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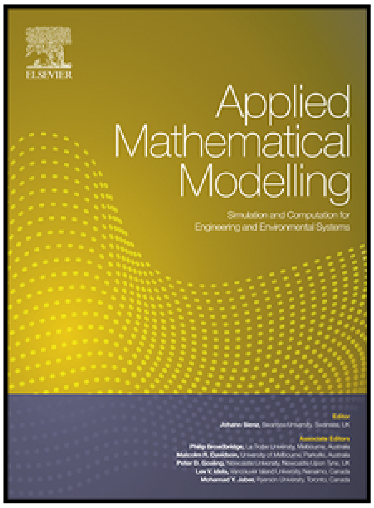


# The digital twin of discrete dynamic systems: Initial approaches and future challenges

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**Highlights**

- The digital twin concept is investigated using a discrete damped dynamic system model
- The concept of "slow time" is proposed to differentiate the evolution of the digital twin from its dynamics
- Stiffness and mass variation in the discrete system is considered
- Sensor errors are modeled in the update process of the digital twin
- The digital twin is expressed as a benchmark closed form solution

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# The digital twin of discrete dynamic systems: Initial approaches and future challenges

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## Abstract

This paper employs a discrete damped dynamic system to investigate the emerging concept of a digital twin. Dynamic systems are well understood across engineering and science domains, and represent a familiar and convenient platform for exploring the various aspects of a digital twin design. The aim is to create a framework accessible to engineering sciences related to the aerospace, electrical, mechanical and computational area. The virtual model of the physical system is expressed as a differential equation in two-time scales, with the concept of a slow time being used to separate the evolution of the system properties from the instantaneous time. Cases involving stiffness variation and mass variation are considered, individually and together. It is assumed that the damped natural frequency and the time response are measured through sensors placed on the physical system. Issues of errors and reduced sampling rate in sensor measurements on the digital twin are investigated. The digital twin is expressed as an analytical solution through closed-form expressions and the effect of sensor errors is brought out through the simulations. Several key concepts introduced in this paper are summarized and ideas for urgent future research needs are proposed. The current work breaks free from the qualitative description of digital twins pervading the literature and can be used as benchmark solutions to validate digital twin of experimental dynamic systems and their implementation using sensors, the internet of things and deployment on the cloud computing systems.



*Keywords:* Digital twin, vibration, response, frequency, modeling

## 1. Introduction

A digital twin is an avatar of a real physical system which exists in the computer. While a computer model of a physical system attempts to closely match the behaviour of a physical system, the digital twin also tracks the temporal evolution of the physical system. Some publications have defined the digital twin at the conceptual level. However, these definitions have been very general due to the attempt to keep a very large number of systems under the ambit of the definitions. Here, we seek to define digital twins of

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discrete dynamic systems, as many physical systems can be expressed in this form. We start with a deterministic single degree of freedom (DOF) system and then extend the concept to systems with random errors in the sensor data.

A recent review of the literature on digital twins is provided by Tao et al [1]. Several definitions of digital twin have been proposed in the literature, two of which have become popular. Reifsnider and Mujumdar [2] consider digital twin to be a high fidelity simulation integrated with an on-board health management system, maintenance history and historical vehicle and aircraft fleet data. They would like the digital twin to mirror the whole flying life of a specific operational physical twin. The availability of such a digital twin will lead to an enormous increase in reliability and safety for aircraft. Another popular definition of the digital twin was given by Glaessgen and Stargel [3]. They defined digital twin as an integrated multiscale, multiphysics probabilistic simulation of a complex product which uses the best available physical model, sensor updates etc. to mirror the life of the physical twin.

The digital twin has theoretical foundations in information science, production engineering, data science and computer science [1]. Tao et al [1] identify four key aspects of digital twin research: (1) modelling and simulation (2) data fusion (3) interaction and collaboration and (4) service. While the focus in [1] was towards information science aspects, we classify these four aspects from the viewpoint of physical systems. The objective of the modelling and simulation phase of the digital twin is to create a virtual model which is a mirror reflection of the physical model [1]. For most physical systems, the virtual model will be a computer program which solves partial differential equations or matrix differential equations. This simulation model of the system must be verified and validated, typically with experimental data. The virtual model may need to be updated at this stage and the model fidelity improved to minimize the discrepancy between the physical and virtual model. Uncertainty analysis of the virtual model may need to be conducted to account for deviations in the physical system properties. Statistical measures can be used to quantify the deviation between the physical model and the virtual model and optimization methods can be used to minimize this difference.

The second phase of the digital twin called data fusion involves the process of collecting data from the system, typically using sensors. Some examples of sensors include pressure sensors, light sensors, accelerometers, gyroscopes and motion sensors. In recent years, industrial sensors have become inexpensive and this has facilitated the development of the digital twin concept. The sensor data is then processed using signal processing, feature extraction, data mining and other methods, typically to amplify significant aspects of the data which reveal more about some desired condition of the physical system. Aspects related to the data having large size (big data) may be encountered at this stage because of a high rate of sampling by the sensors and the presence of a large number of sensors on the physical systems. Sensors can also be connected to prototyping boards such as the Arduino Uno and Raspberry Pi 2 which then allows them to be part of the internet of things. Algorithms based on pattern recognition methods such as machine learning, fuzzy logic etc. can also be used for data processing or feature extraction.

The third phase of the digital twin involving interaction and collaboration implies that there must be information flow between the physical model and the data fusion function of the digital twin. It is also possible that inputs generated by the virtual model are communicated to the physical model via actuators. Thus, the virtual model must incorporate changes in the physical model communicated through sensor data. The virtual model must be synchronized with the physical model at selected time steps and must, therefore, evolve temporally along with the physical model. If possible, this synchronization should be done at frequent intervals and ideally in real-time. Estimation methods such as Kalman filtering or particle filtering are useful during this stage for synchronizing the physical and digital models. Since the virtual model typically exists as a computer program in the cloud, aspects of cloud computing become important along with the placement of the physical model in the context of the sensors as a part of the internet of things [4].

The final function of the digital twin is service which is the reason for which the digital twin exists, such as structure monitoring, lifetime forecasting, in-time manufacturing etc. [1]. Estimation of service conditions and life of the physical system along with the prediction of required maintenance, downtime and replacement are some of the applications of the digital twin concept.

Several aspects of the digital twin concept were addressed in earlier research on prognostics and health monitoring (PHM) [5–7], manufacturing [8–10] and other fields [11, 12]. However, the integration of these aspects is a non-trivial problem but many companies are very interested in this concept due to its ability to track expensive hardware at much-reduced costs. Li et al [5] used a dynamic Bayesian network to monitor the operational state of aircraft wings. They replaced a deterministic model of the structure with a probabilistic model. Haag and Anderl [9] selected a bending beam test bench and attempted to build a digital twin for this system. The test bench consisted of a physical twin, a digital twin and a communication interface between the two.

Worden et al [13] suggest that digital twin is a powerful idea in the computational representation of structures. They mention that there is currently no real consensus attempt to bring order to the subject of digital twin and their paper aims to make an attempt in this direction. To start with, they begin with a structure of the system which is the physical object which is placed in an environment. Both the structure and its environment are described by state-vectors and the digital twin is defined in terms of closeness between the structure and its twin or mirror image using an error bound. They point out that models for digital twin may typically start out as physics-based models. However, black-box models developed from data or grey box models developed from a combination of physics and data are also possible. As an example, one could start with a physics-based model and obtain the parameters using system identification. While this paper presents several interesting ideas, it places them in a fairly abstract context. There is a need to define a digital twin for a simple system, such as a discrete dynamical system, which allows a concrete understanding of this concept. Such an example is also useful as a pedagogical tool which introduces the concept of a digital twin to engineering students and practising engineers.

From this discussion, it is clear that there is a lack of clarity and specificity regarding digital twins. Although there exists a broad consensus about what a digital twin is, there is no detailed methodology on how to develop one for a given system. In this paper, an explicit approach is suggested towards developing a digital twin for a single degree of freedom (SDOF) dynamic system. This is motivated by the need to establish simple mathematical methods which are easily understood and can be extended to more complex systems. As the dynamics of SDOF systems are governed by the natural frequency and damping factor, these quantities are exploited to establish the digital twin. It is considered that an SDOF digital twin is based on the evolution of mass and stiffness parameters. In Section 2 the equation of motion of an SDOF digital twin is introduced using multiple time-scales. The development of a digital twin using only the mass evolution is considered in Section 3. In Section 4 the case only stiffness evolution to establish a digital twin is considered. In Section 5 the combined case of simultaneous mass and stiffness evolution is consider for a digital twin. For the all the three cases, we consider situations when (1) the sensor data is exact, (2) there is a known error in the sensor data, and (3) the sensor data is available with a given error estimate. Numerical examples are given to illustrate the proposed ideas. Some critical discussion on the proposed methodology as well as the overall development of digital twin is given in Section 6 and concluding remarks are given in Section 7. Note that any infinite-dimensional continuous system expressed as partial differential equations can be discretised and expressed as a matrix equation using numerical techniques such as the Galerkin method, finite difference method or finite element method. Such a finite-dimensional system can often be expressed as a system of SDOF systems using orthogonal transformations. Thus, SDOF systems provide a good point to start the analytical study of digital twins, based on which future research on advanced systems can be built upon.

## 2. Introduction to the dynamic model of the digital twin

In this section the nominal dynamic system and the digital twin arising from this model are introduced. The nominal model is the 'initial model' or the 'starting model' of a digital twin. For engineering dynamic systems, the nominal model is a physics based model which has been verified, validated and calibrated. For example, this can be a finite element model of a car or an aircraft when the product leaves the manufacturer facility and ready to go into the service. Its digital twin therefore starts the journey from the nominal model but as time passes, alters the original model such that it reflects the current state of the system. We explain the essential ideas through a single degree of freedom (SDOF) dynamic system.

### *2.1. Single degree of freedom system: The nominal model*

The equation of motion of a single degree of freedom dynamic system [14] is expressed as

$$m_0 \frac{\mathrm{d}^2 u_0(t)}{\mathrm{d}t^2} + c_0 \frac{\mathrm{d}u_0(t)}{\mathrm{d}t} + k_0 u_0(t) = f_0(t) \tag{1}$$

We call the system given by Eq. (1) as the nominal dynamic system. Here $m_0$, $c_0$ and $k_0$ are the nominal mass, damping and stiffness coefficients. The forcing function and the dynamic response is denoted by $f_0(t)$ and $u_0(t)$ respectively. The SDOF model in Eq. (1) can represent a simplified model of a more complex dynamic system or can represent the dynamics a modal coordinate of a multiple degree for freedom system. The formulations in this paper is valid for either of these two cases.

