Matching Spatially Diversified Suppliers with Random Demands

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ABSTRACT

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A fundamental challenge in operations management is to dynamically match spatially diversified supply sources with random demand units. This dissertation tackles this challenge in two major areas: in supply chain management, a company procures from multiple, geographically differentiated suppliers to service stochastic demands based on dynamically evolving inventory conditions; in revenue management of ride-hailing systems, a platform uses operational and pricing levers to match strategic drivers with random, location and time-varying ride requests over geographically dispersed networks.

The first part of this dissertation is devoted to finding the optimal procurement and inventory management strategies for a company facing two potential suppliers differentiated by their lead times, costs and capacities. We synthesize and generalize the existing literature by addressing a general model with the simultaneous presence of (a) orders subject to *capacity limits*, (b) *fixed costs* associated with inventory adjustments, and (c) possible *salvage opportunities* that enable bilateral adjustments of the inventory, both for finite and infinite horizon periodic review models. By identifying a novel, generalized *convexity* property, termed (C_1K_1, C_2K_2)-convexity, we are able to characterize the optimal single-source procurement strategy under the simultaneous treatment of all three complications above, which has remained an open challenge in stochastic inventory theory literature. To our knowledge, we recover almost all existing structural results as special cases of a unified analysis. We then generalize our results to dual-source settings and derive optimal policies under specific lead time restrictions. Based on these exact optimality results, we develop various heuristics and bounds to address settings with fully general lead times.

The second part of this dissertation focuses on a ride-hailing platform's optimal control facing two major challenges: (a) significant *demand imbalances* across the network, and (b) stochastic *demand shocks* at hotspot locations. Towards the first major challenge, which is evidenced by our analysis of New York City taxi trip data, the dissertation shows how the

platform's operational controls—including demand-side admission control and supply-side empty car repositioning—can improve system performance significantly. Counterintuitively, it is shown that the platform can improve the overall value through strategic rejection of demand in locations with ample supply capacity (driver queue).

Responding to the second challenge, a demand shock of uncertain duration, we show how the platform can resort to surge pricing and dynamic spatial matching jointly, to enhance profits in an incentive compatible way for the drivers. Our results provide distinctive insights on the interplay among the relevant timescales of different phenomena, including rider patience, demand shock duration and drivers' traffic delay to the hotspot, and their impact on optimal platform operations.

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To my family

Introduction

One of the fundamental challenges in supply chain and revenue management is the *dynamic matching* of geographically dispersed supply sources and demand units.

In supply chains, production facilities or distribution centers function as the supply sources, servicing a specific network of inventory points over time. Within the broad field of revenue management, consider the area of ride-hailing networks, such as those run by Uber, Lyft and Didi, with a combined revenue of $49.6 billion in the US and $53.5 billion in China alone in 2019[1], here, the challenge is to match a pool of drivers, acting as suppliers of ride capacity, with riders located at the nodes of a given geographically dispersed traffic network.

While the above two planning areas are quite distinct, they share several key features. First, the demand process at the demand points is stochastic. Second, the geographic distance and associated travel time between supply and demand points is a key factor in determining matching strategies, to be traded off with several other factors.

In the supply chain area, these travel times translate into lead times for orders placed by the demand point(s); these lead times, in turn are a key determinant in establishing safety stocks as proper hedges against the uncertainties, embedded in the demand process, in particular when a high level of service is to be provided to service sensitive buyers at the demand points.

Similarly, in ride-hailing systems, the customers (potential riders) are very delay sensitive, as are the suppliers (drivers) for whom pickup times translate into lost earning opportunities. And in both arenas, customers become ever more sensitive to time delays as a key measure of service.

[1] Statista.com: https://www.statista.com/outlook/368/ride-hailing. Accessed on July 22, 2019.

The first part of this dissertation (Chapters 1 and 2) address these problems in a context where a company has access to multiple, geographically dispersed suppliers to procure a single item, the demand for which is stochastic and, possibly, time-dependent. We focus on systems with two potential suppliers (*dual sourcing*). Typically, one supplier charges a lower purchase price but has a relatively large lead time, while the second supplier has a shorter lead time but is more expensive. Besides for their lead time and price, the suppliers may be differentiated by their capacity levels.

Some companies elect a single supplier based on aggregate cost and service considerations; however, significant benefits can, often, be obtained when using the two suppliers in parallel. This then raises the challenge of designing procurement strategies which select when and how much to order from each as a function of the dynamically evolving inventory information.

Chapter 2 develops a tractable planning model for this class of problems, incorporating various practical complications and opportunities. More specifically, we study a finite horizon, single product, periodic review inventory system with two supply sources and salvage options. A challenging trade-off exists between the two sources because the expedited supplier has a shorter lead time but charges a higher per-unit price, while the regular supplier has a longer lead time but lower ordering costs. A further complication is the salvage option that allows for bilateral inventory adjustments. All inventory adjustments involve a fixed cost component in addition to variable costs or revenues and may be subject to capacity limits.

In each period, we show that an optimal policy first determines the size of an order with the expedited supplier, if any, or the size of any salvage quantity, based, exclusively, on the regular full inventory position. Thereafter, the inventory position is adjusted upward (by the expedited supplier order) or downward (by the salvage quantity); any order with the regular supplier is then determined as a function of the *adjusted* inventory position. Moreover, the dependence of the optimal order sizes and/or salvage quantity, on the period's starting inventory position follows a relatively simple structure. In the most general case, the optimal policy is characterized by four critical threshold levels of the inventory position. As far as the second stage ordering decision with the *regular* supplier is concerned, the optimal policy is characterized by two threshold parameters partitioning the *adjusted inventory position* line in up to three regions.

The simultaneous treatment of capacitated inventory adjustments, bilateral inventory adjustments and fixed costs incurred for any such adjustment, is a major challenge even within traditional single sourcing problems. Indeed, since the initiation of stochastic inventory theory by Arrow et al. (1951), various structural results have been identified for various single item models. For example, *base stock policies* are optimal in the *base model*, where order costs are linear in the order sizes, no capacity limits prevail and inventory levels can only be adjusted upward via procurement orders. Double threshold, so-called (s, S) *policies* were shown to be optimal in broad generality when there are additional *fixed order costs*. However, the simultaneous treatment of all of the above complications has remained an open challenge.

Chapter 1 synthesizes and generalizes the existing literature by addressing a general model with the simultaneous presence of (a) orders subject to capacity limits, (b) fixed cost component in addition to variable costs, and (c) possible bilateral adjustment of inventories, both for finite and infinite horizon periodic review models. We provide a full characterization of the optimal procurement strategy by showing that in each period the inventory position line is to be partitioned into (maximally) five regions: in the most far left (right) region, it is optimal to place an order (initiate a salvage sale) of a specific easily calculable magnitude. In the middle region, it is optimal to avoid any inventory adjustment. Finally, in the second region from the left (right), the policy alternates between intervals where one stays put and those where an order is to be placed (a salvage sale is to be initiated). We provide a broad sufficient condition under which the second region from the left (right) vanishes. So that, in particular, the optimal policy is characterized by three regions only.

Our results are obtained by identifying a novel generalized convexity property for the value functions, which we refer to as (C_1K_1, C_2K_2)-convexity. To our knowledge, we recover all existing structural results as special cases of a unified analysis.

In Chapter 3, we shift our attention to ride-hailing platforms such as Uber, Lyft and Didi that match demand (riders) with service capacity (drivers) over a geographically dispersed network. This matching problem is complicated by two challenges. (i) There are significant demand imbalances in the network. (ii) Drivers are self-interested and behave strategically in deciding whether to join, and if so, how to reposition (route) themselves when not transporting riders. To address these challenges, we study the value of two operational controlsdemand-side admission control and supply-side repositioning controlon the performance of a revenue-maximizing ride-hailing platform.

Considering a fluid model of a two-location network in a game-theoretic framework, we characterize the system equilibrium under three control regimes, ranging from minimal control to centralized admission and repositioning control. The results contribute novel insights on the interplay between the platform's admission control and the drivers' strategic routing decisions. We also quantify the impact of control capabilities on platform revenue, participating capacity and per-driver profits. We find that the value of control is largest at moderate utilization and increases with demand imbalances.

Chapter 4, the last chapter of the dissertation, studies an online platform that operates a ride-hailing network with price and delay sensitive riders and strategic drivers that supply processing capacity. Our model jointly considers *surge pricing* (rider price and driver wage) and dynamic *spatial matching* in the platform's profit maximization problem, responding to a demand shock at a hotspot with uncertain magnitude and/or duration. Surge pricing is meant to a) moderate demand and b) incentivize supply to proactively reposition toward the hotspot. Dynamic matching trades off non-hotspot local matches for more lucrative hotspot matches, within service level constraints. Importantly, a surge in driver wages acts after a delay that depends on the distance drivers need to cover until they can get matched at the hotspot, and this heterogeneous repositioning delay may render the drivers' future expected benefit from the current wage surge uncertain, which affects their decisions to react.

We obtain optimal pricing and matching policies under fixed and random shock duration. Our results show the interplay between important timescales, e.g., rider patience, demand shock duration, and drivers' travel delay to the hotspot, and their impact on system performance. The distinctive features of this work lie on the focus of system transient under *non-stationary* demand, the network setting, and drivers' strategic response to surge signals given *delayed incentives*.

Chapter 1

Synthesis and Generalization of Structural Results in Inventory Management: A Generalized Convexity Property

1.1 Introduction.

The seminal papers by Arrow et al. (1951) and Dvoretzky et al. (1953) initiated the field of stochastic inventory theory, more than 65 years ago. These authors proposed a single-item base model with a finite planning horizon in which an order can be placed at the beginning of each period to increase the inventory level. The base model assumes that orders of an arbitrary, unlimited size may be placed and that the associated order costs are *proportional* to the order sizes. Demands are random but independent across time. Additional costs consist of inventory carrying and stockout or backlogging costs, assumed to be *proportional* with the end-of-the-period inventory levels and backlogging sizes, respectively. In the base model, it was shown that a so-called base-stock policy is optimal, in each period. Under such a policy, the inventory level is increased to a "base-stock" level, whenever it is found to be below that level; otherwise, it is optimal not to place any order. Scarf (1960) showed that, under backlogging of stockouts, a base-stock policy continues to be optimal in the presence of an order *lead-time*, except that the policy acts on a different inventory measure, the so-called *inventory position* = inventory level plus all outstanding orders.

It was quickly understood that the base model needed to be generalized to address various complications that arise in practice, for example fixed order costs or capacity limits for individual order sizes. When fixed order costs are included to the base model, Scarf (1960) and Iglehart (1963) showed that, under broad general conditions, an (s, S)-policy is optimal, for finite and infinite horizon models, respectively. Under an (s, S)-policy, it is optimal to elevate the inventory position to an order-up-to level, S, but only if the period's starting inventory position is at or below a second threshold $s < S$ (as opposed to S itself in the absence of fixed order costs). Federgruen and Zipkin (1986a,b) showed that order capacity limits result in the optimality of a so-called *modified* base-stock policy: at the beginning of each period, an order is placed to bring the inventory position as close to the base-stock level as is feasible.

But, what if both complications (fixed order costs and capacity limits for individual orders) prevail simultaneously? As Federgruen and Zipkin (1986b) wrote:

> *"If the production costs have a fixed (as well as a variable) component, it might be reasonable to expect that the modified (s, S) policy would be optimal: when the inventory level falls below a critical number s, produce enough to bring total stock up to S, or as close as possible, given the production capacity; otherwise do not produce."*

However, Wijngaard (1972) and later on Shaoxiang and Lambrecht (1996) and Shaoxiang (2004) identified counterexamples, both in finite and infinite horizon models. Indeed, a more complex structure emerges.

Similarly, some authors, starting with Whisler (1967) and Constantinides and Richard (1978), have considered settings where inventories may be adjusted downwards (as well as upwards) via sales in secondary channels (jobbers, discounters, outlet stores, etc) or returns to the supplier. Several authors have addressed inventory models with *bilateral* inventory adjustment options, i.e., procurement orders along with salvage sales and/or returns to the suppliers, for example Dai and Yao (2013) and Feinberg and Lewis (2005, 2007), see also the references therein. However, to our knowledge, no one has considered settings where the size of the inventory adjustments is subject to capacity limits, for example.

This chapter synthesizes and generalizes the existing literature with exogenously specified demands by addressing a general model with the simultaneous presence of the above-mentioned complications, specifically,

(a) *bilateral* inventory adjustment options, via procurement orders and salvage sales or returns to the supplier;

(b) fixed costs associated with procurement orders and downward inventory adjustments (via salvage sales or returns);

(c) capacity limits associated with upward or downward inventory adjustments.

We provide a full characterization of the optimal inventory adjustment strategy, both for finite and infinite horizon periodic review models, by showing that in each period the inventory position line is to be partitioned into (maximally) five regions: in the most far left (right) region, it is optimal to place an order (initiate a salvage sale) of a specific easily calculable magnitude. In the middle region, it is optimal to avoid any inventory adjustment. Finally, in the second region from the left (right), the policy alternates between intervals where one stays put and those where an order is to be placed (a salvage sale is to be initiated) of a size specified by a given function.

Our results are obtained by identifying a novel generalized convexity property for the value functions, which we refer to as strong (C_1K_1, C_2K_2)-convexity. To our knowledge, we recover most existing structural results for models with exogenous demands as special cases of a unified analysis. (To our knowledge, the exceptions are uncapacitated models with non-linear order costs, of a type, different from the fixed-plus-linear structure.)

The remainder of this chapter is organized as follows: In Section 1.2 we review the related literature. Section 1.3 introduces our general model and the associated notation. Section 1.4 derives the structure of an optimal policy in a single period model. Section 1.5 covers a general finite horizon model; this Section also recovers existing structures in the literature as special cases of our general results. Section 1.6 shows how our structural results extend to stationary infinite horizon models, either under the discounted total cost or the long-run average cost criterion. Section 1.8 ends the chapter with some concluding remarks.

1.2 (C_1K_1, C_2K_2)-convexity: A generalized convexity property and review of existing literature.

The structural results obtained in this chapter are based on our identifying a new generalized concept of convexity.

Definition 1.1 ((C_1K_1, C_2K_2)-convexity). *Given constants $C_1 > 0, K_1 \geq 0$ and $C_2 > 0, K_2 \geq 0$, a real-valued continuous function f is called* strongly (C_1K_1, C_2K_2)-convex *if* *for any* $x \geq y, a \in [0, C_1]$ *and* $b \in (0, C_2]$,

$$f(x+a) + K_1 \geq f(x) + \frac{a}{b}\Big(f(y) - f(y-b) - K_2\Big). \tag{1.1}$$

Denote $SC_{C_1K_1,C_2K_2}$ *as the set of all strongly* (C_1K_1, C_2K_2)*-convex functions. When* (1.1) *is required only for* $x = y$*, we refer to the property as* weak (C_1K_1, C_2K_2)*-convexity.*

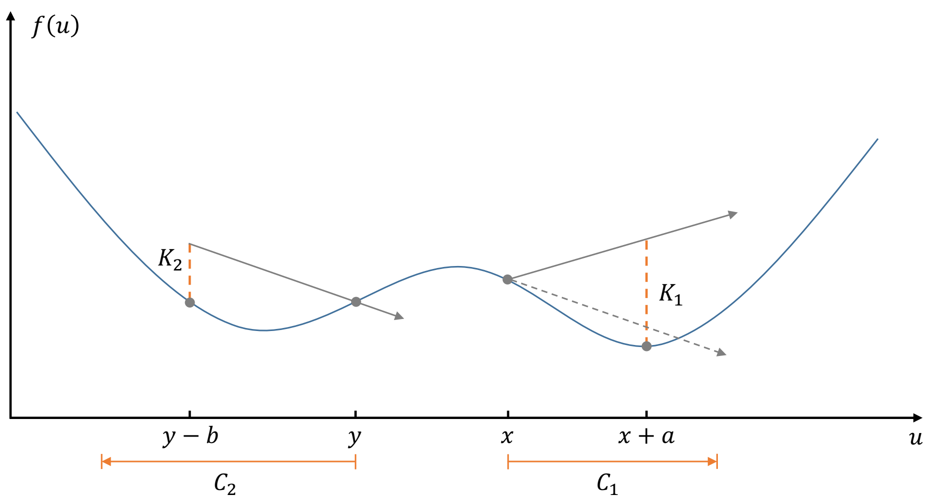


Figure 1.1: Geometric illustration of strongly (C_1K_1, C_2K_2)-convex functions

Figure 1.1 provides an intuitive way of understanding the strong (C_1K_1, C_2K_2)-convexity property. For any two points $y \leq x$, select any point $x + a$ with $a \in (0, C_1]$ and any point $y - b$ with $b \in (0, C_2]$. Raise the function value at point $x + a$ by K_1 and draw a ray from $(x, f(x))$ to $(x+a, f(x+a)+K_1)$. Similarly raise the function value at point $y-b$ by K_2 and draw a ray from $(y-b, f(y-b)+K_2)$ to $(y, f(y))$. Then f is strongly (C_1K_1, C_2K_2)-convex if the slope of the former ray is bigger than or equal to the slope of the latter ray.

The (C_1K_1, C_2K_2)-convexity property generalizes many convexity properties, developed since Scarf (1960) identified K-convexity as the key structural property to establish optimality of the so-called (s, S)-policies. Below, we list these earlier convexity properties in Table 1.1.

Table 1.1: Summary of Commonly Used Convexity Properties

Convexity Property	Definition	Related Papers
convex	$f(x+a) \geq f(x) + \frac{a}{b}[f(y) - f(y-b)]$, $\forall y \leq x,\ a \geq 0,\ b > 0$	Archimedes (3rd Century B.C.E.)
K-convex	$f(x+a) + K \geq f(x) + \frac{a}{b}[f(x) - f(x-b)]$, $\forall a \geq 0,\ b > 0$	Scarf (1960), Veinott (1966), Kolmogorov and Fomin (1970)
CK-convex	$f(x+a) + K \geq f(x) + \frac{a}{b}[f(x) - f(x-b)]$, $\forall a \in [0, C],\ b > 0$	Gallego and Scheller-Wolf (2000)
strongly CK-convex	$f(x+a) + K \geq f(x) + \frac{a}{b}[f(y) - f(y-b)]$, $\forall y \leq x,\ a \in [0, C],\ b > 0$	Gallego and Scheller-Wolf (2000), Shaoxiang and Lambrecht (1996), Shaoxiang (2004)
sym-K-convex	$f(x+a) + \max\{1, \frac{a}{b}\}K \geq f(x) + \frac{a}{b}[f(x) - f(x-b)]$, $\forall a \geq 0,\ b > 0$	Chen and Simchi-Levi (2004a,b)
YD-(K_1, K_2)-convex	$f(x+a) + K_1 - \max\{1, \frac{a}{b}\min\{K_1, K_2\}\} \geq f(x) + \frac{a}{b}[f(x) - f(x-b) - K_2]$, $\forall a \geq 0,\ b > 0$	Ye and Duenyas (2007)
weak (K_1, K_2)-convex or $C(a, b)$-convex	$f(x+a)+K_1 \geq f(x)+\frac{a}{b}[f(x)-f(x-b)-K_2]$, $\forall a \geq 0,\ b > 0$	Gallego and Özer (2001) and Semple (2007)
strongly (C_1K_1, C_2K_2)-convex	$f(x+a)+K_1 \geq f(x)+\frac{a}{b}[f(y)-f(y-b)-K_2]$, $\forall y \leq x,\ a \in [0, C_1],\ b \in (0, C_2]$	This chapter

It appears that the basic convexity property goes back to Archimedes, in his treatise "On the sphere and cylinder" in the third century B.C.E., see also Heath (1912) and Dwilewicz (2009). It arises as a special case of (C_1K_1, C_2K_2)-convexity with $C_1 = C_2 = \infty$ and $K_1 = K_2 = 0$. It is, of course, well known that for basic convexity, the weak and strong versions are equivalent: If the inequality in Table 1.1 holds for all $x = y$—which defines "weak convexity"—it holds for all $x \geq y$, as well. In other words, weak convexity implies strong convexity, and vice versa.

K-convexity corresponds with the special case where $C_1 = C_2 = \infty$ and $K_1 \geq 0, K_2 = 0$. The term was coined by Scarf (1960) to address models with fixed order costs, but no capacity limits or salvage opportunities. Scarf (1960) used the property to show that an (s, S)-policy is optimal under convex holding and backlogging costs. Veinott (1966) subsequently showed this optimality result for holding and backlogging cost functions that are quasi-convex only, but (nearly) increasing over time. See also the recent tutorial by Feinberg (2016). Gallego and Sethi (2005) extended the K-convexity property to functions that are defined on a general Euclidean space $\mathbb{R}^n$, to address multi-product systems with fixed order costs.

Gallego and Scheller-Wolf (2000) addressed models with fixed order costs and capacity limits for individual orders (but no salvage opportunities). These authors introduced the CK-convexity property, again a special case of our general structure where $C_2 = \infty$ and $K_2 = 0$. Gallego and Scheller-Wolf (2000) also pioneered the above distinction between "weak" and "strong" convexity properties.

Chen and Simchi-Levi (2004a,b) addressed a periodic review combined inventory control and pricing model in which each period's demand distribution may be controlled by selecting a unit retail price from a closed price interval. The remaining assumptions are identical to those in the Scarf model, i.e., the base inventory model with fixed order costs. Chen and Simchi-Levi (2004a) covers the finite horizon case, while Chen and Simchi-Levi (2004b) address the long-run average and discounted profit criterion; the models are confined to the case where the order lead time is zero. The authors consider *affine* price-dependent demand functions, specified as:

$$D_n(p) = \alpha_n d_n(p) + \beta_n, \quad n = 1, 2, \ldots, N, \tag{1.2}$$

where $d_n(p)$ is a *deterministic* demand function and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of independent random variables whose distributions are independent of the chosen retail price p_n. In the finite horizon model of Chen and Simchi-Levi (2004a), the authors show that the value functions continue to be K-convex but only in the special case of an *additive* demand model, i.e., when $\alpha_n = 1$ for all n. This implies that an (s, S) policy continues to be optimal in that case. However, K-convexity fails to apply in the general affine demand model (1.2). Indeed, no (s, S) policy is necessarily optimal, contrary to a conjecture by Thomas (1974).

For the more general model, the authors identify the sym-K-convexity property and show that the value functions satisfy this generalized K-convexity property, see Table 1.1. On that basis, they showed that, in each period n, there are two threshold levels $s_n < S_n$ such that no order is placed when the beginning inventory level is above S_n and the inventory level is increased to S_n when it is found to be below s_n. However, when the beginning inventory level is between the two thresholds, it is optimal to either refrain from ordering or to elevate the inventory level to S_n. In Chen and Simchi-Levi (2004b), the author showed that in *infinite* horizon settings, with stationary parameters and distributions, an (s, S)-policy is optimal in the general affine model (1.2), following a different approach, aligned with that of Zheng (1991).

Returning to inventory models with *exogenously* specified demand variables, Chen and Simchi-Levi (2009) addressed a model with bilateral inventory adjustments and fixed costs for each adjustment (but no capacity limits). Their analysis is based on a further generalization of K-convexity, introduced by Ye and Duenyas (2007) which the authors refer to as (K_1, K_2)-convexity. To avoid confusion, we label the property as "YD-(K_1, K_2)-convexity" with YD the initials of the authors. In the special case where $K_1 = K_2 = K$, YD-(K_1, K_2)-convexity reduces to sym-K-convexity. The authors show that all value functions are YD-(K_1, K_2)-convex under minor restrictions for the time-dependence of the fixed adjustment costs K_1 and K_2. Chen and Simchi-Levi (2009) follow Neave (1970) who had addressed the same model but failed to provide a complete analysis for the case where the two fixed costs K_1 and K_2 differ from each other. Similarly, Feinberg and Lewis (2007) employed the $YD - (K_1, K_2)$-convexity property to analyze the infinite horizon version of the stochastic cash balance problem.

Ye and Duenyas (2007) had introduced their YD-(K_1, K_2)-convexity property to analyze a capacity adjustment model, with similar results to those in Chen and Simchi-Levi (2009). Semple (2007) introduced the "weak (K_1, K_2)-convexity" property as a further generalization of YD-(K_1, K_2)-convexity. The author showed, again under the same parameter restrictions as in Ye and Duenyas (2007), that all value functions are weakly (K_1, K_2)-convex if the terminal value function has this property; moreover, all structural results obtained in Ye and Duenyas (2007) can be obtained under this more general convexity property. Clearly, weak (K_1, K_2)-convexity is a special case of our "strong (C_1K_1, C_2K_2)-convexity" property under the special parameter choices $C_1 = C_2 = \infty$ and weakening the definitional inequality (1.1) to hold only for $y = x$. Unbeknownst to Semple, Gallego and Özer (2001) had, six years earlier, introduced the same "weak (K_1, K_2)-convexity", under the name $C(a, b)$-convexity. The authors used this property to establish optimality of a state-dependent (s, S) policy in an inventory model with advanced demand information.

Caliskan-Demirag et al. (2012) introduced a new convexity property that includes the strong CK-convexity property of Gallego and Scheller-Wolf (2000), Shaoxiang and Lambrecht (1996) and Shaoxiang (2004), and the sym-K convexity property of Chen and Simchi-Levi (2004a) as special cases. The authors replace on the right side of inequality (1.1), the fixed cost K, by a general function $\sigma(K, a)$:

$$f(x + a) + \sigma(K, a) \geq f(x) + \frac{a}{b}(f(y) - f(y - b) \quad \text{for any } y \leq x, a \in [0, C], b > 0.$$

The authors employ this property, which they refer to as $\sigma(K, z)$-convexity, to characterize the structure of an optimal policy, when there are two possible fixed order costs, $K_1 < K_2 \leq 2K_1$, with the lower fixed cost K_1 applicable iff the order size is below a given threshold. (The model is uncapacitated and inventory adjustments are in the upward direction only.)

Lu and Song (2014), subsequently, identified another variant of $\sigma(K, z)$-convexity for a model with a convex piecewise-linear order cost function and a fixed cost. These authors refer to their structure as strong (K, c, q)-convexity. K-approximate convexity, introduced in Lu et al. (2016, 2018) is a related approach, in *approximate* rather than *exact* dynamic programming. The fundamental idea is to approximate the exact one-period cost structure and the cost-to-go functions, respectively, with a convex function such that the maximal approximation error is at most K, and derive bounds for the distance between the exact and approximate value functions. See Caliskan-Demirag et al. (2012) and Lu and Song (2014) for a review of other models with a non-linear order cost function, different from the standard fixed-plus-linear structure.

Chapter 2 employs the strong (C_1K_1, C_2K_2)-convexity properly to characterize the structure of the optimal inventory adjustment strategy in a *dual* sourcing setting with salvage opportunities, fixed inventory adjustment costs and capacity limits for orders and salvage batches.

Proposition 1.1 summarizes the above relationships among the various convexity properties.

Proposition 1.1. (a) *convexity* $\Rightarrow$ *K-convexity* $\Rightarrow$ *sym-K-convexity*

$\Rightarrow$ *YD-(K_1, K_2)-convexity* $\Rightarrow$ *weak (K_1, K_2)-convexity* $\Rightarrow$ *strong (C_1K_1, C_2K_2)-convexity*

(b) *convexity* $\Rightarrow$ *strong K-convexity* $\Rightarrow$ *strong CK-convexity*

$\Rightarrow$ *strong (C_1K_1, C_2K_2)-convexity*

Lemma 1.1 establishes various preservation properties for strongly (C_1K_1, C_2K_2)-convex functions.

Lemma 1.1 (Properties of $SC_{C_1K_1,C_2K_2}$)**.**

(i) *If $f(x) \in SC_{C_1K_1,C_2K_2}$, then $f(-x) \in SC_{C_2K_2,C_1K_1}$.*

(ii) *If $f(x) \in SC_{C_1K_1,C_2K_2}$, then $f(x) \in SC_{C_1'K_1',C_2'K_2'}$ for any $C_1' \leq C_1, C_2' \leq C_2$, $K_1' \geq K_1, K_2' \geq K_2$.*

(iii) *If $f(x) \in SC_{C_1K_1,C_2K_2}$ and $g(x) \in SC_{C_1K_1',C_2K_2'}$, then for any $\alpha, \beta \geq 0$, $\alpha f(x) + \beta g(x) \in SC_{C_1(\alpha K_1+\beta K_1'),C_2(\alpha K_2+\beta K_2')}$. As a special case, when $g(x)$ is convex, hence $g(x) \in SC_{C_10,C_20}$, $\alpha f(x) + \beta g(x) \in SC_{C_1(\alpha K_1),C_2(\alpha K_2)}$ for any $\beta \geq 0$.*

(iv) *If* $f(x) \in SC_{C_1K_1,C_2K_2}$*, then* $f(x-a) \in SC_{C_1K_1,C_2K_2}$ *for any real number* a*. Moreover, for any random variable* Y *with* $\mathbb{E}|f(x-Y)| < \infty$*,* $\mathbb{E}f(x-Y) \in SC_{C_1K_1,C_2K_2}$*.*

Proof. Parts (i) and (ii) are immediate.

(iii) Let $h(x) = \alpha f(x) + \beta g(x)$, for any $x \geq y, a \in [0, C_1], b \in (0, C_2]$ we have

$$\begin{aligned}
\Delta &= \alpha K_1 + \beta K_1' + h(x+a) - h(x) - \frac{a}{b}\Big(h(y) - h(y-b) - \alpha K_2 - \beta K_2'\Big) \\
&= \alpha K_1 + \beta K_1' + \alpha f(x+a) + \beta g(x+a) - \alpha f(x) - \beta g(x) \\
&\quad - \frac{a}{b}\Big(\alpha f(y) + \beta g(y) - \alpha f(y-b) - \beta g(y-b) - \alpha K_2 - \beta K_2'\Big) \\
&= \alpha\Big[K_1 + f(x+a) - f(x) - \frac{a}{b}\Big(f(y) - f(y-b) - K_2\Big)\Big] \\
&\quad + \beta\Big[K_1' + g(x+a) - g(x) - \frac{a}{b}\Big(g(y) - g(y-b) - K_2'\Big)\Big] \geq 0
\end{aligned}$$

(iv) Using (iii) this is immediate.

□

1.3 Model.

We consider a single-item periodic review model with a single supplier. Extensions with multiple suppliers are addressed in Chapter 2. At the beginning of each period, an order may be placed with the supplier, possibly subject to a time-dependent capacity limit. In each period, there may also be a (*limited*) *salvage* option to reduce inventory by sales to a secondary channel (discounters, jobbers, outlet stores, etc.) or returns to the supplier. The lead time is L periods, both for ordering and for salvaging, when available as an option. The cost associated with any given order has a fixed and variable component; similarly, a fixed cost is incurred when a salvage sale is initiated, along with revenues that are proportional with the size of the salvage batch. All stockouts are backlogged. In addition to the ordering and salvaging costs and revenues, there are standard holding and backlogging costs, assumed to be proportional or convexly increasing with the end-of-the-period inventory levels and backlog sizes.

We consider a planning horizon of $N \leq \infty$ periods and our objective is to minimize the total expected discounted costs over the full planning horizon. We index the periods *backward* from 1 to N. (Section 1.6 covers the long-run average cost criterion)

The sequence of events in period n is as follows: at the beginning of the period, any order placed [salvage batch initiated] in period $n + L$ is added to [removed from] the inventory. Based on the inventory position (= inventory on hand – backlogs + all outstanding orders), the firm then decides on a new order size, or a salvage quantity to be initiated, if it wants to *reduce* the inventory position. Stochastic demand is then realized and satisfied with on-hand inventory. At the end of the period, any leftover inventory is carried forward to the next period, while any unsatisfied demand is fully backlogged.

We show below that the single inventory position measure suffices to make optimal decisions; moreover, it is never optimal to simultaneously place an order and initiate a salvage batch.

We now state the notation employed in our model:

K_n, C_n = fixed cost and capacity limit for an order placed in period n,

K_n^v, C_n^v = fixed cost and capacity limit for any salvage quantity initiated in period n,

L = order lead time,

c_n = unit price charged by the supplier in period n,

c_n^v = unit revenue received when salvaging inventory in period n,

α = discount factor, $0 \leq \alpha \leq 1$.

The sequence of demands $\{D_n\}$ represents *independent* random variables with general distributions. We make the following assumption.

Assumption 1.1. $c_n \geq c_n^v, \quad n = N, \ldots, 1.$

This ranking is satisfied in all practical settings and precludes it ever being optimal to place an order and initiate a salvage batch in the same period. (Assume, to the contrary, that in some period n, it is optimal to place an order of size q_n, along with the initiation of a salvage batch of size $\bar{q}_n$. Under Assumption 1.1, money is saved by reducing the order to

$(q_n - \bar{q}_n)$ and canceling the salvage batch, if $q_n \geq \bar{q}_n$; alternatively, if $\bar{q}_n > q_n$, money may be saved by reducing the salvage batch to $(\bar{q}_n - q_n)$ and canceling the order.)

Settings without actual salvage opportunities may be represented as having such opportunities, however, with $c_n^v = -M$, where M denotes a sufficiently large constant. This representation allows for a unified treatment of models with and without salvage opportunities.

For $n = N, \ldots, 1$, let

$x_n =$ the inventory position at the beginning of period n, *before* any inventory adjustement;

$y_n =$ the inventory position at the beginning of period n, *after* any inventory adjustmenet.

Inventory and backlogging related costs are assumed to depend on the end-of-period inventory levels only, it is well known since Scarf (1960) that under full backlogging, an equivalent representation of the controllable parts of the total expected discounted cost over the planning horizon is obtained by charging to period $n + L$, the *expected* value of the actual costs incurred at the end of period n. This follows from the sample path relationship between y_n, the inventory position at the beginning of period n, and the inventory level I_{n-L} at the *end* of period $n - L$:

$$I_{n-L} = y_n - D_n^{(L)},$$

where $D_n^{(L)} = D_n + D_{n-1} + \cdots + D_{n-L}$ is the aggregate demand in time interval $[n, n - L]$.

For all $n = N, \ldots, 1$, let

$$\mathcal{L}_n(x_n + q_n) = \text{the expected value of all inventory and backlogging related costs at the end of period } n - L \text{ discounted back to period } n$$

and impose a standard assumption regarding these functions, satisfied for most common cost structures.

Assumption 1.2. (i) *The function $\mathcal{L}_n(\cdot)$ is convex and $\mathcal{L}_n(y) = O(|y|^p)$ for some $p \geq 1$, $n = N, \ldots, 1$. Also, $\mathbb{E}(D_n^p) < \infty$ for $n = N, \ldots, 1$.*

(ii) $c_n^v \leq -\frac{\partial^- \mathcal{L}_n(x)}{\partial x}$ *for* $n = N, \ldots, 1$*, where* $\frac{\partial^- \mathcal{L}_n(x)}{\partial x}$ *denotes the left derivative of the function* $\mathcal{L}_n(\cdot)$*.*[1]

Assumption 1.2 (ii) ensures that, in every period n, the marginal backlogging cost is in excess of the unit salvage value.

Beyond Assumptions 1.1 and 1.2, we need a few additional parameter restrictions.

Assumption 1.3. *For* $n = N, \ldots, 1$*,*

$$K_n \geq \alpha K_{n-1}, \quad K_n^v \geq \alpha K_{n-1}^v, \tag{1.3}$$

$$C_n \leq C_{n-1}, \quad C_n^v \leq C_{n-1}^v. \tag{1.4}$$

The inequalities (1.3) were already recognized as essential in the base model with fixed order costs, see Scarf (1960) and Zipkin (2000). The inequalities (1.4) indicate that capacity limits for order and salvage quantities may not decline over time; this is typically the case in practical applications.

To introduce the dynamic programming formulation, define the following value functions:

$f_n(x) =$ the optimal expected discounted total costs in the last n periods, assuming period n is started with an inventory position of x units;

$f_n^1(x) =$ the optimal expected discounted total costs in the last n periods, assuming period n is started with an inventory position of x units and *no* salvage batch is initiated;

$f_n^2(x) =$ the optimal expected discounted total costs in the last n periods, assuming period n is started with an inventory position of x units and a salvage batch *is* initiated.

Clearly, since, as shown, it is never optimal to place an order and to initiate a salvage

[1] A convex function has left and right derivatives everywhere.

sale in the same period, we have for $n = N, \ldots, 1$:

$$f_n(x) = \min\{f_n^1(x), f_n^2(x)\}, \tag{1.5}$$

$$f_n^1(x) = \min_{x_n \le y_n \le x_n + C_n} \{K_n \delta(y_n - x_n) + c_n(y_n - x_n) + \mathcal{L}_n(y_n) + \alpha \mathbb{E} f_{n-1}(y_n - D_n)\}, \tag{1.6}$$

$$f_n^2(x) = \min_{\min\{[x_n - C_n^v]^+, x_n\} \le y_n \le x_n} \{K_n^v \delta(x_n - y_n) + c_n^v(y_n - x_n) + \mathcal{L}_n(y_n) + \alpha \mathbb{E} f_{n-1}(y_n - D_n)\}, \tag{1.7}$$

for a given terminal value function $f_0(\cdot)$ satisfying:

Assumption 1.4. *The terminal value function* $f_0(\cdot) \in SC_{C_0 K_0, C_0^v K_0^v}$ *and is non-increasing on the negative half-line.*

The dynamic programming formulation exploits the fact that it is never optimal to simultaneously place a procurement order and to initiate a salvage sale. It also utilizes the simple state dynamics $x_{n-1} = y_n - D_n$. The lower bound for y_n in (1.7), i.e., $y_n \ge \min\{[x_n - C_n^v]^+, x_n\}$, reflects the fact that, at least in physical inventory models, there are no salvage opportunities when $x_n \le 0$, while salvage opportunities are bounded by $\min\{x_n, C_n^v\}$ when $x_n > 0$. Instead of analyzing the DP (1.5)–(1.7) directly, we relax the feasible action set in (1.7) to $x_n - C_n^v \le y_n \le x_n$, giving rise to the *relaxed* DP:

$$\widetilde{f}_n(x) = \min\{\widetilde{f}_n^1(x), \widetilde{f}_n^2(x)\}, \tag{1.8}$$

$$\widetilde{f}_n^1(x) = \min_{x_n \le y_n \le x_n + C_n} \{K_n \delta(y_n - x_n) + c_n(y_n - x_n) + \mathcal{L}_n(y_n) + \alpha \mathbb{E} \widetilde{f}_{n-1}(y_n - D_n)\}, \tag{1.9}$$

$$\widetilde{f}_n^2(x) = \min_{x_n - C_n^v \le y_n \le x_n} \{K_n^v \delta(x_n - y_n) + c_n^v(y_n - x_n) + \mathcal{L}_n(y_n) + \alpha \mathbb{E} \widetilde{f}_{n-1}(y_n - D_n)\}. \tag{1.10}$$

We first show that this relaxation can be adopted without affecting the optimal policies.

Theorem 1.1. *For* $i = N, \ldots, 1$, *let* $y_i^*(x_i)$ *denote the optimal inventory policy in the relaxed dynamic program* (1.8)–(1.10) *when the inventory position at the beginning of period* i *is* x_i, *then*

(a) *If* $x_i \le 0$, *then* $y_i^*(x_i) \ge x_i$, *i.e., it is optimal not to salvage;*

(b) *If* $x_i > 0$, *then* $y_i^*(x_i) \ge 0$, *i.e., it is optimal to maintain a non-negative inventory position.*

Proof. (a) Suppose, to the contrary, that $0 < a = x_i - y_i^*(x_i)$. We show that a cost improvement can be achieved on *any* sample path ω, by perturbing the time series $\{y_i^*(x_i), \bar{y}_j = y_j^*(x_j), j = i-1, \dots, 1\}$ to $\{\bar{y}_i = x_i, \bar{y}_j = y_j^*(x_j), j = i-1, \dots, 1\}$. In other words, the perturbation involves the cancellation of the salvage batch in period i, and reducing the inventory adjustment in period $i-1$ by a units. Note that after the inventory adjustment in period $i-1$, the remaining sample path until the end of the planning horizon, remains unaltered. Let Δ denote the incremental costs incurred due to the perturbation,

$$\begin{aligned} \Delta &\leq \left[-K_i^v + ac_i^v + a\frac{\partial^- \mathcal{L}_i(0)}{\partial x} \right] + \alpha[K_{i-1}^v + a\max\{-c_{i-1}, -c_{i-1}^v\}] \\ &= -(K_i^v - \alpha K_{i-1}^v) + a\left(c_i^v + \frac{\partial^- \mathcal{L}_i(0)}{\partial x} \right) - a\max\{c_{i-1}, c_{i-1}^v\} < 0 \end{aligned} \tag{1.11}$$

To justify the first inequality, note that the first term to its right denotes the cost savings in the first period due to the cancellation of the salvage batch in period i. This cancellation results in a saving of K_i^v, the fixed cost of this batch and a reduction of the backlog size at the end of period i, by a units, at a per-unit saving of at least $\frac{\partial^- \mathcal{L}_i(0)}{\partial x}$; on the other hand, a loss of revenues, hence an *increase* in costs of ac_i^v emerges from the canceled salvage transaction.

The second term to the right of the first inequality in (1.11) is an *upper bound* for the additional costs incurred in period $i-1$; here, the decrease in the inventory adjustment may save the fixed cost K_{i-1}, in case this decrease cancels an order or, at worst, it may initiate a salvage batch in period $i-1$, thus adding αK_{i-1}^v to the total cost. In addition, the modified inventory adjustment results in either a reduction of the variable cost c_{i-1} or an additional revenue c_{i-1}^v per unit. The total additional variable cost in period $i-1$ are therefore bounded from above by $-a\max\{c_{i-1}, c_{i-1}^v\}$.

The second inequality in (1.11) follows from Assumptions 1.2 and 1.3.

(b) Suppose, to the contrary, that $y_i^*(x_i) < 0$. Let $b = -y_i^*(x_i) > 0$. Define

$$z_j = -\sum_{k=j+1}^{i} D_k(\omega) \leq 0, \quad j = i, i-1, \dots, 1.$$

Consider the following *modification* to the optimal policy δ^*: in period i reduce the size of the salvage batch by b units; thereafter, stay put until the first period in which $y_j^* \geq z_j$, if any. Let $l = \max\{j \leq i-1 : y_j^*(x_j) \geq z_j\}$, where $l = 0$ when this index set is empty. If

$l \geq 1$, place an order in period l for $y_j^* - z_j$ units. We distinguish between two cases: (b1) $l \geq 1$ and (b2) $l = 0$.

