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A Novel Disturbance Rejection Zeroing Neurodynamic Approach for Robust Synchronization of Chaotic Systems

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ABSTRACT Robust synchronization of chaotic systems becomes a hot topic in scientific and engineering fields because of the ubiquitous existence of time-variant external disturbances in complex application scenarios. In contrast with existing studies that the resultant synchronization error has a supremum or even diverges under the influence of time-variant external disturbances, this paper proposes a disturbance rejection zeroing neurodynamic (DRZN) approach and its related controller for the robust synchronization of chaotic systems against time-variant external disturbances. The controller designed by the proposed DRZN approach distinctively features the rejection of external disturbances with the generated synchronization error being convergence toward zero. Theoretical analyses guarantee that the DRZN approach and its related controller inherently possess robustness. Moreover, numerical studies including three examples substantiate the effectiveness of the proposed DRZN approach and its related controller for the synchronization of chaotic systems against the time-variant external disturbances. Comparisons with existing approaches, e.g., the conventional zeroing neurodynamic (CZN) approach and the linear-active control (LAC) approach, show the superiority of the proposed DRZN approach. Extensive tests further verify that the proposed DRZN approach possesses the outstanding anti-disturbance performance, and thus is suitable for the practical applications with time-variant external disturbances.

INDEX TERMS Zeroing neurodynamic approach, robustness, zeroing neural networks, chaotic systems, external disturbances.

I. INTRODUCTION

Lorenz [1] creatively introduced the study on chaotic attractor half a century ago. After such a pioneering work, a lot of research on the chaotic control has been developed and investigated [2]–[8]. As an interesting and attractive phenomenon in chaotic control, the chaotic synchronization has become a heated research topic, which has attracted attentions of the researchers in recent years [9]–[13]. The synchronization of chaotic systems is a procedure that two (either identical or nonidentical) chaotic systems adjust a provided property of their motion to a frequent behavior by forcing or coupling [14]. In other words, starting with random initial conditions, all states of the slave (or termed response) chaotic system are forced to asymptotically track all states of the master (or termed drive) chaotic system. The synchronization of chaotic systems has numerous practical applications [15], [16], such as secure communications [17], [18], finance systems [19], electronic systems [20], ecological systems [21], and many different engineering systems [22]. For instances, Mobayen and Ma [13] proposed a new nonlinear feedback approach for the outstanding performance synchronization of chaotic systems under the influence of external disturbances, parametric uncertainties, Lipschitz nonlinearities as well as time delays. Yang and Zhang [17] introduced an effective global chaotic synchronization strategy of the identical systems, and then successfully applied to the secure communication. Naderi and Kheiri [18] detailedly investigated the

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exponential synchronization of the chaotic system without linear term, and then successfully applied to the secure communication. In [20], numerical, analog together with digital-circuit models were detailedly investigated with a specific chaotic system. In addition, a novel synaptic weight-update-learning rule of Hermite neural network was proposed in [22], and it was applied to the cryptography via combining Hermite polynomials. Moveover, Chen *et al.* [23] novelly investigated the hybrid synchronization behavior in an array of coupled chaotic systems with ring connection.

Up to now, a large number of approaches and techniques for the synchronization of chaotic systems have been creatively proposed and effectively employed, such as the sliding-mode-control approach [2], the neurodynamic approach [24]–[29], the active-control approach [14], [30], and the adaptive-backstepping control approach [31]. For instances, Ahmad *et al.* [14] studied and investigated a new global chaotic synchronization problem for identical chaotic systems as well as nonidentical chaotic systems by novelly utilizing a linear-active control (LAC) approach. In addition, Li *et al.* [24] presented a relatively simple controller generated by the conventional zeroing neurodynamic (CZN) approach for the synchronization of chaotic systems considering the influence of parameter perturbation, model uncertainty as well as external disturbance. Lin *et al.* [31] proposed an effective systematic approach for modeling and neural adaptive-backstepping control of an uncertain chaotic system by employing only input-and-output data from the underlying-dynamical systems. In the work by Song and Huang [32], the stabilization as well as synchronization of chaotic systems considering the time-variant delays via intermittent control were detailedly investigated. Li *et al.* [33] novelly developed an effective unified approach for impulsive lag-synchronization of chaotic systems in consideration of time-delay by employing the stability theory of impulsive-delayed-differential equations.

Among effective methodologies for the synchronization of chaotic systems, the neurodynamic approach emerges to be a prior alternative for researchers and engineers, due to the advantages of parallelism, distributed storage, as well as adaptive self-learning capability [34]–[44]. As typical kinds of recurrent neural networks (RNN) [45]–[48], a large number of neural network models designed by the CZN approach, are developed as feasible schemes for the time-variant engineering problems solving (including the time-variant synchronization of chaotic systems) [49]–[52]. For instances, Zhang *et al.* [50] detailedly investigated the tracking control problems of Lorenz, Chen and Lu chaotic systems via combining the CZN approach and gradient neurodynamic approach for developing an effective controller. In addition, a simple stabilization control approach of hyper-chaotic Lu system with one control input was proposed in [51] by employing the neurodynamic approach. Jin *et al.* [52] novelly proposed an effective controller design approach for the tracking control of a modified Lorenz nonlinear system in consideration of singularities (or termed division by zero issue) handling.

Although the exhibiting approach, such as the CZN approach [24] and the LAC approach [14], have found extensive applications in different scientific and engineering problems including the synchronization of chaotic systems. However, the research on robustness of the CZN approach for applying to synchronization of chaotic systems still remains deficient. Specifically, in complex application scenarios, there unavoidably exist different forms of external disturbances for chaotic systems such as offset errors, interactions with the environment, electromagnetism interferences in circuit systems, as well as bounded-random noises during signals transmission, etc. [53]. Sometimes those time-variant external disturbances might have negative impacts on both the stability and accuracy of the related controller or model for the synchronization of chaotic systems, and even worse, may cause the failure of the synchronization process [54]. Therefore, a robust approach is urgently needed for the synchronization of chaotic systems to reject the time-variant external disturbances in practice. This is the involved scientific problem and also the motivation that we propose and investigate a novel disturbance rejection approach and its related controller in this work.

As a preliminary attempt, the researchers in reference [24] proposed a relatively simple controller on the basis of the CZN approach to reduce the negative impacts derived from time-variant external disturbances. It was theoretically proven that, with bounded-and-random disturbances, the resultant synchronization error has a supremum by using such a controller [24]. However, the error accumulation phenomenon may occur if such a controller is applied in synchronization of chaotic systems with multiple sub-systems or long synchronization duration finally leading to a high computation burden and low synchronization accuracy. To accelerate the synchronization process of chaotic systems, another work in reference [55] introduced a super-exponential-zeroing neurodynamic approach together with a controller without considering the time-variant external disturbances. It could be theoretically proven that the resultant synchronization error would have a supremum by using the related controller in reference [55] under the influence of disturbances. In addition, Ahmad *et al.* investigated a new global chaotic synchronization problem by novelly utilizing an LAC approach in [14]. However, the impacts of time-variant external disturbances on the accuracy and stability of chaotic systems are not considered in the work [14].

To handle the difficulties and limitations discussed above, unlike the research based on the CZN approach [24] and LAC approach [14] or focusing on the rate of convergence [55], this paper proposes a novel disturbance rejection zeroing neurodynamic (DRZN) approach and its related controller for the robust synchronization of chaotic systems against the time-variant external disturbances. The controller designed by the proposed DRZN approach distinctively features the rejection of time-variant external disturbances with the

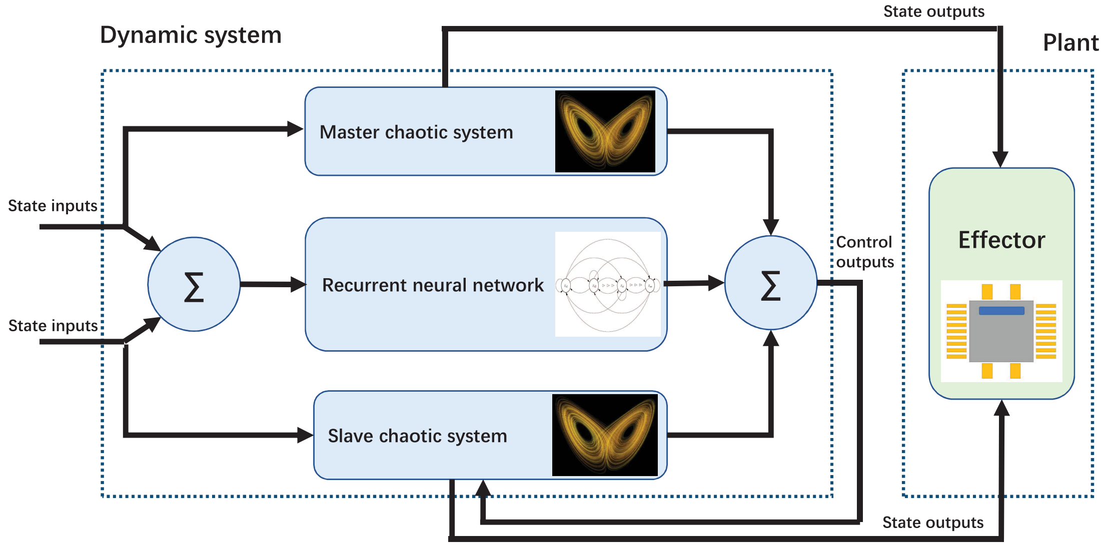


FIGURE 1. **Zeroing neurodynamic architecture for synchronization of chaotic systems.**

synthesized synchronization error being convergence toward zero. Detailed theoretical analyses on the anti-disturbance property and the convergence performance are provided and compared with existing approaches to guarantee the inherent robustness and effectiveness of the proposed DRZN approach and its related controller. Numerical studies including three synchronization examples, comparisons with existing approaches, and extensive tests verify the efficacy and superiority of the proposed DRZN approach and its related controller in practical applications. To the best of the authors' knowledge, such a DRZN approach and its related controller with the outstanding anti-disturbance property for the synchronization of chaotic systems have not been reported in other existing studies. Therefore, this work makes new progresses in both theory and practice for the robust synchronization of chaotic systems. Moreover, the zeroing neurodynamic architecture for designing the controller of chaotic systems is presented in Fig. 1 for better understanding the main principle. As shown in Fig. 1, the whole control system via the proposed DRZN approach is a typical closed-loop control system. The control system uses the initial states of the master and slave chaotic systems as the input information, and the control outputs are used as the system feedback for the slave chaotic system. Both the final state outputs of the master and slave chaotic systems can be integrated to the plant and implemented as the effector for the whole control system.

