q)$ cannot be independent for any configuration $q \in \mathbb{R}^2$, as the column rank is at most $2|V| - \binom{3}{2} = 8 - 3 = 5$ which is strictly less than the number of rows. Therefore since the framework (G_1,p) has a subgraph isomorphic to K_4, the rows of $R(G_1,p)$ will also not be independent, so the framework cannot be isostatic. We may also see that the subgraph isomorphic to C_4 will have a similar continuous motion to that illustrated in Figure 2.1.

(G_2,p) is obtained from (G',p) by adding a diagonal edge across each C_4 subgraph of G'. It is straightforward to see that the rows of $R(G_2,p)$ will be independent, and as $|E_2| = 9 = 2|V_2| - \binom{3}{2}$, (G_2,p) is an isostatic framework. It may also be checked to be a minimally rigid framework, as removing any edge will result in an underconstrained framework and hence a continuous motion, as seen in (G_1,p). Similarly we may see that (G_2,p) is infinitesimally rigid by Theorem 2.4.9, as the independent rows of $R(G_2,p)$ give $\operatorname{rank} R(G_2,p) = 9$ as required.



a (G', p)

b (G_1, p) **c** (G_2, p)

Figure 2.6 Example 2.4.13. (G_1, p) and (G_2, p) are both obtained by adding two bars to the underconstrained framework (G', p), however (G_1, p) has a continuous motion while (G_2, p) is infinitesimally rigid.

To be able to apply combinatorial techniques, we wish to avoid configurations $p \in \mathbb{R}^{dn}$ that lead to row dependencies due to the geometry of p, rather than due to the structure of G, as these lead to situations such as Example 2.4.4. We instead choose to work with the following kind of configurations.

Definition 2.4.14. A configuration $p \in \mathbb{R}^{dn}$ is *regular* if $\operatorname{rank} R(G,p) \geq \operatorname{rank} R(G,q)$ for all $q \in \mathbb{R}^{dn}$.

The set of such configurations is an open, dense subset of $\mathbb{R}^{dn}$ [AR78].

Definition 2.4.15 ([Con87a])**.** When the dn coordinates of $p \in \mathbb{R}^{dn}$ are algebraically independent over $\mathbb{Q}$, the configuration p is *generic.*

If the configuration p is generic, we refer to the framework (G, p) as being *generic.*

The condition for a configuration to be generic is stronger than the condition for a configuration to be regular, however the generic configurations are also a dense subset of $\mathbb{R}^{dn}$ [SW17a].

Theorem 2.4.16 ([AR79])**.** If the d-dimensional framework (G, p) is infinitesimally rigid, then (G, p) is rigid.

This statement is intuitively true, as it is clear that given a non-trivial continuous motion $\{P_i(t)\}$, we may obtain a corresponding non-trivial infinitesimal motion by differentiating and evaluating at $t = 0$. The details of this result are slightly more involved, and in fact Connelly [Con87a] gives three different proof techniques. The first is attributed to Alexandrov [Ale05] and Gluck [Glu75], and also given by Asimow and Roth [AR78, AR79]. These authors apply the implicit function theorem and use differential topology, while an alternate proof by Connelly uses the notion of analytic rigidity to show that the existence of a non-trivial analytic motion implies the existence of a non-trivial infinitesimal motion. The other proof, referenced to Whiteley, relies on what is often referred to as *the averaging method*, and is the method we shall use in Section 3.2 to prove that the same result holds for coordinated frameworks.

There may be situations in which a non-trivial infinitesimal motion does not have a corresponding non-trivial continuous motion, such as the motion in Example 2.4.13. By requiring that the framework (G, p) has a rigidity matrix $R(G, p)$ with maximum possible rank, we may see the following result from Asimow and Roth [AR79].

Theorem 2.4.17 ([AR78, AR79])**.** Let (G, p) be a d-dimensional framework with a regular configuration $p \in \mathbb{R}^{dn}$. Then (G, p) is rigid if and only if (G, p) is infinitesimally rigid.

Asimow and Roth prove in their first paper [AR78] that for regular configurations $p \in \mathbb{R}^{dn}$ that span the space, the framework (G, p) is rigid if and only if the derivative of the edge length function at p has rank $df_G(p) = dn - \binom{d+1}{2}$. The authors then show in their second paper [AR79] that a framework (G, p) is infinitesimally rigid if and only if rank $df_G(p) = dn - \binom{d+1}{2}$, and so rigidity and infinitesimal rigidity coincide for regular configurations $p \in \mathbb{R}^{dn}$.

The following lemma, given by Graver, Servatius and Servatius [GSS93], allows us to transform regular configurations without losing the regular property. The authors state this result in terms of "generic" configurations, which they define as being those configurations for which no non-zero minor of the rigidity matrix $R(G,p)$ has determinant equal to zero. This condition is weaker than our notion of generic, as defined in Definition 2.4.15, however it is a stronger condition than our notion of regular (Definition 2.4.14) and also results in an open dense set of configurations. We shall restrict use of this property to regular configurations, and apply it when proving that coordinated inductive constructions preserve generic rigidity (see Chapter 5).

Lemma 2.4.18 (Lemma 2.2.2 [GSS93])**.** Let $V = \{1,\ldots,n\}$ and let $p : V \to \mathbb{R}^d$ be any configuration of V. Let A be an affine transformation of $\mathbb{R}^d$, and let $K = (V, E_K)$ be the complete graph on V. Then

a. For any edge set $E \subseteq E_K$, the rows of $R(K, A(p))$ corresponding to E are independent if and only if the rows of $R(K,p)$ corresponding to E are independent.

b. $A(p)$ is a regular configuration if and only if p is a regular configuration.

2.5 Static rigidity

We shall now consider the concept of static rigidity, leading up to the standard result that static rigidity and infinitesimal rigidity are equivalent.

Recall that (G,p) is a framework in $\mathbb{R}^d$, with $|V| = n$ and $|E| = m$. We begin with the following standard definitions (see, for example, [Con87b, RW81, Whi78b]).

Definition 2.5.1. A *load* on a d-dimensional framework (G,p) is a set $\mathbf{F} := \{F_1,\ldots,F_n\}$, where each $F_i \in \mathbb{R}^d$.

$\mathbf{F}$ is an *equilibrium load* of (G,p) if

$$\sum_{i=1}^{n} F_i = 0, \tag{2.6}$$

and, for each pair of coordinate directions $1 \leq h < k \leq d$,

$$\sum_{i=1}^{n} \Big[(F_i)_k \, (p_i)_h - (F_i)_h \, (p_i)_k \Big] = 0, \tag{2.7}$$

where $(F_i)_k$ denotes the k^{th} component of the force vector F_i, and $(p_i)_h$ similarly denotes the h^{th} component of the position vector $p_i = p(i)$.

We may consider a load as assigning a force F_i at each distinct joint $p(i)$ of the framework (G, p), for each vertex $i \in V$.

By restricting to the class of equilibrium loads, we remove those loads that would induce a translation or rotation of the whole framework. Equation (2.6) requires that the load apply no net force to the framework as a whole, which is equivalent to there being no rigid body translation of the framework. Equation (2.7) requires that any equilibrium load apply no net rotation to the framework. The space of equilibrium loads therefore has dimension $dn - \binom{d+1}{2}$, when (G, p) affinely spans the space.

At times, it may be useful to consider each force F_i as a vector, and $\mathbf{F}$ as the concatenation of these vectors, $\mathbf{F} := [F_1, \ldots, F_n] \in \mathbb{R}^{dn}$.

Definition 2.5.2. A *stress* on the framework (G, p) is an assignment of a scalar $\omega_{\{i,j\}}$ to each edge $\{i, j\} \in E$. Using the same ordering of the edges as is used to order the rows of the rigidity matrix $R(G, p)$, we may consider the stress as a vector $\boldsymbol{\omega} \in \mathbb{R}^m$.

A stress $\boldsymbol{\omega} \in \mathbb{R}^m$ on (G, p) is a *resolution* of an equilibrium load $\mathbf{F}$ if, for every vertex $i \in V$,

$$\sum_{j:\{i,j\} \in E} \omega_{\{i,j\}} \Big[p(i) - p(j) \Big] = F_i. \tag{2.8}$$

Remark 2.5.3. It is useful to note that by considering every vertex in V, we may write Equation (2.8) in terms of the rigidity matrix, as follows:

$$\boldsymbol{\omega}^\top R(G,p) = \mathbf{F}. \tag{2.9}$$

We consider $\boldsymbol{\omega}$ as a column vector in Equation (2.9).

Definition 2.5.4. We say that the load $\mathbf{F}$ is *resolvable* by (G,p) when a resolution $\boldsymbol{\omega}$ exists.

Definition 2.5.5. A framework (G,p) is *statically rigid* if every equilibrium load on (G,p) has a resolution.

Rather than consider every equilibrium load when checking whether the framework (G,p) is statically rigid, we define the following class of loads that generate the space of equilbrium loads.

Definition 2.5.6. An *edge load* $\mathbf{F}_{ij}$ is defined for any pair of vertices $i,j \in V$, where $F_i = p(i) - p(j)$, $F_j = p(j) - p(i)$, $F_k = 0$ for $k \neq i,j$.

It is straightforward to check that for every edge $\{i,j\} \in E$, the edge load $\mathbf{F}_{ij}$ is an equilibrium load.

Lemma 2.5.7 ([Con87b])**.** Let $p \in \mathbb{R}^{dn}$ be a configuration that affinely spans $\mathbb{R}^d$. Then the set of edge loads $\left\{\mathbf{F}_{ij} : 1 \leq i < j \leq n\right\}$ generates the space of equilibrium loads on (G,p).

We now define a resolution for each edge load as follows.

Definition 2.5.8. An *edge resolution* $\boldsymbol{\rho}_{\{i,j\}}$ for the edge $\{i,j\} \in E$ is given by $\rho_{\{i,j\}_{ij}} = 1$, and $\rho_{\{i,j\}_{kl}} = 0$ for all edges $\{k,l\} \neq \{i,j\}$.

It is clear that the edge resolution $\boldsymbol{\rho}_{\{i,j\}}$ resolves the edge load $\mathbf{F}_{ij}$ for each edge $\{i,j\} \in E$.

Definition 2.5.9. An *equilibrium stress* is a stress $\boldsymbol{\omega} \in \mathbb{R}^{|E|}$ that resolves the zero load, so for every vertex $i \in V$ we have

$$\sum_{j:\{i,j\}\in E} \omega_{\{i,j\}} \Big[p(i) - p(j)\Big] = 0. \tag{2.10}$$

This is equivalent to $\boldsymbol{\omega}^\top R(G,p) = \mathbf{0}$. The space of equilibrium stresses of a framework (G,p) may be denoted by $S(G,p)$.

The following key result allows us to consider the infinitesimal rigidity and static rigidity of a framework as being interchangeable.

Theorem 2.5.10 ([Con87b, Hen11, RW81, Whi78b])**.** Infinitesimal rigidity and static rigidity are equivalent.

We give a sketch of the standard proof here, as we use a similar style of proof for the coordinated analogue to this result (Theorem 3.3.7).

Let (G,p) be a framework that affinely spans $\mathbb{R}^d$. A framework is statically rigid when every equilibrium load has a resolution, and from Lemma 2.5.7 all equilibrium loads can be generated by the set of edge loads $\Big\{\mathbf{F}_{ij} : 1 \leq i < j \leq n\Big\}$. The equilibrium loads exclude those inducing a translation or rotation of the whole framework (G,p), and hence the space of equilibrium loads has dimension $dn - d - \binom{d}{2} = dn - \binom{d+1}{2}$.

An edge load is defined for every pair of vertices $i, j \in V$, and for any $\{i,j\} \in E$ the edge load $\mathbf{F}_{ij}$ corresponds to the associated row of the rigidity matrix $R(G,p)$. Every equilibrium load of (G,p) may be considered as a sum of edge loads, and hence every equilibrium load has a resolution if and only if each edge load $\mathbf{F} \in \Big\{\mathbf{F}_{ij} : 1 \leq i < j \leq n\Big\}$ has a resolution. This is equivalent to the row rank of the rigidity matrix $R(G,p)$ being $dn - \binom{d+1}{2}$, so (G,p) has $dn - \binom{d+1}{2}$ independent edges. From Theorem 2.4.9, a framework (G,p) is infinitesimally rigid if and only if $\operatorname{rank} R(G,p) = dn - \binom{d+1}{2}$, and hence static and infinitesimal rigidity are equivalent.

By considering the space of infinitesimal motions of the framework (G, p), and the set of equilibrium loads on the framework (G, p), Roth and Whiteley [RW81] prove that every infinitesimal motion is a trivial infinitesimal motion if and only if every equilibrium load is resolvable. This leads to the following correspondence, as stated by Connelly [Con87b].

Lemma 2.5.11 ([Con87b])**.** Let (G, p) be a framework in $\mathbb{R}^d$ and let $i, j \in V$ be any pair of vertices. Then the edge load $\mathbf{F}_{ij}$ cannot be resolved if and only if there is an infinitesimal motion p' of (G, p) such that $\Big(p(i) - p(j)\Big) \cdot \Big(p'(i) - p'(j)\Big) \neq 0$.

Lemma 2.5.11 is illustrated in Example 2.5.12. This example also gives some intuition towards Proposition 4.1.2, which shall be required for the coordinated statics viewpoint used in Chapter 4. We also define the following concept.

Example 2.5.12. Let $(G, p) = (G_1, p)$ from Example 2.4.13 (Figure 2.6b). Let the rows of the rigidity matrix $R(G, p)$ be ordered lexicographically, with respect to the labelling of the vertices given in Figure 2.7. We use the same ordering for the entries of $\boldsymbol{\rho} \in \mathbb{R}^9$, which is a resolution of the load $\mathbf{F}$ when $\boldsymbol{\rho}^\top R(G, p) = \mathbf{F}$.

Let $\boldsymbol{\rho} := \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix}^\top \in \mathbb{R}^9$. This is a resolution of the equilibrium load $\mathbf{F} := [(0,0),(0,0),(1,0),(1,0),(-1,-1),(-1,1)] \in \mathbb{R}^{12}$, which is illustrated by grey arrows in Figure 2.7.

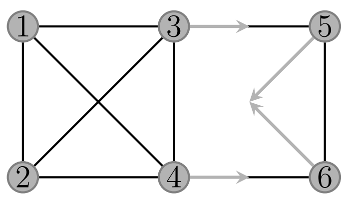


Figure 2.7 Example 2.5.12: the framework (G_1, p) from Example 2.4.13 (Figure 2.6b) with an equilibrium load indicated in grey.

Similarly, let $\boldsymbol{\omega} := \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}^\top \in \mathbb{R}^9$. Clearly $\boldsymbol{\omega} \neq \mathbf{0}$, however $\boldsymbol{\omega}^\top R(G, p) = \mathbf{0}$, and hence $\boldsymbol{\omega}$ is an equilibrium stress of the framework (G, p).

We note that $\boldsymbol{\tau} := \boldsymbol{\rho} - \boldsymbol{\omega} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix}^\top$ may be straightforwardly checked to be another resolution to the equilibrium load $\mathbf{F}$. This gives some intuition towards Proposition 4.1.2.

Let $\mathbf{F}'$ be the edge load $\mathbf{F}_{36} = [(0,0),(0,0),(-1,1),(0,0),(0,0),(1,-1)] \in \mathbb{R}^{12}$. This is an equilibrium load, illustrated in Figure 2.8a, which corresponds to the non-trivial motion shown in Figure 2.8b by Lemma 2.5.11, and hence has no resolution.

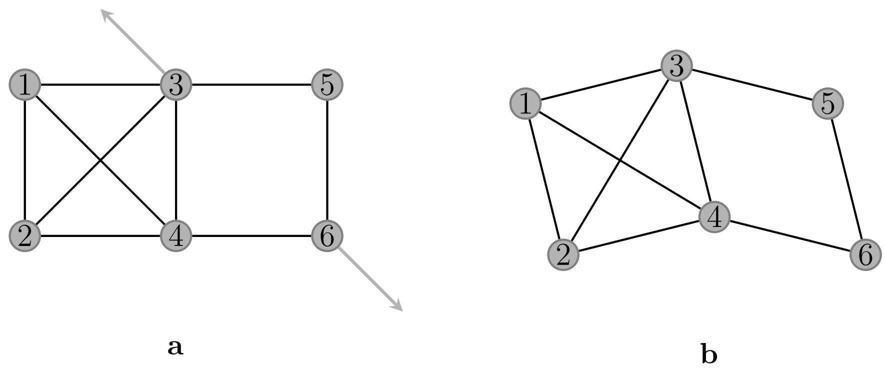


Figure 2.8 Example 2.5.12: an equilibrium load with no resolution, and the corresponding non-trivial motion.

2.6 The rigidity matroid

The structure of a matroid on a set E may be defined by the independent subsets of E, the minimal dependent sets, the closure operator, or in terms of the submodular rank function on all subsets of E. We shall mainly consider the rigidity matroid in terms of the independent and minimally dependent sets.

Definition 2.6.1 ([Oxl06]). A finite set E, and a collection $\mathcal{I}$ of subsets of E, form a *matroid on* E, $\mathcal{M} = (E, \mathcal{I})$, when $\mathcal{I}$ satisfies the following conditions:

(I1) $\emptyset \in \mathcal{I}$;

(I2) If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$;

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_2| < |I_1|$, there there exists $e \in I_1 \setminus I_2$ such that $I_2 \cup \{e\} \in \mathcal{I}$.

E is referred to as the *ground set* of $\mathcal{M}$. A subset $I \subseteq E$ with $I \in \mathcal{I}$ is an *independent set* of $\mathcal{M}$, and any subset of E that is not in $\mathcal{I}$ is *dependent*.

Definition 2.6.2 ([Oxl06])**.** A *circuit of* $\mathcal{M}$ is $C \subseteq E$ such that $C \notin \mathcal{I}$ and, for every proper subset $C' \subsetneq C$, $C' \in \mathcal{I}$. These are the minimal dependent sets of E.

A *basis of* $\mathcal{M}$ is a maximal independent set of E. These are $B \subseteq E$ such that $B \in \mathcal{I}$, and for any other $I \in \mathcal{I}$, $|I| \leq |B|$.

Matroids may be straightforwardly generated by rows of a matrix, where the independent sets $I \subseteq E$ clearly correspond to submatrices with independent rows. The rank of these submatrices corresponds to the submodular rank function (see, for example, [Oxl06]). We may use the rigidity matrix $R(G,p)$ to generate the rigidity matroid as follows.

Definition 2.6.3 ([GSS93])**.** Let V be a set of n vertices and let $K_n = (V, K(V))$ denote the complete graph on the vertex set V. Let the dimension $d \geq 1$ be given, and let $p \in \mathbb{R}^{dn}$ be a generic configuration of V in $\mathbb{R}^d$. The *d-dimensional generic infinitesimal rigidity matroid* for K_n is denoted by $\mathcal{M}_d(K_n)$ (or $\mathcal{M}_d$), and is generated by the rows of the rigidity matrix $R(K_n, p)$.

Let $G = (V, E)$ be a graph on n vertices. The d-dimensional generic rigidity matroid for G, $\mathcal{M}_d(G)$, is generated by the rows of the rigidity matrix $R(G,p)$. This matroid may be viewed as a deletion of $\mathcal{M}_d(K_n)$, since $R(G,p)$ is a submatrix of $R(K_n,p)$.

Remark 2.6.4. Each edge $e \in K(V)$ has a corresponding row in the matrix $R(K_n,p)$. A collection of edges $E \subseteq K(V)$ is *independent* if the rows of $R(K_n,p)$ corresponding to the edges in E are independent. This coincides exactly with the notion of a framework (G,p) being independent, introduced in Section 2.4.

Definition 2.6.5 ([GSS93]). Let $E \subset K(V)$ be an edge set from the complete graph on n vertices, $K_n = (V, K(V))$. An edge $e \in K(V) \setminus E$ is *independent of* E if the row of $R(K_n, p)$ corresponding to the edge e is independent of the rows corresponding to the edges in E. If e is not independent of E, then it is *induced by* E.

The vertex set of $E \subset K(V)$ is denoted by $V(E)$, where $v \in V(E)$ if and only if $\{v, w\} \in E$ for some vertex $w \in V$. When every vertex is contained in $V(E)$ (i.e. $V(E) = V$), E may be referred to as a *spanning edge set* for $K_n = (V, K(V))$.

The graph with edge set $K(V) \setminus E$ may be denoted by $K_n \setminus E$.

Remark 2.6.6 (Lemma 2.5.1 [GSS93]). Let $G = (V, E)$. It may be useful to note that an edge $\{i, j\} \in K(V)$ being independent of the edge set E is equivalent to the regular framework (G, p) having an infinitesimal motion p' such that $[p(i) - p(j)] \cdot [p'(i) - p'(j)] \neq 0$.

Definition 2.6.7 ([GSS93]). An edge set E is rigid if the framework (G, p), where $G = (V(E), E)$ and $p \in \mathbb{R}^{dn}$ is any regular configuration, is infinitesimally rigid.

Lemma 2.6.8 (Lemma 2.5.2 [GSS93]). Let V be a set of n vertices, where $K_n = (V, K(V))$ denotes the complete graph on the vertex set V. Let $p \in \mathbb{R}^{dn}$ be a regular configuration of V, and let $E \subseteq K(V)$ be a spanning edge set (i.e. $V(E) = V$). The edge set E is rigid if and only if every edge $f \in K(V) \setminus E$ is induced by the edge set E.

2.7 Combinatorial rigidity

From Theorem 2.4.9 we know that an infinitesimally rigid framework (G, p) has $\operatorname{rank} R(G, p) = dn - \binom{d+1}{2}$. Definition 2.4.14 defines the regular configurations of a framework to be those $p \in \mathbb{R}^{dn}$ for which $\operatorname{rank} R(G, p)$ is maximal in $\mathbb{R}^{dn}$, so we note that if (G, p) is infinitesimally flexible for some regular $p \in \mathbb{R}^{dn}$ then (G, q) is infinitesimally flexible for all $q \in \mathbb{R}^{dn}$. If instead (G, p) is infinitesimally rigid for some regular $p \in \mathbb{R}^{dn}$, then (G, q) is infinitesimally rigid for all regular $q \in \mathbb{R}^{dn}$. This may be formally stated as follows.

Lemma 2.7.1 (Corollary 2 [AR78])**.** If a framework (G, p) is infinitesimally rigid for some configuration $p \in \mathbb{R}^{dn}$, then (G, q) is infinitesimally rigid for all regular configurations $q \in \mathbb{R}^{dn}$.

It may also be useful to note that by Theorem 2.4.17, we know that for frameworks (G, p) with a regular configuration $p \in \mathbb{R}^{dn}$, infinitesimal rigidity and continuous rigidity are equivalent. When p is regular, rather than consider the rigidity of specific realisations of frameworks, we may consider rigidity as a property of the graph G.

Definition 2.7.2. A graph G is *generically rigid* in dimension d if there is a generic configuration $p \in \mathbb{R}^{dn}$ such that the framework (G, p) is infinitesimally rigid. If G is also independent, we may refer to G as being *d-isostatic*.

We may now characterise frameworks in $\mathbb{R}^d$ that are isostatic for all regular configurations $p \in \mathbb{R}^{dn}$, by characterising d-isostatic graphs. We may therefore apply combinatorial and graph theoretic techniques while avoiding frameworks in singular positions.

From the rigidity matrix $R(G, p)$ and Theorem 2.4.11, it seems useful to define certain sparsity counts for graphs. For d-isostatic graphs, we require that $|E| = d|V| - \binom{d+1}{2}$, and we note that for an independent framework we will also have $|E(V')| \leq d|V'| - \binom{d+1}{2}$ for any subgraph generated by $V' \subset V$ with $|V'| \geq d$. We give the following standard definition in general, noting that the counts in Theorem 2.4.11 are equivalent to the case where $k = d$ and $\ell = \binom{d+1}{2}$.

Definition 2.7.3. A finite graph G is *(k, ℓ)-sparse* if, for all subgraphs $G' = (V', E')$ with $|V'| \geq k$, $|E'| \leq k \cdot |V'| - \ell$. If G also satisfies $|E| = k \cdot |V| - \ell$, then G is *(k, ℓ)-tight*. We may refer to G as a *(k, ℓ)-graph* when G is (k, ℓ)-tight.

A (k, ℓ)-*circuit* is a graph G where the removal of any edge $e \in E$ results in a (k, ℓ)-tight graph $G - e$. This is equivalent to $|E| = k|V| - \ell + 1$, with $|E'| \leq k \cdot |V'| - \ell$ for all proper subgraphs $G' = (V', E')$ with $|V'| \geq k$.

If G is a (k, ℓ)-sparse graph with a (k, ℓ)-tight subgraph G', the subgraph G' is a (k, ℓ)-*block*.

Remark 2.7.4. Recall the definition of a matroid Section 2.6: they may be viewed either as a collection of independent subsets of the ground set E, or in terms of a rank function on the subsets of E, where $D \subseteq E$ is independent if and only if $r(D) = |D|$. The subsets of rows of the rigidity matrix of the complete graph on n vertices, $R(G, p)$, that are independent correspond straightforwardly to the independent sets of the infinitesimal rigidity matroid $\mathcal{M}_d(G)$ (Definition 2.6.3).

The independent sets of the (k, ℓ)-*sparsity matroid* [LS08, Whi96] on an edge set E are precisely all (k, ℓ)-sparse subgraphs. These matroids are characterised for integers k, ℓ such that $0 \leq \ell \leq 2k - 1$ by the (k, ℓ)-pebble game algorithms [LS08].

The minimal dependent sets of a matroid are referred to as the *circuits* of the matroid, and the circuits in the (k, ℓ)-sparsity matroid correspond precisely to the (k, ℓ)-circuits as defined above.

Remark 2.7.5. Following convention, we refer to graphs that are (2,3)-tight as *Laman graphs*, and (2,3)-sparse graphs as *Laman-sparse graphs.* We may also refer to (2,3)-tight subgraphs within a graph as *rigid blocks*, and (2,3)-circuits as *rigidity circuits* (or *circuits* where the context is clear).

Remark 2.7.6. We note that some authors refer to graphs that are $\left(d, \binom{d+1}{2}\right)$-tight as satisfying the *d-dimensional Maxwell-Laman conditions.*

Theorem 2.7.7 ([Max64a, GSS93])**.** If a framework (G, p) that affinely spans $\mathbb{R}^d$ is d-isostatic, the graph G is $\left(d, \binom{d+1}{2}\right)$-tight.

This result follows from Theorem 2.4.11, as an isostatic framework in $\mathbb{R}^d$ must have $|E| = d|V| - \binom{d+1}{2}$, and independence of the rows of $R(G,p)$ will lead to rank $R(G,p) = |E|$. Any submatrix generated by a subset of the rows of $R(G,p)$, equivalent to a subset of the edges $F \subset E$, will also be independent, and so rank $R\Big((V(F), F), p|_{V(F)}\Big) = |F|$. This rank cannot be larger than the number of non-zero columns of the submatrix with the dimension of the space of trivial infinitesimal motions subtracted, and so rank $R\Big((V(F), F), p|_{V(F)}\Big) \leq d|V(F)| - \binom{d+1}{2}$, leading to the required subgraph count.

2.8 Inductive constructions

We shall now consider some inductive graph moves which preserve rigidity, and so may be used to construct larger rigid graphs from smaller graphs that are known to be generically rigid, and are also often used to prove combinatorial rigidity results. We begin with the definitions of the types of extension we shall consider, and state results about them sorted by dimension.

Definition 2.8.1. Let $G = (V, E)$ be a graph. The *d-dimensional 0-extension* of G is the graph $G' = (V', E')$, where $V' := V \cup \{x\}$ for some new vertex x, and $E' := E \cup \Big\{\{x, v_i\} : 1 \leq i \leq d\Big\}$ for some $\{v_i : 1 \leq i \leq d\} \subset V$.

Lemma 2.8.2 ([TW85])**.** Let $G' = (V', E')$ be a graph obtained from $G = (V, E)$ by applying the d-dimensional 0-extension. Then G is generically d-isostatic if and only if G' is generically d-isostatic.

Definition 2.8.3. Let $G = (V, E)$ be a graph with some edge $\{a, b\} \in E$. The *d-dimensional 1-extension* of G on the edge $\{a, b\}$ is $G' = (V', E')$, where $V' := V \cup \{x\}$ for some new vertex x, and $E' := E \setminus \Big\{\{a, b\}\Big\} \cup \Big\{\{x, v_i\} : 1 \leq i \leq d-1\Big\}$ for some $\{v_i : 1 \leq i \leq d-1\} \subset V \setminus \{a, b\}$.

Lemma 2.8.4 ([AR78, TW85])**.** Let $G = (V, E)$ be a graph with an edge $\{a, b\} \in E$ and $d - 1$ distinct vertices $v_i \in V \setminus \{a, b\}$ for $1 \leq i \leq d - 1$, and let $G' = (V', E')$ be a graph obtained from G by applying a d-dimensional 1-extension to the edge $\{a, b\}$. If G is generically d-isostatic, then G' will also be generically d-isostatic.

Remark 2.8.5. We may refer to the reverse of the 0-extension and 1-extension moves as the *0-reduction* and *1-reduction* respectively.

Remark 2.8.6. The 0-extension and 1-extension were described by Henneberg [Hen11], and so are often referred to as *Henneberg moves* of type I and II respectively.

There are also other types of inductive move, such as vertex splitting and extensions applied to higher numbers of edges [NR14, TW85]. We discuss the 2-extension further in Section 2.8.2, as we require a coordinated analogue to this move for the 2-coordinated 2-dimensional constructive characterisation (Theorem 7.2.20), and we also consider some other types of extension in Section 2.9.

2.8.1 Dimension 1

When $d = 1$, continuous rigidity and infinitesimal rigidity coincide [Gra01]. The 1-dimensional Maxwell-Laman conditions are equivalent to requiring that the graph G be a tree, and so a framework (G, p) is isostatic if and only if the graph G is a tree [Gra01]. It is also known that all trees may be constructed from a single vertex by repeated applications of the 1-dimensional 0-extension (see, for example, [Gra01, Theorem 2.20]), and so the 0-extension is sufficient to generate all isostatic frameworks in $\mathbb{R}^1$, though the 1-dimensional 1-extension also preserves isostaticity.

Theorem 2.8.7 ([Gra01])**.** Let G be a graph. The following are equivalent:

1. G is infinitesimally rigid in 1 dimension, and independent;

2. G is continuously rigid in 1 dimension, and independent;

3. G is a tree;

4. G can be constructed from a single vertex by a sequence of 0-extensions.

2.8.2 2 dimensions

When $d = 2$, the conditions for a graph to be $\left(d, \binom{d+1}{2}\right)$-tight are more commonly referred to as the Laman conditions (Remark 2.7.5). Graphs that satisfy these are known as Laman graphs. It is conventional to refer to Laman [Lam70], though many of his results were discovered earlier by Pollaczek-Geiringer [PG27].

Theorem 2.8.8 (Laman's Theorem [Lam70])**.** A graph G is generically infinitesimally rigid, and independent, in 2 dimensions if and only if G is a Laman graph.

Laman's proof of this result rests on the characterisation of what he refers to as "E-graphs": graphs with $|V| \geq 3$, $|E| = 2|V| - 3$, and $|E(V')| \leq 2|V'| - 3$ for any subgraph induced by $V' \subset V$ with $|V'| \geq 2$. These may be seen to be the (2,3)-tight graphs with $|V| \geq 3$, and so the minimal such "E-graph" is K_3, which may be obtained by applying a 0-extension to K_2, the minimal (2,3)-tight graph.

Laman begins by proving that any graph that is infinitesimally rigid in 2 dimensions has a spanning subgraph G', where G' is infinitesimally rigid in 2 dimensions and $|E'| = 2|V'| - 3$, and follows this by proving that any graph G that is infinitesimally rigid in 2 dimensions with $|E| = 2|V| - 3$ is in fact (2,3)-tight.

As the minimally infinitesimally rigid frameworks are characterised, we have the following straightforward consequence.

Corollary 2.8.9. A graph G is infinitesimally rigid in 2 dimensions if and only if G has a spanning Laman subgraph.

Laman proves the following statement (also given by Tay and Whiteley [TW85]), along with an equivalent result to Lemma 2.8.2 in 2 dimensions.

Lemma 2.8.10 (Thm 6.4 [Lam70], Prop 3.3 [TW85])**.** Let $G' = (V', E')$ be a (2,3)-tight graph with a degree 3 vertex x, adjacent to vertices v_1, v_2, v_3, and let $G^* = (V^*, E^*)$ be obtained from G' by removing x and its three associated edges. Then one of the edges $\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}$ may be added to G^*, such that the resulting graph G is also a (2,3)-tight graph.

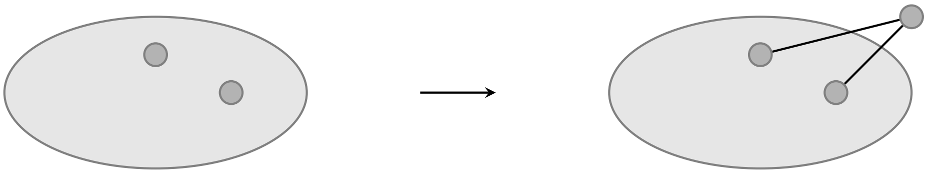
Laman applies Lemma 2.8.2 and Lemma 2.8.10 to give the following inductive proof that any (2,3)-tight graph on at least 3 vertices has an infinitesimally rigid realisation.

It is clear that $G = K_3$ has an infinitesimally rigid realisation in 2-dimensions, and it is easy to see that any (2,3)-tight graph has no vertices of degree 1, and at least one vertex of degree at most 3. As any (2,3)-tight graph G' on $n + 1 \geq 4$ vertices will therefore have a vertex of degree 2 or degree 3, we may apply either Lemma 2.8.2 (in the case of a degree 2 vertex), or Lemma 2.8.10 (in the case of a degree 3 vertex) to reduce to a (2,3)-tight graph G on n vertices. By the inductive hypothesis, G will have an infinitesimally rigid realisation, which we may extend to an infinitesimally rigid realisation of G' by applying either Lemma 2.8.2 or Lemma 2.8.4.

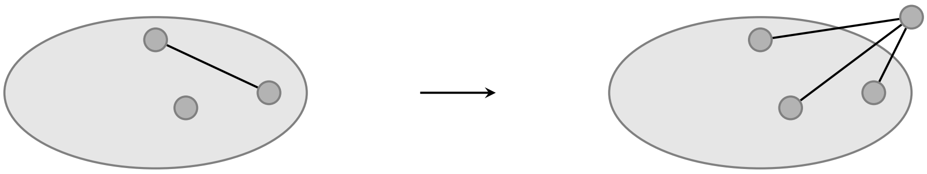
As a consequence of this inductive method of proof, we obtain the following result, often referred to as Henneberg's Theorem. Such a construction of a (2,3)-tight graph may be referred to as a *Henneberg construction*.

Theorem 2.8.11 (Henneberg's Theorem [Lam70])**.** A graph G is generically 2-isostatic if and only if there is a sequence of 0-extensions and 1-extensions from a single edge to create G.

There are other inductive moves in 2 dimensions, but Theorem 2.8.11 shows that only 0-extensions and 1-extensions are necessary to construct all generically isostatic graphs. Multiple surveys of such inductive constructions exist (see, for example, [BLW02, NR14, TW85]), and we shall consider a few further inductive moves that shall be required for the construction of coordinated isostatic graphs.



a 0-extension in 2 dimensions.



b 1-extension in 2 dimensions.

Figure 2.9 Henneberg moves in 2 dimensions.

It is logical to consider extending the definitions of 0-extensions and 1-extensions, to replace increasingly large subsets of edges by vertices of increasing degree. The first such step gives us the following definition.

Definition 2.8.12 ([TW85]). Let $G = (V, E)$ be a graph with a pair of edges $\{v_1, v_2\}, \{v_3, v_4\} \in E$, such that all four end vertices are distinct. Let $G^* = (V, E^*)$ be the graph created by deleting these edges, and let $G' = (V', E')$ be obtained by adding a new vertex x, and four new edges $\{x, v_i\}$ for $1 \leq i \leq 4$. We refer to G' as a *2-extension* of G.

As the two edges removed from G have distinct end vertices, this move is often referred to as an *X-replacement*. Tay and Whiteley [TW85] also consider a similar extension move when the two removed edges share a vertex, known as a *V-extension*, however this move is less useful for our purposes.

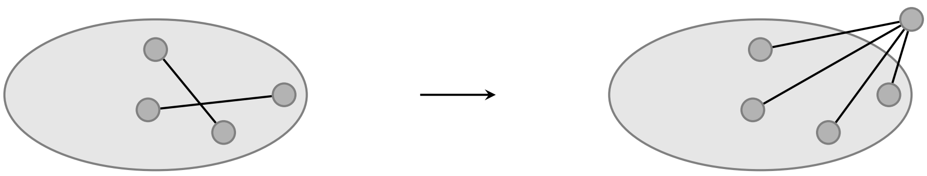


Figure 2.10 A 2-extension in 2 dimensions, or “X-replacement”.

Lemma 2.8.13 (Prop 3.9 [TW85])**.** Let G be a 2-isostatic graph, and let G' be the graph formed by applying an X-replacement on the pair of edges $\{v_1, v_2\}, \{v_3, v_4\} \in E$. Then G' is also 2-isostatic.

Tay and Whiteley [TW85] state the following result in general for 2-isostatic graphs, to permit the removal of a vertex of degree $b+2$. We only require the case where $b = 2$, to allow the removal of a vertex of degree 4 from a Laman graph.

Lemma 2.8.14 (Prop 3.8 [TW85])**.** Let G' be a (2,3)-tight graph, with a vertex x of degree 4, and edges $\{x, v_i\}$ for $1 \leq i \leq 4$. There exists a pair of edges e_1, e_2 between the vertices v_1, v_2, v_3, v_4 such that removing the vertex x and adding e_1, e_2 results in a (2,3)-tight graph G.

This result only gives that some pair of edges may be added, so it may be that the degree 4 vertex arose as the result of either an X-replacement or a V-replacement. As we shall require a coordinated analogue to the X-replacement in Section 7.2, we will need to use some additional structure within the graph to ensure that the two edges may be added to our coordinated framework with distinct end points.

2.9 3 dimensions and higher

For any dimension $d \geq 3$, from Theorem 2.7.7 we have the necessary conditions that a d-isostatic framework that affinely spans $\mathbb{R}^d$ is $\left(d, \binom{d+1}{2}\right)$-tight. The following result is known as folklore, as the proof is straightforward.

Theorem 2.9.1 ([SW17a])**.** If a graph G is generically isostatic in dimension d with $|V| > d$, then G is d-connected.

We have seen in Sections 2.8.1 and 2.8.2 that when $d = 1$ and $d = 2$, the condition that $|E| = d|V| - \binom{d+1}{2}$ together with the associated subgraph condition is equivalent to a graph being isostatic. However when $d = 3$, the conditions given in Theorem 2.7.7

are only necessary and are not sufficient. The canonical example of a (3,6)-tight graph that is not generically isostatic in 3 dimensions is the *double banana*, illustrated in Figure 2.11. There is an intuitive continuous motion of the two parts around the central axis, which may be seen to remain even when the graph is extended to being 3-connected (Figure 63.1.4(c) [SW17a]).

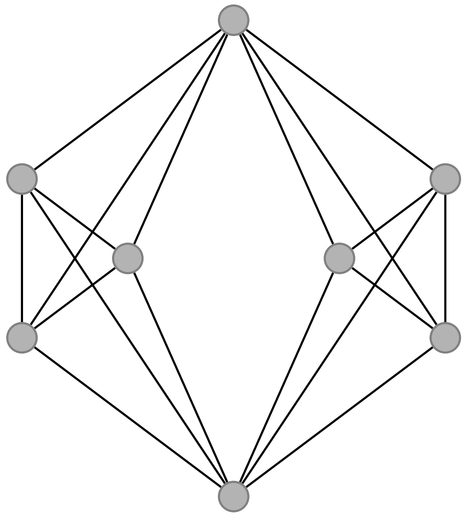


Figure 2.11 The Double Banana.

The following conjecture is often known as the “Sufficient Connectivity Conjecture”.

Conjecture 2.9.2 ([LY82])**.** If a graph G is $d(d+1)$-connected, then G is generically rigid in d-space.

There are significant partial results and conjectures for rigidity in 3 dimensions, in particular for alternative types of framework.

Definition 2.9.3 ([WW87, CJW13])**.** Let $H = (B, E)$ be a multigraph, and let $G_H = (V, E_G)$ be the simple graph induced by H by replacing each vertex $v \in B$ by a complete graph B_v (on $\deg_H(v)$ vertices), where edges $\{v_1, v_2\} \in E$ correspond to edges $\{u_1, u_2\} \in E_G$, where $u_1 \in V(B_{v_1})$ and $u_2 \in V(B_{v_2})$, assigned in such a way that the edges between bodies are pairwise disjoint. G_H is a *body-bar graph*.

Let $b' \in \mathbb{R}^{d|V|}$ be a d-dimensional configuration of the vertices of G_H. Then $b \in \mathbb{R}^{\binom{d+1}{2}|E|}$ is the corresponding *bar configuration* of the multigraph H, where $b_{\{i,j\}} \in \mathbb{R}^{\binom{d+1}{2}}$ is the vector consisting of the Plücker coordinates (2 by 2 minors) of the 2 by $d+1$ matrix $\begin{bmatrix} b'_i & 1 \\ b'_j & 1 \end{bmatrix}$.

Tay [Tay84] gives the following combinatorial characterisation for infinitesimal rigidity of body-bar frameworks.

Theorem 2.9.4 (Tay's Theorem [Tay84, SW17a])**.** For a multigraph H, the following are equivalent:

1. The body-bar framework (H, b) is infinitesimally rigid, for some bar configuration b of H;

2. The body-bar framework (H, b) is infinitesimally rigid for almost all bar configurations b;

3. G_H contains $\binom{d+1}{2}$ edge-disjoint spanning trees.

One class of body-bar frameworks are *body-hinge* frameworks. These consist of rigid bodies, connected by $d-2$-dimensional "hinges", which constrain the relative positions of the bodies. For body-hinge frameworks in 3-dimensions, a hinge between a pair of bodies is equivalent to the bodies being joined by five bars, which all intersect the same line. This line may be referred to as the hinge line between the pair of bodies.

Body-hinge frameworks may also be modelled using bar-joint frameworks by replacing each body by a complete graph, where two bodies connected by a hinge have $d-1$ common vertices, as described below.

Definition 2.9.5 ([JKT16])**.** Let $H = (B, E)$ be a multigraph. The *d-dimensional body-hinge graph* induced by H is $G_H = (V, E_H)$, obtained by replacing each body $v \in B$ with a complete graph $B(v)$ on $(d-1)\deg_H(v) + d + 1$ vertices, where $d+1$

vertices form the *core* of the body, $C(v)$, and the remaining vertices are partitioned such that any pair of bodies $u, v \in B$ that are connected by a hinge (i.e. $\{u, v\} \in E$) share $d-1$ common vertices: $|B(u) \cap B(v)| = d-1$.

We denote the set of common vertices $B(u) \cap B(v)$ by $H(e)$, where $e = \{u, v\} \in E$, and refer to $H(e)$ as the *hinge* between $B(u)$ and $B(v)$.

There is an analogous result to Tay's Theorem (Theorem 2.9.4) for body-hinge frameworks, known as Tay and Whiteley's Theorem [TW84, SW17a].

Molecular frameworks are a special class of 3-dimensional body-hinge frameworks, where all hinges incident to a body intersect at a common point. As the name suggests, these type of frameworks are of interest for modelling the rigidity or flexibility of molecules [Hen95].

Katoh and Tanigawa [KT11] give a proof that the infinitesimally rigid body-hinge frameworks are equivalent to the infinitesimally rigid molecular frameworks, building on work by Jackson and Jordán [JJ08, JJ06]. This confirms the Molecular Conjecture, originally posed by Tay and Whiteley [TW84], and extends the characterisation of body-hinge frameworks to molecular frameworks.

Global rigidity characterisations also exist for body-bar and body-hinge frameworks [CJW13, JKT16].

Chapter 3

Coordinated Rigidity

We shall now define a coordinated framework, which is the main topic for this thesis. We wish to define analogous flavours of rigidity to those discussed in Chapter 2, and develop a similar equivalence between them.

3.1 Definitions

As previously, let the dimension d be fixed and let $G = (V, E)$ be a graph with vertex set V and edge set E. Recall that $n = |V|$ and V will often be identified by $\{1, \ldots, n\} = [n]$, while m may at times be used to denote $|E|$. Edges shall continue to be indicated by unordered pairs, $\{i, j\} \in E$ for $i, j \in V$.

Consider the number of colours k to be fixed. We shall extend our definition of a framework (G, p) to that of a k-coordinated framework (G, c, p, r) as follows.

Definition 3.1.1. For a graph $G = (V, E)$, we define the *edge-colouring function* $c : E \to \{0, 1, \ldots, k\}$. We use $E_0 := c^{-1}(0)$ to denote the set of *uncoloured edges* of the graph G, and define the k *colour classes* to be the induced partitions $c^{-1}(\ell)$ for $1 \leq \ell \leq k$. We denote each colour class by $E_\ell := c^{-1}(\ell)$.

We may then refer to (G, c) as a *k-edge-coloured graph*, or *k-coloured graph*, and refer to edges in $E_1 \cup \cdots \cup E_\ell$ as the *coloured edges* of (G, c).

Remark 3.1.2. Throughout this thesis, we shall represent uncoloured edges of coloured graphs with straight lines. Coloured edges will be indicated by wavy lines, with edges from the same colour class having the same frequency and amplitude of waves. Figure 3.1 shows a graph with $|E_0| = 9$, $|E_1| = 4$ and $|E_2| = 2$. Edges that may be allocated to any colour class, or equally may be uncoloured, will be represented by dashed lines. (These may be seen, for example, in the case of coordinated 0-extensions and 1-extensions, shown in Figures 5.1, 5.2a and 5.2b.)

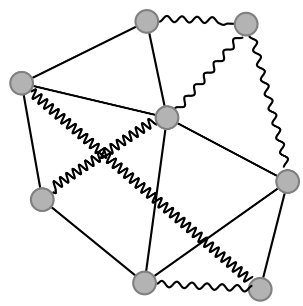


Figure 3.1 A 2-coloured framework with $|E_0| = 9$, $|E_1| = 4$ and $|E_2| = 2$.

In Definition 2.2.1, a d-dimensional framework (G, p) was defined by combining a graph $G = (V, E)$ with a configuration of the vertices $p \in \mathbb{R}^{dn}$, where $p(i) \neq p(j)$ for any edge $\{i, j\} \in E$. We shall now extend this notion to our edge-coloured graph (G, c).

Definition 3.1.3. A *configuration* of the k-edge-coloured graph (G, c) is a pair $(p, r) \in \mathbb{R}^{dn+k}$, where $p \in \mathbb{R}^{dn}$ is a configuration of the uncoloured graph G in $\mathbb{R}^{dn}$ (Definition 2.2.1), and $r \in \mathbb{R}^k$ is a vector.

We refer to (G, c, p, r) as a *k-coordinated framework*.

Remark 3.1.4. In situations where the number of colour classes is clear, we may suppress the k and refer to a *coordinated framework*.

Remark 3.1.5. We may consider $r \in \mathbb{R}^k$ as representing the "strain" $r(\ell)$ in each colour class $1 \leq \ell \leq k$. We shall see in Definition 3.1.16 that infinitesimal motions are defined independently of the choice of r. (See Remark 3.1.20).

Definition 3.1.6. Let (p, r) and (q, s) be two configurations of the k-edge-coloured graph (G, c). The k-coordinated frameworks (G, c, p, r) and (G, c, q, s) are *equivalent* if they satisfy the following:

$$\|p(i) - p(j)\|^2 = \|q(i) - q(j)\|^2 \text{ for all } \{i, j\} \in E_0, \tag{3.1}$$

$$\|p(i) - p(j)\|^2 + r(\ell) = \|q(i) - q(j)\|^2 + s(\ell) \text{ for all } \{i, j\} \in E_\ell,\ \ell \in \{1, \dots, k\}. \tag{3.2}$$

The k-coordinated frameworks (G, c, p, r) and (G, c, q, s) are *congruent* if the frameworks are equivalent and the configurations p and q are congruent. From Definition 2.2.5, the configurations p and q will satisfy

$$\|p(i) - p(j)\|^2 = \|q(i) - q(j)\|^2 \text{ for all } i, j \in V. \tag{3.3}$$

Remark 3.1.7. It is straightforward to observe that the k-coordinated frameworks (G, c, p, r) and (G, c, q, s) being congruent implies that $r = s$.

Definition 3.1.8. The *coordinated edge-length function* for a k-edge-coloured graph (G, c) is $f_{(G,c)} : \mathbb{R}^{dn+k} \to \mathbb{R}^m$, where

$$f_{(G,c)}(p, r)_{\{i,j\}} = \begin{cases} \|p(i) - p(j)\|^2 & \text{for all } \{i, j\} \in E_0, \\ \|p(i) - p(j)\|^2 + r(\ell) & \text{for all } \{i, j\} \in E_\ell,\ \ell \in \{1, \dots, k\}. \end{cases} \tag{3.4}$$

Remark 3.1.9. This is the coordinated analogue to Definition 2.2.4. Two frameworks (G, c, p, r) and (G, c, q, s) are equivalent if and only if they have the same coordinated edge-length function, i.e. $f_{(G,c)}(p, r) = f_{(G,c)}(q, s)$.

We now define equivalent notions to those seen in Definition 2.3.1 and Definition 2.3.2.

Definition 3.1.10. A k-coordinated framework (G, c, p, r) is *(locally) rigid* if there is a neighbourhood U of $(p, r) \in \mathbb{R}^{dn+k}$ such that, if $(q, s) \in U$ and the frameworks (G, c, p, r) and (G, c, q, s) are equivalent, then the frameworks (G, c, p, r) and (G, c, q, s) are congruent.

If a coordinated framework is not rigid, we may refer to it as being *flexible*.

Definition 3.1.11. A *continuous motion* of a k-coordinated framework (G, c, p, r) is a family of continuous functions indexed by the vertex set $V = \{1, \dots, n\}$, $P_i : [0, 1] \to \mathbb{R}^d$, and a family of continuous functions indexed by the colour classes E_ℓ for $1 \leq \ell \leq k$, $R_\ell : [0, 1] \to \mathbb{R}$, such that:

1. $P_i(0) = p(i)$ for all $i \in V$;
   $R_\ell(0) = r(\ell)$ for all $\ell \in \{1, \dots, k\}$;
2. $P_i(t)$ is differentiable on the interval $[0, 1]$ for all $i \in V$;
   $R_\ell(t)$ is differentiable on the interval $[0, 1]$ for all $\ell \in \{1, \dots, k\}$;
3. $\|P_i(t) - P_j(t)\|^2 = \|p(i) - p(j)\|^2$ for all $t \in [0, 1]$ for all edges $\{i, j\} \in E_0$;
   $\|P_i(t) - P_j(t)\|^2 + R_\ell(t) = \|p(i) - p(j)\|^2 + r(\ell)$ for all $t \in [0, 1]$ for all edges $\{i, j\} \in E_\ell$, for all $\ell \in \{1, \dots, k\}$.

A continuous motion is *non-trivial* if there is some $t_0 \in (0, 1]$ such that setting $q(i) = P_i(t_0)$ for $i \in V$ and $s(\ell) = R_\ell(t_0)$ for $1 \leq \ell \leq k$ gives a framework (G, c, q, s) that is equivalent to (G, c, p, r) but not congruent to (G, c, p, r). Otherwise, a continuous motion is *trivial*.

Remark 3.1.12. Let $q_i(t) := P_i(t)$ for $i \in V$ and $s_\ell(t) := R_\ell(t)$ for $1 \le \ell \le k$ for $t \in [0,1]$ for a trivial continuous motion $\{P, R\}$. The sequence of frameworks attained by $\{P, R\}$, $(G, c, q(t), s(t))$ for $t \in [0,1]$, are all congruent to the initial framework (G, c, p, r), and so the configurations $q(t)$ are congruent to p for all $t \in [0,1]$. It follows that $s(t) = r$ for all $t \in [0,1]$. (See Remark 3.1.7.)

The trivial continuous motions of a k-coordinated framework (G, c, p, r) are therefore the trivial continuous motions of the uncoloured framework (G, p), which are the rigid body motions discussed in Remark 2.3.3.

We also have the following equivalent restatement of Definition 3.1.11.

Remark 3.1.13. A *continuous motion* of a k-coordinated framework (G, c, p, r) is a family of frameworks (G, c, p_t, r_t) with $t \in [0,1]$, such that $(p_0, r_0) = (p, r)$ and all the (G, c, p_t, r_t) are equivalent to (G, c, p, r). A continuous motion is *non-trivial* if there is some $t_0 \in (0,1]$ such that (G, c, p_{t_0}, r_{t_0}) is not congruent to (G, c, p, r).

It is equivalent for a k-coordinated framework (G, c, p, r) to be flexible, and for there to be a non-trivial continuous motion of the framework. We may therefore refer to a non-trivial continuous motion as a *flex*.

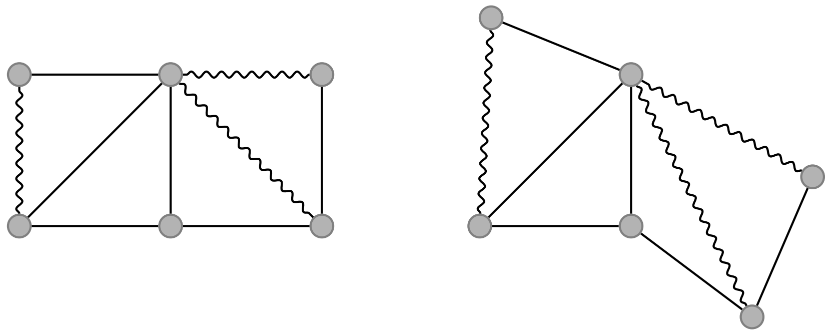
Remark 3.1.14. We note that the difference in edge lengths within a colour class E_ℓ does not depend on $r_t(\ell)$, so it must be constant over the flex:

$$\|p(i) - p(j)\|^2 + r(\ell) - \|p(u) - p(w)\|^2 - r(\ell) = \|p(i) - p(j)\|^2 - \|p(u) - p(w)\|^2.$$

Other types of coordination, such as preserving the ratio of lengths of a pair of edges, may be considered instead, and we shall discuss some of these further in Chapter 9.

Example 3.1.15. The framework (G_2, p) from Example 2.4.13, shown in Figure 2.6c, is infinitesimally rigid. We apply the 1-edge-colouring c with $|E_1| = 3$, shown in Figure 3.2a, to create the 1-coordinated framework (G, c, p, r). There is a continuous

1-coordinated motion between (G, c, p, r) and (G, c, q, s), which are a pair of equivalent but not congruent 1-coordinated frameworks.



a (G, c, p, r) **b** (G, c, q, s)

Figure 3.2 Example 3.1.15. The framework (G, p) is infinitesimally rigid when uncoloured (see (G_2, p) in Example 2.4.13, Figure 2.6c). The 1-edge-coloured graph (G, c) has a continuous 1-coordinated motion between the equivalent configurations (G, c, p, r) and (G, c, q, s).

As in the standard rigidity context, we wish to make it easier to determine the rigidity of a coordinated framework by linearising. We give the following definition, as an analogue to Definition 2.4.1.

Definition 3.1.16. An *infinitesimal motion* of the k-coordinated framework (G, c, p, r) is $(p', r') \in \mathbb{R}^{dn+k}$, where $p' \in \mathbb{R}^{dn}$ is a velocity field supported on p and $r' \in \mathbb{R}^k$ is a vector, such that

$$[p(i) - p(j)] \cdot [p'(i) - p'(j)] = 0 \text{ for all } \{i, j\} \in E_0, \tag{3.5}$$

$$[p(i) - p(j)] \cdot [p'(i) - p'(j)] + r'(\ell) = 0 \text{ for all } \{i, j\} \in E_\ell, \ell \in \{1, \ldots, k\}. \tag{3.6}$$

As with the standard rigidity definition of an infinitesimal motion in Section 2.4, this may be viewed as the first derivative of a continuous motion, evaluated at $t = 0$, but we may also take this as the formal definition of a coordinated infinitesimal motion.

Any standard d-dimensional trivial infinitesimal motion (see Definition 2.4.2) is also a trivial infinitesimal motion of a k-coordinated d-dimensional framework. We therefore define the trivial infinitesimal motions of coordinated frameworks as follows.

Definition 3.1.17. A *trivial infinitesimal motion* (p', r') of a k-coordinated framework (G, c, p, r) is $(p', \mathbf{0})$, where p' is a trivial infinitesimal motion of the uncoloured framework (G, p).

Remark 3.1.18. The trivial infinitesimal motions of a k-coordinated framework (G, c, p, r) will satisfy the following equation:

$$[p(i) - p(j)] \cdot [p'(i) - p'(j)] = 0 \text{ for all } i, j \in V. \tag{3.7}$$

When the infinitesimal motion (p', r') affinely spans $\mathbb{R}^{dn+k}$, this may be taken as the definition of a trivial infinitesimal motion.

Definition 3.1.19. A k-coordinated framework (G, c, p, r) is considered to be *infinitesimally rigid* if all the infinitesimal motions of (G, c, p, r) are trivial infinitesimal motions.

Remark 3.1.20. As Equations (3.5) and (3.6) are both independent of r, in the infinitesimal case we may consider $r = \mathbf{0}$ and refer instead to a k-coordinated framework (G, c, p).

The standard rigidity matrix $R(G, p)$ (Definition 2.4.7) corresponds to the equations of an infinitesimal motion. We now set up the coordinated analogue to the rigidity matrix.

As $|E| = m$, we may consider the edge set of a graph G as an m-dimensional vector, ordered in the same way as the rows of the standard rigidity matrix $R(G, p)$. We denote by $\mathbb{1}_\ell \in \mathbb{R}^m$ the *characteristic vector* of the colour class E_ℓ:

$$\mathbb{1}_\ell(e) = \begin{cases} 1 & \text{if } e \in E_\ell, \\ 0 & \text{if } e \notin E_\ell. \end{cases}$$

We may combine the characteristic vectors for the k colour classes, $E_1, \ldots, E_k$, into the m by k matrix given by $\mathbb{1}(c) := \left[\mathbb{1}_1, \ldots, \mathbb{1}_k\right]$. This gives us a matrix condition for an infinitesimal motion of a k-coordinated framework, equivalent to those given in Definition 3.1.16:

$$R(G,p)p' + \mathbb{1}(c)r' = \mathbf{0}. \tag{3.8}$$

To simplify this notation, we define the coordinated rigidity matrix as follows.

Definition 3.1.21. The *coordinated rigidity matrix* of the k-coordinated framework (G,c,p,r) is $R(G,c,p) := \left[R(G,p), \mathbb{1}(c)\right]$.

$$R(G,c,p) = \begin{array}{c} \begin{array}{ccccccccccc} & & i & & j & & & & E_1 & & E_k \end{array} \\ \left[\begin{array}{ccccccc|ccc} \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \cdots & (p_i - p_j) & \cdots & (p_j - p_i) & \cdots & \mathbf{0} & * & \cdots & * \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \end{array}\right] \end{array} \{i,j\}$$

We take this as the formal definition of the coordinated rigidity matrix, but this may be related to the coordinated edge-length function $f_{(G,c)}$ (Definition 3.1.8) in a similar way to the derivation of the standard rigidity matrix $R(G,p)$ (Definition 2.4.7) using the standard edge-length function f_G (Definition 2.2.4). The Jacobian of the coordinated edge-length function is $df_{(G,c)}(p,r) = \left[2R(G,p), \frac{\partial}{\partial 1}(r(1))\,\mathbb{1}_1, \ldots, \frac{\partial}{\partial k}(r(k))\,\mathbb{1}_k\right]$, where $\mathbb{1}_\ell$ is the characteristic vector for the colour class E_ℓ. Multiplying columns by non-zero scalars preserves the rank of the matrix, and so obtaining $R(G,c,p)$ from $df_{(G,c)}(p,r)$ in this way will give $\operatorname{rank} R(G,c,p) = \operatorname{rank} df_{(G,c)}(p,r)$.

By defining an extension of the rigidity matrix in this way, we may rewrite Equation (3.8) as follows, where $[p', r']$ is considered as a single vector in $\mathbb{R}^{dn+k}$:

$$R(G,c,p)\,[p',r'] = \mathbf{0}. \tag{3.9}$$

We may now state the coordinated analogue to Theorem 2.4.9.

Theorem 3.1.22. Let (G,c,p,r) be a k-coordinated framework that affinely spans $\mathbb{R}^d$. Then (G,c,p,r) is infinitesimally rigid if and only if $\operatorname{rank} R(G,c,p) = dn + k - \binom{d+1}{2}$.

Proof. We note first that any infinitesimal motions of the framework (G,c,p,r) are contained within the kernel of $R(G,c,p)$, and that (G,c,p,r) is infinitesimally rigid if and only if the only infinitesimal motions of the framework are the trivial infinitesimal motions. Recall from Remark 2.4.3 that the trivial infinitesimal motions of $\mathbb{R}^d$ have dimension $\binom{d+1}{2}$. The trivial infinitesimal motions are always contained within the kernel of the coordinated rigidity matrix, and hence $\dim\ker R(G,c,p) \geq \binom{d+1}{2}$ for all frameworks (G,c,p,r).

If (G,c,p,r) is infinitesimally flexible, the kernel of $R(G,c,p)$ contains a non-trivial infinitesimal motion, and hence has $\dim\ker R(G,c,p) > \binom{d+1}{2}$. Therefore $\operatorname{rank} R(G,c,p) < dn+k-\binom{d+1}{2}$. Conversely if (G,c,p,r) is infinitesimally rigid, the only infinitesimal motions of (G,c,p,r) are the trivial infinitesimal motions. The kernel of $R(G,c,p)$ therefore has dimension $\binom{d+1}{2}$, and hence $\operatorname{rank} R(G,c,p) = dn+k-\binom{d+1}{2}$. $\square$

Example 3.1.23. The framework (G,c,p,r) from Example 3.1.15 (Figure 3.2) has the following coordinated rigidity matrix $R(G,c,p)$ (where the vertices are labelled from top to bottom, left to right, as in Figure 2.7 in Example 2.5.12):

$$\begin{bmatrix}
p(1)-p(2) & p(2)-p(1) & 0 & 0 & 0 & 0 & 1 \\
p(1)-p(3) & 0 & p(3)-p(1) & 0 & 0 & 0 & 0 \\
0 & p(2)-p(3) & p(3)-p(2) & 0 & 0 & 0 & 0 \\
0 & p(2)-p(4) & 0 & p(4)-p(2) & 0 & 0 & 0 \\
0 & 0 & p(3)-p(4) & p(4)-p(3) & 0 & 0 & 0 \\
0 & 0 & p(3)-p(5) & 0 & p(5)-p(3) & 0 & 1 \\
0 & 0 & p(3)-p(6) & 0 & 0 & p(6)-p(3) & 1 \\
0 & 0 & 0 & p(4)-p(6) & 0 & p(5)-p(6) & 0 \\
0 & 0 & 0 & 0 & p(5)-p(6) & p(6)-p(5) & 0
\end{bmatrix}$$

The rank of this matrix is $|E| = 9 < 10 = 2|V| + 1 - 3$. This corresponds to the framework (G, c, p, r) being flexible, as seen in Example 3.1.15.