Proof for case (b1): after period l, the modified policy implements the same actions as the original policy δ^*. Let Δ denote the incremental cost due to the policy perturbation. By part (a) and the definition of the time period l, we have for all $j = i-1, \ldots, l+1$ that

$$x_j \leq y_j^* < z_j. \tag{1.12}$$

Note that the sample paths of the modified and the original policies coincide from period l on. Thus, the cost differential Δ arises due to cost differences in the interval $[i, l]$ only. Thus, let $\Delta = \Delta_1 + \Delta_2 + \Delta_3$, where

Δ_1 = difference in procurement and salvage costs in periods $i-1, \ldots, l$;

Δ_2 = lost revenues in period i due to the reduction of the salvage batch in that period by b units;

Δ_3 = difference in backlogging and holding costs in the entire interval $[i, l]$.

Note that, by the definition of the period index l:

$$q_l = y_l^* - x_l = y_l^* - (y_{l+1}^* - D_{l+1}) > y_l^* - z_{l+1} + D_{l+1} = y_l^* - z_l > 0.$$

Thus, the original as well as the modified policy initiate a salvage batch in period i and place an order in period l, and the salvage batch and order size under the modified policy are *smaller* than their counterparts under the original policy δ^*. Since the modified policy avoids inventory adjustments in the intermediate periods in (i, l), it follows that $\Delta_1 \leq 0$. Also $\Delta_2 = bc_i^v$, while $\Delta_3 \leq b\frac{\partial^- \mathcal{L}_i(0)}{\partial x}$, since the backlog size at end of period i is b units smaller under the modified policy and at the end of all remaining periods $j = i-1, \ldots, l+1$, the modified policy has a smaller backlog size than the original policy δ^*, see (1.12). Thus, $\Delta \leq b\left(c_i^v + \frac{\partial^- \mathcal{L}_i(0)}{\partial x}\right) < 0$ by Assumption 1.2.

Proof for case (b2): In this case, the modified policy reduces the salvage batch in period i by b units and stays put for the remainder of the planning horizon, ending the planning horizon with an inventory level $z_1 - D_1$, as opposed to an ending inventory level $y_1^* - D_1$ under the original policy. The proof for case (b1) shows that the modified policy incurs a lower total of procurement, salvage, holding and backlogging costs. However, in this case,

Δ contains the additional differential $f_0(z_1 - D_1) - f_0(y_1^* - D_1) \leq 0$, by Assumption 1.4 and $y_1^* < z_1$. $\square$

In view of Theorem 1.1, we proceed without loss of optimality, with the relaxed dynamic program (1.8)–(1.10), omitting the $\sim$ sign on top of the value functions $\widetilde{f}(\cdot), \widetilde{f}_1(\cdot), \widetilde{f}_2(\cdot)$.

1.4 The single period problem.

It follows from the dynamic programming recursions (1.8)–(1.10) that, in each period n, we face an optimization problem of the following structure

$$g_1(x) = \min_{y \in [x, x+C_1]} \{K_1 \delta(y-x) + \beta_1(y-x) + g(y)\}, \tag{1.13}$$

$$g_2(x) = \min_{y \in [x-C_2, x]} \{K_2 \delta(x-y) + \beta_2(y-x) + g(y)\}, \tag{1.14}$$

$$g_0(x) = \min\{g_1(x), g_2(x)\} \tag{1.15}$$

with $g_1(\cdot) = f_n^1(\cdot), g_2(\cdot) = f_n^2(\cdot), g_0(\cdot) = f_n(\cdot), \beta_1 = c_n, \beta_2 = c_n^v, K_1 = K_n, K_2 = K_n^v, C_1 = C_n, C_2 = C_n^v$ and $g(y) = \mathcal{L}_n(y) + \alpha \mathbb{E} f_{n-1}(y - D_n)$.

We now analyze this single stage optimization problem (1.13)–(1.15), under the assumption that the terminal cost formulation $g(\cdot)$ has the strong (C_1K_1, C_2K_2)-convexity property for specific parameter values C_1, K_1, C_2, K_2.

Define auxiliary functions

$$\widetilde{g}_1(x) = K_1 + \min_{y \in [x, x+C_1]} \{\beta_1(y-x) + g(y)\}, \tag{1.16}$$

$$\widetilde{g}_2(x) = K_2 + \min_{y \in [x-C_2, x]} \{\beta_2(y-x) + g(y)\}, \tag{1.17}$$

as counterparts of $g_1(x)$ and $g_2(x)$, under *definitive* inventory adjustment, i.e., *definitively* incurring fixed costs for ordering or salvaging, respectively, and let $A_i(x) = \widetilde{g}_i(x) - g(x)$ be the increase in minimal cost if forced to order (for $i = 1$) or salvage (for $i = 2$).

To characterize the structure of an optimal policy, we need to define some critical points, with the convention that the infimum (supremum) of an empty set equals $+\infty$ $(-\infty)$.

Definition 1.2 (Critical Points). *For a continuous function $g(\cdot) \in SC_{C_1K_1,C_2K_2}$ and any β_1, β_2, define*

$$B = \inf\big\{\arg\min_y\{\beta_1 y + g(y)\}\big\}, \quad b = \inf\{x: \ A_1(x) \geq 0\}, \quad \bar{b} = \sup\{x: \ A_1(x) < 0\}, \tag{1.18}$$

$$S = \sup\big\{\arg\min_y\{\beta_2 y + g(y)\}\big\}, \quad s = \sup\{x: \ A_2(x) \geq 0\}, \quad \underline{s} = \inf\{x: \ A_2(x) < 0\}. \tag{1.19}$$

These critical points play important roles in the structure of the optimal strategy. By its definition, B is the (smallest) global minimizer of $\widetilde{g}_1(x)$ if $C_1 = \infty$, i.e., the smallest order-up-to level for sufficiently small x if ordering is better than staying put. Similarly, S is the (largest) global minimizer of $\widetilde{g}_2(x)$ if $C_2 = \infty$, .i.e., the biggest salvage-down-to level for sufficiently large x if salvaging is better than staying put; b is the smallest among all inventory levels where ordering is not better than staying put; $\bar{b}$ is the largest among all inventory levels where ordering is better than staying put; s is the largest among all inventory levels where salvaging is not better than staying put; $\underline{s}$ is the smallest among all inventory levels where salvaging is better than staying put.

Note that $b = \bar{b}$ [$\underline{s} = s$] if the function $A_1(\cdot)$ [$A_2(\cdot)$] has a single root. We have observed this single root property to hold in all problem instances we have encountered, see Section 2.6. It can, however, not be guaranteed, for general (C_1K_1, C_2K_2)-convex functions, which may have many local optima, see Figure 1.1.

The Proposition below characterizes the ranking of the critical points, which is important when developing the optimal policy structure.

Proposition 1.2 (Critical Points). *Assume $\beta_1 \geq \beta_2$ and $g(\cdot) \in SC_{C_1K_1,C_2K_2}$, then*

(i) $-\infty \leq b \leq \bar{b} \leq \underline{s} \leq s \leq \infty$;

(ii) $-\infty \leq b \leq B \leq S \leq s \leq \infty$;

(iii) *If $C_2 = \infty$ and $K_1 \geq K_2$, then $\bar{b} \leq B$; if $C_1 = \infty$ and $K_1 \leq K_2$, then $S \leq \underline{s}$;*

(iv) *If $C_1 = \infty$ and $K_2 = 0$, then $b = \bar{b}$; if $C_2 = \infty$ and $K_1 = 0$, then $\underline{s} = s$. If $C_1 = C_2 = \infty$ and $K_1 = K_2 = 0$, then $b = \bar{b} = B, S = \underline{s} = s$.*

In this Proposition, (i) ranks four critical points. (ii) ranks and locates the global minimizers B and S between b and s. (iii) and (iv) lead to simple policy structures, in certain special cases, which will be discussed later.

To prove this Proposition, we first need some auxiliary lemmas. Note that by definition we have

$$g_1(x) = \min\{g(x),\ \widetilde{g}_1(x)\}, \qquad A_1(x) < 0 \quad \forall x < b, \qquad A_1(x) \geq 0 \quad \forall x > \bar{b}, \tag{1.20}$$

$$g_2(x) = \min\{g(x),\ \widetilde{g}_2(x)\}, \qquad A_2(x) < 0 \quad \forall x > s, \qquad A_2(x) \geq 0 \quad \forall x < \underline{s}. \tag{1.21}$$

The following lemma shows that all regions where it is optimal to order (order regions) are to the left of all regions where it is optimal to salvage inventory (salvage regions).

Lemma 1.2 (Separation of Order/Salvage Regions)**.** *Assume* $\beta_1 \geq \beta_2$ *and* $g(\cdot) \in SC_{C_1K_1,C_2K_2}$, *then*

(i) *if* $\widetilde{g}_2(y) < g(y)$ *for some* y, *then* $g(x) \leq \widetilde{g}_1(x)$ *for any* $x \geq y$;

(ii) *if* $\widetilde{g}_1(y) < g(y)$ *for some* y, *then* $g(x) \leq \widetilde{g}_2(x)$ *for any* $x \leq y$.

Proof. (i) Given $\widetilde{g}_2(y) < g(y)$, by the definition of $\widetilde{g}_2(\cdot)$ we have

$$\widetilde{g}_2(y) = K_2 + \beta_2(-b) + g(y-b) < g(y) \quad \text{for some } b \in (0, C_2],$$

where b cannot take the value of 0 because $K_2 \geq 0$. Equivalently,

$$g(y) - g(y-b) - K_2 > -\beta_2 b.$$

Hence by strong (C_1K_1, C_2K_2)-convexity of $g(\cdot)$, for any $x \geq y$ and $a \in [0, C_1]$ we have

$$K_1 + g(x+a) - g(x) \geq \frac{a}{b}\Big(g(y) - g(y-b) - K_2\Big) \geq -\beta_2 a \geq -\beta_1 a,$$

where the last inequality follows from $\beta_1 \geq \beta_2$. Equivalently,

$$K_1 + \beta_1 a + g(x+a) \geq g(x).$$

As this holds for any $a \in [0, C_1]$, we obtain $\widetilde{g}_1(x) \geq g(x)$. It can also be verified that if $K_1 > 0$, we have strict inequality as $\widetilde{g}_1(x) > g(x)$. Case (ii) can be proved in a similar way and the details are omitted here. $\square$

Intuitively, (i) shows that if salvaging is better than staying put at a given level y, then staying put is better than ordering at or above y. In other words, ordering is never optimal above a "salvaging" point. Similarly, (ii) shows that if ordering is better than staying put at a given level y, then staying put is better than salvaging at or below y, i.e., salvaging is never optimal below an "ordering" point.

The following corollary shows that if at a given level y, salvaging is strictly preferred, it is optimal not to order for any inventory level $x > y$. Similarly, if at a given level y, ordering is strictly preferred, it is optimal not to salvage for any inventory level $x < y$.

Corollary 1.1. *Assume $\beta_1 \geq \beta_2$ and $g(\cdot) \in SC_{C_1K_1,C_2K_2}$, then*

(i) *if $g_2(y) < g_1(y)$ for some y, then $g_2(x) \leq g_1(x)$ for any $x \geq y$;*

(ii) *if $g_1(y) < g_2(y)$ for some y, then $g_1(x) \leq g_2(x)$ for any $x \leq y$.*

Proof. To verify (i), notice that $g_2(y) < g_1(y)$ implies $\widetilde{g}_2(y) < g(y)$ since $g_1(y) \leq g(y)$ and $g_2(y) = \min\{g(y), \widetilde{g}_2(y)\}$. By Lemma 1.2 (i), $g(x) \leq \widetilde{g}_1(x)$, which implies $g_2(x) \leq g_1(x)$ since $g_2(x) \leq g(x)$ and $g_1(x) = \min\{g(x), \widetilde{g}_1(x)\}$. Similarly, we can prove (ii): $g_1(y) < g_2(y)$ implies $\widetilde{g}_1(y) < g(y)$ since $g_2(y) \leq g(y)$ and $g_1(y) = \min\{g(y), \widetilde{g}_1(y)\}$. By Lemma 1.2 (ii), $g(x) \leq \widetilde{g}_2(x)$, which implies $g_1(x) \leq g_2(x)$ since $g_1(x) \leq g(x)$ and $g_2(x) = \min\{g(x), \widetilde{g}_2(x)\}$. □

Certain monotonicities of the functions concerned play an important role in formulating optimal policy structure, as are shown in the lemma below.

Lemma 1.3 (Monotonicity)**.** *Assume $g(\cdot) \in SC_{C_1K_1,C_2K_2}$ and finite $|\bar{b}|, |\underline{s}|$,* [2] *then*

(i) *if $K_2 = 0$, $\beta_1 x + g(x)$ is strictly decreasing on $(-\infty, \bar{b})$;*

(ii) *if $K_1 = 0$, $\beta_2 x + g(x)$ is strictly increasing on $(\underline{s}, \infty)$.*

Proof. Here we prove (i) as (ii) can be shown similarly, and we prove the general case where $K_2 \geq 0$ noted by the footnote. Consider $x_1 < x_2 < \bar{b}$ with $x_2 - x_1 \leq C_2$, then there exists

[2] Finite $|\bar{b}|$ and $|\underline{s}|$ can be implied by $A_1(x) < 0$ for some x and $A_2(y) < 0$ for some y, respectively.

$b_0 \in (x_2, \bar{b})$ such that $A_1(b_0) < 0$ by the definition of $\bar{b}$ and the continuity of $A_1(\cdot)$. Hence we have

$$g(b_0) > \widetilde{g}_1(b_0) = K_1 + \beta_1(z - b_0) + g(z),$$

for some $z \in (b_0, b_0 + C_1]$. Note that z cannot take the value of b_0 since otherwise $K_1 < 0$. Equivalently,

$$\beta_1 b_0 + g(b_0) > K_1 + \beta_1 z + g(z).$$

Then by the strong (C_1K_1, C_2K_2)-convexity of $\beta_1 x + g(x)$ we have

$$\beta_1 b_0 + g(b_0) > K_1 + \beta_1 z + g(z) \geq \beta_1 b_0 + g(b_0) + \frac{z - b_0}{x_2 - x_1}\Big((\beta_1 x_2 + g(x_2)) - (\beta_1 x_1 + g(x_1)) - K_2\Big),$$

which implies

$$\beta_1 x_2 + g(x_2) < \beta_1 x_1 + g(x_1) + K_2,$$

i.e., $\beta_1 x + g(x)$ is strictly non-K_2-increasing on $(-\infty, \bar{b})$. Specially, if $K_2 = 0$, $\beta_1 x + g(x)$ is strictly decreasing on $(-\infty, \bar{b})$. □

We are now ready for the proof of Proposition 1.2.

Proof of Proposition 1.2. (i) First, we show $\bar{b} \leq \underline{s}$ by contradiction. Suppose $\bar{b} > \underline{s}$, then by the definition of $\bar{b}$ and $\underline{s}$ in (1.18) and (1.19), respectively, and the continuity of $A_1(\cdot)$ and $A_2(\cdot)$, there exist x and y such that $\underline{s} < x < y < \bar{b}$ for which $A_2(x) < 0$ and $A_1(y) < 0$, or $\widetilde{g}_2(x) < g(x)$ and $\widetilde{g}_1(y) < g(y)$. This contradicts Lemma 1.2 and hence $\bar{b} \leq \underline{s}$. Next, we show $b \leq \bar{b}$ also by contradiction. Assume $b > \bar{b}$, then by the definition of b in (1.18), $A_1(z) < 0$ for any $z \in (\bar{b}, b)$, which contradicts the definition of $\bar{b}$. Hence we have $b \leq \bar{b}$. We can prove $\underline{s} \leq s$ in a similar way.

(ii) First, we show $B \leq S$. Let

$$h_1(y) = \beta_1 y + g(y), \quad h_2(y) = \beta_2 y + g(y) = h_1(y) - (\beta_1 - \beta_2)y,$$

which are both strongly (C_1K_1, C_2K_2)-convex according to Lemma 1.1.(iii). Then by (1.18) and (1.19) we have

$$B = \inf\{\arg\min_y h_1(y)\}, \quad S = \sup\{\arg\min_y h_2(y)\},$$

which imply that $h_1(x) > h_1(B)$ for all $x < B$. Then for any $x < B$, we have

$$h_2(x) = h_1(x) - (\beta_1 - \beta_2)x > h_1(B) - (\beta_1 - \beta_2)x \geq h_1(B) - (\beta_1 - \beta_2)B = h_2(B),$$

where the second inequality follows from $\beta_1 \geq \beta_2$. This implies that $B \leq S$ by the definition of S.

Next, we show $b \leq B$ and $S \leq s$. For $b \leq B$, suppose on the contrary $b > B$, then by (1.20) we have $A_1(B) < 0$, hence

$$g(B) > \widetilde{g}_1(B) = K_1 + \min_{B \leq y \leq B + C_1} \{\beta_1 y + g(y)\} - \beta_1 B = K_1 + g(B),$$

where the last equality follows from the fact that B is a global minimizer of $\beta_1 y + g(y)$. This contradicts $K_1 \geq 0$ and hence it should be $b \leq B$. In a similar way we can show $S \leq s$.

(iii) We prove the case where $C_2 = \infty$ and $K_1 \geq K_2$ by contradiction; the other case where $C_1 = \infty$ and $K_1 \leq K_2$ can be proved in the same way. Assuming $\bar{b} > B$, there exists $x \in (B, \bar{b})$ such that $A_1(x) < 0$ by the definition of $\bar{b}$ in (1.18). Then

$$g(x) > \widetilde{g}_1(x) = K_1 + g(z) + \beta_1(z - x) \tag{1.22}$$

for some $z \in (x, x + C_1]$. Notice that z cannot take value of x because that results in $K_1 < 0$. By the definition of B in (1.18) and $z > x > B$, we have

$$g(z) + \beta_1 z \geq g(B) + \beta_1 B,$$

or equivalently,

$$g(B) - g(z) \leq \beta_1(z - B). \tag{1.23}$$

By strong $(C_1 K_1, \infty K_2)$-convexity of $g(\cdot)$ we have

$$K_1 + g(z) \geq g(x) + \frac{z - x}{x - B}\Big(g(x) - g(B) - K_2\Big),$$

or equivalently,

$$\begin{aligned} g(x) &\leq \frac{x-B}{z-B}\Big(K_1+g(z)\Big)+\frac{z-x}{z-B}\Big(g(B)+K_2\Big) \\ &= K_1+g(z)+\frac{z-x}{z-B}\Big(g(B)+K_2-K_1-g(z)\Big) \\ &\leq K_1+g(z)+\frac{z-x}{z-B}\Big(g(B)-g(z)\Big) \\ &\leq K_1+g(z)+\beta_1(z-x), \end{aligned}$$

where the second inequality follows from the assumption $K_1 \geq K_2$ and the last inequality follows from (1.23). This contradicts (1.22), thus we have shown $\bar{b} \leq B$.

(iv) We first prove by contradiction the case where $K_2 = 0$; the other case where $K_1 = 0$ can be shown in the same way. By part (i), it suffices to show that the assumption $b < \bar{b}$ results in a contradiction. By the definition of b and $\bar{b}$ in (1.18) there exist x and y such that $b \leq x < y < \bar{b}$ and $A_1(x) \geq 0$, $A_1(y) < 0$. Since $K_2 = 0$ and $x < y < \bar{b}$, Lemma 1.3 (i) implies

$$\beta_1 x + g(x) > \beta_1 y + g(y). \tag{1.24}$$

By (iii) of this Lemma, $\bar{b} \leq B$, thus, since $C_1 = \infty$ and since B is a global minimizer of the function $\beta_1 y + g(y)$,

$$\widetilde{g}_1(x) = K_1 + \beta_1 B + g(B) - \beta_1 x, \tag{1.25}$$

$$\widetilde{g}_1(y) = K_1 + \beta_1 B + g(B) - \beta_1 y. \tag{1.26}$$

Noticing the definition of $A_1(\cdot)$ in Definition 1.2, $A_1(x) \geq 0$ and $A_1(y) < 0$ together with (1.24)–(1.26) yield

$$K_1 + \beta_1 B + g(B) \geq \beta_1 x + g(x) > \beta_1 y + g(y) > K_1 + \beta_1 B + g(B),$$

a clear contradiction. Hence, $b = \bar{b}$.

Next, we consider the case where $K_1 = K_2 = 0$. We prove $\bar{b} = B$; the equality $S = \underline{s}$ can be shown in the same way. First notice that $\bar{b} \leq B$ by (iii) of this Lemma. Suppose, to the contrary, $\bar{b} < B$, then by the definition of $\bar{b}$ in (1.18) there exists $x \in (\bar{b}, B)$ that $A_1(x) \geq 0$, or $\widetilde{g}_1(x) \geq g(x)$ by the definition of $A_1(\cdot)$. With $K_1 = 0$, this implies that

$$\widetilde{g}_1(x) = \beta_1 B + g(B) - \beta_1 x \geq g(x),$$

which contradicts the definition of B, as $x < B$. Hence $\bar{b} = B$.

□

We now proceed to the optimal single-period policy structure, in the following Theorem.

Theorem 1.2 (Single Period Optimal Policy Structure)**.** *Assume* $\beta_1 \geq \beta_2$ *and* $g(\cdot) \in SC_{C_1K_1,C_2K_2}$*, then* $g_0(x)$ *and the corresponding minimizer* $y^*(x)$ *are characterized by Table 1.2 and Figure 1.2, in which* $\widetilde{g}_1(\cdot)$ *and* $\widetilde{g}_2(\cdot)$ *are defined by* (1.16) *and* (1.17)*, respectively. If* $y^*(x)$ *is specified as a two-element set* $\{\cdot,\cdot\}$*, either one of the two elements may apply. Let*

$$B(x) = \inf \mathcal{B}(x) \text{ where } \mathcal{B}(x) = \operatorname*{arg\,min}_{x \leq y \leq x+C_1} \{\beta_1 y + g(y)\}, \tag{1.27}$$

$$S(x) = \sup \mathcal{S}(x) \text{ where } \mathcal{S}(x) = \operatorname*{arg\,min}_{x-C_2 \leq y \leq x} \{\beta_2 y + g(y)\} \tag{1.28}$$

denote minimizers of $\widetilde{g}_1(x)$ *and* $\widetilde{g}_2(x)$*, respectively. Let* $b(x) = B(x) - x$ *and* $s(x) = x - S(x)$ *denote the corresponding order and salvage quantity.*

Table 1.2: Single period optimal policy structure

x	$(-\infty, b)$	$[b, \bar{b})$	$[\bar{b}, \underline{s}]$	$(\underline{s}, s]$	(s, ∞)
$g_0(x)$	$\widetilde{g}_1(x)$	$\min\{\widetilde{g}_1(x), g(x)\}$	$g(x)$	$\min\{\widetilde{g}_2(x), g(x)\}$	$\widetilde{g}_2(x)$
$y^*(x)$	$B(x)$	$\{B(x), x\}$	x	$\{S(x), x\}$	$S(x)$

Proof.

- $x \in (-\infty, b)$. $x < b$ implies that $A_1(x) < 0$ by (1.20), so $\widetilde{g}_1(x) < g(x)$ and by Lemma 1.2 $g(x) \leq \widetilde{g}_2(x)$. It follows that $g_0(x) = g_1(x) = \widetilde{g}_1(x)$ and $y^*(x) = B(x)$, the minimizer of $\widetilde{g}_1(x)$.

- $x \in [b, \bar{b})$. By the definition of $\bar{b}$ in (1.18), there exists $y \in (x, \bar{b})$ such that $A_1(y) < 0$, i.e., $\widetilde{g}_1(y) < g(y)$. Then $g(x) \leq \widetilde{g}_2(x)$ by Lemma 1.2. It is therefore optimal to either place an order or to keep the inventory position unaltered. The minimizer $y^*(x)$ therefore equals $B(x)$ or x.

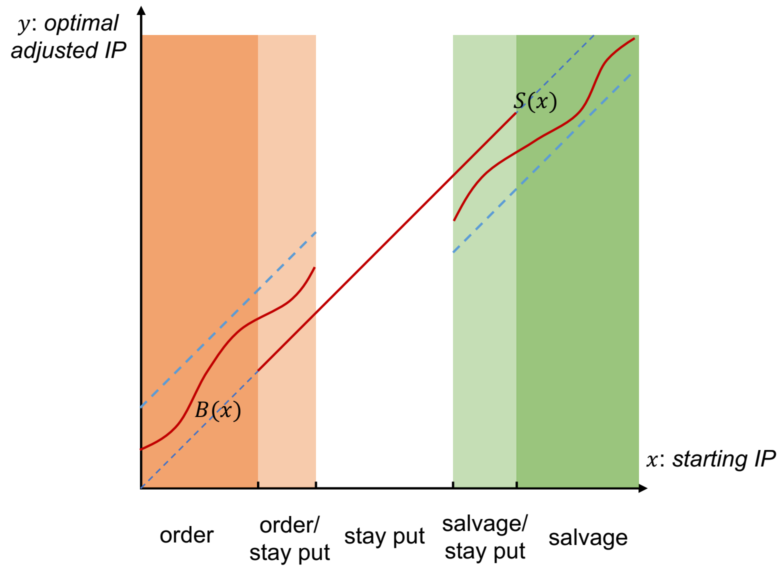


Figure 1.2: Illustration of single period optimal policy structure

- $x \in [\bar{b}, \underline{s}]$. $x \geq \bar{b}$ implies that $A_1(x) \geq 0$ by (1.20), so $\widetilde{g}_1(x) \geq g(x)$. Similarly $x \leq \underline{s}$ implies that $A_2(x) \geq 0$ by (1.21), so $\widetilde{g}_2(x) \geq g(x)$. Therefore $g_0(x) = g_1(x) = g_2(x) = g(x)$ and $y^*(x) = x$.

- $x \in (\underline{s}, s]$. By the definition of $\underline{s}$ in (1.19), there exists $y \in (\underline{s}, x)$ such that $A_2(y) < 0$, i.e., $\widetilde{g}_2(y) < g(y)$. Then $g(x) \leq \widetilde{g}_1(x)$ by Lemma 1.2. Therefore it is optimal to either initiate a salvage batch or stay put, and the minimizer $y^*(x)$ equals $S(x)$ or x.

- $x \in (s, \infty)$. $x > s$ implies that $A_2(x) < 0$ by (1.21), so $\widetilde{g}_2(x) < g(x)$ and by Lemma 1.2 $g(x) \leq \widetilde{g}_1(x)$. It hence follows that $g_0(x) = g_2(x) = \widetilde{g}_2(x)$ and $y^*(x) = S(x)$, the minimizer of $\widetilde{g}_2(x)$.

□

In other words, four critical points partition the inventory position line into five regions. In the two extreme regions, $(-\infty, b)$ and (s, ∞), a positive order or salvage transaction needs to be initiated, respectively; in the middle region, $[\bar{b}, \underline{s}]$, it is optimal to stay put; in the second region, $[b, \bar{b})$, it is optimal to either order or to stay put, and in the fourth region, $(\underline{s}, s]$, it is optimal to either initiate a salvage transaction or to stay put. Within the latter two regions, it is possible that the optimal policy alternates several times between ordering or salvaging versus staying put, a phenomenon already discovered in simpler models without salvage opportunities, see e.g., Shaoxiang and Lambrecht (1996) and Shaoxiang (2004).

As mentioned, if the functions $A_1(\cdot)$ and $A_2(\cdot)$ have a *single* root, $b = \bar{b}$ and $\underline{s} = s$, so that the second and fourth region vanish. In all of our numerical experience, this single root property prevails. In this case, the five-region policy simplifies to a three-region policy, and Table 1.2 and Figure 1.2 simplify to the following Table 1.3 and Figure 1.3. However, Ye and Duenyas (2007), dealing with the special case of our model with unrestricted order sizes, identified an instance where a five-region policy emerges because the functions $A_1(\cdot)$ and $A_2(\cdot)$ have multiple roots.

Table 1.3: Simplified optimal policy structure

x	$(-\infty, b)$	$[b, s]$	(s, ∞)
$g_0(x)$	$\widetilde{g}_1(x)$	$g(x)$	$\widetilde{g}_2(x)$
$y^*(x)$	$B(x)$	x	$S(x)$

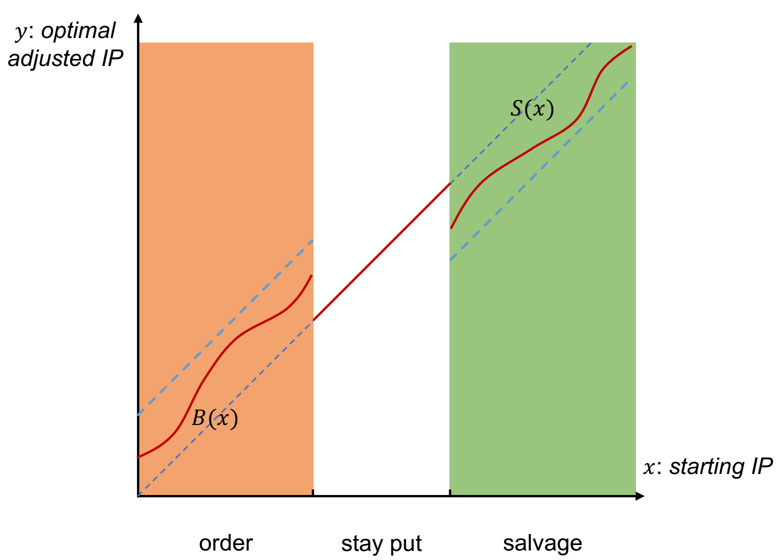


Figure 1.3: Illustration of single period optimal policy structure

The following monotonicity properties enable further simplification when computing an optimal policy.

Proposition 1.3 (Monotonicity). *(a) The functions $B(\cdot)$ and $S(\cdot)$ are increasing for all $n = N, \ldots, 1$.*

(b) The optimal order-up-to level $y^*(x)$ is increasing in x, almost everywhere, for all $n = N, \ldots, 1$. Lack of full monotonicity may occur in terms of downward jumps and these may arise at the, at most finitely many, breakpoint values where the optimal policy switches between ordering and staying put, or between staying put and salvaging.

(c) If the function $g(\cdot)$ is convex, $b(\cdot)$ is decreasing and $s(\cdot)$ is increasing.

Proof. (a) We prove the monotonicity of the function $B(\cdot)$; the proof for the function $S(\cdot)$ is analogous. It suffices to show that the family of sets $\{\mathcal{B}(x) : -\infty < x < \infty\}$ is increasing in the standard partial order $\geq^p$ for subsets of a lattice, see Vives (2001) (p 23): for a pair of sets $\mathcal{B}_1, \mathcal{B}_2$, $\mathcal{B}_1 \geq^p \mathcal{B}_2$ if for any $b_1 \in \mathcal{B}_1$ and $b_2 \in \mathcal{B}_2$, $\sup(b_1, b_2) \in \mathcal{B}_1$ and $\inf(b_1, b_2) \in \mathcal{B}_2$. Note that the feasibility intervals $[x, x + C]$, subsets of the real line $\mathbb{R}$, are increasing in x. Since the minimand in (1.27) is independent of x, hence has decreasing differences in (x, y), the monotonicity of the sets $\{\mathcal{B}(x) : -\infty < x < \infty\}$ follows from Theorem 2.3 (b), in combination with Remark 10, in Vives (2001).
(b) An immediate corollary of part (a) is that $y^*(\cdot)$ is increasing on any interval on which the optimal policy prescribes "ordering" or any interval on which it prescribes "staying put". The remaining characterization of the function $y^*(\cdot)$ is immediate.
(c) We prove that $b(x)$ is decreasing in x. The monotonicity proof for $s(\cdot)$ is analogous. Similar to the proof of part (a), define

$$\Omega(x) = \arg\min_{0 \leq q \leq C_1} \{\beta_1(x + q) + g(x + q)\} \tag{1.29}$$

and note that $b(x) = \inf \Omega(x)$. Since $g(\cdot)$ is convex, it has increasing differences in (x, q). Applying Theorem 2.3 (b) in Vives (2001) to a minimization problem, we get that the sets $\{\Omega(x) : -\infty < x < \infty\}$ are decreasing in the partial order $\geq^p$, defined in the proof of part (a). In particular, $b(\cdot)$ is decreasing as well. $\square$

Downward jumps of the function $y^*(\cdot)$ in a few points, may indeed occur, as exhibited by the common Example in Shaoxiang and Lambrecht (1996) and Shaoxiang (2004): the order-up-to level exhibits a downward jump at $x = 6$. Note that this example pertains to an infinite horizon model with stationary inputs, and hence, a fortiori, in finite horizon models with non-stationary inputs.

In spite of the fact that the order-up-to policy $y^*(\cdot)$ fails to be *perfectly monotone*, the structure in Proposition 1.3 may be exploited to simplify the dynamic programming calculations, no less than in models where perfect monotonicity can be shown. Assuming $y^*(\cdot)$ is calculated on a grid $\{x_1, x_2, \ldots\}$, we may exploit the fact that $y^*(x_i) \in [y^*(x_{i-1}), \infty) \cup \{x_i\}$.

The convexity assumption of the function $g(\cdot)$ is usually satisfied in a true single-period setting, where $g(y) = \mathcal{L}(y)$. Unfortunately, it often fails in multi-period settings. The monotonicity of the function $g(\cdot)$ implies that every interval on which it is optimal to order may be partitioned into two (possibly empty) subintervals: in the first subinterval, it is optimal to order up to capacity; in the second subinterval the order quantity decreases. Similarly, any interval in which it is optimal to salvage, may be partitioned into two (possibly empty) subintervals: in the first subinterval, the salvage quantity increases; if this quantity reaches the capacity level, there is a second subinterval on which the salvage quantity equals the capacity level.

Based on Theorem 1.2 and the previous lemmas, we have the following three corollaries that capture special cases where the optimal policy takes on simpler or more specific forms.

First, as mentioned, a setting without a salvage option corresponds with the parameter choices $\beta_2 = -M, K_2 = 0, C_2 = \infty$. In this case, $\underline{s} = \infty$, and the four-region structure in Table 1.2 reduces to three regions only. Similar simplifications due to $\underline{s} = \infty$ arise in the special cases discussed below.

Corollary 1.2 (No-Salvage Models). *When there is no salvage option and* $g(\cdot) \in SC_{C_1 K_1, \infty 0}$*, the structure of the optimal policy in the one-period problem is displayed by the first three columns in Table 1.2, since* $\underline{s} = \infty$*.*

Corollary 1.3 (Uncapacitated Models). *When* $C_1 = C_2 = \infty$*, part of the optimal policy structure in Theorem 1.2 takes on simpler forms summarized by Table 1.4.*

Table 1.4: Special optimal policy structures when $C_1 = C_2 = \infty$

(a) When $K_1 \geq K_2$ (If $K_2 = 0$, $b = \bar{b}$ and the shaded column disappears)

x	$(-\infty, b)$	$[b, \bar{b})$	$[\bar{b}, \underline{s}]$	$(\underline{s}, s]$	(s, ∞)
$g_0(x)$	$\widetilde{g}_1(x)$	$\min\{\widetilde{g}_1(x), g(x)\}$	$g(x)$	$\min\{\widetilde{g}_2(x), g(x)\}$	$\widetilde{g}_2(x)$
$y^*(x)$	B	$\{B, x\}$	x	$\{S(x), x\}$	S

(b) When $K_1 \leq K_2$ (If $K_1 = 0$, $\underline{s} = s$ and the shaded column disappears)

x	$(-\infty, b)$	$[b, \bar{b})$	$[\bar{b}, \underline{s}]$	$(\underline{s}, s]$	(s, ∞)
$g_0(x)$	$\widetilde{g}_1(x)$	$\min\{\widetilde{g}_1(x), g(x)\}$	$g(x)$	$\min\{\widetilde{g}_2(x), g(x)\}$	$\widetilde{g}_2(x)$
$y^*(x)$	B	$\{B(x), x\}$	x	$\{S, x\}$	S

(c) When $K_1 = K_2 = 0$

x	$(-\infty, B)$	$[B, S]$	(S, ∞)
$y^*(x)$	B	x	S

Proof. In this case we clearly have

$$B(x) = \inf\{\arg\min_{y \geq x}\{\beta_1 y + g(y)\}\} = B, \quad \text{for } x \leq B;$$

$$S(x) = \inf\{\arg\min_{y \leq x}\{\beta_2 y + g(y)\}\} = S, \quad \text{for } x \geq S.$$

By Proposition 1.2 (ii), for $x < b \leq B$, $y^*(x) = B$; for $x > s \geq S$, $y^*(x) = S$. This verifies the structure in the two outer regions for both $K_1 \geq K_2$ and $K_1 \leq K_2$. For the shaded regions in subtable (a) and (b):

- When $K_1 \geq K_2$, ,$\bar{b} \leq B$ by Proposition 1.2 (iii), hence for any $x < \bar{b} \leq B$, $g_0(x) = \widetilde{g}_1(x)$ and $y^*(x) = B$. Specially, if $K_2 = 0$, Proposition 1.2 (iv) indicates $b = \bar{b}$, and the shaded region in Table 1.4 (a) does not exist.

- When $K_1 \leq K_2$, $S \leq \underline{s}$ by Proposition 1.2 (iii), hence for any $x > \underline{s} \geq S$, $g_0(x) = \widetilde{g}_2(x)$ and $y^*(x) = S$. Specially, if $K_1 = 0$, Proposition 1.2 (iv) indicates $s = \underline{s}$, and the shaded region in Table 1.4 (b) does not exist.

For the special case where $K_1 = K_2 = 0$, as given by subtable (c) simply follows from Proposition 1.2 (iv). □

When there are no capacity limits but a fixed cost for ordering or salvaging does exist (as in subtables (a) and (b)), the following simplifications arise: the two outer regions have simple constant order-up-to and salvage-down-to levels B and S, respectively. Depending on the relative size of K_1 and K_2, the second or fourth region also has a specific target adjustment level. Finally, when either K_1 or K_2 is zero, the second or fourth region does not exist. This makes the corresponding ordering or salvaging decision a simple "(s, S)"–type policy. Furthermore, when there are no fixed costs, subtable (c) displays a three-region structure where both ordering and salvaging decisions become "base stock"–type policies.

The characterization in Table 1.4 is similar to that in Theorem 1 in Ye and Duenyas (2007), with additional simplifications indicated when one or both of the fixed costs are zero, see also Semple (2007). Dai and Yao (2013) consider a continuous review variant of this model where the demand process is given by a Brownian motion; the authors also confine themselves to stationary models under the long-run average cost criterion, further assuming that $L = 0$. For this case, they establish optimality of the following 4 threshold policy: there exist threshold $d < D < U < u$, such that inventory is increased (decreased) to D (U) when it reaches the level d (u); no inventory adjustment is made as long as the inventory level resides in (d, u).

Corollary 1.4 (No-Fixed Costs Models). *When either $K_1 = 0$ or $K_2 = 0$, part of the optimal structure can be characterized with more specificity, as is shown in Table 1.5, in which*

$$\bar{B}(x) = \inf\{\underset{\bar{b} \le y \le x + C_1}{\arg\min} \{\beta_1 y + g(y)\}, \qquad \text{for } x \ge \bar{b} - C_1;$$

$$\underline{S}(x) = \sup\{\underset{x - C_2 \le y \le \underline{s}}{\arg\min} \{\beta_2 y + g(y)\}, \qquad \text{for } x \le \underline{s} + C_2;$$

$$\mathbf{1}_b^+ = \mathbf{1}(b > \bar{b} - C_1), \qquad \mathbf{1}_b^- = \mathbf{1}(b < \bar{b} - C_1);$$

$$\mathbf{1}_s^+ = \mathbf{1}(s > \underline{s} + C_2), \qquad \mathbf{1}_s^- = \mathbf{1}(s < \underline{s} + C_2).$$

Proof. We first consider the case where $K_2 = 0$; the case where $K_1 = 0$ is symmetric and can be shown similarly. When $K_2 = 0$, by Lemma 1.3 (i), $\beta_1 x + g(x)$ is strictly decreasing on $(-\infty, \bar{b})$.

Table 1.5: Special optimal policy structures (partly) when $K_1 = 0$ or/and $K_2 = 0$

(a) When $K_2 = 0$. (Structure on $[\bar{b}, \infty)$ same as in Table 1.2)

x	$(-\infty, \min\{\bar{b} - C_1, b\})$	$[\min\{\bar{b} - C_1, b\}, \max\{\bar{b} - C_1, b\})$	$[\max\{\bar{b} - C_1, b\}, \bar{b})$
$g_0(x)$	$\widetilde{g}_1(x)$	$\widetilde{g}_1(x)$	$\min\{\widetilde{g}_1(x), g(x)\}$
$y^*(x)$	$x + C_1$	$\{x + C_1, x\}\mathbf{1}_b^- + \bar{B}(x)\mathbf{1}_b^+$	$\{\bar{B}(x), x\}$

(b) When $K_1 = 0$. (Structure on $(-\infty, \underline{s}]$ same as in Table 1.2)

x	$(\underline{s}, \min\{\underline{s} + C_2, s\}]$	$(\min\{\underline{s} + C_2, s\}, \max\{\underline{s} + C_2, s\}]$	$(\max\{\underline{s} + C_2, s\}, \infty)$
$g_0(x)$	$\min\{\widetilde{g}_2(x), g(x)\}$	$\widetilde{g}_2(x)$	$\widetilde{g}_2(x)$
$y^*(x)$	$\{\underline{S}(x), x\}$	$\{x - C_2, x\}\mathbf{1}_s^+ + \underline{S}(x)\mathbf{1}_s^-$	$x - C_2$

(c) When $K_1 = K_2 = 0$ and $C_1 < \infty, C_2 = \infty$. (Structure on $[\bar{b}, \infty)$ same as in Table 1.2)

x	$(-\infty, \bar{b} - C_1)$	$[\bar{b} - C_1, \bar{b})$
$g_0(x)$	$\widetilde{g}_1(x)$	$\widetilde{g}_1(x)$
$y^*(x)$	$x + C_1$	$\bar{b}$

- $x < \min\{\bar{b} - C_1, b\}$. $x < b$ implies that $g_0(x) = \widetilde{g}_1(x)$ by the general optimal policy in Table 1.2. Since $\beta_1 y + g(y)$ is strictly decreasing on $(-\infty, \bar{b})$ and $x + C_1 < \bar{b}$, clearly $y^*(x) = x + C_1$.

- $\min\{\bar{b} - C_1, b\} \le x < \max\{\bar{b} - C_1, b\}$. It is presumed that $\bar{b} - C_1 \neq b$ since otherwise this interval is empty and there is nothing to show. Then there are two cases to consider:

  (a) $b < \bar{b} - C_1$. The interval is $b \le x < \bar{b} - C_1$. Clearly $x \in [b, \bar{b})$ so $g_0(x) = \min\{\widetilde{g}_1(x), g(x)\}$ by the general optimal policy in Table 1.2. By the same argument as in the previous interval, if an order is placed, it is optimal to place a full capacity order. Therefore $y^*(x) \in \{x + C_1, x\}$.