The rest of the paper is structured as follows. In Section II, the problem formulation of the synchronization between two chaotic systems under the influence of time-variant external disturbances is presented as preliminaries. In Section III, the DRZN approach and its related controller are proposed with theoretical analyses. Section IV shows numerical studies including three synchronization examples, comprehensive comparisons and extensive tests. Section V concludes the paper. The main contributions of the paper are highlighted as below.

- In contrast with existing works that the resultant synchronization error has a supremum or even diverges under the influence of external disturbances, this paper proposes a novel DRZN approach and its related controller for the synchronization of chaotic systems against the time-variant external disturbances.
- The controller generated by the proposed DRZN approach distinctively features the rejection of external disturbances with the synthesized synchronization error being convergence toward zero. It is a breakthrough in the robustness research of neurodynamic approach as well as the synchronization of chaotic systems.
- Theoretical analyses on the anti-disturbance property and the convergence performance are presented and compared to guarantee the inherent robustness and effectiveness of the proposed DRZN approach and its related controller.
- Numerical studies including three synchronization examples, comparisons with existing approaches, as well as extensive tests verify the efficacy and superiority of the proposed DRZN approach and its related controller under the influence of time-variant external disturbances in practical applications.

II. PROBLEM FORMULATION

The problem formulation of the synchronization between two chaotic systems under the influence of time-variant external disturbances are presented in this section. Consider a master chaotic system with a general form as below:

$$\dot{\mathbf{x}}_{\mathrm{m}}(t) = \mathbf{f}_{\mathrm{m}}(\mathbf{x}_{\mathrm{m}}(t)), \tag{1}$$

where $\mathbf{x}_{\mathrm{m}}(t) = [x_{\mathrm{m}1}(t), x_{\mathrm{m}2}(t), \cdots, x_{\mathrm{m}n}(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is the state vector of the master chaotic system, and

$\mathbf{f}_{\mathrm{m}}(\cdot):\mathbb{R}^n\rightarrow\mathbb{R}^n$ is the nonlinear-mapping vector of a specific master chaotic system. Then, the slave chaotic system with a general form is depicted in

$$\dot{\mathbf{x}}_{\mathrm{s}}(t)=\mathbf{f}_{\mathrm{s}}(\mathbf{x}_{\mathrm{s}}(t))+\mathbf{u}(t), \tag{2}$$

where vector $\mathbf{x}_{\mathrm{s}}(t)=[x_{\mathrm{s}1}(t),x_{\mathrm{s}2}(t),\cdots,x_{\mathrm{s}n}(t)]^{\mathrm{T}}\in\mathbb{R}^n$ is the state vector of the slave chaotic system, and mapping $\mathbf{f}_{\mathrm{s}}(\cdot):\mathbb{R}^n\rightarrow\mathbb{R}^n$ is the nonlinear-mapping vector of a specific slave chaotic system. Besides, vector $\mathbf{u}(t)=[u_1(t),u_2(t),\cdots,u_n(t)]^{\mathrm{T}}\in\mathbb{R}^n$ is the control-input vector sent to the slave chaotic system for synchronization.

Note that the exhibiting approaches have found extensive applications in the synchronization of chaotic systems. However, with unexpected impacts, such as the electromagnetism interferences in communication systems and continuous-and-random noises during the signal transmission, which could be deemed as time-variant external disturbances for the related chaotic systems, a more practical form of master chaotic system can be depicted in the following form to describe the involved scientific problem:

$$\dot{\mathbf{x}}_{\mathrm{m}}(t)=\mathbf{f}_{\mathrm{m}}(\mathbf{x}_{\mathrm{m}}(t))+\varrho(t), \tag{3}$$

where $\varrho(t)=[\varrho_1(t),\varrho_2(t),\cdots,\varrho_n(t)]^{\mathrm{T}}\in\mathbb{R}^n$ denotes the bounded external disturbances during the synchronization process, i.e., $|\varrho_i(t)|\leq\varrho_{\max}$ for $t\in[0,T_{\mathrm{d}}]$, and $\varrho_i(t)=0$ for $t>T_{\mathrm{d}}$, with $i=0,1,\cdots,n$, $\varrho_{\max}$ being a constant, and T_{d} being a synchronization duration, of the master chaotic system.

If the initial-state vectors $\mathbf{x}_{\mathrm{m}}(0)$ and $\mathbf{x}_{\mathrm{s}}(0)$ of the master and slave chaotic systems are different from each other, the state trajectories of such two systems may differ very much. The objective in the paper for synchronization between the master chaotic system (3) with external disturbances $\varrho(t)$ and the slave chaotic system (2) is to design a control-input vector $\mathbf{u}(t)$ so that the slave system is forced to track the master system with the synthesized state error $\mathbf{e}(t)=\mathbf{x}_{\mathrm{m}}(t)-\mathbf{x}_{\mathrm{s}}(t)$ converging toward zero.

Remark 1: Due to the complexity of the synchronization process of chaotic systems, the external disturbances may be unavoidable in real-world applications. In addition, most of specific chaotic systems could be formulated as the master chaotic system (3) with a general form of nonlinear-differential equations, such as Lu chaotic systems [50], Chen chaotic systems [51], and Lorenz chaotic systems [52], which covers most common chaotic systems [24].

III. DESIGN APPROACHES AND THEORETICAL ANALYSES

In this section, we propose the DRZN approach for designing the controller of chaotic systems. For further investigation and better comparison, the controller designed by the CZN approach is also provided. In addition, theoretical analyses are presented to guarantee the effectiveness and robustness of the proposed DRZN approach and the related controller.

A. DESIGN APPROACHES

To achieve the synchronization between the slave chaotic system (2) and the master chaotic system (3) under the influence of external disturbances, a controller is designed by the DRZN approach with detailed process shown as below.

Firstly, to monitor the synchronization process of chaotic systems (2) and (3), a vector-valued error function to measure the difference between the states of the master and slave chaotic systems is defined as below:

$$\mathbf{e}(t)=\mathbf{x}_{\mathrm{m}}(t)-\mathbf{x}_{\mathrm{s}}(t). \tag{4}$$

To make each element $e_i(t)$ (with $i=1,2,\cdots,n$) of the synthesized error (4) converge towards zero with respective to time t, the DRZN approach is employed with its dynamic equation described as below:

$$\dot{\mathbf{e}}(t)=-\kappa\Upsilon(\mathbf{e}(t))-\nu\int_0^t\mathbf{e}(\tau)\mathrm{d}\tau+\varrho(t), \tag{5}$$

where design parameters $\kappa\in\mathbb{R}^+$ and $\nu\in\mathbb{R}^+$ are chose for the stability as well as convergence of the neurodynamic model, and vector $\varrho(t)$ denotes the time-variant external disturbances that need to be rejected. In addition, $\Upsilon(\cdot):\mathbb{R}^n\rightarrow\mathbb{R}^n$ is an alternative activation-function vector mapping. Note that the elements of the activation-function vector mapping can be a monotonically-increasing odd function which could be adjusted to improve the convergence performance of the neurodynamic model. Without losing of generality and also for simplicity, a linear activation-function $\Upsilon(\mathbf{e}(t))=\mathbf{e}(t)$ is utilized and investigated in the paper.

By substituting chaotic systems (2) and (3) into dynamic equation (5), the corresponding neurodynamic model for the synchronization of chaotic systems can be depicted as follows:

$$\mathbf{f}_{\mathrm{m}}(\mathbf{x}_{\mathrm{m}}(t))+\varrho(t)-\mathbf{f}_{\mathrm{s}}(\mathbf{x}_{\mathrm{s}}(t))-\mathbf{u}(t)=-\kappa\Upsilon(\mathbf{x}_{\mathrm{m}}(t)-\mathbf{x}_{\mathrm{s}}(t))-\nu\int_0^t(\mathbf{x}_{\mathrm{m}}(\tau)-\mathbf{x}_{\mathrm{s}}(\tau))\mathrm{d}\tau+\varrho(t) \tag{6}$$

with $\dot{\mathbf{e}}(t)=\dot{\mathbf{x}}_{\mathrm{m}}(t)-\dot{\mathbf{x}}_{\mathrm{s}}(t)$. Due to the fact that there unavoidably exist unexpected time-variant external disturbances in complex application scenarios, the synchronization of chaotic systems would be a knotty time-variant problem. As readily founded in neurodynamic model (6), the proposed DRZN approach can effectively handle the disturbance rejection issue in a relatively simple manner by fully exploiting both the time-derivative and integral information of the involved chaotic systems.

According to neurodynamic model (6), the related controller with explicit control-input vector $\mathbf{u}(t)$ is thus designed as follow:

$$\mathbf{u}(t)=\mathbf{f}_{\mathrm{m}}(\mathbf{x}_{\mathrm{m}}(t))-\mathbf{f}_{\mathrm{s}}(\mathbf{x}_{\mathrm{s}}(t))+\kappa\Upsilon(\mathbf{x}_{\mathrm{m}}(t)-\mathbf{x}_{\mathrm{s}}(t))+\nu\int_0^t(\mathbf{x}_{\mathrm{m}}(\tau)-\mathbf{x}_{\mathrm{s}}(\tau))\mathrm{d}\tau. \tag{7}$$

Note that the related controller (7) designed by the proposed DRZN approach does not require any information

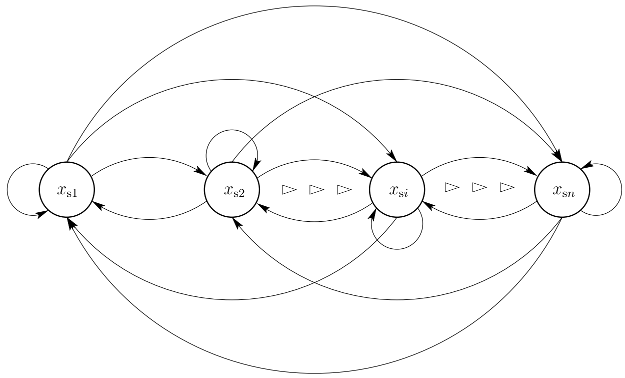


FIGURE 2. Neuron-connection architecture of neurodynamic model (6) for designing the controller of chaotic systems with x_{si} denoting the ith neuron.

of external disturbances which is thus applicable in practical applications. For better understanding of practitioners, the neuron-connection architecture of neurodynamic model (6) for designing the controller of chaotic systems via the proposed DRZN approach is presented in Fig. 2. As we can see from the figure, the neurodynamic model (6) is a typical kind of Hopfield-type RNN [56], which can be developed and implemented easily on analog circuits such as very-large-scale-integration [57].