Dividing by $m_0$, the equation of motion (1) can be expressed as

$$\ddot{u}_0(t) + 2\zeta_0\omega_0\dot{u}_0(t) + \omega_0^2 u_0(t) = \frac{f(t)}{m_0} \tag{2}$$

Here the undamped natural frequency ($\omega_0$) and the damping factor ($\zeta_0$) are expressed as

$$\omega_0 = \sqrt{\frac{k_0}{m_0}} \tag{3}$$

$$\text{and} \quad \frac{c_0}{m_0} = 2\zeta_0\omega_0 \quad \text{or} \quad \zeta_0 = \frac{c_0}{2\sqrt{k_0 m_0}} \tag{4}$$

Th natural time period of the underlying undamped system is given by

$$T_0 = \frac{2\pi}{\omega_0} \tag{5}$$

Taking the Laplace transform of Eq. (2) we have

$$s^2 U_0(s) + s2\zeta_0\omega_0 U_0(s) + \omega_0^2 U_0(s) = \frac{F_0(s)}{m_0} \tag{6}$$

where $U_0(s)$ and $F_0(s)$ are the Laplace transforms of $u_0(t)$ and $f_0(t)$ respectively. Solving the equation associated with coefficient of $U_0(s)$ in Eq. (2) without the forcing term, the complex natural frequencies of the system are given by

$$\lambda_{0_{1,2}} = -\zeta_0\omega_0 \pm \mathrm{i}\omega_0\sqrt{1-\zeta_0^2} = -\zeta_0\omega_0 \pm \mathrm{i}\omega_{d_0} \tag{7}$$

Here the imaginary number $\mathrm{i} = \sqrt{-1}$ and the damped natural frequency is expressed as

$$\omega_{d_0} = \omega_0\sqrt{1-\zeta_0^2} \tag{8}$$

For a damped oscillator, at resonance, the frequency of oscillation is given by $\omega_{d_0} < \omega_0$. Therefore, for positive damping, the resonance frequency of a damped system is always lower than the corresponding underlying undamped system.

*2.2. The digital twin model*

We consider a physical system which can be well approximated by a single degree of freedom spring, mass and damper system as before. Its digital twin equation can be written as

$$m(t_s)\frac{\partial^2 u(t,t_s)}{\partial t^2} + c(t_s)\frac{\partial u(t,t_s)}{\partial t} + k(t_s)u(t,t_s) = f(t,t_s) \tag{9}$$

Here $t$ and $t_s$ are the system time and a "slow time", respectively. In contrast to the nominal system in (1), $u(t,t_s)$ is a function of two variables and therefore the equation of motion is expressed in terms of the partial derivative with respect to the time variable $t$. The slow time or the service time $t_s$ can be considered as a time variable which is much slower than $t$. For example, it could represent the number of cycles in an aircraft. Thus, mass $m(t_s)$, damping $c(t_s)$, stiffness $k(t_s)$ and forcing $F(t,t_s)$ change with $t_s$, for example due to system degradation during its service life. The forcing is also a function of time $t$ and slow time $t_s$, as is the system response $x(t,t_s)$. Equation (9) is considered as a digital twin of a SDOF dynamic system. When $t_s = 0$, that is at the beginning of the service life of the system, the digital twin (9) reduces to the nominal system in (1).

It is assumed that sensors are deployed on the physical system and take measurements at locations of time defined by $t_s$. The functional form of the relationship of mass, stiffness and forcing with $t_s$ is unknown and needs to be estimated from measured sensor data. A general overview of how to construct a digital twin for a single-degree-of-freedom dynamic system is shown in Fig. 1. Based on these discussions we define the digital twin as:

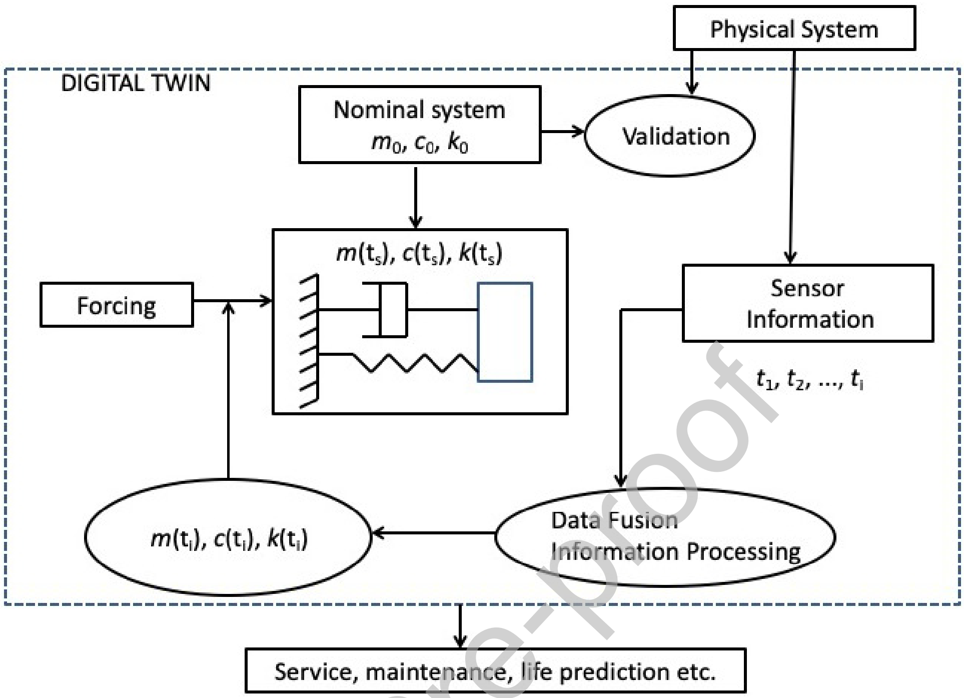



Fig. 1. The overview of constructing a digital twin for a single-degree-of-freedom dynamic system.

**Definition:** *The Digital Twin of a single-degree-of-freedom system is a bi-time-scale model reproducing the dynamics of the physical system at both time-scales.*

From Eq. (9) it can be seen that the digital twin model is effectively embodied in the functions $k(t_s)$, $m(t_s)$ and $c(t_s)$. Clearly a wide variety of functional forms are possible. Considering the stiffness function as an example, $k(t_s)$ can be a deterministic or a random function (i.e., a random process). If $k(t_s)$ is a deterministic function, then the mathematical condition it must satisfy so that Eq. (9) is solvable can be expressed in terms of a suitable norm. Therefore, the necessary conditions are

$$\begin{aligned} k(t_s) = k_0, \quad \text{when } t_s = 0 \\ \text{and} \quad \int_0^{T_s} k^2(t_s)\mathrm{d}t_s < \infty, \quad 0 < T_s < \infty \end{aligned} \tag{10}$$

where $T_s$ is the time upto which the digital twin is to be constructed. If $k(t_s)$ is a random process then its autocorrelation function must be finite at all time. Similar conditions are also imposed on $m(t_s)$ and $c(t_s)$.

One of the key characteristics of a digital twin is its connectivity. The recent development of the Internet of Things (IoT) brings forward numerous new data technologies and consequently drives the development of digital twin technology. This enables connectivity between the physical SDOF system and its digital counterpart. The basis of digital twins is based on this connection, without it, digital twin technology cannot exist. This connectivity is created by sensors on the physical system which obtain data and integrate and communicate this data through various integration technologies. It is reasonable to assume that the sensors sample data intermittently, typically, $t_s$ represents discrete time points. It is assumed that changes in $k(t_s)$, $m(t_s)$ and $c(t_s)$ as so slow that the dynamics of system (9) is effectively decoupled from these functional variations. Therefore, for all practical purposes $k(t_s)$, $m(t_s)$ and $c(t_s)$ are constant as far as the instantaneous dynamics of (9) is concerned as a function of $t$. We also assume that damping is small so that the effect of variations in $c(t_s)$ is negligible. In effect, only variations in the mass and stiffness are considered. Without any loss of generality, the following functional forms are considered

$$\begin{aligned} k(t_s) &= k_0(1 + \Delta_k(t_s)) \\ \text{and} \quad m(t_s) &= m_0(1 + \Delta_m(t_s)) \end{aligned} \tag{11}$$

Due to the conditions in (10), it is clear that $\Delta_k(t_s) = \Delta_m(t_s) = 0$ for $t_s = 0$. In general $k(t_s)$ is expected to be decaying function over a long time to represent a loss in the stiffness of the system. On the other hand, $m(t_s)$ can be an increasing or a decreasing function. For example, in the context of aircraft, it can represent the loading of cargo and passengers and also represent the use of fuel as the flight progresses.

Considering the above constraints and discussions, the following representative functions have been chosen as examples

$$\Delta_k(t_s) = e^{-\alpha_k t_s}\frac{(1 + \epsilon_k \cos(\beta_k t_s))}{(1 + \epsilon_k)} - 1 \tag{12}$$

$$\text{and} \quad \Delta_m(t_s) = \epsilon_m \ \mathrm{SawTooth}(\beta_m(t_s - \pi/\beta_m)) \tag{13}$$

Here SawTooth($\bullet$) represents a sawtooth wave with a period $2\pi$. In Fig. 2 overall variations in stiffness and mass properties arising from these function models have been plotted as function of time normalised to the natural time period of the nominal model. Numerical values used for these examples are $\alpha_k = 4 \times 10^{-4}$, $\epsilon_k = 0.05$, $\beta_k = 2 \times 10^{-2}$, $\beta_m = 0.15$ and $\epsilon_m = 0.25$. The rationale behind the choice of these functions is that the stiffness degrades over time in a periodic manner representing a possible fatigue crack growth in an aircraft over repeated pressurisation. While the mass increases and decreases over the nominal value due to re-fuelling and fuel burn over a flight period. The key consideration is that a digital twin of the dynamical system should be able to track these kinds of changes by exploiting sensor data measured on the system.

## 3. Digital twin via stiffness evolution

Advances in sensor technology are crucial for the development of the digital twin concept. The formulation in this paper assumes that natural frequency and response of the system can be measured online. Some recent papers clearly show that this is feasible. Feng et al [15] proposed a vision-based sensor system for remote measurement of structural displacements. Field tests were carried out on a railway bridge and pedestrian bridge, and the accuracy of the sensor in both time and frequency domains were demonstrated. This vision-based non-contact sensor can extract structural displacement at any point on a structure. Wang et al [16] proposed a fibre Bragg grating sensor which uses vibration-induced strain to measure the natural frequencies of the structure. Experiments were

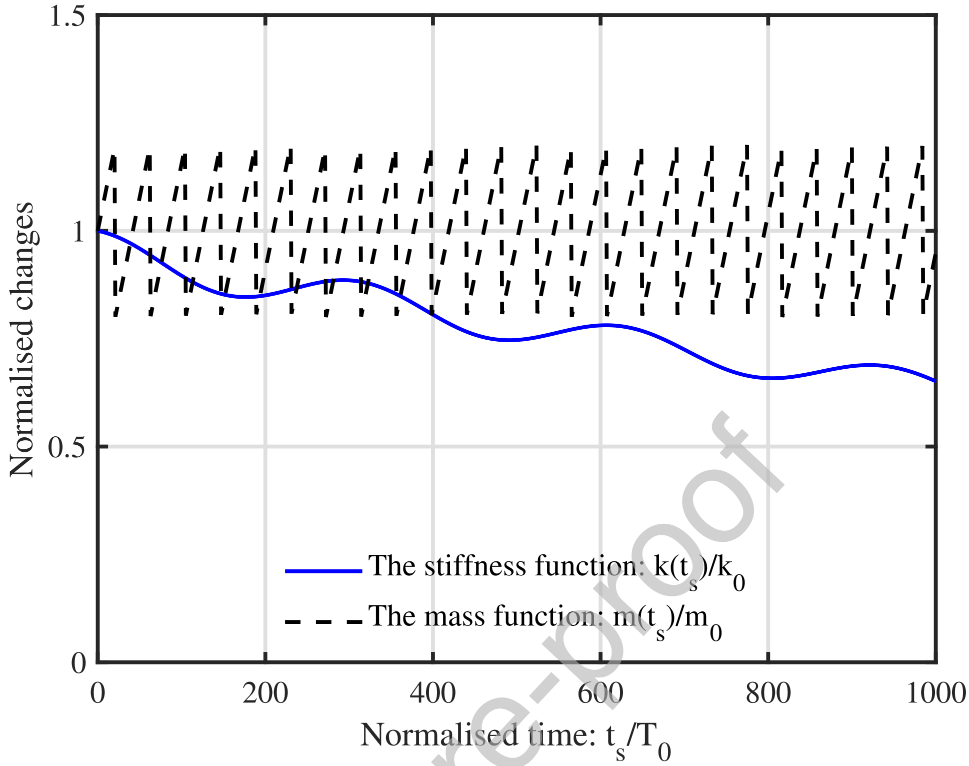



Fig. 2. Examples of model functions representing long term variabilities in the mass and stiffness properties of a digital twin system.

conducted using vibration and impact loads on metal pipes using an electrical strain gage, piezoelectric accelerometer and fibre Bragg grating sensor to obtain natural frequencies. These studies show that natural frequency and displacement can be measured online by sensor systems, a fact which we will use in this study.