Lemma 3.1.24. Let (K_n, c) be a k-coloured copy of the complete graph on n vertices, and let (K_n, c, p, r) affinely span $\mathbb{R}^d$. If (K_n, c, p, r) is infinitesimally rigid, then $k \leq \binom{n-d}{2}$.

Proof. Suppose that (K_n, c, p, r) is infinitesimally rigid. Since (K_n, c, p, r) affinely spans $\mathbb{R}^d$, by Theorem 3.1.22 we have that $\operatorname{rank} R(K_n, c, p) = dn + k - \binom{d+1}{2}$. Since $\operatorname{rank} R(K_n, c, p) \leq |E| = \binom{n}{2}$, we have that $dn + k - \binom{d+1}{2} \leq \binom{n}{2}$. This may be rearranged as follows:

$$\begin{aligned}
k &\leq \binom{n}{2} + \binom{d+1}{2} - dn = \frac{n(n-1)}{2} + \frac{d(d+1)}{2} - dn \\
&= \frac{n^2 - n + d^2 + d - 2dn}{2} = \frac{(n-d)(n-d-1)}{2} = \binom{n-d}{2}. \qquad (3.10)
\end{aligned}$$

$\square$

Lemma 3.1.25. Let (K_n, c) be a k-coloured copy of the complete graph on n vertices. If the k-coordinated complete framework (K_n, c, p, r) that affinely spans $\mathbb{R}^d$ is independent and infinitesimally flexible, then $k > \binom{n-d}{2}$.

Proof. If (K_n, c, p, r) is independent, then $\operatorname{rank} R(G, c, p) = |E| = \binom{n}{2}$. By Theorem 3.1.22, when (K_n, c, p, r) is infinitesimally flexible, $\operatorname{rank} R(G, c, p) < dn + k - \binom{d+1}{2}$. The result follows from rearranging $\binom{n}{2} < dn + k - \binom{d+1}{2}$. □

Through algebraic manipulation of Equation (3.10), we may also obtain the following equivalent result.

Lemma 3.1.26. Let (K_n, c) be a k-coloured copy of the complete graph on n vertices. If the k-coordinated framework (K_n, c, p, r) that affinely spans $\mathbb{R}^d$ is infinitesimally rigid, then $n \geq \frac{1+2d+\sqrt{8k+1}}{2}$.

As we have defined a coordinated rigidity matrix, we may extend Definition 2.4.14 to configurations of k-coordinated frameworks. We also extend Definition 2.4.15 to the k-coordinated context.

Definition 3.1.27. A configuration $(p, r) \in \mathbb{R}^{dn+k}$ of a k-edge-coloured graph (G, c) is *regular* if $\operatorname{rank} R(G, c, p) \geq \operatorname{rank} R(G, c, q)$ for all $q \in \mathbb{R}^{dn}$, which is equivalent to p being regular.

A k-coordinated framework (G, c, p, r) is *regular* if (p, r) is regular, and hence if p is regular.

Definition 3.1.28. A configuration $(p, r) \in \mathbb{R}^{dn+k}$ of the k-edge-coloured graph (G, c) is *generic* if p is generic (as in Definition 2.4.15).

A k-coordinated framework (G, c, p, r) is *generic* if (p, r) is generic, and hence if p is generic.

Remark 3.1.29. As k-coordinated frameworks are generic or regular based only on $p \in \mathbb{R}^{dn}$, rather than the coordinated configuration $(p, r) \in \mathbb{R}^{dn+k}$, this gives further justification to considering $r = \mathbf{0}$ and referring to the framework (G, c, p) (as discussed in Remark 3.1.20).

We may now state the following result, from work with Louis Theran [SST18], which we shall cover in more detail in Section 3.2. We prove one direction using an extension of Whiteley's averaging method, as discussed by Connelly [Con87a] (see Theorem 3.2.6), and adapt work by Asimow and Roth [AR78] for the other direction (see Theorem 3.2.7).

Theorem 3.1.30. Let (G, c, p, r) be a k-coordinated framework. If (G, c, p, r) is infinitesimally rigid in $\mathbb{R}^d$, then it is rigid in $\mathbb{R}^d$. If (G, c, p, r) is regular and infinitesimally flexible in $\mathbb{R}^d$, then it is flexible in $\mathbb{R}^d$.

Definition 3.1.31. A k-coordinated framework (G, c, p, r) is *isostatic in* $\mathbb{R}^d$ if it is infinitesimally rigid in $\mathbb{R}^d$, and the rows of $R(G, c, p)$ are independent.

We note that the framework (G, c, p, r) is isostatic for some $p \in \mathbb{R}^{dn}$ if and only if the framework (G, c, q, s) is isostatic for all generic $q \in \mathbb{R}^{dn}$. This is straightforward to prove for regular configurations $(p, r) \in \mathbb{R}^{dn+k}$, and so holds for generic configurations as these are contained within the set of regular configurations. As in the standard rigidity context, we shall characterise the rigidity of regular k-coordinated frameworks, by characterising the rigidity of k-edge-coloured graphs with regular configurations: we discuss the case of coordinated frameworks in $\mathbb{R}^1$ in Chapter 6, and coordinated frameworks in $\mathbb{R}^2$ are discussed in Chapter 7.

Definition 3.1.32. The k-edge-coloured graph (G, c) is *d-isostatic* if the framework (G, c, p, r) is isostatic for some $(p, r) \in \mathbb{R}^{dn+k}$.

We have the following k-coordinated analogue to Theorem 2.4.11.

Theorem 3.1.33. Let (G, c, p, r) be a k-coordinated framework in $\mathbb{R}^d$ with $|V| \geq d$, where $p \in \mathbb{R}^{dn}$ is a regular configuration. The following are equivalent:

1. (G, c, p, r) is d-isostatic;

2. (G, c, p, r) is independent, and $|E| = d|V| + k - \binom{d+1}{2}$;

3. (G, c, p, r) is infinitesimally rigid, and $|E| = d|V| + k - \binom{d+1}{2}$;

4. (G, c, p, r) is infinitesimally rigid, and the k-coordinated framework $(G', c', p', r') = (G - e, c|_{E\setminus\{e\}}, p, r)$ is infinitesimally flexible for all edges $e \in E_0$ and $e \in E_\ell$ for $1 \leq \ell \leq k$ with $|E_\ell| \geq 2$.

Proof. From Theorem 3.1.22, a framework (G, c, p, r) is infinitesimally rigid if and only if $\operatorname{rank} R(G, c, p) = d|V| + k - \binom{d+1}{2}$, and the rows of $R(G, c, p)$ are independent if and only if $\operatorname{rank} R(G, c, p) = |E|$. Equivalence of the first three conditions follows from these facts.

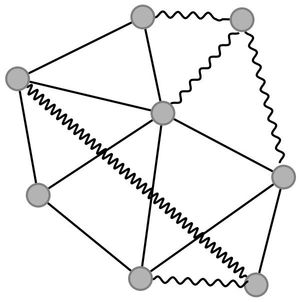
If the framework (G, c, p, r) is d-isostatic, $\operatorname{rank} R(G, c, p) = |E|$ since the rows of the coordinated rigidity matrix are independent. If (G', c', p', r') is formed by removing an edge $e \in E$, the rows of $R(G', p', c')$ will clearly still be independent, and $\operatorname{rank} R(G', c', p') = |E'| = |E| - 1$. If $e \in E$ is chosen such that $E'_\ell \neq \emptyset$ for $1 \leq \ell \leq k$, all columns of $R(G', c', p')$ will remain non-zero, so $d|V| + k - \binom{d+1}{2} > \operatorname{rank} R(G', c', p')$ and hence the framework (G', c', p', r') is infinitesimally flexible. $\square$

a **b**

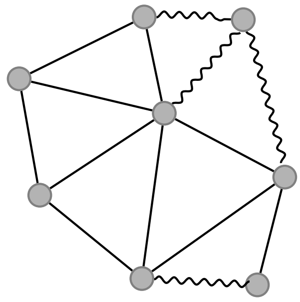
Figure 3.3 The 1 edge-coloured graph (K_4, c) with $|E_0| = 5$ and $|E_1| = 1$ (a) is effectively equivalent to the uncoloured graph $K_4 - e$ (b).

The final condition requires that (G, c, p, r) be "minimally rigid" as a k-coordinated framework. We note that a colour class with $E_\ell = \{e\}$ is equivalent to $e \in E$ being a "non-edge". An example of this situation is shown in Figure 3.3.

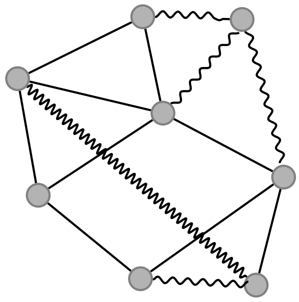
The following example further illustrates Statement 4 of Theorem 3.1.33.



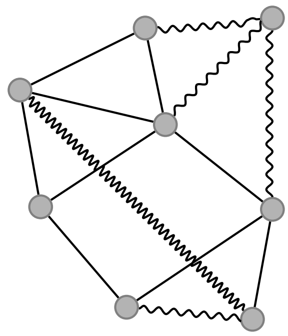
a The isostatic 2-coloured framework (G, c, p, r).



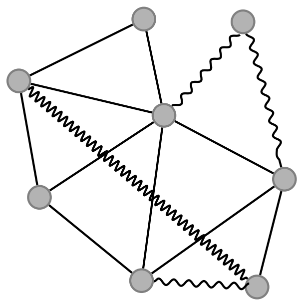
b The isostatic 1-coloured framework (G_1, c_1, p, r).



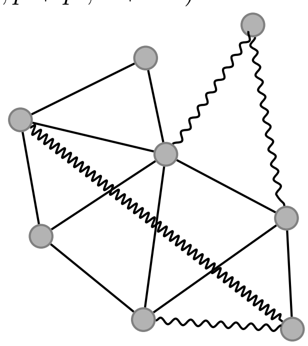
c The flexible 2-coloured framework (G_2, c_2, p, r).



d The flexible 2-coloured framework $(G_2, c_2, p + p', r + 2r')$.



e The flexible 2-coloured framework (G_3, c_3, p, r).



f The flexible 2-coloured framework $(G_3, c_3, p + p'', r + 2r'')$.

Figure 3.4 Example 3.1.34: a 2-coordinated framework (G, c, p, r) that is minimally rigid as a 2-coordinated framework (illustrated by applying the non-trivial flexes (p', r') and (p'', r'') to the reduced 2-coloured frameworks (G_2, c_2, p, r) and (G_3, c_3, p, r)), but remains rigid as a reduced 1-coordinated framework $((G_1, c_1, p, r))$.

Example 3.1.34. The 2-coloured framework illustrated in Figure 3.4a is infinitesimally rigid, with $|E_0| = 10$, $|E_1| = 4$ and $|E_2| = 1$. (The edges of E_1 are indicated by smoothly waved lines, in contrast to the zig-zag indicating the single edge of E_2.)

Removing any uncoloured edge results in a flexible framework, one of which is illustrated in Figure 3.4c, with an equivalent but not congruent realisation shown in Figure 3.4d. Removing an edge from E_1, as seen in Figure 3.4e, similarly creates a framework with a continuous motion. Figure 3.4f illustrates an equivalent realisation that may be achieved through such a motion.

In contrast, Figure 3.4b shows the framework created by removing the single edge from E_2. This results in an infinitesimally rigid 1-coordinated framework, unlike the flexible 2-coordinated frameworks described above.

3.2 Coordinated finite versus infinitesimal rigidity

We now show that infinitesimal rigidity implies local rigidity in the coordinated case, by extending Whiteley's "Averaging Method" to coordinated frameworks. We also give an analogue to a result by Asimow and Roth [AR78] to prove the equivalence in the other direction.

The following result is attributed to Whiteley when stated by Connelly [Con87a], and described as "The Averaging Method".

Theorem 3.2.1 (Proposition 2.41 [Con87a])**.** Let $p, q \in \mathbb{R}^d$ be two configurations in $\mathbb{R}^d$. Then

a. (G, p) is equivalent to (G, q) if and only if $p - q$ is an infinitesimal motion for $(G, \frac{p+q}{2})$,

b. If $p - q$ is a trivial infinitesimal motion of $(G, \frac{p+q}{2})$, then (G, p) is congruent to (G, q),

c. If the affine span $\langle \frac{p+q}{2} \rangle$ contains $p - q$, and (G, p) is congruent to (G, q), then $p - q$ is a trivial infinitesimal motion of $(G, \frac{p+q}{2})$.

The following restatement is also given, which we shall adapt and prove for coordinated frameworks in Theorem 3.2.3. Rather than begin by considering two configurations p and q that may be equivalent, and finding an infinitesimal motion of the framework with configuration $\frac{p+q}{2}$, we begin with a configuration p and a potential infinitesimal motion p', and find pairs of equivalent or congruent frameworks based on the properties of p'. By choosing p' to be a non-trivial infinitesimal motion of very small magnitude, we may find pairs of equivalent frameworks that get arbitrarily close together, but are not congruent.

Theorem 3.2.2 (Remark 2.42 [Con87a])**.** Let $p \in \mathbb{R}^d$ be a configuration in $\mathbb{R}^d$, and let $p' \in \mathbb{R}^d$ be a velocity field supported on p. Then

a. $(G, p + p')$ is equivalent to $(G, p - p')$ if and only if p' is an infinitesimal motion for (G, p),

b. If p' is a trivial infinitesimal motion of (G, p), then $(G, p + p')$ is congruent to $(G, p - p')$,

c. If the affine span $\langle p \rangle$ contains p', and $(G, p + p')$ is congruent to $(G, p - p')$, then p' is a trivial infinitesimal motion of (G, p).

We adapt Theorem 3.2.2 to the coordinated case as follows, which we shall then restate to get an adaptation of Theorem 3.2.1 (Theorem 3.2.5).

Theorem 3.2.3. Let (p, r) be a configuration of the k-edge-coloured graph (G, c), and let $(p', r') \in \mathbb{R}^{dn+k}$ with p' a velocity field supported on p. Then

a. $(G, c, p + p', r + 2r')$ is equivalent to $(G, c, p - p', r - 2r')$ if and only if (p', r') is an infinitesimal motion for (G, c, p, r).

b. If (p', r') is a trivial infinitesimal motion of (G, c, p, r), then $(G, c, p + p', r + 2r')$ is congruent to $(G, c, p - p', r - 2r')$.

c. If the affine span $\langle p \rangle$ contains p', and $(G, c, p + p', r + 2r')$ is congruent to $(G, c, p - p', r - 2r')$, then (p', r') is a trivial infinitesimal motion of (G, c, p, r).

Proof. **a.** Let $\{i, j\} \in E$ be an edge, and consider the difference of the length of $\{i, j\}$ in $(G, c, p{+}p', r{+}2r')$, and the length of $\{i, j\}$ in $(G, c, p{-}p', r{-}2r')$. Recall the coordinated edge-length function $f_{(G,c)}$ from Definition 3.1.8, and note that $f_{(G,c)}(p{+}p', r{+}2r')_{\{i,j\}} = \|(p + p')(i) - (p + p')(j)\|^2 + (r + 2r')(\ell)$ for an edge $\{i, j\} \in E_\ell$, $1 \leq \ell \leq k$. We may extend this to edges $\{i, j\} \in E_0$ by considering $r(0) = r'(0) = 0$, and hence we have the following for all edges in E:

$$
\begin{aligned}
&\Big[\|(p + p')(i) - (p + p')(j)\|^2 + (r + 2r')(\ell)\Big] \\
&\qquad - \Big[\|(p - p')(i) - (p - p')(j)\|^2 + (r - 2r')(\ell)\Big] \\
&= \|p(i) - p(j) + p'(i) - p'(j)\|^2 + (r + 2r')(\ell) \\
&\qquad - \|p(i) - p(j) - p'(i) + p'(j)\|^2 - (r - 2r')(\ell) \\
&= \Big[p(i) - p(j)\Big]^2 + 2\Big[p(i) - p(j)\Big] \cdot \Big[p'(i) - p'(j)\Big] + \Big[p'(i) - p'(j)\Big]^2 + r(\ell) + 2r'(\ell) \\
&\qquad - \Big[p(i) - p(j)\Big]^2 + 2\Big[p(i) - p(j)\Big] \cdot \Big[p'(i) - p'(j)\Big] - \Big[p'(i) - p'(j)\Big]^2 - r(\ell) + 2r'(\ell) \\
&= 4\Big[p(i) - p(j)\Big] \cdot \Big[p'(i) - p'(j)\Big] + 4r'(\ell).
\end{aligned} \tag{3.11}
$$

When (p', r') is an infinitesimal motion, $\Big[p(i) - p(j)\Big] \cdot \Big[p'(i) - p'(j)\Big] + r'(\ell) = 0$, and so when (p', r') is an infinitesimal motion, the final line of Equation (3.11) is equal to 0. From the first line of Equation (3.11), we therefore have that $f_{(G,c)}(p + p', r + 2r')_{\{i,j\}} = f_{(G,c)}(p{-}p', r{-}2r')_{\{i,j\}}$ for all edges $\{i, j\} \in E$ and hence the frameworks are equivalent.

Conversely when the frameworks are equivalent, we have that the coordinated edge-length functions are equal for all edges $\{i, j\} \in E$, and hence the final line of

Equation (3.11) gives $\Big[p(i)-p(j)\Big]\cdot\Big[p'(i)-p'(j)\Big]+r'(\ell)=0$, as required for (p',r') to be an infinitesimal motion.

b. Let (p',r') be a trivial infinitesimal motion of (G,c,p,r). By definition, for any pair of vertices $i,j\in V$ we have $\Big[p(i)-p(j)\Big]\cdot\Big[p'(i)-p'(j)\Big]=0$, and as a consequence we see that $r=r'=\mathbf{0}$.

As in part a, we compare the distance within $(G,c,p+p',r+2r')$ and within $(G,c,p-p',r-2r')$ between any pair of vertices $i,j\in V$, and no longer restrict to only considering pairs of vertices connected by an edge.

$$\begin{aligned}
&\Big[\|(p+p')(i)-(p+p')(j)\|^2\Big]-\Big[\|(p-p')(i)-(p-p')(j)\|^2\Big]\\
&=\|p(i)-p(j)+p'(i)-p'(j)\|^2-\|p(i)-p(j)-p'(i)+p'(j)\|^2\\
&=\Big[p(i)-p(j)\Big]^2+2\Big[p(i)-p(j)\Big]\cdot\Big[p'(i)-p'(j)\Big]+\Big[p'(i)-p'(j)\Big]^2\\
&\qquad-\Big[p(i)-p(j)\Big]^2+2\Big[p(i)-p(j)\Big]\cdot\Big[p'(i)-p'(j)\Big]-\Big[p'(i)-p'(j)\Big]^2\\
&=4\Big[p(i)-p(j)\Big]\cdot\Big[p'(i)-p'(j)\Big]=0.
\end{aligned}\tag{3.12}$$

Hence the frameworks $(G,c,p+p',r+2r')$ and $(G,c,p-p',r-2r')$ are congruent.

c. When the frameworks $(G,c,p+p',r+2r')$ and $(G,c,p-p',r-2r')$ are congruent, for any pair of vertices $i,j\in V$ we have that $\|(p+p')(i)-(p+p')(j)\|^2=\|(p-p')(i)-(p-p')(j)\|^2$. Expanding this equation out, as above, we see that $\Big[p(i)-p(j)\Big]\cdot\Big[p'(i)-p'(j)\Big]=-\Big[p(i)-p(j)\Big]\cdot\Big[p'(i)-p'(j)\Big]$, which gives us that $\Big[p(i)-p(j)\Big]\cdot\Big[p'(i)-p'(j)\Big]=0$ for all vertices $i,j\in V$.

Recall (from Definition 3.1.17) that the trivial infinitesimal motions of a k-coordinated framework (G,c,p,r) are $(p',\mathbf{0})$ for a trivial infinitesimal motion p' of the uncoloured framework (G,p). The standard trivial infinitesimal motions, as discussed in Definition 2.4.2 and Remark 2.4.3, are spanned by the infinitesimal translations, where $p'(i)=p'(j)$ for every pair of vertices $i,j\in V$, and the infinitesimal rotations, for which

$p'(i) - p'(j)$ is orthogonal to $p(i) - p(j)$ for each pair of vertices $i, j \in V$. If p' is not an infinitesimal motion of this type, then p' is not a trivial motion in the span of the framework (G, c, p, r), and hence $p' \notin \langle p \rangle$. As we assumed that $p' \in \langle p \rangle$, p' must be a trivial infinitesimal motion, and so $(p', \mathbf{0}) = (p', r')$ is a trivial infinitesimal motion of the k-coordinated framework (G, c, p, r). $\square$

The following example illustrates a framework with an infinitesimal motion, and the associated pair of equivalent frameworks that are not congruent.

Example 3.2.4. Let (G, c, p, r) be a 1-coordinated framework, with the structure graph G as illustrated in Figure 3.5, $p \in \mathbb{R}^{12}$ defined as follows:

$$\begin{aligned} &p(1) = (-4, 6) \qquad && p(2) = (4, 6) \qquad && p(3) = (0, 4) \\ &p(4) = (0, 0) && p(5) = (-2, 3) && p(6) = (2, 3) \end{aligned}$$

We may consider $r(1)$ as being arbitrary.

We define an infinitesimal motion (p', r') of this framework, by setting p' as follows:

$$\begin{aligned} &p'(1) = (0, 0) \qquad && p'(2) = (0, 0) \qquad && p'(3) = (0, 0) \\ &p'(4) = (0, 0) && p'(5) = \left(-\frac{1}{2}, -\frac{1}{3}\right) && p'(6) = \left(\frac{1}{2}, -\frac{1}{3}\right). \end{aligned}$$

It is straightforward to check that $\left[p(3) - p(5)\right] \cdot \left[p'(3) - p'(5)\right] = \left[2, 1\right] \cdot \left[-\frac{1}{2}, -\frac{1}{3}\right] = \frac{4}{3} = \left[p(3) - p(6)\right] \cdot \left[p'(3) - p'(6)\right]$, and that $\left[p(i) - p(j)\right] \cdot \left[p'(i) - p'(j)\right] = 0$ for all other edges $\{i, j\} \in E$. Thus (p', r') is a non-trivial infinitesimal motion of (G, c, p, r) with $r'(1) = -\frac{4}{3}$, and is illustrated in Figure 3.5.

It is worth noting that rescaling this motion gives another non-trivial infinitesimal motion of (G, c, p, r), such as (p'', r'') with $p'' = -p'$, and $r'' = \frac{4}{3} = -r'$. In fact a

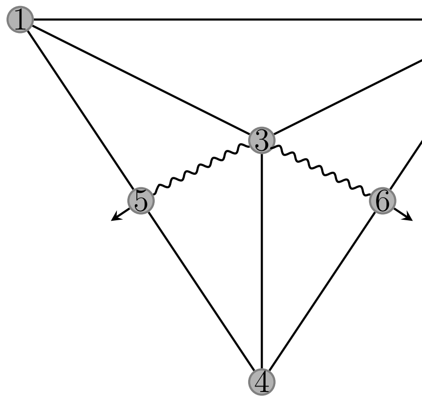
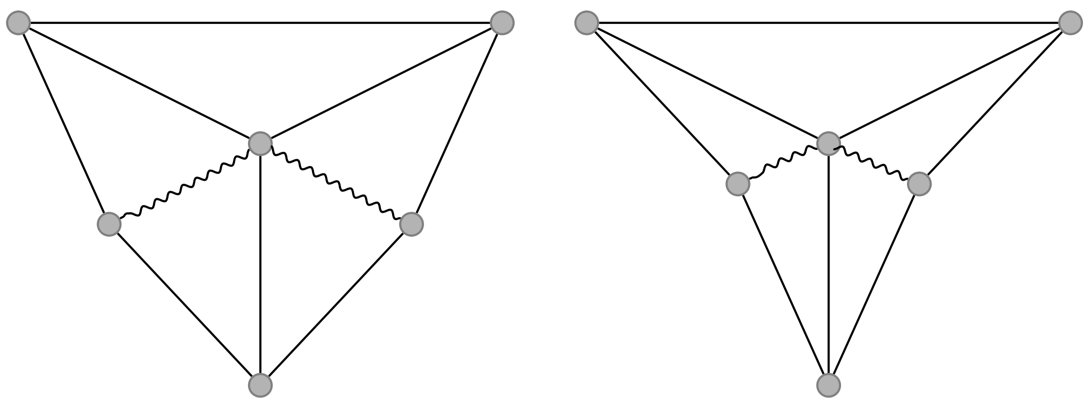


Figure 3.5 Example 3.2.4: (G, c, p, r), with the non-trivial infinitesimal motion p' illustrated. Edges in E_1 are indicated by wavy lines.



a $(G, c, p + p', r + 2r')$ **b** $(G, c, p - p', r - 2r')$

Figure 3.6 Example 3.2.4: a pair of equivalent, but not congruent, frameworks. Both are infinitesimally rigid.

non-trivial infinitesimal motion (p', r') will generate a 1-dimensional space of non-trivial infinitesimal motions of (G, c, p, r).

We may apply the result from Theorem 3.2.3 to get a pair of infinitesimally rigid 1-coordinated frameworks, $(G, c, p + p', r + 2r')$ and $(G, c, p - p', r - 2r')$. These are illustrated in Figure 3.6. The equivalence of these frameworks is straightforward to check: for the uncoloured edges, it is as simple as checking the lengths in both frameworks. For the pair of coloured edges in each framework, it is similarly easy to check that $\|(p+p')(3)-(p+p')(5)\|^2+(r+2r')(1) = \|(p-p')(3)-(p-p')(5)\|^2+(r-2r')(1)$ and $\|(p+p')(3)-(p+p')(6)\|^2+(r+2r')(1) = \|(p-p')(3)-(p-p')(6)\|^2+(r-2r')(1)$ (which holds for any initial definition of $r(1)$).

We may also restate Theorem 3.2.3 as follows, to get an adaptation of Theorem 3.2.1.

Theorem 3.2.5. Let (p, r) and (q, s) be a pair of configurations of the k-edge-coloured graph (G, c). Then

a. (G, c, p, r) is equivalent to (G, c, q, s) if and only if $(\frac{p-q}{2}, \frac{r-s}{4})$ is an infinitesimal motion for $(G, c, \frac{p+q}{2}, \frac{r+s}{2})$,

b. If $(\frac{p-q}{2}, \frac{r-s}{4})$ is a trivial infinitesimal motion of $(G, c, \frac{p+q}{2}, \frac{r+s}{2})$, then (G, c, p, r) is congruent to (G, c, q, s),

c. If the affine span $\langle \frac{p+q}{2} \rangle$ contains $\frac{p-q}{2}$, and (G, c, p, r) is congruent to (G, c, q, s), then $(\frac{p-q}{2}, \frac{r-s}{4})$ is a trivial infinitesimal motion of $(G, c, \frac{p+q}{2}, \frac{r+s}{2})$.

Proof. For clarity of notation, let $(\bar{p}, \bar{r})$ and $(\bar{q}, \bar{s})$ be a pair of configurations of (G, c).

Let $(p, r) := \left(\frac{\bar{p}+\bar{q}}{2}, \frac{\bar{r}+\bar{s}}{2}\right)$ and let $(p', r') := \left(\frac{\bar{p}-\bar{q}}{2}, \frac{\bar{r}-\bar{s}}{4}\right)$. Since (p, r) is the average of two configurations of (G, c), it is also a configuration of (G, c). It is straightforward to confirm that p' is a velocity field supported on p.

We note the following:

$$(p+p', r+2r') = \left(\frac{\bar{p}+\bar{q}}{2} + \frac{\bar{p}-\bar{q}}{2}, \frac{\bar{r}+\bar{s}}{2} + \frac{\bar{r}-\bar{s}}{2}\right) = (\bar{p}, \bar{r}),$$
$$(p-p', r-2r') = \left(\frac{\bar{p}+\bar{q}}{2} - \frac{\bar{p}-\bar{q}}{2}, \frac{\bar{r}+\bar{s}}{2} - \frac{\bar{r}-\bar{s}}{2}\right) = (\bar{q}, \bar{s}).$$

Thus $(G, c, \bar{p}, \bar{r})$ is equivalent to $(G, c, \bar{q}, \bar{s})$ if and only if $(G, c, p+p', r+2r')$ is equivalent to $(G, c, p-p', r-2r')$. By Theorem 3.2.3a,$(G, c, p+p', r+2r')$ is equivalent to $(G, c, p-p', r-2r')$ if and only if the velocity field $(p', r') = \left(\frac{\bar{p}-\bar{q}}{2}, \frac{\bar{r}-\bar{s}}{4}\right)$ is an infinitesimal motion of $(G, c, p, r) = \left(G, c, \frac{\bar{p}+\bar{q}}{2}, \frac{\bar{r}+\bar{s}}{2}\right)$ as required to prove **a**.

The proofs of Theorem 3.2.3b and c may be applied to **b** and **c** with the same substitutions of $(p, r) := \left(\frac{\bar{p}+\bar{q}}{2}, \frac{\bar{r}+\bar{s}}{2}\right)$ and $(p', r') := \left(\frac{\bar{p}-\bar{q}}{2}, \frac{\bar{r}-\bar{s}}{4}\right)$. □

We shall use this statement to prove the equivalence of rigidity and infinitesimal rigidity.

We will find it useful to refer to the set of infinitesimally rigid configurations, $I(G, c) = \{(q, s) :$ the k-coordinated framework (G, c, q, s) is infinitesimally rigid$\}$.

Theorem 3.2.6. Let (G, c, p, r) be a k-coordinated framework. If (G, c, p, r) is infinitesimally rigid, then it is rigid.

Proof. Let $(p, r) \in \mathbb{R}^{dn+k}$ be a configuration of the k-edge-coloured graph (G, c) such that (G, c, p, r) is an infinitesimally rigid framework, i.e. $(p, r) \in I(G, c)$. From Definition 3.1.21, and Theorem 3.1.22, we know that an infinitesimally rigid k-coordinated framework (G, c, p, r) has rigidity matrix $R(G, c, p)$ with rank $dn + k - \binom{d+1}{2}$. (Recall from Remark 3.1.20 that r may be considered to be 0 in the infinitesimal case.)

Since the rank of any matrix is lower semi-continuous, any $(p, r) \in I(G, c)$ will have an open neighbourhood $U_{(p,r)}$ of points $(q, s) \in \mathbb{R}^{dn+k}$ such that $\operatorname{rank} R(G, c, p) = \operatorname{rank} R(G, c, q)$, and hence (G, c, q, s) will be infinitesimally rigid for any $(q, s) \in U_{(p,r)}$.

Therefore $U_{(p,r)} \subset I(G,c)$, and the set of infinitesimally rigid configurations of the k-coloured graph (G,c) is an open set within $\mathbb{R}^{dn+k}$.

Let $U \subset R^{dn+k}$ be an open neighbourhood of (p,r) such that $U \subset I(G,c)$ and, for all $(q,s) \in U$, $(\frac{p+q}{2}, \frac{r+s}{2}) \in U$. Thus the frameworks (G,c,p,r), (G,c,q,s) and $(G, c, \frac{p+q}{2}, \frac{r+s}{2})$ are all infinitesimally rigid.

If (G,c,p,r) is equivalent to (G,c,q,s), then by Theorem 3.2.5(a) the framework $(G, c, \frac{p+q}{2}, \frac{r+s}{2})$ will have the infinitesimal motion $(p-q, \frac{r-s}{2})$. As $(\frac{p+q}{2}, \frac{r+s}{2}) \in I(G,c)$, the only infinitesimal motions of $(G, c, \frac{p+q}{2}, \frac{r+s}{2})$ will be the trivial infinitesimal motions. Since $(p-q, \frac{r-s}{2})$ is a trivial infinitesimal motion of $(G, c, \frac{p+q}{2}, \frac{r+s}{2})$, by Theorem 3.2.5(b) the frameworks (G,c,p,r) and (G,c,q,s) are congruent. Hence, as all frameworks within the neighbourhood U that are equivalent to (G,c,p,r) are congruent to (G,c,p,r), the framework (G,c,p,r) is locally rigid. $\square$

We finally have the following result to prove the other direction of Theorem 3.1.30.

Theorem 3.2.7. Let (G,c,p,r) be a d-dimensional k-coordinated framework, where the configuration p affinely spans $\mathbb{R}^d$. If (G,c,p,r) is regular and infinitesimally flexible in $\mathbb{R}^d$, then the framework (G,c,p,r) is flexible in $\mathbb{R}^d$.

Proof. Let $G = (V,E)$ and let $c : E \to \{0,1,\dots,k\}$ be a k-colouring of the edges of G. This partitions the edge set of G into the edges E_0 that are considered uncoloured, and the coloured edges. Let F denote the coloured edges, i.e. $F := E_1 \cup \cdots \cup E_k$.

Let $(p,r) \in \mathbb{R}^{dn+k}$ be a regular configuration for (G,c), where p affinely spans $\mathbb{R}^d$. We assume that the k-coordinated framework (G,c,p,r) is infinitesimally flexible, and hence $\operatorname{rank} df_{(G,c)}(p,r) < dn + k - \binom{d+1}{2}$.

Let $K_n = (V,K)$ be the complete graph on the vertex set V, and note that the uncoloured framework (K_n, p) is infinitesimally rigid since p affinely spans $\mathbb{R}^d$.

We construct the k-coloured multigraph $(M, \widetilde{c})$ by adding the coloured edges $F \subset E$ to the complete graph. This results in $M = (V, K \cup F)$ with $\widetilde{c}(f) := c(f)$ for all $f \in F$, and we define $\widetilde{c}(e) := 0$ for every edge $e \in K$. We note that $(M, \widetilde{c})$ is a multigraph with pairs of parallel edges exactly where one edge is coloured and one is uncoloured.

The "multi-framework" $(M, \widetilde{c}, p, r)$ contains an uncoloured copy of K_n as a subgraph, with $\operatorname{rank} df_{K_n}(p) = dn - \binom{d+1}{2}$. The k-coordinated "multi-framework" $(M, \widetilde{c}, p, r)$ therefore has $\operatorname{rank} df_{(M,\widetilde{c})}(p, r) = dn + k - \binom{d+1}{2}$. Since this is the maximal possible rank, $(p, r) \in R^{dn+k}$ is a regular configuration for the multigraph $(M, \widetilde{c})$.

By Proposition 2 (and the subsequent discussion) in [AR78], there exist neighbourhoods of $(p, r) \in \mathbb{R}^{dn+k}$, $U_{(p,r)}$ and $U'_{(p,r)}$, such that $f^{-1}_{(G,c)}\left(f_{(G,c)}(p, r)\right) \cap U_{(p,r)}$ is a smooth manifold of dimension $dn+k-\operatorname{rank} df_{(G,c)}(p, r)$, and $f^{-1}_{(M,\tilde{c})}\left(f_{(M,\tilde{c})}(p, r)\right) \cap U'_{(p,r)}$ is a smooth manifold of dimension $dn + k - \operatorname{rank} df_{(M,\tilde{c})}(p, r)$.

Let $U := U_{(p,r)} \cap U'_{(p,r)}$. We note that $f^{-1}_{(G,c)}\left(f_{(G,c)}(p, r)\right) \cap U$ is a strict submanifold of $f^{-1}_{(M,\tilde{c})}\left(f_{(M,\tilde{c})}(p, r)\right) \cap U$ since it has smaller dimension, and hence every neighbourhood of (p, r) will contain elements of $f^{-1}_{(M,\tilde{c})}\left(f_{(M,\tilde{c})}(p, r)\right) \setminus f^{-1}_{(G,c)}\left(f_{(G,c)}(p, r)\right)$. These may be used to construct a continuous non-trivial motion of (G, c, p, r) in the same way as in the standard case, as seen in the discussion after Proposition 2.3.5 [AR78, Proposition 1]. □

3.3 Coordinated static rigidity

We now extend the concept of static rigidity, discussed in Section 2.5, to coordinated frameworks, with the aim of extending the equivalence between infinitesimal rigidity and static rigidity to the coordinated context. We shall also use the static rigidity viewpoint when considering redundant rigidity in Chapter 4.

Definition 3.3.1. A *k-coordinated load* on a k-coordinated framework (G, c, p, r) is given by $(\mathbf{F}, S)$, where $\mathbf{F}$ is a load on the uncoloured framework (G, p) (as described in Definition 2.5.1), and $S \in \mathbb{R}^k$ is any vector.

The k-coordinated load $(\mathbf{F}, S)$ is an *k-coordinated equilibrium load* on the k-coordinated framework (G, c, p, r) when $\mathbf{F}$ is an equilibrium load on the uncoloured framework (G, p).

As before, we may consider the coordinated load $(\mathbf{F}, S)$ as a vector in $\mathbb{R}^{dn+k}$.

Definition 3.3.2. A stress $\boldsymbol{\rho} \in \mathbb{R}^{|E|}$ *resolves* the load $(\mathbf{F}, S)$ if

$$\sum_{j:\{i,j\}\in E} \rho_{ij}(p_i - p_j) = F_i \text{ for all } i \in V, \tag{3.13}$$

$$\sum_{\{i,j\}\in E_\ell} \rho_{ij} = S_\ell \text{ for all } \ell \in \{1, \ldots, k\}. \tag{3.14}$$

Remark 3.3.3. As in Remark 2.5.3, we may rewrite these constraints in terms of the coordinated rigidity matrix as follows:

$$\boldsymbol{\rho}^\top R(G, c, p) = [\mathbf{F}, S]\,. \tag{3.15}$$

Definition 3.3.4. A k-coordinated framework (G, c, p, r) is *statically rigid* if every k-coordinated equilibrium load on (G, c, p, r) has a resolution.

We note that each row of the coordinated rigidity matrix may be considered as an k-coordinated equilibrium load, similarly to each row of the standard rigidity matrix. (Definition 2.5.6.) We shall now define a coordinated analogue to the standard edge loads.

Definition 3.3.5. A *k-coordinated edge load* for any pair of vertices $i, j \in V$ is $(\mathbf{F}_{ij}, S_{ij})$, where $\mathbf{F}_{ij}$ is the standard edge load for the vertices $i, j \in V$, and S_{ij} is a vector in $\{0, 1\}^k$.

If there is an edge $\{i,j\} \in E$, the corresponding row of the coordinated rigidity matrix is a k-coordinated equilibrium load of this form. This may be denoted by $(\mathbf{F}_{ij}, S_\ell)$ for an edge $\{i,j\} \in E_\ell$, $0 \leq \ell \leq k$, where $S_\ell = \mathbf{0}$ for $\{i,j\} \in E_0$, and $S_\ell = \mathbf{e}_\ell$ for $1 \leq \ell \leq k$.

Remark 3.3.6. The edge resolution $\boldsymbol{\rho}_{\{i,j\}}$ in Definition 2.5.8 remains well defined on a k-coordinated framework (G,c,p,r), so we do not need to define a k-coordinated edge resolution.

Theorem 3.3.7. A k-coordinated framework (G,c,p,r) that affinely spans $\mathbb{R}^d$ is statically rigid if and only if it is infinitesimally rigid.

Proof. As the k-coordinated equilibrium loads of a d-dimensional k-coordinated framework (G,c,p,r) exclude any load that induces a translation or rotation of the whole framework, the space of such loads has dimension $dn + k - \binom{d+1}{2}$. Each row of the coordinated rigidity matrix $R(G,c,p)$ corresponds to a k-coordinated edge load, (F_{ij}, S_ℓ), for an edge $\{i,j\} \in E_\ell$ with $0 \leq \ell \leq k$, and each of these k-coordinated edge loads has a corresponding edge resolution $\boldsymbol{\rho}_{\{i,j\}}$ for $\{i,j\} \in E$.

These k-coordinated edge loads generate all k-coordinated equilibrium loads if and only if the rank of the coordinated rigidity matrix is $dn + k - \binom{d+1}{2}$, which is equivalent to the framework (G,c,p,r) being infinitesimally rigid by Theorem 3.1.22. The k-coordinated edge loads of the rigidity matrix generating every k-coordinated equilibrium load means that a resolution for every k-coordinated equilibrium load may be generated in the same way, and hence the framework (G,c,p,r) is statically rigid. $\square$

Chapter 4

Generic Coordinated Rigidity via Redundant Rigidity

In order to define coordinated rigidity in terms of redundant edge sets, we begin by defining the rigidity matroid for the standard rigidity case. The main characterisation of rigid coordinated frameworks in terms of coloured redundant edge sets, discussed in Section 4.1, is also given in a joint paper with Bernd Schulze and Louis Theran [SST18], though the proofs in this thesis contain more detail than is given in that paper.

Recall that as a consequence of Remark 3.1.20, we may simplify notation and refer to the k-coordinated framework (G, c, p), rather than the k-coordinated framework (G, c, p, r).

The d-dimensional generic infinitesimal rigidity matroid $\mathcal{M}_d(G)$ for a generic d-dimensional framework (G, p) is defined in Section 2.6 as a restriction of the d-dimensional generic infinitesimal rigidity matroid for the complete graph on n vertices, $\mathcal{M}_d(K_n) = \mathcal{M}_d(n)$ (Definition 2.6.3). We shall use matroidal techniques to characterise rigidity of k-coordinated frameworks (G, c, p) in terms of the rigidity of the underlying structure graph G.

We recall from Definition 2.6.3 that the complete graph on the vertex set V with $|V| = n$ may be denoted by $K_n = (V, K(V))$, and recall from Definition 2.6.5 that for any $E' \subset K(V)$, the graph with edge set $K(V) \setminus E'$ may be denoted by $K_n \setminus E'$. For consistency, we use similar notation $G \setminus e$ to denote the graph $G = (V, E)$ with the edge $e \in E$ removed, and $G \setminus E'$ to denote the subgraph of $G = (V, E)$ with edge set $E \setminus E'$ and vertex set $V(E \setminus E') \subseteq V$.

4.1 Characterisation using redundant rigidity

We now develop a characterisation of rigidity of k-coordinated frameworks (G, c, p), based on the rigidity of the underlying uncoloured graph G. We use a statics viewpoint to achieve this, by restricting the class of resolutions permitted to resolve equilibrium loads, rather than extending the class of loads that may be applied (as discussed in Section 3.3). Since static and infinitesimal rigidity are equivalent, this allows us to characterise infinitesimal rigidity of regular k-coordinated frameworks.

We recall from Definition 2.5.9 that $S(G, p)$ denotes the space of equilibrium stresses of a framework (G, p).

Definition 4.1.1. Let $G = (V, E)$ be a graph. If the edge $e \in E$ is induced in the matroid $\mathcal{M}_d (G \setminus \{e\})$, then the edge e is *redundant*. If e is not redundant, it is a *bridge*.

Proposition 4.1.2 (Proposition 2.4 [SST18])**.** Let (G, p) be a generic framework and let $\mathbf{F}$ be an equilibrium load. Then

a. Every resolution $\boldsymbol{\rho}$ of $\mathbf{F}$ has a unique decomposition $\boldsymbol{\rho} = \boldsymbol{\omega} + \boldsymbol{\tau}$, where $\boldsymbol{\omega} \in S(G, p)$ is an equilibrium stress and $\boldsymbol{\tau} \in S(G, p)^{\perp}$.

b. The matrices of the linear maps $\boldsymbol{\rho} \mapsto \boldsymbol{\tau}$ and $\boldsymbol{\rho} \mapsto \boldsymbol{\omega}$ are made up of rational functions of the coordinates of p.

c. These rational functions are non-constant on coordinates corresponding to redundant edges in the support of $\boldsymbol{\rho}$.

Proof. **a.** Let $\mathbf{F}$ be a given equilibrium load that is resolvable by (G,p). For any resolution $\boldsymbol{\rho}$ of the equilibrium load $\mathbf{F}$, from Equation (2.9) we have $\boldsymbol{\rho}^\top R(G,p) = \mathbf{F}$, and $\boldsymbol{\omega}^\top R(G,p) = \mathbf{0}$ for any equilibrium stress $\boldsymbol{\omega}$ by definition. Hence $(\boldsymbol{\rho}+\boldsymbol{\omega})^\top R(G,p) = \boldsymbol{\rho}^\top R(G,p) + \boldsymbol{\omega}^\top R(G,p) = \mathbf{F} + \mathbf{0} = \mathbf{F}$, and the space of resolutions of $\mathbf{F}$ has the form $\boldsymbol{\eta} + S(G,p) \subset \mathbb{R}^{|E|}$ for some $\boldsymbol{\eta} \in S(G,p)^\perp$. For a given resolution $\boldsymbol{\rho}$ of $\mathbf{F}$, we may therefore subtract any equilibrium stresses in $\boldsymbol{\rho}$ to give a unique decomposition $\boldsymbol{\rho} = \boldsymbol{\tau}+\boldsymbol{\omega}$, with $\boldsymbol{\omega} \in S(G,p)$ and $\boldsymbol{\tau} \in S(G,p)^\perp$.

b. We note that $S(G,p)$ is the kernel of the matrix $R(G,p)^\top$, and hence that $S(G,p)^\perp$ is the column space of the rigidity matrix $R(G,p)$. Let $D(G,p)$ be the matrix that projects the resolution $\boldsymbol{\rho}$ onto its component in $S(G,p)$, $\boldsymbol{\rho} \mapsto \boldsymbol{\omega}$, and hence $Id - D(G,p)$ is the matrix for the projection $\boldsymbol{\rho} \mapsto \boldsymbol{\tau}$, since $\boldsymbol{\tau} = \boldsymbol{\rho} - \boldsymbol{\omega} = \boldsymbol{\rho} - D(G,p)\boldsymbol{\rho}$. The matrix $D(G,p)$ may be obtained using the singular value decomposition, and both $D(G,p)$ and $Id - D(G,p)$ have rational functions of the coordinates of p as entries.

c. Let (G,p) be a generic framework, and let $\mathbf{F}$ be an equilibrium load on (G,p) with a resolution $\boldsymbol{\rho} = \boldsymbol{\tau} + \boldsymbol{\omega}$ for $\boldsymbol{\omega} \in S(G,p)$ and $\boldsymbol{\tau} \in S(G,p)^\perp$.

Let $\{i,j\} \in E$ be a redundant edge in (G,p), and hence there is an equilibrium stress $\boldsymbol{\omega}^*$ on (G,p) with $\omega^*_{\{i,j\}} \neq 0$. $\boldsymbol{\rho}^* = \boldsymbol{\tau} + \boldsymbol{\omega}^*$ is another resolution for $\mathbf{F}$ on (G,p).

Let p' be another generic configuration of G, where $p'(v) := p(v)$ for all vertices $v \in V \setminus \{i\}$, and $p'(i)$ is a perturbation of $p(i)$ that remains arbitrarily close to $p(i)$. The generic framework (G,p') will have a corresponding equilibrium stress $(\boldsymbol{\omega}^*)'$ that is also non-zero on the edge $\{i,j\}$, with $(\omega^*)'_{\{i,j\}} \neq \omega^*_{\{i,j\}}$. Moving $p'(i)$ along the line $p'(j) - p'(i)$ results in the component $(\omega^*)'_{\{i,j\}}$ of $(\boldsymbol{\omega}^*)'$ increasing, and hence the projection $\boldsymbol{\rho} \mapsto \boldsymbol{\omega}$ is non-constant on the edge $\{i,j\}$. Since the matrix for the projection

$\boldsymbol{\rho} \mapsto \boldsymbol{\tau}$ is obtained from the matrix for $\boldsymbol{\rho} \mapsto \boldsymbol{\omega}$, this will also be non-constant on the edge $\{i, j\}$. $\square$

Lemma 4.1.3 (Proposition 2.5 [SST18])**.** Let (G, p) be a framework that affinely spans $\mathbb{R}^d$. Any equilibrium stress $\boldsymbol{\omega} \in \mathbb{R}^{|E|}$ will have $\omega_{\{i,j\}} = 0$ for all bridges $\{i, j\} \in E$. This equilibrium stress is only supported on redundant edges in E.

Proof. From Definition 2.5.9, any equilibrium stress of (G, p), $\boldsymbol{\omega} \in \mathbb{R}^{|E|}$, has $\sum_{j:\{i,j\}\in E} \omega_{\{i,j\}} \Big[p(i) - p(j)\Big] = 0$ for every vertex $i \in V$. Since each bridge $\{i, j\} \in E$ is independent (in the coordinated rigidity matroid) of the other edges in E, it is clear that there cannot be a linear dependence $\boldsymbol{\omega}$ with $\omega_{\{i,j\}} \neq 0$ where $\sum_{k:\{i,k\}\in E} \omega_{\{i,k\}} \Big[p(i) - p(k)\Big] = 0$ and $\sum_{k:\{j,k\}\in E} \omega_{\{j,k\}} \Big[p(j) - p(k)\Big] = 0$. $\square$

We shall generalise the concept of a bridge and a redundant edge as follows.

Definition 4.1.4. A subset of edges $E' \subseteq E$ of a graph $G = (V, E)$ with $|E'| = k$ is *k-redundant* if the edge set $E \setminus E'$ induces each edge $e' \in E'$. If the graph $G \setminus E'$ does not induce the set E', then E' is a *k-bridge*.

Lemma 4.1.5 (Lemma 4.2 [SST18])**.** Let $G = (V, E)$ be a graph and let (G, p) be a generic framework that affinely spans $\mathbb{R}^d$. A subset of edges $E' \subseteq E$ with $|E'| = k$ is k-redundant if and only if any linear combination of the edge loads $\mathbf{F}_{ij}$ for $\{i, j\} \in E'$ can be resolved by the framework $(G \setminus E', p)$.

Proof. Let $E' = \{e_1, \dots, e_k\}$ be a subset of edges of the generic framework (G, p), with a corresponding set of edge loads $\mathbf{F}_{e_1}, \dots, \mathbf{F}_{e_k}$. Recall that the space of equilibrium loads has dimension $dn - \binom{d+1}{2}$, and that the dimension of the space of equilibrium loads that can be resolved by the framework (G, p) is the rank of the rigidity matrix $R(G, p)$. This space is spanned by the set of edge loads $\{\mathbf{F}_e : e \in E\}$.

Let X be the space of equilibrium loads spanned by the edge loads $\mathbf{F}_{e_1}, \dots, \mathbf{F}_{e_k}$. Since each edge e_i exists in the framework (G, p), each edge load $\mathbf{F}_{e_i}$ has a corresponding edge resolution $\boldsymbol{\rho}_{e_i}$, and hence every equilibrium load in X has a resolution in terms of the edge resolutions $\boldsymbol{\rho}_{e_1}, \ldots, \boldsymbol{\rho}_{e_k}$. If there is some $\mathbf{F} \in X$ that cannot be resolved by the reduced framework $(G \setminus E', p)$, the rank of $R(G \setminus E', p)$ is strictly less than the rank of $R(G, p)$, and so E' is not a k-redundant set of edges.

Conversely, if E' is not a k-redundant set of edges, then $\operatorname{rank} R(G \setminus E', p) < \operatorname{rank} R(G, p)$, and hence there is some equilibrium load $\mathbf{F}$ that is resolvable by (G, p) and not by $(G \setminus E', p)$. Since the edge loads $\mathbf{F}_e$ for $e \in E$ span the space of resolvable equilibrium loads for (G, p), and the edge loads $\mathbf{F}_e$ for $e \in E \setminus E'$ span the equilibrium loads that are resolvable by $(G \setminus E', p)$, the support of any equilibrium load that is not resolvable by $(G \setminus E', p)$ will contain at least one $\mathbf{F}_{e_\ell}$ for $1 \leq \ell \leq k$. $\square$

Lemma 4.1.6 (Lemma 4.3 [SST18])**.** Let $G = (V, E)$ be a graph, let (G, p) be a generic framework and let $E' \subseteq E$ with $|E'| = k$ be a k-redundant set of edges. Then there are linearly independent equilibrium stresses $\{\boldsymbol{\omega}_{e'} : e' \in E'\}$ that remain linearly independent when restricted to the coordinates corresponding to E'.

Proof. Since E' is a k-redundant set of edges and p is generic, by Lemma 4.1.3 there is an equilibrium stress $\boldsymbol{\omega}_{e'}$ supported on the edges of $(G \setminus E') \cup \{e'\}$, for each $e' \in E'$. These will be linearly independent, and remain linearly independent when only the coordinates corresponding to the edges in E' are considered, since for each $e' \in E'$, $(\omega_{e'})_{f'} = 0$ for all $f' \in E' \setminus \{e'\}$. $\square$

Rather than extend the definition of equilibrium loads in $\mathbf{F} \in \mathbb{R}^{dn}$ to k-coordinated equilibrium loads $(\mathbf{F}, S) \in \mathbb{R}^{dn+k}$ (Definition 3.3.1), with resolutions in $\mathbb{R}^{|E|}$, we shall instead define an additional class of equilibrium loads in $\mathbb{R}^{dn}$ with corresponding resolutions.

Definition 4.1.7 ([SST18]). A k-coordinated framework (G, c, p) has a *coordination class load* $\mathbf{F}_\ell$ for each coordination class E_ℓ, $1 \leq \ell \leq k$, defined as

$$\mathbf{F}_\ell := \sum_{\{i,j\} \in E_\ell} \mathbf{F}_{\{i,j\}}, \tag{4.1}$$

where $\mathbf{F}_{\{i,j\}}$ is the edge load for each edge $\{i, j\} \in E_\ell$ (see Definition 2.5.6).

Recall from Definition 2.5.8 that for each edge $\{i, j\} \in E$, the edge load $\mathbf{F}_{\{i,j\}}$ has an edge resolution $\boldsymbol{\rho}_{\{i,j\}}$. The *natural resolution* of each coordination class load $\mathbf{F}_\ell$ is defined to be

$$\boldsymbol{\rho}_\ell := \sum_{\{i,j\} \in E_\ell} \boldsymbol{\rho}_{\{i,j\}}. \tag{4.2}$$

We recall that the coordinated rigidity matrix $R(G, c, p)$ corresponds to the standard rigidity matrix $R(G, p)$, together with the characteristic matrix $\mathbb{1}(c)$. The column span of $\mathbb{1}(c)$ corresponds to the linear span of these natural resolutions.

Definition 4.1.8. The linear span of the natural resolutions of the coordination class loads is $C(G, c, p) := \operatorname{lin}\{\boldsymbol{\rho}_\ell : 1 \leq \ell \leq k\}$.

Lemma 4.1.9 (Lemma 3.6 [SST18]). Let p span $\mathbb{R}^d$. The k-coordinated framework (G, c, p) is infinitesimally rigid if and only if the column space $C(G, c, p)$ of $\mathbb{1}(c)$ intersects the column space $S(G, p)^\perp$ of the rigidity matrix $R(G, p)$ trivially.

Equivalently, (G, c, p) is infinitesimally rigid if and only if the projection of $C(G, c, p)$ onto $S(G, p)$ is k-dimensional.

Proof. We recall (from Equation (3.8)) that $(p', r') \in \mathbb{R}^{dn+k}$ is an infinitesimal motion of the framework (G, c, p) if and only if $R(G, p)p' + \mathbb{1}(c)r' = 0$, and that (G, c, p) is infinitesimally rigid if and only if $\operatorname{rank} R(G, c, p) = dn + k - \binom{d+1}{2}$ by Theorem 3.1.22. Therefore (G, c, p) is infinitesimally rigid if and only if $\operatorname{rank} R(G, c, p) = dn+k-\binom{d+1}{2} \leq$

rank $R(G,p) +$ rank $\mathbb{1}(c) \leq dn - \binom{d+1}{2} + k$. This is equivalent to when $\mathbb{1}(c)$ has rank k, $R(G,p)$ has rank $dn - \binom{d+1}{2}$, and the column spaces $C(G,c,p)$ and $S(G,p)^\perp$ intersect trivially.

Since $C(G,c,p)$ has dimension k, and the projection onto $S(G,p)^\perp$ is trivial, the projection onto $S(G,p)$ will clearly have dimension k. $\square$

Definition 4.1.10. A subset of edges $E' \subseteq E$ of a k-coloured graph (G,c) with $|E'| = k$ is a *rainbow* set of edges if $|E' \cap E_\ell| = 1$ for all colour classes $1 \leq \ell \leq k$.

Theorem 4.1.11 (Theorem 4.1 [SST18])**.** For $d \geq 1$ and $k \geq 1$, the k-coordinated graph (G,c) on n vertices is generically rigid in d dimensions if and only if G is generically rigid in $\mathcal{M}_d(n)$, and some rainbow subset $E' \subseteq E$ is k-redundant in $\mathcal{M}_d(G)$.

Proof. Let $p \in \mathbb{R}^{dn}$ be a generic configuration, so (G,p) is a generic framework in $\mathbb{R}^d$ and (G,c,p) is a generic k-coordinated framework in $\mathbb{R}^d$. We recall from Equation 3.8 that a coordinated infinitesimal motion $(p',r') \in \mathbb{R}^{dn+k}$ of the generic framework (G,c,p) satisfies $R(G,p)p' + \mathbb{1}(c)r' = 0$, which may be restated as

$$R(G,p)p' = -\mathbb{1}(c)r'. \tag{4.3}$$

We recall also that the trivial infinitesimal motions of a k-coordinated framework are $(p', \mathbf{0})$ where p' is a trivial infinitesimal motion of the standard framework (G,p) (Definition 3.1.17).

Suppose first that G is generically rigid in $\mathcal{M}_d(n)$ and that (G,c) contains a k-redundant rainbow subset of edges $E' := \{e_1, \ldots, e_k\}$, where $c(e_i) = i$ for $1 \leq i \leq k$. Since (G,p) is a generic framework, it is infinitesimally rigid, and hence the only infinitesimal motions $p' \in \mathbb{R}^{dn}$ satisfying $R(G,p)p' = \mathbf{0}$ are the trivial infinitesimal motions. This is equivalent to Equation 4.3 having no solution (p',r') where p' is a non-trivial infinitesimal motion and $r' = \mathbf{0}$.

Assume instead that (G,c,p) has a non-trivial infinitesimal motion, (p',r') with $r' \neq \mathbf{0}$, that is a solution to Equation 4.3. The left kernel of $R(G,p)$ is the space of equilibrium stresses of the generic framework (G,p). Considering $r' \neq \mathbf{0}$ as fixed, Equation 4.3 has a solution p' if and only if $-\mathbb{1}(c)r'$ is orthogonal to each equilibrium stress of (G,p).

By Lemma 4.1.6 there is a set of k linearly independent equilibrium stresses $\{\boldsymbol{\omega}_1,\dots,\boldsymbol{\omega}_k\}$ corresponding to the k-redundant edges $E' = \{e_1,\dots,e_k\}$, such that for $1 \leq i \leq k$, $(\omega_i)_{e_i} \neq 0$ and $(\omega_i)_{e_j} = 0$ for $j \neq i$, $1 \leq j \leq k$. Since $\mathbb{1}(c)r'$ is orthogonal to all equilibrium stresses of (G,p), $\mathbb{1}(c)r'$ is orthogonal to each $\boldsymbol{\omega}_i$ for $1 \leq i \leq k$.

Since p is generic, in order for these k linearly independent equilibrium stresses to be orthogonal to $\mathbb{1}(c)r'$, we require that $(\mathbb{1}(c)r')_{e_i} = 0$ for $1 \leq i \leq k$. Since E' is a rainbow set of edges, the submatrix of $\mathbb{1}(c)$ corresponding to the edges E' consists of the k-dimensional basis vectors $\mathbf{e}_1,\dots,\mathbf{e}_k$. This is therefore equivalent to $\mathbf{e}_i \cdot r' = 0$ for $1 \leq i \leq k$, which occurs if and only if $(r')_i = 0$ for $1 \leq i \leq k$, and hence $r' = \mathbf{0}$. A non-trivial infinitesimal motion of (G,c,p) with $r' \neq \mathbf{0}$ therefore cannot exist when (G,c) contains a k-redundant rainbow set of edges.

Hence when G is generically rigid and (G,c) contains a k-redundant rainbow set of edges, the generic framework (G,c,p) is rigid, and therefore (G,c) is generically rigid.

Suppose instead that (G,c,p) is an infinitesimally rigid generic k-coordinated framework. We recall from the proof of Lemma 4.1.9 that (G,c,p) is infinitesimally rigid if and only if $\operatorname{rank} R(G,c,p) = dn + k - \binom{d+1}{2} = \operatorname{rank} R(G,p) + \operatorname{rank} \mathbb{1}(c)$, which is equivalent to $\operatorname{rank} R(G,p) = dn - \binom{d+1}{2}$, $\operatorname{rank} \mathbb{1}(c) = k$, and the projection of the column space of $\mathbb{1}(c)$ onto the space of equilibrium stresses of the framework (G,p) being k-dimensional. This clearly implies that the uncoloured framework (G,p) is infinitesimally rigid (by Theorem 2.4.9), and that all colour classes are non-empty. It remains to show that (G,c,p) contains a k-redundant rainbow subset of edges.

Recall from the proof of Proposition 4.1.2 that the matrix $D(G,p)$ projects a resolution $\boldsymbol{\rho} \in \mathbb{R}^{|E|}$ onto its component in the left kernel of $R(G,p)$, the space of equilibrium stresses $S(G,p)$. The dimension of the projection of $C(G,c,p)$ having dimension k is equivalent to $D(G,p)\mathbb{1}(c)$ having rank k, and hence $D(G,p)\mathbb{1}(c)$ contains a k by k minor with non-zero determinant.

It is clear that a k by k minor with non-zero determinant cannot contains any zero rows, which are the rows of $\mathbb{1}(c)$ corresponding to uncoloured edges of (G,c). Since each row of $\mathbb{1}(c)$ contains at most one non-zero entry, and the rows of this k by k minor must therefore contain exactly one non-zero entry, it is clear that each row of the minor must correspond to an edge of a distinct colour, in order to avoid a zero column within the minor. This minor of $\mathbb{1}(c)$ therefore corresponds to a rainbow subset of edges, E'. We may label the edges of E' as $\{e_1,\ldots,e_k\}$, where $c(e_i)=i$ for $1\le i\le k$.

Each edge in E' has a corresponding edge load on the framework (G,p), $\mathbf{F}_{e_1},\ldots,\mathbf{F}_{e_k}$. For arbitrary $\boldsymbol{\alpha}\in\mathbb{R}^k$, let $\mathbf{F}(\boldsymbol{\alpha})$ be an equilibrium load spanned by the edge loads corresponding to E', $\mathbf{F}(\boldsymbol{\alpha}) := \alpha_1\mathbf{F}_{e_1}+\cdots+\alpha_k\mathbf{F}_{e_k}$. To confirm that E' is a k-redundant subset of edges, we require a resolution $\boldsymbol{\rho}$ of $\mathbf{F}(\boldsymbol{\alpha})$ with $\rho_{e_1}=\cdots=\rho_{e_k}=0$, which is a resolution of $\mathbf{F}(\boldsymbol{\alpha})$ by the framework $(G\setminus E',p)$.