To lay a basis for further investigation and comparison, the controller designed by the CZN approach for the synchronization of chaotic systems is also provided as below [24]:

$$\mathbf{u}(t) = \mathbf{f}_{\mathrm{m}}(\mathbf{x}_{\mathrm{m}}(t)) - \mathbf{f}_{\mathrm{s}}(\mathbf{x}_{\mathrm{s}}(t)) + \gamma\Psi(\mathbf{x}_{\mathrm{m}}(t) - \mathbf{x}_{\mathrm{s}}(t)), \quad (8)$$

where design parameter $\gamma \in \mathbb{R}^+$ is set for the stability and convergence of the above controller, and $\Psi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is an alternative activation-function vector mapping for the CZN approach. For better comparison, design parameters are set to be $\gamma = \kappa$, and activation-function vector mappings $\Upsilon(\cdot)$ and $\Psi(\cdot)$ are both set to be linear activation function.

B. THEORETICAL ANALYSES

In this subsection, to guarantee the effectiveness and robustness of the proposed DRZN approach as well as the related controller (7) for the synchronization of chaotic systems under the influence of time-variant external disturbances, the theoretical analyses are presented. Then, the corresponding theoretical results of the CZN approach are also presented for better comparison.

Definition 1: For the synchronization of chaotic systems (2) and (3), starting with a random initial state $\mathbf{x}_{\mathrm{s}}(0)$, a vector-valued error function $\mathbf{e}(t)$ at time $t \geq 0$ synthesized by a control system is said to be convergent toward zero if it satisfies

$$\lim_{t\to\infty}\sup\{\|\mathbf{e}(t)\|_{\mathrm{E}}\} = 0,$$

where symbol $\sup\{\cdot\}$ denotes the supremum of a sequence, and symbol $\|\cdot\|_{\mathrm{E}}$ denotes the Euclidean norm of a vector.

Definition 2: For the synchronization of chaotic systems (2) and (3), starting with a random initial state $\mathbf{x}_{\mathrm{s}}(0)$, a state trajectory $\mathbf{x}_{\mathrm{s}}(t)$ of slave chaotic system (2) at time $t \geq 0$ synthesized by a control system is said to be convergent toward the state $\mathbf{x}_{\mathrm{m}}(t)$ of master chaotic system (3) with external disturbances if it satisfies

$$\mathbf{x}_{\mathrm{m}}(t) - \mathbf{x}_{\mathrm{s}}(t) = \mathbf{e}(t) \to 0, \quad \text{as } t \to \infty.$$

Theorem 1: For the synchronization of chaotic systems (2) and (3), starting with a random initial state $\mathbf{x}_{\mathrm{s}}(0)$, the vector-valued error function $\mathbf{e}(t)$ at time $t \geq 0$ synthesized by control system equipped with controller (7) converges toward zero.

Proof: Review the neurodynamic model (6) designed by the proposed DRZN approach as

$$\mathbf{f}_{\mathrm{m}}(\mathbf{x}_{\mathrm{m}}(t)) + \varrho(t) - \mathbf{f}_{\mathrm{s}}(\mathbf{x}_{\mathrm{s}}(t)) - \mathbf{u}(t) = -\kappa\Upsilon(\mathbf{x}_{\mathrm{m}}(t) - \mathbf{x}_{\mathrm{s}}(t)) - \nu\int_0^t(\mathbf{x}_{\mathrm{m}}(\tau) - \mathbf{x}_{\mathrm{s}}(\tau))\mathrm{d}\tau + \varrho(t). \quad (9)$$

Within the region of convergence and $\mathbf{e}(t) = \mathbf{x}_{\mathrm{m}}(t) - \mathbf{x}_{\mathrm{s}}(t)$, the Laplace transformation [58] of the ith subsystem of neurodynamic model (9) is depicted as

$$\mathcal{L}(\dot{e}_i(t)) = \mathcal{L}\Big(-\kappa e_i(t) - \nu\int_0^t e_i(\tau)\mathrm{d}\tau + \varrho_i(t)\Big), \quad (10)$$

where operator $\mathcal{L}(\cdot)$ denotes the Laplace transformation with $i = 1, 2, \cdots, n$ with activation-function vector mapping being a linear activation function, i.e., $\Upsilon_i(e_i(t)) = e_i(t)$. From (10), we have

$$se_i(s) - e_i(0) = -\kappa e_i(s) - \frac{\nu}{s}e_i(s) + \int_0^{+\infty}\varrho_i(t)\exp(-st)\mathrm{d}t, \quad (11)$$

where $\int_0^{+\infty}\varrho_i(t)\exp(-st)\mathrm{d}t$ is the Laplace transformation of each element of time-variant external disturbances, i.e., $\varrho_i(t)$. Note that equation (11) could be rewritten as the following form:

$$(s^2 + s\kappa + \nu)e_i(s) = se_i(0) + s\int_0^{+\infty}\varrho_i(t)\exp(-st)\mathrm{d}t.$$

Note that in the field of signals and systems, the neurodynamic model with time-variant disturbances can be skillfully characterized and analyzed in the transform domain by algebraic manipulations.

According to the final-value theorem [58], for bounded external disturbances during the synchronization process, i.e., $|\varrho_i(t)| \leq \varrho_{\max}$ for $t \in [0, T_{\mathrm{d}}]$, and $\varrho_i(t) = 0$ for $t > T_{\mathrm{d}}$ with $i = 0, 1, \cdots, n$, we have

$$\begin{aligned}\Big|\lim_{t\to\infty} e_i(t)\Big| &= \Big|\lim_{s\to 0} se_i(s)\Big| \\ &= \left|\lim_{s\to 0}\frac{s^2\left(e_i(0) + \int_0^{+\infty}\varrho_i(t)\exp(-st)\mathrm{d}t\right)}{s^2 + s\kappa + \nu}\right|\end{aligned}$$

$$= \left| \lim_{s\to 0} \frac{s^2\left(e_i(0)+\int_0^{T_\mathrm{d}} \varrho_i(t)\exp(-st)\mathrm{d}t\right)}{s^2+s\kappa+\nu} \right|$$
$$\leq \left| \lim_{s\to 0}\sup \left\{ \frac{|s^2|\left(|e_i(0)|+\left|\int_0^{T_\mathrm{d}} |\varrho_i(t)|\exp(-st)\mathrm{d}t\right|\right)}{|s^2+s\kappa+\nu|} \right\} \right|$$
$$\leq \left| \lim_{s\to 0}\sup \left\{ \frac{|s^2|\left(|e_i(0)|+\varrho_{\max}\left|\int_0^{T_\mathrm{d}} \exp(-st)\mathrm{d}t\right|\right)}{|s^2+s\kappa+\nu|} \right\} \right|$$
$$= \left| \lim_{s\to 0}\sup \left\{ \frac{|s^2|\,|e_i(0)|+\varrho_{\max}|s|\,|(1-\exp(-sT_\mathrm{d}))|}{|s^2+s\kappa+\nu|} \right\} \right|$$
$$= 0.$$

With each element of error function bing $|\lim_{t\to\infty} e_i(t)| = \lim_{t\to\infty} e_i(t) = 0$ $(i = 1, 2, \cdots, n)$, we finally obtain the following result:

$$\lim_{t\to\infty}\sup\{\|\mathbf{e}(t)\|_\mathrm{E}\} = 0. \quad (12)$$

According to Definition 1, we finally have the result that the vector-valued error function $\mathbf{e}(t)$ synthesized by the control system equipped with controller (7) converges toward zero. The proof is thus completed. □

Corollary 1: For the synchronization of chaotic systems (2) and (3), starting with a random initial state $\mathbf{x}_\mathrm{s}(0)$, the state trajectory $\mathbf{x}_\mathrm{s}(t)$ of slave chaotic system (2) at time $t \geq 0$ synthesized by a control system equipped with controller (7) converges toward the state $\mathbf{x}_\mathrm{m}(t)$ of master chaotic system (3) with external disturbances.

Proof: It can be generalized from Definition 2 and the proof of Theorem 1. □

For better comparison, the corresponding theoretical results of the controller (8) designed by CZN approach for the synchronization of chaotic systems (2) and (3) are also presented as the following lemma [24].

Lemma 1: For the synchronization of chaotic systems (2) and (3), starting with a random initial state $\mathbf{x}_\mathrm{s}(0)$, the vector-valued error function $\mathbf{e}(t)$ at time $t \geq 0$ synthesized by control system equipped with controller (8) converges with a steady-state error having a supremum, i.e., $\lim_{t\to\infty}\sup\{\|\mathbf{e}(t)\|_\mathrm{E}\} = \|\varrho(t)\|_\mathrm{E}/\gamma$.

Proof: Review the neurodynamic model (6) designed by the CZN approach as follows:

$$\mathbf{u}(t) = \mathbf{f}_\mathrm{m}(\mathbf{x}_\mathrm{m}(t)) - \mathbf{f}_\mathrm{s}(\mathbf{x}_\mathrm{s}(t)) + \gamma\Psi(\mathbf{x}_\mathrm{m}(t) - \mathbf{x}_\mathrm{s}(t)). \quad (13)$$

According the chaotic systems (2) and (3), with $\mathbf{e}(t) = \mathbf{x}_\mathrm{m}(t) - \mathbf{x}_\mathrm{s}(t)$, one can obtain the following dynamic equation:

$$\dot{\mathbf{e}}(t) = -\gamma\mathbf{e}(t) + \varrho(t). \quad (14)$$

Solving the above dynamic equation, one can readily obtain:

$$\|\mathbf{e}(t)\|_\mathrm{E} \leq \alpha\exp(-\gamma t) + \frac{\|\varrho(t)\|_\mathrm{E}}{\gamma}, \quad (15)$$

where α is a constant. Therefore, one can finally have:

$$\lim_{t\to\infty}\sup\{\|\mathbf{e}(t)\|_\mathrm{E}\} = \frac{\|\varrho(t)\|_\mathrm{E}}{\gamma}.$$

Therefore, it has the result that the vector-valued error function $\mathbf{e}(t)$ at time $t \geq 0$ synthesized by control system equipped with controller (8) converges with a steady-state error having a supremum $\|\varrho(t)\|_\mathrm{E}/\gamma$. The proof is thus completed. □

IV. NUMERICAL STUDIES

In this section, numerical studies including three synchronization examples, comparisons with existing approaches, and extensive tests are performed to verify the effectiveness, robustness as well as superiority of the proposed DRZN approach and the related controller (7) for the synchronization of chaotic systems with time-variant disturbance rejection.

A. SYNCHRONIZATION EXAMPLES

In the examples, we successively consider the synchronization of two identical Lu chaotic systems, synchronization of two identical autonomous chaotic systems, and synchronization of two nonidentical chaotic systems. Without losing generality, the synchronization duration is selected to be $T_\mathrm{d} = 10$ s. In addition, design parameters are selected to be $\kappa = 3$ and $\nu = 30$. The initial value of each state of both the master chaotic systems and slave chaotic systems is generated randomly between 0 and 1. The time-variant external disturbances in the examples are selected to be $\varrho(t) = [2, 2\sin(0.2t) + 4\exp(-0.5t), 3\cos(0.1t) + 5\exp(-0.2t)]^\mathrm{T}$. The numerical studies are carried out in MATLAB R2012b environment implemented on a personal digital computer with a CPU of Inter(R) Core(TM) i5-7200U @ 2.50 GHz, 4.00 GB memory and a Windows 10 Ultimate operating system.