### *3.1. Exact natural frequency data is available*

It is considered that the mass and damping of the nominal model is unchanged and only the changes in the stiffness properties affect the digital twin. The equation of motion of the digital twin of a single degree of freedom dynamic system with variation only in the stiffness property for a fixed value of $t_s$ is given by

$$m_0 \frac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2} + c_0 \frac{\mathrm{d}u(t)}{\mathrm{d}t} + k_0(1 + \Delta_k(t_s))u(t) = f(t) \tag{14}$$

This equation is a special case of the general equation in (9). As the value of $t_s$ is fixed, the partial derivative terms in equation (9) become total derivatives in this equation. Dividing with $m_0$ and solving the characteristic equation, the damped natural eigenfrequencies can be expressed as

$$\lambda_{s_{1,2}}(t_s) = -\zeta_0\omega_0 \pm \mathrm{i}\omega_0\sqrt{1 + \Delta_k(t_s) - \zeta_0^2} \tag{15}$$

Here the subscript $(\bullet)_s$ denotes the 'measured' values changing with the slow time $t_s$. Rearranging (15) we have

$$\lambda_{s_{1,2}}(t_s) = -\underbrace{\frac{\zeta_0}{\sqrt{1+\Delta_k(t_s)}}}_{\zeta_s(t_s)}\underbrace{\omega_0\sqrt{1+\Delta_k(t_s)}}_{\omega_s(t_s)} \pm \mathrm{i}\,\underbrace{\omega_0\sqrt{1+\Delta_k(t_s)}\sqrt{1-\left(\frac{\zeta_0}{\sqrt{1+\Delta_k(t_s)}}\right)^2}}_{\omega_{d_s}(t_s)} \tag{16}$$

Here $\omega_s(t_s) = \omega_0\sqrt{1+\Delta_k(t_s)}$, $\zeta_s(t_s) = \zeta_0/\sqrt{1+\Delta_k(t_s)}$, and $\omega_{d_s}(t_s) = \omega_s(t_s)\sqrt{1-\zeta_s^2(t_s)}$ are the natural frequency, damping factor and damped natural frequency of the digital twin. These three fundamental properties of system change with the slow time $t_s$.

For an SDOF model, several quantities can be measured and exploited to construct the digital twin. These quantities include but are not limited to transient response, forced response, frequency response natural frequency and damping factor. The natural frequency measurement is one of the most fundamental quantities that can be obtained in a relatively simple manner. It is generally expected that sensor data from the system will be transmitted wirelessly to develop the digital twin. If the digital twin were to be constructed in real-time, it is of paramount importance that the amount of data necessary is minimised. Considering these points, the choice of natural frequency as the 'measured sensor data' will be more effective than considering the time-domain response or the frequency domain response. This is mainly because the natural frequency is only a scalar for an SDOF system, while the response quantities can be long vectors. It should be recalled that unlike the response measurements, the natural frequency is, in general, a derived quantity and not a direct measurement. However, several methods are available [15, 16] to extract frequency in real-time from measured vibration response data which can be implemented efficiently.

Most natural frequency extraction technique obtain the damped natural frequency. It is therefore considered that this data is available for a given time instant $t_s \in [0, T_s]$ where $T_s$ is the upper-bound of the time window over which the digital twin is to be constructed. We define the distance measure between two quantities $A$ and $B$ using the $l_2$ norm as

$$d(A,B) = \sqrt{(A-B)^H(A-B)} = \|A-B\|_2 \tag{17}$$

Here $(\bullet)^H$ denotes the Hermitian transpose. For scalar quantities $(\bullet)^H$ simply becomes the complex conjugate of $(\bullet)$. Considering only the damped natural frequency, the distance measure is given by

$$d_1(t_s) = d(\omega_{d_0}, \omega_{d_s}(t_s)) \tag{18}$$

Using the corresponding expressions, one has

$$\widetilde{d}_1(t_s) = \frac{d_1(t_s)}{\omega_0} = \sqrt{1-\zeta_0^2} - \sqrt{1+\Delta_k(t_s)-\zeta_0^2} \tag{19}$$

The quantity $\widetilde{d}_1(t_s)$ is the distance measure normalised with respect to the undamped natural frequency of the nominal system. We use $\widetilde{d}_1(t_s)$ as *the* input function for developing the digital twin from the original nominal model. In this case the digital twin model is completely described by the function $\Delta_k(t_s)$. This must be calculated only from $\widetilde{d}_1(t_s)$.

This can be achieved by solving Eq. (19) for $\Delta_k(t_s)$ as

$$\Delta_k(t_s) = -\widetilde{d}_1(t_s)\left(2\sqrt{1-\zeta_0^2} - \widetilde{d}_1(t_s)\right) \tag{20}$$

This is the exact solution for the digital twin and it is possible primarily because a SDOF system is considered. For MDOF systems, a similar approach can be perused but most likely exact closed-form solutions may not be possible for the general case. If the damping of the nominal system is small then $1 \gg \zeta_0^2$ and we have the approximate expression

$$\Delta_k(t_s) \approx -\widetilde{d}_1(t_s)\left(2 - \widetilde{d}_1(t_s)\right) \tag{21}$$

If $\omega_{d_0} < \omega_{d_s}(t_s)$ then $\widetilde{d}_1(t_s)$ will be positive and consequently $\Delta_k(t_s)$ from Eq. (20) or (21) will give a negative value if $\widetilde{d}_1(t_s) < 2, \forall t_s \in [0, T_s]$. This is what is physically expected. Recall that the function $\widetilde{d}_1(t_s)$ is a dimensionless quantity denoting the relative shift in the natural frequency of the system as it evolves in time. For most practical cases we impose $\widetilde{d}_1(t_s) < 0.5$, $\forall t_s$. This is because if the natural frequency of a system changes more than 50% over a period of time, then the physical validity of the original nominal model becomes questionable. In such situations one should discard the original nominal model and seek another model which represents the current physics accurately. In other words, it is no longer possible to have digital twin based on the nominal model. As an absolute mathematical condition for the validity of Eqs. (20) and (21), $\widetilde{d}_1(t_s)$ must satisfy

$$\widetilde{d}_1(t_s) < 2,\ \forall t_s \tag{22}$$

For all practical cases, $\widetilde{d}_1(t_s)$ is expected to be significantly smaller than the above mathematical limit. We recommend about 15% of the above, that is, $\widetilde{d}_1(t_s) < 0.3$, for the approached taken here to remain physically meaningful. It should be noted that these bounds and recommendations are given by us based on our physical understanding of dynamic systems and may evolve in the future as the concept of digital twin becomes clearer and more case studies are reported.

### *3.2. Natural frequency data is available with errors*

Data collected and transmitted by sensors onboard of a dynamic system (aerospace-vehicles, automobiles, wind turbines) is susceptible to various kinds of error. They include measurement noise, transmission error, data storage inaccuracies, file corruptions, wireless signal loss, data bandwidth saturation, time-step mismatch, data hacking and alterations to mention a few. Although the expression of $\Delta_k(t_s)$ to construct the digital twin is exact, the input data $\widetilde{d}_1(t_s)$ contain error. Consequently, this error percolates into the digital twin. This error can be quantified following an approach similar to the previous subsection.

Suppose the error is denoted by $\theta(t_s)$. This can be a deterministic function (e.g, a bias in the sensors) or it can be a random function. For the second case, $\theta(t_s)$ becomes a random process [17] and must be defined by a suitable autocorrelation function. Considering this error, the measured damped natural frequency becomes

$$\widehat{\omega}_{d_s}(t_s) = \omega_0\sqrt{1 + \Delta_k(t_s) - \zeta_0^2} + \theta(t_s) \tag{23}$$

Combining this with Eqs. (18) and (19) we have

$$\widetilde{d}_1(t_s) = \frac{d_1(t_s)}{\omega_0} = \sqrt{1-\zeta_0^2} - \sqrt{1+\Delta_k(t_s)-\zeta_0^2} - \theta(t_s)/\omega_0 \tag{24}$$

Solving this equation, the function $\Delta_k(t_s)$ to construct the digital twin can be obtained as

$$\Delta_k(t_s) = -\left(\widetilde{d}_1(t_s) + \theta(t_s)/\omega_0\right)\left(2\sqrt{1-\zeta_0^2} - \left(\widetilde{d}_1(t_s) + \theta(t_s)/\omega_0\right)\right) \tag{25}$$

Although this is the exact solution, the error in the measured data influences the digital twin. Next we propose a different approach through which error in measured data can be addressed in a more robust manner.

### 3.3. Natural frequency data is available with error estimates

To apply the approach in the previous subsection, a value of error in the measured data is necessary for all $t_s$. This value may not be always available or known precisely. A more likely situation is when overall error estimate of the data is available. For example, this can be a manufacturer supplied tolerance of a sensing equipment (say 10%). In this case, without any loss of generality we consider $\theta(t_s)$ is a zero-mean Gaussian white noise with a standard deviation $\sigma_\theta$. Therefore

$$\mathrm{E}\left[\theta^2(t_s)\right] = \sigma_\theta^2 \quad \forall t_s \tag{26}$$

where $\mathrm{E}[\bullet]$ denotes the mathematical expectation operator. Taking the expectation of (18), we redefine the distance measure as

$$\bar{d}_1^2(t_s) = \mathrm{E}\left[d^2\left(\omega_{d_0}, \widehat{\omega}_{d_s}(t_s)\right)\right] \tag{27}$$

Substituting the expression of $\widehat{\omega}_{d_s}(t_s)$ from Eq. (23) in the above equation we have

$$\begin{aligned}\bar{d}_1^2(t_s) &= \mathrm{E}\left[\left(\omega_0\sqrt{1-\zeta_0^2} - \omega_0\sqrt{1+\Delta_k(t_s)-\zeta_0^2} - \theta(t_s)\right)^2\right] \\ &= \mathrm{E}\left[\left(d_1(t_s) - \theta(t_s)\right)^2\right] = d_1^2(t_s) + \sigma_\theta^2\end{aligned} \tag{28}$$

Dividing this equation by $\omega_0^2$, rearranging and taking the square root we have

$$\operatorname{sign}\left(\widetilde{d}_1(t_s)\right)\sqrt{\widetilde{d}_1^2(t_s) - \sigma_\theta^2/\omega_0^2} = \sqrt{1-\zeta_0^2} - \sqrt{1+\Delta_k(t_s)-\zeta_0^2} \tag{29}$$