  (b) $b > \bar{b} - C_1$. The interval is $\bar{b} - C_1 \le x < b$. Again $x < b$ implies that $g_0(x) = \widetilde{g}_1(x)$ by the general optimal policy in Table 1.2. Since $\bar{b} \le x + C_1$ and $\beta_1 y + g(y)$ is strictly decreasing on $(-\infty, \bar{b})$, $y^*(x) = \bar{B}(x)$.

- $\max\{\bar{b} - C_1, b\} \le x < \bar{b}$. Clearly $x \in [b, \bar{b})$ so $g_0(x) = \min\{\widetilde{g}_1(x), g(x)\}$ by the general optimal policy in Table 1.2. Since $\bar{b} \le x + C_1$ and $\beta_1 y + g(y)$ is strictly decreasing on $(-\infty, \bar{b})$, if it is optimal to place an order then $y^*(x) = \bar{B}(x) \in [\bar{b}, x + C_1]$. Thus, $y^*(x) \in \{\bar{B}(x), x\}$.

Next we prove the optimal policy structure given by Table 1.5 (c) under $K_1 = K_2 = 0$ and $C_1 < \infty, C_2 = \infty$. Notice that this is a special case of subtable (a), where we also have $K_1 = 0$ and $C_2 = \infty$. We only need to show $b = \bar{b} = \bar{B}(x), \forall x \in [\bar{b} - C_1, \bar{b}]$ so that subtable (a) becomes subtable (c). First we show $\beta_1 x + g(x)$ is increasing on $(\bar{b}, \infty)$, which directly implies $\bar{B}(x) = \bar{b}, \forall x \in [\bar{b} - C_1, \bar{b}]$ by the definition of $\bar{B}$. To see this, it follows from (1.20) and $K_1 = 0$ that for any $x > \bar{b}$ and $y \in [x, x + C_1]$,

$$A_1(x) \geq 0 \Rightarrow \beta_1 y + g(y) - \beta_1 x \geq g(x) \Leftrightarrow \beta_1 x + g(x) \leq \beta_1 y + g(y).$$

Then we show $b = \bar{b}$. By Lemma 1.3 (i), $\beta_1 x + g(x)$ is strictly decreasing on $(-\infty, \bar{b})$. Therefore for any $x \leq \bar{b} - C_1$,

$$\begin{aligned} \beta_1 x + g(x) > \beta_1 y + g(y) &\Rightarrow \beta_1 y + g(y) - \beta_1 x < g(x), \ \forall y \in (x, x + C_1] \\ &\Rightarrow \widetilde{g}_1(x) < g(x) \Rightarrow A_1(x) < 0. \end{aligned}$$

This implies $b \geq \bar{b} - C_1$ noticing the definition of b in (1.18). Suppose $\bar{b} - C_1 \leq b < \bar{b}$. By the definition of b and $\bar{b}$ in (1.18) there exist x and y such that $\bar{b} - C_1 \leq b \leq x < y < \bar{b}$ and $A_1(x) \geq 0$, $A_1(y) < 0$. It is shown above that $\bar{B}(s) = \bar{b}, \forall s \in [\bar{b} - C_1, \bar{b}]$, hence

$$\widetilde{g}_1(x) = \beta_1 \bar{b} + g(\bar{b}) - \beta_1 x, \quad \widetilde{g}_1(y) = \beta_1 \bar{b} + g(\bar{b}) - \beta_1 y.$$

Therefore

$$\begin{aligned} A_1(x) \geq 0 &\Rightarrow \widetilde{g}_1(x) \geq g(x) \Rightarrow \beta_1 \bar{b} + g(\bar{b}) - \beta_1 x \geq g(x), \\ A_1(y) < 0 &\Rightarrow \widetilde{g}_1(y) < g(y) \Rightarrow \beta_1 \bar{b} + g(\bar{b}) - \beta_1 y < g(y), \end{aligned}$$

which imply the following obvious contradiction:

$$\beta_1 \bar{b} + g(\bar{b}) \geq \beta_1 x + g(x) > \beta_1 y + g(y) > \beta_1 \bar{b} + g(\bar{b}),$$

where the middle inequality follows from Lemma 1.3 (i) as $x < y < \bar{b}$. Hence, $b = \bar{b}$. □

Observe that $B(x)$ $[S(x)]$ denotes the optimal inventory position to order up to [salvage down to] when the period is started with an inventory position of x units and assuming one is committed to initiate an order [a salvage batch]. $\bar{B}(x)$ [$\underline{S}(x)$] restricts the choice for the optimal order-up-to [salvage-down-to] levels to those above [below] $\bar{b}$ [$\underline{s}$]. Corollary 1.4 shows that, when $K_2 = 0$, the (ordering) half line $(-\infty, \bar{b})$ may be partitioned into three intervals, see Table 1.5 (a): in the left most interval, it is optimal to place a maximum size order and in the right most interval, it is optimal to place an order or to stay put (but salvaging is suboptimal). In the middle interval, it is optimal to place an order when $b > \bar{b} - C_1$; when $b \leq \bar{b} - C_1$, it is optimal to either place a maximum size order (C_1) or to stay put. A similar specification may be provided for the (salvage) half line $(\underline{s}, +\infty)$ when $K_1 = 0$, see Table 1.5 (b). When $K_1 = K_2 = 0$ and $C_1 < \infty, C_2 = \infty$, Table 1.5 (c) shows that the (ordering) half line $(-\infty, \bar{b})$ displays a modified base-stock policy for the ordering decision.

1.5 The multi period problem.

The (C_1K_1, C_2K_2)-convexity is preserved under the minimization operations specified by (1.13)–(1.15). This enables us to extend the structural results, above, to general multi-period planning horizons.

Proposition 1.4 (Preservation of strong (C_1K_1, C_2K_2)-convexity)**.** *Assuming $\beta_1 \geq \beta_2$, if $g(\cdot)$ is strongly (C_1K_1, C_2K_2)-convex, then*

$$g_1(x) = \min_{y \in [x, x+C_1']} \{K_1\delta(y-x) + \beta_1(y-x) + g(y)\},$$

$$g_2(x) = \min_{y \in [x-C_2', x]} \{K_2\delta(x-y) + \beta_2(y-x) + g(y)\},$$

$$g_0(x) = \min\{g_1(x), g_2(x)\}$$

are also strongly (C_1K_1, C_2K_2)-convex for any $C_1' \geq C_1, C_2' \geq C_2$.

We are now ready for our main result.

Theorem 1.3 (Multi Period Optimal Policy Structure)**.** (a) *Assume $f_0(\cdot) \in SC_{C_0K_0, C_0^vK_0^v}$ and $f_0(x) = O(|x|^p)$ for some integer $p \geq 1$. Then $f_n(x) \in SC_{C_nK_n, C_n^vK_n^v}$ and $f_n(x) = O(|x|^p)$ for $n = N, N-1, \ldots, 1$.*

(b) *In each period* $n = N, N-1, \ldots, 1$*, the optimal policy structure is as defined in Theorem 1.2 and Proposition 1.3 (a) and (b), with* $g_1(\cdot) = f_n^1(\cdot), g_2(\cdot) = f_n^2(\cdot), g_0(\cdot) = f_n(\cdot), \beta_1 = c_n, \beta_2 = c_n^v, K_1 = K_n, K_2 = K_n^v, C_1 = C_n, C_2 = C_n^v$ *and* $g(y) = \mathcal{L}_n(y) + \alpha \mathbb{E} f_{n-1}(y - D_n)$.

Proof. (a) We prove this theorem by induction. By our assumption, the theorem holds for $n = 0$. Suppose the result holds for period $n-1$, i.e., $f_{n-1}(\cdot) \in SC_{C_{n-1}K_{n-1}, C_{n-1}^v K_{n-1}^v}$ and $f_{n-1}(x) = O(|x|^p)$. We first prove that $f_n(x) = O(|x|^p)$. Since $f_{n-1}(x) = O(|x|^p)$, there exists a constant $A > 0$ such that $|f_{n-1}(x)| \leq A|x|^p$; so that $|\mathbb{E} f_{n-1}(y - D_n)| \leq A\mathbb{E}|y - D_n|^p \leq A\mathbb{E}(|y| + D_n)^p = A\sum_{l=0}^{p} \binom{p}{l} \mathbb{E} D_n^{p-l} |y|^l \leq B \max\{|y|^p, 1\}$ for some constant $B > 0$. Since $\mathcal{L}_n(y) = O(|y|^p)$ by Assumption 1.2 (i), there exists a constant $C > 0$ such that $|\mathcal{L}_n(y)| \leq C|y|^p$. Let $y^*(x)$ achieve the minimum in (1.9), then $|f_n^1(x)| \leq K_n + c_n|y^*| + |\mathcal{L}_n(y^*)| + \alpha B |y^*|^p \leq K_n + c_n(|x| + C_n) + C(|x| + C_n)^p + \alpha B \max\{1, (|x| + C_n)^p\} = O(|x|^p)$, thus $f_n^1(x) = O(|x|^p)$. By similar argument, $f_n^2(x)$ and hence $f_n(x)$ are also $O(|x|^p)$.

We then approve that $f_n(x) \in SC_{C_n K_n, C_n^v K_n^v}$. Since $f_{n-1}(\cdot) \in SC_{C_{n-1}K_{n-1}, C_{n-1}^v K_{n-1}^v}$, by Lemma 1.1 (iii), (iv) and Assumption 1.3,

$$\alpha \mathbb{E} f_{n-1}(y - D_n) \in SC_{C_{n-1}(\alpha K_{n-1}), C_{n-1}^v(\alpha K_{n-1}^v)} \subset SC_{C_n K_n, C_n^v K_n^v}. \tag{1.30}$$

Since $\mathcal{L}_n(\cdot)$ is convex, by Lemma 1.1 (iii) we have

$$g(y) = \mathcal{L}_n(y) + \alpha \mathbb{E} f_{n-1}(y - D_n) \in SC_{C_n K_n, C_n^v K_n^v}. \tag{1.31}$$

It then follows from Proposition 1.4 that $f_n^1(\cdot), f_n^2(\cdot), f_n(\cdot) \in SC_{C_n K_n, C_n^v K_n^v}$.
(b) Immediate from Theorem 1.2 and Proposition 1.3 (a) and (b). □

Pursuant to Proposition 1.3 in Section 1.4, we discussed the implications of the everywhere monotonicity property of Proposition 1.3 (a), and the almost everywhere monotonicity property of Corollary 1.3 (b). The same observations pertain to the general multi-period setting. Proposition 1.3 (c) fails to apply to the general multi-period model, since the convexity assumption, there, typically fails.

1.6 The infinite horizon model: minimizing total expected discounted costs as well as long-run average costs.

In this section, we prove that all of our structural results carry over to *stationary* infinite horizon models, assuming either the present value of all costs and revenues is to be minimized, or the long-run average cost value.

In extending our results from finite horizon to infinite horizon models, we follow the approach in Huh et al. (2011), closely; we therefore adopt much of the notation there.

A deterministic Markov policy δ is a sequence of decision rules $\{\delta_1, \delta_2, \dots, \}$ such that in period t, δ_t prescribes a specific feasible action to any potential state of the system. Under a given Markov policy δ and starting state s, let $\phi(S_t, A_t)$ denote the net costs charged in period t when S_t is the state of the system, and A_t the action (order size, salvage batch) chosen, then. Let $J_\alpha(\delta, s) = \mathbb{E}_\delta[\sum_{t=1}^{\infty} \alpha^t \phi(S_t, A_t)]$ denote the expected infinite-horizon present value of costs under policy δ when starting in state s. A policy δ^α is called *discounted cost optimal* under a given discount factor α, if, simultaneously, for every starting state $s \in S$,

$$J_\alpha(\delta^\alpha, s) = \inf_\delta J_\alpha(\delta, s).$$

The *long-run average cost* under a Markov policy δ and starting state $s \in S$ is defined as

$$\Phi(\delta, s) = \limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \phi(S_t, A_t).$$

A stationary policy δ^* is *long-run average cost optimal* if, simultaneously for all $s \in S$

$$\Phi(\delta^*, s) = \inf_{s' \in S} \inf_\delta \Phi(\delta, s').$$

We show the existence of a stationary discounted cost optimal policy, for any discount factor $\alpha < 1$, as well as the existence of a stationary long-run average cost optimal policy and the even stronger *preservation property* establishing a strong relationship between the two optimality criteria. We show that our model has the preservation property in that there exists a stationary policy δ^* satisfying the following properties.

(i) δ^* is "long-run average cost optimal" stationary in the sense that

$$\Phi(\delta^*, s) = \inf_{s' \in S} \inf_\delta \Phi(\delta, s') \tag{1.32}$$

for all $s \in S$, and

(ii) δ^* is "limit discount optimal" in the following sense: for any starting state s and any $\alpha_m \uparrow 1$, there exit a subsequence $\{\alpha_{m_k}\}$ and a sequence $\{s_k\}$ converging to s such that

$$\delta^*(s) = \lim_{k \to \infty} \delta^{\alpha_{m_k}}(s_k). \tag{1.33}$$

Theorem 1.4 (Infinite Horizon Optimality). (a) *(Discounted Cost Optimality) For every $0 < \alpha < 1$, there exits a sequence of* finite-horizon *optimal policies $\{\delta^\alpha(\cdot)\}$ that converges point-wise to a discounted cost optimal policy $\delta^\alpha(\cdot)$ as T approaches ∞. The discounted optimal policy $\delta^\alpha(\cdot)$ has the structure described in Theorem 1.3.*

(b) *(Long-Run Average Cost Optimality) There exits a stationary long-run average cost optimal policy δ^*. Moreover, the preservation property described in* (1.32) *and* (1.33) *holds.*

Theorem 1.4 corresponds with Theorem 3.1 in Huh et al. (2011) where it is shown to hold for any inventory management Markov Decision Process (MDP) that satisfies Assumptions 1 and 2, as well as Condition (SC) there. The authors show that under these three conditions, the MDP satisfies the conditions in Schäl (1993). The framework addressed in Huh et al. (2011) is very broad and, in some ways, more general than the broad model addressed in this chapter: it allows for demand distributions and capacity values that are Markov modulated, i.e., determined by an underlying world state variable which evolves according to a given Markov chain; it also allows for combined inventory control and pricing problems, where, as discussed in Section 1.2, in each period a price level is chosen along with an inventory adjustment and where the price level may impact the demand distribution. However, Huh et al. (2011) did not allow for *salvage opportunities*, i.e., *bilateral* inventory adjustments.

To ensure that Assumption 1 in Huh et al. (2011) is satisfied, we merely require the additional Assumption:

Assumption 1.5. *In the stationary infinite-horizon model, per definition, the sequence $\{D_n\}$ is assumed to be i.i.d. as a random variable D, and $C_n = C$ for all n. Moreover, $\mathbb{E}D < C$.*

The restriction $\mathbb{E}D < C$ is, of course, necessary to ensure that the inventory process can be governed in a way that it remains stable and the long-run average costs remain finite. See Federgruen and Zipkin (1986b) for a more detailed discussion in the special case where no salvage opportunities exist and no fixed inventory adjustment costs are incurred.

Assumption 2 in Huh et al. (2011) requires us to limit the type of expected holding and backlogging cost functions that may be used:

Assumption 1.6. $\mathcal{L}(y) = \mathbb{E}h((y - D^{l+1})^+) + \mathbb{E}p((D^{l+1} - y)^+)$, *where* $h(\cdot)$ *and* $p(\cdot)$ *are bounded from below and above by affine functions, i.e., strictly positive constants* $\underline{h}, \bar{h}, \underline{p}, \bar{p}$ *exist with*

$$\underline{h} \leq \frac{h(z') - h(z'')}{z' - z''} \leq \bar{h}, \quad \underline{p} \leq \frac{p(z') - p(z'')}{z' - z''} \leq \bar{p}$$

for any pair of distinct nonnegative numbers z' *and* z''.

The holding and backlogging cost structure in Assumption 1.6 is the commonly used structure, both in the literature and in practice. However, some models allow for $h(\cdot)$ and $p(\cdot)$ that grow *superlinearly*, but are bounded by a polynomial function of a higher degree, as in Assumption 1.2. This generalization will be discussed in Section 1.8.

To prove Theorem 1.4, it therefore suffices to be shown that Condition (SC) in Huh et al. (2011) is satisfied. We need some additional notation. Let

$X_t^0 =$ the inventory level at the beginnning of period t after any inventory adjustments initiated L periods earlier

$X_t^l = X_t^0 +$ inventory adjustments to take effect within the next l periods, $l = 1, \ldots, L-1$,

$X_t^L = y_t = X_t^0 +$ all inventory adjustments to take effect within the next L periods.

A function $g : \mathbb{R}^n \to \mathbb{R}$ is a *symmetrically linearly bounded above* (SLBA) function if there exist positive scalers ζ and ρ such that $g(\mathbf{x}) \leq \zeta + \rho\|\mathbf{x}\|$ with $\|\mathbf{x}\|$ the 1-norm of $\mathbf{x}$. A function $g : \mathbb{R}^n \to \mathbb{R}$ is a *symmetrically quadratically bounded above* (SQBA) function if there exist positive scalers ζ, ρ and ξ such that $g(\mathbf{x}) \leq \zeta + \rho\|\mathbf{x}\| + \xi(\|\mathbf{x}\|)^2$.

Condition 1.1 (Condition (SC))**.** *Let* $\mathbf{X} = (X^0, X^1, \ldots, X^{L-1})$ *be an arbitrary vector of inventory levels in any given period. There exist constants* $\bar{M} \geq 0$ *and* $\underline{M} \leq 0$ *satisfying the following.*

(a) *Let* $\mathbf{X}' = (X'^0, X'^1, \ldots, X'^{L-1})$ *denote an inventory vector identical to* $\mathbf{X}$ *except for* one *component, say* $l \in \{0, \ldots, L-1\}$*. There exists a a real-valued function* $\eta^l(\mathbf{X}, \mathbf{X}')$ *with the following properties.*

   (i) *For any Markov policy* δ*, there exists a Markov policy* δ' *such that for all* $N \geq 1$*:*

$$J^N(\delta', \mathbf{X}') \leq J^N(\delta, \mathbf{X}) + \eta^l(\mathbf{X}, \mathbf{X}'),$$

   where $J^N(\delta, \mathbf{X})$ $[J^N(\delta', \mathbf{X}')]$ *denotes the expected total costs over a planning horizon of* N *periods when starting with the inventory vector* $\mathbf{X}$ $[\mathbf{X}]$ *and following policy* δ $[\delta']$*.*

   (ii) *If* $X^l \geq \bar{M}$ *and* $X'^l = I^l - 1$*, then* $\eta^l(\mathbf{X}, \mathbf{X}') \leq 0$*.*

   (iii) *If* $X^l < I'^l \leq \underline{M}$*, then* $\eta^l(\mathbf{X}, \mathbf{X}') \leq 0$*.*

   (iv) *If* $I^l = 0$*, then* $\eta^l(\mathbf{X}, \mathbf{X}')$ *is a SQBA function of* $\mathbf{X}'^l$*.*

(b) *Let* X^L *be such that* $y = X^L > \max\{\bar{M}, X^{L-1}\}$ *and let* δ *be any Markov policy. Then, there exists an action* $X'^L = y'$ *such that* $X^{L-1} \leq X'^L \leq \max\{\bar{M}, X^{L-1}\}$ *and a policy* δ' *such that for any* $N \geq 1$*,*

$$J^N(X'^L, \delta', \mathbf{X}) \leq J^N(X^L, \delta, \mathbf{X}),$$

where $J^N(X^L, \delta, \mathbf{X})$ $[J^N(X'^L, \delta', \mathbf{X})]$ *denotes the expected total costs over a planning horizon of* N *periods when the initial inventory vector is* $\mathbf{X}$ *and the initial inventory position is set to* X^L $[X'^L]$*.*

The following Lemma shows that Condition (SC) is, indeed, satisfied. Together with Assumption 1.6 this provides the proof for Theorem 1.4.

Lemma 1.4. *Condition (SC) holds under Assumptions 1.1–1.6.*

As pointed out in Huh et al. (2011), the preservation property establishes that, for any discount factor $0 < \alpha < 1$, a discounted cost optimal stationary policy exists and that this policy inherits the structural properties established in Theorem 1.3. As far as the long-run average cost policy δ^* is concerned, the preservation property "however, is, in itself, insufficient to show that δ^* inherits the structural properties" in Theorem 1.3. However, the proof of the long-run average cost policy δ^* sharing these properties can be complicated, with similar arguments as those employed in Section 5 of Huh et al. (2011) for the inventory models addressed there.

1.7 Easily implementable heuristics: numerical examples

The structure of the optimal policy may be too complex for implementation, in several managerial settings. This applies, in particular, to the most general model where there may be intervals on which the order-up-to or salvage-down-to quantity is given by general non-linear functions $\{B_n(\cdot), S_n(\cdot)\}$. One recommendation is to replace these functions by a linear (or possibly piecewise linear) function, far more easily understood and accepted.

More specifically, it is easily verified that in any period $n = N, \ldots, 1$, values $L_n < U_n$ exist such that $y_n^*(x_n) = C_n$ for all $x_n < L_n$, and $y_n^*(x_n) = x - C_n^v$ for all $x > U_n$. Procurement models are typically solved on a rolling-horizon basis and only the policy rule pertaining to the first period, period N, needs to be implemented. In case the functions $\{B_N(\cdot)\}$ and $\{S_N(\cdot)\}$ have nonlinear components, replace, on $[L_N, U_N]$, $y_N^*(\cdot)$ by $\tilde{y}_N(\cdot)$ as follows: on any interval $[\underline{x}, \bar{x}]$ in which the optimal policy prescribes an order [salvage quantity], throughout, replace the curve corresponding with $\{y_N^*(\cdot)\}$ by the line connecting $(\underline{x}, y_N^*(\underline{x}))$ and $(\bar{x}, y_N^*(\bar{x}))$. On all other intervals, maintain the policy rule $y_N^*(\cdot)$ without any modifications.

As mentioned, the second and fourth interval in Table 1.2 and Figure 1.2 vanish when the functions $A_1(\cdot)$ and $A_2(\cdot)$ have at most one root. In all of our numerical experience, this is always the case, reducing the policy structure to that in Table 1.3 and Figure 1.3. Moreover, in all of our numerical experience dealing with unimodal demand distributions, the complexity of a nonlinear $B(\cdot)$ or $S(\cdot)$ function never arises, so that the above suggestions for a simplified policy structure never applied, because the structure of the exact optimal policy $\{y_n^*(\cdot)\}$ is already of the desired, simple (piecewise linear) form. The possibility of nonlinear $B(\cdot)$ functions was exemplified by Gallego and Scheller-Wolf (2000) dealing with the special case of our model, where salvaging is not an option. The authors identified one such instance by entertaining an artificial demand distribution with $\{1, 6, 7\}$ as its three-point support such that $\mathbb{P}[D = 1] = \mathbb{P}[D = 7] = 0.15$ and $\mathbb{P}[D = 6] = 0.7$.

We illustrate our results with a set of 13 instances obtained by the systematic variation of 7 key parameters in the model. All instances use stationary data and demand distributions. All demand distributions are Normals truncated at zero. The 13 instances share the parameters $h = 1, \mathbb{E}D = 5, K_v = 2, C_v = 10, c_v = 1.3$. The remaining parameters are specified in Table 1.6.

Table 1.6: Parameter setting for numerical studies

Scenario	K	C	c	α	l	p	σ
base case	2	10	3	1	2	5	2
high fixed ordering cost	**10**	10	3	1	2	5	2
low fixed ordering cost	**0**	10	3	1	2	5	2
large order capacity	2	**20**	3	1	2	5	2
small order capacity	2	**2**	3	1	2	5	2
high unit ordering cost	2	10	**20**	1	2	5	2
low unit ordering cost	2	10	**1.5**	1	2	5	2
small α	2	10	3	**0.7**	2	5	2
long lead time	2	10	3	1	**5**	5	2
zero lead time	2	10	3	1	**0**	5	2
high service level	2	10	3	1	2	**49**	2
volatile demand	2	10	3	1	2	5	**5**
stable demand	2	10	3	1	2	5	**0.5**

The base case example is illustrated by Figure 1.4, in which we display the function $y_N^*(\cdot)$ on the left panel and the value functions $f_N(\cdot), f_N^1(\cdot)$ and $f_N^2(\cdot)$ on the right panel. For $x < 9$, it optimal to place a maximum size order; for $9 \leq x < 16$, it is optimal to order up to the level 19. For $16 \leq x \leq 32$ it is optimal to stay put, and for $32 < x < 38$, it is optimal to salvage down to the level 28. Finally for $x \geq 38$ it is optimal to initiate a maximum salvage quantity. Parallel figures for the remaining 12 instances are contained in the online appendix. Note that $y_N^*(\cdot)$ is piecewise linear in all instances so that the suggested policy simplifications do not need to be undertaken.

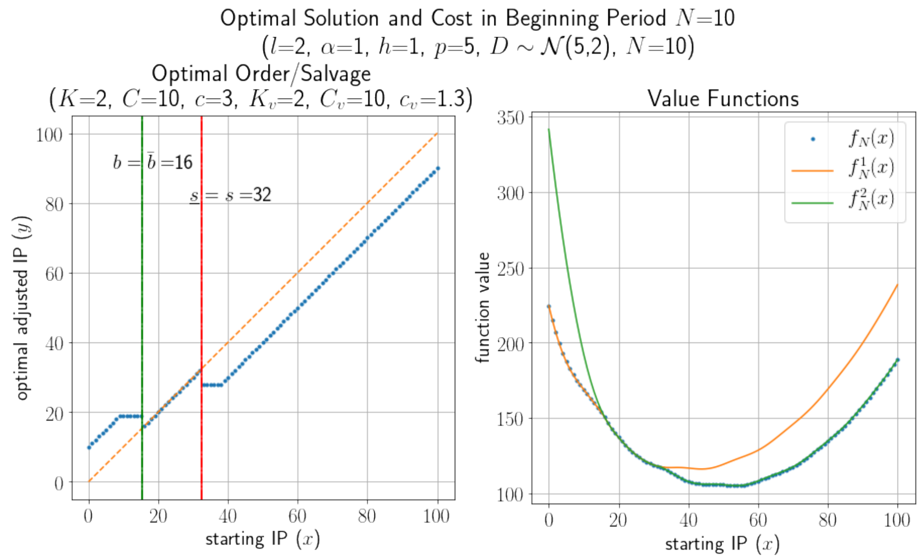


Figure 1.4: A numerical example of optimal policy and value functions

1.8 Concluding remarks.

This chapter analyzes a general periodic review inventory planning model that allows for the simultaneous treatment of three prevalent complicating factors: (a) bilateral inventory adjustments, (b) capacity limits for such adjustments, and (c) fixed costs for any such adjustments. Prior literature has addressed only subsets of these complications. We characterize the structure of an optimal policy, both for finite and infinite horizon models. We also show that earlier structural results can be obtained as corollaries of our general theory. The analyses are enabled by the identification of a new convexity property that generalizes all existing ones, as in Table 1.1.

It is of interest to generalize our results further. Specific directions include combined inventory control and pricing models, i.e., allowing the demand distribution to be endogenously controlled, for example by the dynamic selection of a price level. This would generalize the work of Federgruen and Heching (1999) and Chen and Simchi-Levi (2004a,b) which fail to allow for inventory reductions or capacity limits.

We are also confident that some of the technical restrictions can be relaxed, for example Assumption 1.6. Assumption 1.2 ensures that the $\mathcal{L}_n(\cdot)$ functions are polynomially bounded. It should be possible to eliminate Assumption 1.6 by generalizing Condition (SC) in Huh et al. (2011) to allow for cost differentials $\eta(\cdot, \cdot)$ that are "*symmetrically polynomially bounded above*".

Chapter 2

Dual Sourcing under Capacity Limits, Fixed Costs and Salvage Opportunities

2.1 Introduction and Summary

Manufacturing companies and retail chains often have access to two alternative supply sources for component parts, product modules, finished goods or supply materials. One source is typically low cost but has long lead times, whereas the other provides quicker response but at a higher price. When designing its procurement process, the purchaser may select one of the two sources as its exclusive supplier. Alternatively, it may opt for a dual sourcing strategy which procures from both sources. In the latter case, the challenge is to determine how and when each of the sources is to be used, as a function of the dynamically evolving inventory information. We address this question within a general model that incorporates economies of scale with respect to the order costs, capacity limits for individual orders and opportunities to reduce inventory via salvage sales.

The above strategic dilemmas arise, first and foremost, when firms decide on offshoring vs onshoring options. In the past few decades, there has been a consistent trend to offshore. However, the trend has recently been reversed, as companies have come to realize that, along with other sourcing considerations, price savings associated with offshore options need to be traded off against increased inventory costs and stockout risks due to the larger lead times involved. Longer lead times translate into a need for larger safety stocks, under a given targeted service level, or inferior service levels under given inventory investments. In contrast, onshore procurement from a local or nearby market is fast but typically incurs a higher purchase price or manufacturing cost.

Moreover, many companies have come to realize that a hybrid approach employing two or more suppliers, simultaneously, is, frequently, considerably more effective than one which relies on a *single* supplier, see e.g. Scheller-Wolf et al. (2007).

Within the operations management community, several authors have reported on specific company settings where dual sourcing is employed. These include Beyer and Ward (2002) reporting on Hewlett-Packard's strategy for manufacturing servers and Rao et al. (2000) on Caterpillar's for compact worktools sold in the North American market. Allon and Van Mieghem (2010a) report on a $10 billion high-tech US manufacturer of wireless transmission components with two assembly plants, one in China and one in Mexico. The authors identify a heuristic dual sourcing strategy, which, in their application, saves up to 20% over the best single sourcing strategy. Based on this application, Allon and Van Mieghem (2010b) developed a teaching game, see also Van Mieghem and Allon (2015).

The study of periodic review, dual sourcing inventory models starts with four papers in the early sixties, assuming independent demands, full backlogging of stockouts, and two suppliers with different lead times and different per-unit procurement prices. Other than linear order costs, there are holding and backlogging costs assumed to be convex in the end-of-the-period inventory and backlog levels, respectively.

Fukuda (1964) showed that a *dual base stock* policy, which acts on a *single* inventory measure, is optimal in this model, as long as the lead time of the slower supplier is exactly *one* period longer than that of the expedited supplier. (We refer to this as the *consecutive* lead time case). Under this policy, one starts by determining the size of the order to be placed with the *expedited* supplier, if any. This order is determined by a base-stock (order-up-to) policy acting on the (full) inventory position = the inventory level plus all outstanding orders. After the order with the expedited supplier is added to this inventory position measure, a second base-stock policy is applied to determine the order size with the slower supplier (if any).

Under *arbitrary* lead times, Whittemore and Saunders (1977) showed that no procurement strategy, acting on a single or even two inventory measures (so-called indices) needs to be optimal. As a consequence, many *heuristic* policies were proposed, mostly in the past decade, to handle the general lead time case. See Sun and Van Mieghem (2017) for a recent survey.

However, to our knowledge, little progress was made to include important generalizations and complicating factors that arise in practice and have become part of standard single sourcing inventory models, by themselves or in various combinations; in particular:
(a) fixed order costs and (b) capacity limits associated with the orders with the two suppliers.

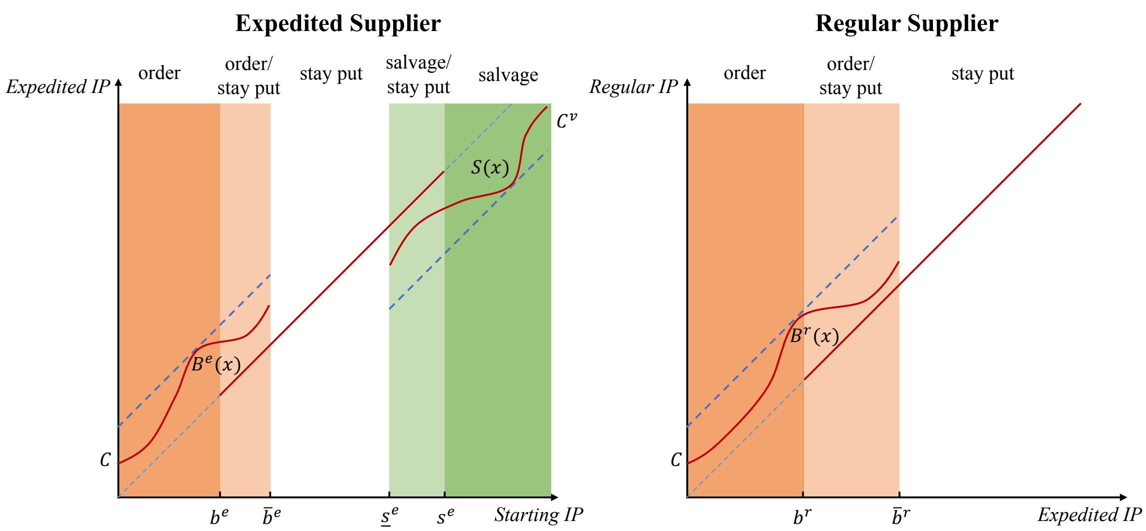
An additional generalization is
(c) the ability to *decrease* the inventory level by salvaging a given quantity through sales in a secondary channel (jobbers, discounters, outlet stores, etc.).

In view of the Whittemore and Saunders (1977) result, we, initially, confine ourselves to the consecutive lead time case. We must also assume that, when applicable, the lead time for a salvage transaction equals the order lead time of one of the two suppliers. This is a natural assumption when salvaging involves returning items to this supplier; in other settings, however, the lead time assumptions may be restrictive. Section 2.5 therefore discusses effective heuristics for general lead time combinations.

We show that a single index policy continues to be optimal for this very general model that incorporates the three complications (a)–(c), simultaneously. (The remaining model assumptions are standard and identical to those employed in the above-mentioned literature on dual sourcing models.) In each period, one first determines the size of an order with the expedited supplier, if any, or the size of any salvage quantity, based, exclusively, on the regular full inventory position. Thereafter, the inventory position is adjusted upward (by the expedited supplier order) or downward (by the salvage quantity); any order with the regular supplier is then determined as a function of the *adjusted* inventory position.

Moreover, under some parameter restrictions, the dependence of the optimal order sizes and/or salvage quantity on the period's starting inventory position follows a relatively simple structure. In the most general case, the optimal policy is characterized by four critical threshold levels of the inventory position: $b^e \leq \bar{b}^e \leq \underline{s}^e \leq s^e$ partitioning the inventory position line into 5 consecutive regions. In the middle range $[\bar{b}^e, \underline{s}^e]$, it is optimal to forgo both an order with the expedited supplier, as well as any salvage activity. In the far-left (-right) region $(-\infty, b^e)$ $[(s^e, \infty)]$, it is optimal to place an order with the expedited supplier [to initiate a salvage sale], the size of which varies as a non-linear function of the inventory position. This leaves us with the remaining pair of intervals $[b^e, \bar{b}^e)$ and $(\underline{s}^e, s^e]$. In the former, one alternates between sub-intervals with a positive order and those where it is optimal to stay put; however, salvaging is not to be considered in this interval $[b^e, \bar{b}^e)$. Similarly, one alternates on $(\underline{s}^e, s^e]$ between sub-intervals where it is optimal to stay put and those where it is optimal to initiate a salvage batch; however, ordering does not need to be considered. We identify a simple condition under which the second and fourth regions vanish, giving rise to an even simpler three-region policy.

As far as the second stage ordering decision with the *regular* supplier is concerned, the optimal policy is characterized by two threshold parameters, $b^r, \bar{b}^r$ partitioning the *adjusted inventory position* line in up to three regions. The structure of the optimal strategy within these regions is identical to those of the first stage decision in the three left most intervals; see Figure 2.1.



Note. The solid line is the 45° line through the origin, shifted by the capacities C and C^v to obtain the dotted lines.

Figure 2.1: General optimal policy structure

Even simpler and more pronounced structures arise in various special cases; where only a subset of the complications (a)–(c) prevails, or where some model parameters take on a specific value, for example, when either orders or salvage sales can be initiated without a fixed cost.

For *general* lead time combinations, we propose and evaluate effective heuristics; see

Section 2.5. We also conduct various numerical studies to illustrate and evaluate these heuristics, and to draw managerial insights.

Here are some of the key managerial insights: The benefits of dual sourcing may be in excess of 10%, and even as large as 30%. They increase as the lead time, and hence the relative value of the expedited supplier, increases. These savings are achieved even though the expedited supplier is used infrequently and for a small part of the total purchase volume. The benefits are larger in capacitated versus uncapacitated systems, and in systems in which the fixed procurement costs are relatively low. These findings explain the current trend to complement offshore production factories with onshore smaller "SpeedFactories", see Boute et al. (2019). Finally, the benefits of dual sourcing increase as a higher service level [demand volatility] is targeted [experienced].

The remainder of this paper is organized as follows. In Section 2.2 we give a review of the literature on multi-sourcing procurement strategies. Section 2.3 shows, under the above lead time restrictions, how the general dual sourcing problem can be reduced to a tractable single sourcing problem. In Section 2.4, we derive under some parameter restrictions, the structure of the optimal dual sourcing policy both in the most general and in various special cases. Based on these structural results, we develop, in Section 2.5, our proposed heuristics for *general* lead time combinations. Numerical studies in Section 2.6 review various managerial insights. Section 2.7 contains concluding remarks.

2.2 Literature Review

In this section, we review the literature on dual sourcing stochastic inventory models. As mentioned in the Introduction, the 60-year-old literature has mostly focused on a base periodic review model with independent demands, full backlogging of stockouts, linear order, holding and backlogging costs, and *two* potential uncapacitated suppliers, differentiated by their per-unit procurement price and lead times. This workhorse model may be viewed as the direct extension of the *seminal* single source model by Arrow et al. (1951).

Three papers in the early sixties, i.e., Barankin (1961), Daniel (1963) and Neuts (1964) focused on the special case where the lead time of the expedited supplier is negligible (or zero), and that of the regular supplier exactly *one* period. Here, the optimal strategy employs *two* base-stock levels: first, the period's starting inventory level—including last period's regular order—is used by a simple base-stock policy to determine whether an order is to be placed with the expedited supplier. Thereafter, a second base-stock policy is used to determine the order with the regular supplier, if any, comparing the adjusted inventory level (inventory level plus the new order with the expedited supplier) with a second base-stock level. Fukuda (1964) extended this result to the case where the lead time of the regular supplier is arbitrary and that of the expedited supplier *one* period shorter. The same structural result continues to apply, except that the base-stock policies act on the period's starting *inventory position*, as opposed to the inventory level.

Whittemore and Saunders (1977) showed that no simple structure prevails for the general case with *non-consecutive* lead times, i.e., lead times that differ by more than a single period. In particular, the optimal policy needs to be based on more than one, or even any constant number of inventory measures, or indices.

With this negative insight, the development of dual sourcing models was interrupted for some 30 years. In the last decade, we have seen a plethora of papers suggesting and comparing *heuristic* policies for the infinite horizon, stationary base model with general lead times. This literature stream started with Veeraraghavan and Scheller-Wolf (2008) proposing the use of *dual index* base-stock polices. Here, the first stage order to the expedited supplier is determined by a base-stock policy acting on the so-called *expedited* inventory position, consisting of the inventory level plus all outstanding orders to arrive within the expedited supplier's lead time from the current period. The second stage order with the regular supplier is determined by a second base-stock policy acting on the *full inventory position*, the current period's order with the expedited supplier included. These authors compared the cost performance of the best policy within their proposed class, with the overall optimal policy. To this end, they conducted a numerical study assuming the demand distribution has support on a few values only., so that the set of possible inventory levels can be limited and the optimal policy can be found via dynamic programming, at least for small lead time values. The authors report that their proposed dual index policy, in the majority of cases, comes within 1% or 2% of optimality.

The authors also consider capacity limits for the two suppliers. In this case, base stock policies need to be replaced by "*modified base-stock policies*", where an order is placed to bring the relative inventory index as close as possible to the base-stock level. In the single sourcing literature, such policies were shown to be optimal, see e.g. Federgruen and Zipkin (1986a,b).

Scheller-Wolf et al. (2007) show that a *single-index* base-stock policy performs comparably or even better than the *dual index* base-stock policy in Veeraraghavan and Scheller-Wolf (2008). Under a single index base-stock policy, the order with the expedited supplier is determined by a base-stock policy acting on the full inventory position, which includes all outstanding orders, rather than the more limited "expedited" inventory position in Veeraraghavan and Scheller-Wolf (2008). In Scheller-Wolf et al. (2007)'s numerical study, the average difference between the cost values of the two heuristics is less than 0.5% and the maximum difference no more than 3%. While the cost performance is almost identical between the two heuristics, the authors point at several major advantages of the single index heuristic, including the fact that it is simpler and that the optimal base-stock levels can be computed *analytically*, at "25–60 times faster" computational times.

Sheopuri et al. (2010) develop six alternative heuristic polices, some of considerably more complex structure (for example, the vector-based base-stock policy and the "best weighted" index policy). Their numerical study shows that each of the heuristics has an average cost performance that comes within 1% of that of the dual index base-stock policy, and hence of the single index base-stock policy, as well. These authors also show that the single source *lost-sales* model can be viewed as a special case as the dual source model with backlogging. Several of the structural properties identified by Zipkin (2008a) for the lost sales model can therefore be generalized in the study of dual sourcing problems. In the same spirit as Sheopuri et al. (2010), Hua et al. (2015) developed another class of heuristic policies, again with comparable cost performance, in their numerical study.

Allon and Van Mieghem (2010a) proposed a Tailored Based-Surge (TBS) heuristic in which a constant size order is placed with the regular supplier and a base-stock policy is used for the expedited supplier. Janakiraman et al. (2015) analyzed this class of policies, deriving optimality gap bounds when the demands consist of a regular base demand plus an infrequent surge demand. In a numerical study, similar to that in Veeraraghavan and Scheller-Wolf (2008), they observed that the optimality gap of the best TBS policy varies between 21% and 3.5%. Xin and Goldberg (2017) showed that a TBS policy is asymptotically optimal when the lead time difference goes to infinity. Xin et al. (2017) extend the results to a setting where the expedited supplier is unreliable, i.e., in any period the supplier is with a given probability unable to fill any order. The authors evaluate the performance of the TBS heuristic based on Walmart data. As mentioned in the Introduction, Allon and Van Mieghem (2010b) developed the Mexico-China teaching game, based on the industrial application which motivated their parallel paper. See also Chapter 7 in the prominent textbook on Operations Strategy by Van Mieghem and Allon (2015) for a treatment of the dual sourcing problem.