1) SYNCHRONIZATION OF TWO IDENTICAL LU CHAOTIC SYSTEMS

Firstly, consider the following Lu chaotic system [24]:

$$\begin{cases} \dot{x}_1(t) = a(x_2(t) - x_1(t)), \\ \dot{x}_2(t) = -x_1x_3(t) + cx_2(t), \\ \dot{x}_3(t) = x_1(t)x_2(t) - bx_3(t), \end{cases} \quad (16)$$

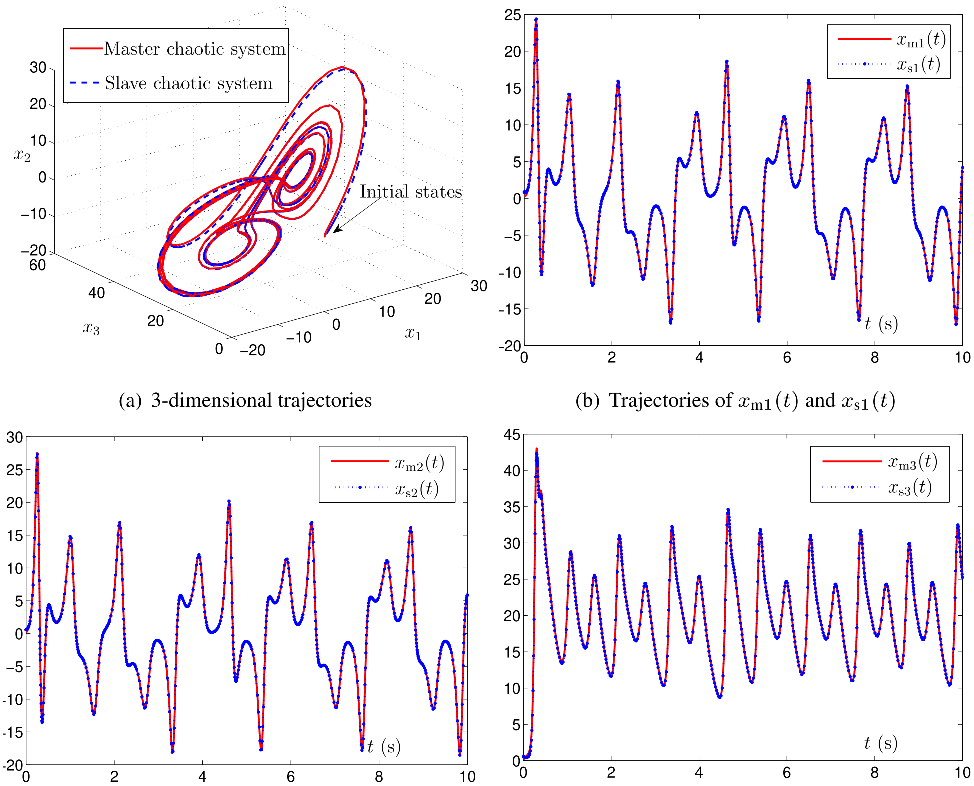
where $a = 36$, $b = 3$ and $c = 20$. For the synchronization of two identical Lu chaotic systems, one can have the master chaotic system with external disturbances is described as

$$\dot{\mathbf{x}}_\mathrm{m}(t) = \begin{bmatrix} a(x_{\mathrm{m}2}(t) - x_{\mathrm{m}1}(t)) \\ -x_{\mathrm{m}1}x_{\mathrm{m}3}(t) + cx_{\mathrm{m}2}(t) \\ x_{\mathrm{m}1}(t)x_{\mathrm{m}2}(t) - bx_{\mathrm{m}3}(t) \end{bmatrix} + \varrho(t), \quad (17)$$

and the slave chaotic system with control-input vector is described as

$$\dot{\mathbf{x}}_\mathrm{s}(t) = \begin{bmatrix} a(x_{\mathrm{s}2}(t) - x_{\mathrm{s}1}(t)) \\ -x_{\mathrm{s}1}x_{\mathrm{s}3}(t) + cx_{\mathrm{s}2}(t) \\ x_{\mathrm{s}1}(t)x_{\mathrm{s}2}(t) - bx_{\mathrm{s}3}(t) \end{bmatrix} + \mathbf{u}(t). \quad (18)$$

The corresponding numerical results of synchronization of two identical Lu chaotic systems (17) and (18) equipped with controller (7) via the proposed DRZN approach are presented in Fig. 3 and Fig. 4. Specifically, Fig. 3(a) shows



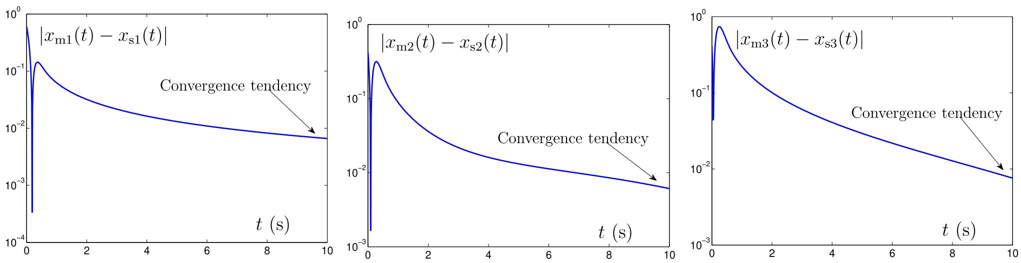
(a) 3-dimensional trajectories

(b) Trajectories of $x_{m1}(t)$ and $x_{s1}(t)$

(c) Trajectories of $x_{m2}(t)$ and $x_{s2}(t)$

(d) Trajectories of $x_{m3}(t)$ and $x_{s3}(t)$

FIGURE 3. Synchronization and disturbance rejection performance between two identical Lu chaotic systems (17) and (18) equipped with controller (7) using the proposed DRZN approach.



(a) $|x_{m1}(t) - x_{s1}(t)|$ (b) $|x_{m2}(t) - x_{s2}(t)|$ (c) $|x_{m3}(t) - x_{s3}(t)|$

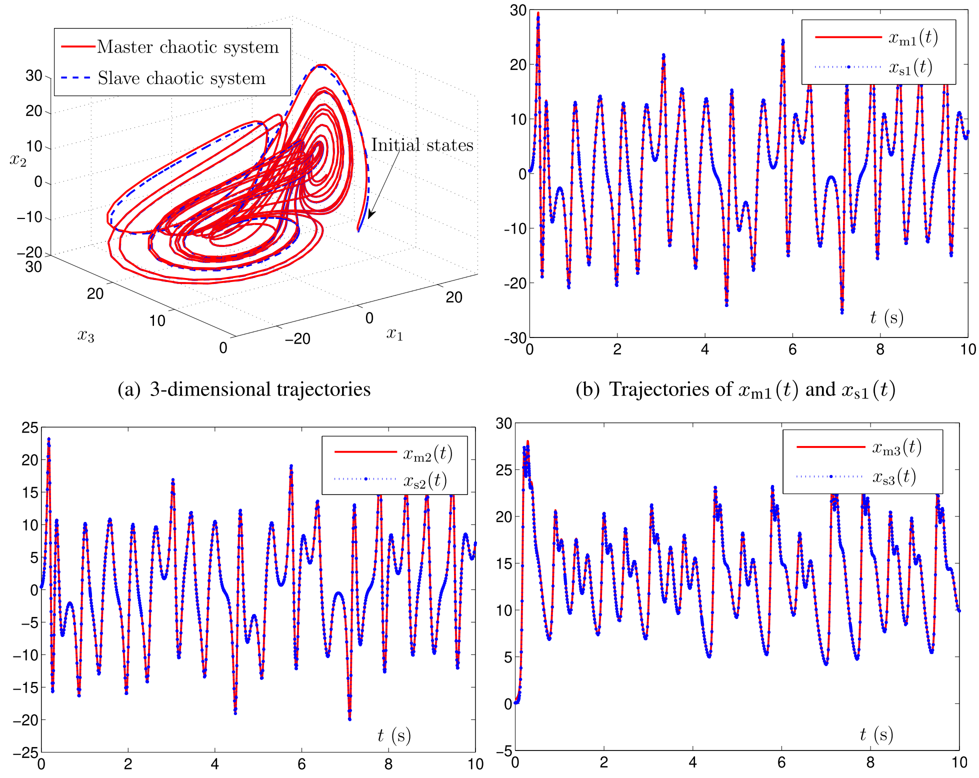
FIGURE 4. Absolute errors between two identical Lu chaotic systems (17) and (18) equipped with controller (7) using the proposed DRZN approach.

real-time synchronization of such two identical Lu chaotic systems (17) and (18) in 3-dimensional space. With initial value of each state randomly generated in [0, 1], the slave Lu chaotic systems (18) quickly synchronizes toward the master Lu chaotic systems (17). In addition, Fig. 3(b) through Fig. 3(d) respectively illustrate each state, i.e., x_{s1}, x_{s2} and x_{s3} of the slave system, which coincides well with each state, i.e., x_{m1}, x_{m2} and x_{m3}, of the master system even under the influence of external disturbances. As detailedly shown in Fig. 4, the absolute values of synchronization errors of all states are relatively small (or to say, ignorable), and quickly converge toward zero. Moreover, the supremum of each error, i.e., $\sup\{|x_{mi} - x_{si}|\}$ (with $i = 1, 2$ and 3), shows convergence tendency, which is consistent with the theoretical result provided in Theorem 1, i.e., with the error function being convergence toward zero. The above results illustrates the great synchronization and disturbance rejection performance of the proposed DRZN approach as well as the related controller (7) for the synchronization of two identical Lu chaotic systems.