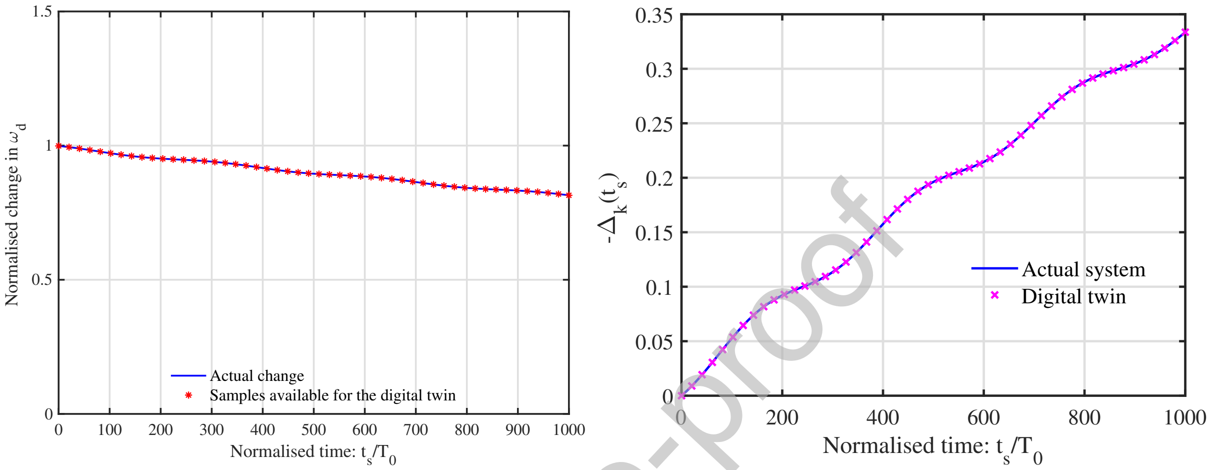
Note that the sign of the square root is taken such that it keeps the sign of $\widetilde{d}_1(t_s)$. Solving this equation, the function $\Delta_k(t_s)$ to construct the digital twin can be obtained as

$$\Delta_k(t_s) = -\operatorname{sign}\left(\widetilde{d}_1(t_s)\right)\sqrt{\widetilde{d}_1^2(t_s) - \sigma_\theta^2/\omega_0^2}\left(2\sqrt{1-\zeta_0^2} - \operatorname{sign}\left(\widetilde{d}_1(t_s)\right)\sqrt{\widetilde{d}_1^2(t_s) - \sigma_\theta^2/\omega_0^2}\right) \tag{30}$$

When the measurement becomes error free, that is, $\sigma_\theta^2 \to 0$, the above equation reduces to the deterministic case in Eq. (20). Therefore, Eq. (30) can be considered as the general expression of the function $\Delta_k(t_s)$ for the digital twin.

### 3.4. Numerical illustrations

To illustrate the applicability of the digital twin equations derived in the previous subsections, we consider an SDOF system with nominal damping factor $\zeta_0 = 0.05$. The physical system evolves continuously in the 'slow time' $t_s$. It is reasonable to consider that sensor data is transmitted intermittently with a certain regular time interval. For this example, the variation in the natural frequency is simulated using the change in the stiffness property of the system shown in Fig. 2. In Fig. 3(a) we show the actual change in the damped natural frequency of the system over time along with sample discrete data points which is available for the digital twin. The frequency of data availability will depend



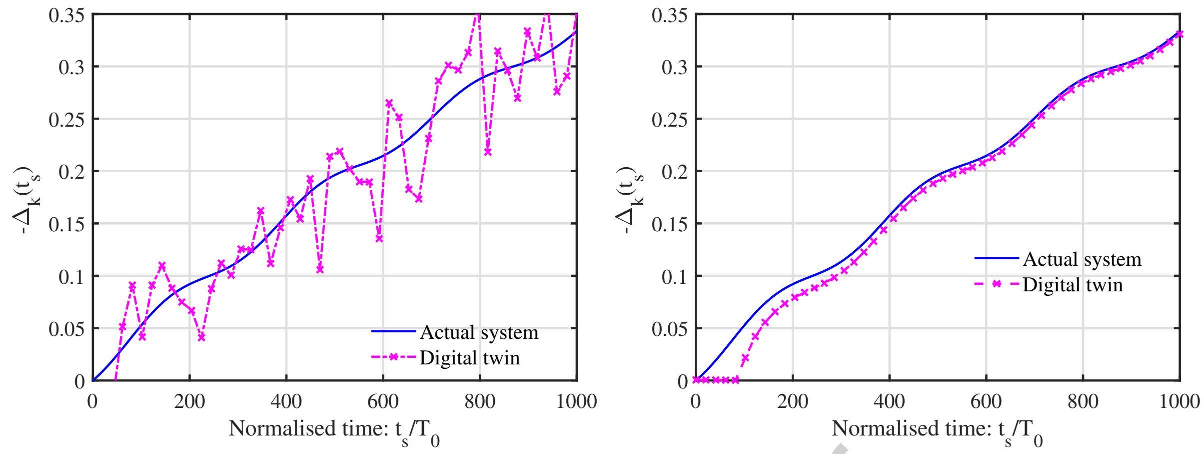

(a) Changes in (damped) natural frequency over time

(b) Digital twin constructed with the exact data

Fig. 3. Changes in the (damped) natural frequency and the digital twin obtained using the exact data plotted as a function of the normalised 'slow time' $t_s/T_0$. The digital twin in (b) is obtained from Eq. (20) using the distance norm calculated from data in (a).

on a number of practical details such as the bandwidth of the wireless data transmission system, energy requirement of data collection, cost of data transmission. The effectiveness of the digital twin will depend on how frequently the data is sampled and transmitted and also how sharp the variation of the measured frequency. For example, if a spike or unusual change in the frequency is missed due to poor sampling, the digital twin as proposed here will not be able to capture that real change in the system. Considering these limitations, in Fig. 3(b) the digital twin model obtained using Eq. (20) is shown. As Eq. (20) is an exact equation, the digital twin perfectly represents the actual system for all values of $t_s$ for which the natural frequency data was available. This verifies the validity of the digital twin given by Eq. (20).

The case of erroneous data is shown in Fig. 4. The digital twin in Fig. 4(a) is obtained using (25) and the one in Fig. 4(b) is obtained using (30). The error in the data is assumed to a discrete zero-mean Gaussian white noise with standard deviation $\sigma_\theta = 0.025$. The case when data error are incorporated in the digital twin, it can show significant deviation from the actual system as observed in Fig. 4(a). On the other hand, when the error estimate is apriori known, the digital twin follows the real system more closely as observed in Fig. 4(b). There are some discrepancies for lower vales of $t_s$ as in this region the value of the standard error is relatively more than that of the distance measure. This numerical example highlights the significant impact of different types of error in the data and how they are processed on the effectiveness of the digital twin.




(a) Digital twin constructed with erroneous data (b) Digital twin constructed with error estimate

Fig. 4. Digital twin constructed with erroneous data and error estimates as a function of the normalised 'slow time' $t_s/T_0$ Error in the form of a discrete zero-mean Gaussian white noise with a standard deviation of 0.025 is considered.

## 4. Digital twin via mass evolution

### *4.1. Exact natural frequency data is available*

It is considered that the stiffness and damping of the nominal model is unchanged and only the changes in the mass properties affect the digital twin. The equation of motion of the digital twin of a SDOF system for a fixed value of $t_s$ with variation only in the mass property is given by

$$m_0(1+\Delta_m(t_s))\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2}+c_0\frac{\mathrm{d}u(t)}{\mathrm{d}t}+k_0u(t)=f(t) \tag{31}$$

Dividing with $m_0$ and solving the characteristic equation, the damped natural eigenfrequencies can be expressed as

$$\lambda_{s_{1,2}}(t_s)=-\frac{\zeta_0\omega_0}{1+\Delta_m(t_s)}\pm\mathrm{i}\frac{\omega_0\sqrt{1+\Delta_m(t_s)-\zeta_0^2}}{1+\Delta_m(t_s)} \tag{32}$$

Rearranging this equation we have

$$\lambda_{s_{1,2}}(t_s)=-\omega_s(t_s)\zeta_s(t_s)\pm\mathrm{i}\omega_{d_s}(t_s) \tag{33}$$

Here

$$\omega_s(t_s)=\omega_0/\sqrt{1+\Delta_m(t_s)} \tag{34}$$

$$\zeta_s(t_s)=\zeta_0/\sqrt{1+\Delta_m(t_s)} \tag{35}$$

$$\text{and}\quad \omega_{d_s}(t_s)=\omega_s(t_s)\sqrt{1-\zeta_s^2(t_s)}=\frac{\omega_0\sqrt{1+\Delta_m(t_s)-\zeta_0^2}}{1+\Delta_m(t_s)} \tag{36}$$

are the natural frequency, damping factor and damped natural frequency of the digital twin. These three fundamental properties of system change with the slow time $t_s$.

Considering the distance measure similar to (18) we define

$$d_2(t_s) = d(\omega_{d_0}, \omega_{d_s}(t_s)) \tag{37}$$

$$\text{or} \quad \widetilde{d}_2(t_s) = \frac{d_2(t_s)}{\omega_0} = \sqrt{1-{\zeta_0}^2} - \frac{\sqrt{1+\Delta_m(t_s)-{\zeta_0}^2}}{(1+\Delta_m(t_s))} \tag{38}$$

Here $\widetilde{d}_2(t_s)$ is the distance measure normalised with respect to the undamped natural frequency of the nominal system. We use $\widetilde{d}_2(t_s)$ as *the* input function for developing the digital twin from the original nominal model. In this case the digital twin model is completely described by the function $\Delta_m(t_s)$ and can be obtained by solving Eq. (38) for $\Delta_m(t_s)$ as

$$\Delta_m(t_s) = \frac{\begin{array}{c} -2\widetilde{d}_2(t_s)^2 + 4\widetilde{d}_2(t_s)\sqrt{1-{\zeta_0}^2} - 1 + 2{\zeta_0}^2 \\ + \sqrt{1 - 4\widetilde{d}_2(t_s)^2{\zeta_0}^2 + 8\widetilde{d}_2(t_s)\sqrt{1-{\zeta_0}^2}{\zeta_0}^2 - 4{\zeta_0}^2 + 4{\zeta_0}^4} \end{array}}{2\left(-\widetilde{d}_2(t_s) + \sqrt{1-{\zeta_0}^2}\right)^2} \tag{39}$$

This is the exact solution and it is valid for any values of $\zeta_0$. If a small damping assumption is made, then $\zeta_0^k \approx 0, k \geq 2$. Using this approximation, Eq. (39) can be simplified as

$$\Delta_m(t_s) \approx \frac{\widetilde{d}_2(t_s)\left(2-\widetilde{d}_2(t_s)\right)}{\left(1-\widetilde{d}_2(t_s)\right)^2} \tag{40}$$

If $\widetilde{d}_2(t_s)$ is positive, then $\Delta_m(t_s)$ should also be positive. From the above equation we can therefore conclude that the absolute mathematical condition for the validity of this analysis is that $\widetilde{d}_2(t_s) < 2$, $\forall t_s$. However, it is recommended $\widetilde{d}_2(t_s) < 0.3$ for the physical relevance of the original nominal model. The original nominal model should be scrutinised if $\widetilde{d}_2(t_s)$ exceeds this value.