Since G is generically rigid, $\mathbf{F}(\boldsymbol{\alpha})$ has a resolution $\boldsymbol{\rho}(\boldsymbol{\alpha})$, and hence by Proposition 4.1.2, $\boldsymbol{\rho}(\boldsymbol{\alpha})=\boldsymbol{\tau}(\boldsymbol{\alpha})+\boldsymbol{\omega}(\boldsymbol{\alpha})$ where $\boldsymbol{\tau}(\boldsymbol{\alpha})=[Id-D(G,p)]\boldsymbol{\rho}(\boldsymbol{\alpha})\in S(G,p)^{\perp}$ is also a resolution for the equilibrium load $\mathbf{F}(\boldsymbol{\alpha})$, and $\boldsymbol{\omega}(\boldsymbol{\alpha})$ is an equilibrium stress. We note that each e_i is in the support of $\boldsymbol{\tau}(\boldsymbol{\alpha})$. If $\rho(\boldsymbol{\alpha})_{e_i}=0$ for $1\le i\le k$, we already have a resolution of $\mathbf{F}(\boldsymbol{\alpha})$ by $(G\setminus E',p)$, hence assume that $\rho(\boldsymbol{\alpha})_{e_\ell}\neq 0$ for some $e_\ell\in E'$.

As the projection of $C(G,c,p)$ onto $S(G,p)$ by $D(G,p)$ is k-dimensional, there are k linearly independent equilibrium stresses corresponding to the edges $e_1,\ldots,e_k$. Let these be denoted by $\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_k\in S(G,p)$, where the support of each $\boldsymbol{\omega}_i$ contains e_i. It is clear that these $\boldsymbol{\omega}_i$ are linearly independent from $\boldsymbol{\tau}(\boldsymbol{\alpha})\in S(G,p)^{\perp}$.

Since $D(G, p)$ consists of rational functions of the coordinates of the generic configuration p and $(\omega_i)_{e_i} \neq 0$ for $1 \leq i \leq k$, the equilibrium stresses $\boldsymbol{\omega}_i$ remain linearly independent when restricted to the coordinates corresponding to the edges in E'. We denote this restriction by $\widehat{\boldsymbol{\omega}_i}$, and let $\widehat{\boldsymbol{\tau}(\boldsymbol{\alpha})} \in \mathbb{R}^k$ denote the restriction of $\boldsymbol{\tau}(\boldsymbol{\alpha})$.

Since both $\widehat{\tau(\boldsymbol{\alpha})}_{e_i} \neq 0$ and $(\omega_i)_{e_i} \neq 0$ for $1 \leq i \leq k$, we may use the linearly independent equilibrium stresses $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_k$ to obtain an equilibrium stress $\boldsymbol{\omega}^*$ such that $(\omega^*)_{e_\ell} + (\tau(\boldsymbol{\alpha}))_{e_\ell} = 0$ for any $e_\ell \in E'$ with $\rho(\boldsymbol{\alpha})_{e_\ell} \neq 0$. This gives a resolution $\boldsymbol{\rho}^* := \boldsymbol{\tau}(\boldsymbol{\alpha}) + \boldsymbol{\omega}^*$ for $\mathbf{F}(\boldsymbol{\alpha})$ with $(\rho^*)_{e_i} = 0$ for $1 \leq i \leq k$, as required for E' to be a k-redundant rainbow subset of edges. $\square$

We note that the coordination class loads $\mathbf{F}_i$ are also equilibrium loads on the framework (G, c, p) for $1 \leq i \leq k$. Since $\mathbf{F}_i$ is generated by the edge loads of the edges $e \in E_i$, and all $e \in E_i \backslash \{e_i\}$ remain in the framework $(G \backslash \{e_1, \ldots, e_k\}, p)$, $\mathbf{F}_i$ is resolvable by $(G \setminus \{e_1, \ldots, e_k\}, p)$ since each edge load $\mathbf{F}_{e_i}$ is resolvable by $(G \setminus \{e_1, \ldots, e_k\}, p)$.

As rigidity in $\mathcal{M}_1(n)$ and $\mathcal{M}_2(n)$ is known, we may apply Theorem 4.1.11 to certify whether or not any k-coordinated graph (G, c) is generically rigid in 1 dimension and in 2 dimensions, however for high k this is not possible in polynomial time. Conversely, since generic rigidity in $\mathcal{M}_d(n)$ has not been classified for $d \geq 3$, we do not gain any increased intuition in higher dimensions from Theorem 4.1.11.

The existence of a k-redundant rainbow subset of edges is not equivalent to every rainbow subset of edges being k-redundant, and the existence of a single k-bridge does not confirm that the k-coordinated framework (G, c) is generically flexible. This is illustrated by Example 4.1.12.

Example 4.1.12. Let (G, c) be the 2-coloured framework illustrated in Figure 4.1. If (G, c) contains a 2-redundant rainbow pair then it is generically rigid, however there are 9 rainbow pairs that may need to be checked.

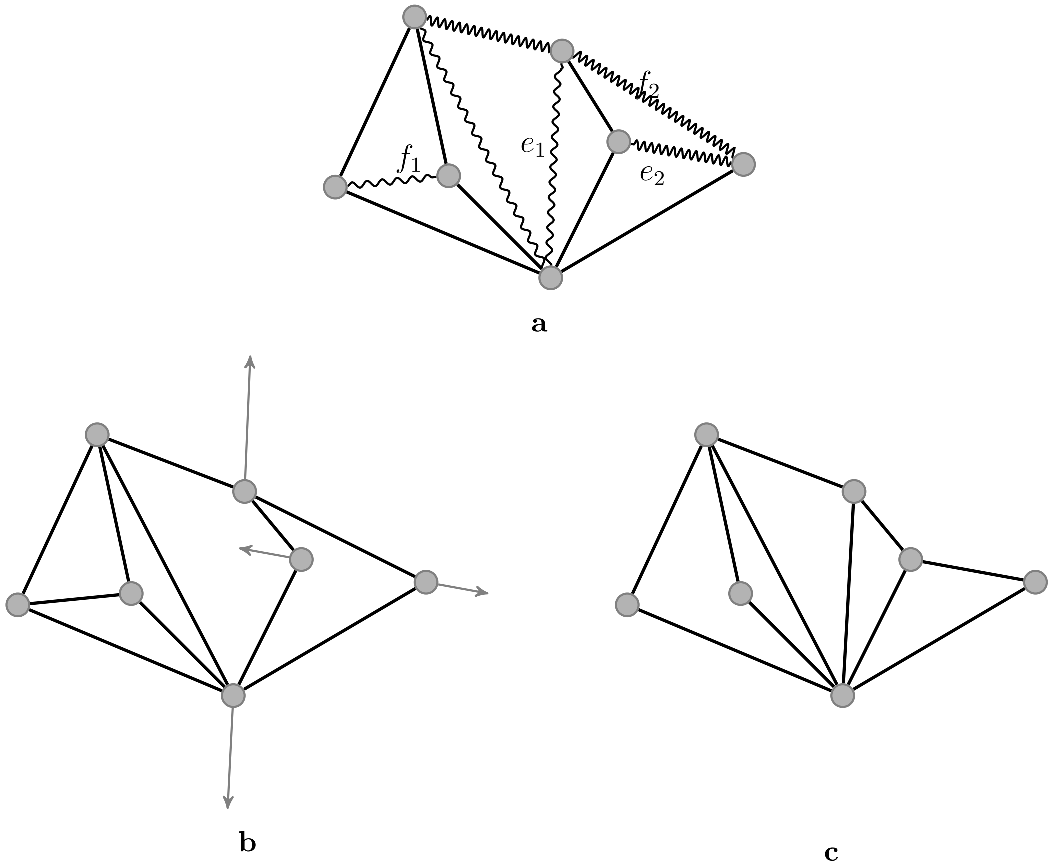


Figure 4.1 A 2-coordinated framework in $\mathbb{R}^2$. The framework in Figure 4.1b, created by removing the rainbow pair $\{e_1, e_2\}$ from the framework in Figure 4.1a, is a flexible uncoloured framework. The motion is indicated by gray arrows. Figure 4.1c shows a rigid uncoloured framework, resulting when the 2-redundant rainbow pair of edges $\{f_1, f_2\}$ is removed from the framework in Figure 4.1a.

The pair of edges labelled e_1 and e_2 are a rainbow pair, with $e_1 \in E_1$ and $e_2 \in E_2$. Removing this rainbow pair results in the uncoloured reduced graph $G \setminus \{e_1, e_2\}$, illustrated in Figure 4.1b. This reduced graph is flexible, and contains a copy of K_4. (A motion is indicated by gray arrows.) The rainbow pair $\{e_1, e_2\}$ is therefore a 2-bridge.

Figure 4.1c shows the uncoloured reduced graph $G \setminus \{f_1, f_2\}$, where $f_1 \in E_1$ and $f_2 \in E_2$. This reduced graph is generically rigid as an uncoloured graph, and hence $\{f_1, f_2\}$ is a 2-redundant rainbow pair, implying that the 2-coordinated framework (G, c) is generically rigid in 2 dimensions.

Chapter 5

Coordinated Inductive Constructions

We considered some standard inductive constructions that preserve generic rigidity of frameworks in Section 2.8. As we have now defined the class of k-coordinated frameworks, we wish to find similar inductive constructions that characterise these k-coordinated frameworks. We shall begin with k-edge-coloured analogues to the standard Henneberg moves.

As in Chapter 3, we consider the uncoloured graph $G = (V, E)$ to have $|V| = n$, and may sometimes denote the number of edges, $|E|$, by m. The k-edge-colouring $c : E \to \{0, 1, \ldots, k\}$ induces a partition of the edges of G into the uncoloured edges E_0, and colour classes E_ℓ for $1 \leq \ell \leq k$. We consider the dimension d and the number of colours k to be fixed.

We note that as a consequence of Remark 3.1.20, we may refer to k-coordinated frameworks as (G, c, p) instead of (G, c, p, r).

We begin by defining the coloured 0-extension and coloured 1-extension in general, and proving that these moves preserve generic isostaticity.

5.1 Coordinated 0-extension

Definition 5.1.1. A d-dimensional *0-extension* of a k-edge-coloured graph (G, c) is applied by creating a new vertex x, and adding d new edges $\{x, v_i\}$ for some distinct vertices $\{v_i : 1 \leq i \leq d\} \subset V$.

The new edges may be allocated in any way to the sets $E_0, E_1, \ldots, E_k$ while preserving the infinitesimal rigidity of the original graph, as we shall see in Lemma 5.1.2.

When all new edges are added to E_0, we refer to this move as the *standard 0-extension*.

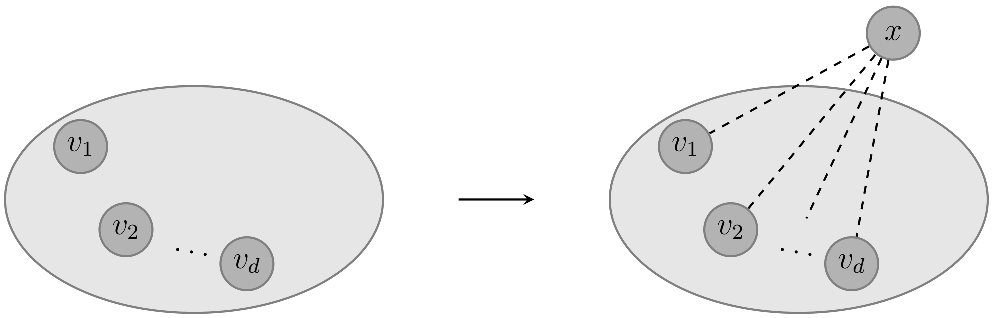


Figure 5.1 A d-dimensional 0-extension.

Lemma 5.1.2. Let (G, c) be a d-isostatic, k-edge-coloured graph, and let (G', c') be obtained by applying a 0-extension of any type to (G, c). Then (G', c') is also d-isostatic.

Proof. Let (G, c, p) be an isostatic k-coordinated framework in $\mathbb{R}^d$ with $|V| = n$, and hence $\operatorname{rank} R(G, c, p) = dn + k - \binom{d+1}{2} = |E|$ by Theorem 3.1.22. Let (G', c') be obtained by applying a d-dimensional 0-extension to (G, c), where $c'(e) := c(e)$ for all $e \in E$, and $c'(e)$ is arbitrary for $e \in E' \setminus E$. Let $\widehat{p} \in \mathbb{R}^{d(n+1)}$ be a configuration of (G', c') such that $\widehat{p}(v) := p(v)$ for $v \in V$, and $\widehat{p}(x)$ is chosen for the new vertex $x \in V' \setminus V$ such that $\widehat{p}(x)$ lies outside the affine span of $\widehat{p}(v_1), \ldots, \widehat{p}(v_d)$.

The coordinated rigidity matrix for the extended graph $R(G', c', \widehat{p})$, contains $R(G, c, p)$ as a submatrix, together with d additional rows and d additional columns.

Each additional row corresponds to an edge $\{x, v_i\}$, and contains the d-dimensional vector $\widehat{p}(x) - \widehat{p}(v_i)$ in the first d columns. For $1 \leq i \leq d$, these vectors are linearly independent, and hence $\operatorname{rank} R(G', c', \widehat{p}) = \operatorname{rank} R(G, c, p) + d = dn + k - \binom{d+1}{2} + d = d(n+1)+k-\binom{d+1}{2}$. Since $R(G', c', \widehat{p})$ has $|E|+d = dn+k-\binom{d+1}{2}+d = d(n+1)+k-\binom{d+1}{2}$ rows, this is the maximal possible rank and hence $\operatorname{rank} R(G', c', \widehat{p}) = d(n+1)+k-\binom{d+1}{2} = |E'|$. Thus $(G', c', \widehat{p})$ is infinitesimally rigid and independent, and hence is isostatic in $\mathbb{R}^d$. $\square$

5.2 Coordinated 1-extension

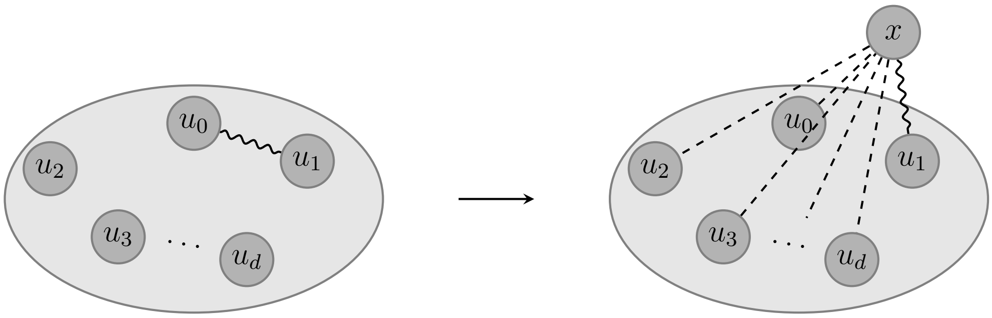
Definition 5.2.1. A d-dimensional *1-extension* of a k-edge-coloured graph (G, c), applied to the edge $\{u_0, u_1\} \in E$, is the creation of a new graph G' by removing the edge $\{u_0, u_1\}$ and creating a new vertex x of degree $d + 1$, with edges $\{x, u_0\}$, $\{x, u_1\}$ and $\{x, u_i\}$ for some $\{u_i : 2 \leq i \leq d\} \subset V \setminus \{u_1, u_2\}$.

If the initial edge is $\{u_0, u_1\} \in E_0$, we require that both edges $\{x, u_0\}$, $\{x, u_1\}$ be added to E'_0. If instead $\{u_0, u_1\} \in E_\ell$ for some $1 \leq \ell \leq k$, we require that at least one of the edges $\{x, u_0\}$ and $\{x, u_1\}$ be added to E'_ℓ. The other may be added to any E_j for $0 \leq j \leq k$. In either case, the additional edges $\{x, u_i\}$ for $2 \leq i \leq d$ may be added to any E_j for $0 \leq j \leq k$.

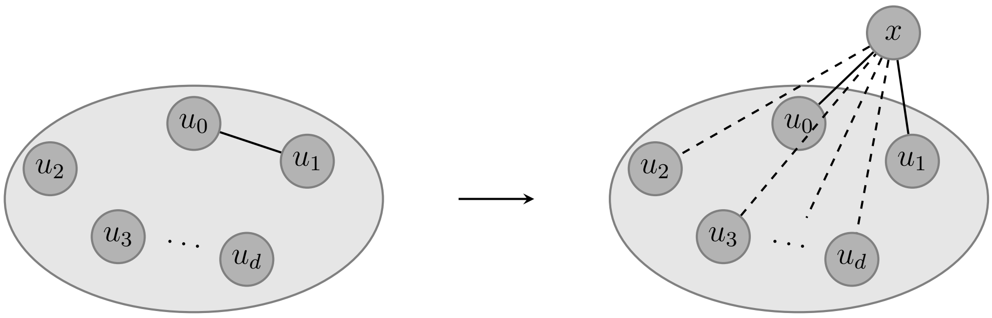
Lemma 5.2.2. Let (G, c) be a d-isostatic, k-edge-coloured graph, and let (G', c') be obtained by applying a 1-extension of any type to an edge of (G, c). Then (G', c') is also d-isostatic.

As previously noted, all vectors within this proof are assumed to be column vectors.

Proof. Let (G', c') be a k-edge-coloured graph on $n + 1$ vertices obtained by applying a 1-extension on the edge $\{u_0, u_1\}$ to the d-isostatic k-edge-coloured graph (G, c). Let $x \in V' \setminus V$ be the new vertex created by this 1-extension, with neighbours $u_0, u_1, \ldots, u_d$.



a A d-dimensional 1-extension applied to an edge in E_1.



b A d-dimensional 1-extension applied to an edge in E_0.

Figure 5.2 Two types of d-dimensional 1-extension.

Let $q \in \mathbb{R}^{dn}$ be a regular configuration of (G, c). By Lemma 2.4.18, $A(q)$ is also a regular configuration of V for any affine transformation A of $\mathbb{R}^d$. We may apply appropriate affine transformations to obtain a regular configuration $A(q) = p \in \mathbb{R}^{dn}$ such that $p(u_0) = \mathbf{0}$ and $p(u_i) = \mathbf{e}_i$ for $1 \leq i \leq d$. This configuration gives a k-coordinated regular framework (G, c, p).

Suppose first that $\{u_0, u_1\} \in E_\ell$ for some $1 \leq \ell \leq k$, and so at least one of $\{x, u_0\} \in E'_\ell$ or $\{x, u_1\} \in E'_\ell$. Since $|V| = n$ and $|V'| = n + 1$, we may relabel the vertices of (G', c') using $\{0, 1, \ldots, n\}$ as follows: let one of u_0 and u_1 that is adjacent to an edge in E'_ℓ be labelled 1, and let the other vertex of this pair be labelled 0. Let the other neighbours of x, $u_2, \ldots, u_d$, be labelled by $2, \ldots, d$, and let x be labelled n. The remaining vertices of (G', c') may be arbitrarily labelled with $d + 1, \ldots, n - 1$.

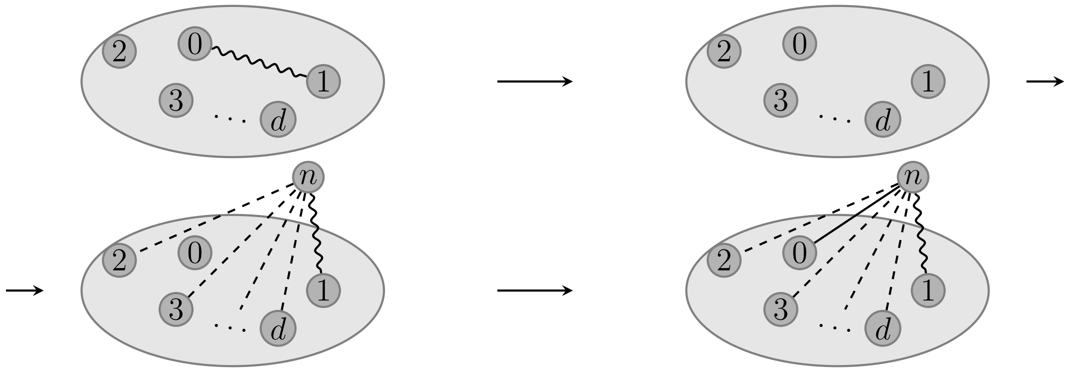


Figure 5.3 The steps of a d-dimensional 1-extension applied to an edge in E_1: the edge $\{0,1\} \in E_1$ is removed from (G,c), resulting in the reduced graph (V,F). The graph $\widehat{G}$ is then obtained by applying a 0-extension to the vertices $\{1,2,\dots,d\}$, where the edge $\{1,n\}$ is required to be added to E_1'. Finally, the edge $\{0,n\}$ is added to E_0' to obtain the 1-extended graph $G' = (V',E')$.

We note that this labelling induces a labelling of the vertices of (G,c) by $\{0,1,\dots,n-1\}$, and also that $\{0,1\} \notin E'$, since this edge is removed from (G,c) by the application of the 1-extension. Let F denote the set of edges that are common to (G,c) and (G',c'), hence $E = F \cup \Big\{\{0,1\}\Big\}$ and $E' = F \cup \Big\{\{0,n\},\{1,n\},\dots,\{d,n\}\Big\}$. Note that $c'(f) = c(f)$ for all edges $f \in F$, since they are unchanged by the 1-extension.

Since (G,c,p) is isostatic, E is independent with rank $R(G,c,p) = d|V| + k - \binom{d+1}{2}$, and it follows that F is also independent. Let $\widehat{E} = F \cup \Big\{\{1,n\},\dots,\{d,n\}\Big\}$, which we note has the structure of applying a 0-extension to F. Let $\widehat{c}$ be the k-edge-colouring of $\widehat{E}$ induced by c', and recall that $\widehat{c}(\{1,n\}) := \ell = c'(\{1,n\})$. Since $\widehat{G} = (V',\widehat{E})$ is obtained by applying a k-edge-coloured 0-extension to F, $\widehat{G}$ is independent by Lemma 5.1.2.

In order to obtain a k-coordinated framework, we require a configuration of $V \cup \{n\} = V'$, $\widehat{p} \in \mathbb{R}^{d(n+1)}$. We may extend the existing regular configuration of V by defining $\widehat{p}(i) := p(i)$ for $0 \le i \le n-1$, and choosing $\widehat{p}(n)$ such that the coordinates of $\widehat{p}(n)$ are algebraically independent over $\mathbb{Q}$.

Since $\{0,1\} \notin F$ and (G,c,p) is isostatic, there exists a coordinated infinitesimal motion $(p',r') \in \mathbb{R}^{dn+k}$ such that $\Big[p(1)-p(0)\Big]^\top \Big[p'(1)-p'(0)\Big] + r'(\ell) \neq 0$, while $\Big[p(i)-p(j)\Big]^\top \Big[p'(i)-p'(j)\Big] + r'\Big(c(\{i,j\})\Big) = 0$ for all $\{i,j\} \in F$. This is a coordinated infinitesimal motion of (V,F) that is not a coordinated infinitesimal motion of the original framework $G=(V,E)$.

To extend (p',r') to an infinitesimal motion of the independent framework $(\widehat{G},\widehat{c},\widehat{p})$, $(p'',r'') \in \mathbb{R}^{d(n+1)+k}$ with $p''(i) := p'(i)$, $0 \le i \le n-1$ and $r''(j) := r'(j)$, $1 \le j \le k$, it remains to determine $p''(n)$. To be an infinitesimal motion of $(\widehat{G},\widehat{c},\widehat{p})$, $p''(n)$ must satisfy $\Big[\widehat{p}(i)-\widehat{p}(n)\Big]^\top \Big[p''(i)-p''(n)\Big] + r''\Big(\widehat{c}(\{i,n\})\Big) = 0$ for each edge $\{i,n\}$ with $1 \le i \le d$. These d constraints determine the d-dimensional vector $\widehat{p}(n)$.

In order to verify that the edge $\{0,n\}$ remains independent over $\widehat{E}$ when added as an uncoloured edge, we must check that $\Big[\widehat{p}(n)-\widehat{p}(0)\Big]^\top \Big[p''(n)-p''(0)\Big] \neq 0$.

For simplicity of notation, let $s'(i,n)$ denote $r''\Big(\widehat{c}(\{i,n\})\Big)$ for $1 \le i \le d$, and for any pair of vertices $x,y \in V$, let

$$\pi(x,y) := \Big[\widehat{p}(x)-\widehat{p}(y)\Big]^\top \Big[p''(x)-p''(y)\Big].$$

Our constraints on $p''(n)$ are therefore equivalent to the infinitesimal motion (p'',r'') of the framework $(\widehat{G},\widehat{c},\widehat{p})$ satisfying $\pi(i,n)+s'(i,n)=0$ for $1 \le i \le d$.

For each i, $1 \le i \le d$, we may expand $\Big[\widehat{p}(i)-\widehat{p}(n)\Big]^\top \Big[p''(i)-p''(n)\Big] + r''\Big(\widehat{c}(\{i,n\})\Big) = 0$ as follows:

$$\begin{aligned}
&\Big[[\widehat{p}(i)-\widehat{p}(0)]-[\widehat{p}(n)-\widehat{p}(0)]\Big]^\top \Big[[p''(i)-p''(0)]-[p''(n)-p''(0)]\Big] + s'(i,n) = 0,\\
&\pi(i,0) - \Big[\widehat{p}(i)-\widehat{p}(0)\Big]^\top \Big[p''(n)-p''(0)\Big]\\
&\qquad - \Big[\widehat{p}(n)-\widehat{p}(0)\Big]^\top \Big[p''(i)-p''(0)\Big] + \pi(n,0) + s'(i,n) = 0. \qquad (5.1)
\end{aligned}$$

Recall that $\widehat{p}(0) = \mathbf{0}$ and $\widehat{p}(i) = \mathbf{e}_i$ for $1 \leq i \leq d$, and so the second component of Equation 5.1 may be rewritten as ${\mathbf{e}_i}^\top \Big[p''(n) - p''(0)\Big]$ for each i, $1 \leq i \leq d$. Note also that for each i, $\Big[\widehat{p}(n)\Big]^\top \Big[p''(i) - p''(0)\Big] = \Big[p''(i) - p''(0)\Big]^\top \Big[\widehat{p}(n)\Big]$, since the components are vectors in $\mathbb{R}^d$. We may therefore rewrite these d equations as the following matrix equation, where Q represents the d by d square matrix with i^{th} column $p''(i) - p''(0)$:

$$\begin{bmatrix} \pi(1,0) \\ \pi(2,0) \\ \vdots \\ \pi(d,0) \end{bmatrix} - \Big[p''(n) - p''(0)\Big] - Q^\top \widehat{p}(n) + \begin{bmatrix} \pi(n,0) \\ \pi(n,0) \\ \vdots \\ \pi(n,0) \end{bmatrix} + \begin{bmatrix} s'(1,n) \\ s'(2,n) \\ \vdots \\ s'(d,n) \end{bmatrix} = 0.$$

We may rearrange and multiply through on the left hand side by $\widehat{p}(n)^\top = \Big[\widehat{p}(n) - \widehat{p}(0)\Big]^\top$ to obtain the following:

$$\begin{aligned} \widehat{p}(n)^\top \begin{bmatrix} \pi(n,0) \\ \vdots \\ \pi(n,0) \end{bmatrix} - \Big[\widehat{p}(n) - \widehat{p}(0)\Big]^\top \Big[p''(n) - p''(0)\Big] =& \widehat{p}(n)^\top Q^\top \widehat{p}(n) \\ &- \widehat{p}(n)^\top \begin{bmatrix} \pi(1,0) + s'(1,n) \\ \vdots \\ \pi(d,0) + s'(d,n) \end{bmatrix}. \end{aligned}$$

This may be simplified further:

$$\pi(n,0) \left(\widehat{p}(n)^\top \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} - 1 \right) = \widehat{p}(n)^\top Q^\top \widehat{p}(n) - \widehat{p}(n)^\top \begin{bmatrix} \pi(1,0) + s'(1,n) \\ \vdots \\ \pi(d,0) + s'(d,n) \end{bmatrix}. \tag{5.2}$$

Since the coordinates of $\widehat{p}(n)$ are algebraically independent over $\mathbb{Q}$, $\widehat{p}(n)^\top \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} - 1$ is a non-zero scalar. Equation 5.2 therefore gives us a polynomial expression in $\widehat{p}(n)$ for $\pi(n,0)$, and hence for $p''(n)$.

Recall that the edge $\{1,n\}$ was added to the ℓ^{th} colour class, and so $s'(1,n) = r'(\ell)$. We noted earlier that $\pi(1,0) + r'(\ell) \neq 0$, and so the first entry of the vector $\begin{bmatrix} \pi(1,0) + s'(1,n) & \pi(2,0) + s'(2,n) & \ldots & \pi(d,0) + s'(d,n) \end{bmatrix}^\top$ is non-zero. This expression for $\pi(n,0)$ is therefore a non-trivial polynomial in $\widehat{p}(n)$ that determines $p''(n) \in \mathbb{R}^d$.

Since the coordinates of $\widehat{p}$ are algebraically independent, $\widehat{p}(n)$ cannot be a solution to this polynomial, and hence $\pi(n,0) \neq 0$. We construct the graph (G', c') from $(\widehat{G}, \widehat{c})$ by adding the edge $\{0,n\}$ to the class of uncoloured edges, E'_0. Since the uncoloured edge $\{0,n\}$ is independent of $(\widehat{G}, \widehat{c}, \widehat{p})$, using the same configuration $\widehat{p}$ gives a framework $(G', c', \widehat{p})$ such that (p'', r'') is no longer a coordinated infinitesimal motion. Thus (G', c') is generically isostatic.

It is straightforward to confirm that the edge $\{0,n\}$ may be added to any colour class. For $w = r''(j)$, $1 \leq j \leq k$, we may add $\widehat{p}(n)^\top \begin{bmatrix} w \\ \vdots \\ w \end{bmatrix} - w$ to both sides of Equation 5.2 in order to obtain the following:

$$\pi(n,0) \left(\widehat{p}(n)^\top \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - 1 \right) + \widehat{p}(n)^\top \begin{bmatrix} w \\ w \\ \vdots \\ w \end{bmatrix} - w =$$

$$\widehat{p}(n)^\top Q^\top \widehat{p}(n) - \widehat{p}(n)^\top \begin{bmatrix} \pi(1,0)+s'(1,n) \\ \pi(2,0)+s'(2,n) \\ \vdots \\ \pi(d,0)+s'(d,n) \end{bmatrix} + \widehat{p}(n)^\top \begin{bmatrix} w \\ w \\ \vdots \\ w \end{bmatrix} - w,$$

$$\Big(\pi(n,0)+w\Big)\left(\widehat{p}(n)^\top \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - 1\right) = \widehat{p}(n)^\top Q^\top \widehat{p}(n) - w - \widehat{p}(n)^\top \begin{bmatrix} \pi(1,0)+s'(1,n)+w \\ \pi(2,0)+s'(2,n)+w \\ \vdots \\ \pi(d,0)+s'(d,n)+w \end{bmatrix}. \tag{5.3}$$

Equation 5.3 is a polynomial in $\widehat{p}(n)$, and hence for $w \neq 0$, $\pi(n,0)+w \neq 0$. (We note that $w = 0$ results in Equation 5.2.) The result follows as above, for all $c'(\{0,n\})$.

Suppose instead that $\{u_0, u_1\} \in E_0$. As previously, let the new vertex x be labelled by n, let u_0 and u_1 be labelled by 0 and 1 respectively, and let the other neighbours of n be labelled by $2, \ldots, d$. Recall that $p(0) = \mathbf{0}$, $p(i) = \mathbf{e}_i$ for $1 \leq i \leq d$, and that $\{0,1\}$ is not an edge in the extended graph (G', c'): we have $E = F \cup \Big\{\{0,1\}\Big\}$ and $E' = F \cup \Big\{\{0,n\}, \{1,n\}, \ldots, \{d,n\}\Big\}$. For all edges $f \in F$, $c'(f) := c(f)$, while $c'(\{0,n\}) := 0$ and $c'(\{1,n\}) := 0$. For $2 \leq i \leq n$, $c'(\{i,n\})$ is arbitrary.

As previously, let $\widehat{G} = (V', \widehat{E})$ be obtained from G by removing the edge $\{0,1\}$ and applying a d-dimensional 0-extension to the vertices $1, \ldots, d$. We define a configuration $\widehat{p} \in \mathbb{R}^{d(n+1)}$ for $(\widehat{G}, \widehat{c})$ by $\widehat{p}(i) := p(i)$ for $0 \leq i \leq n-1$, and choosing $\widehat{p}(n)$ such that the coordinates of $\widehat{p}$ are algebraically independent.

Let $(p', r') \in \mathbb{R}^{dn+k}$ be a coordinated infinitesimal motion of $((V,F), c, p)$ such that $\pi(0,1) \neq 0$, so (p', r') is not a coordinated infinitesimal motion of (G, c, p). We wish to extend this to a coordinated infinitesimal motion $(p'', r'') \in \mathbb{R}^{d(n+1)+k}$ of $(\widehat{G}, \widehat{c}, \widehat{p})$ such that $\pi(0, n) \neq 0$.

Recall that $c(\{0, 1\}) = 0$ and note that since $c'(\{1, n\}) = 0$, $s'(1, n) = 0$. We require that $p''(n)$ satisfies $\pi(i, n) + s'(i, n) = 0$ for $1 \leq i \leq d$. We may therefore obtain the following equivalent constraint to Equation 5.2:

$$\pi(n,0)\left(\widehat{p}(n)^\top \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - 1\right) = \widehat{p}(n)^\top Q^\top \widehat{p}(n) - \widehat{p}(n)^\top \begin{bmatrix} \pi(1,0) \\ \pi(2,0) + s'(2,n) \\ \vdots \\ \pi(d,0) + s'(d,n) \end{bmatrix}. \tag{5.4}$$

Since $\widehat{p}(n)$ is not a solution to this polynomial, which is clearly non-trivial through our initial assumption that $\pi(1, 0) \neq 0$, we therefore have $\pi(n, 0) \neq 0$, as required for the edge $\{0, n\}$ to be independent. The coordinated infinitesimal motion (p'', r'') is therefore not a coordinated infinitesimal motion of $(G', c', \widehat{p})$, and hence (G', c') is generically isostatic. □

The 1-extension applied to an edge $\{u_1, u_2\} \in E_\ell$ may be generalised to allow the edge added to E'_ℓ to be $\{x, v\}$ for any neighbour v of the new vertex x, however the geometric proof given here does not apply in those cases. In Chapter 7 we define an additional type of coloured 1-extension, referred to as the "chosen 1-extension" and defined formally in Definition 7.2.11. This is required for the 2-dimensional 2-coloured characterisation, along with the X-replacement operation (Definition 7.2.17), and the 2-dimensional 2-coloured 0-extension and 1-extension, illustrated in Figure 7.7 and Figure 7.8 respectively. The 2-dimensional 1-coloured 0-extensions and 1-extensions are shown in Figure 7.1, and Figure 7.2.

The 1-dimensional 0-extension and 1-extension will be discussed in more detail in Chapter 6, and the 1-coloured versions are illustrated in Figure 6.1 and Figure 6.2.

Chapter 6

Combinatorial Coordinated Rigidity in 1 Dimension

We begin by characterising 1-dimensional frameworks with coordinated edges, to provide some intuition for the analogous situation in 2 dimensions. Recall from Remark 3.1.20 that coordinated frameworks are denoted (G, c, p).

6.1 One colour class in 1 dimension

We first consider the case of 1 coordination class. From Theorem 4.1.11 a 1-coordinated graph (G, c) will be generically rigid in 1 dimension if and only if the graph G is generically rigid in M_1, and there is at least one coloured edge that is not a bridge in $M_1(G)$. Since any connected graph is generically rigid in 1 dimension (see Theorem 2.8.7 [Gra01, Theorem 2.16]) this is equivalent to (G, c) being a connected graph with at least one edge $e \in E_1$ within a cycle.

Definition 6.1.1. A graph $G = (V, E)$ is *tree-plus-one* if there is an edge $e \in E$ such that $G - e$ is a tree.

Any tree-plus-one graph contains a unique cycle, which we shall often denote by $C = (V(C), E(C))$. It is useful to note that for a cycle $C \subset G$, removing any edge $e \in E(C)$ will result in a tree $G - e$.

It is straightforward to apply Theorem 4.1.11 to show that 1-coordinated frameworks (G, c, p) will be isostatic for any regular configuration $p \in \mathbb{R}^n$ if and only if (G, c) is a tree-plus-one graph with at least one edge from E_1 within the cycle of (G, c), however we also have an inductive construction for such frameworks.

In Section 2.8.1 we noted that continuous rigidity and infinitesimal rigidity coincide in 1 dimension, and that 1-dimensional frameworks are minimally rigid if and only if the graph G is a tree. We also noted that a tree may be constructed from a single vertex using only 0-extensions. For 1-coloured coordinated rigidity, we shall allow both 0-extensions with the edge added to E_0, and 0-extensions with the new edge being added to the coordination class E_1. These are illustrated in Figure 6.1, and will preserve 1-coordinated infinitesimal rigidity by Lemma 5.1.2.

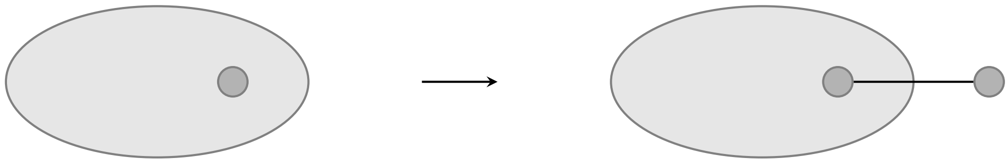
To construct isostatic frameworks with a class of coordinated edges, we also require a 1-dimensional 1-extension. This is applied by removing a single edge $\{u_1, u_2\}$, and adding a new vertex x with two new edges $\{x, u_1\}, \{x, u_2\}$. If $\{u_1, u_2\} \in E_0$, we require that both new edges be added to E_0, as would occur in the non-coordinated situation. If instead $\{u_1, u_2\} \in E_1$, we require that at least one new edge be added to E_1, though the other may be added to either E_0 or E_1. Both types of coloured 1-extension may be seen in Figure 6.2, and will preserve 1-coordinated infinitesimal rigidity by Lemma 5.2.2.

We now state the following main result for this section.

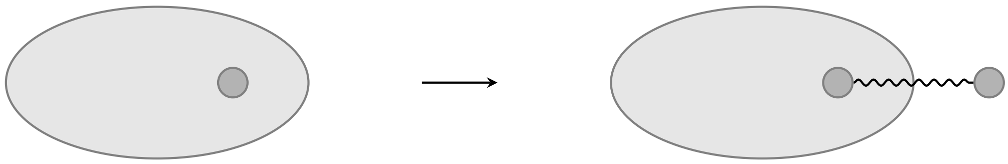
Theorem 6.1.2. Let (G, c, p) be a 1 edge-coloured framework with a regular configuration $p \in \mathbb{R}^n$. The following are equivalent:

1. (G, c, p) is an isostatic framework;

2. The graph G is tree-plus-one, and at least one edge in the cycle of (G, c) is in E_1;

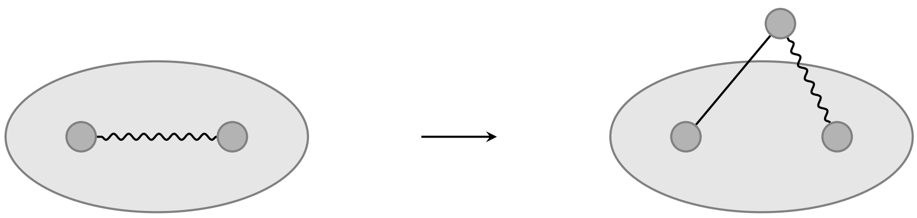


a A 0-extension with the edge added to E_0.

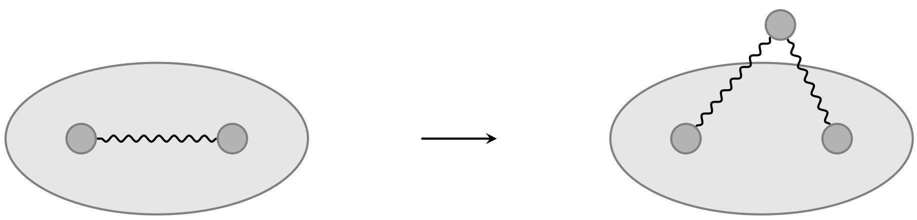


b A 0-extension with the edge added to E_1.

Figure 6.1 Coloured 0-extensions in 1 dimension.



a A 1-extension applied to an edge from E_1, creating a new vertex with one edge in E_0 and one edge in E_1.



b A 1-extension applied to an edge from E_1, creating a new vertex with two edges in E_1.

Figure 6.2 Coloured 1-extensions in 1 dimension.

3. The graph G is tree-plus-one, and for any $D \subseteq E_0$:

$$|D| \leq |V(D)| - 1. \tag{6.1}$$

4. The edge-coloured graph (G, c) can be constructed from a copy of K_3 with at least one edge in E_1, by a sequence of coloured 0-extensions and 1-extensions.

Proof. We begin by proving the equivalence of the second and third statements. Let (G, c) be a tree-plus-one graph and let $C = (V(C), E(C))$ denote the unique cycle of (G, c). Suppose that at least one edge in E_1 is in the cycle of G, and consider the subgraph of G made up of the edges in E_0, denoted by $G_0 = (V, E_0)$. This subgraph will have $|E_0| = |V| - |E_1| \leq |V| - 1$, and will have no cycle since $E_1 \cap E(C) \neq \emptyset$. Thus G_0 is at most a tree, and so will have $|D| \leq |V(D)| - 1$ for any subset $D \subseteq E_0$.

Suppose instead that (G, c) is a tree-plus-one graph with $|D| \leq |V(D)| - 1$ for all $D \subseteq E_0$. The unique cycle $C = (V(C), E(C))$ within G will have $|E(C)| = |V(C)|$, and hence there must be at least one edge in $E_1 \cap E(C)$.

We next prove that the third condition is necessary for the first condition to hold. It is clear that a disconnected graph cannot lead to an isostatic framework, so we may assume that G is connected. It is also a straightforward consequence of Theorem 3.1.33 that a 1 edge-coloured graph (G, c) with $|E| < |V|$ or $|E| > |V|$ cannot be isostatic in 1 dimension, and hence we may assume that (G, c) is a connected 1 edge-coloured graph with $|E| = |V|$.

A connected graph with $|E| = |V|$ is tree-plus-one, so it remains to prove the necessity of Equation (6.1). Suppose that (G, c) contains some $D_0 \subseteq E_0$ such that $|D_0| > |V(D_0)| - 1$. Any subgraph of a tree-plus-one graph, generated by $D \subseteq E$, has $|D| \leq |V(D)|$, so the subgraph generated by these uncoloured edges must have

$|D_0| = |V(D_0)|$. This leads to the conclusion that this subgraph contains all the edges of the unique cycle of (G, c), $E(C) \subseteq D_0$, and so the cycle of (G, c) is uncoloured. The subset of rows of $R(G, c, p)$ associated with the edges of $E(C)$ will contain a dependence, and so rank $R(G, c, p) < |E|$. The framework (G, c, p) therefore will not be independent, and so cannot be isostatic.

We now prove that any graph (G, c) satisfying the third condition may be constructed as described in the fourth condition. We shall prove this by applying induction on $|V| = n$.

The only tree-plus-one graph on 3 vertices is K_3, and all 1 edge-colourings that result in $|D| \leq |V(D)| - 1$ for $D \subseteq E_0$ will give $|E_1| \geq 1$, as required for the set of base graphs (K_3, c) in the fourth condition.

Let (G, c) be a tree-plus-one graph on $n \geq 4$ vertices, which satisfies $|D| \leq |V(D)| - 1$ for every $D \subseteq E_0$. Since $|E| = |V|$, the average degree is two, so (G, c) will either contain at least one vertex of degree strictly less than two, or every vertex will have degree exactly two.

Suppose first that (G, c) contains a vertex u of degree one. If u is adjacent to an edge in E_0, this edge and vertex may clearly be considered as the result of a standard 0-extension move. We note that if instead u is adjacent to an edge in E_1, $|E_1| \geq 2$, since (G, c) satisfies $|D| \leq |V(D)| - 1$ for all $D \subseteq E_0$ and the edge adjacent to u cannot be in any cycle of (G, c). We may therefore also consider the vertex u and its associated edge as the result of a 0-extension.

In both cases, the reverse of the move may be applied to remove the vertex and associated edge. This results in a graph (G', c) on $n - 1$ vertices, which is tree-plus-one and still satisfies Equation (6.1). The coloured 0-extensions are illustrated in Figure 6.1.

Suppose instead that every vertex of (G, c) has degree exactly two, and so the structure of G is a single cycle. As any set of uncoloured edges generates a subgraph that is at most a tree, there will be at least one edge in E_1 within the cycle.

First suppose that there is also at least one edge of the cycle in E_0. There will be at least one vertex u that is adjacent to one edge in E_0 and one edge in E_1. We shall denote this pair of edges by $\{u, x_0\} \in E_0$ and $\{u, x_1\} \in E_1$ respectively. The pair of edges and their common vertex u may be considered as the result of a coloured 1-extension, of the type illustrated in Figure 6.2a. We may apply the reverse of this move to produce a 1 edge-coloured graph on $n-1$ vertices that satisfies the third condition of the Theorem.

Suppose instead that $E_0 = \emptyset$, so (G, c) is a cycle on n vertices with every edge in E_1. Any vertex u and its pair of adjacent edges, $\{u, x_1\}, \{u, x_2\}$, may be viewed as the result of an all coloured 1-extension, as depicted in Figure 6.2b. The reverse of this move may be applied, to replace the vertex and its pair of edges with a single edge, $\{x_1, x_2\}$, and create a cycle of edges in E_1 on $n-1$ vertices. This will clearly still satisfy the third condition.

By the inductive hypothesis, any 1 edge-coloured graph (G, c) satisfying the third condition may be reduced to (K_3, c), where c is an edge-colouring that induces a partition with $|E_1| \geq 1$.

We conclude by proving that the fourth statement implies the first. It is straightforward to confirm that for (K_3, c, p) with $|E_1| \geq 1$ and any regular configuration $p \in \mathbb{R}^{1 \cdot 3}$, the coordinated rigidity matrix will have rank $R(K_3, c, p) = 3$. We may therefore apply Theorem 3.1.33 to see that (K_3, c, p) is isostatic in 1 dimension. From Lemma 5.1.2 and Lemma 5.2.2, applying a coloured 0-extension or coloured 1-extension to an isostatic framework will result in a larger isostatic framework, so any framework constructed in this way from a coloured copy of K_3 with $|E_1| \geq 1$ will be isostatic. $\square$

In Figure 6.3 we illustrate a pair of graphs on five vertices, and their respective reductions to coloured copies of K_3. Both graphs will give isostatic frameworks when combined with a configuration $p \in \mathbb{R}^{1|V|}$.

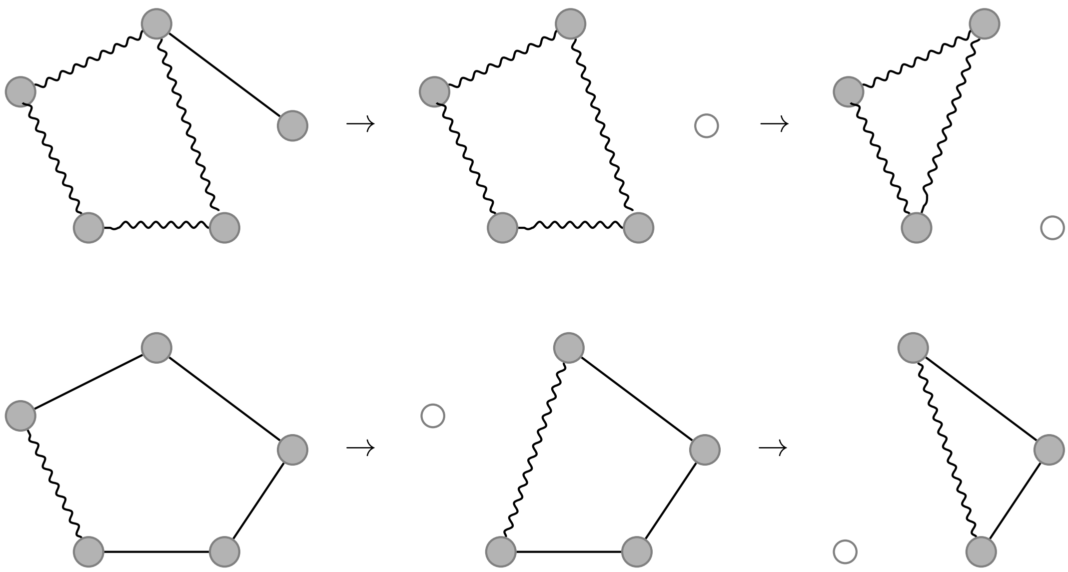


Figure 6.3 Two examples of a reduction of 1-coloured 1-dimensional graphs.

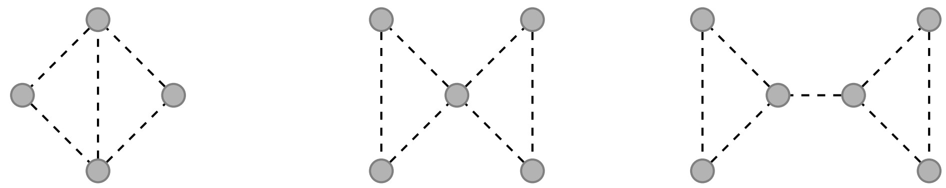
6.2 Two colour classes in 1 dimension

A natural next step is to characterise frameworks with two classes of coordinated edges. We consider edge-coloured graphs (G, c) where the edges are partitioned by c into $E = E_0 \cup E_1 \cup E_2$. We extend Definition 6.1.1 as follows.

Definition 6.2.1. A graph $G = (V, E)$ is *tree-plus-two* if there is a pair of edges $\{e, f\} \subset E$ such that $G - \{e, f\}$ is a tree.

A tree-plus-two graph may contain two edge-disjoint cycles, or may instead contain two intersecting cycles - the disjoint union of which will create a third cycle. We shall refer to the two shortest cycles as C_a and C_b.

We denote the base graphs for our inductive construction by A_4, A_5 and A_6, where $|V(A_i)| = i$ and each base graph contains two copies of C_3. A_4, illustrated in Figure 6.4a, consists of two 3-cycles which intersect on a single edge, and also contains a cycle of length 4. Figure 6.4b shows the base graph A_5, which has two copies of C_3 intersecting on a single vertex, while the base graph A_6 has two completely disjoint cycles, connected by a single edge. This base graph may be seen in Figure 6.4c.



a The base graph A_4. **b** The base graph A_5. **c** The base graph A_6.

Figure 6.4 The base graphs for 1-dimensional frameworks with 2 colour classes.

We allow the 1-dimensional 0-extension, as described in Section 6.1 and illustrated in Figure 6.1, to be applied with the new edge added to any of E_0, E_1 and E_2. We also allow the 1-dimensional 1-extensions on edges in E_1 seen in Figure 6.2, as well as permitting equivalent moves to be applied to an edge in E_2. Along with the standard 1-dimensional 1-extension, wherein an edge from E_0 is replaced by two edges added to E_0, we define an additional 1-extension to be applied to an edge $e = \{u_1, u_2\}$ with either $e \in E_1$ or $e \in E_2$. The edge e is removed and replaced by a new vertex x, and edges $\{x, u_1\}, \{x, u_2\}$ with one edge added to each of E_1 and E_2. An illustration of this move being applied to $e \in E_1$ is shown in Figure 6.5.

We state the following main result for 1-dimensional frameworks with two classes of coordinated edges, which we prove through a sequence of equivalences.

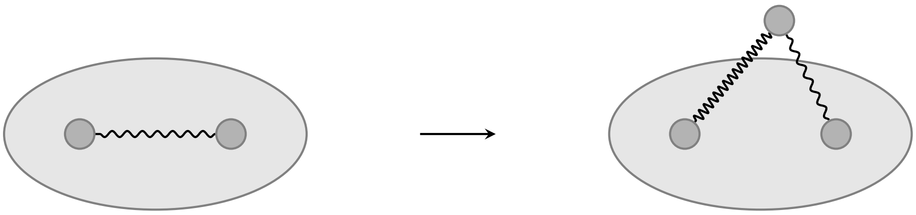


Figure 6.5 1-dimensional 1-extension, replacing an edge in E_1 with one edge in E_1 and one edge in E_2.

Theorem 6.2.2. Let (G, c, p) be a 2 edge-coloured framework with a regular configuration $p \in \mathbb{R}^n$. The following are equivalent:

1. (G, c, p) is an isostatic framework;

2. The graph G is tree-plus-two, there is at least one edge from each of E_1 and E_2 that lies within a cycle of (G, c), and the following counts are satisfied:

$$|D| \leq |V(D)| - 1 \qquad \forall D \subseteq E_0, \tag{6.2}$$

$$|D| \leq |V(D)| \qquad \forall D \subseteq E_0 \cup E_1 \text{ or } D \subseteq E_0 \cup E_2. \tag{6.3}$$

3. The edge-coloured graph (G, c) may be constructed from a coloured copy of one of the three base graphs A_4, A_5, A_6, with at least one coloured edge in each cycle, and at least one edge from each colour class in a cycle, by a sequence of coloured 0-extensions and 1-extensions.

We may also use the language of redundant rigidity to give an equivalent statement to the first two conditions.

Theorem 6.2.3. Let (G, c) be a 2-edge-coloured graph. The graph (G, c) is generically rigid in 1 dimension if and only if G is generically rigid in M_1, and there is a pair of edges $\{e_1, e_2\}$ with $e_1 \in E_1$, $e_2 \in E_2$, that is not a 2-bridge in $M_1(G)$.

This statement is the specific 2-coloured 1-dimensional case of Theorem 4.1.11. However it does not give the inductive construction of such graphs that is obtained from Theorem 6.2.2, which we now prove using a series of Lemmas.

Lemma 6.2.4 ($1 \Rightarrow 2$)**.** If (G, c, p) is an isostatic framework, then (G, c) is a tree-plus-two graph with at least one edge from each of E_1 and E_2 within a cycle, satisfying Equations (6.2) and (6.3).

Proof. It is clear that a disconnected graph cannot be isostatic, and from Theorem 3.1.33 that a graph with $|E| \neq |V| + 1$ will also not be isostatic. We may therefore assume that G is a connected graph with $|E| = |V| + 1$, and hence G has two more edges than a tree. These will create cycles, so G has at least two cycles that may be labelled C_a and C_b. We may choose two edges $e \in E(C_a) \setminus E(C_b)$ and $f \in E(C_b) \setminus E(C_a)$.

Let $G' = G - \{e, f\}$, so $|E'| = |V'| - 1$. The cycles C_a and C_b in G are still connected in G', and shall be denoted by $C'_a := C_a - \{e\}$ and $C'_b := C_b - \{f\}$. If G is not tree-plus-two, G' cannot be a tree, and hence must be disconnected. Let $e = \{u_1, u_2\}$ and $f = \{v_1, v_2\}$. The graph G' still contains a path between u_1 and u_2, in C'_a, and a path in C'_b between v_1 and v_2. Replacing the edges e, f into G' will create the cycles C_a and C_b, and since G' is disconnected, G will remain disconnected. Hence if G with $|E| = |V| + 1$ is not tree-plus-two, then G must be disconnected and so cannot be isostatic.

We may now assume that G is tree-plus-two. Suppose that one colour class, say E_1, does not contain any edges in the cycles of (G, c), so $E_1 \subset E \setminus \Big\{E(C_a) \cup E(C_b)\Big\}$. We define a 1-edge-colouring c' of the graph G, where $c'(e) := 0$ for $e \in E_0 \cup E_2$, and $c'(e) := 1$ for $e \in E_1$. As no edge in E'_1 is in a cycle of G, the 1-coloured graph (G, c') cannot be infinitesimally rigid by Theorem 6.1.2, and this non-trivial infinitesimal motion will remain in the 2-coloured graph (G, c). A similar argument applies when

$E_2 \subset E \setminus \Big\{E(C_a) \cup E(C_b)\Big\}$, and hence both colour classes must have at least one edge within a cycle of G.

Let $D \subseteq E$ be a collection of edges in the tree-plus-two graph G, and so $|D| \leq |V(D)| + 1$. Suppose first that $D \subseteq E_0$ such that $|D| \geq |V(D)|$. In this case, as in the 1-coordinated situation seen in Theorem 6.1.2, the coordinated rigidity matrix $R(G, c, p)$ will contain a submatrix with $|D|$ rows and at most $|V(D)|$ non-zero columns. This submatrix will have rank at most $|V(D)| - 1 < |D|$, and so there will be a dependence preventing G from being isostatic.

Suppose instead that $D \subseteq E_0 \cup E_1$ or $D \subseteq E_0 \cup E_2$ with $|D| = |V(D)| + 1$. The coordinated rigidity matrix $R(G, c, p)$ will contain a submatrix of $|D|$ rows with at most $|V(D)| + 1$ non-zero columns, and hence this submatrix will have rank at most $(|V(D)| + 1) - 1 = |D| - 1$. This submatrix will therefore identify a dependence within the graph G, and so G will not be isostatic. The second condition of Theorem 6.2.2 is therefore necessary for a framework (G, c, p) to be isostatic. $\square$

Lemma 6.2.5 $(2 \Rightarrow 3)$**.** The 2-edge-coloured graph (G, c) can be constructed from some (A_i, c), $i \in \{4, 5, 6\}$, by a sequence of coloured 0-extensions and 1-extensions, when (G, c) is a tree-plus-two graph with at least one edge from each of E_1 and E_2 within a cycle, satisfying Equations (6.2) and (6.3).

Proof. Let G be a tree-plus-two graph, with an edge-colouring c that induces a partition of E into $E_0 \cup E_1 \cup E_2$, such that (G, c) has at least one edge from E_1 and E_2 within a cycle and satisfies Equations (6.2) and (6.3). We note that since any tree-plus-two graph has $|V| \geq 4$, the average degree within the tree-plus-two graph (G, c) is strictly less than three:

$$\frac{1}{|V|} \sum_{v \in V} \deg(v) = \frac{1}{|V|}\Big(2|E|\Big) = \frac{1}{|V|}\Big(2|V| + 2\Big) = 2 + \frac{2}{|V|}. \tag{6.4}$$

The only tree-plus-two graph with $|V| = 4$ is the base graph A_4, so we shall apply induction on V and suppose that (G, c) has $|V| \geq 5$. The minimum degree is either 1 or 2, and we suppose first that (G, c) has a vertex of degree 1, so is clearly not a base graph. Since edges from both E_1 and E_2 are within cycles of (G, c), if a vertex of degree 1 is adjacent to an edge from E_1 then $|E_1| \geq 2$, and similarly if an edge from E_2 is adjacent to a vertex of degree 1 then $|E_2| \geq 2$. Any vertex of degree 1 may therefore be considered as the result of an appropriately coloured 0-extension, so we may remove this vertex and its associated edge.

We now suppose that there are no vertices of degree 1, and so the minimum degree within (G, c) is 2. We shall consider the cycle structure of (G, c). A tree-plus-two graph will contain at least two cycles, which may be vertex disjoint and connected by a path, share a single vertex, or have some path of edges in common. We consider each of these cases separately.

Suppose first that (G, c) contains exactly two cycles C_a and C_b, such that $V(C_a \cap C_b) = \emptyset$. Necessarily $|V| \geq 6$, however if $|V| = 6$ then (G, c) is a coloured copy of the base graph A_6. We assume that $|V| \geq 7$ to apply induction.

Let P denote the subgraph generated by $E \setminus \{E(C_a), E(C_b)\}$, which may be considered as the path between C_a and C_b. There is one vertex $x_a \in V(C_a) \cap V(P)$ and one vertex $x_b \in V(C_b) \cap V(P)$ with $\deg(x_a) = \deg(x_b) = 3$, and all other vertices have degree 2. This is shown in Figure 6.6. As $|V| \geq 7$, either $|V(P)| \geq 3$ or at least one cycle contains at least four edges.

All edges in $E(P)$ are outside the cycles of the edge-coloured graph (G, c). If there exists some $v \in V(P)$ with $\deg(v) = 2$ and associated edges $\{v, u_1\}, \{v, u_2\}$, we may apply the reverse move to a 1-extension to create a reduced graph (G', c'), and replace v and its pair of edges with the single edge $\{u_1, u_2\}$. The edge $\{u_1, u_2\}$ will be in the reduced path P' and hence will remain outside the (unchanged) cycles of

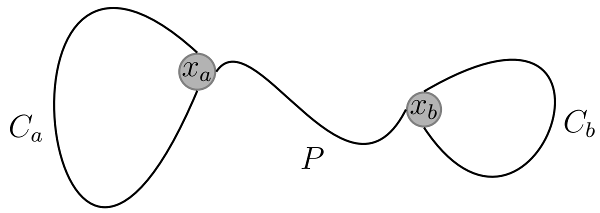


Figure 6.6 The graph (G, c) has two cycles C_a and C_b, linked by a path P.

the reduced graph G'. We may define the edge-colouring c' of G' by $c'(e) := c(e)$ for all edges in $E' \setminus \Big\{\{u_1, u_2\}\Big\}$, and define $c'\Big(\{u_1, u_2\}\Big)$ as appropriate, based on the colours of the removed edges $\{v, u_1\}$ and $\{v, u_2\}$. Since (G, c) satisfied all conditions on the edge-colouring, and the colouring of the edges in the cycles of (G', c') remains unchanged, the reduced graph (G', c') will clearly satisfy the same coloured sparsity conditions.

If instead P is made up of only the edge $\{x_a, x_b\}$, there is at least one cycle made up of at least four edges since $|V| \geq 7$. Without loss of generality, let $|V(C_a)| \geq 4$. If there is an edge in $E_0 \cap E(C_a)$, this edge will be adjacent to at least one vertex of degree 2. Let v be the vertex of degree 2, with $\{v, u_1\} \in E_0$ and $\{v, u_2\} \in E_i$ for some $0 \leq i \leq 2$. We may apply the reverse of a 1-extension move at v, and replace the vertex v and its pair of edges with a single edge $\{u_1, u_2\}$ added to E_i' for the same $0 \leq i \leq 2$. This will either be the reverse of a standard 1-extension, or replace an uncoloured and a coloured edge with a single edge of the same colour. The reduced graph (G', c') will have C_b unchanged, and the same number and type of coloured edges within C'_a as were in C_a.