Sun and Van Mieghem (2017) developed a robust optimization approach for the base dual sourcing problem. Under this approach, the problem can be formulated as a mathematical program, avoiding the curse of dimensionality associated with standard dynamic programming formulations that aim at optimizing aggregate expected costs.

Very few papers address generalizations of the base model. A noted exception is Sethi et al. (2003) who incorporates fixed order costs into the model with the one period lead time difference. They show that, in each period, the order for the expedited supplier and that for the regular supplier are to be determined sequentially, both on the basis of an (s, S)-policy acting on the regular inventory position. We retrieve this result as one of the special cases in our general model. Fox et al. (2006) also allow for fixed order costs but assume zero lead time for the both suppliers. It is easily seen that this model reduces to a single sourcing problem with a piecewise linear concave order cost. Boute and Van Mieghem (2015) add capacity costs and order smoothing considerations into the base model.

A few papers have addressed dual sourcing problems in *continuous review* models. Moinzadeh and Schmidt (1991) and Song and Zipkin (2009) assumed demands are generated by a Poisson process and a dual-index base-stock policy is applied. The former paper assumes the lead times are deterministic while the latter allows for stochastic but exogenous lead times. Moinzadeh and Schmidt (1991) characterize the steady-state inventory level distribution and hence the long-run average cost for any (dual-index) base-stock policy. Song and

Zipkin (2009) models the system as a queueing network with overflow by-passes and obtain the steady-state distribution of the network in product form. Very recently, Zhou and Yang (2016) extended their work by allowing for compound Poisson demands and fixed order costs. These authors propose, as heuristic, a *single index* pair of (R, nQ)-policies. At each demand epoch, one places an order with the expedited supplier iff the full inventory position is below a reorder level R^e; the order is sized as the smallest multiple of an order size Q^e which elevates the inventory position above R^e. With this order added to the inventory position, any order with the regular supplier is determined by a different (R^r, nQ^r)-policy.

All of the above continuous review papers make an upfront restriction to a specific class of heuristic policies. However, the recent paper by Song et al. (2017), characterizes the optimal policies in a setting with linear costs but *endogenously* determined stochastic lead times, under a certain condition, and propose near-optimal heuristic policies when the condition fails.

2.3 Dual Sourcing: Equivalency with a Single Sourcing Model

We consider a single-item periodic review inventory system with two potential suppliers: a regular and an expedited supplier. In each period, there may also be a salvage option to reduce inventory, generating a given per-unit revenue or cost. The lead time for ordering from the expedited supplier [regular supplier, salvaging] is l_e $[l_r, l_s]$. In the base model we focus on the case where $l_r = l_e + 1$ and $l_s = l_e$.[1] (The condition $l_s = l_e$ applies when both expedited orders and salvage transactions occur in negligible time, or when salvaging involves returns to the supplier.) As explained in the Introduction, optimal order and salvaging policies, that act on a single inventory measure each, can only be expected under these lead time restrictions. Moreover, the structural results obtained for this special case suggest effective heuristic adaptations under general lead time combinations. See Section 2.5.

[1] The only other tractable case has $l_s = l_r$, using a similar analysis. Janakiraman and Seshadri (2017) address this specific case, in a model without fixed cost or capacity limits and $c_n^v = 0$.

We index the periods *backward* from 1 to N. The sequence of events in period n is as follows: At the beginning of period n, an order may arrive from the expedited or the regular supplier, or both. Such deliveries are added to the inventory level. The firm now decides on new order sizes to be placed with each of the two suppliers as well as any salvage quantity if it wants to *reduce* its inventory position. Stochastic demand is then realized and satisfied with on-hand inventory. At the end of the period, any unsatisfied demand is fully backlogged while leftover inventory is carried over to the next period. It is easily verified that it is never optimal to simultaneously place an order with the expedited supplier and to sell off a batch of inventory, since both inventory adjustments take effect in the very same period.

We first introduce notation for the model primitives. For period $n = N, N-1, \ldots, 0$, denote:

K_n^e, K_n^r = fixed cost for any expeditied and regular order, respecitively.

C_n^e, C_n^r = capacity limit for any expeditied and regular order, respectively.

c_n^e, c_n^r = unit price charged by the expedited and regular suppliers, respectively. (assuming $c_n^e > c_n^r$)

K_n^v, c_n^v, C_n^v = fixed cost, unit cost and capacity limit, respectively, when salvaging inventory.

D_n = stochastic demand.

α = discount factor, $\alpha \in [0, 1]$.

(Often, $c_n^v < 0$, reflecting salvage revenues.)

Assume $\{D_n, 1 \leq n \leq N\}$ are independent random variables with general distributions. Inventory and backlogging related costs will be introduced below. We impose the following restrictions on the cost parameters. These restrictions are innocuous and merely preclude arbitrage opportunities.

Assumption 2.1. *For any period n, the unit order costs, the unit salvage revenue and backlogging cost satisfy the restrictions: (i) $c_n^e \geq c_n^v$, and (ii) $c_n^r \geq \alpha c_{n-1}^v$.*

Assumption 2.1 (i) precludes the possibility of arbitrage opportunities where goods are procured from the expedited supplier, to be sold at a premium via the salvage channel.

Similarly, Assumption 2.1 (ii) precludes the possibility of buying units form the regular supplier and initiating the sell-off of these units, at a profit, one period later.

The state and action variables, in period n, are given by:

x_n = the inventory position at the beginning of period n

= the beginning inventory level, i.e., on-hand inventory minus backlogs, plus all outstanding orders from both suppliers, exclusive of the current orders.

q_n^e = the size of the expedited-channel inventory adjustment in period n.

q_n^r = the size of the order placed with the regular supplier in period n.

Note that $q_n^r \geq 0$ while $-\infty < q_n^e < \infty$. $q_n^e \geq 0$ [$q_n^e < 0$] represents the size of the expedited order [salvage batch] initiated in period n.

Our ability to aggregate the order with the expedited supplier, with the amount to be salvaged (a common lead time later), follows, by Assumption 2.1 (i), from the fact that it is never optimal to initiate both a negative and a positive expedited-channel inventory adjustment in the same period: if the net inventory position adjustment is positive [negative], one is better off reducing the expedited order [salvage batch] to the level of the *net* inventory adjustment and canceling the salvage batch [expedited order].

The inventory position dynamics are given by

$$x_{n-1} = x_n + q_n^e + q_n^r - D_n.$$

Moreover, the *inventory level* I_{n-l_e} at the *end* of period $n - l_e$ is given by

$$I_{n-l_e} = x_n + q_n^e - D_{n,n-l_e}, \tag{2.1}$$

where $D_{n,m} = D_n + D_{n-1} + \cdots + D_m$ is the aggregate demand in the time interval $[n, m]$ with $m \leq n$.

Assuming the inventory and backlogging related costs depend on the end-of-period inventory level sizes only—as in virtually all single-sourcing inventory models, the (single dimensional) inventory position x_n is a sufficient description of the state of the system. Instead of charging the actual inventory costs that arise at the end of period m ($m = 1, 2, \ldots$)

to *that* period, one obtains an equivalent representation of the controllable part of the total expected discounted cost by charging its *expected value* to the *start* of period $m + l_e$, employing the stochastic identity (2.1). Thus for all $n = 1, 2, \ldots, N$, let

$L_n(x_n + q_n^e)$ = the expected value of all inventory and backlogging related costs at the end of period $n - l_e$ discounted back to period n.

We are now ready to formulate the dynamic programming recursions for our model. Let

$f_n(x)$ = the optimal discounted expected total costs in the last n periods of the planning horizon, when period n is started with an inventory position x.

As mentioned, under Assumption 2.1, it is never optimal to place an order with the expedited supplier along with the initiation of a salvage order. Therefore

$$f_n(x) = \min\{f_n^1(x), f_n^2(x)\}, \tag{2.2}$$

where

$f_n^1(x)$ = the minimum total expected cost in the last n periods, when starting with an inventory position of x units, and assuming an expedited order is placed in period n,

$f_n^2(x)$ = the minimum total expected cost in the last n periods, when starting with an inventory position of x units, and assuming a salvage sale *is* initiated in period n.

These functions satisfy the recursions:

$$\begin{aligned} f_n^1(x) = & \min_{q_n^e \in [0, C_n^e], q_n^r \in [0, C_n^r]} \{K_n^e \delta(q_n^e) + c_n^e q_n^e + K_n^r \delta(q_n^r) + c_n^r q_n^r + L_n(x + q_n^e) \\ & + \alpha \mathbb{E} f_{n-1}(x + q_n^e + q_n^r - D_n)\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} f_n^2(x) = & \min_{q_n^e \in [-C_n^v, 0], q_n^r \in [0, C_n^r]} \{K_n^v \delta(-q_n^e) + c_n^v q_n^e + K_n^r \delta(q_n^r) + c_n^r q_n^r + L_n(x + q_n^e) \\ & + \alpha \mathbb{E} f_{n-1}(x + q_n^e + q_n^r - D_n)\}, \end{aligned} \tag{2.4}$$

where $\delta(u) = 1$ if $u > 0$ and $\delta(u) = 0$ otherwise.

In settings without salvage opportunities, $f_n(x) = f_n^1(x)$. However, to allow for a unified treatment, we model the "no-salvage" case as one in which $c_n^v = -M$, a large *negative* number such that salvaging is completely unattractive. Without loss of generality, we set $K_n^v = 0$ and $C_n^v = \infty$ in this case.

Since (2.3) and (2.4) share the last three terms, and substituting

$y_n = x + q_n^e =$ the beginning inventory position of period n, after inclusion of any expedited supplier order or salvage sale quantity,

$z_n = y_n + q_n^r = x_n + q_n^e + q_n^r =$ the beginning inventory position of period n, after inclusion of all inventory adjustments of period n,

we can rewrite (2.3) and (2.4) as

$$f_n^1(x) = \min_{y \in [x, x+C_n^e]} \{K_n^e \delta(y - x) + c_n^e(y - x) + g_n(y)\}, \tag{2.5}$$

$$f_n^2(x) = \min_{y \in [x-C_n^v, x]} \{K_n^v \delta(x - y) + c_n^v(y - x) + g_n(y)\}, \tag{2.6}$$

where

$$g_n(y) = L_n(y) + f_n^r(y), \tag{2.7}$$

$$f_n^r(y) = \min_{z \in [y, y+C_n^r]} \{K_n^r \delta(z - y) + c_n^r(z - y) + \alpha \mathbb{E} f_{n-1}(z - D_n)\}. \tag{2.8}$$

In other words, the order and salvage decisions in any given period may be thought of as occurring in two stages: first a new order with the expedited supplier/salvage quantity is determined based on x_n—the inventory position, followed by the choice of a supplementary order with the regular supplier based on the augmented inventory position y_n. Our second conclusion is that the general *dual source* inventory planning problem is equivalent to a *single source* model with an "adjusted" future cost function $g_n(\cdot)$. In particular, the optimal combined ordering and salvage policy can be obtained by computing a series of nested *one-dimensional* value functions, and this under general parameters and demand distributions. In particular, one starts with the evaluation of the function $g_n(\cdot)$, followed by that of $f_n^1(\cdot)$ and $f_n^2(\cdot)$. The ultimate value function $f_n(\cdot)$ is obtained as the pointwise minimum of the functions $f_n^1(\cdot)$ and $f_n^2(\cdot)$.

In the next section, we show that under a few parameter restrictions, a specific structure for the optimal policy can be established.

2.4 A Combined Procurement Strategy of Special Structure

Chapter 1 introduces the key convexity concept, (C_1K_1, C_2K_2)-convexity, in 1.1, and shows that (C_1K_1, C_2K_2)-convexity generalizes almost all familiar convexity structures. See Proposition 1.1 and Table 1.1. We show that, under a few restrictions, an optimal policy is of relatively simple structure, exhibited in Theorem 2.2, below. First, in terms of the expected holding and backlogging cost functions $L_n(\cdot)$, we introduce a standard assumption:

Assumption 2.2. *The function $L_n(\cdot)$ is convex and $L_n(y) = O(|y|^p)$ for some $p \geq 1$, $n = 1, 2, \ldots, N$. Also, $\mathbb{E}(D_n^p) < \infty$ for $n = 1, 2, \ldots, N$.*

In addition, we need

Assumption 2.3. *For $n = N, \ldots, 1$:*

(i) $K_n^r = K_n^e = K_n, \quad C_n^r = C_n^e = C_n;$

(ii) $K_n \geq \alpha K_{n-1}, \quad K_n^v \geq \alpha K_{n-1}^v;$

(iii) $C_n \leq C_{n-1}, \quad C_n^v \leq C_{n-1}^v.$

In other words, capacity limits are assumed to weakly increase over time, an assumption satisfied in most practical applications. (Most frequently, capacities are constant over the course of the inventory planning horizon.) The inequalities $K_n \geq \alpha K_{n-1}$ and $K_n^v \geq \alpha K_{n-1}^v$ echo those in the basic single source inventory problem with fixed order costs, see Scarf (1960) or Zipkin (2000). Assumption 2.3 (i) is more restrictive, in particular the assumption that both suppliers have the same capacities.

We first show that all value functions $\{f_n(\cdot)\}, n = N, N-1, \ldots, 1$ satisfy the strong (C_1K_1, C_2K_2)-convexity property, assuming the terminal value function $f_0(\cdot)$ does.

Theorem 2.1. *Assume $f_0(x) \in SC_{C_0K_0, C_0^vK_0^v}$ and $f_0(x) = O(|x|^p)$ for some integer $p \geq 1$. Then $f_n(x) \in SC_{C_nK_n, C_n^vK_n^v}$ and $f_n(x) = O(|x|^p)$ for $n = N, N-1, \ldots, 0$.*

Now, define auxiliary functions

$$\widetilde{g}_n^1(x) = K_n + \min_{y\in[x,x+C_n]}\{c_n^e(y-x) + g_n(y)\}, \qquad A_n^1(x) = \widetilde{g}_n^1(x) - g_n(x),$$

$$\widetilde{g}_n^2(x) = K_n^v + \min_{y\in[x-C_n^v,x]}\{c_n^v(y-x) + g_n(y)\}, \qquad A_n^2(x) = \widetilde{g}_n^2(x) - g_n(x),$$

$$\widetilde{g}_n^r(y) = K_n + L_n(y) + \min_{z\in[y,y+C_n]}\{c_n^r(z-y) + \alpha\mathbb{E}f_{n-1}(z-D_n)\}, \qquad A_n^r(y) = \widetilde{g}_n^r(y) + L_n(y) - \alpha\mathbb{E}f_{n-1}(y-D_n).$$

Here, $\widetilde{g}_n^1(x)$ represents the optimal expected cost from period n on assuming one is committed to place an order with the expedited supplier in this period (possibly combined with an order with the regular supplier). $A_n^1(x) = \widetilde{g}_n^1(x) - g_n(x)$ denotes the cost differential with the optimal expected cost from period n on, assuming one is committed to *forgo* any order with the expedited supplier or a salvage batch in this period. Similarly, $\widetilde{g}_n^2(x)$ represents the optimal expected cost from period n on assuming one is committed to initiate a salvage batch in this period (possibly combined with an order with the regular supplier). $A_n^2(x) = \widetilde{g}_n^2(x) - g_n(x)$ denotes the cost differential with the same benchmark $g_n(\cdot)$ used in the definition of $A_n^1(\cdot)$. Finally, $\widetilde{g}_n^r(y)$ denotes the expected optimal cost until the end of the planning horizon, assuming one is committed to place an order with the regular supplier, while foregoing an order with the expedited supplier or a salvage batch, and $A_n^r(y)$ denotes the cost differential vis-à-vis the optimal expected cost when forgoing *any* inventory adjustments in period n.

Define the following critical points for period n:

$$B_n^e = \inf\{\arg\min_y\{c_n^e y + g_n(y)\}\}, \quad b_n^e = \inf\{x : A_n^1(x) \geq 0\}, \quad \bar{b}_n^e = \sup\{x : A_n^1(x) < 0\}, \tag{2.9}$$

$$S_n = \sup\{\arg\min_y\{c_n^v y + g_n(y)\}\}, \quad s_n = \sup\{x : A_n^2(x) \geq 0\}, \quad \underline{s}_n = \inf\{x : A_n^2(x) < 0\}, \tag{2.10}$$

$$B_n^r = \inf\{\arg\min_z\{c_n^r z + \alpha\mathbb{E}f_{n-1}(z-D_n)\}\}, \; b_n^r = \inf\{x : A_n^r(x) \geq 0\}, \; \bar{b}_n^s = \sup\{x : A_n^r(x) < 0\}. \tag{2.11}$$

Let

$$B_n^e(x) = \inf\{\underset{x \le y \le x+C_n}{\arg\min}\{c_n^e y + g_n(y)\}\}, \quad S_n(x) = \sup\{\underset{x-C_n^v \le y \le x}{\arg\min}\{c_n^v y + g_n(y)\}\},$$

$$B_n^r(y) = \inf\{\underset{y \le z \le y+C_n}{\arg\min}\{c_n^r z + \alpha \mathbb{E} f_{n-1}(z - D_n)\}\}$$

denote minimizers of $\widetilde{g}_n^1(x), \widetilde{g}_n^2(x)$ and $\widetilde{g}_n^r(y)$, respectively.

The following theorem can be proven based on the proof of Theorem 1.2 in Chapter 1.

Theorem 2.2. *(a) The critical points are ranked as follows:*

$$b_n^e \le \bar{b}_n^e \le \underline{s}_n^e \le s_n^e, \quad b_n^r \le \bar{b}_n^r. \tag{2.12}$$

(b) The optimal policy is characterized by Table 2.1, in which $y_n^*(x_n)$ and $z_n^*(y_n^*)$ are either uniquely determined or take a value in a bi-valued set $\{\cdot, \cdot\}$.

Table 2.1: Optimal policy structure for systems with fixed costs and capacity limits

(a) Expedited supplier and salvage

x_n	$(-\infty, b_n^e)$	$[b_n^e, \bar{b}_n^e)$	$[\bar{b}_n^e, \underline{s}_n^v]$	$(\underline{s}_n^v, s_n]$	(s_n, ∞)
$f_n(x_n)$	$\widetilde{g}_n^1(x_n)$	$\min\{\widetilde{g}_n^1(x_n), g(x_n)\}$	$g(x_n)$	$\min\{\widetilde{g}_n^2(x_n), g(x_n)\}$	$\widetilde{g}_n^2(x_n)$
$y_n^*(x_n)$	$B_n^e(x_n)$	$\{B_n^e(x_n), x_n\}$	x_n	$\{S_n(x_n), x_n\}$	$S_n(x_n)$

(b) Regular supplier

y_n^*	$(-\infty, b_n^r)$	$[b_n^r, \bar{b}_n^r)$	$[\bar{b}_n^r, \infty)$
$f_n^r(y_n^*)$	$\widetilde{g}_n^r(y_n^*)$	$\min\{\widetilde{g}_n^r(y_n^*), \alpha \mathbb{E} f_{n-1}(y_n^* - D_n)\}$	$\alpha \mathbb{E} f_{n-1}(y_n^* - D_n)\}$
$z_n^*(y_n^*)$	$B_n^r(y_n^*)$	$\{B_n^r(y_n^*), y_n^*\}$	y_n^*

As shown in Section 2.3, the optimal order and salvage decisions, in period n, can be determined in two steps. In the first step, determine the optimal adjusted inventory position $y_n^*(x_n)$ for the expedited channel (hence the optimal order amount from the expedited supplier or salvage amount) using Table 2.1 (a) after computing all necessary auxiliary functions and critical points; this also involves the calculation of the value function $f_n^r(y)$. In the second step, with y_n^* obtained from the first step, determine the optimal adjusted target $z_n^*(y_n^*)$ for the regular channel (which implies the optimal order amount from the regular supplier) using Table 2.1 (b).

Except for some highly artificial problem instances, the structure of the optimal policy further simplifies to a *three-region* policy and a *two-region* policy in parts (a) and (b), respectively. More specifically, the intermediate regions $[b_n^e, \bar{b}_n^e]$, $[\underline{s}_n^v, s_n^v]$ and $[b_n^r, \bar{b}_n^r]$ invariably vanish, because the functions $A_n^1(\cdot), A_n^2(\cdot)$ and $A_n^r(\cdot)$ have a single root.

2.4.1 Special Settings

In this subsection, we consider several special settings where the optimal policy structure takes on simpler forms.

2.4.1.1 No Fixed Costs, No Capacity Limits, Salvage Opportunities

In this setting $K_n = K_n^v = 0$ and $C_n = C_n^v = \infty$. Based on Table 1.4 (c) in Corollary 1.3, the optimal policy structure for systems without fixed costs and capacity limits is summarized by Table 2.2. The optimal inventory decision for the expedited channel (ordering from the expedited supplier or salvaging) follows a double "base stock"–type policy. The inventory position line is partitioned into three consecutive regions. In the left and right most regions, it is optimal to order up to a base-stock level B_n^e from the expedited supplier or to salvage down to a level S_n, respectively; in the middle region it is optimal to stay put. The optimal inventory decision for the regular supplier is a simple base stock policy with an order-up-to level B_n^r. Note that if salvaging is not allowed, the first stage policy is a simple base stock policy, i.e., $S_n = \infty$, eliminating the last column in subtable (a).

Table 2.2: Optimal policy structure for systems without fixed costs and capacity limits

(a) Expedited supplier and salvage

x_n	$(-\infty, B_n^e)$	$[B_n^e, S_n]$	(S_n, ∞)
$y_n^*(x_n)$	B_n^e	x_n	S_n

(b) Regular supplier

y_n^*	$(-\infty, B_n^r)$	$[B_n^r, \infty)$
$z_n^*(y_n^*)$	B_n^r	y_n^*

2.4.1.2 Fixed Costs, No Capacity Limits, No Salvage.

In this setting $K_n > 0, K_n^v = 0$, $C_n = C_n^v = \infty$. This case was addressed by Sethi et al. (2003), when $l_r = 1$ and $l_e = 0$, but allowing for forecast updates. Based on Table 1.4 (a) from Corollary 1.3, the optimal policy structure is summarized by Table 2.3. Both channels take on the classical "(s, S)"–type policy. More specifically, the expedited channel adopts a (b_n^e, B_n^e) policy and the regular channel follows a (b_n^r, B_n^r) policy.

Table 2.3: Optimal policy structure for systems with fixed costs but without capacity limits

(a) Expedited supplier

x_n	$(-\infty, b_n^e)$	$[b_n^e, \infty)$
$y_n^*(x_n)$	B_n^e	x_n

(b) Regular supplier

y_n^*	$(-\infty, b_n^r)$	$[b_n^r, \infty)$
$z_n^*(y_n^*)$	B_n^r	y_n^*

2.4.1.3 No Fixed Costs, Capacity Limits, No Salvage.

In this setting, $K_n = K_n^v = 0$ and $C_n < \infty, C_n^v = \infty$. Based on Table 1.5 (c) from Corollary 1.4, under the optimal policy, both channels adopt a modified base-stock policy as in Table 2.4.

Table 2.4: Optimal policy structure for systems without salvage option

(a) Expedited supplier

x_n	$(-\infty, \bar{b}_n^e - C_n)$	$[\bar{b}_n^e - C_n, \bar{b}_n^e)$	$[\bar{b}_n^e, \infty)$
$y_n^*(x_n)$	$x_n + C_n$	$\bar{b}_n^e$	x_n

(b) Regular supplier

y_n^*	$(-\infty, \bar{b}_n^r - C_n)$	$[\bar{b}_n^r - C_n, \bar{b}_n^r)$
$z_n^*(y_n^*)$	$y_n^* + C_n$	$\bar{b}_n^r$

2.4.1.4 Fixed Costs and Capacity Limits, No Salvage.

In this setting $K_n > 0, K_n^v = 0$ and $C_n < \infty, C_n^v = \infty$. Based on Table 1.5 (a) from Corollary 1.4, the optimal policy structure for systems with fixed costs and capacity limits but without salvage option is summarized by Table 2.5, where

$$\mathbf{1}_{b_n^e}^+ = \mathbf{1}(b_n^e > \bar{b}_n^e - C_n),\ \mathbf{1}_{b_n^e}^- = \mathbf{1}(b_n^e < \bar{b}_n^e - C_n);\ \mathbf{1}_{b_n^r}^+ = \mathbf{1}(b_n^r > \bar{b}_n^r - C_n),\ \mathbf{1}_{b_n^r}^- = \mathbf{1}(b_n^r < \bar{b}_n^r - C_n).$$

The optimal policy structures for the expedited and regular suppliers are similar in form. There are four regions. In the far-left region, the optimal policy is to order as much as possible—up to the capacity limit, while in the far-right region it is optimal to stay put. In the remaining two intermediate regions, the optimal policy is less categorical: in the third region, it is optimal to either stay put or to place an order; in the second region, one of two cases prevails: in one case it is optimal to either order as much as possible or stay put and in the other it is optimal to order a quantity specified by the aforementioned functions, $\bar{B}^e_n(\cdot)$ or $\bar{B}^r_n(\cdot)$.

2.5 General Lead Times

In this Section, we address the general case, with arbitrary order lead time combinations $l_r > l_e$, rather than the *special* setting where $l_r - l_e = 1$. (In § 2.5.1, we also address general salvage lead time l_v.) Sheopuri et al. (2010) show that a *minimal* state description is of dimension Δl, even in the simplest of models, i.e., in the absence of any fixed order costs, capacity limits or salvage opportunities. One such "minimal" state description is $\mathcal{I}_t \equiv (I_t^{l_e}, I_t^{l_e+1}, \ldots, I_t^{l_r-1})$, where I_t^l = the net inventory level at the beginning of period t plus all outstanding orders that will arrive by the beginning of period $t+l$. While of dimension Δl, this state description is a major simplification, beyond the straightforward state description, which includes the inventory level and each of the $l_e + l_r$ outstanding orders with the two suppliers.

Note that $x_t = I_t^{l_r-1}$ represents the regular full inventory position at the beginning of period t, including *all* outstanding orders with the expedited and regular supplier. The fully optimal strategy is prohibitively difficult to compute, in particular when $\Delta l \geq 3$, say, and even if computable, it would be prohibitively difficult to implement.

For $\Delta l \geq 2$, we therefore propose the following upper and lower bound approximations and heuristics. For the sake of notational simplicity, we confine ourselves to infinite horizon models with stationary inputs (parameters and demand distributions).

Under the lead time pair (l_e, l_r), let $z^*(l_e, l_r)$ $[\pi^*(l_e, l_r)]$ denote the optimal cost value [policy] and $z(\pi | l_e, l_r)$ the cost value of an arbitrary policy π. The following bounds apply with $\hat{l}_r = l_e + 1$ and $\hat{l}_e = l_r - 1$.

$$LB \equiv z^*(l_e, \hat{l}_r) \leq z^*(l_e, l_r) \leq \min\{UB_1 \equiv z^*(\hat{l}_e, l_r), UB_2 \equiv z(\pi^*(\hat{l}_e, l_r)\ |l_e, l_r)\}. \quad (2.13)$$

The inequalities $LB \leq z^*(l_e, l_r) \leq UB_1$ follow from the simple sample path argument in Zipkin (2008b), Section 5. Since $\pi^*(\hat{l}_e, l_r)$ is a feasible policy under (l_e, l_r), $z^*(l_e, l_r) \leq UB_2$.

In settings where the policy $\pi^*(\hat{l}_e, l_r)$ is characterized by a few threshold parameters, a third upper bound $UB_3 \leq UB_2$ can be obtained by searching for the *best* parameter combination within the same policy class.

We illustrate this approach with a study of problem instances without fixed costs or salvage opportunities, but with supplier-specific capacity limits. The lead time pair $(l_e, l_r) = (2, 5)$.

Table 2.5: Optimal policy structure for systems without salvage option

(a) Expedited supplier

x_n	$(-\infty, \min\{\bar{b}_n^e - C_n, b_n^e\})$	$[\min\{\bar{b}_n^e - C_n, b_n^e\}, \max\{\bar{b}_n^e - C_n, b_n^e\})$	$[\max\{\bar{b}_n^e - C_n, b_n^e\}, \bar{b}_n^e)$	$[\bar{b}_n^e, \infty)$
$y_n^*(x_n)$	$x_n + C_n$	$\{x_n + C_n, x_n\}\mathbf{1}_{b_n^e}^- + \bar{B}_n^e(x_n)\mathbf{1}_{b_n^e}^+$	$\{\bar{B}_n^e(x_n), x_n\}$	x_n

(b) Regular supplier

y_n^*	$(-\infty, \min\{\bar{b}_n^r - C_n, b_n^r\})$	$[\min\{\bar{b}_n^r - C_n, b_n^r\}, \max\{\bar{b}_n^r - C_n, b_n^r\})$	$[\max\{\bar{b}_n^r - C_n, b_n^r\}, \bar{b}_n^r)$	$[\bar{b}_n^r, \infty)$
$z_n^*(y_n^*)$	$y_n^* + C_n$	$\{y_n^* + C_n, y_n^*\}\mathbf{1}_{b_n^r}^- + \bar{B}_n^r(y_n^*)\mathbf{1}_{b_n^r}^+$	$\{\bar{B}_n^r(y_n^*), y_n^*\}$	y_n^*

Setting	Capacity limits		LB		UB1			UB2			UB3				Single-Expedited		Single-Regular	
	Ce	Cr	mean	stderr	mean	stderr	Gap 1	mean	stderr	Gap 2	mean	stderr	Gap 3	savings	mean	stderr	mean	stderr
Direct comparison	6	6	**40.38**	0.06	**51.93**	0.10	25%	**47.33**	0.06	16%	**44.39**	0.07	9%	39%	**73.09**	0.27	**89.17**	0.71
	6	8	**40.29**	0.07	**51.31**	0.09	24%	**48.28**	0.06	18%	**44.42**	0.07	10%	26%	**73.09**	0.27	**60.12**	0.18
	6	10	**40.29**	0.07	**50.96**	0.09	23%	**48.36**	0.06	18%	**44.43**	0.07	10%	20%	**73.09**	0.27	**55.67**	0.13
	6	12	**40.29**	0.07	**50.73**	0.09	23%	**48.34**	0.06	18%	**44.42**	0.07	10%	18%	**73.09**	0.27	**53.89**	0.12
	8	8	**40.29**	0.07	**50.74**	0.09	23%	**48.18**	0.06	18%	**44.25**	0.07	9%	26%	**61.26**	0.05	**60.12**	0.18
	8	10	**40.29**	0.07	**50.56**	0.09	23%	**48.21**	0.06	18%	**44.28**	0.07	9%	20%	**61.26**	0.05	**55.67**	0.13
	8	12	**40.29**	0.07	**50.42**	0.09	22%	**48.25**	0.06	18%	**44.31**	0.07	10%	18%	**61.26**	0.05	**53.89**	0.12
	10	10	**40.29**	0.07	**50.31**	0.09	22%	**48.18**	0.06	18%	**44.23**	0.07	9%	21%	**60.93**	0.05	**55.67**	0.13
	10	12	**40.29**	0.07	**50.24**	0.09	22%	**48.19**	0.06	18%	**44.20**	0.07	9%	18%	**60.93**	0.05	**53.89**	0.12
	12	12	**40.29**	0.07	**50.17**	0.09	22%	**48.14**	0.06	18%	**44.18**	0.07	9%	18%	**60.92**	0.05	**53.89**	0.12
Total capacity = 6	6	6													**73.09**	0.27	**89.17**	0.71
	2	4	**58.49**	0.36	**87.34**	0.65	40%	**81.38**	0.52	33%	**71.18**	0.54	20%	3%				
	2.5	3.5	**59.75**	0.35	**87.91**	0.64	38%	**83.32**	0.47	33%	**69.92**	0.51	16%	4%				
	3	3	**61.40**	0.32	**88.89**	0.62	37%	**86.57**	0.43	34%	**69.51**	0.46	12%	5%				
	3.5	2.5	**63.10**	0.32	**90.11**	0.61	35%	**90.77**	0.39	36%	**69.38**	0.42	9%	5%				
	4	2	**64.97**	0.31	**91.47**	0.60	34%	**95.56**	0.35	38%	**69.63**	0.39	7%	5%				
Total capacity = 12	12	12													**60.92**	0.05	**53.89**	0.12
	3	9	**40.32**	0.07	**52.57**	0.10	26%	**49.42**	0.07	20%	**46.27**	0.09	14%	14%				
	4	8	**40.29**	0.07	**52.32**	0.10	26%	**48.90**	0.07	19%	**45.20**	0.08	11%	16%				
	5	7	**40.29**	0.07	**52.09**	0.10	26%	**48.53**	0.07	19%	**44.67**	0.07	10%	17%				
	6	6	**40.38**	0.06	**51.93**	0.10	25%	**47.33**	0.06	16%	**44.39**	0.07	9%	18%				
	7	5	**40.83**	0.06	**52.54**	0.11	25%	**46.72**	0.05	13%	**43.64**	0.07	7%	19%				
	8	4	**42.78**	0.06	**54.26**	0.09	24%	**55.21**	0.03	25%	**43.20**	0.06	1%	20%				
	9	3	**46.57**	0.05	**57.85**	0.10	22%	**66.35**	0.02	35%	**46.59**	0.05	0%	14%				

Note. Gaps defined as $\frac{UB-LB}{(UB+LB)/2}$. Parameters: $c = 5, SL = 95\%\ (h = 5), \mu = 5, \sigma = 2, (l_e, l_r) = (2, 5)$. Mean costs over 1000 simulation runs, each of 1000 periods, with "stderr" representing the standard errors.

Table 2.6: Dual sourcing with general lead times: impact of capacity limits and various bounds

The first panel in Table 2.6 investigates instances in which each supplier has a capacity of 6—only 20% above the mean demand—or larger, so that single sourcing from either supplier is feasible. The second and third panels maintain an aggregate capacity of 6 and 12, respectively, under various allocations to the individual suppliers; in some cases, given this limited capacity, only dual sourcing or sourcing from one of the suppliers is feasible. One period demands have mean [standard deviation] of 5 [2] and a Normal distribution, truncated at 0 and 10.

We display the estimated cost per period under LB, UB_1, UB_2 and UB_3 along with associated standard errors from which confidence intervals can be constructed. As shown, both under LB and UB_1, a pair of modified base-stock policy is optimal, under dual sourcing. We conclude that uniformly, UB_3 is the best of the three upper bounds. The remaining gap between LB and UB_3 is 10%, on average. This is an inflated representation of the true gap since we omit the variable procurement cost, at its base price c_r at the regular supplier, from both cost measures. We conjecture that UB_3 is much closer to the optimal cost value than LB, since it represents the best policy within the class known to contain the optimal policy under consecutive lead times. As a consequence, we report, in the numerical studies in Section 2.6, either the exact optimal cost value—when computable—or UB_3. Throughout, "savings" refers to the % gap between the cost under dual sourcing versus the best single sourcing policy.

2.5.1 Incompatible Salvage Lead Time: the Case $l_v < l_e$

As mentioned, in the presence of salvage opportunities, often $l_v = l_e$ (or $l_v = l_r$) when salvaging involves return shipments to the supplier. Sometimes, $l_v < l_e$, in which case a heuristic approach is required, similar to when order lead times fail to be consecutive. Analogous to the above, we propose as a lower bound, reducing (l_e, l_r) to $(l_v, l_v + 1)$ and finding the optimal policy under these lead times. Upper bounds UB_1, UB_2 and UB_3 are obtained by increasing (l_v, l_e) to $(l_r - 1, l_r - 1)$ as above.

2.6 Numerical Studies: Managerial Insights

In this section, we report on three studies conducted to investigate the impact of various model parameters on the system performance, with a particular focus on the benefits of dual sourcing and the specific ways the expedited supplier is used as a complement to the regular supplier. Some of the studies employ general lead time combinations to appreciate how they impact on the above measures.

Systems with fixed costs (but no capacity limits or salvage opportunities):

As explained, under general lead times, we report on heuristic UB_3, the best in the class of single index pairs of (s, S) policies (s_e, S_e, s_r, S_r), which we have shown to contain an optimal policy when lead times are consecutive. Table 2.7 exhibits our results for a set of 30 instances. The first 7 columns specify the input parameters, in particular: (i) the expedited lead time l_e; (ii) the regular lead time l_r; (iii) the *differential* between the per unit cost rates charged by the expedited and regular suppliers, c; (iv) K, the fixed cost incurred for any order; (v) $SL = p/(h + p)$, the service level in a single source setting when governed by a base-stock level, all with $h = 5$; (vi) [(vii)] the mean [standard deviation] of the Normally distributed one-period demand. The remaining three vertical panels exhibit the results under the optimal dual sourcing policy, within the above class of policies, and the optimal policy under single sourcing with each of the suppliers. Beyond the optimal policy parameters, the mean cost columns exhibit the average over 1000 replicas of the per-period cost under a planning horizon of $N = 1000$ periods, with associated standard errors. The dual sourcing panel, in addition, contains % Savings = the percentage savings realized by the dual sourcing policy, compared with the optimal single sourcing policy; % Expedited = the *percentage* of sales that is procured from the expedited supplier; and *Expedite Freq* = the percentage of periods in which an order is placed with the fast supplier.

Varying parameters	Parameters									Dual Sourcing									Single Sourcing - Expedited				Single Sourcing - Regular			
	le	*lr*	*c*	*K*	*h*	*p*	*SL*	*μ*	*σ*	mean	stderr	savings	*se*	*Se*	*sr*	*Sr*	% Exp.	Exp. Freq	mean	stderr	*se*	*Se*	mean	stderr	*sr*	*Sr*
Lead time difference lr-le	2	**3**	5	10	5	95	95%	5	2	50.3	0.06	1.1%	17	20	24	27	1.3%	1.7%	70.4	0.05	19	21	50.8	0.06	24	27
	2	**4**	5	10	5	95	95%	5	2	54.5	0.07	2.2%	24	25	30	31	6.6%	15.9%	70.4	0.05	19	21	55.7	0.08	30	33
	2	**5**	5	10	5	95	95%	5	2	57.1	0.08	5.1%	29	30	35	36	6.6%	15.9%	70.4	0.05	19	21	60.2	0.11	36	38
	2	**6**	5	10	5	95	95%	5	2	59.1	0.07	7.6%	33	34	38	39	14.2%	31.0%	70.4	0.05	19	21	64.0	0.12	41	44
	2	**7**	5	10	5	95	95%	5	2	60.8	0.07	10.2%	38	39	43	44	14.2%	31.0%	70.4	0.05	19	21	67.8	0.12	47	50
Lead times lr, le (fixing lr-le)	**2**	**3**	5	10	5	95	95%	5	2	50.3	0.06	1.1%	17	20	24	27	1.3%	1.7%	70.4	0.05	19	21	50.8	0.06	24	27
	3	**4**	5	10	5	95	95%	5	2	55.4	0.07	0.6%	23	26	30	33	1.3%	1.7%	75.8	0.07	24	27	55.7	0.08	30	33
	4	**5**	5	10	5	95	95%	5	2	59.7	0.11	0.7%	28	31	35	38	1.3%	1.7%	80.7	0.08	30	33	60.2	0.11	36	38
	5	**6**	5	10	5	95	95%	5	2	63.7	0.11	0.4%	34	37	41	44	1.3%	1.7%	85.2	0.11	36	38	64.0	0.12	41	44
	6	**7**	5	10	5	95	95%	5	2	67.5	0.12	0.4%	39	43	46	50	1.8%	1.8%	88.9	0.12	41	44	67.8	0.12	47	50
Variable cost c	2	5	**1**	10	5	95	95%	5	2	50.4	0.05	0.0%	19	21	19	21	100.0%	93.6%	50.4	0.05	19	21	60.2	0.11	36	38
	2	5	**3**	10	5	95	95%	5	2	56.3	0.08	6.4%	28	29	33	34	14.2%	31.0%	60.4	0.05	19	21	60.2	0.11	36	38
	2	5	**5**	10	5	95	95%	5	2	57.1	0.08	5.1%	29	30	35	36	6.6%	15.9%	70.4	0.05	19	21	60.2	0.11	36	38
	2	5	**10**	10	5	95	95%	5	2	58.7	0.09	2.4%	29	30	36	37	2.5%	6.8%	95.4	0.05	19	21	60.2	0.11	36	38
	2	5	**20**	10	5	95	95%	5	2	59.8	0.10	0.7%	28	30	36	38	0.4%	0.8%	145.4	0.06	19	21	60.2	0.11	36	38
Fixed cost K	2	5	5	**0**	5	95	95%	5	2	44.2	0.07	12.8%	29	29	34	34	15.9%	49.9%	60.9	0.05	21	21	50.7	0.10	38	38
	2	5	5	**5**	5	95	95%	5	2	51.3	0.08	7.6%	28	29	33	34	14.2%	31.0%	65.7	0.05	19	21	55.5	0.11	36	38
	2	5	5	**10**	5	95	95%	5	2	57.1	0.08	5.1%	29	30	35	36	6.6%	15.9%	70.4	0.05	19	21	60.2	0.11	36	38
	2	5	5	**20**	5	95	95%	5	2	67.0	0.09	0.6%	26	33	34	41	1.0%	0.7%	79.0	0.04	18	22	67.4	0.08	35	41
	2	5	5	**40**	5	95	95%	5	2	76.8	0.08	0.2%	26	34	34	42	1.0%	0.6%	89.7	0.05	17	26	77.0	0.08	34	43
Service level (via p)	2	5	5	10	5	**7.5**	**60%**	5	2	31.7	0.03	0.1%	18	24	27	33	0.3%	0.2%	50.5	0.02	12	17	31.7	0.03	27	33
	2	5	5	10	5	**20**	**80%**	5	2	43.2	0.05	0.1%	22	26	31	35	0.2%	0.2%	58.6	0.03	15	18	43.2	0.05	31	35
	2	5	5	10	5	**45**	**90%**	5	2	50.9	0.05	2.7%	28	29	34	35	6.6%	15.9%	64.9	0.04	17	20	52.4	0.07	33	37
	2	5	5	10	5	**95**	**95%**	5	2	57.1	0.08	5.1%	29	30	35	36	6.6%	15.9%	70.4	0.05	19	21	60.2	0.11	36	38
	2	5	5	10	5	**495**	**99%**	5	2	68.9	0.12	8.2%	31	32	36	37	14.2%	31.0%	81.0	0.11	21	23	75.0	0.17	39	42
Demand volatility sigma	2	5	5	10	5	95	95%	5	**0.1**	15.0	0.00	0.0%	16	16	31	31	0.0%	0.0%	40.0	0.00	12	16	15.0	0.00	27	31
	2	5	5	10	5	95	95%	5	**0.5**	22.5	0.02	0.6%	26	27	31	32	0.5%	2.3%	45.3	0.01	15	17	22.7	0.03	30	32
	2	5	5	10	5	95	95%	5	**1**	34.6	0.03	2.0%	27	28	32	33	4.8%	15.8%	53.0	0.02	17	18	35.3	0.05	32	34
	2	5	5	10	5	95	95%	5	**1.5**	46.0	0.06	4.0%	28	29	34	35	3.1%	9.1%	62.0	0.05	18	19	47.9	0.08	34	36
	2	5	5	10	5	95	95%	5	**2**	57.1	0.08	5.1%	29	30	35	36	6.6%	15.9%	70.4	0.05	19	21	60.2	0.11	36	38

Note. Cost estimates are based on optimal DP solution evaluated with 1000 simulation runs of 1000-period problems; (s, S) policies are restricted to integer values; for dual sourcing, a pair of (s, S) policy is considered under the restriction $S_e - s_e = S_r - s_r$.