2) SYNCHRONIZATION OF TWO IDENTICAL AUTONOMOUS CHAOTIC SYSTEMS

Consider the following new autonomous chaotic system proposed in [14]:

$$\begin{cases} \dot{x}_1(t) = p(x_2(t) - x_1(t)) + x_2(t)x_3(t), \\ \dot{x}_2(t) = (r - p)x_1 - x_1x_3(t) + rx_2(t), \\ \dot{x}_3(t) = -qx_3(t) - sx_2(t)x_2(t), \end{cases} \tag{19}$$



(a) 3-dimensional trajectories (b) Trajectories of $x_{m1}(t)$ and $x_{s1}(t)$

(c) Trajectories of $x_{m2}(t)$ and $x_{s2}(t)$ (d) Trajectories of $x_{m3}(t)$ and $x_{s3}(t)$

FIGURE 5. Synchronization and disturbance rejection performance between two identical autonomous chaotic systems (20) and (21) equipped with controller (7) using the proposed DRZN approach.

where $p = 40$, $q = 5$, $r = 30$ and $s \in [0, 10]$. For the synchronization of two identical autonomous chaotic systems with the above form, we have the master chaotic system with external disturbances is described as

$$\dot{\mathbf{x}}_{\mathrm{m}}(t)=\begin{bmatrix} p(x_{\mathrm{m}2}(t)-x_{\mathrm{m}1}(t))+x_{\mathrm{m}2}(t)x_{\mathrm{m}3}(t)\\ (r-p)x_{\mathrm{m}1}-x_{\mathrm{m}1}x_{\mathrm{m}3}(t)+rx_{\mathrm{m}2}(t)\\ -qx_{\mathrm{m}3}(t)-sx_{\mathrm{m}2}(t)x_{\mathrm{m}2}(t)\end{bmatrix}+\varrho(t), \quad (20)$$

and the slave chaotic system with control-input vector is described as

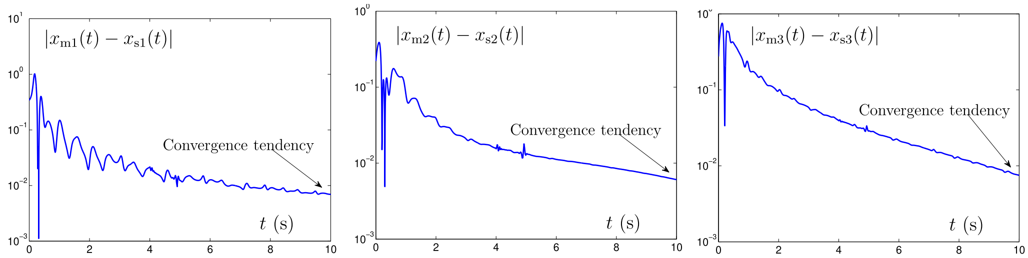
$$\dot{\mathbf{x}}_{\mathrm{s}}(t)=\begin{bmatrix} p(x_{\mathrm{s}2}(t)-x_{\mathrm{s}1}(t))+x_{\mathrm{s}2}(t)x_{\mathrm{s}3}(t)\\ (r-p)x_{\mathrm{s}1}-x_{\mathrm{s}1}x_{\mathrm{s}3}(t)+rx_{\mathrm{s}2}(t)\\ -qx_{\mathrm{s}3}(t)-sx_{\mathrm{s}2}(t)x_{\mathrm{s}2}(t)\end{bmatrix}+\mathbf{u}(t). \quad (21)$$

The corresponding numerical results of synchronization of two identical autonomous chaotic systems (20) and (21) equipped with controller (7) using the proposed DRZN approach are illustrated in Fig. 5 and Fig. 6. Specifically, Fig. 5(a) shows synchronization of such two identical autonomous chaotic systems (20) and (21) in 3-dimensional space. With initial value of each state randomly generated in [0, 1], the slave autonomous chaotic systems (21) also quickly synchronizes toward the master autonomous chaotic systems (20). In addition, Fig. 5(b) through Fig. 5(d) respectively show each state, i.e., x_{s1}, x_{s2} and x_{s3} of the slave system, coincides well with each state, i.e., x_{m1}, x_{m2} and x_{m3}, of the master system even under the influence of external disturbances. As presented in Fig. 6, the absolute values of synchronization errors of all states are also relatively small (or say, ignorable), and quickly converge toward zero. Moreover, the supremum of each error, i.e., $\sup\{|x_{mi}-x_{si}|\}$ (with $i = 1, 2$ and 3), shows convergence tendency, which is consistent with the theoretical result presented in Theorem 1, i.e., with the error function being convergence toward zero.

3) SYNCHRONIZATION OF TWO NONIDENTICAL CHAOTIC SYSTEMS

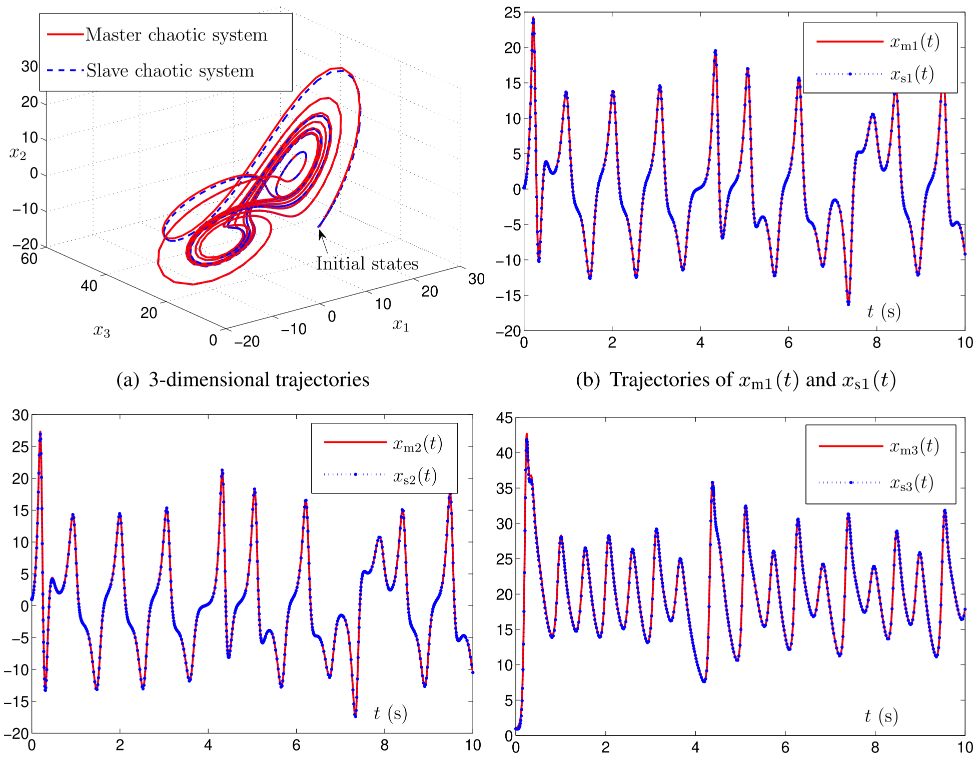
In the example, we further consider and achieve the synchronization of two nonidentical chaotic systems, i.e., with the maser chaotic system being the Lu chaotic system (17) and the slave chaotic system being the autonomous chaotic system (21).

The corresponding numerical results of synchronization of two nonidentical chaotic systems (17) and (21) equipped with controller (7) using the proposed DRZN approach are illustrated in Fig. 7 and Fig. 8. Specifically, the real-time synchronization of such two nonidentical chaotic systems (17) and (21) is shown in Fig. 7(a) in 3-dimensional space. With initial value of each state randomly generated in [0, 1], the slave autonomous chaotic system (21) still quickly synchronizes toward the master Lu chaotic system (17). In addition, Fig. 7(b) through Fig. 7(d) respectively show each state, i.e., x_{s1}, x_{s2} and x_{s3} of the slave autonomous chaotic



(a) $|x_{m1}(t) - x_{s1}(t)|$ (b) $|x_{m2}(t) - x_{s2}(t)|$ (c) $|x_{m3}(t) - x_{s3}(t)|$

FIGURE 6. **Absolute errors between two identical autonomous chaotic systems (20) and (21) equipped with controller (7) using the proposed DRZN approach.**



(a) 3-dimensional trajectories (b) Trajectories of $x_{m1}(t)$ and $x_{s1}(t)$

(c) Trajectories of $x_{m2}(t)$ and $x_{s2}(t)$ (d) Trajectories of $x_{m3}(t)$ and $x_{s3}(t)$

FIGURE 7. **Synchronization and disturbance rejection performance between two nonidentical chaotic systems (17) and (21) equipped with controller (7) using the proposed DRZN approach.**

system (21), almost overlaps each state, i.e., x_{m1}, x_{m2} and x_{m3}, of the master Lu chaotic system (17) even under the influence of external disturbances. As detailedly shown in Fig. 8, the absolute values of synchronization errors of all states are also relatively small and can be ignorable, and quickly converge toward zero. Moreover, the supremum of each error, i.e., $\sup\{|x_{mi} - x_{si}|\}$ (with $i = 1, 2$ and 3), shows convergence tendency, which is also consistent with the theoretical result provided in Theorem 1, i.e., with the error function being convergence toward zero, for the case of synchronization of two nonidentical chaotic systems.

Remark 2: Note that this work investigates the robust synchronization of multiple dimensional chaotic systems against time-variant external disturbances. Specifically, the dimension of the involved chaotic systems is set to be n. Without loss of generality, the dimension n is chosen as 3 in the above synchronization examples of numerical studies. Actually, the dimension of the involved chaotic systems can be chosen as 2 for low dimensional chaotic systems. As shown in the above synchronization examples, the absolute values of synchronization errors of all states are relatively small (or to say, ignorable) with high enough synchronization accuracy, and quickly converge toward zero within shot time.

B. COMPARISONS WITH OTHER APPROACHES

In the subsection, to verify the robustness and superiority of the proposed DRZN approach, we conduct and show the numerical comparisons by using the CZN approach as well as the LAC approach for the synchronization of two identical

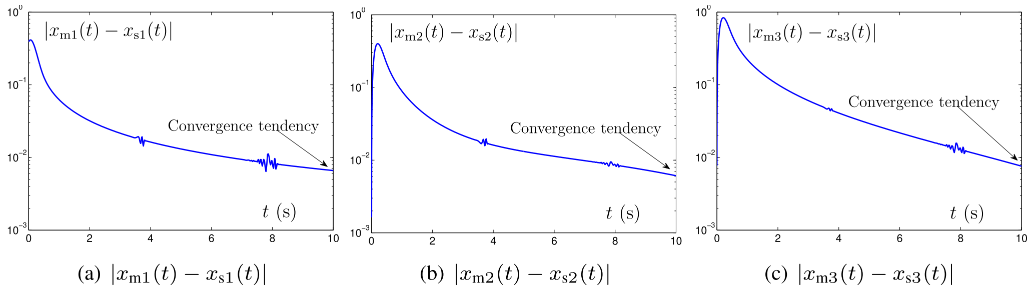


FIGURE 8. **Absolute errors between two nonidentical chaotic systems (17) and (21) equipped with controller (7) using the proposed DRZN approach.**