If $E_0 \cap E(C_a) = \emptyset$, C_a is either made up of only edges from one colour class, or from a mixture of edges from E_1 and E_2. By swapping the labels of the colour classes if necessary, we first suppose that there is a vertex v of degree 2 in $V(C_a)$ with $\{v, u_1\}, \{v, u_2\} \in E_1$. We may replace v and its pair of edges with a single edge $\{u_1, u_2\} \in E'_1$. The reduced graph G' will still have at least one edge from E'_1 within C'_a, and any edges within C_a from E_2 will remain in E'_2 within C'_a. The other cycle C_b will also remain unchanged, and so the reduced graph (G', c') will satisfy the same conditions.

If there is no degree 2 vertex within $V(C_a)$ adjacent to two edges from the same colour class, every vertex $v \in V(C_a)$ with $\deg(v) = 2$ will be adjacent to a pair of edges $\{v, u_1\} \in E_1, \{v, u_2\} \in E_2$. As $|E(C_a)| \geq 4$, we shall have $|E_1 \cap E(C_a)|, |E_2 \cap E(C_a)| \geq 2$. Without loss of generality, suppose that $|E_2 \cap E(C_a)| \geq |E_1 \cap E(C_a)|$. We may remove a vertex v and its pair of adjacent edges, and replace them with a single edge $\{u_1, u_2\}$ which we add to E'_1, in the reverse of the 1-extension illustrated in Figure 6.5. (G', c') has $|E'_1 \cap E(C'_a)| \geq |E'_2 \cap E(C'_a)| \geq 1$, and so edges from both colour classes remain within C'_a. As before, the cycle C_b remains unchanged, and the reduced graph (G', c') has edges from both E_1 and E_2 within cycles, and satisfies Equations (6.2) and (6.3).

Suppose next that (G, c) contains exactly two edge-disjoint cycles, with $V(C_a \cap C_b) = \{x\}$ for some vertex $x \in V$. As the minimum degree within (G, c) is 2, every vertex $v \in V \setminus \{x\}$ will be in exactly one of the cycles, with $\deg(v) = 2$. We assumed that $|V| \geq 5$, however if $|V| = 5$ then $G = A_5$, so we may instead assume $|V| \geq 6$ to apply induction on $|V|$. At least one cycle will have $|E(C)| \geq 4$, and so we may reduce this cycle by applying an identical reduction to that described above for reducing completely disjoint cycles.

Suppose finally that the two shortest cycles within (G, c), labelled C_a and C_b, intersect on some non-empty collection of edges. We denote by P the subgraph $C_a \cap C_b$, and since there are no vertices of degree 1, we have two vertices of degree 3, labelled x and y, and three edge-disjoint paths between them. We denote the paths $C_a \setminus P$ and $C_b \setminus P$ by P_a and P_b respectively, which is illustrated in Figure 6.7.

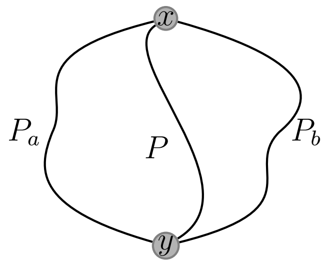


Figure 6.7 Three paths between two vertices. $C_a = P_a \cup P$; $C_b = P_b \cup P$.

The base graph with this cycle structure is A_4, so we may continue assuming that $|V| \geq 5$ to apply induction on $|V|$. Since C_a and C_b are the shortest cycles within (G, c), P will be either the shortest or equal shortest path between x and y.

Since $|E| = |V| + 1 \geq 6$, either all three paths have length at least two, or there is a path of length at least three if there is a path of length one.

There can be at most one path $Q \in \{P, P_a, P_b\}$ made up only of uncoloured edges, as the union of two uncoloured paths would create an uncoloured cycle. Suppose first that there is an uncoloured path of length at least two. Any vertex $v \in V(Q) \setminus \{x, y\}$ will be adjacent to two uncoloured edges, $\{v, u_1\}, \{v, u_2\}$, so these edges and the vertex v may be removed, to be replaced by a single uncoloured edge $\{u_1, u_2\}$ in the reverse of a standard 1-extension. The coloured paths will remain unchanged, and the uncoloured path will have reduced in length by 1, so the reduced graph will still satisfy the conditions of the statement.

If there is no uncoloured path of at least two edges, there is either an uncoloured path of length one, or every path contains at least one coloured edge. If there is a path $Q_1 \in \{P, P_a, P_b\}$ with $|E(Q_1)| = 1$, we shall consider the longest path $Q \in \{P, P_a, P_b\}$, with $|E(Q)| \geq 3$ as noted above.

If there is an uncoloured edge in $E(Q)$, it will be adjacent to at least one vertex of degree 2, v. We label the edges adjacent to v as $\{v, u_1\} \in E_0$ and $\{v, u_2\} \in E_\ell$ for some $\ell \in \{0, 1, 2\}$. We may apply the reverse of a 1-extension at v, and replace this pair of edges by a single edge $\{u_1, u_2\} \in E_\ell$. The number of coloured edges within Q remains unchanged, as do the other two paths between x and y, so the reduced graph will also satisfy the conditions required.

If $E(Q)$ is made up only of edges from $E_1 \cup E_2$, we first check whether there is a vertex $v \in V(Q) \setminus \{x, y\}$ adjacent to a pair of edges $\{v, u_1\}, \{v, u_2\} \in E_\ell$ for $\ell = 1$ or $\ell = 2$. Without loss of generality, suppose that such a vertex exists for $\ell = 1$, so we may apply the reverse of a 1-extension to replace v and its pair of edges with a single edge $\{u_1, u_2\}$ added to E_1. The reduced path Q' will have $|E(Q')| \geq 2$, $|E_2' \cap E(Q')| = |E_2 \cap E(Q)|$ and $|E_1' \cap E(Q')| = |E_1 \cap E(Q)| - 1 \geq 1$. Since the other pair of paths between x and y will remain unchanged, the reduced graph (G', c') will still satisfy the conditions of the statement.

If there is no such vertex adjacent to two edges from the same colour class, every vertex $v \in V(Q) \setminus \{x, y\}$ will be adjacent to a pair of edges $\{v, u_1\} \in E_1, \{v, u_2\} \in E_2$, and so the edges of Q will alternate between the two colour classes. Without loss of generality, suppose that $|E_2 \cap E(Q)| \geq |E_1 \cap E(Q)| \geq 1$. We may apply the reverse of the 1-extension illustrated in Figure 6.5, and replace v and its pair of adjacent edges by a single edge $\{u_1, u_2\}$ added to E_1. This results in $|E_1' \cap E(Q')| = |E_1 \cap E(Q)|$ and $|E_2' \cap E(Q')| = |E_2 \cap E(Q)| - 1$, so overall $|E_1' \cap E(Q')| \geq |E_2' \cap E(Q')| \geq 1$ and edges of both colours remain in the reduced path Q'. The reduced graph (G', c') will still satisfy all the conditions required.

We note that in the case where $|E_1 \cap E(Q)| = 1$, since $|E(Q)| \geq 3$ we have $|E_2 \cap E(Q)| \geq 2$ and hence $|E_2' \cap E(Q')| \geq 1$.

The remaining case to check is when all three paths contain at least one coloured edge. If there is a path $Q \in \{P, P_a, P_b\}$ with $|E(Q)| \geq 3$, we may apply the argument above to reduce the path Q, and hence reduce the graph (G, c) to a smaller graph satisfying the conditions stated. If no such path exists, we have $|E(P)| = |E(P_a)| = |E(P_b)| = 2$, with at least one coloured edge in each path.

If one path is made up of a coloured edge and an uncoloured edge, we may apply the reverse of the 1-extension illustrated in Figure 6.2a. This allows us to replace the vertex and its pair of edges with a single edge $\{x, y\}$ of the same colour as the single coloured edge removed. If instead there are no uncoloured edges within (G, c), but there is a vertex $v \in V \setminus \{x, y\}$ with $\{v, x\}, \{v, y\} \in E_\ell$ for $\ell = 1$ or $\ell = 2$, we may remove this vertex and pair of edges, to replace them with a single edge $\{x, y\}$ added to E_ℓ. This is the reverse of the 1-extension illustrated in Figure 6.2b, in the case $\ell = 1$. In either case, the reduced graph will clearly still satisfy Equations (6.2) and (6.3).

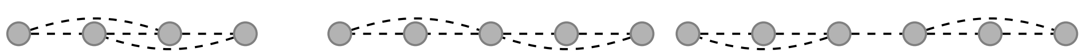
If no vertex of either of these types exists, we have $V = \{x, y, v_1, v_2, v_3\}$ where each vertex v_i is adjacent to one edge in E_1 and one edge in E_2. Without loss of generality, consider the edges $\{v_1, x\} \in E_1$ and $\{v_1, y\} \in E_2$. We may remove the vertex v_1 along with its pair of edges, and create a new edge $\{x, y\}$ to be added to either E_1 or E_2, say E_1. The reduced graph (G', c') will have $|E_1'| = |E_1| = 3$ and $|E_2'| = |E_2| - 1 = 2$, so it is straightforward to confirm that (G', c') will still satisfy Equations (6.2) and (6.3), as well as having at least one edge from both E_1' and E_2' within a cycle of (G', c'), since every edge in (G', c') is within a cycle.

We have shown that any graph (G, c) with at least one edge from each colour class within a cycle of G, that satisfies Equations (6.2) and (6.3), may be reduced to a smaller graph (G', c') that still satisfies Equations (6.2) and (6.3) and retains at least one edge from each colour class within a cycle of (G', c'), by applying the reverse of a coloured 0-extension or 1-extension. Hence by the inductive hypothesis, any coloured graph satisfying these conditions may be constructed from an appropriately coloured copy of a base graph A_4, A_5, A_6 by a sequence of coloured 0-extensions and 1-extensions. □

Lemma 6.2.6 (3 ⇒ 1)**.** Let (G, c) be a 2 edge-coloured graph (G, c) and let $p \in \mathbb{R}^n$ be a regular configuration of the vertex set of G. If (G, c) is constructed from some base graph (A_i, c_i), $i \in \{4, 5, 6\}$, by a sequence of coloured 0-extensions and 1-extensions, then (G, c, p) is an isostatic framework.

Proof. For each base graph $A_i \in \{A_4, A_5, A_6\}$ let c_i be a colouring that induces a partition of E into $E_0 \cup E_1 \cup E_2$, such that $|E_1| \geq 1$, $|E_2| \geq 1$, at least one edge of each colour is in a cycle, and at least one coloured edge is in each cycle.

Let $p_4 = (1, 2, 3, 4)$, $p_5 = (1, 2, 3, 4, 5)$, $p_6 = (1, 2, 3, 4, 5, 6)$, and consider the coordinated rigidity matrices: we have $\operatorname{rank} R(A_4, c_4, p_4) = 5 = |E(A_4)|$, $\operatorname{rank} R(A_5, c_5, p_5) = 6$ and $\operatorname{rank} R(A_6, c_6, p_6) = 7$. Since this is the maximum possible rank for each coordinated rigidity matrix, each p_i is a regular configuration of (A_i, c_i), and all regular configurations have full rank. We may therefore apply Theorem 3.1.33 to confirm that the regular framework (A_i, c_i, p_i) is isostatic for $i \in \{4, 5, 6\}$.



a The base graph (A_4, p_4). **b** The base graph (A_5, p_5). **c** The base graph (A_6, p_6).

From Lemma 5.1.2 and Lemma 5.2.2, any coloured 0-extension or coloured 1-extension will preserve isostaticity of the framework it is applied to, and hence any framework constructed in this way from an appropriately coloured base graph will be isostatic. □

Example 6.2.7. Figure 6.9 shows a 2-coloured graph that results in an isostatic 2-coordinated framework when each vertex is given a placement $p(i) \in \mathbb{R}^1$, $i \in V$. The steps to reduce this graph to a 2-coloured copy of the base graph A_4 are also shown.

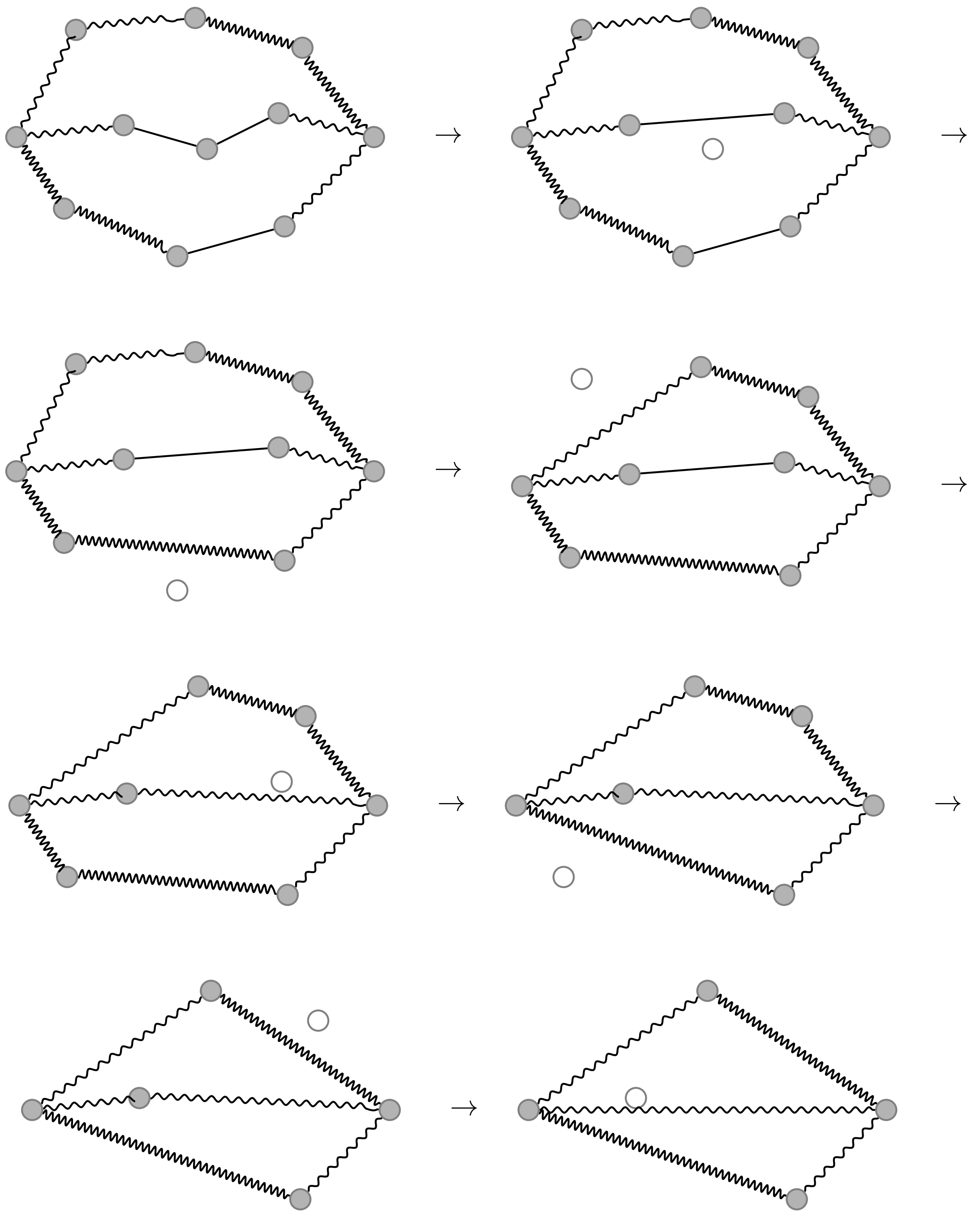


Figure 6.9 Example 6.2.7. A reduction of a 2-coloured graph to the base graph A_4.

6.3 Higher k in 1 dimension

We shall now consider k-coloured graphs with $k \geq 3$.

Definition 6.3.1. The *ℓ-chromatic subgraphs* of a k-coloured graph (G, c) are the subgraphs induced by $E_0 \cup E_{k_1} \cup \cdots \cup E_{k_\ell}$ for all sequences $(k_1, \ldots, k_\ell)$ with $1 \leq k_1 < \cdots < k_\ell \leq k$.

Remark 6.3.2. A k-coloured graph (G, c) has k 1-chromatic subgraphs, induced by $E_0 \cup E_1, E_0 \cup E_2, \ldots, E_0 \cup E_k$.

A 3-coloured graph (G, c) has three 2-chromatic subgraphs, induced by $E_0 \cup E_1 \cup E_2$, $E_0 \cup E_1 \cup E_3$ and $E_0 \cup E_2 \cup E_3$.

It is clearly necessary that, in order for a k-coordinated regular framework (G, c, p) to be isostatic in 1 dimension, the ℓ-chromatic subgraphs of a k-coloured graph (G, c) must be independent. This implies that each ℓ-chromatic subgraph must at most be a tree-plus-ℓ graph, since this is a necessary condition for an ℓ-coloured graph to be independent in 1 dimension.

From Theorem 4.1.11, a k-coordinated graph (G, c) is generically rigid in 1 dimension if and only if the graph G is generically rigid in 1 dimension, and (G, c) contains a k-redundant rainbow subset of edges. We note that redundant edges in a 1-dimensional framework are exactly those edges that lie within at least one cycle.

6.3.1 Three colour classes

We apply Theorem 4.1.11 and Theorem 6.2.3 to prove the following result.

Theorem 6.3.3. Let (G, c, p) be a 3-edge-coloured framework with a regular configuration $p \in \mathbb{R}^n$. (G, c, p) is isostatic if and only if the graph G is tree-plus-three, at least one edge from E_1, E_2 and E_3 lies within a cycle of G, each 1-chromatic subgraph of (G,c) is independent as a 1-coloured graph, and each 2-chromatic subgraph of (G,c) is independent as a 2-coloured graph.

Proof. From Theorem 4.1.11, the 3-edge-coloured graph (G,c) is generically rigid in 1 dimension if and only if the graph G is generically rigid in 1 dimension and contains three edges $\{e_1,e_2,e_3\}$ with $e_i \in E_i$ for $1 \leq i \leq 3$ such that $G \setminus \{e_1,e_2,e_3\}$ is generically rigid as an uncoloured graph in 1 dimension. This implies that for (G,c) to be generically rigid in 1 dimension, G has $|E| \geq |V| - 1 + 3$.

We note that in order for an ℓ-chromatic subgraph to be independent as an ℓ-coloured graph, any subgraph $(V(D),D)$ satisfies $|D| \leq |V(D)| + k(D) - 1$, where $k(D)$ is the number of non-empty colour classes within D. This may be restated as follows for the 1-chromatic and 2-chromatic subgraphs of (G,c):

$$|D| \leq |V(D)| - 1 \quad \forall D \subseteq E_0, \tag{6.5}$$

$$|D| \leq |V(D)| \quad \forall D \subseteq E_0 \cup E_1, D \subseteq E_0 \cup E_2, D \subseteq E_0 \cup E_3, \tag{6.6}$$

$$|D| \leq |V(D)| + 1 \quad \forall D \subseteq E_0 \cup E_1 \cup E_2, D \subseteq E_0 \cup E_1 \cup E_3, D \subseteq E_0 \cup E_2 \cup E_3. \tag{6.7}$$

Suppose that (G,c,p) is isostatic. It is clear from Theorem 3.1.33 that a 3-coloured graph with $|E| \neq |V| + 2$ cannot be isostatic in 1 dimension, and hence (G,c) is a tree-plus-three graph, with the redundant edges $e_1 \in E_1$, $e_2 \in E_2$, $e_3 \in E_3$ within cycles of (G,c).

If (G,c) contains a subset of edges $D \subseteq E_0$ with $|D| > |V(D)| - 1$, this is clearly equivalent to an uncoloured subgraph of (G,c) that is not independent, since the coordinated rigidity matrix $R(G,c,p)$ will contain a submatrix of rank at most $|V(D)| - 1 < |D|$. The graph (G,c) will therefore contain the same dependence, and cannot be isostatic.

If instead (G, c) contains a subgraph generated by a 1-coloured edge set, $D \subseteq E_0 \cup E_1$, with $|D| > |V(D)|$, the submatrix of $R(G, c, p)$ corresponding to these edges will have rank at most $(|V(D)| + 1) - 1 = |V(D)|$. This subgraph will therefore contain a dependence, and hence the graph (G, c) is also not independent. An analogous argument applies for all other 1-coloured subgraphs of (G, c).

Similarly, if (G, c) contains a 2-coloured subgraph, generated by $D \subseteq E_0 \cup E_1 \cup E_2$, with $|D| > |V(D)| + 1$, the corresponding submatrix of $R(G, c, p)$ will have rank at most $(|V(D)| + 2) - 1 = |V(D)| + 1$, and hence this subgraph is not independent. This also applies for subgraphs of the other 2-chromatic subgraphs of (G, c) generated by $D \subseteq E_0 \cup E_1 \cup E_3$ and $D \subseteq E_0 \cup E_2 \cup E_3$.

Suppose finally that (G, c) contains a 2-chromatic subgraph that is not independent as a 2-coloured graph, and without loss of generality let this be the 2-chromatic subgraph generated by $E_0 \cup E_1 \cup E_2$.

Since this subgraph is not independent as a 2-coloured graph, and hence is not isostatic, we consider the conditions of Theorem 6.2.2. This 2-chromatic subgraph is at most tree-plus-two, and all subgraphs satisfy Equations (6.5) and (6.6), which are equivalent to Equations (6.2) and (6.3) respectively. Therefore the 2-chromatic subgraph generated by $E_0 \cup E_1 \cup E_2$ cannot contain edges from both E_1 and E_2 within a cycle.

If this 2-chromatic subgraph contained no cycles or exactly one cycle, it would be independent as a 2-coloured graph. Therefore there are at least two cycles within the 2-chromatic subgraph, and one colour class contains only edges outside the cycles of the 2-chromatic subgraph. If no edge from E_2 is within a cycle of the 2-chromatic subgraph generated by $E_0 \cup E_1 \cup E_2$, and this 2-chromatic subgraph contains at least two cycles, then the 1-chromatic subgraph generated by $E_0 \cup E_1$ also contains at least two cycles. This implies that the 1-chromatic subgraph generated by $E_0 \cup E_1$ is also not independent as a 1-coloured graph. There will therefore be a dependence within the corresponding submatrix of $R(G, c, p)$, and hence (G, c, p) cannot be independent.

Hence the conditions are necessary for the regular framework (G, c, p) to be isostatic.

Suppose instead that the graph (G, c) is a tree-plus-three graph with at least one edge from each colour class E_1, E_2 and E_3 within a cycle of G, where all three 1-chromatic subgraphs are independent as 1-coloured graphs, and all three 2-chromatic subgraphs are independent as 2-coloured graphs. We shall show that the graph (G, c) contains a 3-redundant rainbow set of edges $\{e_1, e_2, e_3\}$, and hence by Theorem 4.1.11 (G, c) is generically rigid in 1 dimension. It then follows from Theorem 3.1.33 that since $|E| = |V| + 3 - 1$, (G, c, p) is isostatic in 1 dimension.

Since the graph G is tree-plus-three, G contains at least three cycles, which may be edge-disjoint or intersect non-trivially with each other, and for any edge within a cycle, $e \in E(C)$, the graph $G \setminus \{e\}$ contains at least two cycles C_1 and C_2. There is at least one edge from each colour class within a cycle of G, and at most one cycle within each of $E_0 \cup E_1$, $E_0 \cup E_2$ and $E_0 \cup E_3$.

A) There is a 1-coloured cycle

Suppose first that there is at least one cycle containing uncoloured edges together with edges from exactly one colour class. We label this cycle C_3, and without loss of generality may assume $E(C_3) \subseteq E_0 \cup E_3$. Let $e_3 \in E_3 \cap E(C_3)$. The graph $(G \setminus \{e_3\}, c)$ will contain at least two cycles, which we may label C_1 and C_2, with edges from both E_1 and E_2 within $E(C_1 \cup C_2)$.

We may define an edge-colouring $c' : E \setminus \{e_3\} \to \{0, 1, 2\}$ for the graph $G \setminus \{e_3\} := G'$ with $c'(e) := c(e)$ for $e \in E_0 \cup E_1 \cup E_2$, and $c'(e) := 0$ for $e \in E_3$. The 2-coloured graph (G', c') is a tree-plus-two graph, with at least one edge from each E_1' and E_2' within a cycle of G'. Since C_3 was a cycle in (G, c) made up of edges from $E_0 \cup E_3$, every cycle remaining in G' contains at least one edge from $E_1 = E_1'$ or $E_2 = E_2'$, and hence $|D| \leq |V(D)| - 1$ for all $D \subseteq E_0'$.

The 2-edge-coloured graph (G', c') therefore satisfies the conditions of Theorem 6.2.2 to be isostatic. We may then apply Theorem 6.2.3 to find a 2-redundant rainbow pair of edges $e_1 \in E_1'$, $e_2 \in E_2'$ such that $G' \setminus \{e_1, e_2\} = G \setminus \{e_1, e_2, e_3\}$ is a tree. This gives a 3-redundant set of edges $e_1 \in E_1$, $e_2 \in E_2$, $e_3 \in E_3$ for the 3-edge-coloured graph (G, c), which is therefore generically rigid by Theorem 4.1.11, and hence (G, c, p) is isostatic.

B) All cycles contain edges from at least two colour classes

We now suppose that all cycles within (G, c) contain edges from at least two colour classes.

Claim 6.3.4. Let C be a cycle of (G, c) with $f_1 \in E_1 \cap E(C)$ and $f_2 \in E_2 \cap E(C)$. Then at least one of the following holds:

- $G \setminus \{f_1\}$ has at least one edge from E_2 within a cycle,
- $G \setminus \{f_2\}$ has at least one edge from E_1 within a cycle.

Proof of Claim 6.3.4. Suppose that the claim is false. Both $G \setminus \{f_1\}$ and $G \setminus \{f_2\}$ contain at least two cycles, where the cycles of $G \setminus \{f_1\}$ are contained within $E_0 \cup E_1 \cup E_3$, and the cycles of $G \setminus \{f_2\}$ are contained within $E_0 \cup E_2 \cup E_3$. This implies that f_1 lies outside all cycles of $G \setminus \{f_2\}$, and f_2 lies outside of all cycles of $G \setminus \{f_1\}$, and hence f_1 and f_2 are contained in exactly the same cycles of (G, c).

Since f_1 and f_2 lie in precisely the same cycles of (G, c), the cycles of $G \setminus \{f_1\}$ and $G \setminus \{f_2\}$ are identical, and hence these cycles are contained within $E_0 \cup E_3$. This contradicts the assumption that every cycle contains edges of at least two colours. Therefore either $G \setminus \{f_1\}$ contains at least one edge from E_2 within a cycle, or $G \setminus \{f_2\}$ contains at least one edge from E_1 within a cycle. $\square$

We note that Claim 6.3.4 holds whether the cycle C contains edges from precisely two colour classes, or edges of all three colours.

Let C_3 be a cycle of (G, c) containing edges from E_2 and E_3, and let $e_3 \in E_3 \cap E(C_3)$. By Claim 6.3.4, if $G \setminus \{e_3\}$ does not contain any edges from E_2 within a cycle, then for some $e_2 \in E_2 \cap E(C_3)$, $G \setminus \{e_2\}$ contains at least one edge from E_3 within a cycle, e_3'. In such a case, we may apply the following argument with e_3' in place of e_3.

Let G' denote $G \setminus \{e_3\}$. Let C_1 and C_2 be two cycles within G', and hence all cycles of G' are contained within $C_1 \cup C_2$. Both C_1 and C_2 are cycles within G, and hence contain edges from at least two colour classes.

B1) C_3 contains no edges from E_1

Since every colour class has an edge within a cycle of (G, c), if there is no edge from E_1 within the cycle C_3 (where $e_3 \in E(C_3)$), then there is at least one edge from E_1 in $E(C_1 \cup C_2)$.

We define an edge-colouring $c' : E \setminus \{e_3\} \to \{0, 1, 2\}$ for the graph G' with $c'(e) := c(e)$ for $e \in E_0 \cup E_1 \cup E_2$, and $c'(e) := 0$ for $e \in E_3$. By assumption, there is at least one edge from E_2' within a cycle of G'. Since every cycle of G contains edges from at least two colour classes, every cycle of G' contains edges from at least one colour class, and hence $|D| \leq |V(D)| - 1$ for all $D \subseteq E_0'$. Since there is at least one edge in $E_1 \cap E(C_1 \cup C_2)$, there is at least one edge from E_1' in a cycle of G', and so the 2-coloured graph (G', c') is a tree-plus-two graph satisfying the conditions of Theorem 6.2.2. Applying Theorem 6.2.3 gives a 2-redundant pair of edges in G', $e_1 \in E_1'$, $e_2 \in E_2'$, such that e_1, e_2, e_3 are a 3-redundant rainbow set of edges for G. Hence (G, c) is generically rigid by Theorem 4.1.11, and (G, c, p) is isostatic.

B2) C_3 contains edges of all three colours

Suppose instead that $E_1 \cap E(C_3), E_2 \cap E(C_3), E_3 \cap E(C_3) \neq \emptyset$, and recall that $G \setminus \{e_3\}$

contains at least one edge from E_2 in a cycle. By Claim 6.3.4, since C_3 contains $e_1 \in E_1$ and $e_3 \in E_3$, then either $G \setminus \{e_1\}$ contains at least one edge from E_3 in a cycle, or $G \setminus \{e_3\}$ contains at least one edge from E_1 in a cycle. In the second case, we have edges from both E_1 and E_2 within cycles of $G \setminus \{e_3\}$, and we may therefore define the colouring $c' : E \setminus \{e_3\} \to \{0, 1, 2\}$ on $G \setminus \{e_3\}$ as discussed in Case B1, and apply Theorem 6.2.3 to find a 2-redundant rainbow pair within the 2-edge-coloured graph (G', c').

Suppose instead that all edges from E_1 lie outside the cycles of $G \setminus \{e_3\}$, and hence $G \setminus \{e_1\}$ contains at least one edge from E_3 within a cycle. Since C_3 contains edges from E_1 and E_2, either $G \setminus \{e_1\}$ has an edge from E_2 within a cycle, or $G \setminus \{e_2\}$ has an edge from E_1 within a cycle. In the first case, we have that $G \setminus \{e_1\}$ has edges from both E_2 and E_3 within a cycle, and as above we may define a colouring $c'' : E \setminus \{e_1\} \to \{0, 2, 3\}$ with $c''(e) := c(e)$ for $e \in E_0 \cup E_2 \cup E_3$, and $c''(e) := 0$ for $e \in E_1$. This gives a 2-edge-coloured graph $(G \setminus \{e_1\}, c'')$ that satisfies the conditions of Theorem 6.2.3, and hence contains a 2-redundant rainbow pair. This pair combines with e_1 to give a 3-redundant rainbow subset of edges for (G, c).

It remains to consider the case in which there are no edges from E_2 within a cycle of $G \setminus \{e_1\}$ and there are no edges from E_1 within a cycle of $G \setminus \{e_3\}$, but $G \setminus \{e_2\}$ contains at least one edge from E_1 within a cycle, and $G \setminus \{e_1\}$ contains at least one edge from E_3 within a cycle. We recall that we are assuming throughout that $G \setminus \{e_3\}$ contains at least one edge from E_2 within a cycle.

We first note that e_1 lies outside the cycles of $G \setminus \{e_3\}$, which implies that e_1 and e_3 lie in the same cycles. Since the cycles of $G \setminus \{e_1\}$ are contained within $E_0 \cup E_1 \cup E_3$, and the cycles of $G \setminus \{e_3\}$ are within $E_0 \cup E_2 \cup E_3$, the fact that these cycles are the same implies that multiple cycles are contained within the 1-coloured subgraph generated by $E_0 \cup E_3$. Since every cycle contains edges from at least two colour classes, this is a contradiction, and hence this case cannot occur.

Remark 6.3.5. We note that the average degree within a tree-plus-three graph is

$$\frac{1}{|V|}\sum_{v\in V}\deg(v) = \frac{2|E|}{|V|} = \frac{2|V|+4}{|V|} = 2 + \frac{4}{|V|}.$$

K_4 is the minimal tree-plus-three graph, while any tree-plus-three graph with $|V| \geq 5$ will have average degree strictly less than 3. The 1-dimensional 0-extensions and 1-extensions will still preserve rigidity, however proving an inductive construction using the same style of proof as that used for Lemma 6.2.5 in the $k = 2$ case seems likely to increase in complexity since a tree-plus-three graph contains at least three cycles. After applying a 1-reduction to a vertex in a tree-plus-three graph, all cycles that vertex was in must be checked to ensure that the reduced graph satisfies the coloured subgraph constraints. Figure 6.10 shows a 3-coloured graph that may be obtained from an appropriately coloured copy of K_4 by applying 1-extensions. Since all cycles of this graph intersect, every vertex of degree 2 lies in four cycles which must be checked after applying a 1-reduction to ensure that the coloured subgraph contraints are still satisfied.

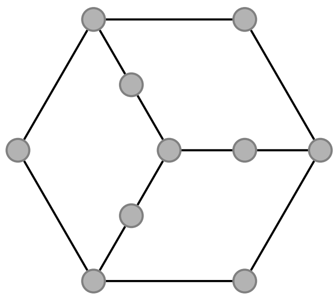


Figure 6.10 A tree-plus-three graph on 10 vertices, obtained by applying repeated 1-extensions to a 3-coloured copy of K_4. Each vertex of degree 2 lies in four cycles. Applying a coloured 1-reduction at any vertex of degree 2 requires checking the colour composition of all four reduced cycles.

6.3.2 More than three colour classes

For any k-edge-coloured graph (G, c) to be generically rigid in 1 dimension, Theorem 4.1.11 requires that (G, c) contain a spanning tree-plus-k subgraph $(H, c|_H)$, with a k-redundant rainbow subset E' within H. It is therefore equivalent for a regular 1-dimensional k-coordinated framework (G, c, p) to be isostatic, and for (G, c) to be a tree-plus-k graph such that at least one set of k-redundant edges E' is a rainbow subset.

For a k-edge-coloured tree-plus-k graph (G, c) to have a k-redundant rainbow subset E', it is clearly necessary for there to be at least one edge from each colour class within a cycle of (G, c). It is also clear that any ℓ-coloured subgraph should be at most a tree-plus-ℓ graph for $1 \leq \ell \leq k-1$, and so inductively it seems necessary that each ℓ-chromatic subgraph is independent as an ℓ-coloured graph, for $1 \leq \ell \leq k$.

The complexity of producing an inductive construction for such k-edge-coloured graphs will get increasingly difficult, due to the increasing permutations in which multiple cycles may interact, and the number of ways that colours may be distributed across the different cycles.

By Lemma 3.1.24, when the complete k-edge-coloured graph (K_n, c) is generically rigid in 1 dimension, we have that $k \leq \binom{n-1}{2}$. This suggests that for larger k, the base graphs for a construction will have correspondingly larger vertex sets. This agrees with the minimal tree-plus-k graphs increasing in size as k increases.

Theorem 6.3.6. Let (G, c, p) be a k-edge-coloured framework with a regular configuration $p \in \mathbb{R}^n$. If (G, c, p) is isostatic, then the k-coloured graph (G, c) satisfies the following conditions, for all $0 \leq \ell \leq k$:

The graph G is tree-plus-k;

At least one edge from each colour class lies within a cycle of (G, c);

Each ℓ-chromatic subgraph is independent as an ℓ-coloured graph.

Proof. We note first that by Theorem 4.1.11, in order for (G, c) to be generically infinitesimally rigid, the k-coloured graph (G, c) must contain a k-redundant rainbow subset of edges. Since the redundant edges within a 1-dimensional framework correspond precisely to the edges within cycles, this implies that every colour class contains an edge that lies in a cycle of (G, c).

Removing this k-redundant rainbow subset results in a graph that is still generically rigid, so (G, c) must contain a spanning tree-plus-k graph. By Theorem 3.1.33, the regular framework (G, c, p) is isostatic if and only if $|E| = |V| + k - 1$, and so if (G, c) is isostatic, then G is a tree-plus-k graph.

Suppose that some ℓ-chromatic subgraph of (G, c) is not independent as an ℓ-coloured graph. The row dependence in the submatrix of the coordinated rigidity matrix for the regular framework (G, c, p) that corresponds to this ℓ-chromatic subgraph remains as a row dependence in the whole coordinated rigidity matrix $R(G, c, p)$, and hence (G, c, p) is not an isostatic framework. $\square$

We conjecture that these necessary conditions are in fact sufficient, but leave this area as an open question to any interested parties.

Conjecture 6.3.7. Let (G, c, p) be a k-edge-coloured framework with a regular configuration $p \in \mathbb{R}^n$. Then (G, c, p) is isostatic, if and only if the k-coloured graph (G, c) satisfies the following conditions, for all $0 \leq \ell \leq k$:

The graph G is tree-plus-k;

At least one edge from each colour class lies within a cycle of (G, c);

Each ℓ-chromatic subgraph is independent as an ℓ-coloured graph.

Chapter 7

Combinatorial Coordinated Rigidity in 2 Dimensions

We shall now consider coordinated frameworks in 2 dimensions. We begin by characterising the isostatic frameworks with one class of coordinated bars, for which we have a Laman-style sparsity condition and a Henneberg-type inductive construction (Theorem 7.1.2). Recall that, as discussed in Remark 3.1.20, coordinated frameworks are denoted (G, c, p).

For frameworks in 2 dimensions with two classes of coordinated bars, we prove the Laman-style characterisation (Theorem 7.2.4) by applying the characterisation of rigid coordinated frameworks using redundant rigidity (Theorem 4.1.11). We require the Laman-style result in order to prove the Henneberg-style inductive construction (Theorem 7.2.20).

7.1 One colour class in 2 dimensions

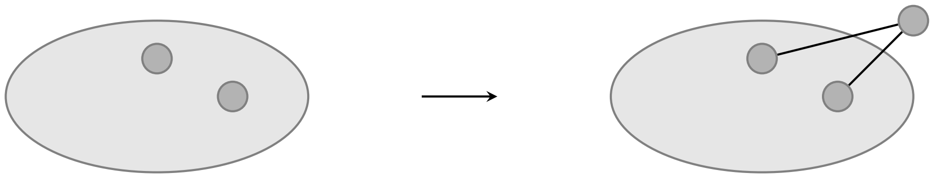
It is reasonable to be interested in extending Theorem 6.1.2 to frameworks in $\mathbb{R}^2$, as we have the standard result of Laman's Theorem (Theorem 2.8.8 [Lam70]). As the isostatic 1-coloured frameworks in 1 dimension are composed of a graph that would be isostatic when uncoloured, with an additional edge and a colouring $c : E \to \{0, 1\}$, we find the following definition useful.

Definition 7.1.1. A graph $G = (V, E)$ is *Laman-plus-one* if there is an edge $e \in E$ such that $G - e$ is a Laman graph. A *(rigidity) circuit* is a Laman-plus-one graph such that $G - e$ is a Laman graph for every edge $e \in E$.

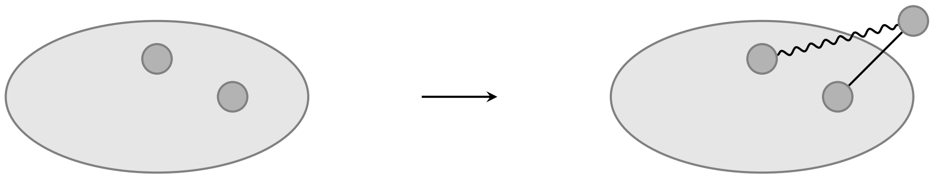
Similarly to the unique cycle within any tree-plus-one graph, every Laman-plus-one graph contains a unique circuit as a subgraph. We shall denote this circuit by $C = (V(C), E(C))$.

Any generic framework that is rigid in 2 dimensions has a spanning Laman graph as a subgraph (Corollary 2.8.9). From Theorem 4.1.11, a 1-coordinated framework with a regular 2-dimensional configuration, (G, c, p), will be infinitesimally rigid if and only if the graph G is generically rigid in M_2, and there is at least one coloured edge that is not a bridge within $M_2(G)$. The bridges within $M_2(G)$ are those edges that do not lie within any circuit, so we require at least one edge from E_1 within a circuit of (G, c) for the framework (G, c, p) to be infinitesimally rigid.

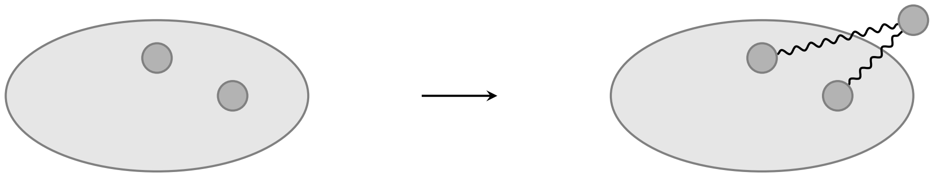
We also have a constructive characterisation of the isostatic 1-coordinated frameworks in 2 dimensions. We require the 2-dimensional 1-coloured 0-extensions and 1-extensions, which were initially defined in general in Definition 5.1.1 and Definition 5.2.1. The 0-extension in 2 dimensions is applied by creating a new vertex x, along with a pair of new edges $\{x, u_1\}, \{x, u_2\}$ for some $u_1, u_2 \in V$. In the standard 0-extension (Definition 2.8.1) both edges are added to E_0, though Lemma 5.1.2 shows that the edges may be added to E_0 or E_1 arbitrarily. All three types of 1-coloured 0-extension in 2 dimensions are shown in Figure 7.1.



a A 0-extension with both edges added to E_0.



b A 0-extension with one edge added to each of E_0 and E_1.



c A 0-extension with both edges added to E_1.

Figure 7.1 2-dimensional 1-coloured 0-extensions.

The 1-extension in 2 dimensions is the replacement of an edge $\{u_1, u_2\}$ by a new vertex x, along with three new edges, $\{x, u_1\}, \{x, u_2\}, \{x, u_3\}$ for some other vertex $u_3 \in V \setminus \{u_1, u_2\}$.

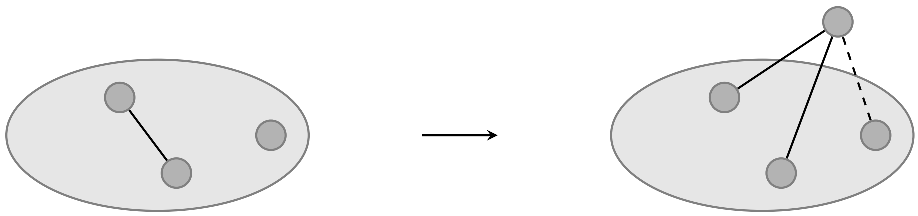
We permit the edge $\{u_1, u_2\}$ to be removed from E_0, and all three new edges be added to E_0, as in the case of the standard 1-extension (Definition 2.8.3). Our 1-coloured 1-extension only requires that the two edges $\{x, u_1\}, \{x, u_2\}$ be added to E_0 if the initial edge $\{u_1, u_2\} \in E_0$, and the third edge may be added to either E_0 or E_1. This extension move is illustrated in Figure 7.2a. If we apply a 1-coloured 1-extension to an edge $\{u_1, u_2\} \in E_1$, this constraint is relaxed further and we require only that one of $\{x, u_1\}$ and $\{x, u_2\}$ be added to E_1. The other of this pair, and the third edge $\{x, u_3\}$, may be added to either E_0 or E_1 arbitrarily. These moves are illustrated in Figure 7.2, and more intuition for these constraints may be found in the proof of Lemma 5.2.2.

Theorem 7.1.2. Let (G, c, p) be a 1-coordinated framework with a regular configuration $p \in \mathbb{R}^{2n}$. The following are equivalent:

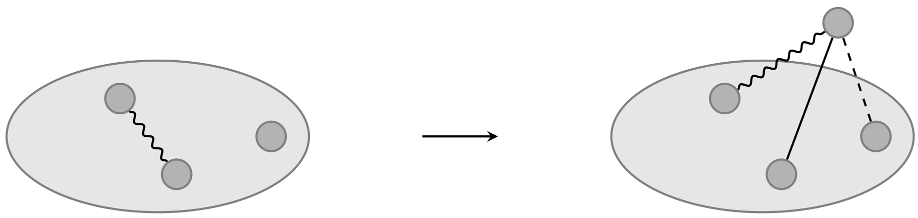
1. (G, c, p) is an isostatic framework;
2. The graph G is Laman-plus-one, and at least one edge in the circuit of (G, c) is in E_1;
3. The graph G is Laman-plus-one, and for any $D \subseteq E_0$:

$$|D| \leq 2|V(D)| - 3. \tag{7.1}$$

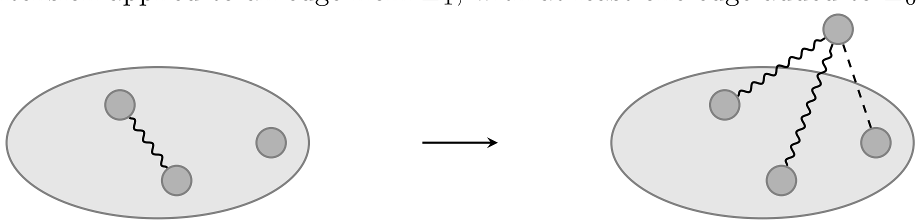
4. The 1-edge-coloured graph (G, c) can be constructed from a copy of K_4 with at least one edge in E_1, by a sequence of coloured 0-extensions and 1-extensions.



a A 1-extension applied to an edge from E_0, with at least two edges added to E_0.



b A 1-extension applied to an edge from E_1, with at least one edge added to E_0 and E_1.



c A 1-extension applied to an edge from E_1, with at least two edges added to E_1.

Figure 7.2 Three kinds of 2-dimensional 1-coloured 1-extensions.

Remark 7.1.3. For the proof of Theorem 7.1.2 we shall require 0-extensions with all combinations of uncoloured and coloured edges, as illustrated in Figure 7.1. We shall also require the standard 1-extension, where all three new edges are added to E_0, along with 1-extensions to replace an edge in E_1 with either one, two or three edges added to E_1 (where the remaining two or one edges are added to E_0). These are shown in Figure 7.2. We note that other combinations of coloured edges may give a valid 1-extension, however these additional moves are not required for our proof.

Proof. We begin by proving the equivalence of the second and third statements. The circuit of the coloured graph (G, c) is defined as the subgraph $C = (V(C), E(C))$ such that $|E(C)| = 2|V(C)| - 2$ and every subgraph of C, generated by $D \subsetneq E(C)$ has $|D| \leq 2|V(D)| - 3$. Any subgraph of G with $|F| = 2|V(F)| - 2$ must contain C, and so if the circuit contains at least one edge in E_1, every collection of edges $D_0 \subseteq E_0$ will have $|D_0| \leq 2|V(D_0)| - 3$. Conversely, if $|D_0| \leq 2|V(D_0)| - 3$ for every $D_0 \subseteq E_0$, then the circuit with $|E(C)| = 2|V(C)| - 2$ must have at least one edge in E_1.

We shall now prove that the third condition is necessary for the first condition. Clearly if $|E| \neq 2|V| - 2$, G cannot be isostatic by Theorem 3.1.33, so we may assume that $|E| = 2|V| - 2$. Suppose first that G is not Laman-plus-one. If G is a circuit, G will be Laman-plus-one, so G instead has a proper subgraph H such that $|E(H)| = 2|V(H)| - 2$. Let $e \in E(H)$ and let $G' = G - e$. G' has $|E'| = 2|V'| - 3$, but G' cannot be a Laman graph as G was not Laman-plus-one, and so G' is not rigid as an uncoloured graph. Since $H' = H - e$ will be a rigid block within G', G will remain flexible when the edge e is added back into H'.

Suppose instead that (G, c) is Laman-plus-one, and contains some $D \subset E_0$ such that $|D| \geq 2|V(D)| - 2$. If $|D| > 2|V(D)| - 2$, G cannot be Laman-plus-one, so we may assume that $|D| = 2|V(D)| - 2$. If this is true, the subset of rows of the rigidity matrix

$R(G, c, p)$ that correspond to the edges of D will form a submatrix with $2|V(D)| - 2$ rows and at most $2|V(D)|$ non-zero columns. This submatrix will hence have rank at most $2|V(D)| - 3$, and so there will be a row dependence within the submatrix. This corresponds to a dependence within the subgraph $(V(D), D)$, and hence (G, c) cannot be isostatic.

We shall use induction on $|V| = n$ to prove that any graph satisfying the third condition may be constructed as described in the fourth condition.

The smallest Laman-plus-one graphs have $|V| = 4$ and $|E| = 6$, which is clearly equivalent to G being K_4. For such graphs to satisfy $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E_0$, we clearly require $|E_1| \geq 1$, which is equivalent to the condition that (G, c) is K_4 with at least one edge in E_1.

Let (G, c) be a Laman-plus-one graph with $|V| \geq 5$ and $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E_0$, and recall that the unique circuit within G is denoted by $C = (V(C), E(C))$. We note that the average degree in a Laman-plus-one graph is $4 - \frac{4}{|V|} < 4$, so G will contain vertices of degree 2 or 3.

Suppose first that G contains a vertex of degree 2. This vertex v will lie outside the circuit of G, as the minimum degree within a circuit is 3. We may view v as the result of a 0-extension of the appropriate type for the edges adjacent to v, and may therefore apply the reverse of this move by removing v and its pair of associated edges.

Suppose instead that G contains no vertices of degree 2, and so the minimum degree is 3. We first show that if $G \neq C$, there is a vertex of degree 3 outside the circuit C.

Let $V(X) = V \backslash V(C)$ generate a subgraph outside the circuit C, $X = (V(X), E(X))$. Since this is a subgraph of G that does not contain C, we shall have $|E(X)| \leq 2|V(X)| - 3$. Let $F \subset E$ denote the set of edges that run between a vertex in C and a vertex in X, so $E = E(C) \cup E(X) \cup F$.

As $|E| = 2|V| - 2 = 2|V(C)| + 2|V(X)| - 2$, and $|E(C)| = 2|V(C)| - 2$, we have that $|E(X)| + |F| = 2|V(X)|$. Therefore $|E(X)| = 2|V(X)| - |F| \leq 2|V(X)| - 3$, and so $|F| \geq 3$. We now consider the total degree of vertices outside C: $\sum_{v \in V(X)} \deg(v) = 2|E(X)| + |F| = 2\Big(2|V(X)| - |F|\Big) + |F| = 4|V(X)| - |F|$. If $V(X) \neq \emptyset$, we have that the average degree of vertices outside C is $\frac{1}{|V(X)|} \sum_{v \in V(X)} \deg(v) = 4 - \frac{|F|}{|V(X)|} < 4$. There are no degree 2 vertices, so there must be degree 3 vertices outside C.

Let $v \in V(X)$ be a vertex of degree 3. There may be at most two neighbours of v in $V(C)$, and $|V(C)| \geq 4$, so there exists an edge $\{x, y\} \in E(C)$ which is not adjacent to a neighbour of v. Removing any edge $e \in E(C)$ from G will result in a Laman graph $G - e$, in which $C - e$ is a rigid block. We may therefore temporarily remove the edge $e = \{x, y\}$ to obtain a Laman graph $G - e$ with a degree 3 vertex v. By standard rigidity results (see, for example, [Lam70, Theorem 6.4], or [TW85, Proposition 3.3]), there exists a well-defined 1-reduction at v that results in a smaller Laman graph.

We may remove v and its three adjacent edges $\{v, u_1\}, \{v, u_2\}, \{v, u_3\}$, and replace them with a new edge $\{u_i, u_j\}$ between some pair of vertices that were neighbours of v, $i, j \in \{1, 2, 3\}$. Since $C - e$ is a rigid block, the new edge $\{u_i, u_j\}$ will not be added between two vertices in $V(C)$, and so $C - e$ will remain unchanged by the 1-reduction. This allows us to replace the temporarily removed edge $e = \{x, y\}$, and obtain a smaller Laman-plus-one graph containing the circuit C. Since $|E_1(C)| \geq 1$ in G, the reduced graph will retain at least one edge in E_1 within the circuit.

Suppose now that no degree 3 vertices exist outside the circuit C, and hence by the argument above the Laman-plus-one graph (G, c) is a circuit. We note that all circuits are at least 2-connected [HOR$^+$03].

If the graph G is a 3-connected circuit, Berg and Jordán [BJ03b] give that the uncoloured graph G has at least three vertices of degree 3 at which a standard 1-reduction may be applied, and will result in a smaller 3-connected circuit. Let

$e = \{x, y\} \in E_1$ be a coloured edge in G. Since there are at least three vertices at which the potential 1-reduction may be applied to obtain a 3-connected circuit G', there will be at least one such vertex that is not adjacent to e. We may apply a 1-reduction at this other vertex. Since $e \in E_1$ will not be touched by the reduction, and will remain within the reduced circuit, we shall retain $|E_1'| \geq 1$.

The final case to consider is when the circuit G is 2-connected. Let $a, b \in V$ be a cut pair, separating G into subgraphs H_1 and H_2. Berg and Jordán [BJ03b] show that $\deg_G(a), \deg_G(b) \geq 4$, and that H_1 and H_2 are Laman subgraphs. They also note that $\{a, b\} \notin E$, and $H_1 + \{a, b\}$, $H_2 + \{a, b\}$ are circuits.

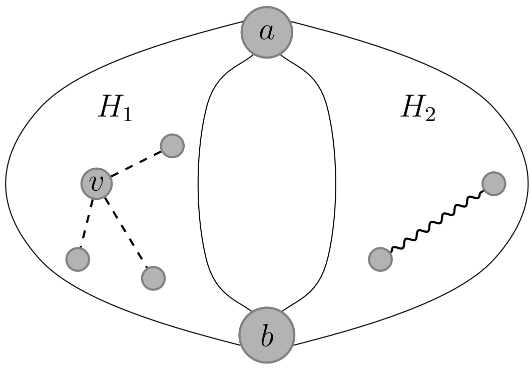


Figure 7.3 A 2-connected circuit with the cut pair $a, b \in V$ identified. This cut pair splits the circuit into Laman subgraphs H_1 and H_2, with a vertex $v \in V(H_1)$ of degree 3 and an edge $e \in E_1 \cap E(H_2)$.

Without loss of generality, let $e \in E_1 \cap E(H_2)$, and consider H_1. Since H_1 is a Laman graph and the minimum degree of vertices in G is 3, there exists $v \in V(H_1)$ such that $\deg_G(v) = \deg_{H_1}(v) = 3$. Let the neighbours of v be u_1, u_2, u_3. As $d(H_1, H_2) = 0$, all three neighbours of v will be in $V(H_1)$, and since H_1 is a Laman graph there is a valid 1-reduction that may be applied at v, resulting in a reduced Laman subgraph.

Let G' be the graph created by applying this 1-reduction at v, removing v and it's three associated edges, and adding the edge $\{x, y\}$ for some $x, y \in \{u_1, u_2, u_3\}$. Let H_1' denote the Laman subgraph of G' corresponding to H_1 within G.

G' is Laman-plus-one, so there exists a unique circuit C' within G'. Clearly C' cannot be confined to a single side of the cut pair a, b, as the subgraphs on either side of the cut pair are Laman, so $C' = J_1 \cup J_2$ where J_1, J_2 are the minimal rigid blocks contained within H_1' and H_2 respectively with $a, b \in V(J_1)$ and $a, b \in V(J_2)$. If H_2 contained a strictly smaller subgraph J_2 with $a, b \in V(J_2)$ and $|E(J_2)| = 2|V(J_2)| - 3$, any vertices in $V(H_2) \setminus V(J_2)$ would be outside the circuit of G, contradicting the assumption that G is a 2-connected circuit. Hence H_2 is the minimal such subgraph within H_2, and $C' = J_1 \cup H_2$.

We note that the minimal rigid block contained within H_1 that contains both a and b is H_1. If $a, b \in \{u_1, u_2, u_3\}$, the 1-reduction may be applied at v by adding the new edge $\{a, b\}$, which will clearly be the minimal rigid block within H_1' containing both a and b. In this case, the circuit within the Laman-plus-one graph H' is $C' = H_2 + \{a, b\}$.

As there was $e \in E_1 \cap E(H_2)$, and H_2 remains within the reduced circuit C' in either case, we shall obtain $|E_1(C')| \geq 1$ as required.

We finish by proving that any graph constructed as described in the fourth statement will create an isostatic framework (G, c, p) with any regular configuration $p \in \mathbb{R}^{2n}$. It is clear from Theorem 3.1.33 that a coloured copy of K_4 with at least one edge in $|E_1|$ will have $|E| = 2|V| - 2$ and be independent, and hence be isostatic. From Lemma 5.1.2 and Lemma 5.2.2, it may be seen that by applying coloured 0-extensions and 1-extensions to a 2-isostatic graph, we will obtain a larger 2-isostatic graph, and so any graph constructed in this way will be 2-isostatic. □

Remark 7.1.4. Theorem 4.1.11 is equivalent to the second statement of Theorem 7.1.2. We note that other coloured versions of the 1-extension will preserve infinitesimal rigidity of a framework, provided that at least one coloured edge is preserved within the circuit of the graph, however we do not have a geometric proof equivalent to Lemma 5.2.2 in these cases.

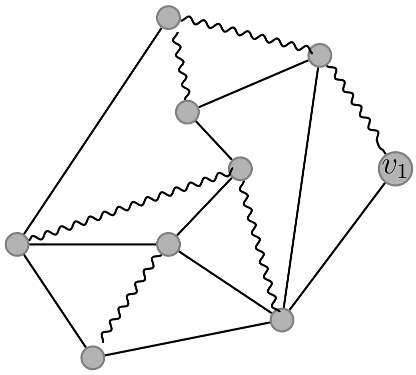
Figure 7.4 illustrates a reduction of a 1-coloured framework in 2 dimensions, as described in this proof. Vertices of degree 2 are straightforwardly removed, and vertices of degree 3 are replaced by an edge between the neighbours of the vertex, coloured appropriately as specified in Remark 7.1.3.

7.2 Two colour classes in 2 dimensions

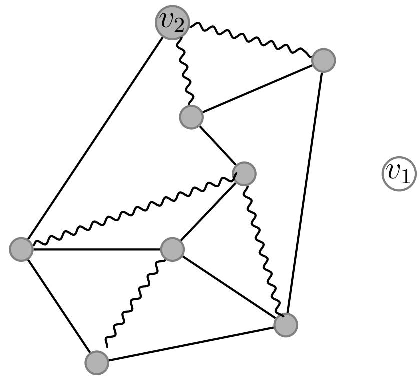
In this section we shall characterise frameworks in 2 dimensions with two classes of coordinated bars. Sections 6.1, 6.2 and 7.1 give a characterisation of each type of coordinated framework simultaneously in terms of graph counts, and inductive constructions. In this section, we shall rely on the characterisation of generically rigid coordinated frameworks in terms of redundant rigidity (Theorem 4.1.11) in order to prove the Laman-style characterisation of rigidity, which we then use to prove our inductive construction. Some smaller classes of rigid 2-coordinated frameworks may in fact be characterised without need for the redundant rigidity result, and this is discussed further towards the end of this section (Remark 7.2.23).

In Section 6.2 we saw that 2-coordinated frameworks in 1 dimension had the structure of tree-plus-two graphs, while the 1-coordinated frameworks considered in Section 6.1 had a tree-plus-one structure. It is therefore logical to define the following class of graphs, an extension to the Laman-plus-one graphs discussed in Section 7.1. These will be seen to be the structure of isostatic 2-coordinated frameworks in 2 dimensions.

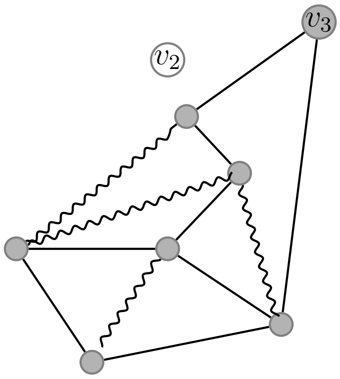
Definition 7.2.1. A graph G is *Laman-plus-two* if there exist a pair of edges $e, f \in E$ such that $G - \{e, f\}$ is a Laman graph.



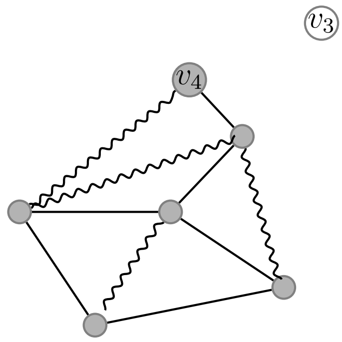
a The framework (G, c, p).



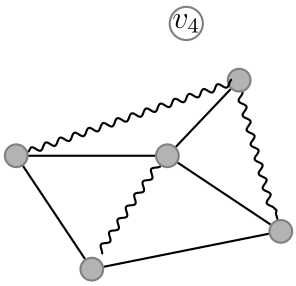
b The reduced framework (G_1, c, p), attained by removing the vertex v_1 of degree 2.



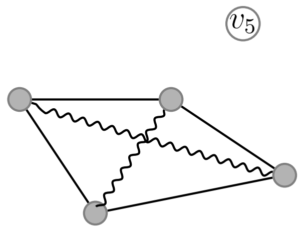
c The reduced framework (G_2, c, p), attained by reducing the vertex v_2 of degree 3.



d The reduced framework (G_3, c, p), attained by removing the vertex v_3 of degree 2.



e The reduced framework (G_4, c, p), attained by removing the vertex v_4 of degree 2. G_4 is a circuit.



f The reduced circuit (G_5, c, p), attained by reducing the vertex v_5 of degree 3.

Figure 7.4 An example reduction of a 1-coloured 2-dimensional framework (G, c, p) to a coloured copy of K_4.

Each of the edges e, f will create a circuit within G, so a Laman-plus-two graph may be straightforwardly seen to contain at least two circuits. These may be disjoint, intersect on a single vertex, or intersect on some non-empty collection of edges.

7.2.1 Structural results

We shall begin by characterising some properties of Laman-plus-two graphs. Section 7.2.4 includes an inductive construction for Laman-plus-two graphs, using 0-extensions, 1-extensions and X-replacements (Theorem 7.2.25).

Laman-plus-two graphs have an overall count of $|E| = 2|V| - 1$, with all subgraphs having $|D| \leq 2|V(D)| - 1$ for $D \subseteq E$. Any subgraph $D \subsetneq E(C)$ for some circuit C has $|D| \leq 2|V(D)| - 3$, while $|E(C)| = 2|V(C)| - 2$. Connected subgraphs of a Laman-plus-two graph with $|D| = 2|V(D)| - 2$ contain all the edges of exactly one circuit C, since they must contain at least one circuit, and the union of any two circuits has $|D| = 2|V| - 1$. Those subgraphs with $|D| = 2|V(D)| - 1$ have $E(C_1) \cup E(C_2) \subseteq D$ for a pair of circuits C_1, C_2 in G.