### *4.2. Natural frequency data is available with errors*

Considering the error function $\theta(t_s)$, the measured damped natural frequency becomes

$$\widehat{\omega}_{d_s}(t_s) = \frac{\omega_0\sqrt{1+\Delta_m(t_s)-\zeta_0^2}}{1+\Delta_m(t_s)} + \theta(t_s) \tag{41}$$

Combining this with Eqs. (37) and (38) we have

$$\widetilde{d}_2(t_s) = \sqrt{1-{\zeta_0}^2} - \frac{\sqrt{1+\Delta_m(t_s)-{\zeta_0}^2}}{(1+\Delta_m(t_s))} - \theta(t_s)/\omega_0 \tag{42}$$

Although the exact solution of this equation is possible, the resulting closed-form expression is not simple. Therefore employing the small damping approximation, the function $\Delta_m(t_s)$ to construct the digital twin can be obtained as

$$\Delta_m(t_s) \approx \frac{\left(\widetilde{d}_2(t_s) + \theta(t_s)/\omega_0\right)\left(2-\widetilde{d}_2(t_s) - \theta(t_s)/\omega_0\right)}{\left(1-\widetilde{d}_2(t_s) - \theta(t_s)/\omega_0\right)^2} \tag{43}$$

For the validity of this expression $\widetilde{d}_2(t_s) + \theta(t_s) < 2$, $\forall t_s$.

*4.3. Natural frequency data is available with error estimates*

We consider $\theta(t_s)$ is a zero-mean Gaussian white noise with a standard deviation $\sigma_\theta$. Taking the expectation of (37), we define the distance measure as

$$\bar{d}_2^2(t_s) = \mathrm{E}\left[d^2\left(\omega_{d_0}, \widehat{\omega}_{d_s}(t_s)\right)\right] \tag{44}$$

Substituting the expression of $\widehat{\omega}_{d_s}(t_s)$ from Eq. (41) in the above equation we have

$$\begin{aligned}\bar{d}_2^2(t_s) &= \mathrm{E}\left[\left(\omega_0\sqrt{1-\zeta_0^2} - \frac{\omega_0\sqrt{1+\Delta_m(t_s)-\zeta_0^2}}{1+\Delta_m(t_s)} - \theta(t_s)\right)^2\right] \\ &= \mathrm{E}\left[\left(d_2(t_s)-\theta(t_s)\right)^2\right] = d_2^2(t_s) + \sigma_\theta^2\end{aligned} \tag{45}$$

Dividing this equation by $\omega_0^2$, rearranging and taking the square root we have

$$\mathrm{sign}\left(\widetilde{d}_2(t_s)\right)\sqrt{\widetilde{d}_2^2(t_s) - \sigma_\theta^2/\omega_0^2} = \sqrt{1-{\zeta_0}^2} - \frac{\sqrt{1+\Delta_m(t_s)-{\zeta_0}^2}}{(1+\Delta_m(t_s))} \tag{46}$$

Note that the sign of the square root is taken such that it keeps the sign of $\widetilde{d}_1(t_s)$. Solving this equation with the small damping approximation, the function $\Delta_m(t_s)$ to construct the digital twin can be obtained as

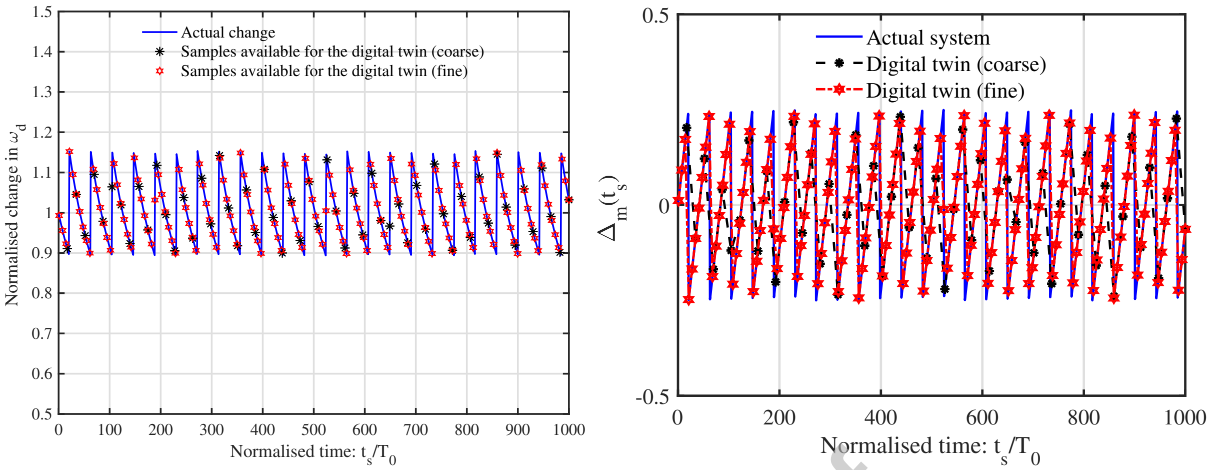
$$\Delta_m(t_s) \approx \frac{\mathrm{sign}\left(\widetilde{d}_2(t_s)\right)\sqrt{\widetilde{d}_2^2(t_s)-\sigma_\theta^2/\omega_0^2}\left(2-\mathrm{sign}\left(\widetilde{d}_2(t_s)\right)\sqrt{\widetilde{d}_2^2(t_s)-\sigma_\theta^2/\omega_0^2}\right)}{\left(1-\mathrm{sign}\left(\widetilde{d}_2(t_s)\right)\sqrt{\widetilde{d}_2^2(t_s)-\sigma_\theta^2/\omega_0^2}\right)^2} \tag{47}$$

When the measurement becomes error free, that is, $\sigma_\theta^2 \to 0$, the above equation reduces to the deterministic case in Eq. (40). Therefore, Eq. (47) can be considered as the general expression of the function $\Delta_m(t_s)$ for the digital twin.

*4.4. Numerical illustrations*

To illustrate the applicability of the digital twin equations derived for mass evolution, we consider an SDOF system with nominal damping factor $\zeta_0 = 0.05$. The physical system evolves continuously in the slow time $t_s$ and for numerical illustrations, it is considered that the sensor data is available at two different sampling rate. The variation in the natural frequency is simulated using the change in the mass property of the system shown in Fig. 2. In Fig. 5(a) we show the actual change in the damped natural frequency of the system over the time along with a coarse and a fine sample of discrete data points which is available for the digital twin. The coarse samples of data miss certain features of the underlying variations. This highlights the role of sampling rate on the digital twin. In Fig. 5(b) the digital twin model obtained using Eq. (40) is shown. As Eq. (40) is an exact equation, the digital twin perfectly represents the actual system for all values of $t_s$ for which the natural frequency data was available. This verifies the validity of the digital twin given by Eq. (40).

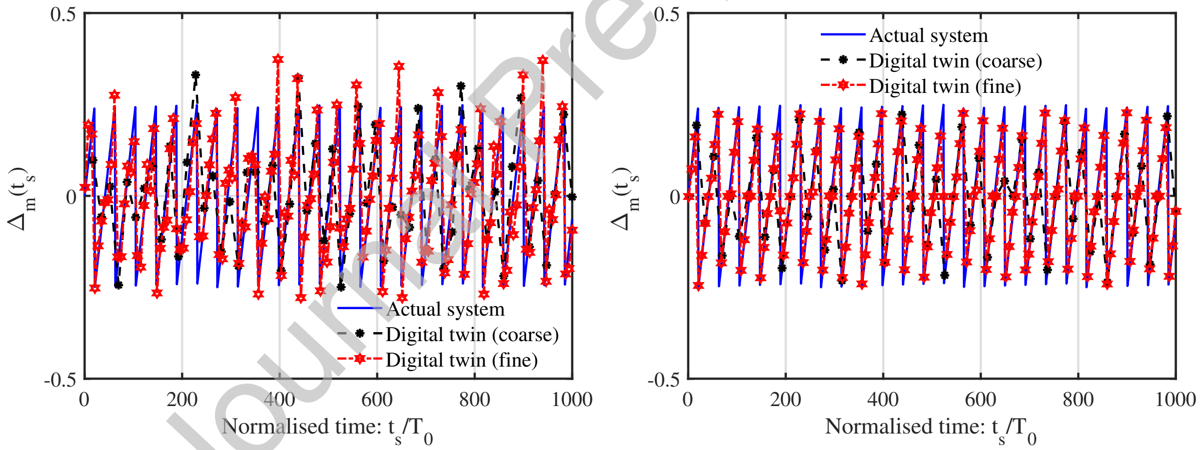
The case of erroneous data is shown in Fig. 6. The digital twin in Fig. 6(a) is obtained using (43) and the one in Fig. 6(b) is obtained using (47). The error in the data is assumed




(a) Changes in (damped) natural frequency over time

(b) Digital twin constructed with the exact data

Fig. 5. Changes in the (damped) natural frequency and the digital twin obtained using the exact data plotted as a function of the normalised 'slow time' $t_s/T_0$. The effect of two different sampling rates is shown. The digital twin in (b) is obtained from Eq. (20) using the distance norm calculated from data in (a).




(a) Digital twin constructed with erroneous data

(b) Digital twin constructed with error estimate

Fig. 6. Digital twin constructed with erroneous data and error estimates as a function of the normalised 'slow time' $t_s/T_0$. Error in the form of a discrete zero-mean Gaussian white noise with a standard deviation of 0.025 is considered.

to a discrete zero-mean Gaussian white noise with standard deviation $\sigma_\theta = 0.025$. The case when data error are incorporated in the digital twin, it can show deviation from the actual system as observed in Fig. 6(a). The deviations become more pronounced when data sampling is coarse.

On the other hand, when the error estimate is apriori known, the digital twin follows the real system more closely as observed in Fig. 6(b). There are some discrepancies for coarse data sampling as expected. This numerical example highlights the significant impact of different types of error in the data and how they are processed on the effectiveness of the digital twin.

## 5. Digital twin via mass and stiffness evolution

It is considered that both the stiffness and mass properties of the system simelteneously change with the slow time-scale $t_s$, while the damping of the nominal model is unchanged. The equation of motion of the digital twin of a SDOF system for a fixed value of $t_s$ with variations in the stiffness and mass properties is given by

$$m_0(1+\Delta_m(t_s))\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2}+c_0\frac{\mathrm{d}u(t)}{\mathrm{d}t}+k_0(1+\Delta_k(t_s))u(t)=f(t) \tag{48}$$

Dividing with $m_0$ and solving the characteristic equation, the damped natural eigenfrequencies can be expressed as

$$\lambda_{s_{1,2}}(t_s)=-\frac{\zeta_0\omega_0}{1+\Delta_m(t_s)}\pm \mathrm{i}\frac{\omega_0\sqrt{(1+\Delta_k(t_s))\,(1+\Delta_m(t_s))-\zeta_0^2}}{1+\Delta_m(t_s)} \tag{49}$$

Rearranging this equation we have

$$\lambda_{s_{1,2}}(t_s)=-\omega_s(t_s)\zeta_s(t_s)\pm \mathrm{i}\omega_{d_s}(t_s) \tag{50}$$

Here

$$\omega_s(t_s)=\omega_0\frac{\sqrt{1+\Delta_k(t_s)}}{\sqrt{1+\Delta_m(t_s)}} \tag{51}$$

$$\zeta_s(t_s)=\frac{\zeta_0}{\sqrt{1+\Delta_k(t_s)}\sqrt{1+\Delta_m(t_s)}} \tag{52}$$

$$\text{and}\quad \omega_{d_s}(t_s)=\omega_s(t_s)\sqrt{1-\zeta_s^2(t_s)}=\frac{\omega_0\sqrt{(1+\Delta_k(t_s))\,(1+\Delta_m(t_s))-\zeta_0^2}}{1+\Delta_m(t_s)} \tag{53}$$

are the natural frequency, damping factor and damped natural frequency of the digital twin. These three fundamental properties of system change with the slow time $t_s$.