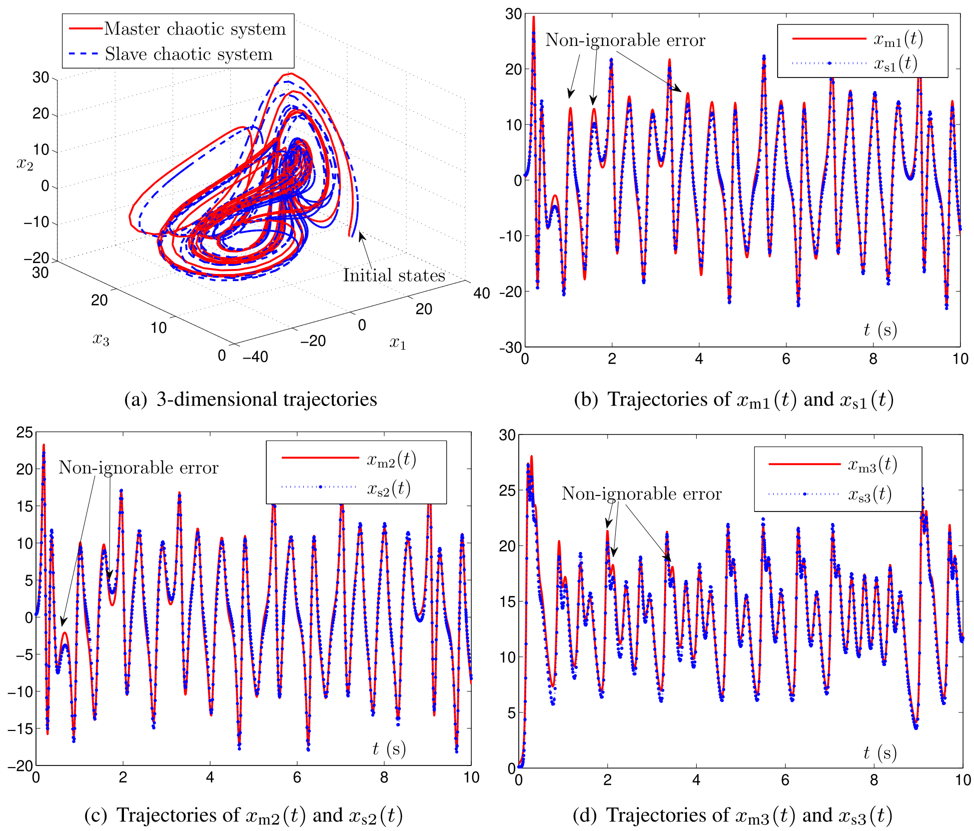
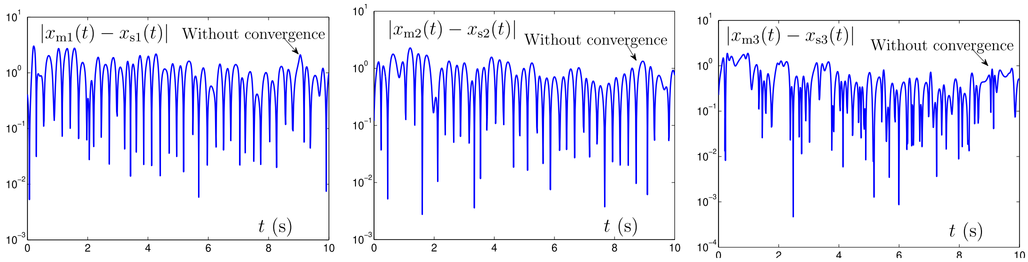


FIGURE 9. **Synchronization performance between two identical autonomous chaotic systems (20) and (21) equipped with controller (8) using the CZN approach.**

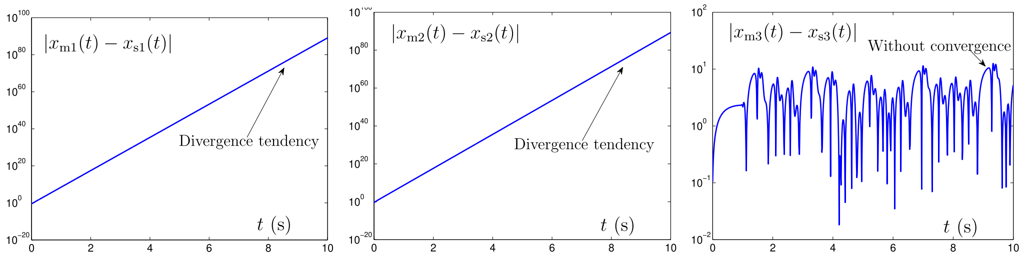
autonomous chaotic systems (20) and (21) under the influence of the same external disturbances.

The CZN approach was derived from Zhang *et al.* [24], and it has been proven to effectively solve the time-variant problems including the synchronization of chaotic systems without external disturbances. To be specifically, for synchronization of chaotic systems, the related controller designed by the CZN approach is depicted as (8). For better comparison, design parameter is set to be $\gamma = 3$. Other numerical conditions are set the same as those in Section IV-A.2. The corresponding numerical results of synchronization performance between two identical autonomous chaotic systems (20) and (21) equipped with controller (8) using the CZN approach are presented in Fig. 9 and Fig. 10. Specifically, with 3-dimensional trajectories illustrated in Fig. 9(a), the slave system (21) can not show an acceptable synchronization performance. With initial value of each state randomly generated in [0, 1], the slave autonomous chaotic systems (21) synchronizes toward the master autonomous chaotic systems (20) with a bounded error. In addition, Fig. 9(b) through Fig. 9(d) shows each state, i.e., x_{s1}, x_{s2} and x_{s3} of the slave system, can not coincide well with each state, i.e., x_{m1}, x_{m2} and x_{m3}, respectively, of the master system under the same influence of external disturbances. As detailedly shown in Fig. 10, all the absolute values of synchronization errors of all states are also non-ignorable, and have an error bound but not



(a) $|x_{m1}(t) - x_{s1}(t)|$ (b) $|x_{m2}(t) - x_{s2}(t)|$ (c) $|x_{m3}(t) - x_{s3}(t)|$

FIGURE 10. Absolute errors between two identical autonomous chaotic systems (20) and (21) equipped with controller (8) using the CZN approach.



(a) $|x_{m1}(t) - x_{s1}(t)|$ (b) $|x_{m2}(t) - x_{s2}(t)|$ (c) $|x_{m3}(t) - x_{s3}(t)|$

FIGURE 11. Absolute errors between two identical autonomous chaotic systems (20) and (21) equipped with controller (22) using the LAC approach.

converge toward zero. Moreover, the supremum of each error, i.e., $\sup\{|x_{mi} - x_{si}|\}$ (with $i = 1, 2$ and 3), does not show convergence tendency but keeps within an error bound, which is also consistent with the theoretical result presented in Lemma 1, i.e., with the error function having a supremum, i.e., $\|\varrho(t)\|_{\mathrm{E}}/\gamma$. Compared with the CZN approach, the above results verify the robustness and superiority of the proposed DRZN approach as well as the related controller (7) for the synchronization and disturbance rejection of chaotic systems.

Recently, as a novel alternative method for the synchronization of chaotic systems, the novel LAC approach [14] is proposed for the design of controller. Specifically, the related controller for the synchronization of two identical chaotic systems (20) and (20) generated by the LAC approach is provided as below:

$$\mathbf{u}(t) = \begin{bmatrix} x_{m2}(t)x_{m3}(t) - x_{s2}(t)x_{s3}(t) + 2(x_{m1}(t) - x_{s1}(t)) \\ x_{s1}(t)x_{s3}(t) - x_{m1}(t)x_{m3}(t) + 3(x_{m2}(t) - x_{s2}(t)) \\ x_{m2}(t)x_{m2}(t) - x_{s2}(t)x_{s2}(t) - 2(x_{m3}(t) - x_{s3}(t)) \end{bmatrix}, \tag{22}$$

For better comparison, other numerical conditions are set the same as those in Section IV-A.2. The simulated results of synchronization between two identical chaotic systems (20) and (21) equipped with controller (22) using the LAC approach are presented in Fig. 11. As one can find in the figure, all the absolute values of synchronization errors of all states are also non-ignorable with the errors $|x_{m1}(t) - x_{s1}(t)|$ and $|x_{m2}(t) - x_{s2}(t)|$ being divergent and $|x_{m3}(t) - x_{s3}(t)|$ having an error bound. Therefore, the time-variant external disturbances would have a negative impact on the accuracy and stability of chaotic systems equipped with controller (22), and even worse, may destroy the corresponding synchronization process.

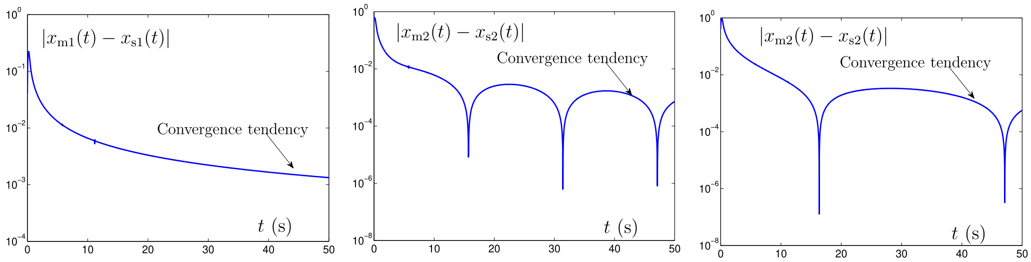
Further, to further highlight the advantages of the proposed DRZN approach and its related controller (7) compared with the related works in recent years, comprehensive comparisons among different approaches for the synchronization of chaotic systems with external disturbances are summarized in Table 1. As seen from the comparisons, the DRZN proposed in this paper shows the anti-disturbance performance and high synchronization accuracy, which is substantiated via three examples in Section IV-A. Compared with other existing approaches in recent years, the vector-valued error $\mathbf{e}(t)$ synthesized by control system equipped with the proposed controller converges toward zero, which is better than other approaches (e.g., the CZN and LAC approaches [14], [24], as well as the other recent approaches in [50]–[52], [55], [59]) with a bounded error or divergence tendency.

C. EXTENSIVE TESTS

To further investigate the anti-disturbance performance of the proposed DRZN approach and the related controller (7), extensive tests of chaotic systems with longer synchronization duration (i.e., 50 s) as well as under different forms of time-variant disturbances are conducted successively. The corresponding numerical results are shown in Fig. 12 and Fig. 13. Specifically, the absolute values of synchronization errors illustrate a persistent convergence tendency for a long

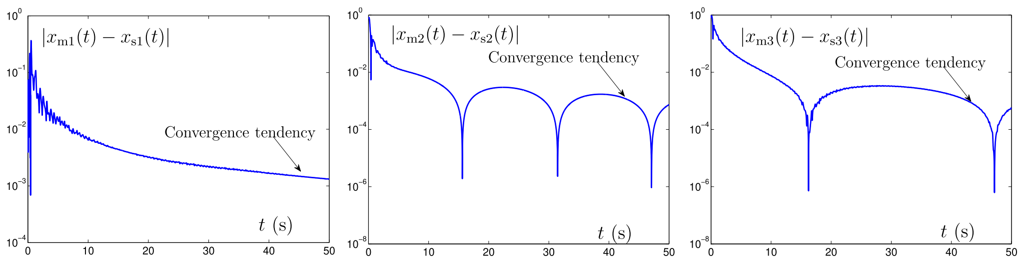
TABLE 1. Comparisons among different approaches for the synchronization of chaotic systems with time-variant external disturbances.