Table 2.7: Dual sourcing with fixed cost: impact of various parameters

The first horizontal panel explains the impact of an increase in lead time differences, fixing $l_e = 2$. The benefit of dual sourcing increases from 1.1% to 10.2%; remarkably the latter very significant savings are achieved even though the expedited supplier is used infrequently—less than once in every three periods—and for only 14% of the total purchase volume. In the second panel, we consider consecutive lead times but increase the lead time values. This panel confirms that the benefit of dual sourcing are relatively small when the *lead time difference* is small and decreasing as the absolute lead times increase. Correspondingly, the expedited supplier is used very sparsely.

The next two panels exhibit the impact of the K/c ratio. In the third [fourth] panel, the c [K] value is varied leaving the K [c] value fixed. Using our base instance as the benchmark, when the c value is reduced, the expedited supplier is, understandably, used for a larger portion of the purchase volume, and the benefits over single sourcing from that supplier decrease as well.

Finally, the service level SL and the demand volatility have intuitive impacts: as either one increases, the expedited supplier is used more frequently and the benefits of dual sourcing increase also. This is exhibited in the last two panels.

Supplier-specific capacity limits:

Varying parameter	Ce	Cr	Dual sourcing						Dual sourcing w/o salvage					Single Exp.		Single Reg.	
			cost mean	stderr	savings	% Exp	Freq Exp	Freq salvage	cost mean	stderr	savings	% Exp	Freq Exp	cost mean	stderr	cost mean	stderr
Exp capacity	2	10	414.9	0.9	17.4%	6.6%	17.1%	14.3%	422.8	0.9	15.9%	6.6%	17.2%	5170.7	6.2	502.4	1.2
	4	10	370.2	0.8	26.3%	11.4%	15.5%	14.3%	378.0	0.8	24.8%	11.5%	15.5%	2973.7	6.2	502.4	1.2
	6	10	342.2	0.7	31.9%	13.8%	13.6%	14.3%	350.0	0.7	30.3%	14.1%	13.9%	964.5	4.3	502.4	1.2
	8	10	323.1	0.6	32.0%	15.4%	12.7%	14.3%	330.9	0.6	30.4%	15.7%	13.0%	475.2	1.4	502.4	1.2
Base case	10	10	308.6	0.6	19.6%	15.9%	11.0%	14.3%	316.4	0.6	17.6%	16.2%	11.3%	383.9	0.9	502.4	1.2
Reg capacity	10	12	305.8	0.6	20.4%	15.2%	10.2%	14.3%	313.6	0.6	18.3%	15.5%	10.5%	383.9	0.9	444.4	1.0
	10	14	304.2	0.5	20.8%	15.1%	10.1%	14.3%	312.0	0.5	18.7%	15.3%	10.5%	383.9	0.9	412.7	0.8
	10	16	303.4	0.5	21.0%	14.7%	10.1%	14.3%	311.2	0.5	18.9%	14.9%	10.5%	383.9	0.9	393.0	0.8
	10	18	302.9	0.5	20.2%	13.8%	10.1%	14.3%	310.7	0.5	18.1%	14.0%	10.5%	383.9	0.9	379.5	0.7
Capacity limits	2	6	587.9	1.7	55.7%	15.1%	39.1%	14.2%	595.7	1.7	55.1%	15.1%	39.1%	5170.7	6.2	1326.8	5.1
	4	8	393.6	0.9	38.7%	13.7%	18.4%	14.3%	401.4	0.9	37.5%	13.7%	18.5%	2973.7	6.2	642.4	1.9
	6	10	342.2	0.7	31.9%	13.8%	13.6%	14.3%	350.0	0.7	30.3%	14.1%	13.9%	964.5	4.3	502.4	1.2
	8	12	317.8	0.6	28.5%	13.3%	11.0%	14.3%	325.6	0.6	26.7%	13.6%	11.3%	475.2	1.4	444.4	1.0
	10	14	304.2	0.5	20.8%	15.1%	10.1%	14.3%	312.0	0.5	18.7%	15.3%	10.5%	383.9	0.9	412.7	0.8
	12	16	296.9	0.5	14.3%	15.2%	10.1%	14.3%	304.7	0.5	12.0%	15.4%	10.5%	346.4	0.7	393.0	0.8

Note. Parameters: $K_e = K_r = 10, K_v = 2; c_e = 15, c_r = 10, c_v = 6; C_v = 10; (l_e, l_r) = (4, 5); SL = 95\%$ $(h = 5); \alpha = 1; \mu = 5, \sigma = 2$. Cost estimates are based on 10,000 simulation runs of 30-period problems; "savings" refers to the % cost gap between dual sourcing and best single sourcing.

Table 2.8: Dual sourcing under fully general model: impact of capacity limits

In Table 2.8, we explore systems in the fully general model in which the two suppliers have different capacity levels; under consecutive lead times, such systems can be solved by the DP method of Section 2.3, even though the special policy structure in Section 2.4 cannot be guaranteed. The first and fourth panels in the table investigate how a smaller expedited supplier may improve the performance of the system. In the first [fourth] panel, the capacity level [advantage] of the regular supplier is kept constant at 10 [4] units. In the third panel, the capacity of the expedited [regular] supplier is kept constant [increased].

The results in the first panel show that the availability of an expedited supplier has major benefits even if she has limited capacity; for example, even when $C_r = 10$ and $C_e = 2$ the cost savings are 17.4%. In general, the benefits of dual sourcing are larger in capacitated systems as opposed to uncapacitated systems, see Table 2.7 (above). As can be expected, additional capacity at either supplier is always beneficial but the marginal benefit decreases rapidly. The same is true for marginal *aggregate* capacity, see the third panel.

Table 2.6 in the previous section sheds light on the impact of differentiated capacity levels in systems with *non-consecutive* lead times. In the first panel, where aggregate capacity is ample, but that of an individual supplier limited, the benefits of dual sourcing are very extensive, ranging between 18% and 39% under the heuristic UB_3. Note, once again, that these major savings are achieved even though the expedited supplier is used, for, on average, only one sixth of the total volume. The second panel has instances where dual sourcing is essential; neither one of the suppliers could be used by herself. In the third panel, one of the suppliers is large enough to procure by herself; the savings due to dual sourcing vary between 14% and 20% in this panel, even though the expedited supplier is used for less than a fifth of the purchase volume.

General salvage lead times:

A final study reported in Appendix B.2 evaluates the benefits of a salvage opportunity in systems with consecutive lead times, fixed order costs and supplier-specific capacity limits. The systems can be solved with the DP method of Section 2.3. Varying model parameters systematically, the salvage opportunity improves costs, on average, by 2% but sometimes by as much as 8.2%. This is in contrast to the special case in Janakiraman and Seshadri (2017) where no benefits were found.

2.7 Concluding Remarks

This paper addresses a general periodic review model, to identify an optimal procurement policy in the presence of *two* suppliers, differentiated by their lead time and per-unit cost price. The model allows for salvage opportunities, capacity limits and fixed costs associated with orders and salvage transactions. We have provided a full characterization of the optimal procurement strategy in the general model as well as various special cases that arise when only part of the above complications prevail. Our exact results are confined to the case where the lead times of the two suppliers differ by a single period only (Even in the base model without fixed cost, capacity limits or salvage opportunities, it is well known that only a one-period lead time difference allows for an optimal policy that acts on a single inventory measure.) However, our structural results for this special case suggest effective heuristics for general lead time combinations, as demonstrated in Section 2.5.

Indeed, significant cost savings can be achieved with dual sourcing. Remarkably, such savings can be obtained even when the expedited supplier is only used for a small part, say 5% of all procurement. The availability of the fast(er) supplier allows one to forgo major inventory investments to achieve a given service level or to prevent costly stockouts under a given inventory investment.

In practice, a second expedited supplier may only be available if a minimum sales volume can be guaranteed, or a minimum frequency with which orders are placed. Future work should address such "participation" constraints.

It is also important to extend our results to settings with more than two suppliers. Generalizations of our structural results are possible when the lead time of any given supplier differs from that of the next fastest supplier by a single period. This extension was carried out by Feng et al. (2005) in the base model. Moreover, we believe that these results continue to suggest effective heuristic strategies under general lead time differences.

Chapter 3

Ride-Hailing Networks with Strategic Drivers: The Impact of Platform Control Capabilities on Performance

3.1 Introduction

We are motivated by the emergence of ride-hailing platforms such as Uber, Lyft and Gett, that face the problem of matching service supply (drivers) with demand (riders) over a spatial network. We study the impact of operational platform controls on the equilibrium performance of such networks, focusing on the interplay with two practically important challenges for this matching problem: (i) Significant demand imbalances prevail across network locations for extended periods of time, as commonly observed in urban areas during rush hour (see Figures 3.1 and 3.2), so that the natural supply of drivers dropping off riders at a location either falls short of or exceeds the demand for rides originating at this location. These mismatches adversely affect performance as they lead to lost demand, to drivers idling at low-demand locations, and/or to drivers repositioning, i.e., traveling without serving a rider, from a low- to a high-demand location. (ii) Drivers are self-interested and decide strategically whether to join the network, and if so, when and where to reposition, trading off the transportation time and cost involved against their matching (queueing) delay at their current location. These decentralized supply decisions may not be optimal for the overall network.

Motivated by these challenges, we consider two platform control levers, demand-side *admission control* and supply-side *capacity repositioning*. Admission control allows the platform to accept or reject rider requests based on their destinations, thereby affecting not only the supply of cars through the network, but also the queueing delays of drivers to be matched at lower-demand locations, and in turn, their decisions to reposition to higher-demand locations. Repositioning control allows the platform to direct drivers to go where they are needed the most, rather than having to incentivize them to do so, thereby alleviating the demand/supply imbalance in the network.

To evaluate these control levers we study the steady-state behavior of a deterministic fluid model of a two-location ride-hailing loss network with four routes (two for local and two for cross-location traffic) in a game-theoretic framework. Demand for each route is characterized by an arrival rate and a deterministic travel time. Three parties interact through this network. Riders generate demand for each route, paying a fixed price per unit travel time. Prices are fixed throughout, and for simplicity assumed to be route-independent, though this is not necessary for our analysis. Drivers decide, based on their outside opportunity cost and their equilibrium expected profit rate from participation, whether to join the network, and if so, whether to reposition from one location to the other, rather than idling. Drivers have heterogeneous opportunity costs but homogenous transportation costs; therefore, participating drivers are homogeneous in their strategies and in the eyes of the platform. The platform receives a fixed commission (fraction) of the fare paid by the rider and seeks to maximize its commission revenue. We consider three control regimes: (i) *Centralized Control* with respect to admission and driver repositioning decisions; (ii) *Minimal Control*: no admission control and decentralized (driver) repositioning decisions; (iii) *Optimal Admission Control* at each location and decentralized repositioning.

Main Results and Contributions. First, we propose a novel model that accounts for the network structure and flow imbalances, the driver incentives, and the interplay of queueing, transportation times, and driver repositioning decisions—all essential features of ride-hailing platforms.

Second, we fully characterize the steady-state system equilibria, including the drivers' repositioning incentive-compatibility conditions, for the three control problems outlined above, relying on the analysis of equivalent capacity allocation problems. The solutions provide insights on how and why platform control impacts the system performance, both financially, in terms of platform revenue and per-driver profits, and operationally, in terms of rider service levels and driver participation, repositioning and queueing. One immediate finding is that when capacity is moderate, accepting rider requests in a pro-rata (or FIFO) manner need not be optimal, and neither is the practice of accepting rider requests at a location as long as it has available driver supply.

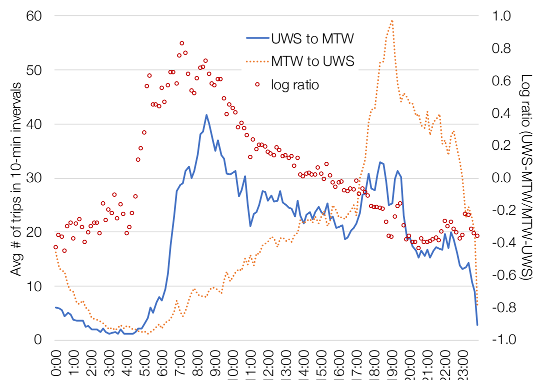
Third, we glean new insights on the interplay between the platform's admission control and drivers' strategic repositioning decisions. Specifically, we show that the platform may find it optimal to *strategically reject demand* at the low-demand location, even though there is an excess supply of drivers, so as to induce repositioning to the high-demand location. We provide intuitive necessary and sufficient conditions for this policy feature in terms of the network characteristics and the driver economics. This deliberate degradation of the rider service level at one location yields more efficient repositioning, and in turn a higher service level at the other location. Thus, operational levers, as opposed to location-specific or origin-destination pricing, can affect repositioning behavior.

Fourth, we derive upper bounds on the performance benefits that the platform and the drivers enjoy due to increased platform control capability. These bounds show that these benefits can be very significant for the platform, of the order of 50%, 100% or even larger improvements, especially when the network operates in a moderate capacity regime, so there are tangible supply/demand trade-offs across the network, and when there are significant cross-location demand imbalances. Per-driver benefits are also most significant when capacity is moderate, but are less significant in their magnitude because driver participation decisions are endogenous—so, if platform controls increase the per-driver profit significantly at a given participation level, more drivers choose to participate, reducing the equilibrium per-driver benefit.

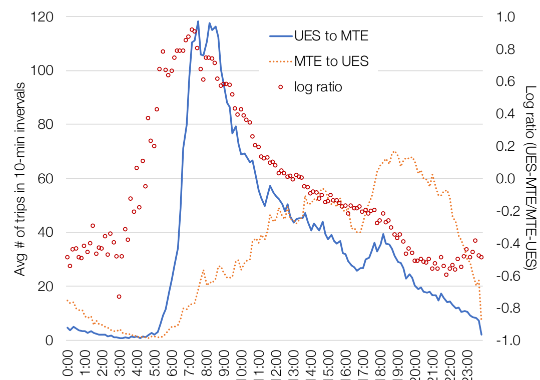
Flow Imbalances: Example Manhattan. We illustrate the magnitude and duration of the demand imbalances alluded to above with data on flow imbalances at the route- and location-level, for taxi rides in Manhattan. These flow imbalances are derived from the publicly available New York City TLC (Taxi & Limousine Commission) Trip Record Data.[1] These data record for each taxi ride the start and end time, the geo pickup and dropoff locations, and the fare paid. Although the data report censored demand (i.e., only realized trips but not rider demand that was not filled), we believe the (uncensored) demand

[1] http://www.nyc.gov/html/tlc/html/about/trip_record_data.shtml

imbalances are likely of the same order of magnitude as the (censored) flow imbalances; indeed, in the high-traffic direction the flow likely underestimates demand by a larger margin than in the opposite direction. We further note that, although our data do not include trips on ride-hailing platforms (this information is not public), these platforms likely experience similar imbalances.



(a) Upper West Side–Midtown West, May 2016

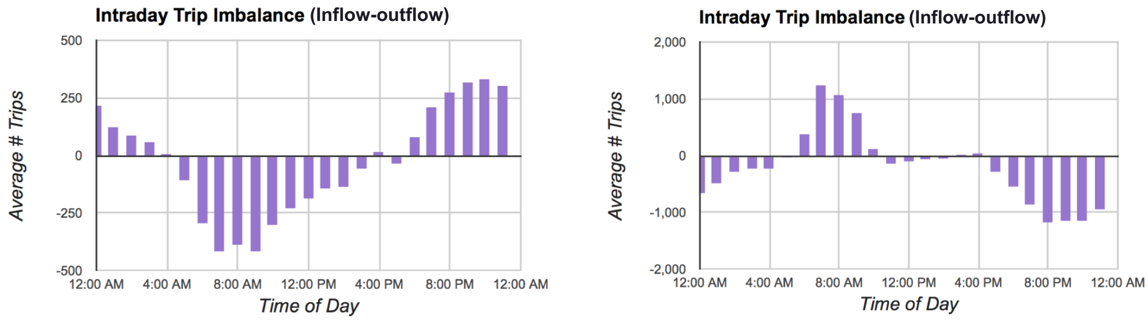


(b) Upper East Side–Midtown East, May 2016

Figure 3.1: Route-level flow imbalances in Manhattan

Figure 3.1 illustrates the route-level realized *flow* imbalances for two origin-destination pairs in Manhattan, NY. Chart (a) shows the average number of trip originations per 10 minute-interval between Upper West Side (UWS) and Midtown West (MW) (to and fro), and the logarithm of the ratio of number of trip originations between the two locations per time bin; for each time bin the average is computed over all weekdays for one month. Chart (b) shows the same quantities between the Upper East Side and Midtown East. We observe a pronounced imbalance of almost one order of magnitude (about 10x) in the morning rush hour and about half an order of magnitude (about 3x) in the evening rush hour in the reverse direction. Therefore, focusing on origin-destination pairs in the network, we observe strong flow imbalances, so the cross traffic between such two locations will not replenish sufficient capacity for the originating trips in the high-demand location (UWS in the morning) and will likely oversupply the low-demand location (MW in the morning).

Figure 3.2 shows that the realized flow imbalances persist even after aggregation to the location-level. That is, the flow of other trips being completed in the high-demand location, e.g., morning arrivals into the UWS from other locations, is insufficient to supply the necessary capacity for the UWS to MW trips: Charts (a) and (b) show the net imbalances (total number of drop-offs minus total number of pickups) at two locations in Manhattan, over the course of a day. These net imbalances are statistically significant and exhibit strong and different intraday patterns.



(a) Upper West Side, May 2016

(b) Midtown, May 2016

Figure 3.2: Location-level flow imbalances in Manhattan

Taken together, these observations on route- and location-level flow imbalances suggest that self-interested strategic drivers actively reposition towards high-demand locations so as to supply the needed capacity that would otherwise not be available at high-demand locations/time-periods. They also suggest that platform controls may be able to use destination information to direct capacity where it is needed (something that does not happen in the taxi market), and to incentivize or completely control a more efficient driver repositioning strategy.

Finally, we note that both Figure 3.1 and Figure 3.2 show that imbalance periods persist for multiple time bins, lasting typically for a couple of hours. To contrast, the typical transportation times between these location pairs are of the order of 10-15min, suggesting that network transients may settle down quickly relative to the duration of these imbalances, which, in part, motivates our focus on the steady state fluid model equations as opposed to a study of the transient process itself.

The paper proceeds as follows. This section concludes with a brief literature review. In §3.2 we outline the system model and formulate the three optimization problems that correspond to the three control regimes mentioned above. §3.3 studies the centralized control regime, and §3.4 studies the two control regimes where drivers make self-interested repositioning decisions. Finally, §3.5 derives upper bounds on the performance benefits to the platform and to the drivers due to increased platform control, and offers some numerical illustrations.

Related Literature. This paper is related to the growing literature on the sharing economy, with a specific focus on the operations of ride-hailing platforms. With a couple of exceptions discussed below, this literature can be broadly grouped into two streams: papers that study (i) a single location (or the system as a whole) with driver incentives, and (ii) a network of locations without strategic drivers or driver incentives. In contrast to these two streams, this paper studies a network model with driver incentives.

In single-location models with incentives, the strategic driver decision is whether to enter the system; driver repositioning is a network consideration that is not captured in these models. Several papers focus on the impact of *surge or dynamic pricing* and its impact on equilibrium performance over static pricing policies. Banerjee et al. (2016) study a stochastic single-location queueing model with price-sensitive riders (demand) and drivers (capacity) and find (consistent with the literature on revenue maximization in large scale queues) that static pricing is asymptotically (first order) optimal. Cachon et al. (2017) compare five pricing schemes, with increasing flexibility in setting consumer price and provider wage, and find that dynamic (surge) pricing significantly increases the platform's profit and benefits drivers and riders. Castillo et al. (2017) show analytically and empirically that surge pricing avoids the "wild goose chase" effect when capacity is scarce—the phenomenon where drivers have to travel far distances to pick up customers. Chen and Sheldon (2015) use Uber data to show empirically that surge pricing incentivizes drivers to work longer and hence increases systemwide efficiency. Hall et al. (2017) in a different empirical study based on Uber data, find that the driver supply is highly elastic to wage changes; a fare hike only has a short-lived effect on raising the driver hourly earnings rate, and is eventually offset by endogenous entry. As mentioned above, our results confirm this empirical phenomenon.

Taylor (2017) and Bai et al. (2017) both study the price and wage decisions of on-demand service platforms that serve delay-sensitive customers and earnings-sensitive independent providers. Taylor (2017) examines the impact of delay sensitivity and provider independence on the optimal price and wage, and shows that the results are different when uncertainty in customer and provider valuations prevails. Bai et al. (2017) extend the model in Taylor (2017) by also considering the impact of demand and service rate and potential capacity, as well as total welfare (in addition to platform profit) as the system performance measures. Some papers consider strategic, self-interested agents (capacity) without queueing considerations. Gurvich et al. (2016) allow the platform to determine the capacity size (staffing), system-state-contingent compensation and a cap on effective capacity. Hu and Zhou (2017) study the optimal price and wage decisions of an on-demand matching platform. They show that a commission contract can be optimal or near-optimal under market uncertainty. Benjaafar et al. (2015) study peer-to-peer product sharing, where individuals with varying usage levels make decisions about whether or not to own; the roles of market participants' as provider or consumer are endogenous, unlike in the above papers (and ours).

In the second stream that studies ride-hailing networks without incentives, papers primarily either focus on routing of cars or on matching riders to drivers. Most papers rely on the analysis of an approximating deterministic fluid model. An important paper is Braverman et al. (2017); they prove an asymptotic limit theorem that provides justification for the use of a deterministic fluid network model (such as the one in this paper), and then study the transient optimization problem of empty car repositioning under centralized control. In contrast to this paper, Braverman et al. (2017) do not consider admission control and focus on centralized repositioning decisions. The underlying model is the BCMP product-form network. Iglesias et al. (2017) also consider centralized matching and repositioning decisions in the context of a BCMP network model. Similar to our paper, both papers model a closed loss network. Banerjee et al. (2017) study optimal dynamic pricing of a vehicle sharing network, and show that state-independent (demand) prices derived through a convex relaxation are near-optimal when capacity grows large; the paper provides explicit approximation guarantees for systems with finite size. He et al. (2017) study the problem of designing the geographical service region for urban electric vehicle sharing systems.

Hu and Zhou (2016) consider dynamic matching for a two-sided, heterogeneous-type, discrete-time system with random arrivals and abandonment. They provide conditions under which the optimal matching policy is a priority rule. Ozkan and Ward (2017) study dynamic matching on a network, and through a linear programming relaxation propose an asymptotically optimal matching policy that outperforms the commonly-used heuristic of matching a rider with the nearest available driver. Their model differs from the one in Hu and Zhou (2016) in that it (i) assumes probabilistic rather than deterministic matching, and (ii) establishes that customers and drivers in the same location may not be matched. Feng et al. (2017) compare customers' average waiting time under two booking systems, on-demand versus street hailing, assuming all trips occur on a circle. Caldentey et al. (2009), Adan and Weiss (2012) and Bušić et al. (2013), among others, study multi-class matching in the context of the infinite bipartite matching model. In the broader dynamic queue matching context, Gurvich and Ward (2014) prove the asymptotic optimality of a discrete review matching policy for a multi-class double-sided matching queue.

In contrast to these two streams, this paper studies a network model with strategic supply, i.e., drivers decide whether to join the system and where to provide service, given the incentives offered by the platform. Bimpikis et al. (2017) study how a ride-sharing platform with strategic drivers should price demand and compensate drivers across a network to optimize its profits, and show that the platform's profits and consumer surplus increase when demands are more balanced across the network; they focus on a discrete-time multi-period model without driver queueing effects, where driver movement between any two locations is costless and takes one time-period. Guda and Subramanian (2017) consider strategic surge pricing and market information sharing in a two-period model with two local markets without cross-location demand (there is no queueing and driver travel takes one period). Buchholz (2017) empirically analyzes, using the NYC taxi data, the dynamic spatial equilibrium in the search and matching process between strategic taxi drivers and passengers. His counterfactual analysis shows that even under optimized pricing, performance can further be improved significantly by more directed matching technology, which supports the value of studying the impact of operational controls.

In contrast to Bimpikis et al. (2017), Guda and Subramanian (2017), and Buchholz (2017), we focus on the impact of operational controls, as opposed to pricing, on system performance, and provides insights and guidelines on the optimal operations of ride-hailing networks. Consistent with the pricing results of Bimpikis et al. (2017), we find that cross-network demand imbalances crucially affect performance, and that the impact of operational platform controls increases in such imbalances.

3.2 Model and Problem Formulations

We consider a deterministic fluid model of a ride-hailing network in steady state. Three parties interact through this network, riders generate demand for rides, drivers supply capacity for the rides, and the platform is instrumental in matching supply with demand. §3.2.1 introduces the operational and financial model primitives; §3.2.2 describes the information that is available to the parties and the controls that determine how supply is matched with demand; §3.2.3 formulates the optimization problems for the three control regimes that we study in this paper. Finally, §3.2.4 explains how to reformulate the problems specified in §3.2.3 in terms of capacity allocation decisions.

Such an approximating fluid model was rigorously justified in Braverman et al. (2017) for a system where drivers were not acting strategically; an adaptation of their arguments could be used in our setting as well. We will focus directly on a set of (motivated) steady state flow equations.

3.2.1 Model Primitives

We describe the model primitives for the network, the riders, the drivers and the platform.

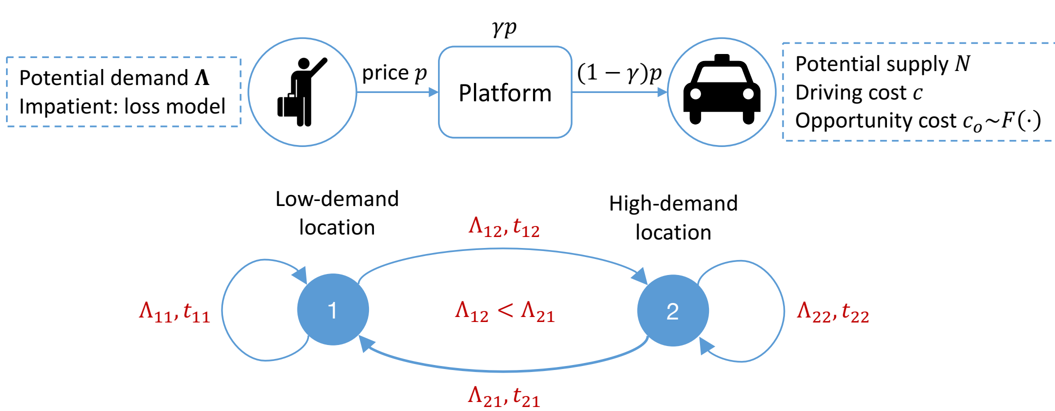


Figure 3.3: Model primitives

Network. The network has two locations (nodes), indexed by $l = 1, 2$, and four routes (arcs), indexed by lk for $l, k \in \{1, 2\}$. Figure 3.3 depicts the network schematic and the model primitives that we define in this section. We denote by t_{lk} the travel time on route lk and by t the travel time vector. We impose no restrictions on travel times; specifically, we allow $t_{12} \neq t_{21}$, to reflect, for example, different uptown/downtown routes. The travel times are constant and, in particular, independent of the number of drivers that serve demand for the platform. This assumes that the number of drivers has no significant effect on road congestion and transportation delays.

Riders. Riders generate demand for trips. We assume that the platform charges a fixed price of \$$p$ per unit of travel time, uniform across all routes.; i.e., a rider pays for a route-lk trip a fee of \$$pt_{lk}$. Given the price p, the potential demand rate for route-lk trips is Λ_{lk}, and Λ denotes the potential demand rate vector. The platform keeps a portion γ of the total fee as commission and drivers collect the remainder. Riders are impatient, i.e., rider requests not matched instantly are lost. We focus on the case $\Lambda_{12} \neq \Lambda_{21}$, i.e., the cross-location demands are not balanced; without loss of generality, we make the following assumption.

Assumption 3.1 (Demand imbalance)**.** $0 < \Lambda_{12} < \Lambda_{21}$.

Drivers. Drivers supply capacity to the network. Let N be the pool of (potential) drivers, each equipped with one car (unit of capacity). Drivers are self-interested and seek to maximize their profit rate per unit time. They decide whether to join the network, and, if so, whether to reposition to the other network location, i.e., travel without serving a rider, in anticipation of a faster match.

Each driver has an idiosyncratic opportunity cost rate, denoted by c_o, that is assumed to be an independent draw from a common distribution F, which is assumed to be continuous with connected support $[0, \infty)$. Drivers join the network if their profit rate per unit time equals or exceeds their outside opportunity cost rate. Therefore, if the per-driver profit rate is \$$x$, then the number of participating drivers, denoted by n, is $n = NF(x) = NP\,(c_o \leq x)$. The per-driver profit rate x is itself a quantity that emerges in equilibrium and depends on n, the drivers' trip-related earnings and cost, the platform's controls, and the driver decisions, to be specified in §3.3 and §3.4.

Participating drivers incur a driving cost rate of \$$c$ per unit of travel time, which is common to all drivers and independent of the car occupancy. Drivers serving rider demand earn revenue at rate \$$\bar{\gamma}p$ per unit travel time, where $\bar{\gamma} = 1 - \gamma$, i.e., the price paid by the rider, net of the platform commission. Therefore, the maximum profit rate that drivers can earn equals $\bar{\gamma}p - c$. However, their actual profit rate is lower if they spend time waiting for riders (accruing zero profit during this wait) and/or repositioning from one location to the other (incurring a cost c per unit time). The following assumption ensures that drivers can earn a positive profit by repositioning[2].

Assumption 3.2 (Positive profit from repositioning)**.** *$ct_{12} < t_{21}(\bar{\gamma}p - c)$ and $ct_{21} < t_{12}(\bar{\gamma}p - c)$.*

Given Assumption 3.1, only the first condition in Assumption 3.2 will prove to be relevant.

Platform. The platform is operated by a monopolist firm that matches drivers with riders with the objective of maximizing its revenue rate.[3] The platform may have two controls: a) demand-side admission control, and b) supply-side capacity repositioning, as detailed in §3.2.3.

Information. Riders and drivers rely on the platform for matching, that is, potential riders cannot see the available driver capacity, and drivers cannot see the arrivals of rider requests.

The platform knows the model primitives introduced above, including the potential demand rates Λ, the destination of each trip request, the travel times t, the driving cost c and the opportunity cost rate distribution F. The driver opportunity cost rates are private information, not known by the platform. Therefore, participating drivers are homogenous to the platform. The platform knows the state of the system, i.e., each driver's location, travel direction and status at each point in time.

Riders do not need any network information since they are impatient—they simply arrive with a trip request and leave if their request is not accepted or cannot be fulfilled immediately.

[2] Assumption 3.2 also implies that $\bar{\gamma}p - c > 0$, hence $F(\bar{\gamma}p - c) > 0$, so at least some drivers choose to join the network.

[3] The platform could, in practice, incorporate additional considerations in its objective or as control constraints, e.g., to reward market penetration or penalize lost demand.

Drivers do not observe the state of the system, but they have (or can infer) the information required to compute their individual expected profit rates, namely: the travel times t; the steady-state delays until they get matched at each location, possibly zero; the destination (routing) probabilities for matches at each location; and the probabilities that they will choose or be instructed to reposition from one location to the other. These delays and the routing and repositioning probabilities are endogenous quantities consistent with the network equilibrium, to be detailed later on.

3.2.2 Matching Supply with Demand

Admission control. Let $\lambda_{lk} \leq \Lambda_{lk}$ denote the *effective* route-lk demand rate, i.e., the rate of trip requests that are served. Let λ be the vector of effective demand rates and $\Lambda - \lambda$ the vector of lost demand rates. A trip request may be lost either if there is no available driver capacity at the time and location of the request (recall that riders are impatient), or if the platform chooses to reject the request (e.g., based on the requested destination), even though driver capacity is available. In the regimes detailed in §3.2.3, the platform exercises either no or optimal admission control.

Matching at each location. At each location drivers that become available (i.e., do not reposition upon arrival) join a single queue, to be matched by the platform with riders originating at this location. (Drivers cannot reject the platform's matching requests.) Throughout we assume that the platform matches drivers to accepted ride requests according to a uniform matching policy, such as First-In-First-Out (FIFO) or random service order. Since participating drivers are homogeneous to the platform, this is not restrictive. This implies that in steady state drivers queueing at location l encounter the same waiting time, denoted by w_l, and are directed to route-lk with probability $\frac{\lambda_{lk}}{\lambda_{l1}+\lambda_{l2}}$. Let q_l denote the steady-state number of drivers queueing at location l. By Little's Law we have $w_l = q_l/(\lambda_{l1} + \lambda_{l2})$ for $l = 1, 2$. Let w and q denote, respectively, the vector of steady-state waiting times and queue lengths.

Repositioning of capacity between locations. Let ν_{12} and ν_{21} be the rates of drivers repositioning from location 1 and 2, respectively, and $\nu = (\nu_{12}, \nu_{21})$. Up to three flows emanate from location l. Drivers that are matched with riders leave at rates λ_{l1} and λ_{l2}, and drivers that reposition to location $k \neq l$ leave at rate ν_{lk} (without queueing at location l). Therefore, letting $\eta(\lambda, \nu)$ denote the corresponding vector of steady-state repositioning fractions, we have

$$\eta_1(\lambda, \nu) = \frac{\nu_{12}}{\lambda_{11} + \lambda_{12} + \nu_{12}} \quad \text{and} \quad \eta_2(\lambda, \nu) = \frac{\nu_{21}}{\lambda_{21} + \lambda_{22} + \nu_{21}}. \tag{3.1}$$

Repositioning decisions are either centralized or decentralized, as detailed in §3.2.3. Under centralized repositioning the platform controls the repositioning flows ν (e.g., drivers are employees or autonomous vehicles) and the fractions η emerge in response through (3.1); in this case we assume the drivers are informed about η. Under decentralized repositioning the participating drivers choose the fractions η to maximize their individual profit rates, and the resulting flow rates ν satisfy (3.1). In both regimes, drivers are homogeneous once they have joined the network, and the resulting repositioning fractions, η, are the same for all participating drivers.

Steady-state system flow balance constraints. Assuming a participating driver capacity equal to n, the effective demand rates λ, repositioning flow rates ν and waiting times w must satisfy: (i) flow balance between locations, $\lambda_{12} + \nu_{12} = \lambda_{21} + \nu_{21}$, and (ii) the capacity constraint $\sum_{l,k=1,2} \lambda_{lk} t_{lk} + (\nu_{12} t_{12} + \nu_{21} t_{21}) + \sum_{l=1,2} w_l (\lambda_{l1} + \lambda_{l2}) = n$, where $\sum_{l,k=1,2} \lambda_{lk} t_{lk}$ is the number of drivers serving riders, $\nu_{12} t_{12} + \nu_{21} t_{21}$ is the number of drivers repositioning between locations, and $\sum_{l=1,2} w_l (\lambda_{l1} + \lambda_{l2})$ is the number of drivers queueing in the two locations.

3.2.3 Three Control Regimes: Problem Formulations

We study three control regimes referred to as *Centralized Control, Minimal Control* and, *Admission Control*, that differ in terms of (i) whether the platform does or does not exercise admission control, and (ii) whether the platform controls or the drivers control repositioning decisions. Each regime yields an optimization problem in terms of the tuple (λ, ν, w, n), the system flow constraints described in §3.2.2, and the incentives of the platform and the drivers that we formalize in this section. In all three regimes drivers make their own participation decision by comparing their outside opportunity cost to their per-driver profit rate from joining the system.

Platform revenue. The platform's steady-state revenue rate is given by $\Pi(\lambda) := \gamma p \lambda \cdot t$, where γp is the platform's commission rate per busy driver and $\lambda \cdot t$ is the number of busy drivers, or equivalently, the number of riders being served. The welfare of riders, as measured by served demand, is therefore proportional to the revenue of the platform.

Driver profit and equilibrium constraints. Drivers are homogeneous once they have joined the network, as explained in §3.2.2, so all participating drivers achieve the same steady-state profit rate. We can compute the per-driver profit rate in two ways: (i) as the per-driver portion of the cumulative driver profits, for the participation equilibrium constraint, and (ii) from the perspective of an individual driver circulating through the network, for the repositioning equilibrium constraint.

First, the per-driver profit rate can be computed as follows

$$\pi(\lambda, \nu, n) = \frac{(\bar{\gamma} p - c) \sum_{l,k=1,2} \lambda_{lk} t_{lk} - c(\nu_{12} t_{12} + \nu_{21} t_{21})}{n},$$

where the numerator is the total profit generated by all n participating drivers. A *participation equilibrium* requires $n = NF(\pi(\lambda, \nu, n))$.

Second, from the perspective of an infinitesimal driver circulating through the network, her profit rate, denoted by $\widetilde{\pi}(\widetilde{\eta}; \lambda, w)$, is a function of her repositioning fractions $\widetilde{\eta}$, the routing probabilities implied by λ and the delays w in the matching queues. The explicit form of $\widetilde{\pi}(\widetilde{\eta}; \lambda, w)$ is discussed in §3.4.1. The effective demand rates λ and delays w emerge as equilibrium quantities that depend on the platform control and the decisions of all drivers. A participating driver's repositioning strategy is a vector of probabilities that specify for each network location the fraction of times that the driver will, upon arrival, immediately reposition to the other location. Since participating drivers are homogeneous, it suffices to focus on symmetric strategies, where drivers symmetrically choose the fractions $\widetilde{\eta}$ to maximize $\widetilde{\pi}(\widetilde{\eta}; \lambda, w)$. In equilibrium, we require that the unique repositioning fractions $\eta(\lambda, \nu)$ induced by the aggregate flow rates (λ, ν) through (3.1) must agree with individual drivers' profit-maximizing repositioning decisions, i.e.,

$$\eta(\lambda, \nu) \in \arg\max_{\widetilde{\eta}} \; \widetilde{\pi}(\widetilde{\eta}; \lambda, w). \tag{3.2}$$

Since every driver chooses $\eta(\lambda, \nu)$, each earns the same profit rate, so that $\widetilde{\pi}(\eta(\lambda, \nu); \lambda, w) = \pi(\lambda, \nu, n)$ for all (λ, ν, w, n) that satisfy the system flow constraints described in §3.2.2.

Centralized Control (C). In the centralized benchmark the platform has "maximum" control, over both demand admission and driver repositioning decisions. The platform solves:

$$\text{(Problem C)} \quad \max_{\lambda,\nu,w,n} \quad \Pi(\lambda) \tag{3.3}$$

$$\text{s.t} \quad \lambda_{12} + \nu_{12} = \lambda_{21} + \nu_{21}, \tag{3.4}$$

$$\sum_{l,k=1,2} \lambda_{lk} t_{lk} + \nu_{12} t_{12} + \nu_{21} t_{21} + \sum_{l=1,2} w_l \left(\lambda_{l1} + \lambda_{l2}\right) = n, \tag{3.5}$$

$$0 \le \lambda \le \Lambda, \ \nu \ge 0, \ w \ge 0, \tag{3.6}$$

$$\pi(\lambda,\nu,n) = \frac{(\bar{\gamma}p - c)\sum_{l,k=1,2} \lambda_{lk} t_{lk} - c(\nu_{12}t_{12} + \nu_{21}t_{21})}{n}, \tag{3.7}$$

$$n = NF\left(\pi(\lambda,\nu,n)\right), \tag{3.8}$$

where (3.4)–(3.6) are flow balance conditions and (3.7) and (3.8) enforce the participation equilibrium.

Admission Control (A). This regime differs from the centralized benchmark in that repositioning decisions are decentralized, i.e., controlled by drivers. The platform must therefore also account for the repositioning equilibrium constraint (3.2) and solves:

$$\text{(Problem A)} \quad \max_{\lambda,\nu,w,n} \left\{\Pi(\lambda) \ : \ (3.1)-(3.2), \ (3.4)-(3.8)\right\}. \tag{3.9}$$

Minimal Control (M). In this regime the platform does not exercise any demand admission control, and drivers control repositioning decisions. The platform simply matches rider trip requests in a pro-rata (or FIFO) manner to drivers, and never turns away requests when there are drivers available to serve them. In addition to (3.1)–(3.2) and (3.4)–(3.8), the network flows should satisfy the additional conditions (3.10)–(3.12) that we discuss next.

First, at each location the effective demand rates are proportional to the corresponding potential demand rates:

$$\frac{\lambda_{l1}}{\Lambda_{l1}} = \frac{\lambda_{l2}}{\Lambda_{l2}}, \quad l = 1,2, \tag{3.10}$$

i.e., routes originating at a location receive equal service probabilities (pro-rata service).

Second, drivers cannot be repositioning out of location l if the potential rider demand at that location has not been fully served, i.e.,

$$(\Lambda_{l1} + \Lambda_{l2} - \lambda_{l1} - \lambda_{l2})\nu_{lk} = 0, \quad l = 1, 2, k \neq l, \tag{3.11}$$

and demand requests originating at a location l can only be lost if this location has no supply buffer, so no drivers are waiting, i.e.,

$$(\Lambda_{l1} + \Lambda_{l2} - \lambda_{l1} - \lambda_{l2})w_l = 0, \quad l = 1, 2. \tag{3.12}$$

In the Minimal Control Regime, the set of feasible tuples is given by

$$\mathcal{M} = \{(\lambda, \nu, w, n) \ : \ (3.1) - (3.2)\,,\ (3.4) - (3.8),\ (3.10) - (3.12)\}\,. \tag{3.13}$$

We will show in §3.4.2 that for fixed participating capacity n, the set $\mathcal{M}$ is a singleton.

3.2.4 Reformulation to Capacity Allocation Problems

It is intuitive and analytically convenient to reformulate the above problems in terms of the driver capacities allocated to serving riders, repositioning (without riders), and queueing for riders.