Unlike the previous two cases considered, here we have two unknown functions, namely, $\Delta_k(t_s)$ and $\Delta_m(t_s)$, which define the digital twin. To obtain these two unknown functions uniquely, two independent equations are necessary for all $t_s$. From Eqs. (51) – (53) observe that all the three key dynamic quantities, namely, the natural frequency, damping factor and damped natural frequency change as functions of both $\Delta_k(t_s)$ and $\Delta_m(t_s)$. Therefore, it is sufficient to consider two out of these three dynamic quantities to measure for establishing the digital twin. The choice of which two quantities should be considered depends on what data is available as neither of these quantities are 'direct measurements' for a real-life dynamic system. In the following, we again consider three physically realistic cases through which a digital twin can be established.

### *5.1. Exact natural frequency data is available*

The damped natural eigenfrequencies give in Eq. (49) are complex valued quantities. We consider the real and imaginary parts separately to develop two equations through which the digital twin may be established. Considering the distance measure similar to

(18) and applying it separately to the real and imaginary parts we define

$$d_{\Re}(t_s) = d(\Re(\lambda_0), \Re(\lambda_s(t_s))) \tag{54}$$

$$\text{and} \quad d_{\Im}(t_s) = d(\Im(\lambda_0), \Im(\lambda_s(t_s))) \tag{55}$$

Dividing the distance measures by $\omega_0$ the normalised error measured are expressed as

$$\widetilde{d}_{\Re}(t_s) = \frac{d_{\Re}(t_s)}{\omega_0} = \frac{\zeta_0}{1+\Delta_m(t_s)} - \zeta_0 \tag{56}$$

$$\text{and} \quad \widetilde{d}_{\Im}(t_s) = \frac{d_{\Im}(t_s)}{\omega_0} = \sqrt{1-{\zeta_0}^2} - \frac{\sqrt{(1+\Delta_k(t_s))\,(1+\Delta_m(t_s)) - \zeta_0^2}}{1+\Delta_m(t_s)} \tag{57}$$

We use $\widetilde{d}_{\Re}(t_s)$ and $\widetilde{d}_{\Im}(t_s)$ as the input functions for developing the digital twin from the original nominal model. In this case the digital twin model is completely described by the functions $\Delta_k(t_s)$ and $\Delta_m(t_s)$ and can be obtained by solving the above two equations simultaneously as

$$\Delta_m(t_s) = -\frac{\widetilde{d}_{\Re}(t_s)}{\zeta_0 + \widetilde{d}_{\Re}(t_s)} \tag{58}$$

$$\text{and} \quad \Delta_k(t_s) = \frac{\zeta_0\,\widetilde{d}_{\Re}^2(t_s) - \left(1-2\,{\zeta_0}^2\right)\widetilde{d}_{\Re}(t_s) - 2\sqrt{1-{\zeta_0}^2}\zeta_0\,\widetilde{d}_{\Im}(t_s) + \zeta_0\,\widetilde{d}_{\Im}^2(t_s)}{\zeta_0 + \widetilde{d}_{\Re}(t_s)} \tag{59}$$

This is the exact solution and it is valid for any values of $\zeta_0$. It is evident that to establish a digital twin with simultaneous mass and stiffness evolutions, one needs distance measures for both real and imaginary parts of the complex natural frequency.

*5.2. Natural frequency data is available with errors*

The real and imaginary parts of a complex natural frequency physically correspond to significantly different aspects of dynamics even they are part of a singe eigenvalue. The real part corresponds to the decay rate of the free vibration response while the imaginary part corresponds to the oscillation frequency. The techniques to measure these quantities [18] also differ considerably. Therefore, it is reasonable to consider that the error corresponding to these two quantities are in general different and independent to each other.

Considering the error functions $\theta_{\Re}(t_s)$ and $\theta_{\Im}(t_s)$, the measured complex eigenfrequency becomes

$$\widehat{\lambda}_{s_{1,2}}(t_s) = -\left(\frac{\zeta_0\omega_0}{1+\Delta_m(t_s)} + \theta_{\Re}(t_s)\right) \pm \mathrm{i}\left(\frac{\omega_0\sqrt{(1+\Delta_k(t_s))\,(1+\Delta_m(t_s)) - \zeta_0^2}}{1+\Delta_m(t_s)} + \theta_{\Im}(t_s)\right) \tag{60}$$

Combining this with Eqs. (54) – (57) we have

$$\widetilde{d}_{\Re}(t_s) = \frac{\zeta_0}{1+\Delta_m(t_s)} - \zeta_0 + \theta_{\Re}(t_s)/\omega_0 \tag{61}$$

$$\text{and} \quad \widetilde{d}_{\Im}(t_s) = \sqrt{1-{\zeta_0}^2} - \frac{\sqrt{(1+\Delta_k(t_s))\,(1+\Delta_m(t_s)) - \zeta_0^2}}{1+\Delta_m(t_s)} - \theta_{\Im}(t_s)/\omega_0 \tag{62}$$

Solving the above two equations simultaneously, the functions $\Delta_k(t_s)$ and $\Delta_m(t_s)$ and can be obtained to describe the digital twin model with error in the data as

$$\Delta_m(t_s) = -\frac{\widetilde{d}_{\Re}(t_s) - \theta_{\Re}(t_s)/\omega_0}{\zeta_0 + \widetilde{d}_{\Re}(t_s) - \theta_{\Re}(t_s)/\omega_0} \tag{63}$$

and

$$\Delta_k(t_s) = \frac{\begin{array}{c}\zeta_0 \left(\widetilde{d}_{\Re}(t_s) - \theta_{\Re}(t_s)/\omega_0\right)^2 - \left(1 - 2\,{\zeta_0}^2\right)\left(\widetilde{d}_{\Re}(t_s) - \theta_{\Re}(t_s)/\omega_0\right) \\ -\,2\sqrt{1-{\zeta_0}^2}\zeta_0 \left(\widetilde{d}_{\Im}(t_s) + \theta_{\Im}(t_s)/\omega_0\right) + \zeta_0 \left(\widetilde{d}_{\Im}(t_s) + \theta_{\Im}(t_s)/\omega_0\right)^2\end{array}}{\zeta_0 + \widetilde{d}_{\Re}(t_s) - \theta_{\Re}(t_s)/\omega_0} \tag{64}$$

The error functions $\theta_{\Re}(t_s)$ and $\theta_{\Im}(t_s)$ can be deterministic or random functions. If they are random functions, then the complete description should be given cross correlation function of these two random processes. In this paper we assume them to be statistically uncorrelated functions.

*5.3. Natural frequency data is available with error estimates*

Following on from the discussion in the previous subsection, we consider $\theta_{\Re}(t_s)$ and $\theta_{\Im}(t_s)$ are statistically uncorrelated zero-mean Gaussian white noises with a standard deviations $\sigma_{\theta_{\Re}}$ and $\sigma_{\theta_{\Im}}$. Taking the expectation of (54) and (55), we define new distance measures as

$$\widetilde{d}^2_{\Re}(t_s) = \mathrm{E}\left[d^2(\Re(\lambda_0), \Re(\widehat{\lambda}_s(t_s)))\right] \tag{65}$$

$$\text{and} \quad \widetilde{d}^2_{\Im}(t_s) = \mathrm{E}\left[d^2(\Im(\lambda_0), \Im(\widehat{\lambda}_s(t_s)))\right] \tag{66}$$

Substituting the expression of $\widehat{\lambda}_s(t_s)$ from Eq. (60) in the above equation and following the procedure introduced in the previous sections we have

$$\operatorname{sign}\left(\widetilde{d}_{\Re}(t_s)\right)\sqrt{\widetilde{d}^2_{\Re}(t_s) - \sigma^2_{\theta_{\Re}}/\omega_0^2} = \frac{\zeta_0}{1+\Delta_m(t_s)} - \zeta_0 \tag{67}$$

$$\text{and} \quad \operatorname{sign}\left(\widetilde{d}_{\Im}(t_s)\right)\sqrt{\widetilde{d}^2_{\Im}(t_s) - \sigma^2_{\theta_{\Im}}/\omega_0^2} = \sqrt{1-{\zeta_0}^2} - \frac{\sqrt{(1+\Delta_k(t_s))\,(1+\Delta_m(t_s)) - \zeta_0^2}}{1+\Delta_m(t_s)} \tag{68}$$

Solving these two coupled equations, the functions $\Delta_k(t_s)$ and $\Delta_m(t_s)$ and can be obtained as

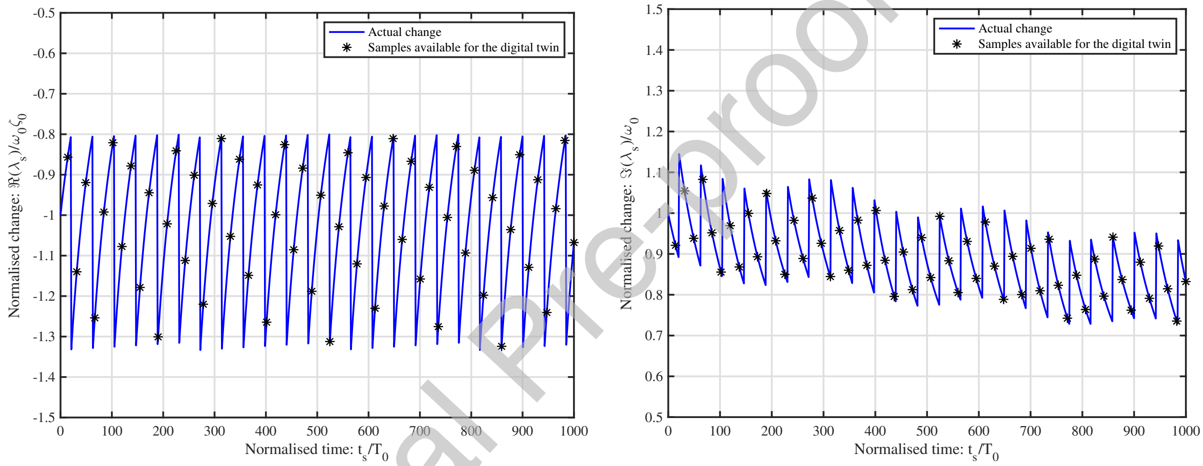
$$\Delta_m(t_s) = -\frac{\operatorname{sign}\left(\widetilde{d}_{\Re}(t_s)\right)\sqrt{\widetilde{d}^2_{\Re}(t_s) - \sigma^2_{\theta_{\Re}}/\omega_0^2}}{\zeta_0 + \operatorname{sign}\left(\widetilde{d}_{\Re}(t_s)\right)\sqrt{\widetilde{d}^2_{\Re}(t_s) - \sigma^2_{\theta_{\Re}}/\omega_0^2}} \tag{69}$$

$$\text{and} \quad \Delta_k(t_s) = \frac{\begin{array}{c}\zeta_0\left(\widetilde{d}^2_{\Re}(t_s) - \sigma^2_{\theta_{\Re}}/\omega_0^2\right) - \left(1-2\,{\zeta_0}^2\right)\operatorname{sign}\left(\widetilde{d}_{\Re}(t_s)\right)\sqrt{\widetilde{d}^2_{\Re}(t_s) - \sigma^2_{\theta_{\Re}}/\omega_0^2} \\ -\,2\sqrt{1-{\zeta_0}^2}\zeta_0\operatorname{sign}\left(\widetilde{d}_{\Im}(t_s)\right)\sqrt{\widetilde{d}^2_{\Im}(t_s) - \sigma^2_{\theta_{\Im}}/\omega_0^2} + \zeta_0\left(\widetilde{d}^2_{\Im}(t_s) - \sigma^2_{\theta_{\Im}}/\omega_0^2\right)\end{array}}{\zeta_0 + \operatorname{sign}\left(\widetilde{d}_{\Re}(t_s)\right)\sqrt{\widetilde{d}^2_{\Re}(t_s) - \sigma^2_{\theta_{\Re}}/\omega_0^2}} \tag{70}$$

This is the exact solution and it is valid for any values of $\zeta_0$. When the measurement becomes error free, that is, $\sigma^2_{\theta_{\Re}}, \sigma^2_{\theta_{\Im}} \to 0$, the above equations reduce to the deterministic case in Eqs. (58) and (59).