Approach	Anti-disturbance	Convergence property	Synchronization accuracy
Proposed DRZN	**Yes**	**Convergence**	**High**
CZN	No	With bounded error	High
LAC	No	Not convergence	Low
[50]	No	Not convergence	Low
[51]	No	Not convergence	Low
[52]	No	Not convergence	Low
[55]	No	Not convergence	Low
[59]	No	Not convergence	Low



(a) $|x_{m1}(t) - x_{s1}(t)|$ (b) $|x_{m2}(t) - x_{s2}(t)|$ (c) $|x_{m3}(t) - x_{s3}(t)|$

FIGURE 12. Absolute errors between two identical Lu chaotic systems (17) and (18) for 50 s duration equipped with controller (7) using the proposed DRZN approach.



(a) $|x_{m1}(t) - x_{s1}(t)|$ (b) $|x_{m2}(t) - x_{s2}(t)|$ (c) $|x_{m3}(t) - x_{s3}(t)|$

FIGURE 13. Absolute errors between two identical autonomous chaotic systems (20) and (21) for 50 s duration equipped with controller (7) using the proposed DRZN approach.

TABLE 2. Maximal absolute values of errors at different time instants during the synchronization between two identical Lu chaotic systems (17) and (18) under different forms of disturbances equipped with controller (7) using the proposed DRZN approach.

External disturbance	$\max\{\mathbf{e}(10)\}$	$\max\{\mathbf{e}(20)\}$	$\max\{\mathbf{e}(30)\}$	$\max\{\mathbf{e}(40)\}$	$\max\{\mathbf{e}(50)\}$
Constant form	1.7778×10^{-2}	8.9313×10^{-3}	5.9638×10^{-3}	4.4765×10^{-3}	3.5829×10^{-3}
Sine form	8.8171×10^{-3}	6.0842×10^{-3}	5.5171×10^{-3}	3.5321×10^{-3}	1.7287×10^{-3}
Exponential-decay form	3.6205×10^{-3}	4.8459×10^{-4}	1.1151×10^{-4}	3.0499×10^{-5}	8.9685×10^{-6}
Bounded random form	1.8122×10^{-2}	7.0518×10^{-3}	4.6501×10^{-3}	3.8660×10^{-3}	3.4830×10^{-3}

TABLE 3. Maximal absolute values of errors at different time instants during the synchronization between two identical autonomous chaotic systems (20) and (21) under different forms of disturbances equipped with controller (7) using the proposed DRZN approach.

Disturbance Form	$\max\{\mathbf{e}(10)\}$	$\max\{\mathbf{e}(20)\}$	$\max\{\mathbf{e}(30)\}$	$\max\{\mathbf{e}(40)\}$	$\max\{\mathbf{e}(50)\}$
Constant form	1.7942×10^{-2}	8.9887×10^{-3}	5.9792×10^{-3}	4.4418×10^{-3}	3.5866×10^{-3}
Sine form	8.8214×10^{-3}	6.0283×10^{-3}	5.4971×10^{-3}	3.5322×10^{-3}	1.7210×10^{-3}
Exponential-decay form	3.7484×10^{-3}	4.8314×10^{-4}	1.1115×10^{-4}	3.0448×10^{-5}	8.9549×10^{-6}
Bounded random form	1.3072×10^{-2}	7.4995×10^{-3}	5.6238×10^{-3}	4.1222×10^{-3}	2.6855×10^{-3}

synchronization duration (see both Fig. 12 and Fig. 13). It means that the synchronization errors can decrease to sufficiently small as time tends to appropriately large enough.

In order to evaluate the synchronization performance in finite time and also to monitor the convergence process during the whole synchronization under different forms of external disturbances, the maximal absolute value of synchronization error, i.e., max{$\mathbf{e}(t)$}, synthesized by the proposed DRZN approach and its related controller (7) at different time instants, i.e., t = 10 s, 20 s, 30 s, 40 s, 50 s, is presented in Table 2 and Table 3 for synchronization of Lu and autonomous chaotic systems, respectively. As both shown in Table 2 and Table 3, all the maximal absolute values of synchronization errors show convergence properties with different forms of time-variant disturbances (i.e., the constant form, sine form, exponential-decay form, and bounded random form). The above graphical results are also consistent with theoretical results. In other words, control system using the DRZN approach and its related controller (7) can converge toward the solution rapidly. All the numerical results of extensive tests verify that the proposed DRZN approach possesses the outstanding and inherent anti-disturbance performance, and thus is suitable for practical applications in the complex environment with time-variant external disturbances.

V. CONCLUSION AND FUTURE WORKS

In this paper, a DRZN approach and its related controller (7) have been proposed for the synchronization of chaotic systems against the time-variant external disturbances. Differing from existing works that the resultant synchronization error has a supremum or even diverge under the influence of external disturbances, the proposed DRZN approach can effectively suppress the external disturbances with the synthesized synchronization errors being convergence toward zero. It has been proven that the proposed DRZN approach and its related controller (7) inherently possess robustness. Moreover, numerical studies via three different examples have substantiated the effectiveness of the proposed DRZN approach as well as its related controller (7) for the synchronization of chaotic systems against the time-variant external disturbances. Comparisons with existing approaches, e.g., the CZN approach and LAC approach, have shown the superiority of the proposed DRZN approach. Meanwhile, extensive tests have further shown that the proposed DRZN approach possesses the outstanding and inherent anti-disturbance performance, and thus is suitable for the practical applications in complex application scenarios with time-variant external disturbances.

Future works and improvements lie in the following facts: i) investigation on different dimensional and different types of chaotic systems under the influence of time-variant external disturbances via the proposed DRZN approach and its related controller; ii) investigation on the finite-time convergence property of the proposed DRZN approach under the influence of time-variant external disturbances; iii) extension and implementation of synchronization of chaotic systems with internal system uncertainty with complete theoretical analyses on robustness of the proposed DRZN approach; and iv) development of the proposed DRZN approach and its related controller on an electrical systems to verify the physical realizability. As a final remark of this paper, to the best of authors' knowledge, this is the first work in the framework of zeroing neurodynamic which is able to elegantly handle the synchronization of chaotic systems under the influence of time-variant external disturbances with zero-oriented convergence performance.

REFERENCES

[1] E. N. Lorenz, "Deterministic nonperiodic flow," *J. Atmos. Sci.*, vol. 20, no. 2, pp. 130–141, 1963.

[2] X. Chen, J. H. Park, J. Cao, and J. Qiu, "Adaptive synchronization of multiple uncertain coupled chaotic systems via sliding mode control," *Neurocomputing*, vol. 273, nos. 9–21, Jan. 2018.

[3] Y. Li and C. Li, "Complete synchronization of delayed chaotic neural networks by intermittent control with two switches in a control period," *Neurocomputing*, vol. 173, pp. 1341–1347, Jan. 2016.

[4] X. Wang, C. Li, T. Huang, and L. Chen, "Dual-stage impulsive control for synchronization of memristive chaotic neural networks with discrete and continuously distributed delays," *Neurocomputing*, vol. 149, pp. 621–628, Feb. 2015.

[5] P. Muthukumar, P. Balasubramaniam, and K. Ratnavelu, "Sliding mode control design for synchronization of fractional order chaotic systems and its application to a new cryptosystem," *Int. J. Dyn. Control*, vol. 5, no. 1, pp. 115–123, Mar. 2017.

[6] B. Vaseghi, A. M. Pourmina, and S. Mobayen, "Secure communication in wireless sensor networks based on chaos synchronization using adaptive sliding mode control," *Nonlinear Dyn.*, vol. 89, no. 3, pp. 1689–1704, Aug. 2017.

[7] F. Zouari, A. Boulkroune, and A. Ibeas, "Neural adaptive quantized output-feedback control-based synchronization of uncertain time-delay incommensurate fractional-order chaotic systems with input nonlinearities," *Neurocomputing*, vol. 237, pp. 200–225, May 2017.

[8] M. Siddique, M. Rehan, M. Bhatti, and S. Ahmed, "Delay-range-dependent local adaptive and robust adaptive synchronization approaches for time-delay chaotic systems," *Nonlinear Dyn.*, vol. 88, no. 4, pp. 2671–2691, Jun. 2017.

[9] D. Lin and X. Wang, "Observer-based decentralized fuzzy neural sliding mode control for interconnected unknown chaotic systems via network structure adaptation," *Fuzzy Sets Syst.*, vol. 161, no. 15, pp. 2066–2080, Aug. 2010.

[10] C. Liu, C. Li, and C. Li, "Quasi-synchronization of delayed chaotic systems with parameters mismatch and stochastic perturbation," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 16, no. 10, pp. 4108–4119, Oct. 2011.

[11] X. Chen, J. Cao, J. H. Park, T. Huang, and J. Qiu, "Finite-time multi-switching synchronization behavior for multiple chaotic systems with network transmission mode," *J. Franklin Inst.*, vol. 355, no. 5, pp. 2892–2911, Mar. 2018.

[12] O. Mofid and S. Mobayen, "Adaptive synchronization of fractional-order quadratic chaotic flows with nonhyperbolic equilibrium," *J. Vib. Control*, vol. 24, no. 21, pp. 4971–4987, 2018.

[13] S. Mobayen and J. Ma, "Robust finite-time composite nonlinear feedback control for synchronization of uncertain chaotic systems with nonlinearity and time-delay," *Chaos, Solitons Fractals*, vol. 114, pp. 46–54, Sep. 2018.

[14] I. Ahmad, A. B. Saaban, A. B. Ibrahim, and M. Shahzad, "Global chaos synchronization of new chaotic system using linear active control," *Complexity*, vol. 21, no. 1, pp. 379–386, 2015.

[15] O. Mofid, S. Mobayen, and M.-H. Khooban, "Sliding mode disturbance observer control based on adaptive synchronization in a class of fractional-order chaotic systems," *Int. J. Adapt. Control Signal Process.*, vol. 33, no. 3, pp. 462–474, 2019.

[16] M. Afshari, S. Mobayen, R. Hajmohammadi, and D. Baleanu, "Global sliding mode control via linear matrix inequality approach for uncertain chaotic systems with input nonlinearities and multiple delays," *J. Comput. Nonlinear Dyn.*, vol. 13, Mar. 2018, Art. no. 031008.