For route lk, let S_{lk} denote the offered load of trips, and s_{lk} denote the (effective) capacity serving riders. Let S and s denote the corresponding vectors, $\overline{S} = \sum_{lk} S_{lk}$ the total offered load, and $\overline{s} = \sum_{lk} s_{lk}$ the total service capacity. From Little's Law,

$$S_{lk} = \Lambda_{lk} t_{lk} \quad \text{and} \quad s_{lk} := \lambda_{lk} t_{lk}, \quad l, k \in \{1, 2\}. \tag{3.14}$$

Let r_{lk} be the capacity repositioning from location l to k, $r = (r_{12}, r_{21})$, and $\overline{r} = r_{12} + r_{21}$, where

$$r_{lk} = \nu_{lk} t_{lk}, \quad l \neq k, \tag{3.15}$$

and q_l be the capacity queueing at location l. Let $q = (q_1, q_2)$ and $\overline{q} = q_1 + q_2$, where

$$q_l = w_l(\lambda_{l1} + \lambda_{l2}), \quad l = 1, 2. \tag{3.16}$$

Using (3.14)–(3.16) we can transform the formulations in §3.2.3 into equivalent problems with respect to (s, r, q, n). With some abuse of notation, we write the platform profit function as $\Pi(s) = \gamma p \overline{s}$ instead of $\Pi(\lambda)$, and similarly the per-driver profit functions as $\widetilde{\pi}(\widetilde{\eta}; s, q)$ instead of $\widetilde{\pi}(\widetilde{\eta}; \lambda, w)$ and $\pi(s, r, n)$ instead of $\pi(\lambda, \nu, n)$, and the repositioning fractions in (3.1) as $\eta(s, r)$ instead of $\eta(\lambda, \nu)$. For instance, the driver participation constraints (3.7)–(3.8) are represented by

$$\pi(s, r, n) = \frac{(\bar{\gamma} p - c)\overline{s} - c\overline{r}}{n}, \tag{3.17}$$

$$n = NF(\pi(s, r, n)). \tag{3.18}$$

3.2.5 Two-Step Solution Approach

We propose a two-step solution approach to solve the platform's revenue maximization problem in each regime $X \in \{M, A, C\}$[4]. In the above formulations, denote by $\mathcal{C}_X$ the set of decision variables (s, r, q, n) that satisfy all the constraints for regime X except the driver participation constraints (3.17)–(3.18). That is, $\mathcal{C}_C = \{(s, r, q, n) : (3.4) - (3.6), (3.14) - (3.16)\}$, $\mathcal{C}_A = \{(s, r, q, n) : (3.1) - (3.2), (3.4) - (3.6), (3.14) - (3.16)\}$, and $\mathcal{C}_M = \{(s, r, q, n) : (3.1) - (3.2), (3.4) - (3.6), (3.10) - (3.12), (3.14) - (3.16)\}$. The platform's optimization problem and the associated optimal revenue rate in regime X are given by

$$\Pi_X^* := \max_{s,r,q,n} \{\Pi(s) : (s, r, q, n) \in \mathcal{C}_X, (3.17) - (3.18)\}. \tag{3.19}$$

We solve (3.19) in the following two steps.

Step 1 Solve for the optimal capacity allocation for fixed capacity of participating drivers, n:

$$\Pi_X(n) := \max_{s,r,q} \{\Pi(s) : (s, r, q, n) \in \mathcal{C}_X\}. \tag{3.20}$$

Let $\pi_X(n)$ be the resulting reduced form of the per-driver profit $\pi(s, r, n)$ given by (3.17).

Step 2 Solve for the equilibrium capacity of participating drivers, n_X^*, from (3.18), i.e.,

$$n_X^* = NF(\pi_X(n_X^*)). \tag{3.21}$$

The platform revenue and per-driver profit at equilibrium are given by $\Pi_X(n_X^*)$ and $\pi_X(n_X^*)$, respectively.

[4]For regime M, this simplifies to a feasibility problem.

The following lemma provides a validity condition for this two-step approach to indeed solve (3.19). (All proofs are included in the Appendix.)

Lemma 3.1 (Validity condition for two-step approach)**.** *Under constraints* $(s, r, q, n) \in \mathcal{C}_X$*, if for each* n *the per-driver profit is maximized under the platform's optimal capacity allocation, i.e., from Step 1:*

$$\pi_X(n) = \max_{s,r,q}\{\pi(s, r, n) : (s, r, q, n) \in \mathcal{C}_X\}, \tag{3.22}$$

and $\pi_X(n)$ *is continuously decreasing in* n*, then* $\Pi_X^* = \Pi_X(n_X^*)$ *from Step 2.*

We show below that regimes C and M always satisfy condition (3.22), whereas this is not the case for regime A, in which case we modify the two-step solution approach accordingly. The implication is that when condition (3.22) holds, the welfare of riders is in fact consistent with the revenue of the platform.

3.3 Centralized Control (C)

In the centralized control regime (C) the platform controls demand admission and driver repositioning so as to maximize its revenue, and drivers make participation decisions in response to the resulting profit rate. The optimization problem for this regime is (3.3)–(3.8), which can be reformulated using (3.14)–(3.16) as:

$$\text{(Problem C)} \quad \max_{s,r,q,n} \quad \Pi\left(s\right) \tag{3.23}$$

$$\text{s.t} \quad \frac{s_{12} + r_{12}}{t_{12}} = \frac{s_{21} + r_{21}}{t_{21}}, \tag{3.24}$$

$$\overline{s} + \overline{r} + \overline{q} = n, \tag{3.25}$$

$$0 \leq s \leq S,\ r \geq 0,\ q \geq 0, \tag{3.26}$$

$$\pi\left(s, r, n\right) = \frac{(\bar{\gamma}p - c)\overline{s} - c\overline{r}}{n}, \tag{3.27}$$

$$n = NF\left(\pi\left(s, r, n\right)\right). \tag{3.28}$$

Following the solution approach outlined in §3.2.5, we present the solution of (3.23)–(3.28) in two steps. First, in Proposition 3.1 we solve for

$$\Pi_C(n) = \max_{s,r,q}\{\Pi(s) : (3.24) - (3.26)\}, \tag{3.29}$$

the optimal capacity allocation assuming an exogenously given capacity of participating drivers, n. Then, we characterize the resulting reduced form of the per-driver profit $\pi(s,r,n)$ in (3.27) as $\pi_C(n)$, and establish in Corollary 3.1 that there exists a unique equilibrium capacity of participating drivers, n_C^*, as the solution of $n_C^* = NF(\pi_C(n_C^*))$.

Proposition 3.1 (Allocation of fixed driver capacity under regime C)**.** *Consider the problem* (3.29) *for fixed driver capacity n. Define the constants*

$$n_1^C := \overline{S} - (\Lambda_{21} - \Lambda_{12})\, t_{21} \qquad \textit{and} \qquad n_2^C := \overline{S} + (\Lambda_{21} - \Lambda_{12})\, t_{12}, \tag{3.30}$$

where $n_1^C < \overline{S} < n_2^C$. The optimal capacity utilization has the following structure. (Figure 3.4 (a))

(1) Scarce capacity ($n \le n_1^C$). All drivers serve riders: $\overline{s} = n;\ r = 0;\ q = 0$.

(2) Moderate capacity ($n_1^C < n \le n_2^C$). Drivers serve riders or reposition from the low- to the high-demand location: $\overline{s} + r_{12} = n$ where $r_{12} = t_{12}/\left(t_{12} + t_{21}\right)(n - n_1^C)$, $r_{21} = 0;\ q = 0$.

(3) Ample capacity ($n > n_2^C$). Drivers serve all riders, reposition from the low- to the high-demand location, or wait in queue: $\overline{s} = \overline{S};\ r_{12} = n_2^C - \overline{S}, r_{21} = 0;\ \overline{q} = n - n_2^C$.

Note that n_1^C is the maximum offered load that can be served without repositioning, n_2^C is the minimum capacity level required to serve the total offered load, $\overline{S}$, i.e., with the minimum amount of repositioning, and $n_1^C < \overline{S} < n_2^C$ due to the demand imbalance $\Lambda_{21} > \Lambda_{12}$ (Assumption 3.1).

The results in Proposition 3.1 are intuitive. In the scarce-capacity zone 1 ($n \le n_1^C$) drivers are serving riders 100% of the time. The exact allocation of n is arbitrary except $s_{12}/t_{12} = s_{21}/t_{21}$ implied by $r = 0$. For $n = n_1^C$ the platform only loses the excess demand from the high- to the low-demand location, $\Lambda_{21} - \Lambda_{12}$, whereas destination-based admission control allows the platform to selectively direct capacity to serve all local requests (route-22).

In the moderate-capacity zone, the platform uses destination-based admission control and capacity repositioning to optimize performance; all drivers are moving around, so no capacity is "wasted" due to idling. Repositioning is only needed from the low- to the high-demand location.[5]

In the ample-capacity zone 3 ($n_2^C < n$), in addition to serving all riders and repositioning as needed, there is spare capacity that will ultimately idle waiting to be matched with riders. The zone-3 solution in Proposition 3.1 is not unique; multiple allocations of r and q support serving all demand. However, the solution with $r_{21} = 0$, i.e., no repositioning from location 2 to 1, maximizes the drivers' profit rate for given n, and supports the maximum achievable equilibrium capacity.

Let $\pi_C(n)$ denote the per-driver profit under the optimal capacity allocation in Proposition 3.1, as a function of the number of drivers n. Substituting $\overline{s}$ and $\overline{r}$ from Proposition 3.1 into (3.27) yields

$$\pi_C(n) = \frac{(\bar{\gamma}p - c)\overline{s} - c\overline{r}}{n} = \begin{cases} \bar{\gamma}p - c, & \text{zone 1 } (n \le n_1^C), \\ \frac{1}{n}\bar{\gamma}p\left(n_1^C + (n - n_1^C)\frac{t_{21}}{t_{12}+t_{21}}\right) - c, & \text{zone 2 } (n_1^C < n \le n_2^C), \\ \frac{1}{n}(\bar{\gamma}p\overline{S} - cn_2^C), & \text{zone 3 } (n > n_2^C). \end{cases} \tag{3.31}$$

This profit rate reflects the drivers' utilization profile: in zone 1 they serve riders all the time (with revenue rate $\bar{\gamma}p$ and cost rate c); in zone 2 they serve riders only a fraction of the time but still drive around all the time; and in zone 3 they also queue a fraction of the time.

Corollary 3.1 (Driver participation equilibrium under regime C)**.** *Under centralized admission control and repositioning,*

(i) the platform's optimal capacity allocation maximizes the per-driver profit, i.e., condition (3.22) *in Lemma 3.1 holds;*

(ii) there exists a unique equilibrium capacity of participating drivers, denoted by n_C^**, which solves* $n_C^* = NF(\pi_C(n_C^*))$*, where* $\pi_C(n)$ *is the continuously decreasing per-driver profit in* (3.31) *and* $\lim_{n\to\infty} \pi_C(n) = 0$.

[5] The capacity dedicated into repositioning captures, in a discrete, two-location network, the "wild goose chase" phenomenon described in Castillo et al. (2017).

Corollary 3.1 proves the validity condition in Lemma 3.1, so that the two-step approach yields an optimal solution to Problem C given by (3.23)–(3.28). The uniqueness of the equilibrium participating capacity n_C^* follows from the monotonicity of $\pi_C(n)$.

3.4 Regimes with Decentralized Repositioning

In §3.4.1 we provide an explicit characterization of a driver repositioning equilibrium. Subsequently we characterize the system equilibria for two regimes: in §3.4.2 for Minimal Control (M) where the platform exercises no admission control, and in §3.4.3 for Admission Control (A) where the platform optimizes over admission control decisions. Finally, in §3.4.4 we summarize the key differences between the equilibria of all three regimes, C, M, and A.

3.4.1 Driver Repositioning Equilibrium

Recall from §3.2.2 that under decentralized repositioning, drivers symmetrically choose their repositioning fractions $\widetilde{\eta}$ to maximize their profit rate $\widetilde{\pi}(\widetilde{\eta}; \lambda, w)$, and by (3.1) and (3.2) a driver repositioning equilibrium requires $\eta(\lambda, \nu) \in \arg\max_{\widetilde{\eta}} \widetilde{\pi}(\widetilde{\eta}; \lambda, w)$. That is, a set of flow rates (λ, ν) and delays w admit a driver repositioning equilibrium if, and only if, the unique repositioning fractions $\eta(\lambda, \nu)$ that are consistent with (λ, ν) are every driver's best response to (λ, w). Using (3.14)–(3.16) to map (λ, ν, w) to (s, r, q), we henceforth express the functions $\eta(\lambda, \nu)$ and $\widetilde{\pi}(\widetilde{\eta}; \lambda, w)$ as $\eta(s, r)$ and $\widetilde{\pi}(\widetilde{\eta}; s, q)$, respectively. Lemma 3.2 characterizes the profit rate function $\widetilde{\pi}(\widetilde{\eta}; s, q)$. Its explicit expression is given in the proof in the Appendix.

Lemma 3.2 (Per-driver profit rate)**.** *Let the function $T(\widetilde{\eta}; s, q)$ denote a driver's expected steady-state cycle time through the network, i.e., the average time between consecutive arrivals to the same location. Let $T^s(\widetilde{\eta}; s)$, $T^r(\widetilde{\eta})$ and $T^q(\widetilde{\eta}; s, q)$ denote the expected time a driver spends during a cycle serving riders, repositioning and queueing, respectively, where $T^s(\widetilde{\eta}; s) + T^r(\widetilde{\eta}) + T^q(\widetilde{\eta}; s, q) = T(\widetilde{\eta}; s, q)$. The drivers' expected steady-state profit rate is an explicit function that satisfies*

$$\widetilde{\pi}(\widetilde{\eta}; s, q) = \frac{(\bar{\gamma} p - c) T^s(\widetilde{\eta}; s) - c T^r(\widetilde{\eta})}{T^s(\widetilde{\eta}; s) + T^r(\widetilde{\eta}) + T^q(\widetilde{\eta}; s, q)}. \tag{3.32}$$

We have the following formal definition of a driver repositioning equilibrium.

Definition 3.1 (Driver repositioning equilibrium)**.** *A capacity allocation* (s, r, q) *forms a driver repositioning equilibrium if, and only if,* $\eta(s, r)$ *is every driver's best response, that is*

$$\eta_1(s,r) = \frac{r_{12}}{s_{11}\frac{t_{12}}{t_{11}} + s_{12} + r_{12}} \quad \textit{and} \quad \eta_2(s,r) = \frac{r_{21}}{s_{21} + s_{22}\frac{t_{21}}{t_{22}} + r_{21}}, \tag{3.33}$$

$$\eta(s,r) \in \arg\max_{\widetilde{\eta}} \widetilde{\pi}(\widetilde{\eta}; s, q). \tag{3.34}$$

Observe that under Assumption 3.1, it is not optimal to reposition from the high-demand location (2) to the low-demand location (1). Therefore, we focus hereafter on driver repositioning equilibria with $\eta_2 = 0$ and $r_{21} = 0$. Condition (3.34) in Definition 3.1 yields an explicitly defined set of capacity allocations (s, r, q) that admit a driver repositioning equilibrium. We call this the set of *driver-incentive compatible* capacity allocations, denoted by $\mathcal{D}$, and specify it in Proposition 3.2.

Proposition 3.2 (Driver-incentive compatibility)**.** *There exists a driver repositioning equilibrium with* $\eta_2 = 0$ *if, and only if, the capacity allocation* (s, r, q) *is driver-incentive compatible, that is:*

$$(s,r,q) \in \mathcal{D} := \left\{ (s,r,q) \geq 0 : r_{21} = 0,\ q_1 \begin{cases} \leq q_1^*(s) + k(s) q_2 & \textit{if } r_{12} = 0 \\ = q_1^*(s) + k(s) q_2 & \textit{if } r_{12} > 0 \end{cases} \right\}, \tag{3.35}$$

where $q_1^*(s)$ *and* $k(s)$ *are specified in* (C.7)*, and* $q_1^*(s) = k(s) = 0$ *if* $s_{11} + s_{12} = 0$*, and* $q_1^*(s), k(s) > 0$ *if* $s_{11} + s_{12} > 0$*. For* $(s, r, q) \in \mathcal{D}$ *the unique driver repositioning equilibrium is given by* (3.33)*.*

The structure of the driver-incentive compatible capacity allocation set $\mathcal{D}$ is intuitive. Inducing drivers not to reposition from location 1 (i.e., $r_{12} = 0$) requires a relatively short location-1 queue, i.e., $q_1 \leq q_1^*(s) + k(s)q_2$. Inducing drivers to reposition from location 1 (i.e., $r_{12} > 0$) requires appropriately balanced queues in the two locations[6], i.e., $q_1 = q_1^*(s) + k(s)q_2$; in this case, drivers are indifferent between repositioning to location 2

[6]Observe that the conditions $r_{12} > 0$ and $q_1 > q_1^*(s) + k(s)q_2$ are mutually incompatible. The latter condition implies that every driver repositions from location 1, but then no location-1 demand is served so that $q_1 = 0$.

and queueing at location 1, and (3.33) identifies the unique randomization probability for the corresponding symmetric driver repositioning equilibrium. Proposition 3.2 also foreshadows the critical role that demand admission control plays in shaping drivers' repositioning incentives, to be discussed in detail in §3.4.3.

In light of Proposition 3.2, constraint (3.35) alone suffices to account for decentralized repositioning in the problem formulations of regimes M and A: Given a driver-incentive compatible allocation, the corresponding repositioning equilibrium fractions are immediately determined by (3.33).

3.4.2 Minimal Control (M)

In the minimal control regime (M) the platform exercises no admission control and drivers make their own repositioning and participation decisions. We express the feasible set $\mathcal{M}$, given by (3.13) as the set of feasible capacity allocations (s, r, q, n). Using (3.14)–(3.16) the flow constraints (3.10)–(3.12) are equivalent to

$$\frac{s_{l1}}{S_{l1}} = \frac{s_{l2}}{S_{l2}}, \quad l = 1, 2, \tag{3.36}$$

$$(S_{l1} + S_{l2} - s_{l1} - s_{l2})r_{lk} = 0, \quad l = 1, 2, k \neq l, \tag{3.37}$$

$$(S_{l1} + S_{l2} - s_{l1} - s_{l2})q_l = 0, \quad l = 1, 2. \tag{3.38}$$

Substituting (3.35) for (3.1)–(3.2) based on Proposition 3.2, and noting that (3.4)–(3.8) correspond to (3.24)–(3.28), the set of feasible capacity allocations under minimal control is given by

$$\mathcal{M} = \{(s, r, q, n) \ : \ (3.24) - (3.28), \ (3.36) - (3.38), \ (3.35)\}. \tag{3.39}$$

Following the solution approach outlined in §3.2.5, we specify the system equilibrium in two steps. First, we characterize in Proposition 3.3 the unique equilibrium capacity allocation of a fixed number n of participating drivers, and get

$$\Pi_M(n) = \max_{s,r,q}\{\Pi(s) : (3.24) - (3.26), (3.36) - (3.38), (3.35)\} \tag{3.40}$$

Then, we establish in Corollary 3.2 that there exists a unique equilibrium capacity of participating drivers.

Proposition 3.3 (Allocation of fixed capacity under regime M). *Consider the problem* (3.40) *for fixed driver capacity n. Define the constants*

$$n_1^M := n_1^C - \left(1 - \frac{\Lambda_{12}}{\Lambda_{21}}\right) S_{22}, \quad n_2^M := n_1^M + q_1^*(S), \quad \text{and} \quad n_3^M := n_2^C + q_1^*(S), \tag{3.41}$$

where n_1^C *and* n_2^C *are defined in* (3.30) *and* $n_1^M < n_1^C < \overline{S} < n_2^C < n_3^M$. *There is a unique feasible driver capacity utilization* (s, r, q) *to* (3.40) *with the following structure. (Figure 3.4 (b))*

(1) Scarce capacity ($n \leq n_1^M$). *All drivers serve riders:* $\overline{s} = n$; $r = 0$; $q = 0$.

(2) Moderate capacity—no repositioning but queueing ($n_1^M < n \leq n_2^M$). *Drivers serve all riders at the low- and a fraction* $\frac{\Lambda_{12}}{\Lambda_{21}}$ *of riders at the high-demand location, or queue at the low-demand location:* $\overline{s} = n_1^M$ *where* $s_{1k} = S_{1k}$, $s_{2k} = S_{2k}\frac{\Lambda_{12}}{\Lambda_{21}}$ *for* $k = 1, 2$; $r = 0$; $q_1 = n - n_1^M < q_1^*(S)$, $q_2 = 0$.

(3) Moderate capacity—repositioning and queueing ($n_2^M < n \leq n_3^M$). *Drivers serve all riders at the low- and more than a fraction* $\frac{\Lambda_{12}}{\Lambda_{21}}$ *of riders at the high-demand location, reposition from the low- to the high-demand location, or queue at the low-demand location:* $\overline{s} > n_1^M$ *where* $s_{1k} = S_{1k}$ *for* $k = 1, 2$; $r_{12} > 0$, $r_{21} = 0$; $q_1 = q_1^*(S)$, $q_2 = 0$.

(4) Ample capacity ($n > n_3^M$). *Drivers serve all riders, reposition from the low- to the high-demand location, or queue at both locations in an incentive-compatible split:* $\overline{s} = \overline{S}$; $r_{12} = n_2^C - \overline{S}$, $r_{21} = 0$; $q_1 = q_1^*(S) + k(S)q_2$, $q_2 > 0$.

Proposition 3.3 differs in two ways from Proposition 3.1 for Centralized Control (regime C). First, because the platform exercises no admission control in regime M, the maximum offered load that can be served without repositioning, given by n_1^M, is lower than under optimal admission control (regime C), i.e., $n_1^M = n_1^C - (1 - \Lambda_{12}/\Lambda_{21}) S_{22}$. Specifically, with destination-based admission control the platform could direct the scarce capacity to serve all local traffic at the high-demand location. Second, and most importantly, because repositioning is decentralized in regime M, drivers can only be induced to reposition to the high-demand location if there is a sufficiently long queueing delay at the low-demand location. So, at lower capacity (zone 2), the low-demand queue builds up, but is not sufficient to incentivize drivers to reposition to the high-demand location. Only at sufficiently high capacity levels (zone 3) is the queue long enough for drivers to reposition. Since queueing is required to get sufficient repositioning to occur, the minimum capacity level required to serve the total offered load, n_3^M, exceeds the corresponding requirement under centralized control, n_2^C, by exactly the size of the queue that will provide the repositioning incentive.

Let $\pi_M(n)$ denote the per-driver profit under the equilibrium capacity allocation of Proposition 3.3, as a function of the number of drivers n. (See the Proof of Corollary 3.2 in the Appendix for an explicit expression of $\pi_M(n)$.) It is easy to verify that $\pi_M(n)$ is continuously decreasing in n and $\lim_{n\to\infty} \pi_M(n) = 0$.

Corollary 3.2 (Driver participation equilibrium under regime M)**.** *Under no admission control and decentralized repositioning,*

(i) the platform's optimal capacity allocation maximizes the per-driver profit, i.e., condition (3.22) *in Lemma 3.1 holds;*

(ii) there exists a unique equilibrium capacity of participating drivers, denoted by n_M^*, which solves $n_M^* = NF(\pi_M(n_M^*))$, where $\pi_M(n)$ is the continuously decreasing per-driver profit.

Corollary 3.2 proves the validity condition in Lemma 3.1, so that the two-step approach yields an optimal solution at which $\mathcal{M}$ as a singleton given by (3.39) is obtained. The uniqueness of the equilibrium participating capacity n_M^* follows from the monotonicity of $\pi_M(n)$.

3.4.3 Admission Control (A)

3.4.3.1 Optimal Allocation of Fixed Capacity

In regime A the platform is free to choose the optimal admission control policy, while drivers make their own repositioning and participation decisions. The corresponding capacity allocation problem, the analog of (3.9), is given by

$$\text{(Problem A)} \quad \max_{s,r,q,n} \left\{ \Pi\left(s\right) \;:\; (3.24)-(3.28),\; (3.35) \right\}. \tag{3.42}$$

Contrasting with (3.39), the platform's control need not satisfy (3.36)–(3.38). We proceed as before in two steps, but will modify the two-step solution approach in §3.4.3.3 to handle a special feature of the optimal platform control in this regime: the platform's potential *strategic demand rejection* discussed in §3.4.3.2. In that case the validity condition in Lemma 3.1 may not hold and the exact equilibrium is more complicated. In Proposition 3.4, we first solve for

$$\Pi_A(n) = \max_{s,r,q}\{\Pi(s) : (3.24) - (3.26), (3.35)\}, \tag{3.43}$$

the optimal capacity allocation assuming an exogenously given capacity of participating drivers, n.

Proposition 3.4 (Allocation of fixed capacity under regime A)**.** *Consider the problem* (3.43) *for fixed driver capacity n. Define the constants*

$$n_1^A := n_1^C \qquad \textit{and} \qquad n_3^A := n_2^C + q_1^*(S), \tag{3.44}$$

where n_1^C and n_2^C are defined in (3.30) and $n_1^A = n_1^C < \overline{S} < n_2^C < n_3^A$. There exists a threshold n_2^A such that $n_1^A < n_2^A < n_3^A$ and the optimal capacity utilization has the following structure. (Figure 3.4 (c))

(1) Scarce capacity ($n \le n_1^A$). All drivers serve riders: $\overline{s} = n$; $r = 0$; $q = 0$.

(2) Moderate capacity—no repositioning but queueing ($n_1^A < n \le n_2^A$). Drivers serve all riders except a fraction $1 - \frac{\Lambda_{12}}{\Lambda_{21}}$ from the high- to the low-demand location, and queue at the low-demand location: $\overline{s} = n_1^A$; $r = 0$; $q_1 = n - n_1^A < q_1^*(s)$, $q_2 = 0$.

(3) Moderate capacity—repositioning, with or without queueing ($n_2^A < n \le n_3^A$). Compared to zone 2, drivers serve more riders at the high- but possibly fewer riders at the low-demand location, they reposition from the low- to the high-demand location, and may queue at the low-demand location: $\overline{s} > n_1^A$; $r_{12} > 0$, $r_{21} = 0$; $q_1 = q_1^*(s) \ge 0$, $q_2 = 0$.

(4) Ample capacity ($n > n_3^A$): Drivers serve all riders, reposition from the low- to the high-demand location, or queue at both locations in an incentive-compatible split: $\overline{s} = \overline{S}$; $r_{12} = n_2^C - \overline{S}$, $r_{21} = 0$; $q_1 = q_1^*(S) + k(S)q_2$, $q_2 > 0$.

Proposition 3.4 differs in two ways from Proposition 3.3 for Minimal Control (regime M). First, because the platform exercises admission control in regime A, the maximum offered load that can be served without repositioning is the same as under Centralized Control, i.e., $n_1^A = n_1^C$, and exceeds its counterpart under Minimal Control, i.e., $n_1^M < n_1^A$, as discussed in §3.4.2. The exact allocation of n at scarce capacity (zone 1) is arbitrary except $s_{12}/t_{12} = s_{21}/t_{21}$, the same as in regime C.

Second and more importantly, whereas under Minimal Control the demand served on each route increases in the total available driver capacity n, this may *not* hold under optimal admission control: Compared to zone 2, in zone 3 the platform may *reject*, and therefore serve *fewer*, rider requests at the low-demand location even if there are available drivers, and even though there is more capacity in the network. The idea is to make it less attractive for drivers to queue at the low-demand location by decreasing the served demand rate, rather than by relying on the buildup of a long queue. This frees cars to reposition and subsequently serve riders.[7] We term this policy feature *strategic demand rejection* as it reduces the revenue at the low-demand location, in order to incentivize drivers to reposition and generate more revenue at the high-demand location. We elaborate on the rationale and specify optimality conditions for this key result in §3.4.3.2 below.

3.4.3.2 Strategic Demand Rejection to Induce Driver Repositioning

Proposition 3.4 states that under moderate-capacity conditions (zone 3, $n_2^A < n \leq n_3^A$), drivers may serve *fewer*riders at the low-demand location (1), compared to when there is less capacity in the network (zone 2). In this case the optimal admission control policy exhibits a somewhat counterintuitive behavior, whereby the platform rejects some or all rider requests in the low-demand location, even though there is an excess supply of drivers, that is, empty cars are leaving and possibly also waiting to get matched at this location. This strategic demand rejection sacrifices revenue at the low-demand location, so as to incentivize drivers to reposition from the low- to the high-demand location where they can generate more revenue for themselves and for the platform. Specifically, rejecting rider requests at the low-demand location creates an artificial demand shortage that drivers offset by choosing to reposition more frequently to the high-demand location, rather than joining the low-demand matching queue. The end result is a shorter queue at the low-demand location (the waiting time may increase or decrease). In terms of Proposition 3.2,

[7]This property of the optimal platform's control seems to be in contrast to the literature in ride-hailing networks.

rejecting demand at location 1 alters the driver-incentive compatible capacity allocation by reducing the queue-length threshold $q_1^*(s)$, which frees up driver capacity to reposition and serve riders at the high-demand location. By controlling congestion, the platform has an operational lever to incentivize drivers to reposition, as opposed to, for example, increasing their wage.

This control action can only be relevant when capacity is moderate; when capacity is scarce all drivers are busy; when capacity is ample, all riders are served. For the moderate-capacity zone (3), Proposition 3.5 identifies a necessary and sufficient condition for the optimality of strategic demand rejection in terms of the model primitives. To simplify notation and highlight the structural imbalances, define the following ratios:

$$\rho_1 := \frac{S_{11}}{S_{11}+S_{12}},\ \rho_2 := \frac{S_{22}}{S_{21}+S_{22}},\ \tau := \frac{t_{21}}{t_{12}},\ \kappa := \frac{c}{\bar{\gamma}p} < 1, \tag{3.45}$$

where ρ_1 and ρ_2 are the shares of the local-demand offered load at location 1 and 2, respectively, τ is the ratio between cross-location travel times, and κ is the ratio of driving cost to drivers' service revenue ("relative driving cost"). Assumption 3.2 requires that $\kappa < \tau/(1+\tau)$.

Proposition 3.5 (Optimality of strategic demand rejection in regime A)**.** *Under optimal platform admission control and decentralized repositioning, it is optimal at moderate capacity, i.e., for some $n \in (n_2^A, n_3^A]$, to strategically reject rider requests at the low-demand location so as to induce repositioning to the high-demand location only if the following condition holds:*

$$\frac{\Lambda_{12}}{\Lambda_{21}}\frac{1-\rho_1\kappa}{1-\rho_1} < \frac{\tau-(\tau+1-\rho_2)\kappa}{1-\rho_2}\left(\kappa\frac{1+\tau}{\tau}\frac{\tau+1-\rho_2}{\rho_2}-1\right). \tag{3.46}$$

Note that condition (3.46) is necessary but not sufficient, for demand rejection to be optimal at equilibrium. §3.4.3.3 explains the reason in detail and provides a sufficient condition for demand rejection to be optimal at equilibrium. To gain some intuition we discuss (3.46) under the simplifying assumption that $\tau = 1$.

Effect of ρ_2, the share of the local-demand offered load at the high-demand location.[8] Intuitively, if the local-demand share at the high-demand location ρ_2 is low, then this

[8] For $\tau = 1$ and $\kappa > 0$ the right-hand side (RHS) of (3.46) decreases in ρ_2 from $+\infty$ for $\rho_2 = 0$ to $-\infty$ as $\rho_2 \to 1$, so that (3.46) holds for ρ_2 below some threshold.

location is less attractive because of the higher likelihood that a driver will get matched to a rider going back to the low-demand location; drivers have a weak "natural" incentive to reposition to the high-demand location, so encouraging them to do so requires rejecting demand at the low-demand location.

Effect of ρ_1, the share of the local-demand offered load at the low-demand location.[9] Holding ρ_2 fixed, which fixes the RHS of (3.46), the condition cannot hold for sufficiently large ρ_1, i.e., if the local demand at the low-demand location is dominant: In this case drivers in the low-demand location may get stuck serving local requests and queueing in between, which adversely affects their profit rate and makes repositioning naturally more attractive.

Effect of $\Lambda_{12}/\Lambda_{21}$, the cross-location demand imbalance. The LHS of (3.46) is positive and decreases to zero as Λ_{21} increases from Λ_{12} to ∞, where $\Lambda_{12}/\Lambda_{21} < 1$ by Assumption 3.1. Therefore, (3.46) holds for sufficiently large Λ_{12}, provided the RHS is strictly positive (i.e., the local-demand share at the high-demand location, ρ_2, is below some threshold, as discussed above). Intuitively, more cross-location demand at the high-demand location increases the value of rejecting demand at the low-demand location in order to induce drivers to reposition.

Effect of κ, the relative driving cost. (3.46) can only hold for sufficiently large $\kappa > 0$. When repositioning becomes significantly more expensive than queueing (for which drivers incur no direct cost), the platform needs to strengthen the incentive for repositioning over queueing at the low-demand location by rejecting demand there.

3.4.3.3 Equilibrium Characterization

As discussed above, when the platform exerts strategic demand rejection, there is a shorter queue at the low-demand location and more drivers are incentivized to reposition from the low- to the high-demand location. The per-driver profit at a given participation level may therefore go down, as drivers pay for repositioning but not for queueing. Therefore, even when the necessary condition for strategic demand rejection, (3.46), holds, the platform

[9] The left-hand side (LHS) of (3.46) increases in ρ_1 from $\Lambda_{12}/\Lambda_{21}$ for $\rho_1 = 0$ to ∞ as $\rho_1 \to 1$, so that condition (3.46) holds if both local-demand shares, ρ_1 and ρ_2, are below some threshold.

may be able to generate more revenue without rejecting location-1 demand. In particular, because at fixed participation level, strategic demand rejection maximizes revenue per driver but may reduce per-driver profit, the platform faces the following trade-off: increase revenue through strategic demand rejection at the expense of reducing the per-driver profits and limiting driver participation; or increase revenue with less (or no) strategic demand rejection, to boost per-driver profits and driver participation.

In this section we present a modification to the two-step approach that identifies how to find the solution in regime A when Lemma 3.1 does not hold, and we obtain a sufficient condition for strategic demand rejection to be optimal in equilibrium.

We propose the following modified two-step solution approach that bounds the equilibrium participating capacity n_A^* between two participation levels, the *lower* participation that is obtained if the platform exercises strategic demand rejection to maximize its revenue at each given participation level, and the *higher* participation that is obtained if the platform completely forgoes strategic demand rejection.

Modified Two-Step Solution Approach

Step 1 Solve for the optimal capacity allocation for fixed capacity of participating drivers, n

$$\Pi_A(n) = \max_{s,r,q}\{\Pi(s) : (s, r, q, n) \in \mathcal{C}_A\}, \tag{3.47}$$

and solve for the optimal capacity allocation in the *absence* of demand rejection in the low-demand location when $n \in (n_1^A, n_3^A]$:

$$\hat{\Pi}_A(n) = \max_{s,r,q}\{\Pi(s) : (s, r, q, n) \in \mathcal{C}_A, s_{1k} = S_{1k} \text{ for } k = 1, 2,\ n \in (n_1^A, n_3^A]\}. \tag{3.48}$$

Let $\pi_A(n)$ and $\hat{\pi}_A(n)$ be the resulting per-driver profit functions, respectively.

Step 2 Solve for the equilibrium capacity of participating drivers, n_A, from

$$NF(\pi_A(n_A^+)) \leq n_A \leq NF(\pi_A(n_A^-)), \tag{3.49}$$

where $n_A^+ = \lim_{\epsilon\downarrow 0} n_A+\epsilon$ and $n_A^- = \lim_{\epsilon\downarrow 0} n_A-\epsilon$, and solve for the equilibrium capacity of participating drivers in the absence of strategic demand rejection, $\hat{n}_A$, from

$$\hat{n}_A = NF(\hat{\pi}_A(\hat{n}_A)). \tag{3.50}$$

The actual equilibrium participating capacity $n_A^* \in [n_A, \hat{n}_A]$ can be identified by solving for Π_A^* in (3.19) with n restricted to $n \in [n_A, \hat{n}_A]$.

As reasoned above, the per-driver profit need not be maximized under the platform's optimal capacity allocation when it involves strategic demand rejection. In this case the original two-step approach is not valid by Lemma 3.1 and we propose the above modified version. Moreover, while $\hat{\pi}_A(n)$ is continuously decreasing in n to a limit value of zero, $\pi_A(n)$ also decreases in n to zero but is *discontinuous* under strategic demand rejection. Either policy ensures a unique participation equilibrium. These observations and the modified two-step approach are formalized by the following lemma.

Lemma 3.3 (Driver participation equilibrium under regime A)**.** *Under optimal admission control and decentralized repositioning:*

(i) Allowing strategic demand rejection, the platform's optimal capacity allocation may not maximize per-driver profit, i.e., condition (3.22) *in Lemma 3.1 does not hold; there exists a unique equilibrium participating capacity n_A which solves* (3.49)*.*

(ii) Disallowing strategic demand rejection, the platform's optimal capacity allocation maximizes per-driver profit, i.e., condition (3.22) *in Lemma 3.1 holds; there exists a unique equilibrium participating capacity $\hat{n}_A$ which solves* (3.50)*.*

(iii) The actual equilibrium participating capacity is $n_A^* \in [n_A, \hat{n}_A]$. When regime A does not involve strategic demand rejection, i.e., condition (3.46) *does not hold, $n_A^* = n_A = \hat{n}_A$.*

(iv) A sufficient condition for strategic demand rejection to be optimal is $\hat{\Pi}_A(\hat{n}_A) < \Pi_A(n_A)$.

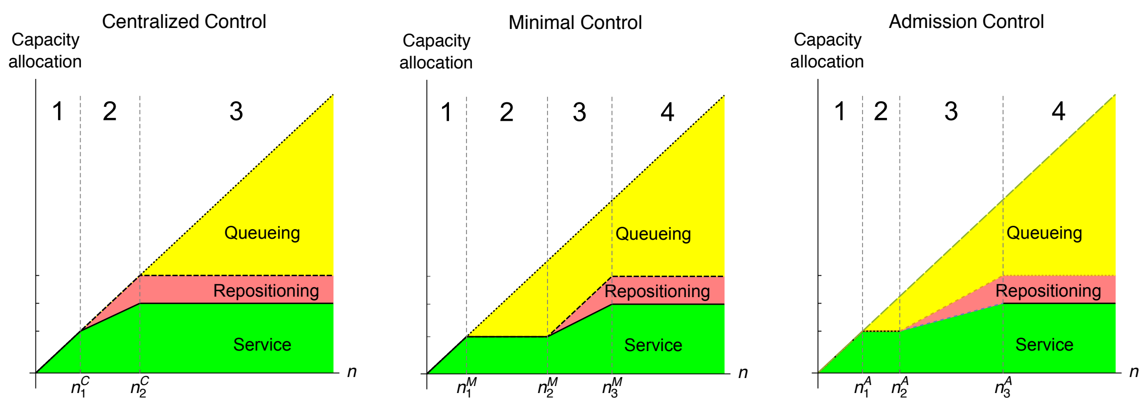
Proposition 3.6 identifies a plausible sufficient condition for strategic demand rejection to be optimal in equilibrium under regime A, i.e., for Lemma 3.3 (iv) to hold.

Proposition 3.6 (Sufficient condition for strategic demand rejection in regime A)**.** *There exists a positive threshold level of local-demand offered load at the high-demand location, denoted by $\hat{S}_{22}$, such that if $S_{22} \in [0, \hat{S}_{22})$, then strategic demand rejection is optimal under any equilibrium with moderate participating capacity $n_A^* \in (n_2^A, n_3^A)$.*

Proposition 3.6 relies on the following intuition that holds in the absence of local demand at the high-demand location 2. For drivers at the low-demand location 1, repositioning to the high-demand location 2 is maximally unattractive if they find no local demand there ($S_{22} = 0$), because this condition minimizes their utilization between the time when they leave and the time when they return to the low-demand location. Therefore, giving drivers the incentive to reposition without strategic demand rejection requires a very long queue at the low-demand location and therefore substantial excess capacity, which reduces their profits significantly. As a result, strategic demand rejection allows the platform to boost revenues with only modest adverse effect on driver profits.

3.4.4 Graphical Depiction of Control Regimes C, M, and A

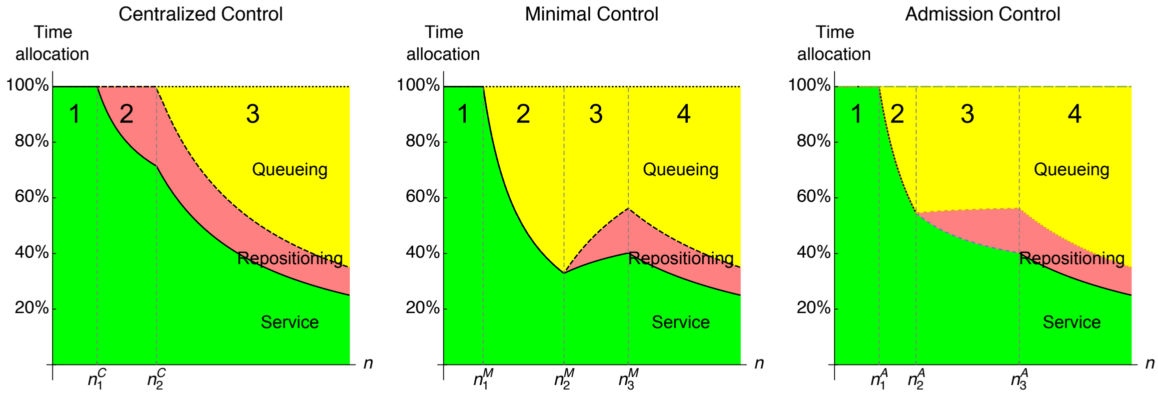
Figures 3.4 and 3.5 depict the broad features that are identified in Propositions 3.1, 3.3 and 3.4 for the three control regimes. (1) When capacity is scarce, all drivers are busy serving



(a) Regime C (b) Regime M (c) Regime A

Figure 3.4: Optimal capacity allocation in three control regimes

riders under all three control regimes. (2) When capacity is ample, all riders are served, and the three control regimes agree again. But, importantly, (3) with optimal admission control and centralized repositioning the platform can serve the entire rider demand with less capacity and without any drivers queueing at any location; in contrast, with decentralized repositioning, in order to create the appropriate incentives, drivers need to queue. (4) With admission control, the platform can (i) prioritize rider demand at the high-demand location based on their destination, and therefore increase driver utilization in the absence



(a) Regime C (b) Regime M (c) Regime A

Figure 3.5: Driver time allocation in three control regimes

of repositioning, and (ii) reject rider demand at the low-demand location to incentivize driver repositioning.

Switching to the driver's view (Figure 3.5), additional platform control capability ensures that drivers are busy for a larger fraction of their time, and thus more profitable.