### 5.4. Numerical illustrations

We employ the previously used SDOF model to gain an understanding of the digital twin equations derived for simultaneous mass and stiffness evolution. The variation in the natural frequency as a function of the slow time $t_s$ is simulated using the change in the mass and stiffness properties of the system as is shown in Fig. 2. In Fig. 7 we show the actual change in the real and imaginary parts of the natural frequency of the system over time. The reduced number of samples available to construct the digital twin is also shown in the figure. The damping factor is assumed to be $\zeta_0 = 0.05$. Unlike the



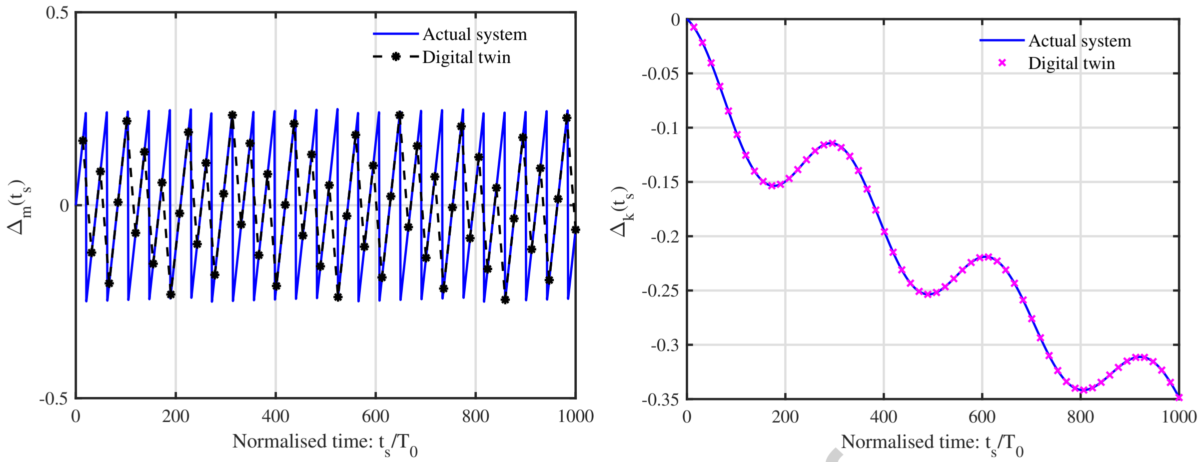

(a) Changes in the real part of the natural frequency

(b) Chnges in the imaginary part of the natural frequency

Fig. 7. Normalised changes in the real and imaginary parts of the natural frequency as a function of the normalised 'slow time' $t_s/T_0$. The reduced number of samples available to construct the digital twin are are marked by a '*'.

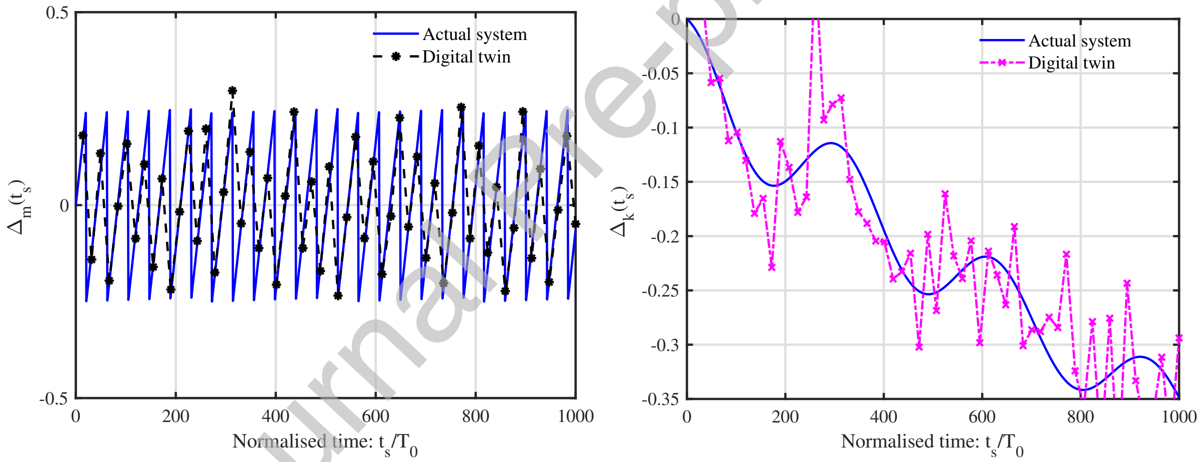
previous two cases, the presence of damping is crucial to construct the digital twin using the mass and stiffness evolution. In Fig. 8 we show the actual simultaneous changes in the mass and stiffness properties and the values obtained using the digital twin equations (58) and (59). As the data is coarse, one needs to interpolate the digital twin between the data points. The interpolation of the stiffness function resembles the actual system quite well as the functional variation is smooth and mathematically continuous in nature. However, the mass function is not interpolated with the same accuracy as the variations are sharp and not mathematically continuous in nature. This illustrates that different aspects of the same digital twin can be affected in a very different manner under the same data-resolution.

Next, we consider the effect of random errors in the data on the digital twin. In Fig. 9 we show the mass and stiffness properties using the digital twin equations (63) and (64) when the standard deviation of the errors in the imaginary and real parts of the complex natural frequencies are 0.025 and $0.025\zeta_0$ respectively. It can be observed that the impact




(a) The mass function for the digital twin (b) The stiffness function for the digital twin

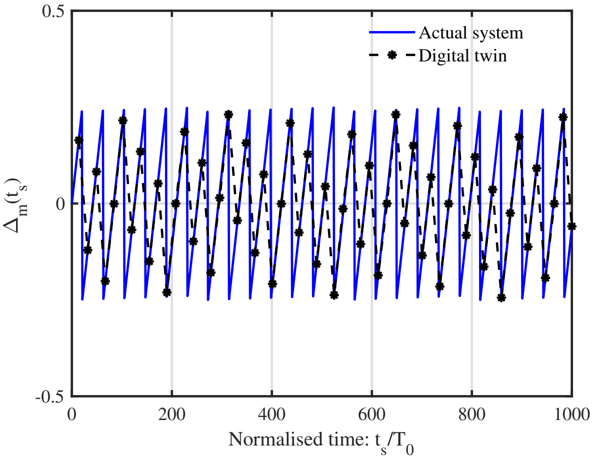
Fig. 8. Digital twin obtained from coarse but exact data via simultaneous mass and stiffness evolution as a function of the normalised 'slow time' $t_s/T_0$. The mass function in (a) is obtained using (58) while the stiffness function in (b) is obtained using (59).



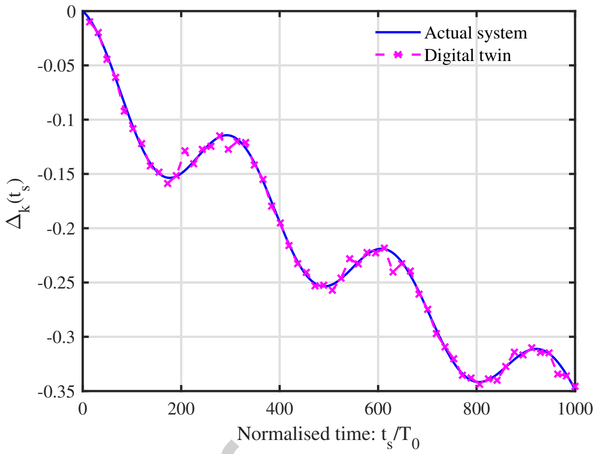

(a) The mass function for the digital twin (b) The stiffness function for the digital twin

Fig. 9. Digital twin obtained from erroneous data via simultaneous mass and stiffness evolution as a function of the normalised 'slow time' $t_s/T_0$. The mass function in (a) is obtained using (63) while the stiffness function in (b) is obtained using (64). Errors in the form of a discrete zero-mean Gaussian white noise with standard deviations of 0.025 and $0.025\zeta_0$ are considered for the imaginary and real parts.

of the error in the data is most severe on the stiffness evolution function. The mass evolution function of the digital twin, on the other hand, is less affected by the error in the data. In Fig. 10 the digital twin obtained with error estimates is shown as a function of the normalised 'slow time' $t_s/T_0$. The evolution of the mass and stiffness properties are obtained from (69) and (70). The standard deviation of the errors in the imaginary and real parts of the complex natural frequencies are considered as before. We observe from Fig. 10(a) that the mass evolution function of the digital twin is not significantly perturbed by the error in the data compared to what observed in Fig. 9(a). On the other hand, a significant difference is observed for the stiffness evolution function between what




(a) The mass function for the digital twin




(b) The stiffness function for the digital twin

Fig. 10. Digital twin obtained with error estimates via simultaneous mass and stiffness evolution as a function of the normalised 'slow time' $t_s/T_0$. The mass function in (a) is obtained using (69) while the stiffness function in (b) is obtained using (70). Errors in the form of a discrete zero-mean Gaussian white noise with standard deviations of 0.025 and $0.025\zeta_0$ are considered for the imaginary and real parts.

is obtained in Fig. 9(b) and Fig. 10(b). It is evident that if an error estimate of the data is available, the application of equation (70) leads to a realistic digital twin.

## 6. Discussions

A digital twin of physical systems can be achieved in various ways with the increasing demand of data and complexity of processing algorithms. The majority of existing works focus on broader conceptual aspects of a digital twin. The aim of this paper is to consider the specific case of structural dynamic systems. In particular, a single degree of freedom system is considered. Key ideas introduced in this paper include:

- The equation of motion is in terms of two time variables which are independent of each other. Time $t$ denotes the (fast) time describing the system dynamics. Time $t_s$ denotes a (slow) time describing the evolution of the digital twin. This separation of the time-scale is crucial for the practical development of digital twins of engineering dynamic systems.

- There must be a nominal model of the system from which a digital twin evolves. The nominal model is a validated, calibrated and verified the model of the system at time $t_s = 0$.

- Variations in the key response descriptors (natural frequency and damping factor in this case) should not deviate from the nominal system by more than 25%.

- Noise in the sensor data can be absorbed in two different ways - either directly or by statistical means.

- Exact and approximate closed-form mathematical expressions to explicitly obtain the digital twin of an SDOF dynamical system under different physically realistic contexts have been derived.
- Model functions describing stiffness and mass variations, coarse sampling and associated uncertainty models have been introduced in the numerical illustrations.