Proposition 7.2.2. Let G be a Laman-plus-two graph, with two circuits within G labelled C_1 and C_2. Let X be the subgraph induced by $V(X) = V \setminus \{V(C_1) \cup V(C_2)\}$. The following properties hold:

- **a.** If $E(C_1) \cap E(C_2) \neq \emptyset$, any other circuit C_3 is contained within $C_1 \cup C_2$;
- **b.** A vertex $v \in V(X)$ has at most two neighbours within a circuit C;
- **c.** If $|V(C_1 \cap C_2)| \geq 2$, the subgraph $C_1 \cap C_2$ is a Laman graph.

Proof. **a.** Let C_1 and C_2 be two circuits such that $|E(C_1 \cap C_2)| \geq 1$, and let C_3 denote a third circuit within G.

Suppose first that C_3 is edge-disjoint with C_1. Removing any $e \in E(C_3)$ and $f \in E(C_1 \setminus C_2)$ will result in a graph $G' = G - \{e, f\}$ with $|E'| = 2|V'| - 3$, that still contains the circuit C_2. Removing instead any edge $f \in E(C_1 \cap C_2)$ will leave $G - \{e, f\}$ with a subgraph $(C_1 \cup C_2)'$ which has $|E\left((C_1 \cup C_2)'\right)| = |E(C_1 \cup C_2)| - 1 = 2|V(C_1 \cup C_2)| - 2$, so no pair of edges e, f can remove all three circuits of the Laman-plus-two graph G to result in a Laman graph $G - \{e, f\}$. Hence the third circuit C_3 must intersect non-trivially with C_1. A symmetric argument shows that C_3 must intersect non-trivially with C_2.

The union of any pair of circuits that intersect non-trivially will have $|E(C_i \cup C_j)| = 2|V(C_i \cup C_j)| - 1$. If C_3 is not contained within $C_1 \cup C_2$, the subgraph containing all the vertices of $(C_1 \cup C_2) \cup C_3$ would contain more than $2|V(C_1 \cup C_2 \cup C_3)| - 1$ edges, which is impossible within the Laman-plus-two graph G. Hence the union $C_1 \cup C_2$ contains all other circuits within the Laman-plus-two graph G.

b. Let v be a vertex in X of degree δ, with $\delta_1 \leq \delta$ neighbours within $V(C_1)$. The subgraph B induced by $V(C_1) \cup \{v\}$ will have $|E(B)| = |E(C_1)| + \delta_1 = 2|V(C_1)| - 2 + \delta_1$. Since $|V(B)| = |V(C_1)| + 1$, we have $|E(B)| = 2|V(B)| + \delta_1 - 4$. Since $v \in V(X)$ lies outside all circuits of G by **a.**, B contains exactly one circuit, C_1. Therefore $|E(B)| \leq 2|V(B)| - 2$. This gives $\delta_1 - 4 \leq -2$, and so $\delta_1 \leq 2$ as required.

c. Suppose that $|V(C_1 \cap C_2)| \geq 2$. The union of the pair of circuits, $C_1 \cup C_2$, is a subgraph of a Laman-plus-two graph and so $|E(C_1 \cup C_2)| \leq 2|V(C_1 \cup C_2)| - 1$. Individually, each circuit has $|E(C_i)| = 2|V(C_i)| - 2$, with $|D| \leq 2|V(D)| - 3$ for any $D \subsetneq E(C_i)$. We note that $C_1 \cap C_2$ is a subgraph of both C_1 and C_2, so we have the following:

$$\begin{aligned}
|E(C_1 \cup C_2)| &= |E(C_1)| + |E(C_2)| - |E(C_1 \cap C_2)| \\
&= 2|V(C_1)| - 2 + 2|V(C_2)| - 2 - |E(C_1 \cap C_2)| \\
&\geq 2|V(C_1)| - 2 + 2|V(C_2)| - 2 - 2|V(C_1 \cap C_2)| + 3 \\
&= 2\Big(|V(C_1)| + |V(C_2)| - |V(C_1 \cap C_2)|\Big) - 2 - 2 + 3
\end{aligned}$$

$$= 2|V(C_1 \cup C_2)| - 1.$$

Hence $|E(C_1 \cup C_2)| = 2|V(C_1 \cup C_2)| - 1$, and $|E(C_1 \cap C_2)| = 2|V(C_1 \cap C_2)| - 3$. Any subgraph of $C_1 \cap C_2$ will also be a subgraph of both circuits, and so $|D| \leq 2|V(D)| - 3$ for any $D \subset E(C_1 \cap C_2)$, as required for $C_1 \cap C_2$ to be a Laman graph. $\square$

By Proposition 7.2.2a, any subgraph of G with $|D| = 2|V(D)| - 1$ will contain all circuits of G.

Proposition 7.2.3. Let G be a Laman-plus-two graph, with two circuits within G labelled C_1 and C_2. Let X be the subgraph induced by $V(X) = V \setminus \{V(C_1) \cup V(C_2)\}$, and suppose that the minimum degree within $V(X)$ is at least 4. The following properties hold:

- **a.** The circuits C_1 and C_2 are disjoint;
- **b.** Either $X = \{x\}$ for some vertex $x \in V$ with $\deg(x) = 4$, or $X = H$ for some Laman graph H with exactly three edges from each circuit to X;
- **c.** Every vertex in $V(X)$ has degree exactly 4.

Proof. **a.** Suppose that C_1 and C_2 are not disjoint, and consider the subgraph induced by $V(C_1 \cup C_2)$. Let F_α denote the set of edges that run between vertices in $V(C_1 \setminus C_2)$ and vertices in $V(C_2 \setminus C_1)$. The subgraph $C_1 \cup C_2$ will therefore satisfy the following:

$$\begin{aligned} |E(C_1 \cup C_2)| &= |E(C_1)| + |E(C_2)| + |F_\alpha| - |E(C_1 \cap C_2)| \\ &= 2|V(C_1)| - 2 + 2|V(C_2)| - 2 + |F_\alpha| - |E(C_1 \cap C_2)|. \end{aligned} \tag{7.2}$$

We suppose first that $|V(C_1 \cap C_2)| \geq 2$, so by Proposition 7.2.2c $C_1 \cap C_2$ is a Laman graph with $|E(C_1 \cap C_2)| = 2|V(C_1 \cap C_2)| - 3$. In this case, Equation (7.2) becomes the following:

$$
\begin{aligned}
|E(C_1 \cup C_2)| &= 2\Big(|V(C_1)| + |V(C_2)| - |V(C_1 \cap C_2)|\Big) - 2 - 2 + 3 + |F_\alpha| \\
&= 2|V(C_1 \cup C_2)| - 1 + |F_\alpha|.
\end{aligned}
$$

Since G is a Laman-plus-two graph, $C_1 \cup C_2$ must have $|E(C_1 \cup C_2)| \leq 2|V(C_1 \cup C_2)| - 1$, so $F_\alpha = \emptyset$.

If instead $C_1 \cap C_2$ is a single vertex, we have $|E(C_1 \cap C_2)| = 0 = 2|V(C_1 \cap C_2)| - 2$, and so Equation (7.2) gives

$$
\begin{aligned}
|E(C_1 \cup C_2)| &= 2\Big(|V(C_1)| + |V(C_2)| - |V(C_1 \cap C_2)|\Big) - 2 - 2 + 2 + |F_\alpha| \\
&= 2|V(C_1 \cup C_2)| - 2 + |F_\alpha|.
\end{aligned}
$$

We therefore have $|F_\alpha| \leq 1$. If $|F_\alpha| = 1$, we have $|E(C_1 \cup C_2)| = 2|V(C_1 \cup C_2)| - 1$ as above, or we have $|E(C_1 \cup C_2)| = 2|V(C_1 \cup C_2)| - 2$ when $F_\alpha = \emptyset$.

The Laman-plus-two graph G has an overall constraint that $|E| = 2|V| - 1$. Let F denote the edges between vertices in $V(C_1 \cup C_2)$ and vertices in $V(X)$, so G has $|E| = |E(C_1 \cup C_2)| + |E(X)| + |F|$, while recalling that $|V| = |V(C_1 \cup C_2)| + |V(X)|$ by the definition of the subgraph X.

We begin by considering the cases where $|E(C_1 \cup C_2)| = 2|V(C_1 \cup C_2)| - 1$. We note that $|E| = |E(C_1 \cup C_2)| + |E(X)| + |F| = 2|V(C_1 \cup C_2)| - 1 + |E(X)| + |F|$. Since $|E| = 2|V| - 1 = 2|V(C_1 \cup C_2)| + 2|V(X)| - 1$, we obtain

$$
|E(X)| = 2|V(X)| - |F|. \tag{7.3}
$$

Since X is a subgraph of G that lies outside both circuits, when $|V(X)| \geq 2$ we have $|E(X)| = 2|V(X)| - |F| \leq 2|V(X)| - 3$, and so $|F| \geq 3$.

We note that from standard graph theory, the total degree of a graph is at least the minimum degree multiplied by the number of vertices. The total degree within X is the following:

$$\sum_{v\in V(X)} \deg(v) = 2|E(X)| + |F| = 2\Big(2|V(X)| - |F|\Big) + |F|$$
$$= 4|V(X)| - |F|.$$

As the minimum degree within X is 4, we have $\sum_{v\in V(X)} \deg(v) \geq 4 \cdot |V(X)|$, which leads to a contradiction with $|F| \geq 3$.

If instead $|V(X)| = 1$, $|E(X)| = 0 = 2|V(X)| - 2$ and so $|F| = 2$. This gives X as a single vertex of degree 2, which clearly does not have minimum degree 4.

We now consider the case where $|E(C_1 \cup C_2)| = 2|V(C_1 \cup C_2)| - 2$. The following statement may be derived similarly to Equation (7.3):

$$|E(X)| = 2|V(X)| + 1 - |F|.$$

If $|V(X)| \geq 2$, we see that $|F| \geq 4$, and the total degree within X will be $4|V(X)| + 2 - |F|$. When the minimum degree within X is 4, this implies that $|F| \leq 2$, and so we have another contradiction.

As above, if $|V(X)| = 1$ we obtain that X is a single vertex of degree 3, from the requirement that $0 = 2|V(X)| + 1 - |F| = 3 - |F|$.

Therefore when $V(C_1 \cap C_2) \neq \emptyset$, the minimum degree within X must be strictly less than 4, and when $\min_{v\in V(X)} \deg(v) = 4$ we have $C_1 \cap C_2 = \emptyset$.

b. By (a) the circuits C_1 and C_2 are disjoint. As before, we denote by F the set of edges with one end point in X and one end point in $C_1 \cup C_2$, and we use F_α to denote any edges $\{u_1, u_2\}$ with $u_1 \in V(C_1)$, $u_2 \in V(C_2)$.

Let C denote the subgraph induced by $V(C_1) \cup V(C_2)$, so $|E(C)| = |E(C_1)| + |E(C_2)| + |F_\alpha| = 2|V(C_1)| - 2 + 2|V(C_2)| - 2 + |F_\alpha| = 2|V(C)| + |F_\alpha| - 4$. We therefore have $|F_\alpha| \leq 3$, since $|E(C)| \leq 2|V(C)| - 1$.

The overall constraint that $|E| = 2|V| - 1$ may be broken down into $|E| = |E(C)| + |E(X)| + |F|$ and $|V| = |V(C)| + |V(X)|$ to give the following:

$$\begin{aligned} |E(X)| + |F| &= |E| - |E(C)| \\ &= \big(2|V(C)| + 2|V(X)| - 1\big) - \big(2|V(C)| + |F_\alpha| - 4\big) \\ &= 2|V(X)| + 3 - |F_\alpha|. \end{aligned} \tag{7.4}$$

Suppose first that $|V(X)| \geq 2$. Since X is a subgraph of a Laman-plus-two graph that lies outside both circuits, $|E(X)| \leq 2|V(X)| - 3$, and so $|E(X)| + |F| \leq 2|V(X)| - 3 + |F|$. This gives that $3 - |F_\alpha| \leq |F| - 3$, and hence $|F| \geq 6 - |F_\alpha|$.

We now consider the average degree within X:

$$\begin{aligned} \frac{1}{|V(X)|} \sum_{v \in V(X)} \deg_G(v) &= \frac{2|E(X)| + |F|}{|V(X)|} \\ &= \frac{4|V(X)| + 6 - 2|F_\alpha| - 2|F| + |F|}{|V(X)|} \\ &= \frac{4|V(X)| + 6 - |F| - 2|F_\alpha|}{|V(X)|} \\ &= 4 + \frac{6 - |F| - 2|F_\alpha|}{|V(X)|}. \end{aligned}$$

As the minimum degree within X is 4, the average degree must be at least 4, so we require that $6 - |F| - 2|F_\alpha| \geq 0$. This implies that $|F| \leq 6 - 2|F_\alpha|$, therefore $6 - |F_\alpha| \leq |F| \leq 6 - 2|F_\alpha|$ and $|F_\alpha| \leq 0$. When $|V(X)| \geq 2$ and the minimum degree within X is 4, there can be no edges running directly between the two circuits, and $|F| = 6$. This gives an overall constraint that $|E(X)| = 2|V(X)| + 3 - |F_\alpha| - |F| = 2|V(X)| + 3 - 0 - 6$, and so $|E(X)| = 2|V(X)| - 3$. Since any subgraph of X is a subgraph of a Laman-plus-two graph that does not contain any circuits, any such subgraph $(V(D), D)$ will satisfy $|D| \leq 2|V(D)| - 3$, and so X is a Laman subgraph of G.

Let F be partitioned into F_1 and F_2, where $F_i = \Big\{\{u, w\} \in E : u \in V(C_i), w \in V(X)\Big\}$. As $|F| = 6$, we have $|F_1| + |F_2| = 6$. Without loss of generality, consider the subgraph A_1 induced by $V(C_1) \cup V(X)$. $|E(A_1)| = |E(C_1)| + |E(X)| + |F_1| = 2|V(C_1)| - 2 + 2|V(X)| - 3 + |F_1| = 2|V(A_1)| + |F_1| - 5$. Since A_1 contains exactly one circuit of the Laman-plus-two graph G, $|E(A_1)| = 2|V(A_1)| - 2$, and hence $|F_1| = 3 = |F_2|$.

Suppose instead $|V(X)| = 1$. Then $|E(X)| = 0 = 2|V(X)| - 2$, and so by Equation (7.4) $2|V(X)|-2+|F| = 2|V(X)|+3-|F_\alpha|$. This implies that $|F|+|F_\alpha| = 5$. Since by (a), a vertex in $V(X)$ can have at most two neighbours within each circuit, when the minimum degree within X is 4 and $|V(X)| = 1$, we have a single vertex of degree 4 and $|F_\alpha| = 1$.

c. By (a) the circuits C_1 and C_2 are disjoint, and by (b) the subgraph X outside both circuits is either a single vertex, or a Laman graph with exactly three edges between vertices in $V(X)$ and vertices in $V(C_i)$ for each circuit C_i.

Suppose first that $X = \{x\}$ for some single vertex $x \in V$. As the minimum degree in X is 4, we have $\deg_G(x) = 4$ and so trivially every vertex in X has degree 4.

Suppose instead that X is some Laman graph H, and let $|F|$ denote the number of edges from vertices in $V(C_1 \cup C_2)$ to vertices in $V(X)$. The average degree within X is

$$\begin{aligned}\frac{1}{|V(X)|}\sum_{v\in V(X)} \deg_G(v) &= \frac{2|E(X)| + |F|}{|V(X)|} = \frac{2(2|V(X)| - 3) + 6}{|V(X)|} \\ &= \frac{4|V(X)| - 6 + 6}{|V(X)|} = 4.\end{aligned}$$

Since the average degree is exactly 4, and the minimum degree within $V(X)$ is 4, every vertex in $V(X)$ has degree 4. □

These properties will be useful for characterising the 2-edge coloured isostatic graphs in 2 dimensions, and for the general characterisation of Laman-plus-two graphs given in Section 7.2.4 (Theorem 7.2.25).

7.2.2 Laman-type result

We shall now use the result from Chapter 4 to give a characterisation of 2-edge-coloured graphs that give isostatic 2-dimensional frameworks when combined with a regular configuration.

Theorem 7.2.4. Let (G, c, p) be a 2-coloured framework with a regular configuration $p \in \mathbb{R}^{2n}$. Then (G, c, p) is an isostatic framework if and only if the graph G is a Laman-plus-two graph, the 2-edge-coloured graph (G, c) has at least one edge from each of E_1 and E_2 within a circuit, and the following counts are satisfied:

$$|D| \leq 2|V(D)| - 3 \quad \forall D \subseteq E_0, \tag{7.5}$$

$$|D| \leq 2|V(D)| - 2 \quad \forall D \subseteq E_0 \cup E_1, D \subseteq E_0 \cup E_2. \tag{7.6}$$

Proof. By Theorem 4.1.11, the 2-coloured graph (G, c) is generically rigid if and only if the graph G is generically rigid, and (G, c) contains a 2-redundant rainbow pair of edges $e_1 \in E_1, e_2 \in E_2$. This is equivalent to $G \setminus \{e_1, e_2\}$ containing a spanning Laman subgraph. In order for (G, c) to be isostatic, $G \setminus \{e_1, e_2\}$ must be precisely a Laman graph, where replacing each of the edges from the rainbow pair e_1, e_2 creates a circuit within (G, c).

If (G, c) does not satisfy Equation (7.5), for all regular configurations $p \in \mathbb{R}^{2n}$ the submatrix of the coordinated rigidity matrix $R(G, c, p)$ corresponding to the uncoloured subgraph will contain a dependence by Laman's Theorem [Lam70] (Theorem 2.8.8). Similarly if (G, c) does not satisfy Equation (7.6), Theorem 7.1.2 implies that the submatrix of $R(G, c, p)$ corresponding to this 1-chromatic subgraph will contain a dependence for all regular configurations p.

Suppose now that (G, c) satisfies the conditions stated. We shall show that there is a 2-redundant rainbow pair $\{e, f\} \subset E$, and so the sufficiency of these conditions will follow from Theorem 4.1.11.

As noted previously, the Laman-plus-two graph G will contain at least two circuits, so we shall label two of the circuits C_1 and C_2.

Suppose first that C_1 and C_2 are edge-disjoint, and so these are the only two circuits within G. There is at least one edge of each colour within a circuit, and each circuit contains at least one coloured edge. We may therefore choose a coloured edge from each circuit, one from each colour class, to create the desired rainbow pair of edges.

Suppose next that $C_1 \cap C_2$ contains at least one edge. If one circuit contains only edges from one colour class, say $E(C_1) \subseteq E_0 \cup E_1$, then $C_2 \setminus C_1$ contains at least one edge from E_2, since there is at least one edge from E_2 within a circuit of (G, c), and all circuits of (G, c) are contained within $C_1 \cup C_2$ (Proposition 7.2.2a).

Suppose finally that both circuits C_1 and C_2 contain edges from both colour classes, and let $e_1 \in E_1 \cap E(C_1)$. If $e_1 \in E(C_1 \setminus C_2)$, there is an edge $e_2 \in E_2 \cap E(C_2)$, and hence e_1, e_2 is the rainbow 2-redundant pair of edges required. If instead $e_1 \in E(C_1 \cap C_2) \cap E_1$, by the circuit elimination axiom there is a circuit C_3 with $E(C_3) \subseteq E(C_1 \cup C_2) \setminus \{e_1\}$. If C_3 contains edges from only one colour class, we may apply the argument above.

If instead C_3 contains edges from both colour classes, we may choose an edge $f_2 \in E_2 \cap E(C_3)$ to form a rainbow pair e_1, f_2, that is clearly 2-redundant. □

Corollary 7.2.5. Let (G, c, p) be a 2-coordinated isostatic framework, with a regular configuration $p \in \mathbb{R}^{2n}$. Then the 2-coloured Laman-plus-two graph (G, c) contains at most one circuit within each of $E_0 \cup E_1$ and $E_0 \cup E_2$.

Proof. Since (G, c, p) is a 2-coordinated isostatic framework in 2 dimensions, by Theorem 7.2.4, the 2-coloured graph (G, c) is Laman-plus-two, with at least one edge from each colour class within a circuit, and Equations (7.5) and (7.6) are satisfied.

If (G, c) contains exactly two edge-disjoint circuits C_1 and C_2, it is clear from the condition that each colour class has at least one edge within a circuit that $E(C_1 \cup C_2) \not\subseteq E_0 \cup E_1$ and $E(C_1 \cup C_2) \not\subseteq E_0 \cup E_2$.

Suppose instead that $E(C_1) \cap E(C_2) \neq \emptyset$, and so by Proposition 7.2.2a, all circuits of the Laman-plus-two graph (G, c) are contained within $C_1 \cup C_2$, and $C_1 \cap C_2$ is a Laman graph. If $E(C_1) \subseteq E_0 \cup E_1$ and $E(C_2) \subseteq E_0 \cup E_1$, the subgraph $C_1 \cup C_2$ has

$$\begin{aligned} |E(C_1 \cup C_2)| &= |E(C_1)| + |E(C_2)| - |E(C_1 \cap C_2)| \\ &= 2|V(C_1)| - 2 + 2|V(C_2) - 2 - 2|V(C_1 \cap C_2) + 3 \\ &= 2|V(C_1 \cup C_2)| - 1. \end{aligned}$$

Since $E(C_1 \cup C_2) \subseteq E_0 \cup E_1$, this contradicts Equation 7.6. A symmetric argument applies for $E(C_1), E(C_2) \subseteq E_0 \cup E_2$. □

Figure 7.5 shows two different generically flexible 2-coloured Laman-plus-two graphs.

7.2.3 Henneberg type result

In Section 7.1, we were able to characterise 1-coordinated frameworks in 2 dimensions simultaneously by coloured sparsity conditions, and with an inductive construction.

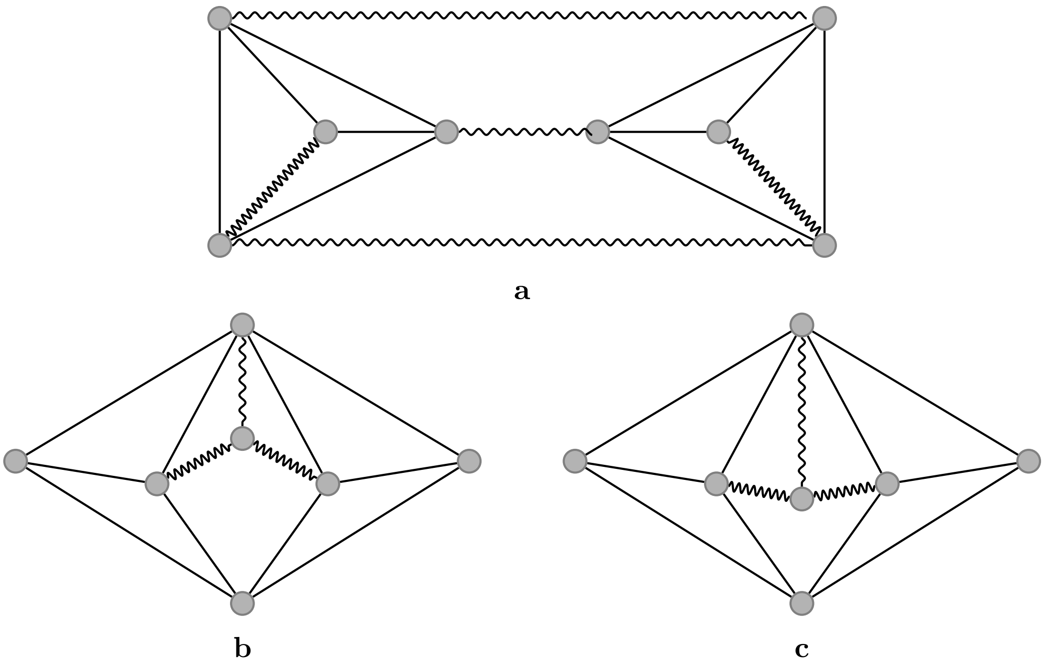


Figure 7.5 Two examples of generically flexible 2-coloured Laman-plus-two graphs. The graph in Figure 7.5(a) contains two disjoint circuits, and each circuit contains one edge from E_1, however all edges in E_2 are bridges. Since no 2-redundant rainbow pair can exist, generic flexibility of the graph follows from Theorem 4.1.11.
The graph in Figure 7.5(b) contains multiple intersecting circuits. Removing any coloured edge results in a Laman-plus-one graph with a circuit made only of uncoloured edges, contradicting Equation 7.5. Figure 7.5(c) shows a congruent configuration of the same graph, which may clearly be obtained from Figure 7.5(b) through a continuous 2-coordinated flex.

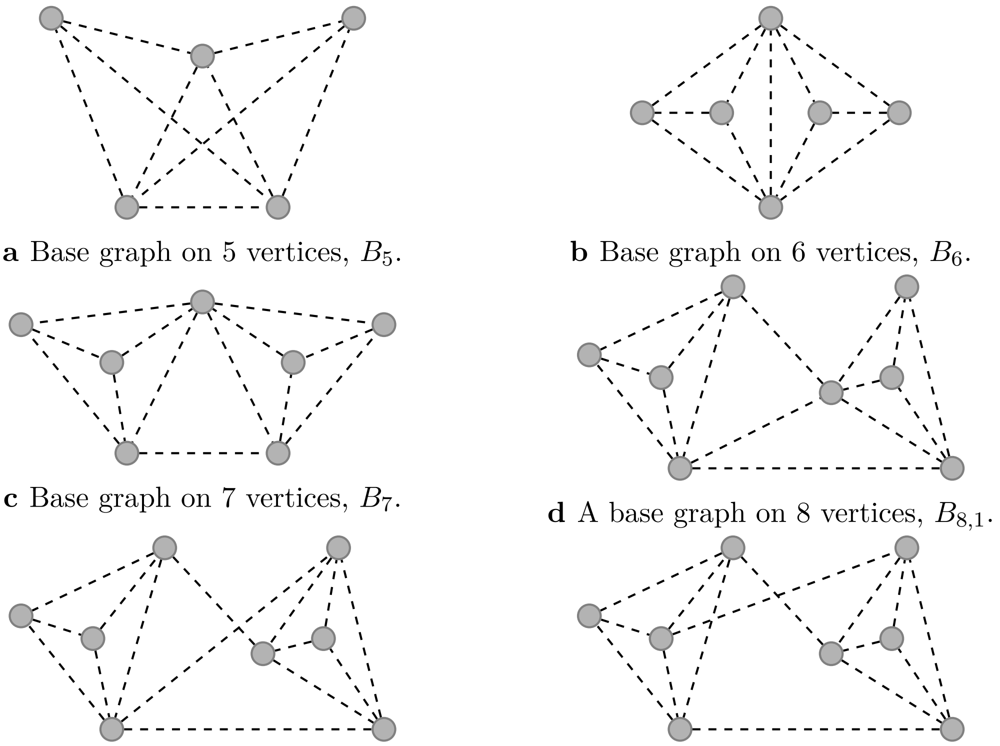
For frameworks with two coordination classes, we rely on our characterisation in terms of redundant rigidity (Theorem 4.1.11 and Theorem 7.2.4) in order to give a combinatorial proof of an inductive construction. In the case where the circuits of our 2-edge-coloured graph (G, c) remain disjoint throughout the construction, we are not reliant on Theorem 4.1.11, as discussed in Remark 7.2.23.

To develop a Henneberg-style construction of 2-dimensional generically isostatic graphs with two coordination classes, in the style of the construction for 1-coordinated frameworks in 2 dimensions given in Section 7.1, we begin by defining a class of base graphs. Let $\mathcal{B}$ be the set of base graphs (B, c), where the graph B is one of the smallest Laman-plus-two graphs, for which the minimal two circuits are copies of K_4. These graphs are illustrated and labelled in Figure 7.6, and we require $B \in \left\{B_5, B_6, B_7, B_{8,1}, B_{8,2}, B_{8,3}\right\}$ for $(B, c) \in \mathcal{B}$. We also require that the colouring c of B has at least one edge of each colour within a circuit, and at least one coloured edge within each circuit.

Lemma 7.2.6. Any base graph $(B, c) \in \mathcal{B}$ is isostatic.

Proof. Every graph $B \in \left\{B_5, B_6, B_7, B_{8,1}, B_{8,2}, B_{8,3}\right\}$ contains exactly two copies of K_4, along with other circuits in the case of B_5 and B_6. Removing one edge from each copy of K_4 will destroy any subgraphs with $|D| > 2|V(D)| - 3$, as all circuits are contained within the union of any pair of circuits by Proposition 7.2.2a. Thus B is a Laman-plus-two graph.

For $(B, c) \in \mathcal{B}$ to be a base graph, we require that every circuit contains at least one coloured edge, and so any uncoloured subgraph $D \subseteq E_0$ must have $|D| \leq 2|V(D)| - 3$. Any subgraph with $|D| = 2|V(D)| - 1$ will contain at least two circuits, and so by Proposition 7.2.2a will contain all circuits. As we require that the colouring c of a base graph has at least one edge from each of E_1 and E_2 within a circuit, any subgraph with $|D| = 2|V(D)| - 1$ will contain edges from both E_1 and E_2. Any subgraph



a Base graph on 5 vertices, B_5.

b Base graph on 6 vertices, B_6.

c Base graph on 7 vertices, B_7.

d A base graph on 8 vertices, $B_{8,1}$.

e A base graph on 8 vertices, $B_{8,2}$.

f A base graph on 8 vertices, $B_{8,3}$.

Figure 7.6 The graphs for $\mathcal{B}$.

containing only one colour of edges, so $D \subseteq E_0 \cup E_1$ or $D \subseteq E_0 \cup E_2$, will therefore have $|D| \leq 2|V(D)| - 2$.

Hence $(B, c) \in \mathcal{B}$ satisfies the conditions of Theorem 7.2.4, and so is isostatic. $\square$

Remark 7.2.7. An alternative proof for Lemma 7.2.6 exists by applying checking the rank of the rigidity matrix $R(B, c, p)$ for any regular $p \in \mathbb{R}^{2n}$ to verify that the base graph is infinitesimally rigid by Theorem 3.1.22, and applying Theorem 3.1.33. This avoids the need to rely on Theorem 7.2.4.

Geometric moves

As with the 1-coordinated frameworks discussed in Section 7.1, we shall require the coloured 0-extensions and 1-extensions described in Chapter 5. A 0-extension is applied by creating a new vertex x of degree 2, where the new edges $\{x, v_1\}, \{x, v_2\}$, for some distinct $v_1, v_2 \in V$, may be uncoloured or coloured in any combination. This is illustrated in Figure 7.7, where the dashed lines indicate that edges may be either uncoloured, or coloured with any colour. Any such 0-extension will preserve generic rigidity by Lemma 5.1.2.

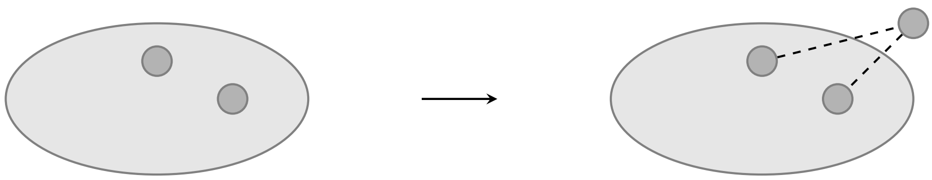
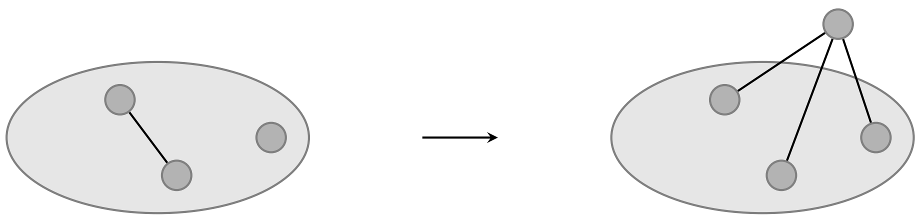


Figure 7.7 The 2-coloured 0-extensions.

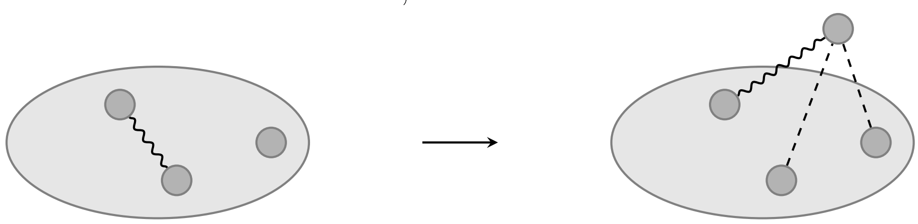
A 1-extension is applied to the coloured graph (G, c) by removing an edge $\{v_1, v_2\}$, and replacing this edge with a new vertex of degree 3, with edges $\{x, v_1\}, \{x, v_2\}, \{x, v_3\}$ for some vertex v_3 distinct from both v_1 and v_2. If the edge $\{v_1, v_2\}$ is uncoloured, the new edges are added to E_0. If instead the removed edge was coloured, at least one of the edges $\{x, v_1\}$ and $\{x, v_2\}$ must be given the same colour as $\{v_1, v_2\}$, though the other two edges may be coloured in any way. A selection of these coloured 1-extensions are shown in Figure 7.8. Figure 7.9 illustrates that 1-extensions applied to edges of different colours, may result in vertices of degree 3 that are adjacent to the same number of edges of each colour.

Remark 7.2.8. We note that the proof of Lemma 5.2.2 permits the replacement of an uncoloured edge, $\{v_1, v_2\} \in E_0$, with a vertex x of degree 3 with the requirement that only $\{x, v_1\}$ and $\{x, v_2\}$ are added to E_0, while the third edge may be coloured arbitrarily. To simplify the proof, in this case we require that all three edges be added to E_0.

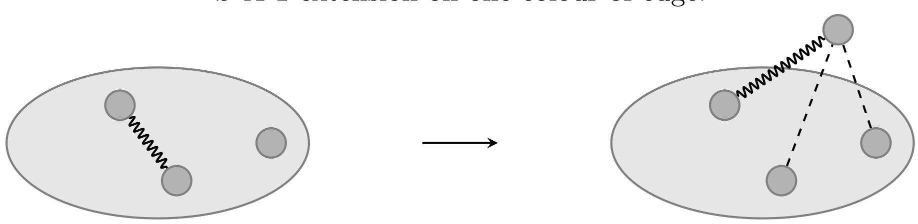
We note that by Lemma 5.2.2, these 2-coloured 1-extensions will preserve generic rigidity.



a The standard, uncoloured 1-extension.



b A 1-extension on one colour of edge.



c A 1-extension on the other colour of edge.

Figure 7.8 The 2-coloured 1-extensions.

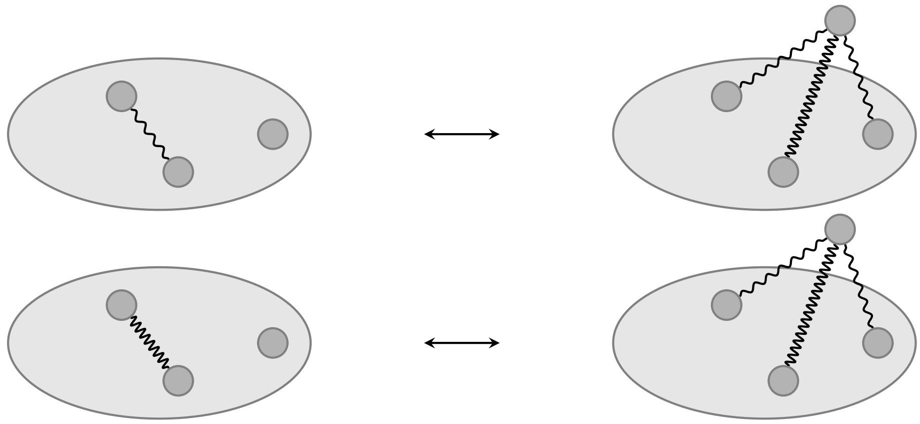


Figure 7.9 A degree 3 vertex adjacent to edges of both colours could be the result of either type of 1-extension.

The geometric proof of Lemma 5.2.2 requires that when applying a coloured 1-extension to an edge $\{v_1, v_2\} \in E_1$, at least one of the new edges $\{x, v_1\}$ and $\{x, v_2\}$ must be added to E_1'. We wish to generalise the 1-extension move to allow an edge $\{v_1, v_2\} \in E_1$ within a circuit of (G, c) to be replaced by a new vertex x of degree 3, with $\{x, v_1\}, \{x, v_2\} \in E_0' \cup E_2'$, in situations where there will still be an edge from E_1' within the corresponding circuit of (G', c'). This shall be called the *"chosen 1-extension"*, defined formally in Definition 7.2.11. In order to prove that this move preserves infinitesimal rigidity, we rely on Theorem 4.1.11.

Figure 7.10 shows an example of a graph that requires the reverse of such a move. There are exactly two vertices of degree 3, labelled u and v. The vertex u lies within a copy of K_4, and so no 1-reduction may be applied at u, while applying the reverse of a coloured 1-extension at v would result in an uncoloured copy of K_4 on the left-hand side. To preserve the counts of Theorem 7.2.4, and hence obtain a smaller isostatic graph, we require a 1-reduction at v that allows us to place an edge of colour 1 within the new copy of K_4.

We begin by defining notation in Remarks 7.2.9 and 7.2.10, before formally defining the *"chosen 1-extension"* in Definition 7.2.11.

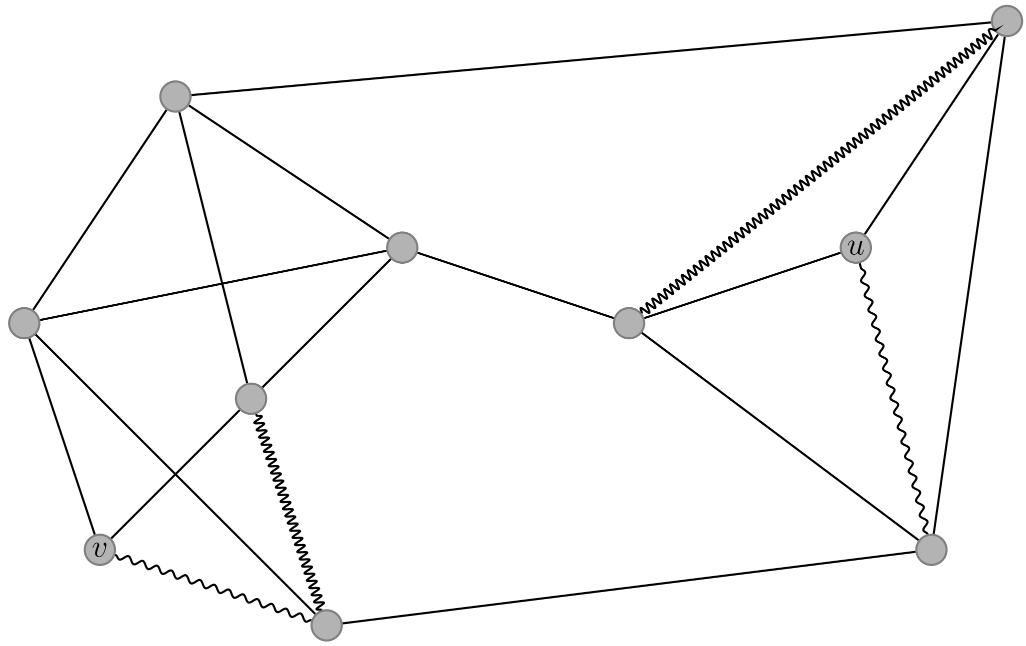


Figure 7.10 A graph that requires the reverse of a "chosen 1-extension".

Remark 7.2.9. By relabelling the circuits, we may assume without loss of generality that the "chosen 1-extension" is applied to an edge $f \in E(C_1)$. If C_1 contains edges from both colour classes, we may relabel the colour classes as necessary to assume that $E_2 \cap E(C_2) \neq \emptyset$, and if C_1 contains edges from only one colour class, we may assume without loss of generality that this colour class is E_1. We may therefore assume that $f \in E_1 \cap E(C_1)$, and so we require that $E_1' \cap E(C_1') \neq \emptyset$ after applying the "chosen 1-extension".

Remark 7.2.10. The extended graph (G', c') may contain more than two circuits. When applying a "chosen 1-extension" to an edge $f = \{v_1, v_2\} \in E_1 \cap E(C_1)$, we create a vertex x and three edges $\{x, v_1\}, \{x, v_2\}, \{x, v_3\}$ for some $v_3 \in V \setminus \{v_1, v_2\}$. We shall show that there exists a circuit C_1' of (G', c') such that $V(C_1') \supseteq V(C_1) \cup \{x\}$ and $E(C_1') \supseteq E(C_1) \setminus \{f\} \cup \Big\{\{x, v_1\}, \{x, v_2\}, \{x, v_3\}\Big\}$. Since the vertices v_1 and v_2 are within the circuit C_1 of (G, c), the graph $G \setminus \{f\}$ contains a Laman subgraph with vertex set $V(C_1)$. If $v_3 \in V(C_1)$, then $V(C_1) \cup \{x\}$ is the vertex set of a circuit within

G', and this circuit is denoted C_1'. If instead $v_3 \notin V(C_1)$, $V(C_1) \cup \{x\}$ is the vertex set of a Laman subgraph of G', with the structure of a 0-extension applied to the Laman subgraph $C_1 \setminus \{f\}$. Adding the third edge $\{x, v_3\}$ creates a circuit C' within G', with $v_1, v_2, v_3, x \in V(C')$. This circuit will consist of x, the three additional edges, and a minimal Laman subgraph of G that contains v_1, v_2, v_3. Since C_1 is a circuit, $C_1 \setminus \{f\}$ is the minimal Laman subgraph of $G \setminus \{f\}$ containing v_1 and v_2, and any non-empty subgraph of a Laman subgraph is a Laman subgraph, $C_1 \setminus \{f\}$ will be contained within the circuit C'. Hence C_1' is of the desired form.

When $f \in E(C_1 \setminus C_2)$, the circuit C_2 will remain unchanged, and we shall label this circuit with C_2'.

If instead $f \in E(C_1 \cap C_2)$, we shall confirm that every vertex of G' remains in a circuit. We note first that $V(C_1) \cup \{x\} \subseteq V(C_1')$, and hence these vertices all lie in at least one circuit of G'. It remains to consider the vertices $V(C_2 \setminus C_1)$.

Suppose that there is some vertex $u \in V(C_2 \setminus C_1)$ that does not lie in any circuit of G'. We note that since u lies outside the intersection $C_1 \cap C_2$ within the graph G, u is unchanged by the application of the structure of a 1-extension on the edge $f \in E(C_1 \cap C_2)$. Since u lies outside all circuits of G', the graph $G\prime \setminus \{u\}$ must contain at least two circuits, however $\deg(u) \geq 3$ since $u \in V(C_2)$. We note that $G' \setminus \{u\}$ has $|E(G' \setminus \{u\})| = |E'| - \deg(u) = |E| - 1 + 3 - \deg(u) = 2|V| - 1 - 1 + 3 - \deg(u) = 2|V' \setminus \{u\}| + 1 - \deg(u) \leq 2|V' \setminus \{u\}| - 2$. It is clearly impossible for such a graph to contain more than one circuit, therefore every vertex in $V(C_2 \setminus C_1)$, and hence every vertex in G', lies in at least one circuit.

We may label by C_2' a circuit within G' that contains the unchanged Laman subgraph $C_2 \setminus C_1$. C_1' and C_2' are not necessarily the minimal pair of circuits within (G', c'), as illustrated in Example 7.2.14 (Figure 7.13), however by Proposition 7.2.2a, any other circuit within (G', c') is contained within the union $C_1' \cup C_2'$.

We formally define the "chosen 1-extension" as follows, for an edge $f \in E_1$. An analogous move may be defined for an edge $f' \in E_2$.

Definition 7.2.11. Let (G, c) be a 2-edge-coloured Laman-plus-two graph, containing at least two circuits C_1 and C_2. By relabelling the circuits and colours if necessary, there exists an edge $f \in E_1 \cap E(C_1)$ where $|E_1| \geq 2$. We apply a *"chosen 1-extension"* to (G, c) on the edge $f = \{v_1, v_2\}$ by creating a new vertex x, and extending G into G', where $V' = V \cup \{x\}$ and $E' = E \setminus \Big\{\{v_1, v_2\}\Big\} \cup \Big\{\{x, v_1\}, \{x, v_2\}, \{x, v_3\}\Big\}$ for some distinct vertex $v_3 \in V \setminus \{v_1, v_2\}$. We define a k-colouring c' for the extended G' where $c'(e) := c(e)$ for all edges $e \in E \cap E'$, $c'(\{x, v_1\}), c'(\{x, v_2\}) \neq 1$, and $c'(\{x, v_3\}) \in \{0, 1, \ldots, k\}$ may be chosen.

We require that every vertex of G' be within at least one circuit, and require that the coloured graph (G', c') has $E_1' \cap E(C_1') \neq \emptyset$.

Remark 7.2.12. A "chosen 1-extension" is applied to an edge $f \in E_1 \cap E(C_1)$, and from Remark 7.2.9, the original graph (G, c) has $E_2 \cap E(C_2) \neq \emptyset$. Since C_2' is defined to have $E(C_2') \supset E(C_2) \setminus \{f\}$ (Remark 7.2.10), the extended graph (G', c') has $E_2' \cap E(C_2') \neq \emptyset$.

Remark 7.2.13. Since $v_1, v_2 \in V(C_1)$, we shall have $x, v_1, v_2, v_3 \in V(C_1')$, and so the three edges $\{x, v_1\}, \{x, v_2\}, \{x, v_3\}$ will be contained within $E(C_1')$ in the extended graph G'. For this to be a valid "chosen 1-extension", we require only that $E_1 \cap E(C_1') \neq \emptyset$, so if there is already another edge in $E_1 \cap E(C_1)$, we may choose for all three edges adjacent to x to be added to either E_0 or E_2. We may also choose $c'(\{x, v_3\}) = 1$, and hence guarantee that there will be at least one edge from E_1 within the extended circuit C_1'.

The following example illustrates how the circuits of the extended graph may be much larger than those within the original graph. This depends on how many circuits

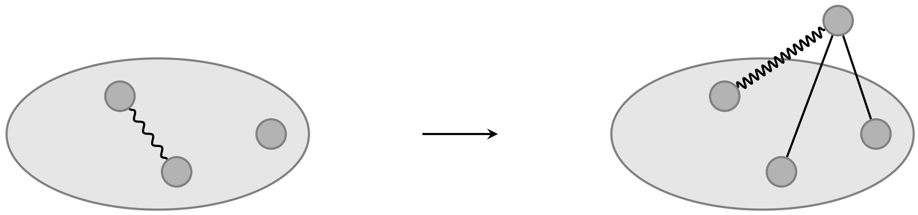


Figure 7.11 An example of a "chosen 1-extension".

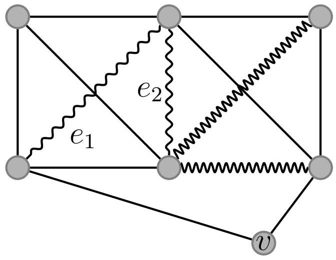
the edge that is extended lies within, and how any vertices outside the circuits of the original graph are connected to the circuits.

Example 7.2.14. Let (G, c) be a coloured copy of the graph B_6, with a 0-extension applied to create an additional vertex $v \in V(X)$. The minimal circuits of (G, c) are the two copies of K_4, of which the left-hand copy will be considered C_1 and the right-hand copy will be C_2. We apply a "chosen 1-extension" to the edge $e_1 \in E_1 \cap E(C_1)$ by creating a new vertex x, and requiring that there be at least one edge from E_1' within the circuit C_1' of the extended graph G'. We ensure this by defining $c'(\{x, v\}) = 1$, though the edge $e_2 \in E_1 \cap E(C_1) \cap E(C_2)$ would satisfy this condition. Figure 7.12 shows the graph (G, c) and the extended graph (G', c'), together with the circuits C_1' and C_2' of (G', c').

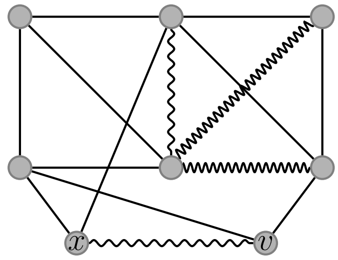
As $e_1 \in E(C_1) \setminus E(C_2)$, the circuit C_2 remains unchanged in the extended graph (G', c'). We may instead apply the "chosen 1-extension" to the edge $e_2 \in E_1 \cap E(C_1) \cap E(C_2)$, to create an extended graph (G'', c''). This is illustrated in Figure 7.13. The circuits C_1'' and C_2'', with $V(C_i) \subset V(C_i'')$, are illustrated, along with the minimal circuit within (G'', c''), C_3''.

Lemma 7.2.15. The 2-coloured graph (G', c'), obtained by applying a "chosen 1-extension" to an isostatic 2-coloured graph (G, c), will be an isostatic 2-coloured graph.

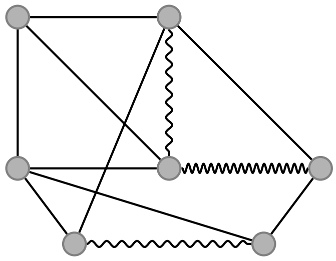
Proof. Let (G, c) be an isostatic 2-coloured graph, so by Theorem 4.1.11 (G, c) is a Laman-plus-two graph with a 2-redundant rainbow pair of edges. As discussed in



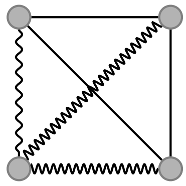
a The graph (G, c).



b The extended graph (G', c').

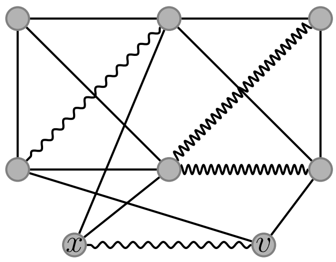


c The circuit C'_1.

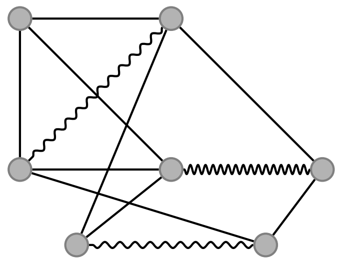


d The circuit $C'_2 = C_2$.

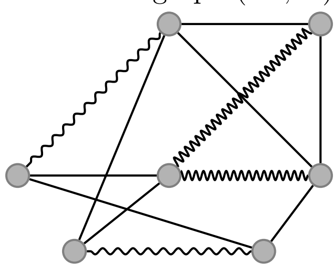
Figure 7.12 Example 7.2.14: the graph (G, c), and the graph (G', c') obtained by applying a "chosen 1-extension" on $e_1 \in E(C_1)$, along with two circuits of (G', c').



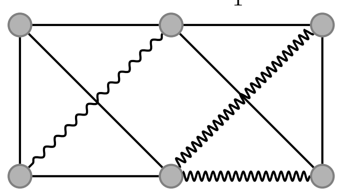
a The extended graph (G'', c'').



b The circuit C''_1.



c The circuit C''_2.



d The circuit C''_3.

Figure 7.13 Example 7.2.14: the extended graph (G'', c'') obtained by applying a "chosen 1-extension" to $e_2 \in E(C_1) \cap E(C_2)$, along with three circuits within (G'', c'').

Remark 7.2.9, we may suppose that one such 2-redundant rainbow pair is e_1, e_2 with $e_1 \in E_1 \cap E(C_1)$ and $e_2 \in E_2 \cap E(C_2)$.

Suppose first that (G', c') is obtained by applying a "chosen 1-extension" to an edge $f = \{v_1, v_2\} \in E_1 \cap E(C_1 \cap C_2)$. The circuits C_1' and C_2' both contain x and the three new edges $\{x, v_1\}, \{x, v_2\}, \{x, v_3\}$ created by the "chosen 1-extension" (see Remark 7.2.10), and hence $x, v_1, v_2, v_3 \in V(C_1' \cap C_2')$. By definition, $E_1' \cap E(C_1') \neq \emptyset$, and since $E_2 \cap E(C_2) \neq \emptyset$ and $E(C_2') \supseteq E(C_2) \setminus \{f\} \cup \Big\{\{x, v_1\}, \{x, v_2\}, \{x, v_3\}\Big\}$ there is at least one edge in $E_2' \cap E(C_2')$.

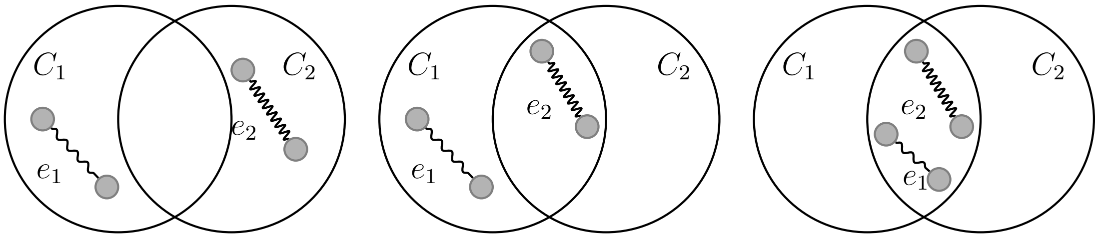
Suppose that no rainbow pair of edges in (G', c') is 2-redundant, and let $e_1' \in E_1' \cap E(C_1')$, $e_2' \in E_2 \cap E(C_2')$ be one such rainbow pair. Since e_1', e_2' is not 2-redundant, there is a circuit C_3' within (G', c') that does not contain either e_1' or e_2'. There is at least one coloured edge within C_3', say $f_1' \in E_1' \cap E(C_3')$, which forms another rainbow pair f_1', e_2'. By assumption there is a circuit C_4' that does not contain either $f_1' \in E_1'$ and $e_2' \in E_2'$. However this implies that $e_2' \notin E(C_3' \cup C_4')$, and hence the circuit C_2' is not contained within $C_3' \cup C_4'$. This contradicts Proposition 7.2.2a, since every circuit is contained within the union of any other pair of circuits, and hence (G', c') contains a 2-redundant rainbow pair of edges.

Suppose instead that (G', c') is obtained by applying a "chosen 1-extension" to an edge $f \in E_1 \cap E(C_1 \setminus C_2)$. The circuit C_2 remains unchanged in (G', c'), and is labelled by C_2', while C_1' is the circuit of (G', c') such that $V(C_1') \supseteq V(C_1) \cup \{x\}$ and $E(C_1') \supseteq E(C_1) \setminus \{f\} \cup \Big\{\{x, v_1\}, \{x, v_2\}, \{x, v_3\}\Big\}$. By definition, the circuit C_1' contains at least one edge in E_1', and there is at least one edge in $E_2' \cap E(C_2')$ since $C_2' = C_2$.

By a similar argument to that given above, the extended graph (G', c') contains a 2-redundant rainbow pair of edges $e_1' \in E_1'$, $e_2' \in E_2'$.

Since e_1', e_2' is a 2-redundant pair within (G', c'), the graph $G' \setminus \{e_1', e_2'\}$ contains no circuits, and hence $|D| \leq 2|V(D)| - 3$ for all subgraphs $(V(D), D)$ of $G' \setminus \{e_1', e_2'\}$.

Since $|E'| - 2 = 2|V'| - 3$, $G' \setminus \{e'_1, e'_2\}$ is a Laman graph, and hence (G', c') is a Laman-plus-two graph containing a 2-redundant rainbow pair. Therefore the extended graph (G', c') is isostatic by Theorem 4.1.11. □



a Case A: both outside the intersection of C_1 and C_2.
b Case B: one outside the intersection of the circuits and one within $C_1 \cap C_2$.
c Case C: both within the intersection of C_1 and C_2.

Figure 7.14 Possible placements for a 2-redundant rainbow pair in two intersecting circuits.

Remark 7.2.16. We note that the "chosen 1-extension" could be defined more generally for any edge such that the extended graph retains a 2-redundant rainbow pair of edges, however we only require the move to be applied to an edge within a circuit. (See Remark 7.2.24.)

Along with these types of 0-extension and 1-extension, we shall also require a move to replace two edges with a vertex of degree 4. This style of move may be referred to as a *2-extension* (Definition 2.8.12), though we shall use the more intuitive name of *X-replacement* in the style of Tay and Whiteley [TW85], to make clear that the four end vertices of the pair of edges are distinct.

Definition 7.2.17. Let (G, c) be an isostatic 2-coloured graph, which contains two disjoint circuits C_1 and C_2. A *coloured X-replacement* is applied on a pair of edges $\{v_1, v_2\}, \{v_3, v_4\} \in E \setminus \left\{E(C_1) \cup E(C_2)\right\}$ with distinct vertices v_1, v_2, v_3, v_4. The edges may be either uncoloured or coloured with either colour. The pair of edges is removed and replaced by a new vertex x of degree 4, with new edges $\{x, v_1\}, \{x, v_2\}, \{x, v_3\}, \{x, v_4\}$. Each new edge may be either uncoloured, or coloured arbitrarily.

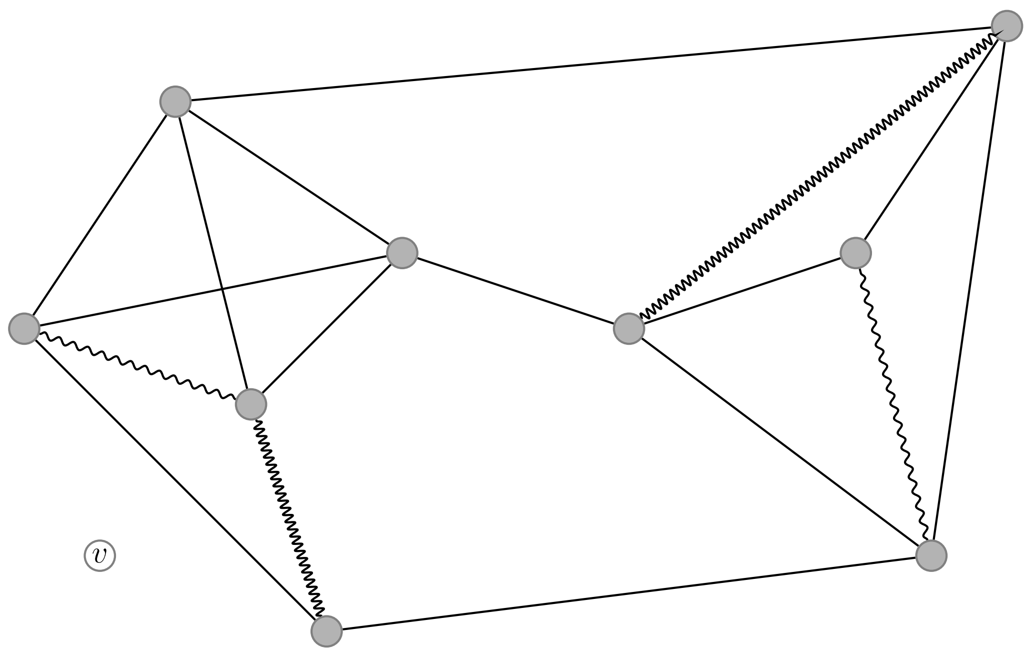
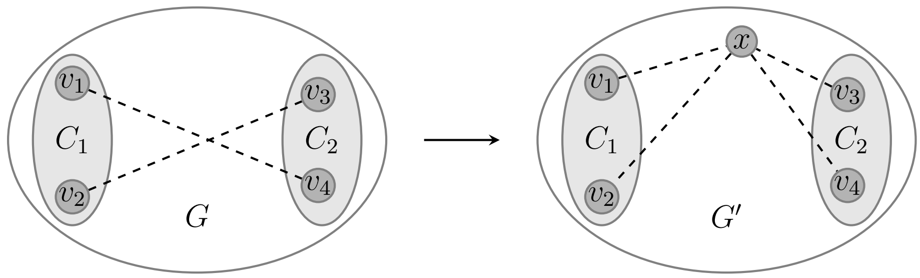
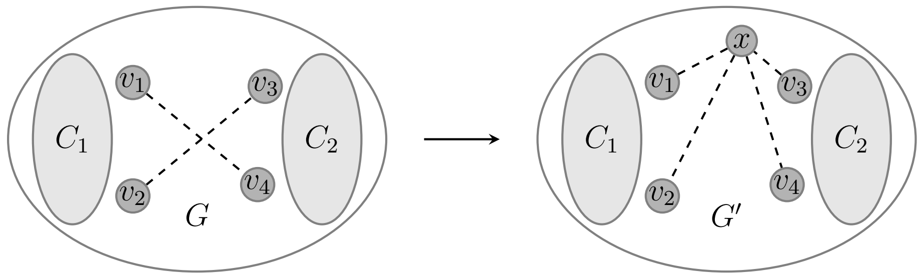


Figure 7.15 Applying the reverse to a "chosen 1-extension" to the graph in Figure 7.10 at v results in a smaller isostatic graph. Figure 7.10 may similarly be viewed as the result of a "chosen 1-extension" applied to this graph on the coloured edge within the left-hand circuit.



a An X-replacement applied to a pair of edges that run between the circuits of (G, c).



b An X-replacement applied to a pair of edges $\{v_1, v_2\}, \{v_3, v_4\}$ with all four vertices outside the circuits of (G, c).

Figure 7.16 Two potential types of X-replacement.

As we apply this move only to (G, c) where the circuits are disjoint, and only to edges that lie outside both circuits, the colours of the edges involved in the X-replacement may be considered as arbitrary due to the following result.

Lemma 7.2.18. Let (G, c) be a 2-edge-coloured graph, and let (G, c') be another colouring of the same graph, such that the colourings c and c' differ only on a single edge outside the circuits of G. For any regular $p \in \mathbb{R}^{2d}$, (G, c, p) is infinitesmally rigid if and only if (G, c', p) is infinitesimally rigid.

Proof. Let $e_1 \in E \setminus \Big\{E(C_1) \cup E(C_2)\Big\}$ and suppose that the colourings c and c' differ only on e_1, so $c'(e_1) \neq c(e_1)$, and $c'(e) = c(e)$ for all $e \in E \setminus \{e_1\}$.

Suppose that (G, c, p) is infinitesimally rigid and (G, c', p) is infinitesimally flexible. We therefore have a non-zero equilibrium stress $\boldsymbol{\omega}$ of $R(G, c', p)$ such that $\boldsymbol{\omega}^\top R(G, c', p) = \mathbf{0}$ and $\boldsymbol{\omega}^\top R(G, c, p) \neq \mathbf{0}$. Since the only difference between $R(G, c, p)$ and $R(G, c', p)$ is on the row corresponding to e_1, $\omega_{e_1} \neq 0$.

By Lemma 4.1.3, any equilibrium stress $\boldsymbol{\rho}$ of the standard rigidity matrix $R(G, p)$ is zero on all bridges of G. Since $\boldsymbol{\omega}$ is an equilibrium stress of $R(G, c, p)$, $\boldsymbol{\omega}$ is also an equilibrium stress of the standard rigidity matrix $R(G, p)$, and hence $\omega_{e_1} = 0$ since e_1 lies outside the circuits of (G, c). Hence (G, c', p) has no equilibrium stress with $\omega_{e_1} \neq 0$, and so (G, c', p) is also infinitesimally rigid.