3.5 The Impact of Platform Controls on System Performance

In this section we compare the equilibrium performance of the three control regimes $\{M, A, C\}$ from the viewpoint of the platform, the drivers, and the rider service level.

3.5.1 Ranking of Platform Revenue, Per-Driver Profit, and Driver Capacity

We start by ranking the platform revenue, the per-driver profit, and the driver capacity under the three control regimes. Let $\Pi_X(n)$ denote the platform revenue rate under the equilibrium capacity allocation in regime $X \in \{M, A, C\}$, as a function of the participating driver capacity n. Proposition 3.7 establishes that for fixed participating capacity both the platform and the drivers are better off with increasing platform control capabilities.

Proposition 3.7 (Ranking of equilibrium profits for fixed capacity)**.** *For fixed participating capacity n control capabilities have the following impact on profits:*

(1) The platform revenue rate increases with increasing platform control capability ($M \to A \to C$):

$$\Pi_M(n) \leq \Pi_A(n) \leq \Pi_C(n). \tag{3.51}$$

(2) Centralized control maximizes the per-driver profit rate:

$$\max\{\pi_M(n), \pi_A(n)\} \leq \pi_C(n). \tag{3.52}$$

(3) Under decentralized repositioning, if (3.46) *is not satisfied then optimal admission control increases the per-driver profit rate:*

$$\pi_M(n) \leq \pi_A(n). \tag{3.53}$$

Parts (1) and (2) of Proposition 3.7 are as expected. Part (1) of Proposition 3.7 also implies that riders benefit from increasing platform control capability: From the riders' viewpoint, an important performance metric is the network-wide service level, defined as the fraction of the total rider demand that is served. This service level equals the ratio of total service capacity to total offered load, i.e., $\overline{s}/\overline{S}$. Since the platform revenue rate is proportional to the total service capacity, i.e., $\Pi(s) = \gamma p \overline{s}$ by (3.23), the network-wide service level is proportional to the platform revenue rate, and therefore increases with increasing platform control capability by Part (1) of Proposition 3.7.

Part (3) of Proposition 3.7 establishes that optimal admission control typically benefits drivers, provided that condition (3.46) is *not* satisfied. Conversely, if condition (3.46) holds then by Proposition 3.5 the platform chooses to strategically reject demand in the low-demand location at moderate capacity levels to induce driver repositioning; as a result, drivers incur higher driving costs and may be worse off than without admission control (the platform is still better off).

Proposition 3.8 establishes that the equilibrium driver participation increases with increasing platform control capabilities, and as under fixed capacity, so do the resulting platform revenue and per-driver profits. Let $\Pi_X^* := \Pi_X(n_X^*)$ and $\pi_X^* := \pi_X(n_X^*)$ denote, respectively, the platform revenue rate and the per-driver profit rate, under the equilibrium capacity allocation and levels in regime $X \in \{M, A, C\}$.

Proposition 3.8 (Ranking of equilibrium profits and capacity)**.** *(1) The equilibrium platform revenue rate increases with increasing platform control capability ($M \to A \to C$):*

$$\Pi_M^* \le \Pi_A^* \le \Pi_C^*. \tag{3.54}$$

(2) Centralized control maximizes the equilibrium driver participation and per-driver profit rate:

$$\max\{n_M^*, n_A^*\} \le n_C^* \quad \textit{and} \quad \max\{\pi_M^*, \pi_A^*\} \le \pi_C^*. \tag{3.55}$$

(3) Under decentralized repositioning, if (3.46) *is not satisfied, then optimal admission control increases drivers' participation and profit rate:*

$$n_M^* \le n_A^* \quad \textit{and} \quad \pi_M^* \le \pi_A^*. \tag{3.56}$$

Parts (2) and (3) follow from Parts (2) and (3) of Proposition 3.7, respectively, and because the marginal opportunity cost function, defined as $c_o(n) := F^{-1}\left(\frac{n}{N}\right)$, strictly increases in the participating capacity n. Part (1) follows because the platform revenue functions $\Pi_C(n)$, $\Pi_M(n)$ and $\Pi_A(n)$ are increasing in n. Propositions 3.1, 3.3, and 3.4 imply that the performance improvements reported in Proposition 3.8 are strictly positive if, and only if, the equilibrium driver capacity is in the moderate-capacity zone where increased control capability is effective.Corollary 3.3 makes these conditions precise.

Corollary 3.3 (Conditions for performance gains)**.** *Platform controls strictly improve profits under the following equilibrium conditions:*

(1) Admission control (regime A over M): If (3.46) *is not satisfied then*

$$\Pi_M^* < \Pi_A^*,\ \pi_M^* < \pi_A^* \textit{ and } n_M^* < n_A^* \quad \textit{if and only if} \ \ n_M^* \in (n_1^M, n_3^M), \tag{3.57}$$

where n_1^M and n_3^M are defined in (3.41)*.*

(2) Centralized repositioning control (regime C over A):

$$\Pi_A^* < \Pi_C^*,\ \pi_A^* < \pi_C^* \textit{ and } n_A^* < n_C^* \quad \textit{if and only if} \ \ n_A^* \in (n_1^A, n_3^A), \tag{3.58}$$

where n_1^M and n_3^M are defined in (3.44)*.*

The equilibrium iff capacity conditions in (3.57) are satisfied when the potential driver supply and the outside opportunity cost distribution are such that (i) more drivers are willing to participate at the maximum profit rate ($\overline{\gamma}p - c$) than are needed to meet the rider demand that can be served without repositioning (so $n_M^* > n_1^M$), but (ii) not enough drivers are willing to participate at the reduced profit rate resulting from the queueing delays and repositioning costs that are necessary to serve all riders (so $n_M^* < n_3^M$). Both of these conditions seem plausible, and as such we expect admission control to benefit the platform and the drivers in practical settings. A similar argument indicates that in practical parameter regimes the iff capacity conditions in (3.58) also hold, i.e., that centralized repositioning is beneficial.

3.5.2 Upper Bounds on the Gains in Platform Revenue and Per-Driver Profit

In this section we provide upper bounds on the gains in platform revenue and per-driver profit due to increased platform control as explicit functions of the network primitives ρ_1, ρ_2, and τ defined in (3.45) across all feasible driver outside opportunity cost rate distributions.

Proposition 3.9 (Upper bounds on platform revenue gains)**.** *Fix $N \geq n_3^M = n_3^A$.*
(1) Platform revenue gain due to admission control (regime A over M): If (3.46) *is not satisfied,*

$$\max_{F(\cdot)} \frac{\Pi_A^* - \Pi_M^*}{\Pi_M^*} \leq \frac{\overline{S}}{n_1^M} - 1 = \left(\frac{\Lambda_{21}}{\Lambda_{12}} - 1\right) \frac{1}{1 + \frac{1-\rho_2}{1-\rho_1}\frac{1}{\tau}}. \tag{3.59}$$

(2) Platform revenue gain due to centralized repositioning control (regime C over A):

$$\max_{F(\cdot)} \frac{\Pi_C^* - \Pi_A^*}{\Pi_A^*} \leq \frac{\overline{S}}{n_1^A} - 1 = \left(\frac{\Lambda_{21}}{\Lambda_{12}} - 1\right) \frac{1}{1 + \frac{1}{1-\rho_1}\frac{1}{\tau} + \frac{\rho_2}{1-\rho_2}\frac{\Lambda_{21}}{\Lambda_{12}}}. \tag{3.60}$$

The key insight is that the potential revenue gains from both admission control and centralized repositioning increase as a function of the cross-location demand imbalance ($\Lambda_{21}/\Lambda_{12}$), and the cross-location travel time imbalance ($\tau = t_{21}/t_{12}$). The condition $N \geq n_3^M = n_3^A$ requires that the potential driver supply is large enough to serve all riders under decentralized repositioning. The upper bound on the gain from admission control (the RHS in (3.59)) is attained if, under minimal control the platform can only meet the rider demand that does not require repositioning, and admission control allows the platform to increase driver participation enough to serves all riders. The upper bound on the gain from repositioning control in (3.60) has a similar interpretation. The upper bound in (3.59) exceeds the bound in (3.60), because regime A allows the platform to serve more riders without repositioning. The bounds in (3.59) and (3.60) can be approached arbitrarily closely for specific choices of the opportunity cost distribution $F(\cdot)$, as discussed in Section S2 of the Supplemental Materials.

Table 3.1 numerically illustrates the magnitude of the upper bounds in (3.59) and (3.60) as a function of the cross-location demand imbalance $\Lambda_{21}/\Lambda_{12}$.

Table 3.1: Upper bounds in (3.59) and (3.60) on platform revenue gain ($t_{lk} = 1, \forall lk$, $\Lambda_{12} = \Lambda_{22} = 1$)

(a) Balanced cross-local demand at low-demand location ($\rho_1 = 0.5$)

cross demand imbalance ($\frac{\Lambda_{21}}{\Lambda_{12}}$)	1	2	5	10
from admission control	0%	43%	150%	319%
from central. repositioning	0%	25%	100%	225%

(b) Imbalanced cross-local demand at low-demand location ($\rho_1 = 0.25$)

($\frac{\Lambda_{21}}{\Lambda_{12}}$)	1	2	5	10
from admission control	0%	53%	189%	407%
from central. repositioning	0%	30%	120%	270%

Turning attention to the per-driver profit gain, we have the following result.

Proposition 3.10 (Upper bound on per-driver profit gains). *Fix $N \geq n_3^M = n_3^A$ and assume that* (3.46) *is not satisfied. The per-driver profit gain from admission control (under regime A or C) satisfies:*

$$\max_{F(\cdot)} \frac{\pi_A^* - \pi_M^*}{\pi_M^*} = \max_{F(\cdot)} \frac{\pi_C^* - \pi_M^*}{\pi_M^*} \leq \frac{1 - \rho_2}{\tau - (1 - \rho_2 + \tau)\kappa}. \tag{3.61}$$

In contrast to the bounds on the platform revenue gains in Proposition 3.9 that can only be attained when more control yields repositioning, the bound in (3.61) can only be attained in the *absence* of repositioning, i.e., when admission control increases drivers' utilization to 100% with only a small increase in their participation. In the online appendix we illustrate this tension between the drivers' and the platform's gains from control, along with the properties of the opportunity cost distribution $F(\cdot)$ that are required to attain the bound in (3.61). In a nutshell, if a small change in per-driver profit rate increases the capacity of participating drivers significantly, then the platform may extract significant gains while drivers are only marginally better off; conversely, if it takes a large change in per-driver profit rate to attract incremental participating driver capacity, then drivers extract significant incremental profits, while the platform benefit is moderate.

3.5.3 Value of Platform Control: Numerical Illustration

We conclude with a numerical illustration of the value of platform control. Taking Minimal Control with commission rate $\gamma = 0$ as the base case, we consider the performance effects of Centralized Control with $\gamma = 0$ and $\gamma = 0.2$. We assume unit travel times, i.e., $t_{lk} = 1$ for $l, k = 1, 2$, fix the price per unit time $\$p = 3$, the driver cost rate $\$c = 1$, the offered load vector $S = (1, 1, x, x)$ and consider the cases $x = 2$ and $x = 5$, corresponding to moderate and large demand imbalance, respectively. We assume that drivers' outside opportunity costs are uniformly distributed on $[0, p - c]$.

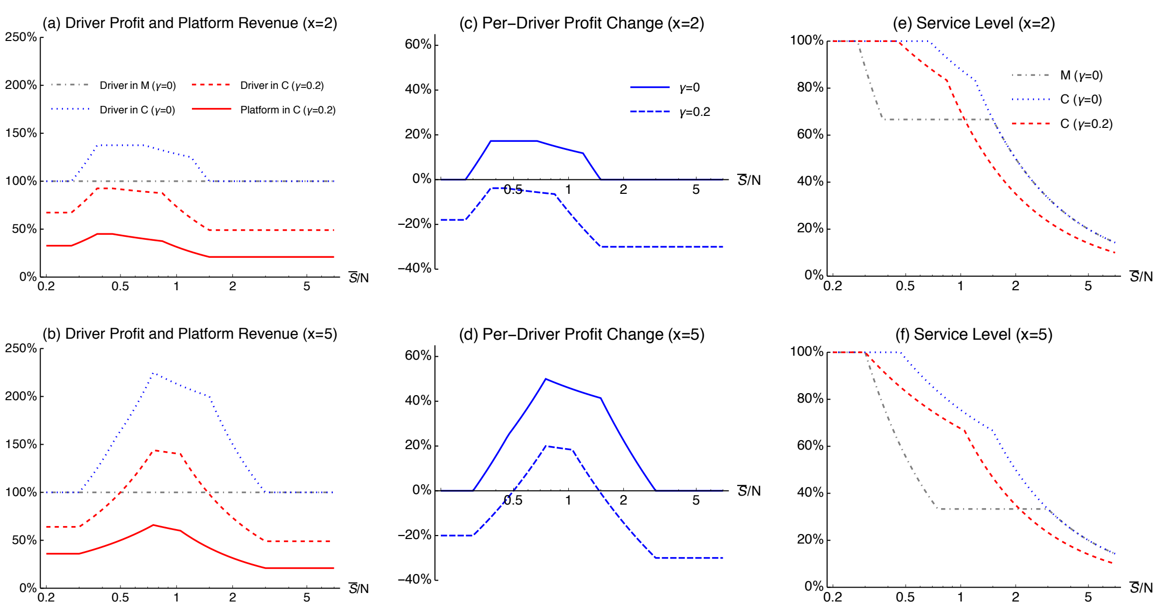


Figure 3.6: Impact of Centralized vs. Minimal control on performance ($x = S_{2k}/S_{1k}, k = 1, 2$)

The graphs in Figure 3.6 depict the effects of Centralized Control on four equilibrium performance measures, the total driver profit and platform revenue (left panel), the per-driver profit rate (center panel), and the rider service level (right panel), in the top row for cross-location imbalance $S_{21} = 2$ and in the bottom row for $S_{21} = 5$. These measures are shown as functions of the offered-load-to-driver-pool ratio $\overline{S}/N$, where $\overline{S}$ is fixed. Because $F(p - c) = 1$, all drivers in the pool would participate at the maximum achievable profit rate $p - c$, i.e., if they could serve riders all the time and the platform extracted no commission ($\gamma = 0$). However, due to the cross-location demand imbalance, serving all riders involves repositioning (plus queueing under Minimal Control) which reduces the per-driver profit rate below $p - c$, and requires more participating cars than demand (so $n > \overline{S}$), so that serving all riders in equilibrium is only feasible if $\overline{S}/N < 1$.

We make the following observations, which appear robust to other system parameters (and possibly to other general network structures).

Total driver profit and platform revenue. Panels (a) and (b) show the effects of Centralized Control on the total driver profit and platform revenue, in percentage terms relative to Minimal Control with $\gamma = 0$ (100%). Irrespective of the commission rate, both the drivers and the platform are best off at intermediate values of the offered-load-to-driver-pool ratio $\overline{S}/N$. At intermediate values of $\overline{S}/N$, drivers are always better off under Centralized Control with a zero commission rate ($\gamma = 0$), but are only better off when the cross network imbalance is significant ($x = 5$); it is natural to assume that $\overline{S}/N$ is itself moderate, say in the interval $(.5, 2)$, since otherwise the platform may choose to change the price that riders pay to increase or to reduce demand, respectively.

Per-driver profit vs. total driver profit. Panels (c) and (d) show the effects of Centralized Control on the per-driver profit, in percentage terms relative to Minimal Control with $\gamma = 0$. Comparing with the total driver profits, we see that the change in the per-driver profit is always significantly lower in magnitude relative to the change in total driver profits, because participation is endogenous and large relative profit gains will be spread among an increasing supply of participating drivers, or relative profit losses will be moderated by a decreasing supply.

Rider service level. Panels (e) and (f) show the rider service level. In the Minimal Control regime, the service level drops significantly when the load ratio $\overline{S}/N$ increases; this is aggravated when the demand imbalance is higher. Centralized platform control boosts the service level significantly at moderate values of $\overline{S}/N$. However, at at higher values of $\overline{S}/N$, when capacity is scarce, the adverse effect of the commission ($\gamma = 0.2$) on driver participation may dominate the benefits of Centralized Control, resulting in a lower service level; e.g., see panel (e) for $\overline{S}/N > 1$.

In summary, in the presence of demand imbalance, platform control capabilities can generate significant value for all parties at moderate load ratios, provided that the commission rate is commensurate with the extra gross revenue that better control yields for the drivers.

Chapter 4

Surge Pricing and Dynamic Matching for Hotspot Demand Shock in Ride-Hailing Networks

4.1 Introduction

Ride-hailing platforms such as Uber, Lyft and Via face non-stationary demand that exhibits not only predictable time-of-day and day-of-week fluctuations, but also unpredictable *demand shocks* that occur at a hotspot due to special events (e.g., the end of a concert, a random disruption in public transportation, etc.). In presence of such shocks, the problem of dynamically pricing and spatially matching rider demand and driver supply faces four challenges: (i) The demand shock is of uncertain magnitude and/or duration. (ii) The riders are price- and delay- sensitive, and specifically they only accept matching delays that do not exceed their idiosyncratic delay tolerance. (iii) The drivers are geographically dispersed and behave strategically in deciding whether to proactively reposition toward the hotspot, given their distance to and their expected payoff at the hotspot. (iv) Due to their travel delay and payoff risk, the drivers' response to a hotspot wage surge is both delayed and uncertain. Motivated by these challenges, we study the platform's optimal dynamic pricing and matching control problem, focusing on two questions:

(i) What is the platform's optimal surge pricing policy (in terms of rider price and driver wage) and matching policy under a demand shock of uncertain duration?

(ii) How do the optimal policy and the system performance depend on the interplay of three key timescales—rider patience, demand shock duration, and drivers' travel delay to the hotspot?

This work contributes to the literature on the sharing economy, with focus on ride-hailing networks. The key novelty of this chapter is that it jointly considers (i) *non-stationary* *demand*, (ii) *geographically dispersed supply* and (iii) *delayed incentives*. Existing studies focus on networks without incentives (e.g., Hu and Zhou (2015), O'Mahony and Shmoys (2015), Braverman et al.(2016), Iglesias et al. (2016), Ozkan and Ward (2017), Freund et al. (2017)), on single-location models with incentives (e.g., Bai et al. (2016), Benjaafar et al. (2016), Cachon et al. (2016), Castillo et al. (2017), Gurvich et al. (2016), Riquelme et al. (2016), Taylor (2016)), or on stationary networks with incentives (Afche et al. (2018), Bimpikis et al. (2017)). Guda and Subramanian (2017) study surge pricing under non-stationary demand in a two-location, two-period model. Castro et al. (2017) study surge pricing and spatial supply response without timescale considerations.

We study the transient behavior of a ride-hailing network over a geographic area with non-stationary demand at a hotspot. Riders are price- and delay-sensitive: they only accept matches that cost less than their willingness-to-pay and arrive within their delay tolerance. Drivers are dispersed over the geographic area and decide, upon receiving a hotspot wage surge signal, whether to reposition toward the hotspot. They trade off local and certain matches at the regular wage with hotspot matches at the surged wage that are subject to the risk of being left unmatched when the shock ends. The platform seeks to maximize its extra profit from the demand shock by jointly determining the pricing and matching policy. Specifically, the platform responds to a demand shock with uncertain magnitude and/or duration, (i) by dynamically controlling the rider price and driver wage, at a constant or varying level over a fixed or random surge duration, where the features of these policies are designed to moderate demand and incentivize drivers to proactively reposition toward the hotspot; (ii) by spatially matching riders with drivers, subject to drivers' incentive-compatibility constraints (they trade off non-hotspot local matches with more profitable hotspot matches), riders' delay tolerance, and overall service level constraints.

We formulate and solve the platform's profit maximization problem under two system regimes: (1) centralized system: drivers are treated like "autonomous vehicles" and there is no wage decision; (2) decentralized system: drivers are strategic and need to be incentivized by a hotspot wage surge. We also characterize the impact of the key timescale parameters, namely, the rider patience, the demand shock duration, and drivers' travel delays, on the optimal policy and the system performance.

We contribute to the literature on three dimensions.

(i) *Novel model of dynamic pricing and spatial matching under non-stationary demand.* The model we propose captures the interplay between non-stationary demand, geographically dispersed supply, and delayed and risky supply response and incentives. The uncertain demand shock leads to a dynamic and stochastic hotspot surge price and wage policy, and given the driver dispersion, yields delayed and uncertain supply and incentives: a wage surge only affects the available supply after some (travel) delay; and drivers only benefit from repositioning if they reach the hotspot "on time".

(ii) *Optimal dynamic pricing and spatial matching policy.* We study and compare the performance of various pricing policies under centralized and decentralized regimes. For policies that equate the price/wage surge duration to the demand surge duration, the decentralized system matches drivers from within a *nearer* distance, offers a higher hotspot wage and charges riders a *higher* price. In responding to the wage surge, drivers account for their risk of missing out by reaching the hotspot too late, so that only nearby drivers reposition to the hotspot. Motivated by this result, we study price policies that improve performance by optimally controlling the duration of the wage surge vis- -vis the demand surge duration to mitigate drivers' repositioning risk.

(iii) *Impact of key timescales on the optimal policy and system performance.* The optimal rider price and driver wage at the hotspot, and the matching distance of non-hotspot supply, depend crucially on the rider patience. Our results, contrasted with extreme timescale parameter settings, provide intuition on asymptotic system behaviors where riders are impatient (loss model) or infinitely patient (static planning), drivers that can relocate to the hotspot quickly (light traffic) or slowly (heavy traffic), and the demand shock duration is short (unpredictable events) or relatively long (predictable intraday pattern). In the decentralized system, farther drivers are matched when the rider patience is higher, the demand shock duration is longer, and drivers' travel delays to the hotspot are shorter; the hotspot surge price is higher when the rider patience is lower, the demand shock duration is shorter, and drivers' travel delays are shorter.

4.2 Model and Problem Formulation

4.2.1 Hotspot Demand Shock

Hotspot Demand Shock. The hotspot demand shock has a magnitude, $\Lambda(p) \leq \bar{\Lambda}$, a decreasing function of the rider price charged at the hotspot. We consider the shock duration to be deterministic or exponential with mean $\bar{T}$, $T \sim \exp(1/\bar{T})$.

Matching Decision. The platform offers wage surge signal to drivers within distance u_t, the wage surge range, at time t. The offered wage within u_t should be high enough to incentivize drivers to reposition to the hotspot. The surge duration is denoted by τ, which can be the same as the demand shock duration T or longer.

Drivers can be matched if they are in the rider patience zone—within distance θ from the hotspot. Therefore the actual matching rate at time t is the sum of instantaneous driver available rate within distance θ and the driver entering rate at distance θ. Let ϕx_t be the matching rate at time t. We have

$$x_t = \min\{u_t, \theta\} + \int_0^t \mathbf{1}(u_s \geq \theta + t - s)ds, \quad t \in [0, T], \tag{4.1}$$

where the first term is the instant matching rate and the second term counts the entering rate through the θ-zone. To get the second term, refer to Figure 4.1 (a). First note that drivers entering the θ-zone at time t may be those that start repositioning at time s and distance $t - s$ from the θ-zone, for $s \in [0, t]$, which is depicted by the -45 degree dashed line; then integrating over $[0, t]$ the time ds when the wage surge range at time s is larger than $\theta + t - s$ (the orange solid segments on the -45 degree dashed line) yields this entering rate.

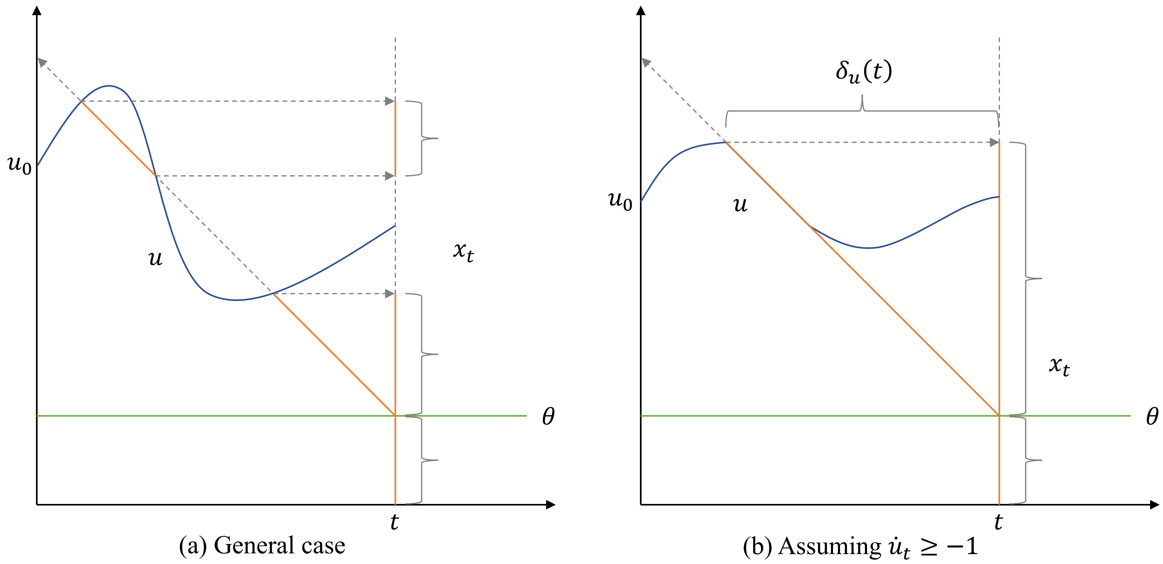


Figure 4.1: Derivation of x_t

Notice from Figure 4.1 (a) that, in the general case, x_t involves summing the time segments before t where u is above or on the -45 degree line, which can be mathematically cumbersome. To simplify the exposition without loss of generality, we focus on the control policies where u is continuous and does not decrease too fast so that x_t has a straightforward form. This is formalized by the following assumption and proposition.

Assumption 4.1. *u_t is continuous in t and $\dot{u}_t \geq -1$, $\forall t \geq 0$.*

Under Assumption 4.1, we can derive x_t given $u_s, s \in [0, t]$ in the following proposition. (All proofs are in the Appendix.)

Proposition 4.1. *Given the surge range profile $u_s \geq 0, s \in [0, t]$ and assuming Assumption 4.1, the matching rate at time t is given by*

$$x_t = \begin{cases} u_t & \text{if } 0 \leq u_t < \theta \\ \theta + t & \text{if } u_t \geq \theta \text{ and } t < u_0 - \theta \\ \theta + \delta_u(t) & \text{if } u_t \geq \theta \text{ and } t \geq u_0 - \theta \end{cases} \tag{4.2}$$

$$= (u_t \wedge \theta) + \delta_u(t), \tag{4.3}$$

where

$$\delta_u(t) := \max\{0 \leq s \leq t : u_{t-s} \geq \theta + s\} \tag{4.4}$$

is the "delay of control" at time t—the state at time t is determined by the control $\delta_u(t)$ time units ago. When $\{u_{t-s} \geq \theta + s\} = \emptyset$, we follow the convention that $\delta_u(t) = 0$.

Figure 4.1 (b) illustrates the derivation of x_t under Assumption 4.1. When $\dot{u}_t \geq -1, \forall t \geq 0$, the integral in (4.1) involves at most one time segment of length $\delta_u(t)$ instead of several disconnected time segments. As a result, x_t is only determined by the control $\delta_u(t)$ time units ago.

The following corollary of Proposition 4.1 gives the derivative of x as a function of u and its derivative.

Corollary 4.1. *Under Assumption 4.1, the derivative of the matching rate at time t is given by*

$$\dot{x}_t = \begin{cases} \dot{u}_t & \text{if } 0 \leq u_t < \theta \\ 1 & \text{if } u_t \geq \theta \text{ and } t < u_0 - \theta \\ 1 - \frac{1}{1+\dot{u}_{t-\delta_u(t)}} & \text{if } u_t \geq \theta \text{ and } t \geq u_0 - \theta \end{cases}, \tag{4.5}$$

where $\dot{u}_{t-\delta_u(t)}$ denotes the derivative of u at time $t - \delta_u(t)$. If $\dot{u}_{t-\delta_t} = -1$, x is discontinuous at t with a downward jump and $\dot{x}_t$ does not exist.

The third case in (4.5) captures the monotonic relationship between $\dot{u}$ and $\dot{x}$ (with time delay) and is summarized in Table 4.1.

$\dot{u}_{t-\delta_u(t)}$	$\dot{x}_t$
$+\infty$	1
$(0, +\infty)$	$(0, 1)$
0	0
$(-1, 0)$	$(-\infty, 0)$
-1	$-\infty$

Table 4.1: Relationship between $\dot{u}_{t-\delta_u(t)}$ and $\dot{x}_t$ when $u_t \geq \theta$ and $t \geq u_0 - \theta$

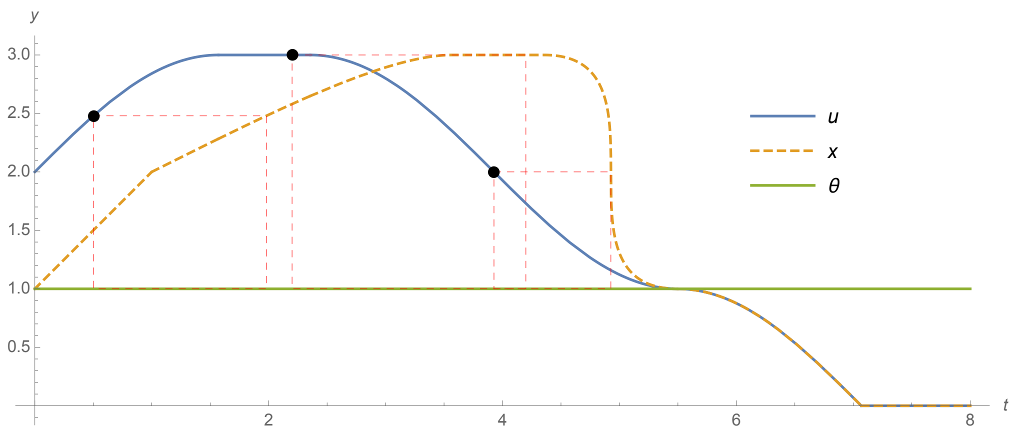


Figure 4.2: Exemplary control and state trajectories

Figure 4.2 shows exemplary control and state trajectories that contain all three cases in Proposition 4.1 and Corollary 4.1. The delayed effect of u on x, $x_t = u_{t-\delta_u(t)}$, is depicted by the three red dashed squares as examples. To distinguish the timelines of control u and state x and avoid ambiguity, we use t^u and t^x respectively, instead of t, when necessary. Let t^x be the timeline of x (so x_{t^x} is an explicit form of x_t), then let

$$t^u := t^x - \delta_u(t^x) \tag{4.6}$$

be the timeline of u. The following lemma relates x_{t^x}, t^x and u_{t^u}, t^u in terms of their value and derivatives.

Lemma 4.1. *For any $t^x \geq (u_0 - \theta)^+$, $x_{t^x} = u_{t^u}$. Furthermore,*
(i) If $0 \leq u_{t^x} < \theta$, $\frac{dt^x}{dt^u} = 1$.
(ii) If $u_{t^x} \geq \theta$ and $\dot{u}_{t^u} > -1$, then

$$\frac{dt^x}{dt^u} = 1 + \dot{u}_{t^u} \equiv \frac{1}{1 - \dot{x}_{t^x}}. \tag{4.7}$$

If $u_{t^x} \geq \theta$ and $\dot{u}_{t^u} = -1$, then x_t is discontinuous at t^x and $dt^x = 0$.

Using Lemma 4.1, we can prove the following key lemma which relates the integral of functions of u_{t^u} and x_{t^x}.

Lemma 4.2. *For any function $f(\cdot)$ and $0 \leq a < b$,*

$$\int_a^b f(u_t)dt = \int_{a+(u_a-\theta)^+}^{b+(u_b-\theta)^+} f(x_t)dt - \int_{u_a \vee \theta}^{u_b \vee \theta} f(x)dx. \tag{4.8}$$

4.2.2 Rider Price and Driver Wage

Let p_0 and w_0 be the constant rider price and driver wage at non-hotspot locations, respectively. We assume the platform *does not* change non-hotspot pricing at any time. Next we consider the pricing at the hotspot.

Hotspot rider price p_t is a function of the matching rate, $p_t = p(x_t)$, where $p(x)$ is the highest price to get $\phi x \leq \bar{\Lambda}$ hotspot demand:

$$\Lambda(p) = \phi x \Rightarrow p(x) = \Lambda^{-1}(\phi x). \tag{4.9}$$

At time t, the platform's hotspot revenue rate is therefore $\phi p(x_t) x_t$.

Let $w_t(y)$ be the *incentive compatible* (IC) hotspot wage at time t that makes a (marginal) driver at distance y indifferent between staying local and repositioning to the hotspot. The wage is quoted when the driver starts to reposition but is paid when the driver is matched. When the IC condition—hence drivers' repositioning decision—is *time independent* (which is true under deterministic or exponential surge duration), the IC wage is also time independent, denoted by $w(y)$. Thereafter we are restricted to $w(y)$ without explicit notification.

We consider two classes of wage quoting policies: *broadcast* and *personalized message*. Under the broadcast policy, a common wage is quoted to all drivers (but paid individually upon matching). Given that the platform chooses wage surge range u_t and the IC wage $w(y)$ is increasing in y, all drivers that start repositioning at t will be paid $w(u_t)$ if matched eventually. Hence the platform's wage payment rate at time t is

$$\phi W_t^b = \phi \left[\int_0^{u_t \wedge \theta} w_t(u_t) dy + \int_0^t \mathbf{1}(u_s \geq \theta + t - s) w_s(u_s) ds \right]. \tag{4.10}$$

Under time-independent wage $w(y)$ and using Proposition 4.1, we can show (DETAILS can be a lemma) that (4.10) is simplified as

$$\phi W_t^b = \phi \int_0^{x_t} w(u_{t-(y-\theta)^+}) dy. \tag{4.11}$$

Note that if all repositioning drivers can be matched (which is true under deterministic shock duration), we may adopt the wage-payable rate $\widetilde{W}_t^b = w(u_t) u_t$. In this case the common wage $w(u_t)$ is IC to the marginal drivers at u_t but overpaying drivers at $x < u_t$.

Under the personalized message policy, each driver at x is quoted his/her own IC wage $w(x)$. Hence the platform's wage payment rate at time t is

$$\phi W_t^p = \phi \left[\int_0^{u_t \wedge \theta} w_t(y)dy + \int_0^t \mathbf{1}(u_s \geq \theta + t - s) w_s(\theta + t - s) ds \right]. \tag{4.12}$$

Under time-independent wage $w(y)$ and using Proposition 4.1, we can show (DETAILS can be a lemma) that (4.12) is simplified as

$$\phi W_t^p = \phi \int_0^{x_t} w(x)dx. \tag{4.13}$$

4.2.3 Sequence of Events

In this continuous time model, the sequence of events is as follows.

(1) Platform determines wage surge range u_t at time t.

(2) Platform quotes surged wage to drivers at $u \in [0, u_t]$ at time t using broadcast or personalized message policy.

(3) Drivers within $[0, u_t]$ reposition toward the hotspot and are matched immediately (if within $[0, \theta]$) or later (if within $(\theta, u_t]$ by the end of the surge duration).

(4) Matching rate ϕx_t realized (x_t yielded by $u_s, 0 \leq s \leq t$ through (4.1))

(5) ϕx_t riders are matched and charged total price $\phi p(x_t)x_t$;
ϕx_t drivers are matched and paid total wage ϕW_t^b or ϕW_t^p.

4.3 Deterministic Demand Shock Duration

Given deterministic shock duration $T > 0$, the platform determines the wage surge range $u_t > 0, t \in [0, \tau]$ and quotes the implied IC wage using the broadcast or personalized message policy. As T is deterministic, optimal surge duration $\tau = T$ since (i) $\tau \geq T$—if it is optimal to surge ($\tau > 0$), then it is optimal to surge at least until T, otherwise there is lost extra hotspot profit; and (ii) $\tau \ngtr T$—there is no hotspot demand after $T \Rightarrow$ no matching needed $\Rightarrow$ no wage surge.

Since the demand shock duration and surge duration are both T known to all drivers, only drivers that can be within the patience zone θ before the shock ends will reposition. This gives rise to the IC wage determined by the following equation, where the wage is used to cover drivers' lost revenue during repositioning:

$$\frac{w_0}{m} = \frac{w_t(y)}{y+m}, \quad y \leq \theta + T - t. \tag{4.14}$$

It follows that the effective wage surge range needs to satisfy $u_t \leq \theta + T - t$, for $t \in [0, T]$. Therefore the platform's profit maximization problem can be formulated as

$$\max_{u_t: t\in[0,T]} \quad \phi \int_0^T \{[p(x_t)x_t - W_t] - (p_0 - w_0)u_t\}\, dt \tag{4.15}$$

$$\text{s.t.} \quad 0 \leq x_t \leq \theta + t, \quad t \in [0, T] \tag{4.16}$$

$$0 \leq u_t \leq \theta + T - t, \quad t \in [0, T] \tag{4.17}$$

$$x_t = \min\{u_t, \theta\} + \int_0^t \mathbf{1}(u_{t-s} \geq \theta + s) ds, \quad t \in [0, T]. \tag{4.18}$$

We next consider three wage policies with increasing flexibility:

(i) Static broadcast: set u_t as a constant u in the above optimization problem and choose W_t^b given by (4.11) in (4.15).

(ii) Dynamic broadcast: choose W_t^b given by (4.11) in (4.15).

(iii) Dynamic personalized message: choose W_t^p given by (4.13) in (4.15).

4.3.1 Static Broadcast Wage Policy

Under this policy the platform chooses a constant wage level $w(u_0)$ that is incentive compatible with surge range u_0. The effective surge range u_t hence follows the structure:

$$u_t = \begin{cases} \min\{u_0, \theta + T - t\} & \text{if } u_0 \geq \theta \\ u_0 & \text{if } u_0 < \theta \end{cases}.$$

The optimization problem is reduced to

$$\max_{u_t:t\in[0,T]} \quad \phi \int_0^T \{[p(x_t)x_t - w(u_0)u_t] - (p_0 - w_0)u_t\}\, dt \tag{4.19}$$

$$\text{s.t.} \quad u_t = \begin{cases} \min\{u_0, \theta + T - t\} & \text{if } u_0 \geq \theta \\ u_0 & \text{if } u_0 < \theta \end{cases}, \quad t \in [0, T] \tag{4.20}$$

$$x_t = (u_t \wedge \theta) + \delta_t, \quad t \in [0, T]. \tag{4.21}$$

Proposition 4.2 (Static Broadcast)**.** *Under the static broadcast wage policy, the optimal wage surge range is* $u_t^* = \min\{u_0^*, \theta + T - t\},\ t \in [0, T]$ *in which*

$$\begin{aligned} u_0^* = \arg\max_{u_0 \geq 0} & \, G(u_0)u_0 T \mathbf{1}(u_0 < \theta) \\ & + \left[G(u_0)u_0[T - (u_0 - \theta)] + \int_\theta^{u_0} ((p(u) - w(u_0)) - (p_0 - w_0))\, u du \right] \mathbf{1}(u_0 \geq \theta), \end{aligned} \tag{4.22}$$

where $G(u)$ *is the extra profit per hotspot match defined as*

$$G(u) \coloneqq (p(u) - w(u)) - (p_0 - w_0). \tag{4.23}$$

4.3.2 Dynamic Broadcast Wage Policy

Under this policy the optimization problem is

$$\max_{u_t:t\in[0,T]} \quad \phi \int_0^T \{[p(x_t)x_t - w(u_t)u_t] - (p_0 - w_0)u_t\}\, dt \tag{4.24}$$

$$\text{s.t.} \quad 0 \leq u_t \leq \theta + T - t, \quad t \in [0, T] \tag{4.25}$$

$$x_t = (u_t \wedge \theta) + \delta_t, \quad t \in [0, T]. \tag{4.26}$$

Proposition 4.3 (Dynamic Broadcast)**.** *Under the dynamic broadcast wage policy, the optimal wage surge range is*

$$u_t^* = \arg\max_{0 \leq u \leq \theta + T - t} G(u)u, \quad t \in [0, T], \tag{4.27}$$

where $G(\cdot)$ *is defined in (4.23).*

4.3.3 Dynamic Personalized Message Wage Policy

Under this policy the optimization problem is

$$\max_{u_t:t\in[0,T]} \quad \phi \int_0^T \left\{ \left[p(x_t)x_t - \int_0^{x_t} w(x)dx \right] - (p_0 - w_0)u_t \right\} dt \tag{4.28}$$

$$\text{s.t.} \quad 0 \le u_t \le \theta + T - t, \quad t \in [0,T] \tag{4.29}$$

$$x_t = (u_t \wedge \theta) + \delta_t, \quad t \in [0,T]. \tag{4.30}$$

Proposition 4.4 (Dynamic Personalized Message)**.** *Under the dynamic personalized message wage policy, the optimal wage surge range is*

$$u_t^* = \arg \max_{0\le u\le \theta+T-t} G^p(u)u, \quad t \in [0,T], \tag{4.31}$$

where $G^p(u)$ *is the extra profit per hotspot match defined as*

$$G^p(u) := \left(p(u) - \frac{\int_0^u w(s)ds}{u} \right) - (p_0 - w_0). \tag{4.32}$$

4.3.4 Comparing the Three Wage Policies

Theorem 4.1. *Let* u_t^s, u_t^b, u_t^p *be the optimal solution in Propositions 4.2, 4.3 and 4.4, respectively, then* $u_t^s \le u_t^b \le u_t^p$*.*

Theorem 1 compares/ranks the optimal surge range. Next to compare performance measures and possible bounds on the improvement.

4.4 Exponential Demand Shock Duration

In this section we consider exponential demand shock duration, $T \sim \exp(1/\bar{T})$. The platform determines the wage surge range $u_t > 0$, $t \in [0,\tau]$ and quotes the implied wage using the personalized message policy. We consider two types of surge duration τ: (i) *restricted* surge duration where the surge ends when the demand shock ends, i.e., $\tau = T$, and (ii) *relaxed* surge duration where the surge may extend beyond the demand shock duration, i.e., $\tau \ge T$, in order to incentivize faraway drivers.

The platform's profit maximization problem can be formulated as

$$
\begin{aligned}
\max_{u_t:t\in[0,\tau],\tau\geq T} \quad & \mathbb{E}_{T\sim\exp(1/\bar{T})}\phi\int_0^T \left[p(x_t)x_t-(p_0-w_0)u_t\right]dt \\
& \qquad\qquad -\phi\int_0^\tau\int_0^{u_t}\mathbf{1}(\tau-t\geq(y-\theta)^+)w_t(y)dydt \\
\text{s.t.} \quad & 0\leq x_t\leq\theta+t, \quad t\in[0,T] \\
& x_t=\min\{u_t,\theta\}+\int_0^t\mathbf{1}(u_{t-s}\geq\theta+s)ds, \quad t\in[0,T] \\
& \frac{w_0}{m}=\frac{w_t(y)}{y+m}\bar{F}_{\tau-t}((y-\theta)^+)+\int_0^{(y-\theta)^+}\frac{\frac{w_0}{m}(y+m-s)}{y+m}f_{\tau-t}(s)ds.
\end{aligned}
$$

We next consider the two surge duration types.