The system studied here is a simple single-degree-of-freedom dynamic system. This is expressed by a second-order ordinary differential equation. The closed-form digital twin expressions are directly applicable to other physical problems (e.g., electrical circuits) governed by this kind of equations. Although the work presented here considers only a single-degree-of-freedom system, the underlying conceptual framework can form the basis for more rigorous theoretical investigations encompassing a wider variety of practical problems. Future possibilities will include, but not limited to, investigating the following urgent issues:

1. *Multi time-scale digital twins*: This paper introduces the idea of one time-scale for the evolution of the digital twin. However, there is no physical or mathematical reason as to why this must be restricted to only one time scale. It is perfectly possible that various factors in a complex digital twin evolve at different time scales. For example, the mass of a system can change due to corrosion, while the stiffness of a system can degrade due to fatigue. These two processes will have very different time scale of evolution. Therefore, in a more general setting, the equation of motion of a dynamic digital twin can be expressed as
$$m(t_{s_1}, t_{s_2}, t_{s_3}, \cdots)\frac{\partial^2 u(t, t_{s_1}, t_{s_2}, t_{s_3}, \cdots)}{\partial t^2} + c(t_{s_1}, t_{s_2}, t_{s_3}, \cdots)\frac{\partial u(t, t_{s_1}, t_{s_2}, t_{s_3}, \cdots)}{\partial t} + k(t_{s_1}, t_{s_2}, t_{s_3}, \cdots)u(t, (t_{s_1}, t_{s_2}, t_{s_3}, \cdots) = f(t, t_{s_1}, t_{s_2}, t_{s_3}, \cdots) \quad (71)$$
Here $u(t, t_{s_1}, t_{s_2}, t_{s_3}, \cdots)$ is a multivariate function of not only the system time $t$ but also the independent multiple slow times $t_{s_1}, t_{s_2}, t_{s_3}, \cdots$.
2. *Digital twins from the response in the time or frequency domain*: Natural frequency and damping factor measurements are used here for establishing the digital twin. These are discrete scalar numbers which are normally derived/estimated from response measurements in the time or frequency domain. Methods need to be developed which will establish a digital twin directly from the response measurements. This is necessary because real dynamic systems may not have onboard capabilities for reliable extraction of natural frequencies and damping factors. In other words, digital twin should be developed by using continuous real-time dynamic measurements.
3. *Multiple-degrees-of-freedom (MDOF) digital twins*: The single-degree-of-freedom model is a simple idealisation of complex multiple-degrees-of-freedom (MDOF) systems. For an effective digital twin with realistic predictive capabilities, damped MDOF models (see for example [14, 19]) should be considered. The equation of motion of a MDOF digital twin can be expressed by
$$\mathbf{M}(t_s)\frac{\partial^2 \mathbf{u}(t, t_s)}{\partial t^2} + \mathbf{C}(t_s)\frac{\partial \mathbf{u}(t, t_s)}{\partial t} + \mathbf{K}(t_s)\mathbf{u}(t, t_s) = \mathbf{f}(t, t_s) \quad (72)$$

Here $\mathbf{M}(t_s)$, $\mathbf{C}(t_s)$ and $\mathbf{K}(t_s)$ are $N \times N$ matrices and $\mathbf{u}(t, t_s)$ and $\mathbf{f}(t, t_s)$ are $N$ dimensional vectors. A set of eigenvalues and corresponding eigenvectors can be utilised to construct the digital twin model.

4. *Stochastically parametrised digital twins*: The mass $m(t_s)$, damping $c(t_s)$, stiffness $k(t_s)$ functions are assumed to be deterministic. However, it was established that there are uncertainties in the measured and transmitted data available to construct the digital twin. In this paper, the effect of these uncertainties has been mitigated by using statistical averages. While this can be considered as a first step, a more rigorous approach will be to model $m(t_s)$, $c(t_s)$ and $k(t_s)$ themselves as random quantities. As they are functions of the time variable $t_s$, each of them must be modelled as a random process. One can start with a stationary Gaussian random process model with an assumed autocorrelation function. In this case, the establishment of the digital twin will be through the estimation of the parameters of the autocorrelation function.
5. *Nonlinear digital twins*: The evolution of the mass $m(t_s)$, damping $c(t_s)$, stiffness $k(t_s)$ functions are considered as nonlinear functions of $t_s$ in this paper. However, the dynamics of the system in $t$ is considered as linear. For many digital twins, a linear dynamic assumption may not be physically accurate or realistic. One example is dynamic systems with large-amplitude vibrations. Several types of nonlinearities can be considered for the digital twin. An illustration of a cubic-type nonlinearity (see for example [20]) can be achieved using a Duffing's oscillator digital twin as

$$m(t_s)\frac{\partial^2 u(t,t_s)}{\partial t^2} + c(t_s)\frac{\partial u(t,t_s)}{\partial t} + k(t_s)\left(u(t,t_s) - \epsilon\, u^3(t,t_s)\right) = f(t,t_s) \tag{73}$$

This is a softening type nonlinear system and $\epsilon$ is a fixed constant which quantifies the 'strength' of the nonlinearity. Note that $\epsilon$ can also evolve with $t_s$. In that case, $\epsilon(t_s)$ needs to be identified from the sensor data. The establishment of this digital twin model therefore requires nonlinear dynamic analysis in the context of inverse problems.
6. *Digital twins for continuum systems*: Many engineering dynamic problems are modelled and subsequently analysed using continuum models such as beams, plates and shells. The advantage of using a continuum model, as opposed to discrete or discretised models, is that it can give simple and physically insightful results. Digital twins for relevant real systems can make use of the continuum models for simplicity and practical relevance. As an example, digital twin of a damped Euler-Bernoulli beam [21] can be expressed as

$$EI(t_s)\frac{\partial^4 U(x,t,t_s)}{\partial x^4} + c_1(t_s)\frac{\partial^5 U(x,t,t_s)}{\partial x^4 \partial t} + m(t_s)\frac{\partial^2 U(x,t,t_s)}{\partial t^2} + c_2(t_s)\frac{\partial U(x,t,t_s)}{\partial t} = F(x,t,t_s) \tag{74}$$

In the above equation $x$ is the coordinate along the length of the beam, $t$ is the time variable representing the beam dynamics, $EI(t_s)$ is the bending rigidity, $m(t_s)$ is the mass per unit, $c_1(t_s)$ is the strain-rate-dependent viscous damping coefficient, $c_2(t_s)$ is the velocity-dependent viscous damping coefficient, $F(x,t,t_s)$ is the applied spatial dynamic forcing and $U(x,t,t_s)$ is the transverse displacement. Similar to the

SDOF system considered here, the digital twin will be realised by establishing the quantities which are functions of the slow time $t_s$.

7. *Predictive response of digital twins*: Identifying the coefficient functions varying with $t_s$ is the first step towards exploiting a digital twin for dynamic systems. A potential advantage of having a digital twin is to use this for response predictions and subsequent engineering decisions. New efficient computational methods are needed for multiple-degrees-of-freedom (MDOF) digital twins where the mass, damping and stiffness matrices can be large and also varying as functions of $t_s$. Additionally, if the forcing function is random, the dynamic response will be a random process evolving in $t_s$. This type of problems can be addressed using the framework of random vibration [22, 23]. New analytical formulations must be developed to characterise random processes involving multiple time-scales.
8. *Uncertainty quantification for digital twins*: Quantification and management of uncertainties have a fundamental role in the development of digital twins irrespective of its complexity, depth and range. Further research is necessary not only to model uncertainties in the coefficient functions and the forcing functions, but also to propagate such uncertainties in an efficient manner. The equation of motion of a stochastic multiple-degrees-of-freedom digital twin can be expressed as

$$\mathbf{M}(t_s,\xi)\frac{\partial^2 \mathbf{u}(t,t_s,\xi)}{\partial t^2} + \mathbf{C}(t_s,\xi)\frac{\partial \mathbf{u}(t,t_s,\xi)}{\partial t} + \mathbf{K}(t_s,\xi)\mathbf{u}(t,t_s,\xi) = \mathbf{f}(t,t_s,\xi) \tag{75}$$

where $\xi \in \Xi$ denotes the sample space. Therefore, $\mathbf{M}(t_s,\xi)$, $\mathbf{C}(t_s,\xi)$ and $\mathbf{K}(t_s,\xi)$ are $N \times N$ random matrices and $\mathbf{u}(t,t_s,\xi)$ and $\mathbf{f}(t,t_s,\xi)$ are $N$ dimensional random vectors. Stochastic finite element method can be adapted to be used for systems with time-varying coefficients. Reduced order uncertainty propagation techniques such as polynomial chaos [24] and other surrogate modelling approaches [25, 26] need to be developed for stochastic digital twins.
9. *Digital twins via Bayesian frameworks*: Bayesian approaches in general start with a prior probability density function and then update it based on the available data. There are many theoretical and computational methods to achieve this. For example, several powerful filtering methods such as ensemble Kalman filter, extended Kalman filter and particle filter (see for example [27, 28]) have been proposed. Due to the availability of large data, often continuously in time $t_s$, digital twin technology is perfectly positioned to exploit Bayesian methods. There are significant opportunities to develop new Bayesian methodologies to develop probabilistic models of digital twins. A Bayesian digital twin is likely to instil more confidence on the users than their non-Bayesian counterparts.
10. *Machine learning and big data in digital twins*: Many modern systems are integrated with a large number of sensors. Moreover, these sensors often have a high sampling rate and data is continuously communicated from the system to the cloud where the information processing algorithms extract features from the data. This combination of a large number of sensors and a high sampling rate produces an enormous amount of data. This is typically called a big data problem. The data transmitted to the cloud needs to be used for updating the digital twin. The data is also polluted with uncertainty from various sources and machine learning methods are useful in ensuring the digital twin matches the evolution of the physical system as closely as

possible. Sensor data may be numerical for example acceleration, strain or pressure measurements. However, data may also be in the form of text, speech and images obtained from recordings of noise from machinery, commands given by the pilot or maintenance engineer and images obtained by digital cameras/smart phones about changes in the system state. Deep learning methods provide one approach to fuse such data and to extract features from it.

## 7. Conclusions

The recent rise of digital twins motivated academic and industrial researchers to formalise and standardise underlying procedures and methodologies. The authors of this paper clearly see the benefit of such an approach due to the possibility of wider adoption of digital twins across several industrial sectors. However, a major drawback of such an approach is jargon-dominated literature, which many early adopters of digital twins may find it difficult to tailor for their specific application sector. Keeping this background in mind, this paper proposes a specific but original approach to develop digital twins for structural dynamic systems. The physics-based model of a single-degree-of-freedom dynamic system is governed by second-order differential equations. The main scientific proposition is that the digital twin evolves at a time-scale which is much slower than the dynamics of the system. This makes it possible to identify crucial system parameters as a function of the 'slow-time' from continuously measured data. Based on the quantity, quality and the nature of the available data, two broad cases are considered. They include (a) only the imaginary part of the complex natural frequency is available., and (b) both the real and imaginary parts of the complex natural frequency are available. For each of these cases, three possible practical situations are envisaged, namely, (1) measured data is exact, (2) measured data contain explicit error, (3) measured data is available with error estimates. It is proposed that the stiffness and mass of the single-degree-of-freedom digital twin evolve with the slow-time. Exact closed-form expression has been derived to establish the digital twin for these different cases.

The application of the new mathematical expressions has been demonstrated using numerical examples. Simulated functions representing mass and stiffness variation as functions of the slow time have been proposed considering aircraft dynamic systems. The case when data is available at a lower sampling rate has been investigated numerically. Although only single-degree-of freedom dynamic systems are considered, several conceptual extensions directly followed from this work are described in details. The work presented is intended to be considered as a mathematical framework for establishing digital twins for which the underlying physics is governed by differential equations.

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