A symmetrical argument applies in the converse direction, and so the infinitesimal rigidity of (G, c, p) and (G, c', p) is equivalent. □

Lemma 7.2.19. Applying a coloured X-replacement on edges outside the circuits of an isostatic 2-coloured graph (G, c) will result in an isostatic 2-coloured graph (G', c').

Proof. Let (G, c) be a 2-coloured graph that is generically isostatic in 2 dimensions, so (G, c) satisfies the conditions of Theorem 7.2.4. Since (G, c) is an appropriately coloured Laman-plus-two graph, we may remove a 2-redundant pair of edges e_1, e_2

from the circuits of G to create an uncoloured Laman graph $G^* = G \setminus \{e_1, e_2\}$. By Theorem 2.8.8, G^* will be generically isostatic in 2 dimensions as an uncoloured graph, containing rigid blocks C_1^* and C_2^* that correspond to the circuits C_1 and C_2 within the Laman-plus-two graph G.

Lemma 2.8.13 states that applying a standard X-replacement to an isostatic uncoloured graph H will result in a larger isostatic uncoloured graph H'. Let $\{v_1, v_2\}, \{v_3, v_4\}$ be a pair of edges in G^* that lie outside the rigid blocks C_1^* and C_2^*, and so also lie outside the circuits C_1 and C_2 within G, with distinct vertices v_1, v_2, v_3, v_4. We may apply a standard X-replacement to create a larger graph $G^{*\prime}$, where $V(G^{*\prime}) = V(G) \cup \{x\}$ and $E(G^{*\prime}) = E(G^*) \setminus \Big\{\{v_1, v_2\}, \{v_3, v_4\}\Big\} \cup \Big\{\{x, v_1\}, \{x, v_2\}, \{x, v_3\}, \{x, v_4\}\Big\}$. By Lemma 2.8.13 $G^{*\prime}$ will also be generically isostatic, and will be a larger Laman graph.

As the rigid blocks C_1^* and C_2^* are unchanged by the X-replacement, we may replace the edges e_1, e_2 to create a larger G' containing unchanged circuits C_1 and C_2, where $V(G') = V(G) \cup \{x\}$ and $E(G') \setminus \Big\{\{v_1, v_2\}, \{v_3, v_4\}\Big\} \cup \Big\{\{x, v_1\}, \{x, v_2\}, \{x, v_3\}, \{x, v_4\}\Big\}$. As $G^{*\prime}$ is a Laman graph, G' is clearly a Laman-plus-two graph.

Since the edges $\{v_1, v_2\}, \{v_3, v_4\}$ are outside the circuits of G, there can be at most two vertices out of v_1, v_2, v_3, v_4 within each circuit. The new vertex x will therefore be outside the circuits of the extended graph G', and so the new edges $\{x, v_1\}, \{x, v_2\}, \{x, v_3\}, \{x, v_4\}$ will be outside the circuits of the Laman-plus-two graph G'. We wish to define a colouring c' of G', and by Lemma 7.2.18 we may define $c'(\{x, v_i\})$ arbitrarily for $i = 1, 2, 3, 4$.

We may extend this to define a colouring c' for the graph G' with $c'(e) = c(e)$ for all edges $e \in E(G) \cap E(G')$. This gives a 2-coloured graph (G', c') with coloured circuits $(C_1', c') = (C_1, c)$ and $(C_2', c') = (C_2, c)$. As (G, c) satisfied the conditions of

Theorem 7.2.4 to be generically isostatic, the extended graph (G', c') will also satisfy the same conditions, and hence be generically isostatic in 2 dimensions. □

Henneberg-type construction

We may now give a construction for all generically isostatic 2-coordinated graphs in 2 dimensions.

Theorem 7.2.20. A 2-coloured graph (G, c) is a generically isostatic graph if and only if it may be constructed from a 2-coloured base graph $(B, c) \in \mathcal{B}$ (as illustrated in Figure 7.6) by a sequence of coloured 0-extensions, coloured 1-extensions, coloured X-replacements applied outside disjoint circuits, and "chosen 1-extensions".

Since the proof of this result is lengthy, we first give an outline of the proof steps required.

It is straightforward to prove that a graph constructed in this way is generically isostatic, first by verifying that any base graph $(B, c) \in \mathcal{B}$ is isostatic (Lemma 7.2.6), and verifying that the coloured extension moves listed preserve isostaticity (Lemma 5.1.2, Lemma 5.2.2, Lemma 7.2.19 and Lemma 7.2.15). We note that the proofs of Lemmas 7.2.6 and 7.2.19 rely on Theorem 7.2.4, which in turn relies on Theorem 4.1.11, while the proof of Lemma 7.2.15 relies directly on Theorem 4.1.11.

We shall prove the converse using induction on the number of vertices in (G, c), and checking that any isostatic graph may be reduced to a smaller isostatic graph. We apply the characterisation from Theorem 7.2.4 that a 2-coloured graph is generically isostatic if and only if G is Laman-plus-two, each circuit contains at least one coloured edge, each colour class contains an edge within a circuit and certain coloured sparsity conditions (Equations (7.5) and (7.6)) are satisfied.

The minimum degree within a Laman-plus-two graph is at least 2, and any vertex of degree 2 will lie outside both circuits, and so may be straightforwardly reduced. We next confirm that any vertex of degree 3 outside both circuits may be reduced. We finally consider the case in which the circuits are disjoint, and every vertex outside the circuits has degree 4 (see Proposition 7.2.3) and confirm that vertices of degree 4 outside the circuits may be reduced. In these cases, we verify that the reduced graph contains circuits that still satisfy the conditions of of Theorem 7.2.4, and so the reduced graph is isostatic.

We continue by showing that when every vertex is within a circuit, if a circuit C has $|V(C)| \geq 5$, there is a vertex $v \in V(C)$ with $\deg_G(v) = \deg_C(v) = 3$ that may be reduced to produce a smaller graph that is isostatic by Theorem 7.2.4. We do this in two cases, first where (G, c) contains exactly two edge-disjoint circuits, and finally when the two minimal circuits intersect on some non-empty collection of edges.

Proof of Theorem 7.2.20. Lemma 5.1.2 and Lemma 5.2.2 give that the coloured 0-extensions and 1-extensions will preserve isostaticity for any k-coloured graph in d dimensions. From Lemma 7.2.15, we see that the "chosen 1-extension" will preserve isostaticity of a 2-coloured graph in 2 dimensions, and a coloured X-replacement applied to edges outside the circuits of a 2-coloured graph (G, c) will preserve isostaticity by Lemma 7.2.19. From Lemma 7.2.6 the base graphs are isostatic, therefore any 2-coloured graph constructed from a base graph using a sequence of these moves will be isostatic.

By Theorem 7.2.4, a 2-coloured graph (G, c) is generically isostatic in 2 dimensions if and only if the graph G is a Laman-plus-two graph, the 2-edge-coloured graph (G, c) has at least one edge from each of E_1 and E_2 within a circuit, and the following conditions (Equations (7.5) and (7.6)) are satisfied:

$$|D| \leq 2|V(D)| - 3 \quad \forall D \subseteq E_0, \tag{7.5}$$

$$|D| \leq 2|V(D)| - 2 \quad \forall D \subseteq E_0 \cup E_1, D \subseteq E_0 \cup E_2. \tag{7.6}$$

We shall apply induction on the number of vertices to prove that any 2-edge-coloured Laman-plus-two graph (G, c), which has at least one edge of each colour within a circuit and satisfies Equations (7.5) and (7.6), may be constructed from a base graph $(B, c) \in \mathcal{B}$ by a sequence of coloured 0-extensions, 1-extensions, "chosen 1-extensions", and X-replacements applied outside disjoint circuits.

Let (G, c) be a 2-edge-coloured Laman-plus-two graph that satisfies the conditions of Theorem 7.2.4. We begin by considering the average degree within G:

$$\frac{1}{|V|}\sum_{v\in V}\deg(v) = \frac{2|E|}{|V|} = \frac{4|V|-2}{|V|} = 4 - \frac{2}{|V|}.$$

Since the average degree is strictly less than 4, there will be vertices of degree 2 or 3.

Throughout this proof we shall refer to the two smallest circuits of (G, c) as C_1 and C_2, and denote by X the subgraph generated by $V(X) = V \setminus \{V(C_1) \cup V(C_2)\}$. The vertices of X will be outside all circuits of (G, c) by Proposition 7.2.2a.

Case 1: Vertices outside circuits

Suppose first that $V(X) \neq \emptyset$. The minimum degree within (G, c) is at least 2, and any vertices of degree 2 will lie outside the circuits of G, as the minimum degree within a circuit is 3. As the 0-extension permits the addition of a vertex of degree 2 with any combination of coloured or uncoloured edges, any vertex of degree 2 may be considered as the result of a 0-extension, and so the reverse move may be applied. The circuits of (G, c) will remain unchanged in the reduced graph (G', c'), so (G', c') will be isostatic.

We may now suppose that the minimum degree within (G, c) is 3, and that $V(X)$ remains non-empty. Let $x \in V(X)$ be a vertex of degree 3. By Proposition 7.2.2b, x may have at most two neighbours within any circuit of (G, c).

Let $e_1 \in E(C_1 \setminus C_2)$ and $e_2 \in E(C_2 \setminus C_1)$ be a pair of edges within the circuits of (G, c). Since both edges lie outside the intersection of the circuits C_1 and C_2, it is straightforward to confirm that they are a 2-redundant pair. The graph $G^* = G - \{e_1, e_2\}$ will be a Laman graph, with each circuit of G corresponding to a rigid block within G^*. We may apply the structure of a 1-reduction at x, which will create a reduced Laman graph $(G^*)'$ with a new edge between some pair of vertices u_1, u_2 that were neighbours of x in G^*. The new edge will be added outside the rigid blocks of G^*, so the edges e_1, e_2 may be replaced to create a reduced graph G' with identical circuits to G. We may define a colouring c' for the Laman-plus-two graph G' with $c'(e) = c(e)$ for any edge $e \in E(G) \cap E(G')$, and $c'(\{u_1, u_2\})$ induced by $c(\{x, u_1\})$ and $c(\{x, u_2\})$, which were two of the edges removed by the 1-reduction. The reduced graph (G', c') will be isostatic by Theorem 7.2.4, since the circuits remain unchanged.

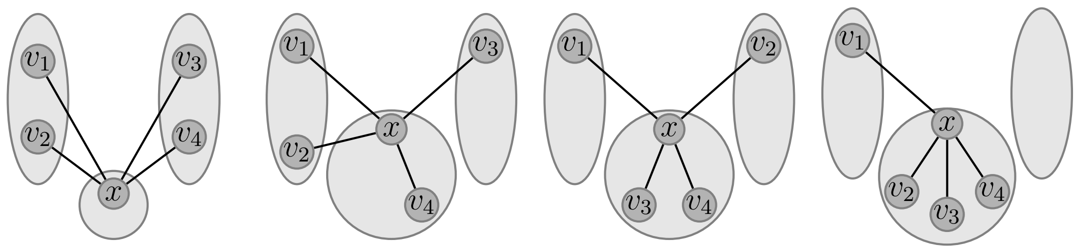
We finally suppose that the minimum degree within X is 4. By Proposition 7.2.3a the circuits C_1 and C_2 are disjoint, by Proposition 7.2.3c every vertex in $V(X)$ has degree 4, and by Proposition 7.2.3b X is either a single vertex or a Laman graph. Note that the minimum degree within (G, c) remains 3, so there are at least two vertices of degree 3 within $V(C_1) \cup V(C_2)$.

Case 1A: Vertices of degree 4

Let $x \in V(X)$ be a vertex of degree 4, with $N(x) = \{v_1, v_2, v_3, v_4\}$. We may partition the neighbours of x into those in $V(C_1)$, those in $V(C_2)$, and those that also lie in $V(X)$. We shall denote the "type" of the vertex x by $\tau(x) := (|N(x) \cap V(C_1)|, |N(x) \cap V(C_2)|, |N(x) \cap V(X)|) = (q_1, q_2, q_X)$, where $q_1 + q_2 + q_X = 4$. By Proposition 7.2.2b, any vertex in $V(X)$ may have at most two neighbours within a circuit, so $q_1, q_2 \leq 2$. We shall only need to consider those cases where $q_1 \geq q_2$, as a relabelling of the argument for $\tau(x) = (a, b, c)$ will apply for any vertex y with $\tau(y) = (b, a, c)$.

When $|V(X)| \geq 2$, there are exactly three edges between vertices in $V(X)$ and vertices in each circuit by Proposition 7.2.3b. We may therefore assume that the vertex we wish to reduce has at least one neighbour within a circuit, and disregard the case where $\tau(x) = (0, 0, 4)$. Since there are exactly three edges from each circuit to vertices in X, a vertex of type (2,0,2) implies the existence of a vertex of type (1,1,2) or (1,2,1), which we may reduce instead, and so we may disregard any vertices $x \in V(X)$ with $\tau(x) = (2, 0, 2)$.

In the case where X is a single vertex x of degree 4, it is clear that $\tau(x) = (2, 2, 0)$. When $|V(X)| \geq 2$, the cases that remain to be considered are $\tau(x) = (2, 1, 1), (1, 1, 2), (1, 0, 3)$. These are illustrated in Figure 7.17.



a $\tau(x) = (2, 2, 0)$. **b** $\tau(x) = (2, 1, 1)$. **c** $\tau(x) = (1, 1, 2)$. **d** $\tau(x) = (1, 0, 3)$.

Figure 7.17 The four types of degree 4 vertex to be considered.

Given a vertex $x \in V(X)$ with $\deg_G(x) = 4$, we wish to apply the reverse of an X-replacement at x. This requires removing the vertex x and its adjacent edges, and creating two new edges $\{u_1, u_2\}, \{u_3, u_4\}$ for distinct vertices $u_1, u_2, u_3, u_4 \in N(x)$. We denote the graph $G - x$ by G^*, and the reduced graph with the new edges added by G'.

The four neighbours of x will have three potential pairs of edges that could be created: $\Big\{\{v_1, v_2\}, \{v_3, v_4\}\Big\}$, $\Big\{\{v_1, v_3\}, \{v_2, v_4\}\Big\}$, $\Big\{\{v_1, v_4\}, \{v_2, v_3\}\Big\}$. For each type of vertex x, we shall verify that at least one potential pair of edges $\{u_1, u_2\}, \{u_3, u_4\}$ may be added to create a smaller Laman-plus-two graph G', where $V(G') = V \setminus \{x\}$ and $E(G') = E(G) \setminus \Big\{\{x, v_1\}, \{x, v_2\}, \{x, v_3\}, \{x, v_4\}\Big\} \cup \Big\{\{u_1, u_2\}, \{u_3, u_4\}\Big\}$. We may refer to the potential pair of edges $\{u_1, u_2\}, \{u_3, u_4\}$ as being *valid* when such a graph G' is Laman-plus-two.

Note that we may define a colouring c' for G' by $c'(e) = c(e)$ for any edge $e \in E(G') \cap E(G)$. As the new edges $\{u_1, u_2\}, \{u_3, u_4\}$ lie outside both circuits, they may be coloured arbitrarily by Lemma 7.2.18. We shall therefore have a 2-edge-coloured graph (G', c') with $|V'| = |V| - 1$ that satisfies the conditions of Theorem 7.2.4, and is hence also isostatic.

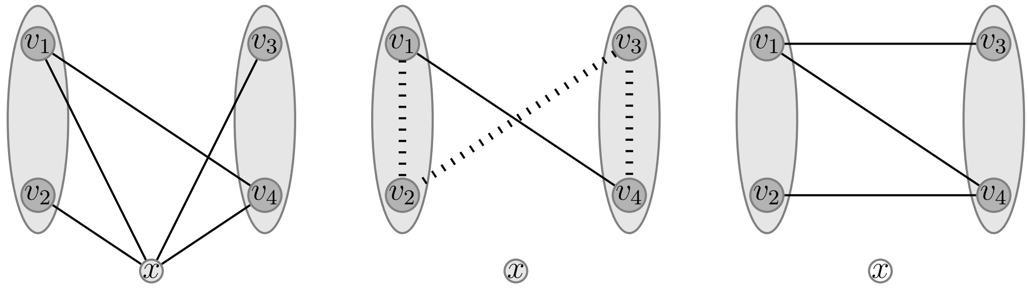
Case 1A(i): $\boldsymbol{\tau}(\mathbf{x}) = (\mathbf{2}, \mathbf{2}, \mathbf{0})$

When $V(X) = \{x\}$ and the minimum degree within X is 4, we have that $\tau(x) = (2, 2, 0)$ and there is a single edge $\{w_1, w_2\}$, where $w_1 \in V(C_1)$ and $w_2 \in V(C_2)$. This is illustrated in Figure 7.17a and Figure 7.18a.

Suppose that $v_1, v_2 \in V(C_1)$ and $v_3, v_4 \in V(C_2)$. We note that the potential edge pair $\{v_1, v_2\}, \{v_3, v_4\}$ clearly cannot be valid, as both edges would be added within the existing circuits of (G, c). There is exactly one edge with an end point in each circuit, which could prevent exactly one of the other edge pairs from being valid, and so at least one of the edge pairs $\{v_1, v_3\}, \{v_2, v_4\}$ and $\{v_1, v_4\}, \{v_2, v_3\}$ will be valid. The steps of such a reduction are shown in Figure 7.18, where existing edges are denoted by solid lines, and potential edges that cannot be part of a valid edge pair are denoted by angled dashes.

Case 1A(ii): $\boldsymbol{\tau}(\mathbf{x}) = (\mathbf{2}, \mathbf{1}, \mathbf{1})$

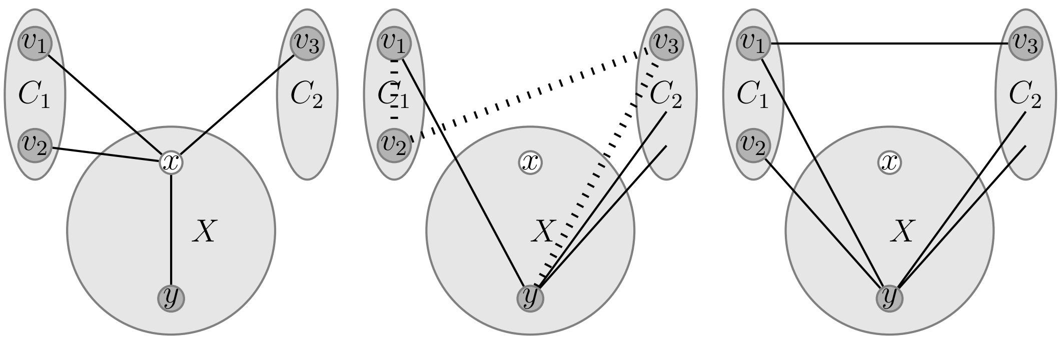
Suppose next that $|V(X)| \geq 2$, and $x \in V(X)$ has $\tau(x) = (2, 1, 1)$. As $\deg_X(x) = 1$ and X is a Laman graph by Proposition 7.2.3b, X will be a pair of vertices connected by a single edge, as that is the only Laman graph containing vertices of degree 1. Let $V(X) = \{x, y\}$, and so by a similar argument $\tau(y) = (1, 2, 1)$. This type of vertex is illustrated in Figure 7.17b and Figure 7.19a. Note that $N(x) = \{v_1, v_2, v_3, y\}$.



a The graph G. **b** The graph $G - x$, with edges that cannot be part of a valid edge pair marked. **c** A permitted X-replacement of x.

Figure 7.18 The steps of a reduction of a degree 4 vertex x, with $\tau(x) = (2, 2, 0)$.

As two neighbours of x lie in C_1, say $v_1, v_2 \in V(C_1)$, the potential edge pair $\{v_1, v_2\}, \{v_3, y\}$ will not be valid. There is exactly one other edge from X to C_1, say $\{y, w_1\}$ for some $w_1 \in V(C_1)$, which can only prevent one of the two other potential edge pairs if w_1 is also a neighbour of x. Without loss of generality, suppose that $w_1 = v_1$, and so the potential edge pair $\{v_1, y\}, \{v_2, v_3\}$ cannot be valid. The final potential edge pair, $\{v_1, v_3\}, \{v_2, y\}$ will however be valid, as there are no other edges from C_1 into X. This reduction is illustrated in Figure 7.19.



a The graph G. **b** The graph $G - x$, with edges that cannot be in a valid edge pair marked. **c** The reduced graph G'.

Figure 7.19 Reduction of a degree 4 vertex x with $\tau(x) = (2, 1, 1)$.

Case 1A(iii): $\boldsymbol{\tau}(\mathbf{x}) = (\mathbf{1}, \mathbf{1}, \mathbf{2})$

We may now suppose that $|V(X)| \geq 3$. Let $x \in V(X)$ with $N(x) = \{v_1, v_2, v_3, v_4\}$

and $\deg_X(x) = 2$, so the graph $G^* = G - x$ will have a Laman subgraph $X^* = X - x$. We wish to add a pair of edges to G^* to create a Laman-plus-two graph G' and since G^* contains a pair of circuits, we do not wish to create a new circuit by adding one of these edges within the Laman graph G^*. Preventing an edge from being added between the neighbours of x within X, say $v_3, v_4 \in V(X)$, corresponds to the edge pair $\{v_1, v_2\}, \{v_3, v_4\}$ not being valid, where $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$.

By Proposition 7.2.3b, there are exactly three edges from each circuit into X, and each set of three edges will have at least two distinct end points within X and within the circuit, as otherwise G would contain a cut vertex, which cannot occur in a Laman graph, and hence does not occur in a Laman-plus-two graph. Therefore v_3 and v_4 cannot both be neighbours of v_1, and cannot both be neighbours of v_2.

Without loss of generality, suppose first that v_3 is not adjacent to v_1, and v_4 is not adjacent to v_2, and let H_1 be a subgraph of G^* containing both v_1 and v_3. For H_1 to be a Laman subgraph with $|E(H_1)| \geq 2$, it must contain both edges in G^* that run from a vertex in C_1 to a vertex in X, along with a subgraph B_1 of C_1 and a subgraph B_X of X. Let $\{w_1, w_2\}, \{w_3, w_4\} \in E(G^*)$ be the pair of edges with $w_1, w_3 \in V(C_1)$, $w_2, w_4 \in V(X^*)$, and not both $w_1 = w_3, w_2 = w_4$. The subgraph B_1 must contain v_1, w_1, w_3, and $v_3, w_2, w_4 \in V(B_X)$, so both subgraphs contain at least two vertices. Since B_X lies in the Laman subgraph X^*, and B_1 is strictly contained in the circuit C_1 (as H_1 cannot be a Laman subgraph if it contains the circuit C_1), we have $|E(B)| \leq 2|V(B)| - 3$ for $B = B_1$ and $B = B_X$. This gives the following:

$$\begin{aligned}
|E(H_1)| &= |E(B_1)| + |E(B_X)| + 2 \\
&\leq 2|V(B_1)| - 3 + 2|V(B_X)| - 3 + 2 \\
&= 2|V(H_1)| - 4.
\end{aligned}$$

Hence the subgraph H_1 cannot be a Laman subgraph.

We note that this argument is equivalent to proving that a subgraph containing a pair of edges between two non-trivial subgraphs cannot be Laman.

By a similar argument, any subgraph H_2 of G^* containing both v_2 and v_4 cannot be a Laman subgraph of G^*, and so the potential edge pair $\{v_1, v_3\}, \{v_2, v_4\}$ is valid. Such a reduction is shown in Figure 7.20.

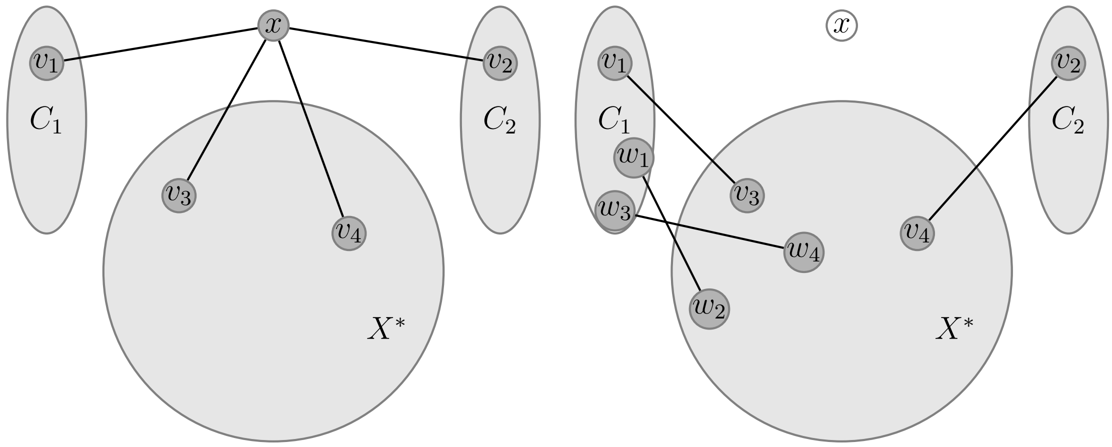


Figure 7.20 A straightforward reduction of a vertex x with $\tau(x) = (1, 1, 2)$.

Suppose next that v_3 is adjacent to v_1, which prevents the potential edge pair $\{v_1, v_3\}, \{v_2, v_4\}$ from being valid, whether or not the edge $\{v_2, v_4\}$ exists in G. The third edge from C_1 into X will be $\{w_1, w_2\}$ for some $w_2 \in V(X)$, $w_1 \in V(C_1) \setminus \{v_1\}$, since the three edges from a circuit to X must be adjacent to at least two vertices in $V(C_1)$. This situation is illustrated in Figure 7.21a. We shall consider a subgraph H_3 of G^* that contains v_3 and v_2, and a subgraph H_4 of G^* that contains both v_1 and v_4.

Since the third edge from C_1 to X is adjacent to a vertex $w_1 \neq v_1$, H_4 cannot be the single edge $\{v_1, v_4\}$, and $v_1, w_1, v_3, v_4 \in V(H_4)$ since H_4 must contain both edges from C_1 into X. By a similar argument to that used for H_1, no subgraph H_4 can be a Laman subgraph, and so the edge $\{v_1, v_4\}$ will not prevent a potential edge pair from being valid.

We consider now a subgraph H_3 with $v_2, v_3 \in V(H_3)$. G^* contains two edges from vertices in C_2 to vertices in X, which we shall denote by $\{w_5, w_6\}, \{w_7, w_8\}$ with

$w_5, w_7 \in V(C_2)$ and $w_6, w_8 \in V(X^*)$. These edges are shown in Figure 7.21b. There must be at least two vertices in C_2 adjacent to vertices in X, so at most one of w_5, w_7 may be the vertex v_2. Since $|V(X)| \geq 3$, no vertex in X may be adjacent to more than one vertex in either circuit, and hence v_3 may be the same as at most one of the vertices w_6, w_8.

We first consider the situation in which $\{v_2, v_3\} \notin E$, and then consider the case in which the edge $\{v_2, v_3\}$ exists separately. This is illustrated in Figure 7.22. We shall show that although no potential edge pair among the neighbours of x is valid, there is an alternative vertex within $V(X)$ at which a valid edge pair exists.

Suppose first that the edge $\{v_2, v_3\}$ does not exist, and hence there cannot be a Laman subgraph of G^* containing both v_2 and v_3 made up of the single edge $\{v_2, v_3\}$. Let H_3 be a subgraph of G^* with $v_2, v_3 \in V(H_3)$. We note that in order for H_3 to be a Laman subgraph, both edges from X^* into C_2 must be within H_3, and hence $v_3, w_6, w_8 \in V(X^*) \cap V(H_3)$ and $v_2, w_5, w_7 \in V(C_2) \cap V(H_3)$. This would be equivalent to H_3 containing a pair of edges whose removal results in two non-trivial subgraphs, and hence H_3 cannot be a Laman subgraph of G^*. This shows that the potential edge pair $\{v_2, v_3\}, \{v_1, v_4\}$ will be valid in this case, and the reduction shown in Figure 7.21c may be applied.

If instead the edges in G from C_2 into X are $\{v_2, x\}, \{v_2, v_3\}, \{w_7, w_8\}$ with $w_7 \neq v_2$ and $w_8 \neq x$, it is clear that there is no valid edge pair that can be added. (This situation is illustrated in Figure 7.22.) As X is a Laman subgraph and $|V(X)| \geq 3$, $\tau(v_3) = (2, 1, 1)$ would imply the existence of a degree 1 vertex within X, and hence $\tau(v_3) = (1, 1, 2)$. A similar argument shows that no potential edge pair between the neighbours of v_3 will be valid.

Since there are exactly three edges from each circuit into X, there will either be a vertex y with $\tau(y) = (1, 1, 2)$, or two vertices y_1, y_2 with $\tau(y_1) = (1, 0, 3)$,

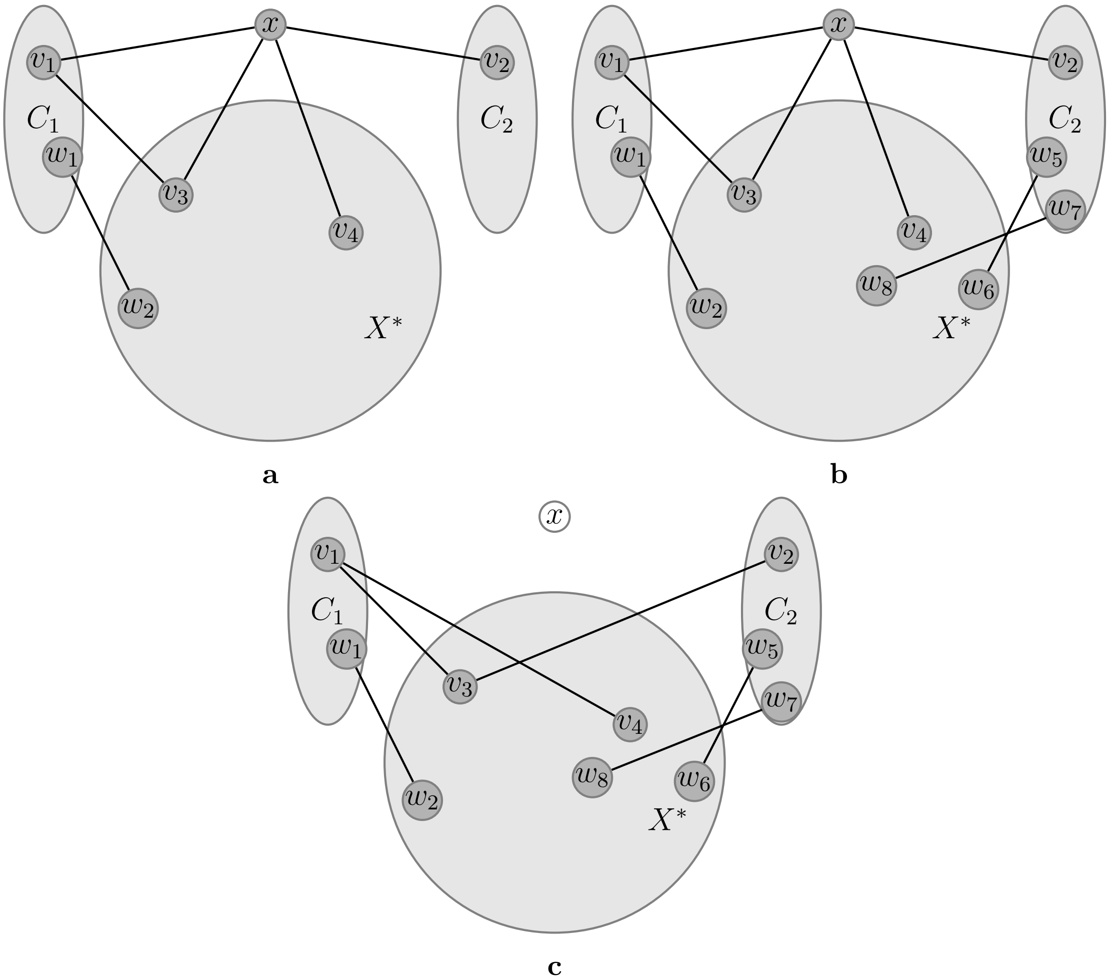


Figure 7.21 Reduction of a vertex x with $\tau(x) = (1, 1, 2)$.

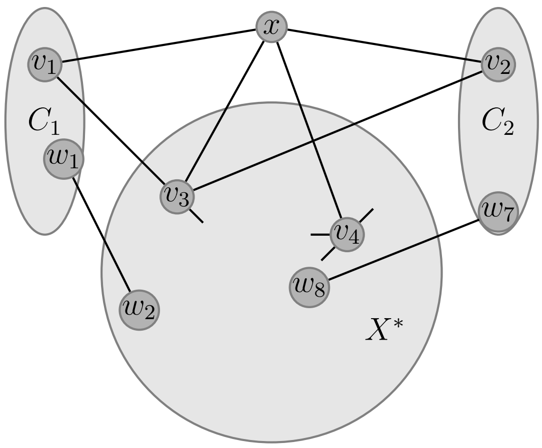


Figure 7.22 A vertex x with $\tau(x) = (1, 1, 2)$ where no valid reduction exists.

$\tau(y_2) = (0, 1, 3)$. A vertex of type (1,0,3) will be seen to be reducible below, so we must only consider y with $\tau(y) = (1, 1, 2)$.

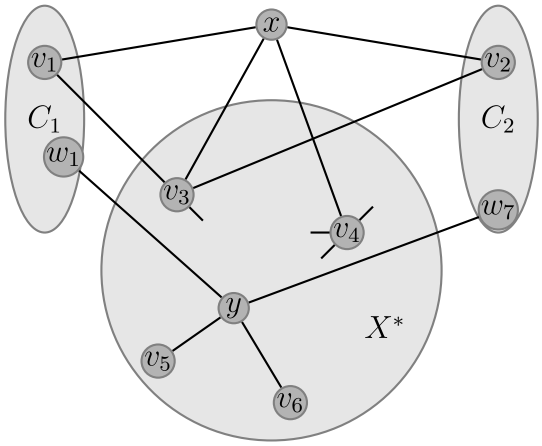


Figure 7.23 Another vertex y with $\tau(y) = (1, 1, 2)$, with a valid reduction at y indicated.

For a vertex $y \in V(X)$ to have $\tau(y) = (1, 1, 2)$, we have $y = w_8 = w_2$. This is shown in Figure 7.23, where $N(y) = \{w_1, w_7, v_5, v_6\}$ for some $v_5, v_6 \in V(X) \setminus \{x\}$. As all six edges from X into C_1 and C_2 are known, neither of v_5 and v_6 can be adjacent to either of w_1 or w_7. Thus two potential edge pairs amongst the neighbours of y will be valid: $\{w_1, v_5\}, \{w_7, v_6\}$ and $\{w_1, v_6\}, \{w_7, v_5\}$.

Case 1A(iv): $\boldsymbol{\tau}(\mathbf{x}) = (\mathbf{1}, \mathbf{0}, \mathbf{3})$

Let $x \in V(X)$ with $N(x) = \{v_1, v_2, v_3, v_4\}$, where $v_1 \in V(C_1)$, $v_2, v_3, v_4 \in V(X)$. As X is a Laman subgraph of G, by Proposition 7.2.3b, it is clear that not all three edges can exist between v_2, v_3, v_4, and $X^* = X - x$ will have $|E(X^*)| = 2|V(X^*)| - 4$ so X cannot contain three rigid blocks that each contain a distinct pair of neighbours of x.

Since there are exactly three edges between C_1 and X, and G must be 2-connected, at most one of v_2, v_3, v_4 may also be a neighbour of $v_1 \in V(C_1)$. Suppose first that none of the edges $\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}$ exist. At least one pair of neighbours of x within $V(X)$, say v_2, v_3, is not contained within a rigid block of X^*, so the potential edge pair $\{v_1, v_4\}, \{v_2, v_3\}$ will be valid.

Suppose instead that the edge $\{v_1, v_4\}$ already exists, and that there are Laman subgraphs of X^*, H_1 and H_2, that contain v_2, v_4 and v_3, v_4 respectively. (See Figure 7.25a.) If both Laman subgraphs are non-trivial, so neither of the edges $\{v_2, v_4\}, \{v_3, v_4\}$ exist, and $|V(H_1 \cap H_2)| \geq 2$, then the subgraph H of X, induced by $V(H_1) \cup V(H_2) \cup \{x\}$, would have $|E(H)| = 2|V(H)| - 2$. This contradicts the fact that X is Laman, hence $H_1 \cap H_2 = \{v_4\}$.

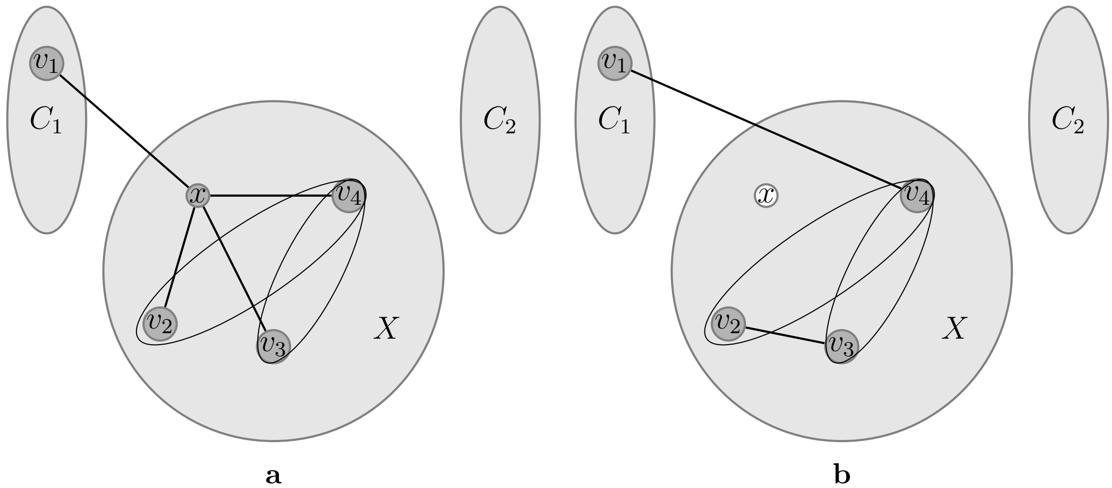


Figure 7.24 Reduction of a vertex x with $\tau(x) = (1, 0, 3)$.

Since $\{v_4, x\}, \{v_4, v_1\} \in E$, v_4 must be adjacent to exactly one edge in each of the Laman subgraphs H_1 and H_2. This implies that H_1 and H_2 are both single edges, $E(H_1) = \Big\{\{v_4, v_2\}\Big\}$ and $E(H_2) = \Big\{\{v_4, v_3\}\Big\}$. This situation is illustrated in Figure 7.25b, and it is straightforward to see that no potential edge pair between the neighbours of x will be valid (and by a similar argument, there will be no valid reduction at v_4). We shall instead find an alternative vertex at which to reduce.

There are exactly three edges from $V(C_1)$ to $V(X)$, two of which are $\{v_1, x\}$ and $\{v_1, v_4\}$. The third edge must be adjacent to some $w_1 \in V(C_1)$, $w_1 \neq v_1$, since G must be 2-connected. This edge may either be adjacent to another neighbour of x (v_2 or v_3), or adjacent to some other $w \in V(X) \setminus \{x, v_2, v_3, v_4\}$. These are illustrated in Figure 7.26.

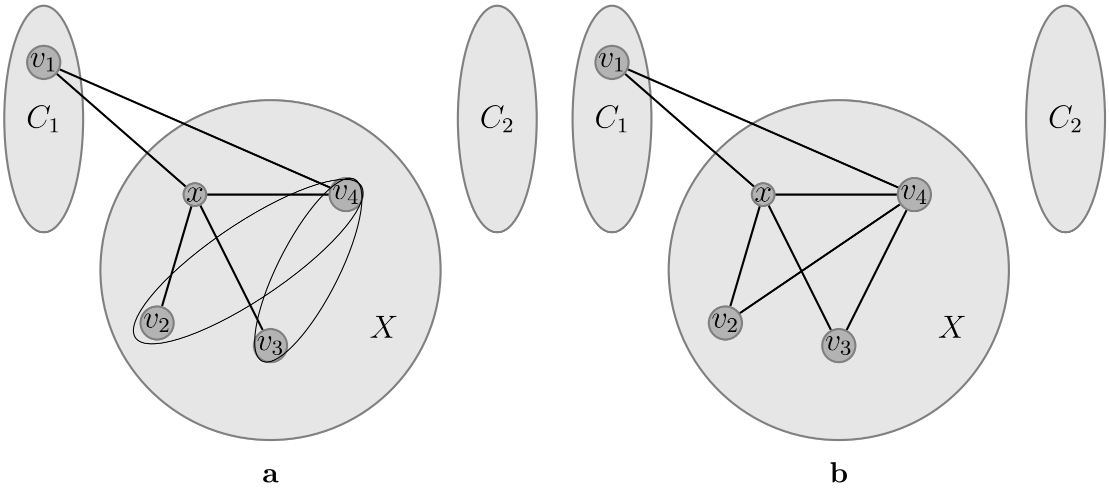


Figure 7.25 A vertex x with $\tau(x) = (1, 0, 3)$, and rigid blocks indicated.

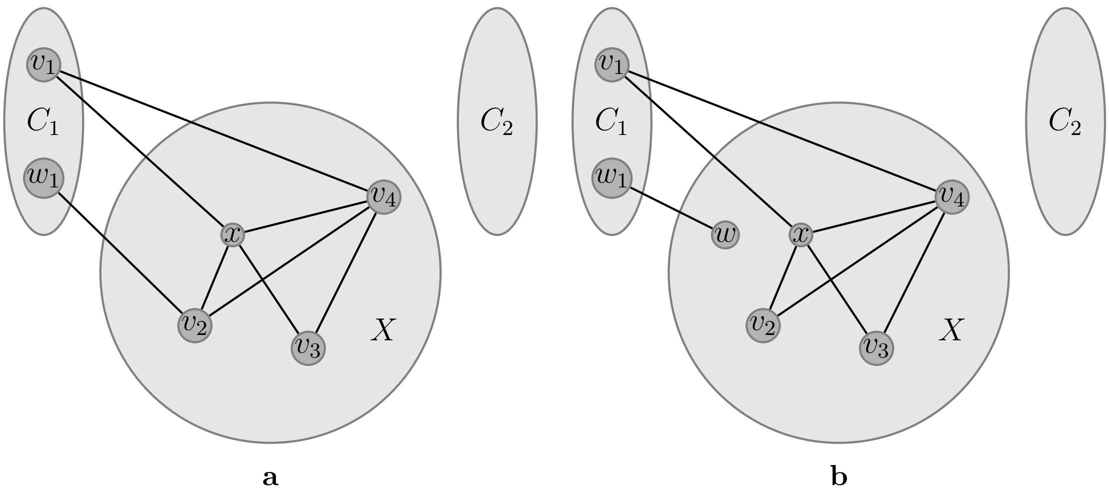
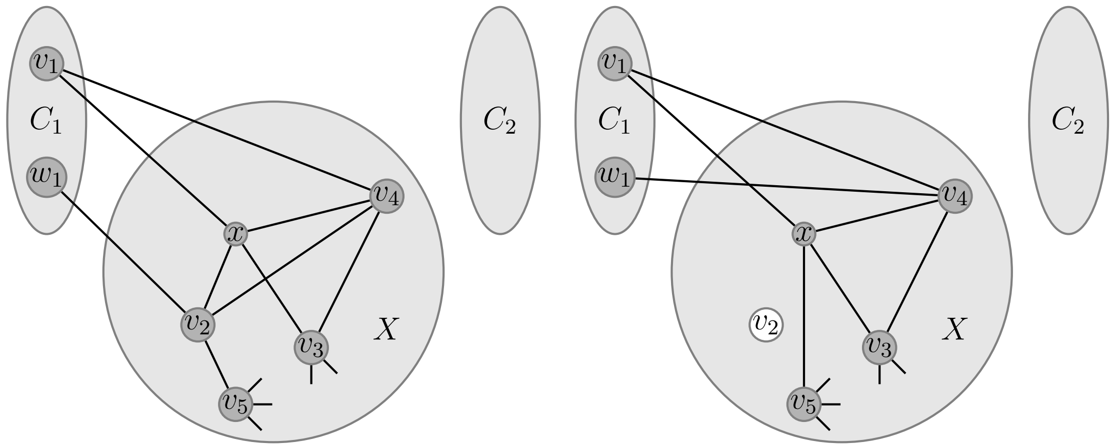


Figure 7.26 The two potential cases for the third edge from C_1 to X.

Suppose first that $\{w_1, v_2\} \in E$ (Figure 7.26a). If $\tau(v_2) = (1, 1, 2)$, we may apply a reduction at v_2 as already discussed, since neither neighbour of v_2 within X may be adjacent to w_1. If instead $\tau(v_2) = (1, 0, 3)$, let $v_5 \in N(v_2) \cap V(X)$, so $N(v_2) = \{w_1, x, v_4, v_5\}$. We consider the potential Laman subgraph H_3 within $X - v_2$ containing both x and v_5: there cannot be an edge $\{x, v_5\}$, so x must have degree at least 2 within H_3 for it to be a Laman subgraph. Within $X - v_2$, x has degree 2, so both edges $\{x, v_3\}, \{x, v_4\}$ must be within H_3. If such a subgraph containing x, v_5, v_4 was a Laman subgraph, X would contain an overconstrained subgraph generated by $V(H_3) \cup \{v_2\}$, so no such Laman subgraph can exist to block the potential edge $\{x, v_5\}$. The other edge of this potential edge pair, $\{w_1, v_4\}$, is clearly not prevented from being added by any Laman subgraph since $w_1 \in V(C_1)$ and $v_4 \in V(X)$, and hence adding the edge pair $\{x, v_5\}, \{w_1, v_4\}$ is valid. This is illustrated in Figure 7.27.



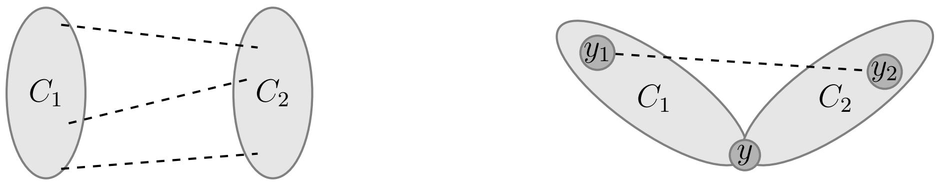
a The neighbours of v_2. **b** The reduction applied at v_2.

Figure 7.27 A reduction applied at v_2 with $\tau(v_2) = (1, 0, 3)$.

Suppose finally that the third edge between $V(C_1)$ and $V(X)$ is $\{w_1, w\}$ for some $w_1 \in V(C_1) \setminus \{v_1\}$, $w \in V(X) \setminus \{x, v_2, v_3, v_4\}$ (Figure 7.26b). It is clear that no neighbour of w can be adjacent to w_1, as the three edges from C_1 to X are known, so it is straightforward to see that there will be a valid potential edge pair at w, whether $\tau(w) = (1, 1, 2)$ or $\tau(w) = (1, 0, 3)$.

Case 2A: Vertices in edge-disjoint circuits

As any vertex outside the circuits of the Laman-plus-two graph (G, c) may be reduced, we may now assume that every vertex in G lies in at least one circuit. We shall begin by considering the case where $E(C_1 \cap C_2) = \emptyset$. We shall either have $C_1 \cap C_2 = \emptyset$, in which case (G, c) consists of two circuits connected by exactly three edges, or we shall have $C_1 \cap C_2 = \{y\}$ for some single vertex $y \in V(C_1) \cap V(C_2)$, and so there is precisely one edge $\{y_1, y_2\} \in E$ with $y_1 \in V(C_1) \setminus \{y\}$, $y_2 \in V(C_2) \setminus \{y\}$. These two cases are illustrated in Figure 7.28. The Laman-plus-two graph (G, c) will contain at least two vertices of degree 3.



a A Laman-plus-two graph containing two disjoint circuits.

b A Laman-plus-two graph containing two circuits with a single common vertex, y.

Figure 7.28 The two types of edge-disjoint Laman-plus-two graphs, with every vertex in at least one circuit.

We first consider the case in which $C_1 \cap C_2 = \emptyset$. The base graph in this situation has $|V| = 8$, so we may assume that $|V| \geq 9$, and hence at least one circuit contains at least five vertices. We therefore consider $|V(C_1)| \geq 5$.

Since the minimum degree within a circuit is 3, $V(C_1)$ contains at least four vertices with $\deg_{C_1}(v) = 3$. Exactly two or three vertices within C_1 are adjacent to vertices outside C_1, so there is at least one vertex $x \in V(C_1)$ with $\deg_G(x) = \deg_{C_1}(x) = 3$.

Suppose instead that $C_1 \cap C_2 = \{y\}$ for some single vertex y. We assume that (G, c) is not a base graph, and hence at least one circuit is not a copy of K_4, so we may assume that $|V(C_1)| \geq 5$. Each circuit contains exactly two vertices with $\deg_G(v) > \deg_C(v)$, and hence since the circuit C_1 contains at least four vertices of degree 3, there are at least two vertices with $\deg_G(v) = \deg_{C_1}(v) = 3$.

We may now assume, in either case, that we have a vertex $x \in V(C_1)$ with $\deg_G(x) = \deg_{C_1}(x) = 3$, where $N(x) = \{v_1, v_2, v_3\} \subseteq V(C_1)$. Since C_1 is a circuit on at least 5 vertices, we may apply the structure of a 1-reduction at x, and replace x and its three associated edges by a single edge between two neighbours of x. We assume that the edge to be added is $\{v_1, v_2\}$.

Let G' denote the reduced graph, within which $\widehat{C_1'}$ is the Laman-plus-one graph corresponding to the circuit C_1 in G. $\widehat{C_1'}$ contains a unique circuit C_1', which consists of the new edge $\{v_1, v_2\}$ together with the minimal Laman subgraph of C_1 containing both v_1 and v_2. We shall denote this Laman subgraph by $\widetilde{C_1}$, and so $C_1' = \widetilde{C_1} + \{v_1, v_2\} \subseteq \widehat{C_1'}$.

We require a colouring c' of the reduced graph G' such that (G', c') satisfies the conditions of Theorem 7.2.4, and hence (G', c') is generically isostatic. Since the structure of a 1-reduction is applied at a vertex within C_1, the circuit C_2 remains unchanged within (G', c'). It remains to verify that an edge from each colour class remains within a circuit of (G', c'), and that the circuit C_1' (and hence the Laman-plus-one subgraph $\widetilde{C_1'}$) satisfy Equations (7.5) and (7.6).

Suppose first that C_1 contains edges from only one colour class, so we may assume that $E(C_1) \subseteq E_0 \cup E_1$ and $E_2 \cap E(C_2) \neq \emptyset$. Since (G', c') requires that C_1' contains at least one coloured edge, we require that $E_1' \cap E(C_1') \neq \emptyset$ for (G', c') to contain edges from both colour classes within a circuit. If there is at least one edge in $E_1 \cap E(\widetilde{C_1})$, we may apply a 1-reduction with the colour of $\{v_1, v_2\}$ induced by the colours of the initial edges $\{x, v_1\}, \{x, v_2\}$. If instead $\widetilde{C_1}$ is uncoloured, we may create a new edge $\{v_1, v_2\} \in E_1'$, either in the reverse of a coloured 1-extension if either $\{x, v_1\}$ or $\{x, v_2\}$ is in E_1, or in the reverse of a "chosen 1-extension" if $\{x, v_1\}, \{x, v_2\} \in E_0$.

Suppose instead that C_1 and C_2 both contain edges from both colour classes. Since C_2 remains unchanged, we require only that C_1' contains at least one coloured edge, and (G', c') will retain at least one edge from each colour class within a circuit. If either edge $\{x, v_1\}$ or $\{x, v_2\}$ is coloured, we may apply an appropriately coloured 1-reduction to create an edge $\{v_1, v_2\}$ of the same colour within (G', c'). If instead both $\{x, v_1\}, \{x, v_2\} \in E_0$ and there is at least one coloured edge within $\widetilde{C_1}$, we may apply a uncoloured 1-reduction as there will straightforwardly be at least one coloured edge within C_1'. If instead $\{x, v_1\}$, $\{x, v_2\}$ and $\widetilde{C_1}$ are all uncoloured, we may apply the reverse of a "chosen 1-extension" to create a new edge added to either E_1' or E_2'.

Suppose finally that C_1 contains edges from both colour classes, and C_2 contains edges from only one colour class. We may assume that $E(C_2) \subseteq E_0 \cup E_2$, and so we require that the colouring c' of G' induces $E_1' \cap E(C_1') \neq \emptyset$. If there is an edge from E_1 within $\widetilde{C_1}$, we may apply an appropriately coloured 1-reduction based on the edges $\{x, v_1\}$ and $\{x, v_2\}$, and similarly if either $\{x, v_1\}$ or $\{x, v_2\}$ is in E_1, we may apply the coloured 1-reduction to add the edge $\{v_1, v_2\} \in E_1'$. If instead $\widetilde{C_1} \subseteq E_0 \cup E_2$ and $\{x, v_1\}, \{x, v_2\} \notin E_1$, we may apply the reverse of a "chosen 1-extension" to add the edge $\{v_1, v_2\}$ to E_1', and hence $E_1' \cap E(C_1') \neq \emptyset$.

Case 2B: Vertices in intersecting circuits

If every vertex within the Laman-plus-two graph (G, c) is within at least one circuit, the minimum degree within (G, c) is 3, and (G, c) contains at least two vertices of degree 3. Let C_1 and C_2 be the two minimal circuits, and let H denote the subgraph $C_1 \cap C_2$, which is a Laman subgraph by Proposition 7.2.2c. Suppose that $|V| \geq 7$, so at least one of C_1 and C_2 has $|V(C)| \geq 5$. Without loss of generality, suppose that this is C_1.

If $C_2 = K_4$, the intersection of C_1 and C_2 must either be K_2 or K_3. There will therefore be at most three vertices in $V(C_1)$ with $\deg_G(v) > \deg_{C_1}(v)$. As the circuit C_1 contains at least four vertices with $\deg_{C_1}(v) = 3$, there will be at least one vertex x in $V(C_1) \setminus V(C_2)$ with $\deg_G(x) = \deg_{C_1}(x) = 3$.

Suppose instead that both circuits contain at least five vertices, and note that if C_1 contains only one vertex outside $C_1 \cap C_2$, this vertex will have degree 3.

Claim 7.2.21. If $|V(C_1)|, |V(C_2)| \geq 5$ and $|V(C_1) \setminus V(C_2)| \geq 2$, there is at least one vertex in $V(C_1) \setminus V(C_2)$ with $\deg_G(v) = \deg_{C_1}(v) = 3$.

Proof of Claim 7.2.21. Recall that H denotes the Laman subgraph $C_1 \cap C_2$, and let J denote the subgraph of C_1 induced by $V(C_1) \setminus V(C_2)$. $E(C_1) = E(H) \cup E(J) \cup F$, where F denotes the set of edges $\{u_1, u_2\}$ with $u_1 \in V(C_1) \setminus V(C_2)$, $u_2 \in V(C_1) \cap V(C_2)$.

Since J is a proper subgraph of C_1 containing at least two vertices, $|E(J)| \leq 2|V(J)| - 3$. Since C_1 is a circuit, $|E(C_1)| = 2|V(C_1)| - 2$, and we have that $|E(C_1)| = |E(H)| + |E(J)| + |F| \leq 2|V(H)| - 3 + 2|V(J)| - 3 + |F| = 2|V(C_1)| + |F| - 6$. Hence $-2 \leq |F| - 6$ and so $|F| \geq 4$.

We now consider the average degree within J:

$$\begin{aligned}\frac{1}{|V(J)|} \sum_{v \in V(J)} \deg_G(v) &= \frac{2|E(J)| + |F|}{|V(J)|} \\ &\leq \frac{4|V(J)| - 6 + 4}{|V(J)|} = 4 - \frac{2}{|V(J)|}.\end{aligned}$$

The average degree within J is therefore strictly less than 4. As $\deg_G(v) = \deg_{C_1}(v)$ for all $v \in V(J)$, and the minimum degree within a circuit is 3, J contains at least one vertex with $\deg_G(v) = \deg_{C_1}(v) = 3$. $\square$

We may therefore suppose that there is a degree 3 vertex $x \in V(C_1) \setminus V(C_2)$, and note that $|V(C_1)| \geq 5$.

Let the neighbours of $x \in V(C_1)$ be denoted by $\{v_1, v_2, v_3\} \in V(C_1)$. Since C_1 is a circuit on at least five vertices, and hence does not contain a copy of K_4, the subgraph $C_1 \setminus \{x\}$ of G is a Laman subgraph on at least four vertices, and at least one pair of neighbours of x is not connected by an edge. We may apply the structure of a 1-reduction to the circuit C_1 by removing x and its three associated edges, and replacing them by a single edge between the neighbours of x. Reducing x in the circuit C_1 results in a Laman-plus-one subgraph $\widehat{C_1'}$ of (G', c'), containing a unique circuit C_1'.

Without loss of generality, suppose that the edge created by the 1-reduction is $\{v_1, v_2\}$. The reduced circuit C_1' will be $\widetilde{C_1} + \{v_1, v_2\}$, where $\widetilde{C_1}$ denotes the minimal Laman subgraph of $C_1 \setminus \{x\}$ containing both v_1 and v_2 (with $|V(\widetilde{C_1})| \geq 4$). We require a colouring c' of the reduced graph G' such that (G', c') satisfies the conditions of Theorem 7.2.4, in order for this to be a valid reduction of (G, c). By Theorem 4.1.11, this is equivalent to (G', c') being a Laman-plus-two graph containing a 2-redundant rainbow pair of edges.

Suppose first that at least one of v_1 and v_2 lies in $V(C_1 \setminus C_2)$. Applying the structure of a 1-reduction at x will leave C_2 unchanged, and hence $C_2' = C_2$ is a circuit within the reduced graph (G', c').

We may relabel the colour classes as required, in order to assume that (G, c) has $E_1 \cap E(C_1) \neq \emptyset$ and $E_2 \cap E(C_2) \neq \emptyset$: if either circuit contains only edges from one colour class, we may assign that colour class the appropriate label (noting that if both circuits contain edges from only one colour class, the colours in each circuit will be distinct, and $E(C_1 \cap C_2) \subseteq E_0$). Alternatively, if both circuits contain edges of both colours, no relabelling is required. This allows us to assume that we wish to obtain a 2-redundant rainbow pair with $e_1 \in E_1' \cap E(C_1')$ and $e_2 \in E_2' \cap E(C_2')$.

We apply the structure of a 1-reduction at x to create the new edge $\{v_1, v_2\}$, and we require that $c'(\{v_1, v_2\}) = 1$: if both $c(\{x, v_1\}), c(\{x, v_2\}) \neq 1$, we may apply the reverse of the "chosen 1-extension" to add the new edge to E_1', however if either $c(\{x, v_1\}) = 1$ or $c(\{x, v_2\}) = 1$, we may apply the reverse of the coloured 1-extension. This guarantees that $E_1' \cap E(C_1') \neq \emptyset$.

Since $\{v_1, v_2\} \in C_1' \setminus C_2'$, removing $\{v_1, v_2\}$ from the reduced graph (G', c') results in a 2-coloured Laman-plus-one graph, with the unique circuit $C_2' = C_2$. Since $E_2 \cap E(C_2) \neq \emptyset$, there is an edge $e_2 \in E_2' \cap E(C_2')$. Removing any edge from the circuit of a Laman-plus-one graph results in a Laman graph, and we therefore obtain the necessary 2-redundant rainbow pair of edges, $\{v_1, v_2\} \in E_1'$, $e_2 \in E_2'$, for (G', c') to be generically rigid by Theorem 4.1.11. Since (G', c') is a Laman-plus-two graph, (G', c') is generically isostatic.

Suppose instead that $v_1, v_2 \in V(C_1 \cap C_2)$. We apply the structure of a 1-reduction at x to add the edge $\{v_1, v_2\}$ and create the graph G'. Since $|V(C_1 \cap C_2)| \geq 2$, $C_1 \cap C_2$ is a Laman subgraph of (G, c) by Proposition 7.2.2c, and by Proposition 7.2.2a, all other circuits within (G, c) are contained within $C_1 \cup C_2$, since $C_1 \cap C_2$ is a non-empty Laman subgraph and hence $E(C_1 \cap C_2) \neq \emptyset$.

We note that the Laman subgraph $C_1 \cap C_2$ of (G, c) remains a Laman subgraph when the vertex x and its three associated edges are removed. We shall denote this subgraph by H. Completing the application of the 1-reduction by adding the edge $\{v_1, v_2\}$ results in a Laman-plus-one subgraph within the reduced graph G'. This subgraph contains a unique circuit C, consisting of the edge $\{v_1, v_2\}$ together with the minimal Laman subgraph of H containing both v_1 and v_2, denoted by H'. Requiring that $c'(\{v_1, v_2\}) = 1$ (by either applying the reverse of the standard 1-reduction if either $c(\{x, v_1\}) = 1$ or $c(\{x, v_2\}) = 1$, and applying the reverse of the "chosen 1-extension" otherwise) ensures that $E_1 \cap E(C) \neq \emptyset$, with $e_1 = \{v_1, v_2\} \in E_1' \cap E(C)$.

As $x \in V(C_1) \setminus V(C_2)$, removing x and its three associated edges leaves the circuit C_2 unchanged in the reduced graph G', and we may denote this circuit by $C_2' = C_2$. We note that (G', c') therefore retains the edge $e_2 \in E_2' \cap E(C_2')$.

Since the circuit C consists of a subgraph of $H = C_1 \cap C_2$ together with one additional edge, $E(C \cap C_2') \neq \emptyset$. By Proposition 7.2.2a, any other circuits within

(G', c') are contained within $C \cup C_2'$. This confirms that $\{v_1, v_2\} \in E_1' \cap E(C)$ and $e_2 \in E_2' \cap E(C_2')$ are a 2-redundant rainbow pair of edges, and hence the Laman-plus-two graph (G', c') is generically isostatic as required. $\square$

Remark 7.2.22. Example 7.2.26 shows a Laman-plus-two graph that has vertices of degree 3 in $V(C_1 \setminus C_2)$ and $V(C_1 \cap C_2)$.

Remark 7.2.23. When (G, c) and (G', c') contain exactly two disjoint circuits, Lemma 7.2.15 may be proved straightforwardly using the coordinated rigidity matrix, without relying on Theorem 7.2.4. The 2-coordinated isostatic graphs in 2 dimensions with disjoint circuits may therefore be characterised by an inductive construction, without relying on the redundant rigidity results (Theorem 4.1.11).

Remark 7.2.24. The "chosen 1-extension" (Definition 7.2.11) may be defined more broadly. In fact, we could permit any coloured 1-extension that results in a 2-edge-coloured Laman-plus-two graph with a 2-redundant rainbow pair, which is clearly generically rigid by Theorem 4.1.11. This would simplify the stated conditions, however we have prioritised a construction that is easily verified to be valid at each step, rather than requiring a 2-redundant rainbow pair to be found at each step.