[17] L.-X. Yang and J. Zhang, "Synchronization of three identical systems and its application for secure communication with noise perturbation," in *Proc. Int. Conf. Inf. Eng. Comput. Sci.*, Dec. 2009, pp. 1–4.

[18] B. Naderi and H. Kheiri, "Exponential synchronization of chaotic system and application in secure communication," *Optik*, vol. 127, no. 5, pp. 2407–2412, 2016.

[19] H. Yu, G. Cai, and Y. Li, "Dynamic analysis and control of a new hyperchaotic finance system," *Nonlinear Dyn.*, vol. 67, no. 3, pp. 2171–2182, Feb. 2012.

[20] M. Tuna and C. B. Fidan, "Electronic circuit design, implementation and FPGA-based realization of a new 3D chaotic system with single equilibrium point," *Optik*, vol. 127, no. 24, pp. 11786–11799, Dec. 2016.

[21] J. Brindley, T. Kapitaniak, and L. Kocarev, "Controlling chaos by chaos in geophysical systems," *Geophys. Res. Lett.*, vol. 22, no. 10, pp. 1257–1260, May 1995.

[22] X. Shi, S. Duan, L. Wang, T. Huang, and C. Li, "A novel memristive electronic synapse-based Hermite chaotic neural network with application in cryptography," *Neurocomputing*, vol. 166, pp. 487–495, Oct. 2015.

[23] X. Chen, J. Qiu, J. Cao, and H. He, "Hybrid synchronization behavior in an array of coupled chaotic systems with ring connection," *Neurocomputing*, vol. 173, pp. 1299–1309, Jan. 2016.

[24] J. Li, M. Mao, and Y. Zhang, "Simpler ZD-achieving controller for chaotic systems synchronization with parameter perturbation, model uncertainty and external disturbance as compared with other controllers," *Optik*, vol. 131, pp. 364–373, Feb. 2017.

[25] C. Li, X. Liao, and R. Zhang, "Impulsive synchronization of nonlinear coupled chaotic systems," *Phys. Lett. A*, vol. 328, no. 1, pp. 47–50, Jul. 2004.

[26] Z. Zhang, Z. Fu, L. Zheng, and M. Gan, "Convergence and robustness analysis of the exponential-type varying gain recurrent neural network for solving matrix-type linear time-varying equation," *IEEE Access*, vol. 6, pp. 57160–57171, 2018.

[27] D. Chen, S. Li, Q. Wu, and X. Luo, "New disturbance rejection constraint for redundant robot manipulators: An optimization perspective," *IEEE Trans. Ind. Informat.*, to be published. doi: 10.1109/TII.2019.2930685.

[28] Y. Lei, B. Liao, and Q. Yin, "A noise-acceptable ZNN for computing complex-valued time-dependent matrix Pseudoinverse," *IEEE Access*, vol. 7, pp. 13832–13841, 2019.

[29] Q. Xiang, W. Li, B. Liao, and Z. Huang, "Noise-resistant discrete-time neural dynamics for computing time-dependent Lyapunov equation," *IEEE Access*, vol. 6, pp. 45359–45371, 2018.

[30] I. Ahmad, A. B. Saaban, A. B. Ibrahima, M. Shahzadb, and N. Naveed, "The synchronization of chaotic systems with different dimensions by a robust generalized active control," *Optik*, vol. 127, no. 11, pp. 4859–4871, Jun. 2016.

[31] D. Lin, X. Wang, F. Nian, and Y. Zhang, "Dynamic fuzzy neural networks modeling and adaptive backstepping tracking control of uncertain chaotic systems," *Neurocomputing*, vol. 73, no. 16, pp. 2873–2881, 2010.

[32] Q. Song and T. Huang, "Stabilization and synchronization of chaotic systems with mixed time-varying delays via intermittent control with non-fixed both control period and control width," *Neurocomputing*, vol. 154, pp. 61–69, Apr. 2015.

[33] C. Li, X. Liao, and R. Zhang, "A unified approach for impulsive lag synchronization of chaotic systems with time delay," *Chaos, Solitons Fractals*, vol. 23, no. 4, pp. 1177–1184, Feb. 2005.

[34] P. S. Stanimirović, S. Srivastava, and D. K. Gupta, "From Zhang neural network to scaled hyperpower iterations," *J. Comput. Appl. Math.*, vol. 331, pp. 133–155, Mar. 2018.

[35] M. D. Marco, M. Forti, P. Nistri, and L. Pancioni, "Discontinuous neural networks for finite-time solution of time-dependent linear equations," *IEEE Trans. Cybern.*, vol. 46, no. 11, pp. 2509–2520, Nov. 2016.

[36] M. D. Marco, M. Forti, P. Nistri, and L. Pancioni, "Nonsmooth neural network for convex time-dependent constraint satisfaction problems," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 27, no. 2, pp. 295–307, Feb. 2016.

[37] P. S. Stanimirović, I. S. Zivković, and Y. Wei, "Recurrent neural network approach based on the integral representation of the Drazin inverse," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 27, no. 10, pp. 2107–2131, 2015.

[38] D. Fu, H. Wang, X. Xiao, S. Liao, and L. Jin, "Adaptive zeroing-gradient controller for ship course tracking with near singularity considered and zero theoretical tracking error," *IEEE Access*, vol. 7, pp. 38205–38212, 2019.

[39] D. Chen, S. Li, W. Li, and Q. Wu, "A multi-level simultaneous minimization scheme applied to jerk-bounded redundant robot manipulators," *IEEE Trans. Automat. Sci. Eng.*, to be published. doi: 10.1109/TASE.2019.2931810.

[40] L. Jin, Z. Huang, Y. Li, Z. Sun, H. Li, and J. Zhang, "On modified multi-output chebyshev-polynomial feed-forward neural network for pattern classification of wine regions," *IEEE Access*, vol. 7, pp. 1973–1980, 2019.

[41] L. Xiao, Y. Zhang, Z. Hu, and J. Dai, "Performance benefits of robust nonlinear zeroing neural network for finding accurate solution of Lyapunov equation in presence of various noises," *IEEE Trans. Ind. Informat.*, to be published. doi: 10.1109/TII.2019.2900659.

[42] L. Jin, Y. Zhang, S. Li, and Y. Zhang, "Noise-tolerant ZNN models for solving time-varying zero-finding problems: A control-theoretic approach," *IEEE Trans. Autom. Control*, vol. 62, no. 2, pp. 992–997, Feb. 2017.

[43] D. Chen, S. Li, F.-J. Lin, and Q. Wu, "New super-twisting zeroing neural-dynamics model for tracking control of parallel robots: A finite-time and robust solution," *IEEE Trans. Cybern.*, to be published. doi: 10.1109/TCYB.2019.2930662.

[44] L. Jin, Y. Zhang, and S. Li, "Integration-enhanced Zhang neural network for real-time-varying matrix inversion in the presence of various kinds of noises," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 27, no. 12, pp. 2615–2627, Dec. 2016.

[45] J. Na, X. Ren, and D. Zheng, "Adaptive control for nonlinear pure-feedback systems with high-order sliding mode observer," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 24, no. 3, pp. 370–382, Mar. 2013.

[46] S. Li and Y. Li, "Nonlinearly activated neural network for solving time-varying complex Sylvester equation," *IEEE Trans. Cybern.*, vol. 44, no. 8, pp. 1397–1407, Aug. 2014.

[47] S. Li, J. He, Y. Li, and M. U. Rafique, "Distributed recurrent neural networks for cooperative control of manipulators: A game-theoretic perspective," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 2, pp. 415–426, Feb. 2017.

[48] S. Li, M. Zhou, X. Luo, and Z.-H. You, "Distributed winner-take-all in dynamic networks," *IEEE Trans. Autom. Control*, vol. 62, no. 2, pp. 577–589, Feb. 2017.

[49] L. Xiao, K. Li, and M. Duan, "Computing time-varying quadratic optimization with finite-time convergence and noise tolerance: A unified framework for zeroing neural network," *IEEE Trans. Neural Netw. Learn. Syst.*, to be published. doi: 10.1109/TNNLS.2019.2891252.

[50] Y. Zhang, Z. Xiao, D. Guo, M. Mao, and Y. Yin, "Singularity-conquering tracking control of a class of chaotic systems using Zhang-gradient dynamics," *IET Control Theory Appl.*, vol. 9, no. 6, pp. 871–881, Apr. 2015.

[51] Y. Zhang, T. Qiao, D. Zhang, H. Tan, and D. Liang, "Simple effective Zhang-dynamics stabilization control of the 4th-order hyperchaotic Lu system with one input," in *Proc. 12th Int. Conf. Natural Comput., Fuzzy Syst. Knowl. Discovery (ICNC-FSKD)*, Aug. 2016, pp. 325–330.

[52] L. Jin, Y. Zhang, T. Qiao, M. Tan, and Y. Zhang, "Tracking control of modified Lorenz nonlinear system using ZG neural dynamics with additive input or mixed inputs," *Neurocomputing*, vol. 196, no. 4, pp. 82–94, 2016.

[53] D. Chen, Y. Zhang, and S. Li, "Tracking control of robot manipulators with unknown models: A jacobian-matrix-adaption method," *IEEE Trans. Ind. Informat.*, vol. 14, no. 7, pp. 3044–3053, Jul. 2018.

[54] D. Chen and Y. Zhang, "Robust zeroing neural-dynamics and its time-varying disturbances suppression model applied to mobile robot manipulators," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 9, pp. 4385–4397, Sep. 2018.

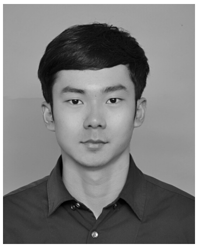
[55] D. Chen, S. Li, and Q. Wu, "Rejecting chaotic disturbances using a super-exponential-zeroing neurodynamic approach for synchronization of chaotic sensor systems," *Sensors*, vol. 19, no. 1, p. 74, 2019.

[56] Z. Zhang, Z. Yan, and T. Fu, "Varying-parameter RNN activated by finite-time functions for solving joint-drift problems of redundant robot manipulators," *IEEE Trans. Ind. Informat.*, vol. 14, no. 12, 5359–5367, Dec. 2018.

[57] Z. Zhang, L. Zheng, J. Weng, Y. Mao, W. Lu, and L. Xiao, "A new varying-parameter recurrent neural-network for online solution of time-varying Sylvester equation," *IEEE Trans. Cybern.*, vol. 48, no. 11, pp. 3135–3148, Nov. 2018.

[58] A. V. Oppenheim and A. S. Willsky, *Signals and Systems*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1997.

[59] Z. Yunong, L. Ming, J. Long, Z. Yinyan, and T. Hongzhou, "Synchronization of two chaotic systems with three or two inputs via ZG method," in *Proc. 34th Chin. Control Conf. (CCC)*, Jul. 2015, pp. 325–330.



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