7.2.4 Further comments

Laman-plus-two characterisation

It is straightforward to see that a similar reduction to that described in the proof of Theorem 7.2.20 may be applied to characterise all Laman-plus-two graphs, using only a set of standard inductive moves.

Theorem 7.2.25. *G is a Laman-plus-two graph if and only if G may be constructed from a base graph $B \in \{B_5, B_6, B_7, B_{8,1}, B_{8,2}, B_{8,3}\}$ by a sequence of 0-extensions, 1-extensions, and X-replacements applied outside the circuits of G.*

Proof. The base graphs are illustrated in Figure 7.6, and may be straightforwardly checked to confirm that each is a Laman-plus-two graph. As 0-extensions and 1-extensions result in a larger Laman graph when applied to a Laman graph, it is clear that applying these moves to a Laman-plus-two graph will produce a Laman-plus-two graph. We note that applying X-replacements to edges outside the circuits of a Laman-plus-two graph also results in a larger Laman-plus-two graph. Therefore any graph constructed from a base graph $B \in \{B_5, B_6, B_7, B_{8,1}, B_{8,2}, B_{8,3}\}$ by applying 0-extensions and 1-extensions, and by applying X-replacements to pairs of disjoint edges that lie outside the circuits of the smaller graph, will be a Laman-plus-two graph.

To prove that any Laman-plus-two graph may be constructed in this way, we apply induction on the number of vertices. This proof follows the same structure as the proof of Theorem 7.2.20. Let $G = (V, E)$ be a Laman-plus-two graph that is not one of the base graphs.

The Laman-plus-two graph G will contain at least two circuits. Let two of these circuits be labelled C_1 and C_2. Removing any edge $e_1 \in E(C_1) \setminus E(C_2)$ results in a Laman-plus-one graph that still contains the circuit C_2, so C_2 is the unique circuit within $G - e_1$. Any edge $e_2 \in E(C_2) \setminus E(C_1)$ may be removed to leave the Laman graph $G^* = G - \{e_1, e_2\}$.

If G contains any vertices of degree 2, they will lie outside all circuits within G. The vertex and its pair of edges may be removed without changing the circuits of G, so the reduced graph G' will remain as a Laman-plus-two graph. By a similar argument to that seen in the proof of Theorem 7.2.20, any vertex of degree 3 that lies outside the circuits of the Laman-plus-two graph G may be reduced while leaving the circuits unchanged, and so this 1-reduction will result in a smaller Laman-plus-two graph.

By Proposition 7.2.3, if the Laman-plus-two graph G contains two disjoint circuits and the vertices outside these circuits have minimum degree 4, every vertex outside the circuits of G will have degree exactly 4. The section "Vertices of degree 4" of the proof of Theorem 7.2.20 (page 160) shows that in such a situation, there will be a vertex of degree 4 outside the circuits of G that may be removed and replaced by a pair of edges with distinct end points, such that the reduced graph G' will also be a Laman-plus-two graph.

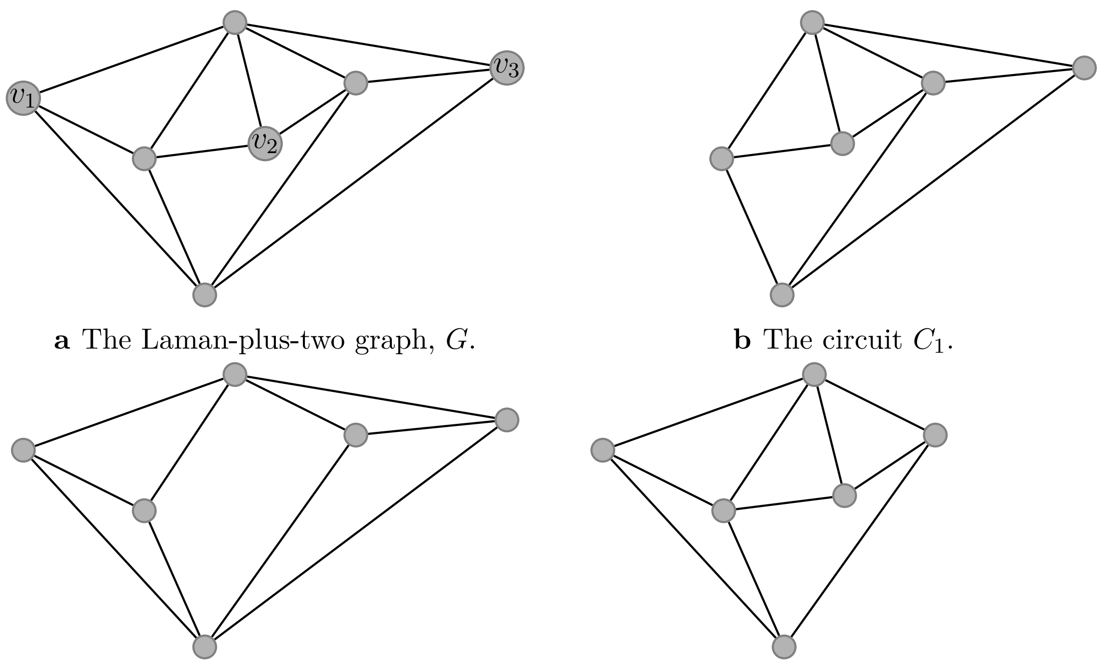
It remains to show that when every vertex of the Laman-plus-two graph G is within a circuit, the reverse of an inductive move may be applied to create a smaller Laman-plus-two graph. As the minimum degree within a circuit is 3, the minimum degree within G is 3.

There will either be exactly two edge-disjoint circuits within G, at least one of which has $|V(C)| \geq 5$, or there will be two circuits with $E(C_1) \cap E(C_2) \neq \emptyset$. All other circuits within G will be contained within $C_1 \cup C_2$, by Proposition 7.2.2a, and so we may suppose that any other circuit C_3 has $|V(C_3)| \geq |V(C_1)|$, $|V(C_3)| \geq |V(C_2)|$, and that at least one of C_1 and C_2 has $|V(C)| \geq 5$. Without loss of generality, suppose that $|V(C_1)| \geq 5$.

The Laman-plus-two graph G contains at least two vertices with $\deg_G(x) = 3$, and a circuit contains at least four vertices with $\deg_{C_1}(x) = 3$. If G contains exactly two edge-disjoint circuits, there will be either two or three vertices within C_1 with $\deg_G(v) > \deg_{C_1}(v)$, and so there will be at least one vertex with $\deg_G(x) = \deg_{C_1}(x) = 3$. If instead C_1 has a non-trivial intersection with C_2, and C_2 is a copy of K_4, there will also be either two or three vertices with $\deg_G(v) > \deg_{C_1}(v)$, and at least one vertex in $V(C_1) \setminus V(C_2)$ of degree 3. From Claim 7.2.21, if neither of C_1 and C_2 is a copy of K_4 and $|V(C_1 \cap C_2)| \geq 2$, there is at least one vertex of degree 3 within $V(C_1) \setminus V(C_2)$.

Removing the vertex $x \in V(C_1)$ with $\deg_G(x) = \deg_{C_1}(x) = 3$, and its three associated edges, results in a graph $\widehat{G}$ with $|\widehat{E}| = |E| - 3 = 2|V| - 1 - 3 = 2|\widehat{V}| - 2$, which is a Laman-plus-one graph. Since the removed vertex was not within a copy of K_4, at least one pair of neighbours of this vertex will not be joined by an edge in $\widehat{G}$. We may add this edge to complete the reverse of the 1-extension, and create a Laman-plus-two graph G'. $\square$

Example 7.2.26. Figure 7.29 shows a Laman-plus-two graph G where every vertex lies within at least one circuit. Three circuits C_1, C_2, C_3 are illustrated below, and it is straightforward to see that any circuit lies within the union of the other two (Proposition 7.2.2a). Each of the vertices labelled v_1, v_2 and v_3 has degree 3 and the reverse of a 1-extension may be applied at any of these vertices. We note that removing the vertex v_i and its three associated edges from G results in precisely the circuit C_i $(1 \leq i \leq 3)$.



a The Laman-plus-two graph, G.

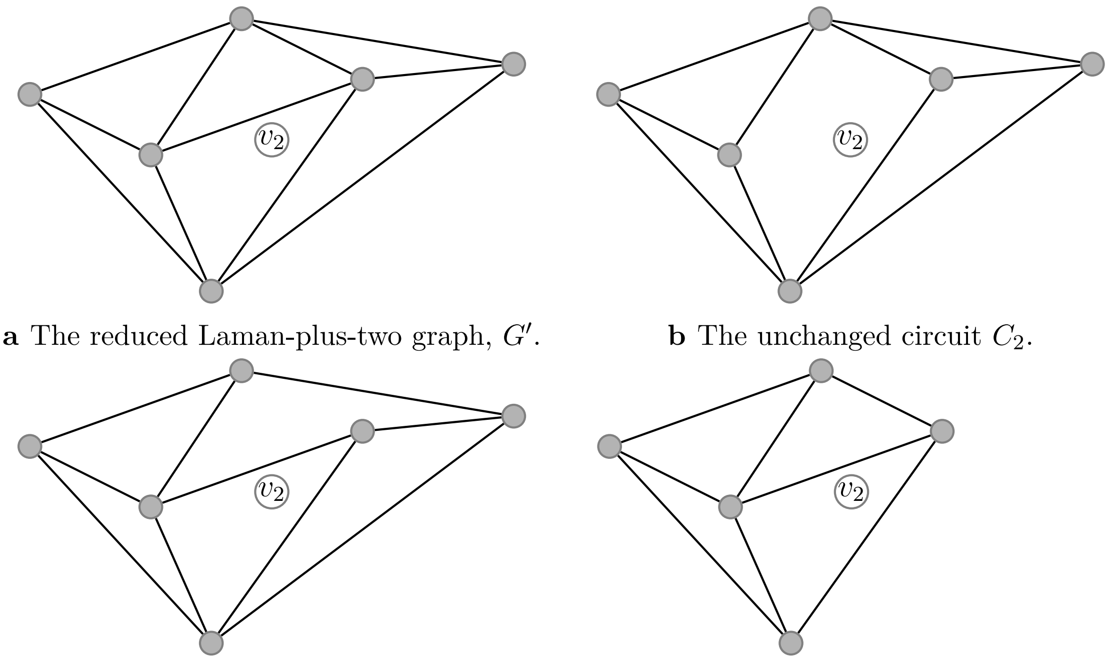
b The circuit C_1.

c The circuit C_2.

d The circuit C_3.

Figure 7.29 A Laman-plus-two graph and three of its circuits.

Reducing at the vertex $v_2 \in V(C_1) \cap V(C_3)$ produces the reduced graph G' illustrated in Figure 7.30. The circuit C_2 remains unchanged in the reduced graph G', which also contains the two reduced circuits C_1' and C_3' shown below, among others.



a The reduced Laman-plus-two graph, G'.

b The unchanged circuit C_2.

c A reduced circuit C_1'.

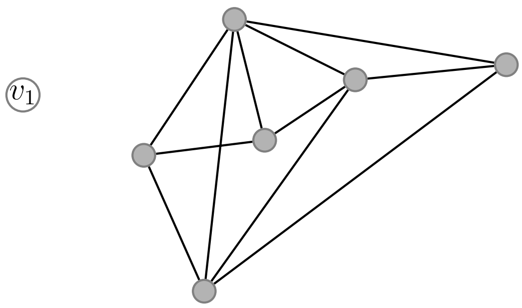
d A reduced circuit $C_3' = W_4$.

Figure 7.30 A reduced Laman-plus-two graph and three of its circuits.

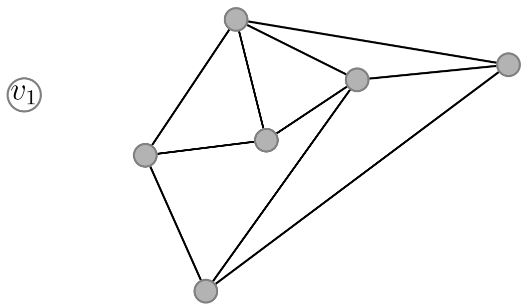
Reducing instead at the vertex $v_1 \in V(C_2) \cap V(C_3)$ results in a different reduced Laman-plus-two graph, which is isomorphic to the reduced graph produced by applying the reverse of a 1-extension at $v_3 \in V(C_1) \cap V(C_2)$. The reduced graph G'' is illustrated in Figure 7.31, along with five of its circuits. We note that the union of any pair of circuits illustrated here is the graph G''.

7.3 Higher k in 2 dimensions

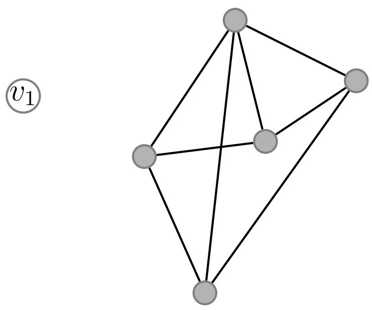
For a regular k-coordinated framework (G, c, p) to be isostatic in 2 dimensions, it is clearly necessary for (G, c) to contain a k-redundant rainbow set (Theorem 4.1.11),



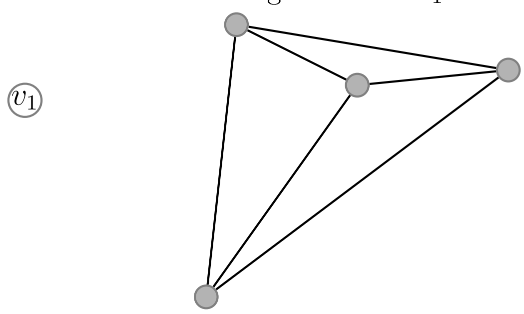
a The reduced Laman-plus-two graph, G''.



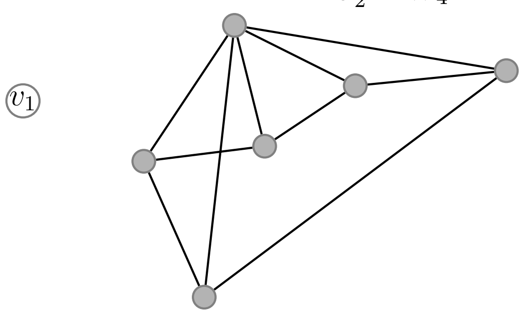
b The unchanged circuit C_1.



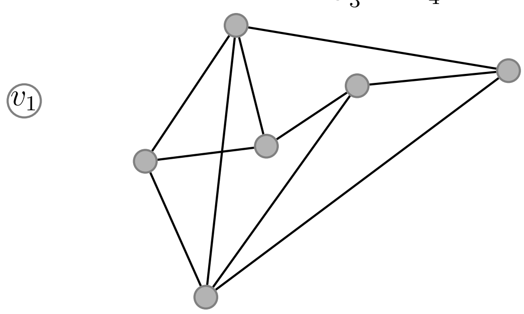
c A reduced circuit $C_2'' = W_4$.



d A reduced circuit $C_3'' = K_4$.



e A reduced circuit C_4''.



f A reduced circuit C_5''.

Figure 7.31 Another reduced Laman-plus-two graph, and five of its circuits.

and so (G, c) must be at least a Laman-plus-k graph. This condition is equivalent to every colour class E_ℓ having at least one edge within a circuit of (G, c).

Recall from Definition 6.3.1 that the 2-chromatic subgraphs of the k-coloured graph (G, c) are generated by the edge sets $E_0 \cup E_1 \cup E_2$, $E_0 \cup E_1 \cup E_3, \ldots, E_0 \cup E_1 \cup E_k$, $E_0 \cup E_2 \cup E_3, \ldots, E_0 \cup E_2 \cup E_k, \ldots, E_0 \cup E_{k-1} \cup E_k$. It is clearly necessary for each 2-chromatic subgraph of (G, c) to be independent as a 2-coloured graph, in order for (G, c) to be generically rigid. This is equivalent to each 2-chromatic subgraph being at most a Laman-plus-two graph, at least one edge from each colour class within a circuit, and satisfying Equations (7.5) and (7.6).

It seems natural to extend the sparsity conditions from Theorem 7.1.2 and Theorem 7.2.4 to k-edge-coloured graphs. As in the 1-dimensional case, each ℓ-chromatic subgraph should be independent as an ℓ-coloured graph. This implies that the ℓ-chromatic subgraphs are at most Laman-plus-ℓ graphs, where any subgraph of the ℓ-chromatic subgraph generated by $D \subseteq E$ satisfies $|D| \leq 2|V(D)| + \ell - 3$.

A Laman-plus-k graph will contain at least k circuits, for which each pair of circuits may be edge-disjoint or intersect non-trivially. This increase in potential combinations of circuits leads to increased complexity of proving a constructive characterisation similar to those given in Section 7.1 and Section 7.2.

The interaction between circuits also seems likely to increase the difficulty of proving that the necessary conditions we conjecture here are equivalent to the existence of k-redundant rainbow subset of edges in (G, c), which implies generic rigidity of (G, c) by Theorem 4.1.11.

Theorem 7.3.1. Let (G, c, p) be a k-edge-coloured framework with a regular configuration $p \in \mathbb{R}^{2n}$. If (G, c, p) is isostatic, then the k-coloured graph (G, c) satisfies the following conditions, for all $0 \leq \ell \leq k$:

The graph G is Laman-plus-k;

At least one edge from each colour class lies within a circuit of (G, c);

Each ℓ-chromatic subgraph is independent as an ℓ-coloured graph.

Proof. By Theorem 3.1.33, the k-coloured regular framework (G, c, p) cannot be isostatic with $|E| \neq 2|V| + k - 3$. If the uncoloured graph G has $|E| = 2|V| + k - 3$ but is not a Laman-plus-k graph, then G cannot contain a k-redundant set of edges, and hence (G, c) does not contain a k-redundant rainbow subset of edges. By Theorem 4.1.11, such a k-coloured graph is not infinitesimally rigid, and hence is not isostatic.

By Theorem 4.1.11, any generically rigid k-coloured graph (G, c) contains a k-redundant rainbow subset of edges. We note that any edge in a k-redundant subset of edges is itself a redundant edge, and that the redundant edges within a Laman-plus-k graph G are exactly those within the circuits of G. Therefore the existence of a k-redundant rainbow subset of edges in (G, c) implies that every colour class contains an edge within a circuit of (G, c).

Suppose that some ℓ-chromatic subgraph of (G, c) is not independent as an ℓ-coloured graph. The submatrix of the coordinated rigidity matrix corresponding to this subgraph of (G, c) contains a row dependence, which remains as a row dependence in the coordinated rigidity matrix $R(G, c, p)$. Hence (G, c, p) is not isostatic. $\square$

Conjecture 7.3.2. Let (G, c, p) be a k-edge-coloured framework with a regular configuration $p \in \mathbb{R}^{2n}$. Then (G, c, p) is isostatic, if and only if the k-coloured graph (G, c) satisfies the following conditions, for all $0 \leq \ell \leq k$:

The graph G is Laman-plus-k;

At least one edge from each colour class lies within a circuit of (G, c);

Each ℓ-chromatic subgraph is independent as an ℓ-coloured graph.

7.4 Algorithms

Since we have obtained inductive characterisations for 1-coloured Laman-plus-one graphs and 2-coloured Laman-plus-two graphs that give isostatic frameworks in 2 dimensions for regular configurations, we have polynomial time algorithms for checking that a given graph satisfies these constraints.

For a given graph G with $|V| = n$ and $|E| = m$, the standard pebble-game algorithm [LS08, LST05, BJ03a] may be applied to find the maximal (2,3)-sparse subgraph of G in time $O(n^2)$, and, for each additional edge outside this rigid subgraph, the circuit containing this edge can be identified in $O(mn)$ time.

In order to verify that a 1-coloured graph (G, c) is generically isostatic in 2 dimensions (Theorem 7.1.2), we require a spanning Laman subgraph of G and an overall count of $|E| = 2|V| - 2$, which allows us to characterise (G, c) as Laman-plus-one. The additional edge outside the spanning Laman subgraph induces a unique circuit which we require to contain at least one coloured edge. The circuit may be identified in $O(mn)$ time, while $O(m)$ time is required to check the colours of the edges within this circuit. This gives an overall requirement of $O(mn)$.

To characterise a 2-coloured graph (G, c) as being generically isostatic in 2 dimensions, we require that (G, c) is a Laman-plus-two graph with a 2-redundant rainbow pair (Theorem 4.1.11, Theorem 7.2.4). A Laman-plus-two graph contains an edge e such that $G \setminus e$ is a Laman-plus-one graph, for which the removal of an additional edge f results in a Laman graph $G \setminus \{e, f\}$. In order to verify the existence of a 2-redundant rainbow pair, we first check whether $G \setminus e_1$ is a Laman-plus-one graph for each $e_1 \in E_1$, and then confirm whether the unique circuit of $G \setminus e_1$ contains an edge $e_2 \in E_2$. This procedure may be applied in $O(mn)$ time for each $e_1 \in E_1$, which results in an overall requirement of $O(m^2n)$ time. It seems likely that this requirement may be reduced by applying improved algorithms or data structures [SST18, LS08].

Chapter 8

Symmetry

Frameworks with symmetric realisations may have infinitesimal motions that preserve all symmetries of the framework, and motions that preserve only a subgroup of the symmetries of the initial framework. Frameworks for which any motion preserves all symmetries of the framework, known as forced symmetric rigid frameworks, may be considered as a subset of the incidentally symmetric rigid frameworks, for which a motion may preserve only a subgroup of the overall symmetry group. We introduce some definitions and concepts for analysing symmetric rigidity, for which further details and references may be found in [SW17b] and [Sch17], and include some initial extensions to symmetric frameworks with collections of coordinated bars.

As previously, we model coordinated frameworks by edge-coloured graphs, with the additional constraint that symmetric copies of an edge are all coloured identically. We shall use Schoenflies notation for symmetry groups, where $\mathcal{C}_s$ denotes the reflection group (of order 2), $\mathcal{C}_n$ denotes the group generated by an n-fold rotation, and $\mathcal{C}_{nv}$ denotes the dihedral group of order $2n$, consisting of a reflection and an n-fold rotation.

Recall that as a consequence of Remark 3.1.20, a coordinated framework (G, c, p, r) may simply be denoted by (G, c, p).

8.1 Rigidity of frameworks with forced symmetry

Throughout this chapter, let $G = (V, E)$ be a graph and Γ be a group, where id denotes the identity element of Γ.

Definition 8.1.1 ([Ser77, ST15])**.** An *automorphism* of the graph G is a permutation of the vertex set, $\alpha : V \to V$, such that $\{v_1, v_2\} \in E$ if and only if $\{\alpha(v_1), \alpha(v_2)\} \in E$. The set of all automorphisms of G form a group, denoted by $\mathrm{Aut}(G)$.

A group homomorphism $\theta : \Gamma \to \mathrm{Aut}(G)$ is an *action* of the group Γ on the graph G. If, for all $\gamma \in \Gamma \setminus \{\mathrm{id}\}$, $\theta(\gamma)(v) \neq v$ for all $v \in V$, the action θ is *free* (on the vertices of G).

The following standard definitions may be found in (among others) [ST15] and [SW17b].

Definition 8.1.2. A graph G is Γ-*symmetric* (with respect to θ) when the group Γ acts on G by θ.

Definition 8.1.3. The Γ-symmetric graph $G = (V, E)$ has *vertex orbits* which partition the vertex set, $\Gamma v := \{\theta(\gamma)(v) : \gamma \in \Gamma\}$ for $v \in V$. Similarly, an *edge orbit* of G is $\Gamma\{v_1, v_2\} := \{\theta(\gamma)(\{v_1, v_2\}) : \gamma \in \Gamma\}$ for an edge $\{v_1, v_2\} \in E$.

Definition 8.1.4. The *quotient graph* $G/\Gamma = (V^*, E^*)$ for the Γ-symmetric graph G is the multigraph with vertex set $V^* := \{\Gamma v : v \in V\}$, and edge set $E^* := \{\Gamma\{v_1, v_2\} : \{v_1, v_2\} \in E\}$.

Definition 8.1.5. Let G be a Γ-symmetric graph with quotient graph $G/\Gamma = (V^*, E^*)$. A representative vertex may be chosen for each vertex orbit, $\Gamma v = \{\theta(\gamma)v : \gamma \in \Gamma\}$. Each edge orbit $\Gamma\{v_1, v_2\}$ in E^*, connecting the vertex orbits Γv_1 and Γv_2, may be viewed as $\{\{\theta(\gamma)(v_1), \theta(\gamma) \circ \theta(\alpha)(v_2)\} : \gamma \in \Gamma\}$ for some unique $\alpha \in \Gamma$. We orient the edge orbit $\Gamma\{v_1, v_2\}$ from Γv_1 towards Γv_2, and label it with *gain* α. The labelling of all edge orbits E^* is denoted by $\psi : E^* \to \Gamma$.

This gives the directed, labelled, *quotient* Γ*-gain graph* of G, $(G/\Gamma, \psi)$.

Remark 8.1.6. The quotient Γ-gain graph $(G/\Gamma, \psi)$ is unique up to the choice of representative vertex for each vertex orbit, and the choice of orientation of directed edges. We may reverse the orientation of an edge $e \in E^*$ by replacing $\psi(e)$ with $(\psi(e))^{-1}$.

Remark 8.1.7. For simplicity, in figures any edges with $\psi(e) = \mathrm{id}$ will usually be left unlabelled.

Definition 8.1.8. Let $v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1} = v_1$ be a closed walk W within the quotient Γ-gain graph $(G/\Gamma, \psi)$. This closed walk is *balanced* if

$$\psi(W) := \prod_{i=1}^{k} \psi(e_i)^{\mathrm{sign}(e_i)} = \mathrm{id},$$

where $\mathrm{sign}(e_i) = 1$ if the edge $e_i = \{v_i, v_{i+1}\}$ is directed from v_i to v_{i+1}, and $\mathrm{sign}(e_i) = -1$ if the edge e_i is directed from v_{i+1} to v_i.

Let $D \subseteq E^*$ be a (possibly disconnected) subset of edges of the quotient Γ-gain graph $(G/\Gamma, \psi)$. $\mathcal{W}(D, v)$ is the set of closed walks in G/Γ starting at $v \in V(D)$ using only edges of D, and $\langle D \rangle_{\psi,v} := \{\psi(W) : W \in \mathcal{W}(D, v)\}$ is the *subgroup of* Γ *induced by* D.

A connected component F of $D \subseteq E^*$ is *balanced* if $\langle F \rangle_{\psi,v} = \mathrm{id}$ for some $v \in V(F)$. If each connected component of $D \subseteq E^*$ is balanced, then the edge set D is *balanced.* If not, D is *unbalanced.*

Remark 8.1.9. If Γ is an additive group, we define instead $\psi(W) := \sum_{i=1}^{k} \psi(e_i)^{\mathrm{sign}(e_i)}$.

Lemma 8.1.10 ([JKT12])**.** Let $(G/\Gamma, \psi)$ be a quotient Γ-gain graph where $G/\Gamma = (V^*, E^*)$. For a connected component $F \subseteq E^*$, $\langle F \rangle_{\psi,v} = \mathrm{id}$ for some $v \in V(F)$ is equivalent to $\langle F \rangle_{\psi,v} = \mathrm{id}$ for all $v \in V(F)$.

We have the following extension of Definition 2.7.3.

Definition 8.1.11 ([ST15]). Let $G = (V, E)$ be a Γ-symmetric graph, and let $\psi : E^* \to \Gamma$ be a gain labelling of the quotient graph $G/\Gamma = (V^*, E^*)$. The quotient Γ-gain graph $(G/\Gamma, \psi)$ is (k, ℓ, m)-*gain-sparse* (for non-negative integers k, ℓ, m with $m \leq \ell$) when

$$|F| \leq \begin{cases} k|V(F)| - \ell \text{ for all non-empty balanced } F \subseteq E^*, \\ k|V(F)| - m \text{ for all non-empty } F \subseteq E^*. \end{cases}$$

If $(G/\Gamma, \psi)$ also satisfies $|E^*| = k|V^*| - m$, then $(G/\Gamma, \psi)$ is (k, ℓ, m)-*gain-tight*.

We wish to extend standard rigidity results to Γ-symmetric frameworks, which requires some extension from these definitions on Γ-symmetric graphs. We note that $O(\mathbb{R}^d)$ denotes the *orthogonal group* of $\mathbb{R}^d$, consisting of the isometries of $\mathbb{R}^d$ that preserve the origin.

Definition 8.1.12 ([SW17b, ST15]). Let (G, p) be a framework, where $G = (V, E)$ is a Γ-symmetric graph with respect to the homomorphism $\theta : \Gamma \to \mathrm{Aut}(G)$ and $p : V \to \mathbb{R}^d$ is a configuration of the vertices. If $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$ is a homomorphism such that $\tau(\gamma)\big(p(v)\big) = p\big(\theta(\gamma)(v)\big)$ for all $v \in V$, (G, p) is a Γ-*symmetric framework*.

The Γ-symmetric framework (G, p) has *symmetry group* $\tau(\Gamma) := \{\tau(\gamma) : \gamma \in \Gamma\} \subset \mathcal{O}(\mathbb{R}^d)$.

For simplicity, we assume that the action $\theta : \Gamma \to \mathrm{Aut}(G)$ is free on the vertices of G in the following definition.

Definition 8.1.13 ([SW11, SW17b, ST15]). Let (G, p) be a Γ-symmetric framework (with respect to the free action $\theta : \Gamma \to \mathrm{Aut}(G)$ and the homomorphism $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$), with quotient Γ-gain graph $(G/\Gamma, \psi)$. Let $V^* = \{v_1, \ldots, v_k\}$ be a set of representative vertices, one for each vertex orbit of V^*, and recall that each edge orbit $\Gamma\{u_1, u_2\}$ may be represented by $\{\theta(\gamma)(v_i), \theta(\gamma) \circ \theta(\alpha)(v_j)\}$, where $u_1 \in \Gamma(v_i)$, $u_2 \in \Gamma(v_j)$ for representative vertices v_i, v_j.

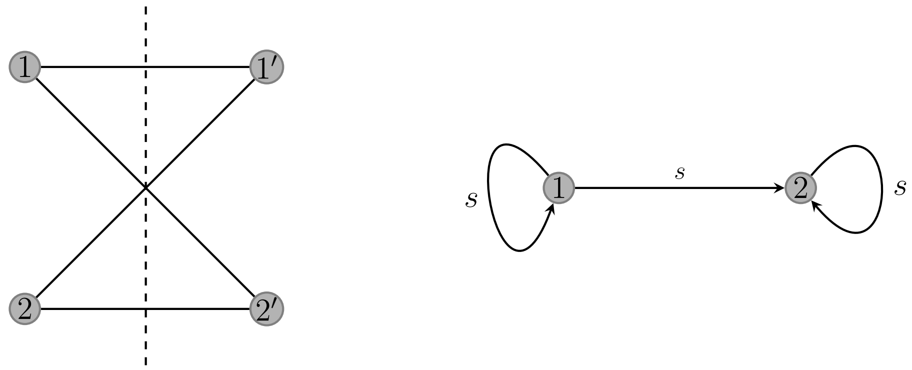
The *orbit rigidity matrix* $O(G/\Gamma, \psi, p)$ consists of $|E^*|$ rows, where the row corresponding to the edge orbit $\Gamma\{v_i, v_j\}$ is

$$\begin{bmatrix} \dots & 0 & p(v_i) - \tau(\psi(\{v_i, v_j\}))p(v_j) & 0 & \dots & 0 & p(v_j) - \tau(\psi(\{v_i, v_j\}))^{-1}p(v_i) & 0 & \dots \end{bmatrix}.$$

The row corresponding to a loop at v_i is

$$\begin{bmatrix} 0 & \dots & 0 & 2p(v_i) - \tau(\psi(\{v_i, v_i\}))p(v_i) - \tau(\psi(\{v_i, v_i\}))^{-1}p(v_i) & 0 & \dots & 0 \end{bmatrix}.$$

Example 8.1.14. Figure 8.1 shows a $\mathcal{C}_s$-symmetric realisation of the graph C_4, together with the corresponding quotient $\mathcal{C}_s$-gain graph $(G/\Gamma, \psi, p)$.



a (G, p) **b** $(G/\Gamma, \psi, p)$

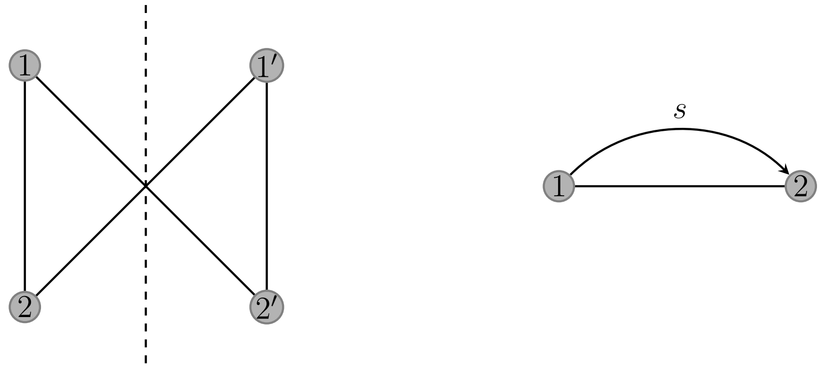
Figure 8.1 Example 8.1.14: a $\mathcal{C}_s$-symmetric realisation of the graph C_4, together with the corresponding quotient $\mathcal{C}_s$-gain graph. (G, p) is forced $\mathcal{C}_s$-symmetric infinitesimally rigid, since the only non-trivial infinitesimal motion is not symmetric.

The quotient $\mathcal{C}_s$-gain graph $(G/\Gamma, \psi, p)$ has the following orbit rigidity matrix:

$$O(G/\Gamma, \psi, p) = \begin{bmatrix} p(1) - \tau(s)p(2) & p(2) - \tau(s)^{-1}p(1) \\ 2p(1) - \tau(s)p(1) - \tau(s)^{-1}p(1) & 0 \\ 0 & 2p(2) - \tau(s)p(2) - \tau(s)^{-1}p(2) \end{bmatrix}.$$

Figure 8.2 shows another $\mathcal{C}_s$-symmetric realisation of the graph C_4, which is not forced-symmetric rigid. The C_s-symmetric framework (G', p') has orbit rigidity matrix

$$O(G'/\Gamma, \psi', p') = \begin{bmatrix} p'(1) - p'(2) & p'(2) - p'(1) \\ p'(1) - \tau(s)p'(2) & p'(2) - \tau(s)^{-1}p'(1) \end{bmatrix}.$$



a (G', p') **b** $(G'/\Gamma, \psi', p')$

Figure 8.2 Example 8.1.14: an alternative $\mathcal{C}_s$-realisation of the graph C_4, with corresponding quotient $\mathcal{C}_s$-gain graph. (G', p') has a symmetry-preserving non-trivial infinitesimal motion, and hence has a corresponding continuous symmetry-preserving motion by Theorem 8.1.17.

Definition 8.1.15 ([ST15])**.** The Γ-symmetric framework (G, p) is Γ-*regular* when the orbit rigidity matrix $O(G/\Gamma, \psi, p)$ has $\operatorname{rank} O(G/\Gamma, \psi, p) \geq \operatorname{rank} O(G/\Gamma, \psi, q)$ for all Γ-symmetric realisations q of G.

Definition 8.1.16 ([SW17b, ST15])**.** Let (G, p) be a Γ-symmetric framework with respect to $\theta : \Gamma \to \operatorname{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$. An infinitesimal motion p' of (G, p) is a *fully* Γ-*symmetric infinitesimal motion* of the framework (G, p) when, for all $\gamma \in \Gamma$ and all $v \in V$,

$$\tau(\gamma)p'(v) = p'\Big(\theta(\gamma)(v)\Big).$$

A trivial infinitesimal motion that is also a fully Γ-symmetric infinitesimal motion is a *fully* Γ-*symmetric trivial infinitesimal motion.* We denote the space of fully Γ-

symmetric trivial infinitesimal motions of (G, p) by $\mathcal{T}_\Gamma(G, p)$, or $\mathcal{T}_\Gamma$ when the framework (G, p) is clear, with dimension $\mathrm{triv}_{\tau(\Gamma)}$.

A Γ-symmetric framework (G, p) for which every fully Γ-symmetric infinitesimal motion p' is a trivial infinitesimal motion is *forced* Γ*-symmetric infinitesimally rigid*.

We have the following result, which motivates the consideration of forced symmetric rigidity.

Theorem 8.1.17 ([Sch10d])**.** A Γ-regular framework (G, p) has a non-trivial fully Γ-symmetric infinitesimal motion if and only if (G, p) has a non-trivial symmetry-preserving continuous motion.

The orbit rigidity matrix allows us to check for forced symmetric rigidity as follows:

Theorem 8.1.18 ([SW11])**.** Let (G, p) be a Γ-symmetric framework with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$. The orbit rigidity matrix $O(G/\Gamma, \psi, p)$ has the following properties:

- **a.** The kernel of $O(G/\Gamma, \psi, p)$ is isomorphic to the space of fully Γ-symmetric infinitesimal motions of (G, p);
- **b.** (G, p) is forced Γ-symmetric infinitesimally rigid if and only if $\operatorname{rank} O(G/\Gamma, \psi, p) = d|V/\Gamma| - \mathrm{triv}_{\tau(\Gamma)}$ (where $\mathrm{triv}_{\tau(\Gamma)}$ is the dimension of the space of trivial fully Γ-symmetric infinitesimal motions of (G, p)).

Definition 8.1.19. If the rows of the orbit rigidity matrix $O(G/\Gamma, \psi, p)$ are also independent, then (G, p) is *forced* Γ*-symmetric isostatic.*

There are necessary conditions for a Γ-symmetric framework (G, p) to be forced Γ-symmetric isostatic [SW17b, Theorem 64.1.4], and Laman-type characterisations of forced Γ-symmetric infinitesimally rigid frameworks when Γ is $\mathcal{C}_2$, $\mathcal{C}_s$, or $\mathcal{C}_{nv}$ for odd n [JKT12, MT14, MT15], in terms of (k, ℓ, m)-gain-sparsity of the quotient Γ-gain graph $(G/\Gamma, \psi, p)$.

8.2 Incidental symmetry

Γ-symmetric frameworks may have some infinitesimal motions that preserve the whole symmetry group, but may also have infinitesimal motions that preserve only some symmetries. The orbit rigidity matrix identifies infinitesimal motions that are fully Γ-symmetric, and in order to identify other infinitesimal motions, we decompose the rigidity matrix into smaller submatrices. Each of these may be considered as being associated with an irreducible representation of the group Γ.

We note that many of the following standard definitions are collected in [SW17b].

Definition 8.2.1 ([Ser77])**.** A *group representation* is a homomorphism $\rho : \Gamma \to \mathrm{GL}(X)$, where the linear space X is the *representation space* of ρ. A subspace $U \subseteq X$ is *ρ-invariant* if $\rho(\gamma)(U) \subseteq U$ for all $\gamma \in \Gamma$.

A representation ρ is *irreducible* if the only ρ-invariant subspaces of X are X and the trivial subspace $\{0\}$.

The *character* of a representation $\rho : \Gamma \to \mathrm{GL}(X)$ is the row vector $\chi(\rho) := \left[\operatorname{trace}\left(\rho(\gamma_1)\right) \quad \dots \quad \operatorname{trace}\left(\rho(\gamma_{|\Gamma|})\right)\right]$, for some fixed ordering of the elements of Γ, $\gamma_1, \dots, \gamma_{|\Gamma|}$.

Let $\rho_0, \dots, \rho_r$ denote the irreducible representations of the group Γ, where ρ_0 is the trivial irreducible representation.

Definition 8.2.2 ([ST15])**.** Let (G, p) be a Γ-symmetric framework (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$). Recall that the Kronecker delta is defined as $\delta_{i,j} = 1$ for $i = j$, $\delta_{i,j} = 0$ for $i \neq j$.

The *external representation* of (G, p) is the group representation $\tau \otimes P_V : \Gamma \to \mathbb{R}^{dn}$, where $P_V : \Gamma \to \mathrm{GL}(\mathbb{R}^n)$ is the permutation matrix of the vertex set V by $\theta(\gamma)$, given by $P_V(\gamma) := \left[\delta_{i,\theta(\gamma)(j)}\right]_{i,j}$.

The *internal representation* of (G, p) is the group representation $P_E : \Gamma \to \mathrm{GL}(\mathbb{R}^m)$, where $P_E(\gamma)$ is the permutation matrix of the edge set E by $\theta(\gamma)$.

Definition 8.2.3 ([ST15])**.** Let ρ_1, ρ_2 be matrix representations of the group Γ, and let $R : \mathbb{R}^s \to \mathbb{R}^t$ be a matrix. If $R\rho_1(\gamma) = \rho_2(\gamma)R$ for all $\gamma \in \Gamma$, R is a Γ-*linear map* of ρ_1 and ρ_2. The linear space of all such Γ-linear maps is denoted by $\mathrm{Hom}_\Gamma(\rho_1, \rho_2)$.

Theorem 8.2.4 ([ST15, Sch10a])**.** Let (G, p) be a Γ-symmetric framework (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$). Then the rigidity matrix $R(G, p)$ lies in $\mathrm{Hom}_\Gamma(\tau \otimes P_V, P_E)$.

Since $R(G, p) \in \mathrm{Hom}_\Gamma(\tau \otimes P_V, P_E)$, Schulze [Sch10a] shows that by Schur's Lemma $R(G, p)$ may be block-diagonalised as follows.

Corollary 8.2.5 ([Sch10a])**.** Let $R(G, p)$ be the rigidity matrix for the framework (G, p), which is Γ-symmetric (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$). There exist non-singular matrices S and T such that $T^\top R(G, p)S$ is in block diagonalised form, with each block corresponding to an irreducible representation of Γ.

We refer to $\widetilde{R}(G, p) := T^\top R(G, p)S$ as the *block diagonalised rigidity matrix.*

Remark 8.2.6 ([Sch10a, SW17b])**.** The matrices S and T correspond to the external representation and internal representation respectively. These decompose $\mathbb{R}^{dn}$ into the direct sum of $(\tau \otimes P_V)$-invariant subspaces $X_0 \oplus \ldots \oplus X_r$, and $\mathbb{R}^m$ into the direct sum of the P_E-invariant subspaces $Y_0 \oplus \ldots \oplus Y_r$. Each block $\widetilde{R}_i(G, p)$ for $0 \le i \le r$ consists of $\dim(X_i)$ columns and $\dim(Y_i)$ rows, and corresponds to an irreducible representation ρ_i of Γ.

$$\widetilde{R}(G, p) := T^\top R(G, p)S = \begin{bmatrix} \widetilde{R}_0(G, p) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \widetilde{R}_r(G, p) \end{bmatrix}.$$

As ρ_0 denotes the trivial irreducible representation, the submatrix $\widetilde{R_0}(G, p)$ is equivalent to the orbit rigidity matrix.[SW11]

Definition 8.2.7 ([Sch10a])**.** The space of trivial infinitesimal motions of a framework (G,p) is denoted by $\mathcal{T}(G,p)$, or $\mathcal{T}$ when the framework (G,p) is clear.

Schulze proves that $\mathcal{T}(G,p)$ is a $(\tau\otimes P_V)$-invariant subspace of $\mathbb{R}^{dn}$ [Sch10a, Lemma 4.2]. We may decompose $\mathcal{T}(G,p)$ into the direct sum $T_0\oplus\ldots\oplus T_r$, and for each irreducible representation ρ_i, $0\le i\le r$, it is necessary that $\dim(Y_i)=\dim(X_i)-\dim(T_i)$ for the Γ-symmetric framework (G,p) to be infinitesimally rigid and independent. We note that the fully Γ-symmetric trivial infinitesimal motions (Definition 8.1.16) are the trivial infinitesimal motions that correspond to the trivial irreducible representation, $T_0=\mathcal{T}_\Gamma(G,p)$.

We may restate the necessary conditions discussed above in the following way, where $(\tau\otimes P_V)^{(\mathcal{T})}$ denotes the subrepresentation of $\tau\otimes P_V$ with representation space $\mathcal{T}(G,p)$.

Theorem 8.2.8 ([OP10, Sch10a, SW17b])**.** Let G be a Γ-symmetric graph with respect to $\theta:\Gamma\to\mathrm{Aut}(G)$ and $\tau:\Gamma\to\mathcal{O}(\mathbb{R}^d)$. Then when (G,p) is isostatic, the following holds:

$$\chi(P_E)=\chi(\tau\otimes P_V)-\chi\left((\tau\otimes P_V)^{(\mathcal{T})}\right).$$

Definition 8.2.9 ([SW17b])**.** If the action θ is not free on the vertex set V, and hence there is a vertex $v\in V$ such that $\theta(\gamma)(v)=v$ for some $\gamma\in\Gamma$, we refer to v as a *fixed vertex.*

An edge $\{v_1,v_2\}\in E$ such that either $\theta(\gamma)(v_1)=v_1$ and $\theta(\gamma)(v_2)=v_2$, or $\theta(\gamma)(v_1)=v_2$ and $\theta(\gamma)(v_2)=v_1$, will have $\theta(\gamma)(\{v_1,v_2\})=\{v_1,v_2\}$, and shall be referred to as a *fixed edge.*

We denote the set of vertices fixed by $\gamma \in \Gamma$ by V_γ, and the set of edges fixed by $\gamma \in \Gamma$ by E_γ.

Theorem 8.2.10 ([SW17b, CFG$^+$09])**.** Let (G, p) be a Γ-symmetric isostatic framework (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$). For each $\gamma \in \Gamma$, we have the following:

$$|E_\gamma| = \mathrm{trace}(\tau(\gamma)) \cdot |V_\gamma| - \mathrm{trace}((\tau \otimes P_V)^{(\mathcal{T})}(\gamma))$$

Table 8.1 contains the standard 2-dimensional characters which have been calculated for various elements of symmetry groups [SW17b, AH94], where C_n denotes a rotation around the origin by $\frac{2\pi}{n}$.

	id	s	C_2	C_n, $n > 2$
$\chi\left(P_E\right)$	$\lvert E\rvert$	$\lvert E_s\rvert$	$\lvert E_{C_2}\rvert$	$\lvert E_{C_n}\rvert$
$\chi\Big(\tau \otimes P_V\Big)$	$2\lvert V\rvert$	0	$-2\lvert V_{C_2}\rvert$	$\left(2\cos\frac{2\pi}{n}\right)\lvert V_{C_n}\rvert$
$\chi\Big((\tau \otimes P_V)^{(\mathcal{T})}\Big)$	3	-1	-1	$2\cos\frac{2\pi}{n} + 1$

Table 8.1 The 2-dimensional characters calculated for elements of the symmetry group $\tau(\Gamma)$ in 2 dimensions [SW17b]

For isostatic incidentally Γ-symmetric frameworks in $\mathbb{R}^2$, we may use the character equations to obtain additional necessary conditions for such frameworks to be isostatic, in terms of the number of elements fixed by a given symmetry element.

Theorem 8.2.11 (Theorem 64.2.5 [SW17b])**.** Let (G, p) be a 2-dimensional framework that is Γ-symmetric with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$, and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^2)$, and let $|V_\gamma|$, $|E_\gamma|$ denote the number of vertices and edges fixed by $\gamma \in \Gamma$ respectively. If (G, c, p) is isostatic then the following are satisfied:

a. if $s \in \tau(\Gamma)$, then $|E_s| = 1$;

b. if $C_2 \in \tau(\Gamma)$, then $|V_{C_2}| = 0$ and $|E_{C_2}| = 1$;

c. if $C_3 \in \tau(\Gamma)$, then $|V_{C_3}| = 0$;

d. $C_n \notin \tau(\Gamma)$ for any $n > 3$.

When θ is a free action on the vertex set V, infinitesimal rigidity may be characterised for Γ-symmetric frameworks where $\Gamma \in \{\mathcal{C}_2, \mathcal{C}_s, \mathcal{C}_n : n \text{ odd}\}$ [ST15, Ike15]. The counts in Theorem 8.2.11 are sufficient for (G, p) to be Γ-symmetric isostatic, when when G also satisfies the Laman conditions and (G, p) is Γ-generic (i.e. $R(G, p)$ has maximum rank among Γ-symmetric realisations of G) [Sch10c, Sch10b, CFG$^+$09, SW17b]. Such a characterisation remains an open problem for the dihedral groups $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$. By Theorem 8.2.11 there can be no isostatic frameworks in $\mathbb{R}^2$ with C_n symmetry for $n > 3$, hence there are no isostatic Γ-symmetric frameworks with $\Gamma \in \{\mathcal{C}_n, \mathcal{C}_{nv} : n > 3\}$.

8.3 Coordinated symmetry

In order to consider coordinated frameworks with symmetry, we require that all symmetric copies of an edge have the same colour. We give the following definition.

Definition 8.3.1. Let G be a Γ-symmetric graph with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$. A Γ-*symmetric k-edge-colouring* for G is a k-edge-colouring $c : E \to \{0, 1, \dots, k\}$ such that for every edge $\{u_1, u_2\} \in E$, $c(\gamma(\{u_1, u_2\})) = c(\{u_1, u_2\})$ for all $\gamma \in \Gamma$.

Definition 8.3.2. The k-coloured graph (G, c) is *k-coloured* Γ-*symmetric* (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$) if G is a Γ-symmetric graph (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$), and $c : E \to \{0, 1, \dots, k\}$ is a Γ-symmetric k-edge-colouring.

Since the colouring c is consistent for all edges within each edge orbit, we may define an equivalent k-edge-colouring on the quotient Γ-gain graph.

Definition 8.3.3. Let (G, c) be a Γ-symmetric graph (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$) with a Γ-symmetric k-edge-colouring $c : E \to \{0, 1, \ldots, k\}$. The *$k$-coloured quotient Γ-gain graph* $(G/\Gamma, \psi, c_\Gamma)$ is the directed ψ-labelled quotient graph $G/\Gamma = (V^*, E^*)$, with a k-edge-orbit-colouring $c_\Gamma : E^* \to \{0, 1, \ldots, k\}$, where $c_\Gamma(\Gamma\{v_1, v_2\}) = c(\{v_1, v_2\})$.

Definition 8.3.4. Let (G, c) be a k-coloured Γ-symmetric graph with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$, and let $p : V \to \mathbb{R}^d$ be a configuration of the vertices of G. If $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$ satisfies $\tau(\gamma)\Big(p(v)\Big) = p\Big(\theta(\gamma)(v)\Big)$ for all $v \in V$, (G, c, p) is a *k-coordinated Γ-symmetric framework*.

Remark 8.3.5. We may equivalently define a k-coordinated Γ-symmetric framework to be a Γ-symmetric framework (G, p) with a Γ-symmetric k-edge-colouring, since in both cases the edge colours are preserved by the symmetry operation.

8.3.1 Forced coordinated symmetry

We may extend the notion of the orbit rigidity matrix to k-coordinated Γ-symmetric frameworks as follows.

Definition 8.3.6. Let $\mathbb{1}_\Gamma(c)$ denote the matrix consisting of $|E^*|$ rows, where the row corresponding to the edge orbit $\Gamma\{v_1, v_2\}$ is $\mathbf{e}_{c_\Gamma(\Gamma\{v_1,v_2\})} = \mathbf{e}_{c(\{v_1,v_2\})} \in \mathbb{R}^k$ when $c_\Gamma(\Gamma\{v_1, v_2\}) = c(\{v_1, v_2\}) \in \{1, \ldots, k\}$, and the row is the k-dimensional zero vector if $c_\Gamma(\Gamma\{v_1, v_2\}) = c(\{v_1, v_2\}) = 0$. This may be referred to as the *orbit characteristic matrix*.

The *coordinated orbit rigidity matrix* for a Γ-symmetric k-coordinated framework (G, c, p) is $O(G/\Gamma, \psi, c_\Gamma, p) := \Big[O(G/\Gamma, \psi, p)\Big|\mathbb{1}_\Gamma(c)\Big]$.

This allows us to define the following class of infinitesimal motions of a k-coordinated Γ-symmetric framework (G, c, p).

Definition 8.3.7. Let (G, c, p) be a k-coordinated Γ-symmetric framework with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$, and let (p', r') be a k-coordinated infinitesimal motion of (G, c, p). Then (p', r') is a *fully* Γ*-symmetric* k*-coordinated infinitesimal motion* of (G, c, p) if, for all $\gamma \in \Gamma$ and all $v \in V$,

$$\tau(\gamma)p'(v) = p'\Big(\theta(\gamma)(v)\Big).$$

Remark 8.3.8. We note that since (G, c, p) is a Γ-symmetric k-coordinated framework, $\tau(\gamma)p(v) = p\Big(\theta(\gamma)(v)\Big)$ for all $v \in V$ and $\gamma \in \Gamma$, and that $\tau(\gamma)p'(v) = p'\Big(\theta(\gamma)(v)\Big)$ for all $v \in V$ and $\gamma \in \Gamma$ for the fully Γ-symmetric k-coordinated infinitesimal motion $(p', r') \in \mathbb{R}^{d|V|+k}$.

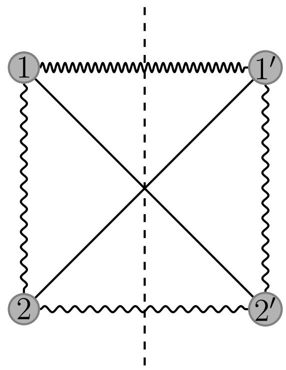
Recall that $G/\Gamma = (V^*, E^*)$ and that $(G/\Gamma, \psi, c_\Gamma, p)$ is the k-coordinated quotient Γ-gain graph of (G, c, p). We may define an equivalent fully Γ-symmetric infinitesimal motion $(p^*, r^*) \in \mathbb{R}^{d|V^*|+k}$ on $(G/\Gamma, \psi, c_\Gamma, p)$ by $p^* := p'|_{V^*}$ and $r^* := r'$, since any edge orbit may be written as $\Big\{\{\theta(\gamma)(v_i), \theta(\gamma) \circ \theta(\psi_{ij})(v_j)\} : \gamma \in \Gamma\Big\}$ for some unique $\psi_{ij} \in \Gamma$. It is equivalent for $O(G/\Gamma, \psi, c_\Gamma, p)(p^*, r^*) = [\mathbf{0}, \mathbf{0}]$ and $R(G, c, p)(p', r') = [\mathbf{0}, \mathbf{0}]$.

Remark 8.3.9. Since the trivial infinitesimal motions of a coordinated framework correspond exactly to the trivial infinitesimal motions of a standard framework (Definition 3.1.17), the *fully* Γ*-symmetric trivial infinitesimal motions* of the k-coordinated framework (G, c, p) are the standard fully Γ-symmetric trivial infinitesimal motions of the uncoloured framework (G, p) (Definition 8.1.16). We denote the space of fully Γ-symmetric trivial infinitesimal motions of (G, c, p) by $\mathcal{T}_\Gamma(G, c, p)$, and the dimension remains $\mathrm{triv}_{\tau(\Gamma)}$.

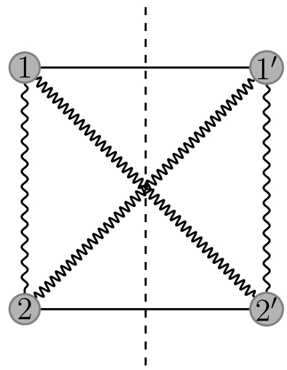
Definition 8.3.10. The k-coordinated Γ-symmetric framework (G, c, p) is *forced* Γ-*symmetric infinitesimally rigid* when every fully Γ-symmetric infinitesimal motion of (G, c, p) is a trivial fully Γ-symmetric infinitesimal motion.

(G, c, p) is *forced* Γ-*symmetric isostatic* if (G, c, p) is forced Γ-symmetric infinitesimally rigid and the rows of the coordinated orbit rigidity matrix $O(G/\Gamma, \psi, c_\Gamma, p)$ are independent.

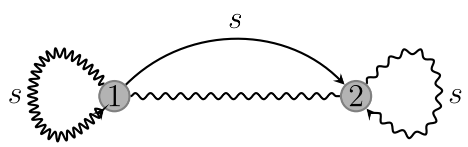
Example 8.3.11. Figure 8.3 contains two 2-coordinated $\mathcal{C}_s$-symmetric realisations of K_4, labelled (G, c, p) and (G', c', p'), and their corresponding 2-coloured quotient $\mathcal{C}_s$-gain graphs, $(G/\Gamma, \psi, c_\Gamma)$ and $(G'/\Gamma, \psi', c'_\Gamma)$. Both are forced $\mathcal{C}_s$-symmetric flexible.



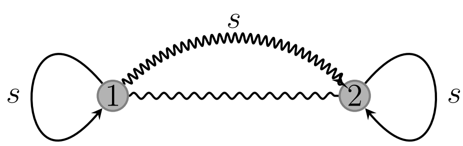
a The 2-coordinated $\mathcal{C}_s$-symmetric framework (G, c, p).



b The 2-coordinated $\mathcal{C}_s$-symmetric framework (G', c', p').



c The 2-coordinated quotient $\mathcal{C}_s$-gain graph $(G/\Gamma, \psi, c_\Gamma, p)$.



d The 2-coordinated quotient $\mathcal{C}_s$-gain graph $(G'/\Gamma, \psi', c'_\Gamma, p')$.

Figure 8.3 Example 8.3.11: two different 2-coordinated $\mathcal{C}_s$-symmetric realisations of the graph K_4, and the corresponding quotient $\mathcal{C}_s$-gain graphs.

(G, c, p) has the following coordinated orbit rigidity matrix $O(G/\Gamma, \psi, c_\Gamma, p)$:

$$\begin{bmatrix} p(1)-p(2) & p(2)-p(1) & 1 & 0 \\ 2p(1)-\tau(s)p(1)-\tau(s)^{-1}p(1) & 0 & 0 & 1 \\ p(1)-\tau(s)p(2) & p(2)-\tau(s)^{-1}p(1) & 0 & 0 \\ 0 & 2p(2)-\tau(s)p(2)-\tau(s)^{-1}p(2) & 1 & 0 \end{bmatrix},$$

while (G', c', p') has coordinated orbit rigidity matrix $O(G'/\Gamma, \psi', c'_\Gamma, p')$:

$$\begin{bmatrix} p'(1)-p'(2) & p'(2)-p'(1) & 1 & 0 \\ 2p'(1)-\tau(s)p'(1)-\tau(s)^{-1}p'(1) & 0 & 0 & 0 \\ p'(1)-\tau(s)p'(2) & p'(2)-\tau(s)^{-1}p'(1) & 0 & 1 \\ 0 & 2p'(2)-\tau(s)p'(2)-\tau(s)^{-1}p'(2) & 0 & 0 \end{bmatrix}.$$

Both coordinated orbit rigidity matrices consist of four rows and six columns, however there is only one trivial motion preserved by $\mathcal{C}_s$. This makes it clear that both 2-coordinated 2-dimensional $\mathcal{C}_s$-symmetric frameworks will be forced $\mathcal{C}_s$-symmetric flexible.

Example 8.3.12. Figure 8.4 shows a 1-coordinated $\mathcal{C}_s$-symmetric realisation of K_4 that is forced $\mathcal{C}_s$-symmetric rigid. This may be obtained from (G', p') in Example 8.1.14 (Figure 8.2) by adding two coloured edges, which remove the fully $\mathcal{C}_s$-symmetric motion.

The coordinated orbit rigidity matrix is

$$\begin{bmatrix} p'(1)-p'(2) & p'(2)-p'(1) & 0 \\ p'(1)-\tau(s)p'(2) & p'(2)-\tau(s)^{-1}p'(1) & 0 \\ 2p'(1)-\tau(s)p'(1)-\tau(s)^{-1}p'(1) & 0 & 1 \\ 0 & 2p'(2)-\tau(s)p'(2)-\tau(s)^{-1}p'(2) & 1 \end{bmatrix}.$$

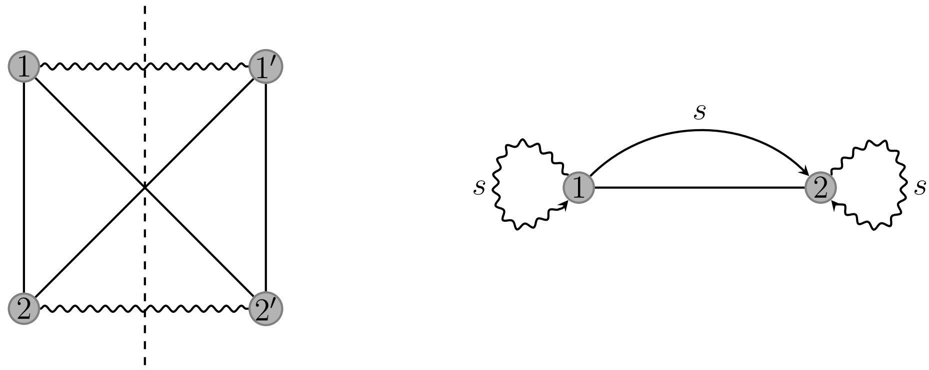


Figure 8.4 A 1-coordinated realisation of K_4 that is forced $\mathcal{C}_s$-symmetric rigid, and the associated quotient $\mathcal{C}_s$-gain graph.

This matrix has rank four, which corresponds to the 1-coordinated 2-dimensional $\mathcal{C}_s$-symmetric framework being forced $\mathcal{C}_s$-symmetric rigid, and forced $\mathcal{C}_s$-symmetric isostatic.

Theorem 8.3.13. Let (G, c, p) be a k-coordinated framework that affinely spans $\mathbb{R}^d$, and is Γ-symmetric with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$. Then (G, c, p) is forced Γ-symmetric infinitesimally rigid if and only if the coordinated orbit rigidity matrix $O(G/\Gamma, \psi, c_\Gamma, p)$ satisfies

$$\operatorname{rank} O(G/\Gamma, \psi, c_\Gamma, p) = d|V/\Gamma| + k - \mathrm{triv}_{\tau(\Gamma)}.$$

Proof. Let (G, c, p) be a k-coordinated Γ-symmetric framework. The coordinated orbit rigidity matrix $O(G/\Gamma, \psi, c_\Gamma, p)$ consists of $|E/\Gamma|$ rows and $d|V/\Gamma| + k$ columns, and hence $\operatorname{rank} O(G/\Gamma, \psi, c_\Gamma, p) = d|V/\Gamma| + k - \dim\ker\Big(O(G/\Gamma, \psi, c_\Gamma, p)\Big)$.

We recall that by Theorem 8.1.18a, the kernel of the standard orbit rigidity matrix $O(G/\Gamma, \psi, p)$ is isomorphic to the space of fully Γ-symmetric infinitesimal motions of (G, p). The kernel of $O(G/\Gamma, \psi, c_\Gamma, p)$ is therefore isomorphic to the space of fully Γ-symmetric coordinated infinitesimal motions of (G, c, p).

Since the space of trivial coordinated infinitesimal motions has equal dimension to the standard space of trivial infinitesimal motions (see Definition 3.1.17), the space of fully Γ-symmetric trivial coordinated infinitesimal motions of (G,c,p) will have the same dimension as the space of fully Γ-symmetric trivial infinitesimal motions, $\mathrm{triv}_{\tau(\Gamma)}$. Therefore (G,c,p) is forced Γ-symmetric infinitesimally rigid if and only if $\dim\ker O(G/\Gamma,\psi,c_\Gamma,p) = \mathrm{triv}_{\tau(\Gamma)}$, and hence $\mathrm{rank}\, O(G/\Gamma,\psi,c_\Gamma,p) = d|V/\Gamma| + k - \mathrm{triv}_{\tau(\Gamma)}$. □

Theorem 8.3.14. Let (G,c,p) be a k-coordinated framework that is Γ-symmetric with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$. If (G,c,p) is forced Γ-symmetric isostatic then the k-coordinated quotient Γ-gain graph $(G/\Gamma,\psi,c_\Gamma,p)$ satisfies

$$
\begin{aligned}
&|E/\Gamma| = d|V/\Gamma| + k - \mathrm{triv}_{\tau(\Gamma)}\\
&\quad |F| \leq d|V(F)| + k(F) - \mathrm{triv}_{\tau\left(\langle F\rangle_{\psi,v}\right)}(p(F)) \text{ for all } F \subseteq E/\Gamma \text{ and all } v \in V(F),
\end{aligned}
$$

where $v \in V(F)$ is identified with its representative vertex in V/Γ, $k(F)$ denotes the number of non-empty colour classes ($F \cap E_i \neq \emptyset$ for $1 \leq i \leq k$), $p(F) = \Big\{\tau(\gamma)\big(p(v)\big) : v \in V(F), \gamma \in \Gamma\Big\}$, and $\mathrm{triv}_{\tau\left(\langle F\rangle_{\psi,v}\right)}(p(F))$ denotes the dimension of the space of fully $(\langle F\rangle_{\psi,v})$-symmetric trivial infinitesimal motions of the configuration $p(F)$.

Proof. Let (G,c,p) be a k-coordinated framework that is forced Γ-symmetric isostatic. By Theorem 8.3.13, $\mathrm{rank}\, O(G/\Gamma,\psi,c_\Gamma,p) = d|V/\Gamma| + k - \mathrm{triv}_{\tau(\Gamma)}$ when (G,c,p) is forced Γ-symmetric infinitesimally rigid, and $\mathrm{rank}\, O(G/\Gamma,\psi,c_\Gamma,p) = |E/\Gamma|$ when (G,c,p) is an independent framework. This gives the overall constraint that $|E/\Gamma| = d|V/\Gamma| + k - \mathrm{triv}_{\tau(\Gamma)}$.

We consider subgraphs of the k-coordinated quotient Γ-gain graph $(G/\Gamma, \psi, c_\Gamma, p)$ as being generated by a set of edges $F \subset E/\Gamma$, where $V(F)$ is the set of vertices adjacent to edges in F, identified with their representative vertices in V/Γ.

Suppose that there is a subgraph $(V(F), F)$ such that $|F| > d|V(F)| + k(F) - \mathrm{triv}_{\tau\left(\langle F\rangle_{\psi,v}\right)}(p(F))$. The submatrix of $O(G/\Gamma, \psi, c_\Gamma, p)$ consisting of the rows corresponding to the edges in F will have a kernel with dimension $\mathrm{triv}_{\tau\left(\langle F\rangle_{\psi,v}\right)}(p(F))$, since it consists of fully $(\langle F\rangle_{\psi,v})$-symmetric trivial infinitesimal motions of the configuration $p(F)$, and hence the rank of this submatrix is equal to $d|V(F)| + k(F) - \mathrm{triv}_{\tau\left(\langle F\rangle_{\psi,v}\right)}(p(F))$. This rank is strictly smaller than $|F|$, and hence the submatrix contains a row dependence.

The row dependence within this submatrix is a row dependence within the coordinated orbit rigidity matrix $O(G/\Gamma, \psi, c_\Gamma, p)$. The existence of such a subgraph implies that (G, c, p) is not independent, and hence cannot be k-coordinated forced Γ-symmetric isostatic. $\square$

We also conjecture that for $k = 1$ and either $\Gamma \in \{\mathcal{C}_s, \mathcal{C}_n, \mathcal{C}_{mv} : n, m \in \mathbb{N}, n \geq 2, m \text{ odd}\}$, the conditions stated in Theorem 8.3.14 are both necessary and sufficient for a Γ-regular 1-coordinated Γ-symmetric framework to be forced Γ-symmetric isostatic in $\mathbb{R}^2$. It seems likely, however, that inductive proofs similar to those that exist in the uncoloured case will be hard to produce, since $\Gamma = \mathcal{C}_2$ and $\Gamma = \mathcal{C}_s$ will require characterising the 1-coordinated quotient Γ-gain graph with $|E/\Gamma| = 2|V/\Gamma|$, and so a vertex of degree 3 is no longer guaranteed. For $\Gamma = \mathcal{C}_{mv}$ with m odd, the 1-coordinated Γ-gain graph satisfying these counts has $|E/\Gamma| = 2|V/\Gamma| + 1$. It is possible that a matroid union argument may gain more traction than an inductive argument in these cases, if an appropriate matroid can be identified.

8.3.2 Incidental coordinated symmetry

We first prove the following key result, related to the standard "intertwining property", which allows us to block-decompose the coordinated rigidity matrix.

Theorem 8.3.15. Let (G, c, p) be a Γ-symmetric k-coordinated framework (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$). Then the coordinated rigidity matrix $R(G, c, p)$ lies in $\mathrm{Hom}_\Gamma((\tau \otimes P_V) \oplus I_k, P_E)$.

Proof. Let $R(G, c, p)$ be the coordinated rigidity matrix for the k-coordinated Γ-symmetric d-dimensional framework (G, c, p). Recall that for any edge $e = \{v_i, v_j\} \in E$, the row of $R(G, c, p)$ corresponding to e will contain $p(v_i) - p(v_j)$ in the d columns corresponding to v_i, and $p(v_j) - p(v_i)$ in the d columns corresponding to v_j. For $e \in E_\ell$, $1 \leq \ell \leq k$, the final k columns of the row corresponding to e may be considered as the k-dimensional basis vector $\mathbf{e}_\ell$, while for $e \in E_0$, the final k columns of the row corresponding to e will contain only zeros.

Let R_e denote the matrix obtained from $R(G, c, p)$ by replacing all entries outside the row corresponding to the edge e with zeros, and hence $R(G, c, p) = \sum_{e \in E} R_e$. Each R_e may be viewed as the combination of a pair of matrices, $\left[\widehat{R}_e, \mathbb{1}_e\right]$, where $\widehat{R}_e$ is the $|E|$ by $d|V|$ matrix obtained from the standard rigidity matrix $R(G, p)$ by deleting all non-zero entries outside the row corresponding to e, and $\mathbb{1}_e$ is the $|E|$ by k matrix obtained similarly from the characteristic matrix $\mathbb{1}(c)$. We note that for all $e \in E_0$, $\mathbb{1}_e$ is the zero matrix.

We may construct a directed graph $\vec{G}$ from G by assigning an arbitrary direction to each edge $e \in E$. Let $I_{\vec{G}}$ denote the directed incidence matrix for $\vec{G}$. For each $e \in E$, we obtain $I_{\vec{e}}$ from $I_{\vec{G}}$ as above, by replacing all non-zero entries outside the row corresponding to e with zeros. As discussed by Schulze and Tanigawa [ST15], each matrix $\widehat{R}_e$ for $e = \{v_i, v_j\} \in E$ may be viewed as the Kronecker product $p(e)^\top \otimes I_{\vec{e}}$, where $p(e) := p(v_i) - p(v_j) \in \mathbb{R}^d$. We therefore have $R_e = \Big[p(e)^\top \otimes I_{\vec{e}}, \mathbb{1}_e\Big]$ for each $e \in E$.

Recall that for each $\gamma \in \Gamma$, $P_V(\gamma)$ is the permuation matrix of the vertex set V by $\gamma(\theta)$ and $P_E(\gamma)$ is the permutation matrix of the edge set E by $\theta(\gamma)$, i.e. $P_V(\gamma) = \Big[\delta_{i,\theta(\gamma)(j)}\Big]_{i,j}$ is a square matrix of dimension $|V|$.

Since $R(G,c,p) = \sum_{e \in E} R_e$, we consider $P_E(\gamma) R_e \Big[\Big((\tau \otimes P_V) \oplus I_k\Big)(\gamma)\Big]^\top$ for each $e \in E$ and $\gamma \in \Gamma$:

$$\begin{aligned}
P_E(\gamma) R_e \Big[\Big((\tau \otimes P_V) \oplus I_k\Big)(\gamma)\Big]^\top &= P_E(\gamma)\Big[p(e)^\top \otimes I_{\vec{e}}, \mathbb{1}_e\Big]\Big[\Big(\tau(\gamma)^\top \otimes P_V(\gamma)^\top\Big) \oplus I_k\Big] \\
&= P_E(\gamma)\Big[\, \Big(p(e)^\top \otimes I_{\vec{e}}\Big)\Big(\tau(\gamma)^\top \otimes P_V(\gamma)^\top\Big), (\mathbb{1}(e))\,(I_k)\,\Big] \\
&= P_E(\gamma)\Big[\Big(p(e)^\top \tau(\gamma)^\top\Big) \otimes \Big(I_{\vec{e}} P_V(\gamma)^\top\Big), \mathbb{1}_e\Big] \\
&= \Big[P_E(\gamma)\Big(\Big(\tau(\gamma)p(e)\Big)^\top \otimes \Big(I_{\vec{e}} P_V(\gamma)^\top\Big)\Big), P_E(\gamma)\mathbb{1}_e\Big] \\
&= \Big[\Big(\tau(\gamma)p(e)\Big)^\top \otimes \Big(P_E(\gamma) I_{\vec{e}} P_V(\gamma)^\top\Big), P_E(\gamma)\mathbb{1}_e\Big].
\end{aligned}$$

Since (G,c,p) is Γ-symmetric with respect to $\theta : \Gamma \to Aut(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$, for $e = \{v_i, v_j\} \in E$, $\tau(\gamma)p(e) = \tau(\gamma)\Big(p(v_i) - p(v_j)\Big) = \tau(\gamma)p(v_i) - \tau(\gamma)p(v_j) = p\Big(\theta(\gamma)(v_i)\Big) - p\Big(\theta(\gamma)(v_j)\Big) = p\Big(\theta(\gamma)(e)\Big)$, where $\theta(\gamma)(e) = \{\theta(\gamma)(v_i), \theta(\gamma)(v_j)\} \in E$.

It is straightforward to confirm that since $P_V(\gamma)$ and $P_E(\gamma)$ are permutation matrices for the vertex set and edge set of (G,c,p) respectively, and G is Γ-symmetric with respect to θ, $P_E(\gamma) I_{\vec{e}} P_V(\gamma)^\top = I_{\theta(\vec{\gamma)}(e)}$.

Recall that since (G,c,p) is a Γ-symmetric k-coordinated graph, the colouring c is consistent for all edges within each edge orbit. For each $\gamma \in \Gamma$, the permutation matrix $P_E(\gamma)$ will map $e \in E$ to $\theta(\gamma)(e) \in E$, with $c(\theta(\gamma)(e)) = c(e)$. Thus $P_E(\gamma)\mathbb{1}_e = \mathbb{1}_{\theta(\gamma)(e)}$.

We therefore have $P_E(\gamma)R_e\Big[\Big((\tau\otimes P_V)\oplus I_k\Big)(\gamma)\Big]^\top = \Big[p\left(\theta(\gamma)(e)\right)^\top \otimes I_{\overrightarrow{\theta(\gamma)(e)}}, \mathbb{1}_{\theta(\gamma)(e)}\Big] = R_{\theta(\gamma)(e)}$ for all $e\in E$ and $\gamma\in\Gamma$. Hence for each $\gamma\in\Gamma$,

$$\begin{aligned} P_E(\gamma)R(G,c,p)\Big[\Big((\tau\otimes P_V)\oplus I_k\Big)(\gamma)\Big]^\top &= \sum_{e\in E} P_E(\gamma)R_e\Big[\Big((\tau\otimes P_V)\oplus I_k\Big)(\gamma)\Big]^\top \\ &= \sum_{e\in E} R_{\theta(\gamma)(e)} = \sum_{e\in E} R_e = R(G,c,p). \end{aligned}$$

Therefore $R(G,c,p)\in \mathrm{Hom}_\Gamma\left((\tau\otimes P_V)\oplus I_k, P_E\right)$. $\square$

By applying Schur's Lemma [Ser77], we obtain the following result.

Corollary 8.3.16. Let $R(G,c,p)$ be the coordinated rigidity matrix for the Γ-symmetric k-coordinated framework (G,c,p). There exist non-singular matrices $\overline{P}$ and $\overline{Q}$ such that $\overline{R}(G,c,p) := \overline{Q}^\top R(G,c,p)\overline{P}$ is block-diagonalised, where each block is associated to an irreducible representation of Γ.

Remark 8.3.17. We may refer to $\overline{R}(G,c,p) = \overline{Q}^\top R(G,c,p)\overline{P}$ as the *block-diagonalisation* of $R(G,c,p)$. As discussed in Remark 8.2.6, the matrices $\overline{P}$ and $\overline{Q}$ decompose $\mathbb{R}^{dn+k}$ and $\mathbb{R}^m$ into direct sums of $(\tau\otimes P_V)\oplus I_k$-invariant and P_E-invariant subspaces respectively. For irreducible representations $\rho_0,\dots,\rho_r$ of Γ, let $\mathbb{R}^{dn+k} = X_0'\oplus\dots\oplus X_r'$ and $\mathbb{R}^m = Y_0'\oplus\dots\oplus Y_r'$, where each X_i' and Y_i' correspond to the irreducible representation ρ_i. Each block $\overline{R}_i(G,c,p)$ within $\overline{R}(G,c,p)$ will have dimension $\dim(Y_i')$ by $\dim(X_i')$.

$$\overline{R}(G,c,p) = \begin{bmatrix} \overline{R}_0(G,c,p) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \overline{R}_1(G,c,p) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \overline{R}_r(G,c,p) \end{bmatrix}.$$

This block-diagonalisation $\overline{R}(G,c,p)$ gives similar necessary conditions for a Γ-symmetric coordinated framework to be infinitesimally rigid and independent (as discussed after Definition 8.2.7). These are discussed in Remark 8.3.26.

We note that this block-diagonalised form of $R(G,c,p)$ loses the separation between the standard rigidity matrix $R(G,p)$ and the characteristic matrix $\mathbb{1}(c)$. By reordering the basis of $\mathbb{R}^{dn+k}$, we may obtain an alternative block decomposition of $R(G,c,p)$ which retains this property of the coordinated rigidity matrix.

Corollary 8.3.18. Let $R(G,c,p)$ be the coordinated rigidity matrix for the Γ-symmetric k-coordinated framework (G,c,p) (with respect to $\theta:\Gamma\to\mathrm{Aut}(G)$, and $\tau:\Gamma\to\mathcal{O}(\mathbb{R}^d)$). There exist non-singular matrices $\widehat{S}$ and $\widehat{T}$ such that $\widetilde{R}(G,p):=\widehat{T}^\top R(G,c,p)\widehat{S}$ has the following form:

$$\widetilde{R}(G,c,p)=\begin{bmatrix}\widetilde{R}_0(G,c,p) & \mathbf{0} & \dots & \mathbf{0} & \mathbb{1}_\Gamma(c)\\ \mathbf{0} & \widetilde{R}_1(G,c,p) & \dots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \dots & \widetilde{R}_r(G,c,p) & \mathbf{0}\end{bmatrix}.$$

Proof. By Schur's Lemma [Ser77] $\widehat{S}$ and $\widehat{T}$ must be matrices of basis transformation for $\mathbb{R}^{dn+k}$ and $\mathbb{R}^m$ respectively. For the irreducible representations $\rho_0,\dots,\rho_r$ of Γ, $\widehat{S}$ is a transformation matrix from the standard basis to the basis of the $(\tau\otimes P_V)\oplus I_k$-invariant subspaces of $\mathbb{R}^{dn+k}$. We label these subspaces $X'_0,\dots,X'_r$, and similarly label the P_E-invariant subspaces of $\mathbb{R}^m$ by $Y'_0,\dots,Y'_r$. Each X'_i and Y'_i corresponds to the irreducible representation ρ_i. Let $X_0,\dots,X_r$ be the $(\tau\otimes P_V)$-invariant subspaces of $\mathbb{R}^{dn}$.

Recall that a vector $v\in\mathbb{R}^p$ is fully Γ-symmetric with respect to a representation $\phi:\Gamma\to\mathbb{R}^p$ if $\phi(\gamma)v=v$ for every $\gamma\in\Gamma$. We note that for any $v\in\mathbb{R}^{dn}$ that is fully

Γ-symmetric with respect to $(\tau \otimes P_V)$, $(v, r) \in \mathbb{R}^{dn+k}$ is fully Γ-symmetric with respect to $(\tau \otimes P_V) \oplus I_k$ for all $r \in \mathbb{R}^k$. The subspace X_0' of $\mathbb{R}^{dn+k}$, corresponding to the trivial representation ρ_0, therefore has $\dim(X_0') = \dim(X_0) + k$.

For $1 \leq i \leq r$, the $(\tau \otimes P_V) \oplus I_k$-invariant subspace $X_i' \subset \mathbb{R}^{dn+k}$ is isomorphic to the $(\tau \otimes P_V)$-invariant subspace $X_i \subset \mathbb{R}^{dn}$, so we may take the same basis for X_i' as for X_i.

The standard basis vectors $\mathbf{e}_{dn+1}, \dots, \mathbf{e}_{dn+k}$ of $\mathbb{R}^{dn+k}$ are clearly basis vectors of X_0'. An appropriate ordering of the basis vectors of $X_0, \dots, X_r$ gives the non-singular matrix S that is applied to the standard rigidity matrix $R(G, p)$ to obtain the standard block-diagonalised rigidity matrix $\widetilde{R}(G, p)$ (as discussed in Remark 8.2.6). We may therefore apply an appropriate ordering of the basis vectors of $X_0', \dots, X_r'$ to get $\widehat{S} = S \oplus I_k = \begin{bmatrix} S & \mathbf{0} \\ \mathbf{0} & I_k \end{bmatrix}$.

For $0 \leq j \leq r$, each Y_j' is identical to the equivalent P_E-invariant subspace Y_j for the uncoloured framework (G, p), so we may take the same orthonormal basis of $\mathbb{R}^m$ to construct the change of basis matrix T (as discussed in Remark 8.2.6). We therefore obtain the following, where M is a $\dim(Y_0)$ by k matrix:

$$
\begin{aligned}
T^\top R(G, c, p)\widehat{S} &= T^\top \Big[R(G, p), \mathbb{1}(c)\Big]\Big(S \oplus I_k\Big) \\
&= T^\top \Big[R(G, p)S, \mathbb{1}(c)\Big] = \Big[T^\top R(G, p)S, T^\top \mathbb{1}(c)\Big] \\
&= \begin{bmatrix} \widetilde{R}_0(G, p) & \mathbf{0} & \dots & \mathbf{0} & M \\ \mathbf{0} & \widetilde{R}_1(G, p) & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \widetilde{R}_r(G, p) & \mathbf{0} \end{bmatrix}.
\end{aligned}
$$

Since T is the matrix used in the uncoloured case, the first dn columns contain the standard block-diagonalised rigidity matrix $\widetilde{R}(G, p)$.

Our claim is that $M = \mathbb{1}_\Gamma(c)$ for an appropriate choice of basis for Y_0. We may multiply the rows of $T^\top$ by appropriate scalars to obtain a matrix $\widehat{T}^\top$ such that $\widehat{T}^\top \mathbb{1}(c)$ results in the coordinated orbit characteristic matrix $\mathbb{1}_\Gamma(c)$, augmented by an appropriate number of rows containing only zeros. We therefore obtain

$$\widehat{T}^\top R(G,c,p)\widehat{S} = \begin{bmatrix} \widetilde{R}_0(G,c,p) & \mathbf{0} & \dots & \mathbf{0} & \mathbb{1}_\Gamma(c) \\ \mathbf{0} & \widetilde{R}_1(G,c,p) & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \widetilde{R}_r(G,c,p) & \mathbf{0} \end{bmatrix}$$

as required. □

Remark 8.3.19. We may refer to $\widetilde{R}(G,c,p)$ as the *"quasi"-block-diagonalisation* of $R(G,c,p)$. We note that each block within the first dn columns of $\widetilde{R}(G,c,p)$, $\widetilde{R}_i(G,c,p)$ is a similar matrix to the equivalent block $\widetilde{R}_i(G,p)$ of the block-diagonalisation of the standard rigidity matrix $R(G,p)$, but they are not necessarily identical due to the scalar multiplication applied to obtain $\widehat{T}^\top$ from $T^\top$.

Remark 8.3.20. Since each pair $\widetilde{R}_i(G,p)$ and $\widetilde{R}_i(G,c,p)$ are similar matrices, they will have the same dimension and rank, including the rank of submatrices. For $1 \leq i \leq r$, these blocks are also similar to the blocks within the block-diagonalisation of $R(G,c,p)$, $\overline{R}_i(G,c,p)$, since $\widetilde{R}(G,c,p)$ and $\overline{R}(G,c,p)$ are obtained through identical processes, apart from a reordering of the basis of $\mathbb{R}^{dn+k}$ (and subsequent scalar multiplication of the rows of T).

The first block of $\overline{R}(G,c,p)$, $\overline{R}_0(G,c,p)$ may be checked to be the coordinated orbit rigidity matrix $O(G/\Gamma,\psi,c_\Gamma,p)$, which is made up of the standard orbit rigidity matrix $O(G/\Gamma,\psi,p) = \widetilde{R}_0(G,p)$ and the orbit characteristic matrix $\mathbb{1}_\Gamma(c)$. Since $\widetilde{R}_0(G,p)$ is similar to $\widetilde{R}_0(G,c,p)$, the first $\dim(Y_0)$ rows of $\overline{R}(G,c,p)$ and $\widetilde{R}(G,c,p)$ are also similar matrices: both may be considered to contain two blocks of non-zero entries, together with $dn - \dim(X_0)$ zero columns. The k columns corresponding to $\mathbb{1}_\Gamma(c)$ are identical in both, and the other non-zero blocks $\widetilde{R}_0(G,p)$ and $\widetilde{R}_0(G,c,p)$ are similar matrices. These are the blocks that correspond to the trivial irreducible representation ρ_0.

Remark 8.3.21. We note that the block decomposition of $\mathbb{1}(c)$ occurs due to the constraint that $c(\gamma(e)) = c(e)$ for all $\gamma \in \Gamma$: since all symmetric copies of an edge are coloured identically, any infinitesimal motion of the Γ-symmetric k-coordinated framework (G,c,p) with a non-trivial coordination component must either be fully symmetric, or the linear combination of infinitesimal motions in multiple irreducible representations.

Example 8.3.22. Example 8.3.11 discussed two 2-coordinated $\mathcal{C}_s$-symmetric realisations of K_4, illustrated together with their quotient $\mathcal{C}_s$-gain graphs in Figure 8.3. Through an appropriate choice of basis, we obtain the following matrices which may be applied to obtain the block-decompositions $\widetilde{R}(G,c,p)$ and $\widetilde{R}(G',c',p')$. Since both frameworks have the same structure, $R(G,p) = R(G',p')$, and the coordinated rigidity matrices differ only in $\mathbb{1}(c)$ and $\mathbb{1}(c')$.

$$R(G,p) = \begin{bmatrix} 0 & 4 & 0 & -4 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & -4 & -4 & 4 & 4 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & -4 \end{bmatrix}, \quad \mathbb{1}(c) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbb{1}(c') = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$\widehat{S} = S \oplus I_2 = \left[\begin{array}{cccccccc|cc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$\widehat{T} = T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad \widehat{T}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{bmatrix}.$$

The block-decomposed coordinated rigidity matrices for the 2-coordinated Γ-symmetric frameworks (G, c, p) and (G', c', p') are therefore the following:

$$\widetilde{R}(G, c, p) = \left[\begin{array}{cccc|cccc|cc} 0 & 4 & 0 & -4 & 0 & 0 & 0 & 0 & 1 & 0 \\ -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & 4 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 4 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 4 & 4 & 4 & 0 & 0 \end{array}\right],$$

$$\widetilde{R}(G',c',p') = \left[\begin{array}{cccc|cccc|cc} 0 & 4 & 0 & -4 & 0 & 0 & 0 & 0 & 1 & 0 \\ -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 4 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 4 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 4 & 4 & 4 & 0 & 0 \end{array}\right].$$

In each matrix, the first four rows may be noted to be the coordinated orbit rigidity matrix (with four additional columns of zeros), as stated in Example 8.3.11.

Definition 8.3.23. Let $\rho_0, \dots, \rho_r$ denote the irreducible representations of Γ, and recall that these decompose $\mathbb{R}^{dn+k}$ and $\mathbb{R}^m$ into $X'_0 \oplus \dots \oplus X'_r$ and $Y'_0 \oplus \dots \oplus Y'_r$ respectively, where the subspaces X'_i and Y'_i correspond to the irreducible representation ρ_i.

For an irreducible representation ρ_j with $1 \leq j \leq r$, an infinitesimal motion (p', r') of the k-coordinated Γ-symmetric framework (G, c, p) is ρ_j*-symmetric* if $p' \in X'_j$.

The ρ_0*-symmetric infinitesimal motions* are contained in the kernel of the coordinated orbit rigidity matrix $O(G/\Gamma, \psi, c_\Gamma, p) = \left[\widetilde{R}_0(G,p), \mathbb{1}_\Gamma(c)\right]$. This space is isomorphic to the space of fully Γ-symmetric infinitesimal motions.

Remark 8.3.24. Recall that the coordinated rigidity matrix $R(G, c, p)$ may be decomposed into blocks corresponding to the irreducible representations of Γ:

$$\widetilde{R}(G,c,p) = \begin{bmatrix} \widetilde{R}_0(G,p) & \mathbf{0} & \dots & \mathbf{0} & \mathbb{1}_\Gamma(c) \\ \mathbf{0} & \widetilde{R}_1(G,p) & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \widetilde{R}_r(G,p) & \mathbf{0} \end{bmatrix}.$$

Each block $\widetilde{R}_i(G,p)$ consists of $\dim(Y'_i)$ rows and $\dim(X'_i)$ columns. The kernel of this block is isomorphic to the space of ρ_i-symmetric infinitesimal motions.

Remark 8.3.25. Recall from Definition 8.2.7 that $\mathcal{T}(G,p)$ denotes the space of trivial infinitesimal motions of the framework (G,p), and let $\mathcal{T}(G,c,p)$ denote the space of trivial infinitesimal motions of the k-coordinated framework (G,c,p). We note that from Definition 3.1.17, $\dim\Big(\mathcal{T}(G,c,p)\Big) = \dim\Big(\mathcal{T}(G,p)\Big)$.

Remark 8.3.26. Recall that $\mathcal{T}(G,p)$ may be decomposed into the direct sum $T_0 \oplus \ldots \oplus T_r$, where each T_i corresponds to the irreducible representation ρ_i, and $T_0 = \mathcal{T}_\Gamma(G,p)$. We note that for $1 \leq i \leq k$, the blocks $\widetilde{R}_i(G,c,p)$ and $\overline{R}_i(G,c,p)$, in $\widetilde{R}(G,c,p)$ and $\overline{R}(G,c,p)$, are identical matrices. These are the blocks corresponding to the non-trivial irreducible representations of Γ, and each block is made up of of $\dim(Y_i')$ rows and $\dim(X_i')$ columns. We therefore have $\dim(Y_i') = \dim(X_i') - \dim(T_i')$ for each block, $1 \leq i \leq r$, as a necessary condition for (G,c,p) to be isostatic as a symmetric framework.

From the block-decomposition of the coordinated rigidity matrix, $\widetilde{R}(G,c,p)$, we obtain the condition $\dim(Y_i) = \dim(X_i) - \dim(T_i)$ for $1 \leq i \leq r$, and $\dim(Y_0) = \dim(X_0) + k - \dim(T_0)$ for the trivial irreducible representation ρ_0.

Let $((\tau \otimes P_V) \oplus I_k)^{(\mathcal{T})}$ denote the subrepresentation of $(\tau \otimes P_V) \oplus I_k$ with representation space $\mathcal{T}(G,c,p) = \mathcal{T}(G,p)$, and recall that $(\tau \otimes P_V)^{(\mathcal{T})}$ denotes the subrepresentation of $\tau \otimes P_V$ with representation space $\mathcal{T}(G,p)$. We may restate the conditions obtained from the block-decomposed coordinated rigidity matrix in the following way.

Theorem 8.3.27. Let (G,c,p) be a k-coordinated framework that is Γ-symmetric with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$, and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$. If (G,c,p) is isostatic, then the following character equation holds:

$$\begin{aligned}\chi(P_E) &= \chi\Big((\tau \otimes P_V) \oplus I_k\Big) - \chi\Big(\,((\tau \otimes P_V) \oplus I_k)^{(\mathcal{T})}\,\Big) \\ &= \chi\Big(\tau \otimes P_V\Big) + \chi\left(I_k\right) - \chi\Big((\tau \otimes P_V)^{(\mathcal{T})}\Big). \qquad (8.1)\end{aligned}$$

Proof. We recall that in the standard case, $\chi(P_E) = \chi(\tau \otimes P_V) - \chi\left((\tau \otimes P_V)^{(\mathcal{T})}\right)$ when the uncoloured framework (G, p) is isostatic (Theorem 8.2.8). The representation P_E remains unchanged in the coordinated case, while the representation $(\tau \otimes P_V)$ is augmented by the k-dimensional identity matrix (see Theorem 8.3.15). We may therefore apply a similar method to the standard result in order to obtain .$\chi(P_E) = \chi\Big((\tau \otimes P_V) \oplus I_k\Big) - \chi\Big(((\tau \otimes P_V) \oplus I_k)^{(\mathcal{T})}\Big)$.

We note that the character of a direct sum is equal to the sum of the characters. This allows us to simplify $\chi\Big((\tau \otimes P_V) \oplus I_k\Big)$ to $\chi\Big(\tau \otimes P_V\Big) + \chi\left(I_k\right)$, which are known. Similarly $\chi\Big(((\tau \otimes P_V) \oplus I_k)^{(\mathcal{T})}\Big) = \chi\Big((\tau \otimes P_V)^{(\mathcal{T})}\Big) + \chi\Big((I_k)^{(\mathcal{T})}\Big)$. $(I_k)^{(\mathcal{T})}$ is empty, and hence $\chi\Big(((\tau \otimes P_V) \oplus I_k)^{(\mathcal{T})}\Big) = \chi\Big((\tau \otimes P_V)^{(\mathcal{T})}\Big)$ as required. □

We note that, since $\chi(I_k)(\gamma) = k$ for all $\gamma \in \Gamma$, all components of Equation 8.1 may be obtained from the calculations of characters for the rigidity of uncoloured Γ-symmetric frameworks in $\mathbb{R}^2$ and $\mathbb{R}^3$. For clarity, Table 8.2 contains the standard results stated in Table 8.1, with the addition of $\chi(\mathcal{I}_k)$.

We restrict our statements to $d = 2$ and $d = 3$ here, since for symmetric frameworks within these spaces, all elements of the symmetry group may be listed, making the problem tractable, and the trivial motions are known from point group tables [AH94]. We may therefore apply the existing standard results for symmetric 2-dimensional or 3-dimensional frameworks [Sch10a] to obtain the entries for the final row of Table 8.2, without need for further calculation.

	id	s	C_2	$C_n,\, n > 2$
$\chi\left(P_E\right)$	$\lvert E\rvert$	$\lvert E_s\rvert$	$\lvert E_{C_2}\rvert$	$\lvert E_{C_n}\rvert$
$\chi\Big(\tau \otimes P_V\Big)$	$2\lvert V\rvert$	0	$-2\lvert V_{C_2}\rvert$	$\left(2\cos\frac{2\pi}{n}\right)\lvert V_{C_n}\rvert$
$\chi\left(I_k\right)$	k	k	k	k
$\chi\Big((\tau \otimes P_V)^{(\mathcal{T})}\Big)$	3	-1	-1	$2\cos\frac{2\pi}{n} + 1$

Table 8.2 Character calculations for k-coordinated frameworks in 2 dimensions.

Example 8.3.28. Let (G,c,p) be the 2-coordinated $\mathcal{C}_s$-symmetric framework considered in Example 8.3.11a, and illustrated in Figure 8.3. From standard rigidity results [SW17b], we have $\chi\Big((\tau\otimes P_V)^{(\mathcal{T})}\Big)=(3,-1)$, and we note that $\chi\left(I_2\right)=(2,2)$. Since $\chi(\tau)=(2,-2)$ and $\chi(P_V)=(4,0)$, $\chi\left(\tau\otimes P_V\right)=(8,0)$, and we have that $\chi(P_E)=(6,2)$. This implies that

$$\begin{aligned}&\chi\Big(\tau\otimes P_V\Big)+\chi\left(I_k\right)-\chi\Big((\tau\otimes P_V)^{(\mathcal{T})}\Big)=\\&(8,0)+(2,2)-(3,-1)=(7,3)\neq(6,2)=\chi(P_E)\end{aligned}$$

which confirms that (G,c,p) is not 2-coordinated Γ-symmetric rigid.

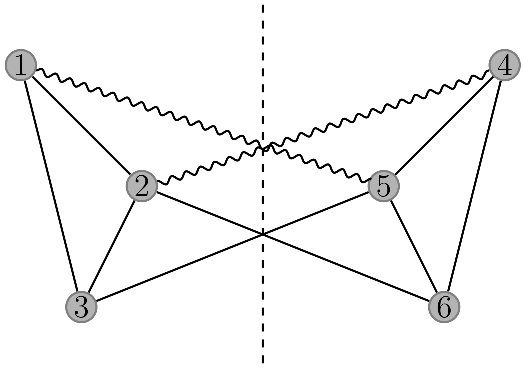
The standard irreducible representations for $\mathcal{C}_s$ are $\rho_0=(1,1)$ and $\rho_1=(1,-1)$. It is straightforward to obtain that $\chi(P_E)=(6,2)=4\rho_0+2\rho_1$, while $\chi\Big(\tau\otimes P_V\Big)+\chi\left(I_k\right)-\chi\Big((\tau\otimes P_V)^{(\mathcal{T})}\Big)=(7,3)=5\rho_0+2\rho_1$, which implies that (G,c,p) has a fully $\mathcal{C}_s$-symmetric infinitesimal motion.

Example 8.3.29. Let (G,c,p) be the 1-coordinated $\mathcal{C}_s$-symmetric framework shown in Figure 8.5a. (G,c,p) has $\chi(P_E)=(10,0)$ and $\chi(\tau\otimes P_V)=(12,0)$, and hence

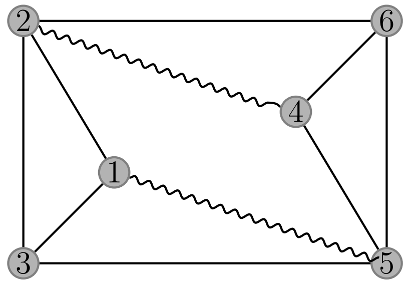
$$\begin{aligned}&\chi\Big(\tau\otimes P_V\Big)+\chi\left(I_k\right)-\chi\Big((\tau\otimes P_V)^{(\mathcal{T})}\Big)=\\&(12,0)+(1,1)-(3,-1)=(10,2)\neq(10,0)=\chi(P_E).\end{aligned}$$

Since $\rho_0=(1,1)$ and $\rho_1=(1,-1)$ are the standard irreducible representations for $\mathcal{C}_s$, we have $\chi(P_E)=(10,0)=5\rho_0+5\rho_1$, while $\chi\Big(\tau\otimes P_V\Big)+\chi\left(I_k\right)-\chi\Big((\tau\otimes P_V)^{(\mathcal{T})}\Big)=(10,2)=6\rho_0+4\rho_1$. This implies that (G,c,p) has a fully $\mathcal{C}_s$-symmetric infinitesimal motion, and an "anti-symmetric" equilibrium stress.

Figure 8.5b shows a $\mathcal{C}_2$-symmetric realisation of the 1-coordinated graph (G,c), (G,c,p'). We note that (G,c,p') also has a fully $\mathcal{C}_2$-symmetric flex, and an "anti-



a The 1-coordinated $\mathcal{C}_s$-symmetric framework (G, c, p).



b The 1-coordinated $\mathcal{C}_2$-symmetric framework (G, c, p').

Figure 8.5 Example 8.3.29: two realisations of the 1-edge-coloured graph (G, c), where (G, c, p) is a $\mathcal{C}_s$-symmetric framework and (G, c, p') is a $\mathcal{C}_2$-symmetric framework.

symmetric" equilibrium stress since, as above, $\chi\Big(\tau \otimes P_V\Big) + \chi\left(I_k\right) - \chi\Big((\tau \otimes P_V)^{(\mathcal{T})}\Big) = (12, 0) + (1, 1) - (3, -1) = (10, 2) = 6\rho_0 + 4\rho_1$ and $\chi(P_E) = (10, 0) = 5\rho_0 + 5\rho_1$.

We note that the realisation in Figure 8.5b makes it clear that the 1-edge-coloured graph (G, c) is a 2-connected circuit, and hence (G, c) is clearly generically rigid as a non-symmetric 1-coordinated framework in $\mathbb{R}^2$.

Since the statement of Theorem 8.3.27 is equivalent to a condition for each $\gamma \in \Gamma$, we obtain Corollary 8.3.30, based on the standard counts (Table 8.1) and the fact that $\chi(I_k)(\gamma) = k$ for all $\gamma \in \Gamma$. We state results for some specific symmetry groups in Theorem 8.3.31.

Corollary 8.3.30. Let (G, c, p) be a k-coordinated framework that is Γ-symmetric with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$, and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^d)$, and let $|V_\gamma|$, $|E_\gamma|$ denote the number of vertices and edges fixed by $\gamma \in \Gamma$ respectively. If (G, c, p) is isostatic then, for each $\gamma \in \Gamma$,

$$|E_\gamma| = \mathrm{trace}(\tau(\gamma)) \cdot |V_\gamma| + k - \mathrm{trace}\Big((\tau \otimes P_V)^{(\mathcal{T})}(\gamma)\Big).$$

Theorem 8.3.31. Let (G, c, p) be a k-coordinated 2-dimensional framework that is Γ-symmetric with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$, and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^2)$, and let $|V_\gamma|$, $|E_\gamma|$ denote the number of vertices and edges fixed by $\gamma \in \Gamma$ respectively. If (G, c, p) is isostatic then we have the following:

a. if $s \in \tau(\Gamma)$, then $|E_s| = k + 1$;

b. if $C_2 \in \tau(\Gamma)$, then $|V_{C_2}| = 0$ and $|E_{C_2}| = k + 1$, or $|V_{C_2}| = 1$ and $|E_{C_2}| = k - 1$;

c. if $C_3 \in \tau(\Gamma)$, then $|E_{C_3}| = 0$, $|V_{C_3}| = 1$ and $k = 1$;

d. if $C_4 \in \tau(\Gamma)$, then $|E_{C_4}| = 0$, $k = 1$ and either $|V_{C_4}| = 0$ or $|V_{C_4}| = 1$;

e. if $C_n \in \tau(\Gamma)$ for $n \geq 5$, then $|E_{C_n}| = 0$, $|V_{C_n}| = 1$ and $k = 1$.

Proof. From Theorem 8.3.30 and Table 8.2 we have the following necessary conditions for each potential symmetry element $\gamma \in \Gamma$ in order for the k-coordinated framework (G, c, p) to be Γ-symmetric isostatic:

$$|E| = 2|V| + k - 3, \qquad |E_s| = 0 + k + 1, \qquad |E_{C_2}| = -2|V_{C_2}| + k + 1,$$

and, for $n \geq 3$,

$$|E_{C_n}| = 2\cos\frac{2\pi}{n}\left(|V_{C_n}| - 1\right) + k - 1.$$

It is clear that if (G, c, p) is isostatic and $s \in \tau(\Gamma)$, then $|E_s| = k + 1$.

We note that for a rotation C_n, $n \geq 2$, there can be at most one vertex fixed by C_n, since the configuration p of the framework is assumed to be injective.

Suppose that (G, c, p) is Γ-symmetric isostatic and $C_2 \in \tau(\Gamma)$. If $|V_{C_2}| = 0$, then $|E_{C_2}| = k + 1$. If instead $|V_{C_2}| = 1$, $|E_{C_2}| = -2 + k + 1 = k - 1$.

Suppose next that (G, c, p) is Γ-symmetric isostatic and $C_3 \in \tau(\Gamma)$. Since no edge can be fixed by a rotation C_n with $n > 2$, $|E_{C_3}| = 0 = 2\cos\frac{2\pi}{3}(|V_{C_3}| - 1) + k - 1$. If $|V_{C_3}| = 1$, $k - 1 = 0$, and hence the only isostatic coordinated frameworks with $C_3 \in \tau(\Gamma)$, $|E_{C_3}| = 0$ and $|V_{C_3}| = 1$ are 1-coordinated frameworks. If instead $|E_{C_3}| = 0$ and $|V_{C_3}| = 0$, we have $0 = k - 1 - 2\cos\frac{2\pi}{3} = k$. This implies that (G, c, p) is an uncoloured isostatic Γ-symmetric framework, a contradiction.

Suppose next that $C_4 \in \tau(\Gamma)$. As above, $|E_{C_4}| = 0$, and hence $0 = 2\cos\frac{2\pi}{4}(|V_{C_4}| - 1) + k - 1 = k - 1$, since $\cos\frac{\pi}{2} = 0$. Since p is assumed to be injective, $|V_{C_4}| = 0$ or $|V_{C_4}| = 1$.

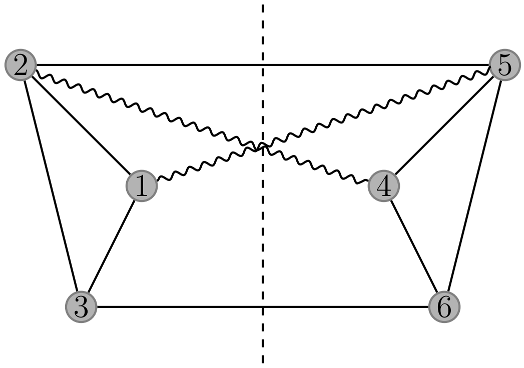
We suppose finally that $C_n \in \tau(\Gamma)$ for $n \geq 5$, and note that $|E_{C_n}| = 0 = 2\cos\frac{2\pi}{n}(|V_{C_n}| - 1) + k - 1$. When $|V_{C_n}| = 1$, this implies that $k = 1$ as before.

If instead $|V_{C_n}| = 0$, $0 = k - 1 - 2\cos\frac{2\pi}{n}$. For $n \neq 6$, $2\cos\frac{2\pi}{n} \notin \mathbb{Z}$, which would imply that (G, c, p) has a non-integer number of colour classes, a contradiction. When $n = 6$, $2\cos\frac{2\pi}{n} = 1$, and hence $k = 2$. We also note that $C_6 \in \tau(\Gamma)$ implies that $C_2, C_3 \in \tau(\Gamma)$, and $|V_{C_6}| = 0$ implies that $|V_{C_2}| = |V_{C_3}| = 0$. From the initial condition that $|E_{C_n}| = 2\cos\frac{2\pi}{n}(|V_{C_n}| - 1) + k - 1$ we obtain that $|E_{C_3}| = k$, which gives a clear contradiction between the statements $|E_{C_3}| = 0$ and $k = 2$. Hence no such isostatic framework with $C_6 \in \tau(\Gamma)$ and $|V_{C_6}| = 0$ exists. $\square$

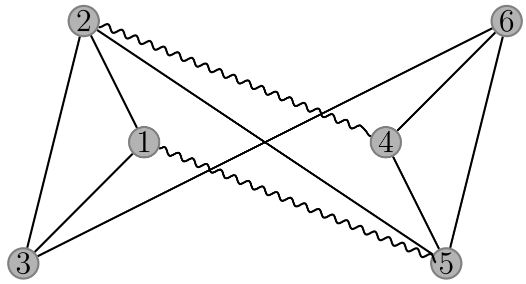
We note that Theorem 8.3.31 implies that the only symmetry groups for which isostatic frameworks with more than one class of coordinated edges exist are $\mathcal{C}_s$, $\mathcal{C}_2$ and $\mathcal{C}_{2v}$.

In contrast to the standard symmetric result (Theorem 8.2.11), there are 1-coordinated isostatic frameworks with all n-fold rotational symmetries with one central fixed vertex, while 1-coordinated isostatic frameworks with no fixed central vertex may have a 4-fold rotation.

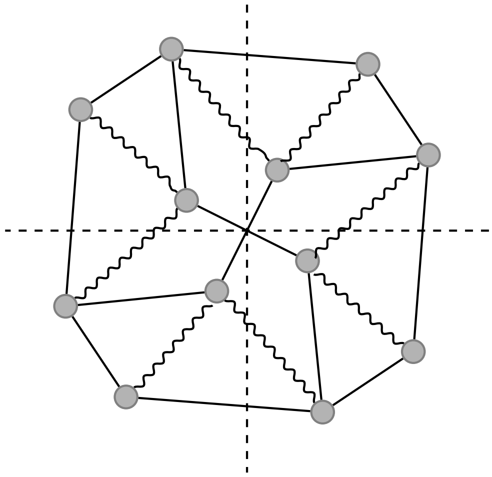
One family of examples of 1-coordinated frameworks with $C_n \in \tau(\Gamma)$ are the 1-coordinated wheel graphs (W_n, c_n) for $n \geq 3$, where c_n is a $\mathcal{C}_n$-symmetric 1-edge-



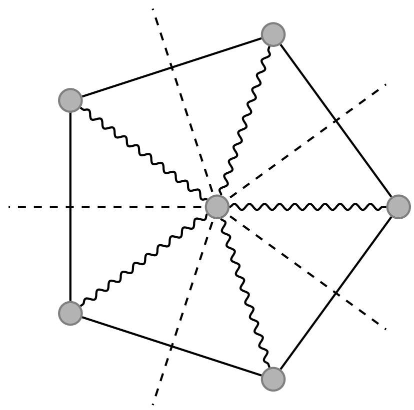
a The 1-coordinated $\mathcal{C}_s$-symmetric isostatic framework (G, c, p).



b The 1-coordinated $\mathcal{C}_2$-symmetric isostatic framework (G, c, p').



c The 1-coordinated $\mathcal{C}_4$-symmetric isostatic framework $(\widehat{G}, \widehat{c}, \widehat{p})$.



d The 1-coordinated $\mathcal{C}_{5v}$-symmetric isostatic framework $(\overline{G}, \bar{c}, \bar{p})$.

Figure 8.6 Figure 8.6a and Figure 8.6b show a $\mathcal{C}_s$-symmetric and a $\mathcal{C}_2$-symmetric 2-dimensional realisation of the same 1-edge-coloured graph (G, c). Both are Γ-symmetric isostatic.
Figure 8.6c is an example of an isostatic 2-dimensional 1-coordinated $\mathcal{C}_4$-symmetric framework, $(\widehat{G}, \widehat{c}, \widehat{p})$, while Figure 8.6d shows an example of an isostatic 2-dimensional 1-coordinated $\mathcal{C}_{5v}$-symmetric framework.

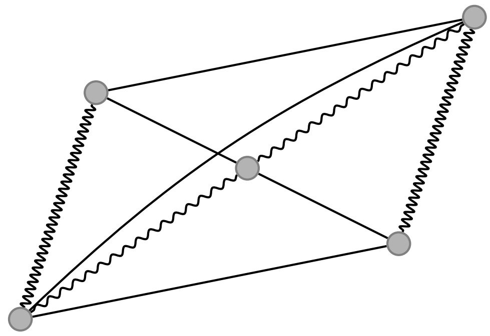


Figure 8.7 A $\mathcal{C}_2$-symmetric 2-dimensional 2-coordinated framework, (G, c, p). We note that the graph G is the 2-coordinated 2-dimensional base graph B_5 (Figure 7.6a). The curved edge should be considered as a direct edge fixed by C_2, however it is shown curved for clarity.

colouring of W_n. We note that W_n is a circuit with $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n = 2|V(W_n)| - 2$.

Let p_n be a $\mathcal{C}_n$-symmetric realisation of (W_n, c_n), and note that p_n is also a $\mathcal{C}_s$-symmetric realisation of (W_n, c_n). The quotient $\mathcal{C}_n$-gain graph of W_n contains two vertex orbits and two edge orbits, which implies that there are three potential $\mathcal{C}_n$-symmetric 1-edge-colourings. These are illustrated in Figure 8.8, and all may be checked to be $\mathcal{C}_s$-symmetric 1-edge-colourings for the appropriate axes of symmetry of (W_n, p_n): when n is even, (W_n, c_n, p_n) will have axes of $\mathcal{C}_s$ symmetry with $|V_s| = 1$, $|E_s| = 2$, and axes with $|V_s| = 3$, $|E_s| = 2$. Every axis of $\mathcal{C}_s$-symmetry for (W_n, c_n, p_n) with n odd has $|V_s| = 2$, $|E_s| = 2$. (W_n, c_n, p_n) also has $|E_{C_n}| = 0$ and $|V_{C_n}| = 1$.

The 1-coordinated framework (W_n, c_n, p_n) is therefore $\mathcal{C}_{nv}$-symmetric. It is straightforward to confirm, by the rank of the coordinated rigidity matrix $R(W_n, c_n, p_n)$ or otherwise, that (W_n, c_n, p_n) is $\mathcal{C}_{nv}$-symmetric isostatic. We may also apply a geometric argument when one edge orbit (with respect to C_n) is coloured, and the other is uncoloured, and it seems likely that a similar geometric argument applies when all edges are coloured.

We conjecture that we may extend the sufficient conditions for standard Γ-symmetric isostatic frameworks to 1-coordinated Γ-symmetric isostatic frameworks, and potentially

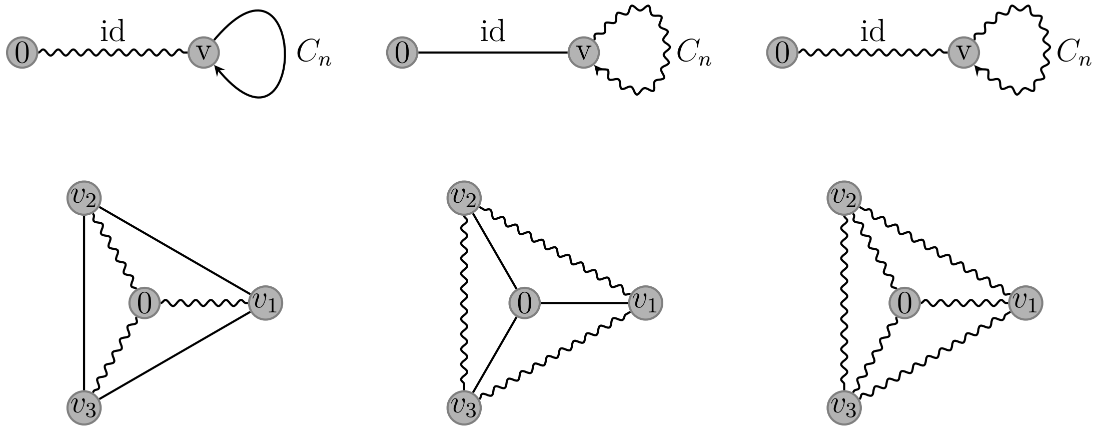


Figure 8.8 The three potential quotient $\mathcal{C}_n$-gain graphs for the 1-coordinated wheel framework (W_n, c_n, p_n), and the corresponding realisations for $n = 3$.

extend such characterisations for higher k in the case $\Gamma = \mathcal{C}_s$, $\Gamma = \mathcal{C}_2$ and $\Gamma = \mathcal{C}_3$. We formally state the following conjecture when $k = 1$ for a subset of groups.

Conjecture 8.3.32. Let (G, c, p) be a 1-coordinated 2-dimensional incidentally-Γ-symmetric framework (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to \mathcal{O}(\mathbb{R}^2)$. Then (G, c, p) is isostatic if and only if G is a Laman-plus-one graph such that $|D| \leq 2|V(D)| - 3$ for all uncoloured subgraphs $D \subseteq E_0$, and G (and any symmetric Laman-plus-one subframeworks of G) satisfy the following:

- **a.** when $\tau(\Gamma) = \mathcal{C}_s$, $|E_s| = 2$;
- **b.** when $\tau(\Gamma) = \mathcal{C}_2$, either $|V_{C_2}| = 0$ and $|E_{C_2}| = 2$, or $|V_{C_2}| = 1$ and $|E_{C_2}| = 0$;
- **c.** when $\tau(\Gamma) = \mathcal{C}_3$, $|E_{C_3}| = 0$ and $|V_{C_3}| = 1$;
- **d.** when $\tau(\Gamma) = \mathcal{C}_4$, $|E_{C_4}| = 0$, and either $|V_{C_4}| = 0$ and $|E_{C_2}| = 2$, or $|V_{C_4}| = 1$ and $|E_{C_2}| = 0$.

Chapter 9

Further directions

We now briefly discuss some additional open questions related to the rigidity of coordinated frameworks.

The bulk of this thesis considers infinitesimal rigidity of k-coordinated frameworks, leaving global rigidity of such frameworks as a natural area of additional interest. It may also be of interest to extend our work to higher dimensions or to alter the class of coordinated frameworks considered: either by considering alternative types of framework in place of the standard bar-joint framework, or by redefining the type of coordinated motion permitted.

9.1 Higher dimensions

As a combinatorial characterisation for rigidity of standard bar-joint frameworks in $\mathbb{R}^3$ has not yet been found, there is also no three-dimensional characterisation of rigid k-coordinated bar-joint frameworks. Instead, it may be of interest to consider alternative types of framework within $\mathbb{R}^3$.

One class of frameworks for which a complete combinatorial characterisation of rigidity exists is the class of body-bar frameworks. As discussed in Section 2.9, these consist of rigid bodies linked by bars, which may be represented using a copy of the complete graph K_n for each body (on an appropriate number of vertices to ensure that the bars between rigid bodies remain vertex disjoint). The bodies within a k-coordinated body-bar framework would still be required to be rigid blocks, and may be modelled by uncoloured complete bar-joint frameworks, while edges between bodies may be coloured arbitrarily.

Generically rigid body-bar frameworks in $\mathbb{R}^d$ are characterised to be those containing $\binom{d+1}{2}$ edge-disjoint spanning trees [Tay84, Tay's Theorem]. Since this characterisation has an inductive proof, it seems possible that we may be able to obtain a similar inductive proof to characterise generically rigid k-coordinated body-bar frameworks.

Multiple authors have considered the rigidity of frameworks restricted to surfaces within $\mathbb{R}^3$ [NOP12, JMN14, JN16]. It is possible that our results for the rigidity of coordinated frameworks in $\mathbb{R}^2$ would extend similarly to characterise the rigidity of coordinated frameworks on 2-dimensional surfaces within $\mathbb{R}^3$. In the plane we were able to utilise the X-replacement construction. On surfaces other than the plane or sphere it seems to be a difficult problem to show that X-replacement preserves rigidity without relying on existing combinatorial characterisations (e.g. [NOP12]), and hence alternative techniques would need to be developed. The counting conditions for rigid frameworks on surfaces are also different than the standard conditions in $\mathbb{R}^2$, so there may also be additional difficulties that arise in adding constraints for coloured edges to these.

Frameworks that lie on concentric spheres that may expand or contract [NSTW18] may be modelled as vertex-coloured frameworks, where each colour class contains the vertices that lie on one surface. This may be considered as being equivalent to an edge-coloured coned framework, obtained by creating a new central vertex x adjacent to every existing vertex, where the edge $\{x, v\}$ for each existing vertex v is coloured identically to the vertex-colouring of v. Since every edge within a colour class will have equal length, frameworks that lie on expanding concentric spheres may be considered as a special type of our coordinated frameworks, with the additional constraint that every coloured edge is adjacent to a central vertex x, and all edges within a colour class are required to have the same length. This type of construction could result in some partial results for rigidity of coordinated frameworks in $\mathbb{R}^3$, however this is a highly non-generic type of coordinated framework, compared to those considered throughout this thesis, and hence our combinatorial results do not apply to such frameworks.

9.2 Periodic frameworks

Another type of framework for which it may be of interest to consider coordinated analogues is periodic frameworks. These are infinite frameworks with a symmetry group Γ (isomorphic to $\mathbb{Z}^d$), such that every vertex has finite degree and the quotient Γ-gain graph G/Γ is finite [BS10, MT13, KST16].

As with symmetric frameworks, we require each copy of a given edge to be coloured identically by the k-edge-colouring c of the periodic framework (G, Γ, p), and so each edge orbit will be contained within a single colour class. This naturally induces a k-edge-colouring c_Γ of the quotient Γ-gain graph G/Γ, as discussed for symmetric frameworks.

Periodic frameworks may be characterised with respect to their quotient frameworks. Ross [Ros11] and Malestein and Theran [MT13] characterise rigidity of periodic frameworks through Laman-type characterisations of the quotient graph. Ross considers the quotient framework of a periodic framework in the plane as a framework on the fixed torus, proving that minimally rigid frameworks on the fixed torus are equivalent to

(2,2)-tight frameworks and providing an inductive construction for such frameworks. It seems likely that an analageous characterisation for 1-coordinated minimally rigid frameworks on the fixed torus would require $|E| = 2|V| + 1 - 2 = 2|V| - 1$, and a construction may be possible since a (2,1)-tight quotient graph will contain at least one vertex of degree 2 or degree 3. This would allow us to study the coordinated rigidity of infinite periodic frameworks by considering the rigidity of the corresponding finite coordinated quotient framework.

Crystallographic frameworks are a subset of the periodic frameworks, for which the symmetry group Γ is a crystallographic symmetry group [KP18, BS14]. Since there are many applications for crystallographic frameworks, it is natural to ask whether coordinated crystallographic frameworks may also have similar applications.

9.3 Alternative constraint systems

Many questions may be asked about extending the class of coordinated frameworks for which rigidity is considered. The type of coordination constraint may be adapted, and coordination constraints may be added to alternative types of framework, as discussed previously for body-bar frameworks.

9.3.1 Alternative types of framework

It is possible that existing work on frameworks within non-Euclidean geometries (see, for example, [KP14]) may be extended to k-coordinated frameworks within spaces with non-Euclidean norms, by adapting the initial definitions of Chapter 3 to alternative norms.

Another alternative type of framework that may be of interest to apply coordination to is the class of direction-length frameworks.

Definition 9.3.1 ([JK11])**.** A *mixed graph* $G = (V; D, L)$ is a graph $G = (V, E)$ for which the edge set E is decomposed into disjoint sets D and L, denoting direction edges and length edges respectively. Parallel edges are permitted within G if one is of each type. A *d-dimensional direction-length framework* is (G, p) for a mixed graph $G = (V; D, L)$ and a configuration of the vertices $p \in \mathbb{R}^{d|V|}$.

Two direction-length frameworks (G, p) and (G, q) are *equivalent* if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for all $\{u, v\} \in L$, and $p(u) - p(v)$ is a scalar multiple of $q(u) - q(v)$ for all $\{u, v\} \in D$. (G, p) and (G, q) are *congruent* if, for all $u, v \in V$, $p(u) - p(v) = \pm 1 \cdot \Big(q(u) - q(v)\Big)$.

We note that length edges within a direction-length framework are equivalent to bars in a standard bar-joint framework, so it is straightforward to define coordinated length edges. Since the length of a direction edge is not defined, it would be possible to define a k-coordinated direction-length framework to be a direction-length framework with a k-edge-colouring of the length edges only, $c : L \to \{0, 1, \dots, k\}$, which induces a partition $L_0 \cup \cdots \cup L_k$. It may also be natural in some circumstances to define coordinated changes to direction edges, perhaps by allowing all direction edges within a colour class to change angle by the same amount.

9.3.2 Alternative coordination constraints

This thesis mainly concerns itself with coordinated frameworks in which the coordinated edges increase or decrease length by the same amount, which is equivalent to preserving the pairwise differences of edges within a coordination class (see Note 3.1.14). This perspective allowed for straightforward notation for equivalent frameworks in the context of continuous rigidity, where (G, c, p, r) and (G, c, q, s) are equivalent when

$\|p(i) - p(j)\|^2 + r(\ell) = \|q(i) - q(j)\|^2 + s(\ell)$ for all edges $\{i,j\} \in E_\ell$. We were also able to define a reasonable notation for infinitesimal rigidity of this type of coordinated framework, by using the characteristic vectors $\mathbb{1}_\ell$ for each colour class to define the coordinated rigidity matrix.

It may be intuitive to define coordinated motions of frameworks as preserving ratios of the lengths of edges within a coordination class. If this type of coordinated motion is permitted, it is possible to define equivalence of two realisations of a k-coloured graph, (G,c,p) and (G,c,q), as follows:

$$\begin{aligned} \|p(i) - p(j)\|^2 &= \|q(i) - q(j)\|^2 && \text{for all } \{i,j\} \in E_0, \\ \frac{\|p(i) - p(j)\|^2}{\|p(u) - p(w)\|^2} &= \frac{\|q(i) - q(j)\|^2}{\|q(u) - q(w)\|^2} && \text{for all pairs } \{i,j\},\{u,w\} \in E_\ell,\ \ell \in \{1,\dots,k\}. \end{aligned}$$

Another alternative type of coordination constraint that seems natural is to require that the sum of the lengths of edges within a coordination class remains constant. This may be considered as a generalisation of the concept of a pulley system, within which the total length of the rope cannot change.

Figure 9.1 Two positions of a pulley system. The length of the rope connecting two joints via the third remains fixed, though their positions relative to one another change.

In the context of continuous rigidity, two such frameworks (G,c,p) and (G,c,q) would be *equivalent* when

$$\begin{aligned} \|p(i) - p(j)\|^2 &= \|q(i) - q(j)\|^2 && \text{for all } \{i,j\} \in E_0, \\ \sum_{\{i,j\}\in E_\ell} \|p(i) - p(j)\|^2 &= \sum_{\{i,j\}\in E_\ell} \|q(i) - q(j)\|^2 && \text{for } \ell \in \{1,\dots,k\}. \end{aligned}$$

In both these contexts, the length constraint for a single edge within a colour class depends on the position vectors of every vertex adjacent to an edge in this colour class. Any analogue to the rigidity matrix in either case would therefore have a very different structure to the standard rigidity matrix, since it would contain rows involving more than two vertices. This suggests that the underlying combinatorial structure in these situations will be a hypergraph. Defining infinitesimal rigidity for either of these types of framework, and obtaining combinatorial characterisations for generic rigidity for them, remain as open questions.

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