4.4.1 Restricted Surge Duration

The platform offers surged price and wage during the demand shock duration. As is mentioned in 4.2.2, since the surge duration coincides with the exponential shock duration, the drivers' repositioning decision is time independent but only location dependent, hence the IC wage is also time independent, denoted by $w(x)$. Using $\tau=T$ and (4.3), we can obtain

$$
\int_0^T\int_0^{u_t}\mathbf{1}(T-t\geq(y-\theta)^+)w(y)dydt=\int_0^T\int_0^{x_t}w(x)dxdt, \tag{4.33}
$$

and the platform's problem is simplified to

$$
\begin{aligned}
\max_{u_t:t\in[0,T]} \quad & \mathbb{E}_{T\sim\exp(1/\bar{T})}\phi\int_0^T\left\{\left[p(x_t)x_t-\int_0^{x_t}w(x)dx\right]-(p_0-w_0)u_t\right\}dt \\
\text{s.t.} \quad & x_t=\begin{cases} u_t & \text{if } 0\leq u_t<\theta \\ \theta+t & \text{if } u_t\geq\theta \text{ and } t<u_0-\theta \\ \theta+\max\{\delta\geq 0: u_{t-\delta}-\theta\geq\delta\} & \text{if } u_t\geq\theta \text{ and } t\geq u_0-\theta \end{cases}, \quad t\in[0,T] \\
& \frac{w_0}{m}=\frac{w(y)}{y+m}\bar{F}_T((y-\theta)^+)+\int_0^{(y-\theta)^+}\frac{\frac{w_0}{m}(y+m-s)}{y+m}f_T(s)ds.
\end{aligned}
$$

The following Lemma from Klinger (1969) transforms the expectation of an integral over a random duration to an integral over the infinite horizon.

Lemma 4.3 (Klinger 1969)**.** *If $T\sim F$ is a positive random variable and the lhs of (4.34) is finite, then*

$$
\mathbb{E}_{T\sim F}\int_0^T g(\mathbf{x},\mathbf{u},t)dt=\int_0^\infty g(\mathbf{x},\mathbf{u},t)(1-F(t))dt. \tag{4.34}
$$

Applying this Lemma, the objective function becomes

$$\phi \int_0^\infty \left\{ \left[p(x_t)x_t - \int_0^{x_t} w(x)dx \right] - (p_0 - w_0)u_t \right\} e^{-\frac{t}{T}} dt. \tag{4.35}$$

Using Lemma 4.1, we get

$$\begin{aligned} \int_0^\infty e^{-\frac{t}{T}} u_t dt &= \int_{(u_0-\theta)^+}^\infty e^{-\frac{t-(x_t-\theta)^+}{\bar{T}}} \frac{dt^u}{dt^x} x_t dt \\ &= \int_{(u_0-\theta)^+}^\infty e^{-\frac{t-(x_t-\theta)^+}{\bar{T}}} (1 - \dot{x}_t \mathbf{1}(x_t \geq \theta)) x_t dt \\ &= \int_{(u_0-\theta)^+}^\infty e^{\frac{(x_t-\theta)^+}{\bar{T}}} (1 - \dot{x}_t \mathbf{1}(x_t \geq \theta)) x_t e^{-\frac{t}{T}} dt, \end{aligned}$$

so the objective function can be written as

$$\begin{aligned} \max_{u_0 \geq 0, x_t : \dot{x}_t \leq 1} &\left\{ \phi \int_0^{(u_0-\theta)^+} \left[p(x_t)x_t - \int_0^{x_t} w(x)dx \right] e^{-\frac{t}{T}} dt \right. \\ &\left. + \phi \int_{(u_0-\theta)^+}^\infty \left\{ \left[p(x_t)x_t - \int_0^{x_t} w(x)dx \right] - (p_0 - w_0) e^{\frac{(x_t-\theta)^+}{\bar{T}}} (1 - \dot{x}_t \mathbf{1}(x_t \geq \theta)) x_t \right\} e^{-\frac{t}{T}} dt \right\}. \end{aligned} \tag{4.36}$$

Let

$$R(x) := p(x)x - \int_0^x w(s)ds, \quad L(x) := (p_0 - w_0) e^{\frac{(x-\theta)^+}{\bar{T}}} x, \tag{4.37}$$

and use (4.2) in Proposition 4.1, the objective function becomes

$$\max_{x_t : \dot{x}_t \leq 1} \int_0^\infty \phi \left[R(x_t) - (1 - \dot{x}_t \mathbf{1}(x_t \geq \theta)) L(x_t) \right] e^{-\frac{t}{T}} dt. \tag{4.38}$$

Define

$$V(x,t) := \max_{x_s \geq 0, \dot{x}_s \leq 1} \int_t^\infty \phi \left[R(x_s) - (1 - \dot{x}_s \mathbf{1}(x_s \geq \theta)) L(x_s) \right] e^{-\frac{s}{T}} ds \quad \text{and} \quad V(x) := V(x,0).$$

Since the cost function depends on time only through the discounting factor $e^{-s/\bar{T}}$ and the control $\dot{x}_s$ does not depend on time, the Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\begin{aligned} \frac{1}{\bar{T}} V(x) &= \max_{\dot{x} \leq 1} \{ \phi \left[R(x) - (1 - \dot{x} \mathbf{1}(x \geq \theta)) L(x) \right] + V'(x)\dot{x} \} \\ &= \max_{\dot{x} \leq 1} \{ [\phi \mathbf{1}(x \geq \theta)) L(x) + V'(x)] \dot{x} \} + \phi [R(x) - L(x)]. \end{aligned}$$

Hence

$$V(x) = \begin{cases} \bar{T}\{\phi R(x) + V'(x) + \phi[\mathbf{1}(x \geq \theta)) - 1]L(x)\} & \text{if } \phi\mathbf{1}(x \geq \theta))L(x) + V'(x) > 0 \\ \bar{T}\phi[R(x) - L(x)] & \text{if } \phi\mathbf{1}(x \geq \theta))L(x) + V'(x) = 0 \\ \infty & \text{if } \phi\mathbf{1}(x \geq \theta))L(x) + V'(x) < 0 \end{cases}$$

and

$$\dot{x}^* = \begin{cases} 1 & \text{if } \phi\mathbf{1}(x \geq \theta))L(x) + V'(x) > 0 \\ \text{any value} \leq 1 & \text{if } \phi\mathbf{1}(x \geq \theta))L(x) + V'(x) = 0 \cdot \\ -\infty & \text{if } \phi\mathbf{1}(x \geq \theta))L(x) + V'(x) < 0 \end{cases}$$

Therefore the optimal state trajectory $\{x_t : t \geq 0\}$ is one where x_t first increases at speed 1 from θ to $\bar{x}$ which solves the second case:

$$\mathbf{1}(x \geq \theta))L(x) + \bar{T}[R'(x) - L'(x)] = 0, \tag{4.39}$$

and then stays at $\bar{x}$. It follows that $u_0^* = \bar{x}$.

4.4.1.1 Static Personalized Message Wage Policy

Given $u_0 \geq \theta$, the key maximization problem (second term in (4.38)) is

$$\max_{x_t : -\infty < \dot{x}_t \leq 1} \int_{u_0 - \theta}^{\infty} [R(x_t) - (1 - \dot{x}_t)L(x_t)]\, e^{-\frac{t}{T}}\, dt. \tag{4.40}$$

Assuming $R(x)$ is concave and unimodal (e.g., a linear price function $p(x)$ and convex total wage rate $\int_0^x w(s)ds$ can imply this), since $L(x)$ is convex, $R(x) - (1 - \dot{x}_t)L(x)$ is concave and unimodal for any fixed $\dot{x}_t$. Therefore, for policy class with $\dot{x}_t = 0$, the optimal solution is

$$x_t^* \equiv x^* = \arg\max_x \{R(x) - L(x)\}, \quad t \geq u_0 - \theta. \tag{4.41}$$

4.4.1.2 Dynamic Personalized Message Wage Policy

We consider if any deviation from the optimal static wage policy x^* can yield extra profit. In specific, we consider the deviation pattern that first increases x from x^* at a constant speed to x', and then decreases x at (another) constant speed back to x^*. The intuition is as follows: when increasing x, $\dot{x}_t > 0$ makes the profit rate in (4.40) higher than $R(x) - L(x)$, while when decreasing x, $\dot{x}_t < 0$ makes the profit rate lower. However due to the exponential decaying factor $e^{-\frac{t}{\bar{T}}}$, the profit gain in the increasing phase may exceed the profit loss in the decreasing phase, resulting in a net gain.

The following lemma characterizes the profit gain (loss) from the aforementioned wage deviation pattern.

Lemma 4.4. *The profit gain from deviation $x^* \xrightarrow{v} x' \xrightarrow{u} x^*$ (increase x from x^* to x' at speed v and decrease x back to x^* at speed u) at time $t > u_0 - \theta$ where $x' > x^*$ is*

$$\Pi(x') = e^{-\frac{t}{\bar{T}}} \int_{x^*}^{x'} \left\{ \left[\frac{1}{v}\left[(R(x) - L(x)) - (R(x^*) - L(x^*))\right] + L(x)\right] e^{-\frac{1}{\bar{T}}\frac{x-x^*}{v}} \right. \\ \left. - \left[\frac{1}{u}\left[(R(x^*) - L(x^*)) - (R(x) - L(x))\right] + L(x)\right] e^{-\frac{1}{\bar{T}}\left[\frac{x'-x^*}{v} + \frac{x'-x}{u}\right]} \right\} dx. \tag{4.42}$$

When $\bar{T} \to \infty$, the limit is

$$-\int_{x^*}^{x'} \left\{ \left(\frac{1}{v} + \frac{1}{u}\right) \left[(R(x^*) - L(x^*)) - (R(x) - L(x))\right] \right\} dx < 0, \tag{4.43}$$

where the inequality follows for any $v, u > 0$ since x^* maximizes $R(x) - L(x)$.

Proposition 4.5. *If $\bar{T} = \infty$, the static policy is optimal; if $\bar{T} < \infty$, there exists $x' > x*$ and $v \geq 0, 0 \leq u \leq 1$ such that repeated deviation $x^* \xrightarrow{v} x' \xrightarrow{u} x^*$ yields higher expected profit than the static policy.*

The intuition is that when the expected shock/surge duration is short, increasing x to capture more matching may benefit more than the potential loss from the decreasing phase, when the shock is more likely to have ended already. Based on the equivalent formulation with an exponential decaying factor as in (4.35), the extra gain from the increasing phase is discounted less than the extra loss from the decreasing phase.

4.4.2 Relaxed Surge Duration

Now we consider relaxed surge duration where the platform may surge longer than the demand shock duration, i.e., $\tau \geq T$, in order to incentivize faraway drivers. By surging longer, the platform shares some or all fail-to-match risk with the drivers and pays less hotspot wage to incentivize drivers. We study two specific policies of relaxed surge durations: (1) guaranteed wage and (2) minimum surge duration.

With $\tau \geq T$, since it is profitless to offer surge signal (and wage) to any driver after the demand shock, we make the following assumption on the wage surge range u_t.

Assumption 4.2. *$u_t = 0$ for $T \leq t \leq \tau$.*

4.4.2.1 Guaranteed Wage

Under this policy the platform guarantees that all drivers who reposition before the demand shock ends will be paid a certain guaranteed wage, depending on the starting position of the drivers. This implies a time-varying leftover surge duration $\tau(t) = \max\{T - t, (u_t - \theta)^+\}$ seen by drivers at time t. With guaranteed wage payment all repositioning drivers face no fail-to-match risk, and hence their repositioning decision is time independent but only location dependent. Denote $\tilde{w}(x)$ as the incentive compatible hotspot wage for drivers at distance x under the guaranteed wage policy. Consequently the incentive compatible wage is lower than that under the restricted surge duration policy.

Lemma 4.5. *$\tilde{w}(x) \leq w(x)$ for $x \geq 0$. More specifically, $\tilde{w}(x) = w(x)$ for $0 \leq x \leq \theta$ and $\tilde{w}(x) < w(x)$ for $x > \theta$.*

Using $\tau(t) = \max\{T - t, (u_t - \theta)^+\}$ and (4.3), we have $\tau = T + (u_T - \theta)^+$ and hence obtain

$$\int_0^\tau \int_0^{u_t} \mathbf{1}(\tau - t \geq (y - \theta)^+)\tilde{w}(x)dxdt = \int_0^T \int_0^{u_t} \tilde{w}(x)dxdt. \tag{4.44}$$

Applying Lemma 4.3, the platform's problem is simplified as

$$\max_{u_t:t\geq 0} \quad \phi \int_0^\infty \left\{ \left[p(x_t)x_t - \int_0^{u_t} \tilde{w}(x)dx \right] - (p_0 - w_0)u_t \right\} e^{-\frac{t}{T}}dt \tag{4.45}$$

$$\text{s.t.} \quad x_t = \begin{cases} u_t & \text{if } 0 \leq u_t < \theta \\ \theta + t & \text{if } u_t \geq \theta \text{ and } t < u_0 - \theta \\ \theta + \max\{\delta \geq 0 : u_{t-\delta} - \theta \geq \delta\} & \text{if } u_t \geq \theta \text{ and } t \geq u_0 - \theta \end{cases}, \quad t \in [0, T]$$

$$\frac{w_0}{m} = \frac{\tilde{w}(y)}{y+m}\bar{F}_T((y-\theta)^+) + \int_0^{(y-\theta)^+} \frac{\frac{\tilde{w}(y)}{m}(y+m-s)}{y+m} f_T(s)ds.$$

The following proposition shows that under *certain* static personalized message wage policy, the optimal choice between restricted surge duration and guaranteed wage depends on the expected demand shock duration $\bar{T}$.

Proposition 4.6. *Under static personalized message wage policy with control $u_t \equiv u_0 \geq \theta, t \geq 0$, there exists a threshold level $\bar{T}^* > 0$ such that the guaranteed wage yields lower expected platform profit than the restricted surge duration if $\bar{T} < \bar{T}^*$ and vice versa if $\bar{T} > \bar{T}^*$.*

4.4.2.2 Minimum Surge Duration

Under this policy the platform announces a minimum surge duration $\tau_0 > 0$ such that the actual surge duration is never shorter than τ_0, i.e., $\tau = \max\{T, \tau_0\}$. The time dependent incentive compatible hotspot wage $w_t(x)$ is given by:

$$w_t(x) = \begin{cases} \tilde{w}(x) & \text{if } (x-\theta)^+ \leq (\tau_0 - t)^+ \\ w(x) & \text{if } (x-\theta)^+ > (\tau_0 - t)^+ \end{cases}, \tag{4.46}$$

where $\tilde{w}(x)$ and $w(x)$ denote the (time independent) IC hotspot wage for drivers at distance x that have guaranteed wage payment (no fail-to-match risk) and that face exponential surge/shock duration (hence full fail-to-match risk risk), respectively.

Depending on the relative length of T and τ_0, the platform's wage payment to repositioning drivers can be derived as

$$\begin{aligned} \int_0^\tau \int_0^{u_t} \mathbf{1}(\tau - t \geq (x-\theta)^+) w_t(x) dx dt &= \int_0^T \int_0^{u_t \wedge (\theta + \tau - t)} w_t(x) dx dt \\ &= \begin{cases} \int_0^T \int_0^{u_t \wedge (\theta + \tau_0 - t)} \tilde{w}(x) dx dt & \text{if } \tau = \tau_0 > T \\ \int_0^T \int_0^{x_t} \left[\mathbf{1}(t \leq \tau_0)\tilde{w}(x) + \mathbf{1}(t > \tau_0) w(x)\right] dx dt & \text{if } \tau = T \geq \tau_0 \end{cases}, \end{aligned} \tag{4.47}$$

where the first equation uses Assumption 4.2 and the second equation follows from (4.46). Hence the expected wage payment is $\phi \mathbb{E}_{T \sim \exp(1/\bar{T})}$ Eq.(4.47).

Lemma 4.6. *Under the minimum surge duration policy, there exists a unique $\tau_0^* \geq 0$ that minimizes the platform's expected wage payment to repositioning drivers.*

Applying Lemma 4.3, the platform's problem is simplified as

$$\max_{u_t:t\geq 0} \quad \phi \int_0^\infty \{p(x_t)x_t - (p_0 - w_0)u_t\} \, e^{-\frac{t}{\bar{T}}} dt - \phi \mathbb{E}_{T\sim \exp(1/\bar{T})} \text{ Eq.(4.47)}$$

$$\text{s.t.} \quad x_t = \begin{cases} u_t & \text{if } 0 \leq u_t < \theta \\ \theta + t & \text{if } u_t \geq \theta \text{ and } t < u_0 - \theta \ , \\ \theta + \max\{\delta \geq 0 : u_{t-\delta} - \theta \geq \delta\} & \text{if } u_t \geq \theta \text{ and } t \geq u_0 - \theta \end{cases} \quad t \in [0, T]$$

$$\frac{w_0}{m} = \frac{\tilde{w}(y)}{y+m} \bar{F}_T((y-\theta)^+) + \int_0^{(y-\theta)^+} \frac{\frac{\tilde{w}(y)}{m}(y+m-s)}{y+m} f_T(s) ds$$

$$\frac{w_0}{m} = \frac{w(y)}{y+m} \bar{F}_T((y-\theta)^+) + \int_0^{(y-\theta)^+} \frac{\frac{w_0}{m}(y+m-s)}{y+m} f_T(s) ds.$$

Under certain static personalized message wage policy, the following proposition compares the minimum surge duration policy with the guaranteed wage and restricted surge duration policies at different levels of demand shock duration.

Proposition 4.7. *Under static personalized message wage policy with control* $u_t \equiv u_0 \geq \theta, t \geq 0$*, the minimum surge duration policy yields higher platform profit than the guaranteed wage policy when* $T \leq \tau_0^*$*, and than the restricted surge duration policy when* $T \geq \tau_0^*$*.*

Corollary 4.2. *When* $\bar{T} \to 0$*, restricted surge duration policy yields the highest platform profit; when* $\bar{T} \to \infty$*, guaranteed wage policy yields platform profit; when* $\bar{T}$ *is intermediate, minimum surge duration yields the highest platform profit.*

4.5 Summary

Chapter 3 focuses on operational controls to manage the steady-state equilibrium; this chapter, on the other hand, addresses transient but significant demand shocks at a hotspot, and focus on the drivers' strategic response to surge signals given delayed incentives. The platform responds to a demand shock with uncertain magnitude and/or duration at a hotspot, by optimizing (i) surge pricing, which is meant to moderate demand, and surge wages meant to incentivize drivers to proactively reposition toward the hotspot, and (ii) dynamic matching, which trades off non-hotspot local matches for more profitable hotspot matches.

The distinctive features of this chapter lie on the focus of system transient under *non-stationary* demand, the network setting, and drivers' strategic response to surge signals given *delayed incentives*. Our focus on the time effects sheds light on the interplay between rider patience, demand shock duration and driver's travel delay, which together play a crucial rule on the optimal operations of ride-hailing platforms.

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Appendices

Appendix A

Appendix for Chapter 1

A.1 Proofs

The proof of Proposition 1.4 is rather involved and lengthy and requires us to first demonstrate the preservation result in the special case where $\beta_1 = \beta_2 = 0$, as per the following Lemma.

Lemma A.1 (Preservation Property)**.** *If a function $g(\cdot)$ is strongly (C_1K_1, C_2K_2)-convex, then*

$$f_1(x) = \min_{y\in[x,x+C_1']}\{K_1\delta(y-x)+g(y)\},$$

$$f_2(x) = \min_{y\in[x-C_2',x]}\{K_2\delta(x-y)+g(y)\},$$

$$f(x) = \min\{f_1(x), f_2(x)\}.$$

are also strongly (C_1K_1, C_2K_2)-convex for any $C_1' \geq C_1, C_2' \geq C_2$.

Proof of Lemma A.1. (I) First, we show $f_1(\cdot) \in SC_{C_1K_1,C_2K_2}$. Let

$$\Delta_1 = K_1 + f_1(x+a) - f_1(x) - \frac{a}{b}\Big(f_1(y) - f_1(y-b) - K_2\Big). \tag{A.1}$$

It suffices to show that $\Delta_1 \geq 0$ for $y \leq x, a \in [0, C_1]$ and $b \in (0, C_2]$. To this end, we consider the following four different cases for the pair of values $f_1(x+a)$ and $f_1(y-b)$.

(a) $\underline{f_1(x+a) = g(x+a) \text{ and } f_1(y-b) = g(y-b)}$. In this case, we have

$$\begin{aligned}\Delta_1 &= K_1 + g(x+a) - f_1(x) - \frac{a}{b}\Big(f_1(y) - g(y-b) - K_2\Big)\\ &\geq K_1 + g(x+a) - g(x) - \frac{a}{b}\Big(g(y) - g(y-b) - K_2\Big) \geq 0,\end{aligned}$$

where the first inequality follows from the definition of $f_1(\cdot)$, and the second inequality follows from the strong (C_1K_1, C_2K_2)-convexity of $g(\cdot)$.

(b) $f_1(x+a) = g(x+a)$ and $f_1(y-b) = g(y-b+u) + K_1$ with some $u \in [0, C_1']$. In this case, we have

$$\Delta_1 = K_1 + g(x+a) - f_1(x) - \frac{a}{b}\Big(f_1(y) - g(y-b+u) - K_1 - K_2\Big).$$

Based on the value of $f_1(y)$ we consider the following two subcases:

(b.1) $f_1(y) \leq g(y-b+u) + K_1 + K_2$. Since $a \in [0, C_1] \subset [0, C_1']$ we know $f_1(x) \leq g(x+a) + K_1$, hence

$$\Delta_1 \geq K_1 + g(x+a) - f_1(x) \geq K_1 + g(x+a) - (g(x+a) + K_1) \geq 0.$$

(b.2) $f_1(y) > g(y-b+u) + K_1 + K_2$. Knowing that $0 \leq u \leq C_1'$, first we show $u < b$. Obviously, this is true if $C_1' < b$. Otherwise consider $b \leq C_1'$, suppose on the contrary that $b \leq u \leq C_1'$, then $y \leq y - b + u \leq y + C_1'$ and by the definition of $f_1(\cdot)$ we have

$$f_1(y) \leq g(y-b+u) + K_1,$$

which together with the subcase assumption $f_1(y) > g(y-b+u) + K_1 + K_2$ implies that $K_2 < 0$, contradicting the fact that $K_2 \geq 0$. Hence we have shown that $0 \leq u < b$, which implies $b - u \in (0, C_2]$. Therefore

$$\begin{aligned}\Delta_1 &\geq K_1 + g(x+a) - f_1(x) - \frac{a}{b-u}\Big(f_1(y) - g(y-b+u) - K_1 - K_2\Big)\\ &\geq K_1 + g(x+a) - g(x) - \frac{a}{b-u}\Big(g(y) - g(y-b+u) - K_2\Big) \geq 0,\end{aligned}$$

where the second inequality follows from $f_1(x) \leq g(x)$ and $f_1(y) \leq g(y) + K_1$ by the definition of $f_1(\cdot)$, and the last inequality follows from the strong (C_1K_1, C_2K_2)-convexity of $g(\cdot)$.

(c) $f_1(x+a) = g(x+a+u) + K_1$ with some $u \in [0, C_1']$ and $f_1(y-b) = g(y-b)$. Since $u \in [0, C_1']$, we have $f_1(x) \leq g(x+u) + K_1$ and therefore

$$\begin{aligned}\Delta_1 &= K_1 + g(x+a+u) + K_1 - f_1(x) - \frac{a}{b}\Big(f_1(y) - g(y-b) - K_2\Big)\\ &\geq K_1 + g(x+u+a) - g(x+u) - \frac{a}{b}\Big(g(y) - g(y-b) - K_2\Big) \geq 0,\end{aligned}$$

where the last inequality follows from the strong (C_1K_1, C_2K_2)-convexity of $g(\cdot)$.

(d) $\underline{f_1(x+a) = g(x+a+u) + K_1 \text{ and } f_1(y-b) = g(y-b+w) + K_1 \text{ with some}}$ $\underline{u, w \in [0, C_1']}$. In this case, Δ_1 defined by (A.1) can be written as

$$\Delta_1 = K_1 + g(x+a+u) + K_1 - f_1(x) - \frac{a}{b}\Big(f_1(y) - g(y-b+w) - K_1 - K_2\Big). \quad \text{(A.2)}$$

Based on the value of $f_1(y)$ we consider the following two subcases:

(d.1) $\underline{f_1(y) \le g(y-b+w) + K_1 + K_2}$. In this case if $a + u \le C_1'$, we know $f_1(x) \le g(x+a+u) + K_1$ and hence by (A.2) we have

$$\begin{aligned}\Delta_1 &\ge K_1 + g(x+a+u) + K_1 - f_1(x) \\ &\ge K_1 + g(x+a+u) + K_1 - (g(x+a+u) + K_1) = K_1 \ge 0.\end{aligned}$$

If $a + u > C_1'$, again by (A.2) we have

$$\begin{aligned}\Delta_1 &\ge K_1 + g(x+a+u) + K_1 - (g(x+C_1') + K_1) \\ &\quad - \frac{a}{b}\Big(f_1(y) - g(y-b+w) - K_1 - K_2\Big) \\ &\ge K_1 + g(x+a+u) - g(x+C_1') \\ &\quad - \frac{a+u-C_1'}{b}\Big(f_1(y) - g(y-b+w) - K_1 - K_2\Big) \\ &\ge K_1 + g(x+a+u) - g(x+C_1') \\ &\quad - \frac{a+u-C_1'}{b}\Big(g(y+w) - g(y+w-b) - K_2\Big) \\ &\ge 0,\end{aligned}$$

where the first inequality is from the definition of $f_1(\cdot)$, the second inequality follows from the fact that $0 < a + u - C_1' \le a$ and the case assumption $f_1(y) \le g(y-b+w) + K_1 + K_2$, the third inequality is again implied by the definition of $f_1(\cdot)$ such that $f_1(y) \le g(y+w) + K_1$, and the last inequality follows from the strong (C_1K_1, C_2K_2)-convexity of $g(\cdot)$ noticing that $0 < a+u-C_1' \le a \le C_1$ and $x + C_1' \ge y + w$.

(d.2) $\underline{f_1(y) > g(y-b+w) + K_1 + K_2}$. With the same proof as in (b.2) we can show that $0 \le w < b$, implying $b - w \in (0, C_2]$, hence using $u \in [0, C_1']$ we have,

$$\begin{aligned}\Delta_1 &\ge K_1 + g(x+a+u) + K_1 - f_1(x) - \frac{a}{b-w}\Big(f_1(y) - g(y-b+w) - K_1 - K_2\Big) \\ &\ge K_1 + g(x+a+u) + K_1 - (g(x+u) + K_1) \\ &\qquad - \frac{a}{b-w}\Big(g(y) - g(y-b+w) - K_2\Big) \\ &\ge K_1 + g(x+u+a) - g(x+u) - \frac{a}{b-w}\Big(g(y) - g(y-b+w) - K_2\Big) \ge 0,\end{aligned}$$

where the first inequality follows from $\frac{a}{b} \le \frac{a}{b-w}$, since $0 < b - w \le b$, the second inequality follows from the definition of $f_1(\cdot)$ such that $f_1(x) \le g(x+u) + K_1$ and $f_1(y) \le g(y) + K_1$, and the last inequality follows from the strong (C_1K_1, C_2K_2)-convexity of $g(\cdot)$.

Combining (a)-(d) we have shown that $f_1(x) \in SC_{C_1K_1, C_2K_2}$.

(II) Next we prove that $f_2(x) \in SC_{C_1K_1, C_2K_2}$. We first re-denote f_1 and f_2 more precisely as

$$f^g_{1,CK}(x) = \min_{y \in [x, x+C]} \{K\delta(y-x) + g(y)\}, \quad f^g_{2,CK}(x) = \min_{y \in [x-C, x]} \{K\delta(x-y) + g(y)\},$$

where both f_1 and f_2 are functions of g, C, K, and x. In part (I) we have essentially proved that $g(\cdot) \in SC_{C_1K_1, C_2K_2}$ implies $f^g_{1,C_1'K_1}(\cdot) \in SC_{C_1K_1, C_2K_2}$ for $C_1' \ge C_1$, and in this part we want to show that $g(\cdot) \in SC_{C_1K_1, C_2K_2}$ also implies $f^g_{2,C_2'K_2}(\cdot) \in SC_{C_1K_1, C_2K_2}$ for $C_2' \ge C_2$.

Applying Lemma 1.1 (i), if $g(x) \in SC_{C_1K_1, C_2K_2}$, then $h(x) := g(-x) \in SC_{C_2K_2, C_1K_1}$, and hence by part (I) we know $f^h_{1,C_2'K_2}(x) \in SC_{C_2K_2, C_1K_1}$. We can make further manipulations as

$$f^h_{1,C_2'K_2}(x) = \min_{y \in [x, x+C_2']} \{K_2\delta(y-x) + h(y)\} = \min_{-y \in [-x-C_2', -x]} \{K_2\delta(-x-(-y)) + h(y)\}.$$

Transforming variable $y' = -y$, we get

$$\begin{aligned}f^h_{1,C_2'K_2}(x) &= \min_{y' \in [-x-C_2', -x]} \{K_2\delta(-x-y') + h(-y')\} \\ &= \min_{y' \in [-x-C_2', -x]} \{K_2\delta(-x-y') + g(y')\} \\ &= f^g_{2,C_2'K_2}(-x),\end{aligned}$$

implying that $f^g_{2,C_2'K_2}(-x) \in SC_{C_2K_2,C_1K_1}$. Applying Lemma 1.1 (i) again we see $f^g_{2,C_2'K_2}(x) \in SC_{C_1K_1,C_2K_2}$, confirming the strong (C_1K_1, C_2K_2)-convexity of $f_2(x)$.

(III) Finally we show $f(x) \in SC_{C_1K_1,C_2K_2}$. Let

$$\Delta = K_1 + f(x+a) - f(x) - \frac{a}{b}\Big(f(y) - f(y-b) - K_2\Big). \tag{A.3}$$

We consider the following four different cases for the pair of values $f(x+a)$ and $f(y-b)$ to show that $\Delta \geq 0$ for all $y \leq x, a \in [0, C_1]$ and $b \in (0, C_2]$. Notice that the definition of $f(x)$ implies $f(x) \leq f_1(x)$ and $f(x) \leq f_2(x)$.

(a) $\underline{f(x+a) = f_1(x+a) \text{ and } f(y-b) = f_1(y-b)}$. We have

$$\begin{aligned}\Delta &= K_1 + f_1(x+a) - f(x) - \frac{a}{b}\Big(f(y) - f_1(y-b) - K_2\Big) \\ &\geq K_1 + f_1(x+a) - f_1(x) - \frac{a}{b}\Big(f_1(y) - f_1(y-b) - K_2\Big) \geq 0,\end{aligned}$$

where the last inequality follows directly from the strong (C_1K_1, C_2K_2)-convexity of $f_1(x)$.

(b) $\underline{f(x+a) = f_1(x+a) \text{ and } f(y-b) = f_2(y-b)}$. We can rewrite (A.3) as

$$\Delta = K_1 + f_1(x+a) - f(x) - \frac{a}{b}\Big(f(y) - f_2(y-b) - K_2\Big). \tag{A.4}$$

Per definition, $f_2(y-b) = g(y-b)$ or $f_2(y-b) = g(y-b-u) + K_2$ with some $u \in (0, C_2']$. We consider these two subcases:

(b.1) $\underline{f_2(y-b) = g(y-b)}$. Since $f_1(y-b) \leq g(y-b)$ by the definition, we have $f(y-b) = f_1(y-b)$ and this subcase becomes case (a) and $\Delta \geq 0$ follows.

(b.2) $\underline{f_2(y-b) = g(y-b-u) + K_2 \text{ for some } u \in (0, C_2']}$. Then (A.4) becomes

$$\begin{aligned}\Delta &= K_1 + f_1(x+a) - f(x) - \frac{a}{b}\Big(f(y) - g(y-b-u) - K_2 - K_2\Big) \\ &\geq K_1 + f_1(x+a) - f(x) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big),\end{aligned}$$

where the inequality follows from $f(y) \leq f_2(y) \leq g(y-u) + K_2$, by the definitions of f and f_2. Now if $f_1(x+a) = g(x+a)$, we have

$$\begin{aligned}\Delta &\geq K_1 + g(x+a) - f(x) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big) \\ &\geq K_1 + g(x+a) - g(x) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big) \geq 0,\end{aligned}$$

where the second inequality follows from $f(x) \leq g(x)$ and the last inequality follows from the strong (C_1K_1, C_2K_2)-convexity of $g(x)$. Otherwise if $f_1(x+a) = g(x+a+w) + K_1$ for some $w \in (0, C_1']$, we have

$$\begin{aligned}
\Delta &\geq K_1 + g(x+a+w) + K_1 - f(x) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big) \\
&\geq K_1 + g(x+a+w) + K_1 - f_1(x) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big) \\
&\geq K_1 + g(x+a+w) + K_1 - (g(x+w) + K_1) \\
&\qquad - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big) \\
&\geq K_1 + g(x+w+a) - g(x+w) - \frac{a}{b}\Big(g(y-u) - g(y-u-b) - K_2\Big) \geq 0,
\end{aligned}$$

where the second and third inequalities follow from $f(x) \leq f_1(x)$ and $f_1(x) \leq g(x+w) + K_1$ with $w \in (0, C_1']$, respectively, and the last inequality follows from the strong (C_1K_1, C_2K_2)-convexity of $g(x)$.

(c) <u>$f(x+a) = f_2(x+a)$ and $f(y-b) = f_2(y-b)$</u>. The proof is analogous to case (a).

(d) <u>$f(x+a) = f_2(x+a)$ and $f(y-b) = f_1(y-b)$</u>. We can rewrite (A.3) as

$$\Delta = K_1 + f_2(x+a) - f(x) - \frac{a}{b}\Big(f(y) - f_1(y-b) - K_2\Big). \tag{A.5}$$

By its definition, $f_1(y-b) = g(y-b)$ or $f_1(y-b) = g(y-b+w) + K_1$ with some $w \in (0, C_1']$. We consider these two subcases:

(d.1) <u>$f_1(y-b) = g(y-b)$</u>. Since $f_2(y-b) \leq g(y-b)$, per definition, we have $f(y-b) = f_2(y-b)$ and this subcase becomes case (c) and $\Delta \geq 0$ follows.

(d.2) <u>$f_1(y-b) = g(y-b+w) + K_1$ with some $w \in (0, C_1']$</u>. Then (A.5) becomes

$$\Delta = K_1 + f_2(x+a) - f(x) - \frac{a}{b}\Big(f(y) - g(y-b+w) - K_1 - K_2\Big). \tag{A.6}$$

We first show that $f(y) \leq g(y-b+w) + K_1 + K_2$ always holds in this subcase. To this end, note that $b \in (0, C_2]$ and $w \in (0, C_1']$, if $w \geq b$, then $w - b \in [0, C_1')$ and hence

$$f(y) \leq f_1(y). = \inf_{u \in [y, y+C_1']} \{K_1\delta(u-y) + g(u)\} \leq g(y+(w-b)) + K_1 \leq g(y-b+w) + K_1 + K_2,$$

otherwise if $w < b$, then $b - w \in (0, C_2)$ and hence

$$f(y) \le f_2(y) = \inf_{u \in [y - C_2', y]} \{K_2 \delta(y-u) + g(u)\} \le g(y-(b-w)) + K_2 \le g(y-b+w) + K_1 + K_2.$$

We have thus proved that given $b \in (0, C_2]$ and $w \in (0, C_1']$,

$$f(y) \le g(y - b + w) + K_1 + K_2. \tag{A.7}$$

Similarly we can prove that given $a \in [0, C_1]$ and $u \in (0, C_2']$,

$$f(x) \le g(x + a - u) + K_1 + K_2. \tag{A.8}$$

Next we consider the possible values of $f_2(x + a)$. If $f_2(x + a) = g(x + a)$, then by (A.6) and (A.7),

$$\Delta = K_1 + g(x+a) - f(x) - \frac{a}{b}\Big(f(y) - g(y-b+w) - K_1 - K_2\Big) \ge K_1 + g(x+a) - f_1(x) \ge 0,$$

where the last inequality follows from the definition of $f_1(x)$. On the other hand if $f_2(x + a) = g(x + a - u) + K_2$ with some $u \in (0, C_2']$, then by (A.6), (A.7) and (A.8)

$$\Delta = K_1 + g(x + a - u) + K_2 - f(x) - \frac{a}{b}\Big(f(y) - g(y - b + w) - K_1 - K_2\Big) \ge 0.$$

Consequently, combining (a)-(d) we have proved that $f(x) \in SC_{C_1K_1,C_2K_2}$. The proof of this proposition is also completed. □

Proof of Proposition 1.4. We first prove that $g_0(\cdot)$ is continuous. Note that

$$g_0(x) = \min\left\{ \min_{x \le y \le x+C_1} \{K_1 + \beta_1(y - x) + g(y)\}, \min_{x-C_2 \le y \le x} \{K_2 + \beta_2(y - x) + g(y)\}, g(x) \right\}.$$

Thus, continuity of $g_0(\cdot)$ follows by showing that $\min_{x \le y \le x+C_1}\{K_1 + \beta_1(y - x) + g(y)\}$ is continuous in x and $\min_{x-C_2 \le y \le x}\{K_2 + \beta_2(y - x) + g(y)\}$ is continuous in x. Both continuity results follow from Berge's Maximum Theorem result, since the minimands are continuous functions and the feasible sets are continuous correspondences of x, see e.g. Theorem 9.14 in Sundaram (1996).

It is not hard to see $g(\cdot) \in SC_{C_1K_1,C_2K_2} \Rightarrow g_1(\cdot), g_2(\cdot) \in SC_{C_1K_1,C_2K_2}$: Lemma 1.1 (iii) shows that $g(y) \in SC_{C_1K_1,C_2K_2} \Rightarrow g(y) + \beta_1 y \in SC_{C_1K_1,C_2K_2}$ for any β_1; then by Lemma A.1,

$g_1(x) + \beta_1 x = \min_{y\in[x,x+C_1']}\{K_1\delta(y-x) + \beta_1 y + g(y)\} \in SC_{C_1K_1,C_2K_2}$, and hence $g_1(x) \in SC_{C_1K_1,C_2K_2}$ using Lemma 1.1 (iii) again. Similarly we can show $g_2(x) \in SC_{C_1K_1,C_2K_2}$.

Note that if $\beta_1 \neq \beta_2$ we cannot directly apply Lemma A.1 to claim strong (C_1K_1, C_2K_2)-convexity of $g_0(\cdot)$.

For any $x \geq y$, $u \in [0, C_1]$ and $t \in (0, C_2]$, we need to show that

$$0 \leq \Delta = K_1 + g_0(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_0(y-t) - K_2\Big). \tag{A.9}$$

As is in the proof of Lemma A.1 (III), we consider the following four cases for the pair of values $g_0(x+u)$ and $g_0(y-t)$:

(a) $g_0(x+u) = g_1(x+u),\ g_0(y-t) = g_1(y-t)$;

(b) $g_0(x+u) = g_1(x+u),\ g_0(y-t) = g_2(y-t)$;

(c) $g_0(x+u) = g_2(x+u),\ g_0(y-t) = g_2(y-t)$;

(d) $g_0(x+u) = g_2(x+u),\ g_0(y-t) = g_1(y-t)$.

First note that both $g_1(\cdot), g_2(\cdot) \in SC_{C_1K_1,C_2K_2}$, therefore case (a) and (c) can be easily proved in the same way as (a) and (c) of Lemma A.1 (III), respectively.

For case (b), given $g_0(x+u) = g_1(x+u) \leq g_2(x+u)$, if $g_1(x+u) = g_2(x+u)$, this becomes case (c). Otherwise, $g_1(x+u) < g_2(x+u)$; then, by Corollary 1.1 (ii), we have $g_1(y-t) \leq g_2(y-t)$ since $y - t \leq x + u$. Thus, $g_0(y-t) = g_1(y-t) = g_2(y-t)$, so that case (a) applies.

Thus, only case (d) remains to be proven. Notice that if $g_2(x+u) = g(x+u)$, then the relations

$$g_2(x+u) = g_0(x+u) \leq g_1(x+u) \leq g(x+u) = g_2(x+u)$$

implies $g_0(x+u) = g_1(x+u) = g_2(x+u)$, so that case (a) applies. Similarly if $g_1(y-t) = g(y-t)$, we can deduct that $g_0(y-t) = g_1(y-t) = g_2(y-t)$ and case (c) applies. Therefore we only need to consider the distinct situations where

$$g_1(y-t) = \widetilde{g}_1(y-t) = K_1 + g(B(y-t)) + \beta_1(B(y-t) - y + t) < g(y-t), \tag{A.10}$$

$$g_2(x+u) = \widetilde{g}_2(x+u) = K_2 + g(S(x+u)) + \beta_2(S(x+u) - x - u) < g(x+u), \tag{A.11}$$

where $B(\cdot)$ and $S(\cdot)$ are defined by (1.27) and (1.28) with C_1 and C_2 replaced by C_1' and C_2', respectively. For notational simplicity, we henceforth denote $B(y-t)$ and $S(x+u)$ by $\widetilde{B}$ and $\widetilde{S}$, respectively. Noticing that $\widetilde{B} \in (y-t, y-t+C_1']$, $\widetilde{S} \in [x+u-C_2', x+u)$ and $u \in [0, C_1]$, $t \in (0, C_2]$, it is easy to see that

$$g_0(y) \le \begin{cases} g_1(y) \le K_1 + \beta_1(\widetilde{B} - y) + g(\widetilde{B}), & \text{if } y \le \widetilde{B}, \text{ (since } \widetilde{B} \le y - t + C_1' < y + C_1') \\ g_2(y) \le K_2 + \beta_2(\widetilde{B} - y) + g(\widetilde{B}), & \text{if } y > \widetilde{B}; \text{ (since } \widetilde{B} > y - t \ge y - C_2') \end{cases} \tag{A.12}$$

$$g_0(x) \le \begin{cases} g_1(x) \le K_1 + \beta_1(\widetilde{S} - x) + g(\widetilde{S}), & \text{if } x < \widetilde{S}, \text{ (since } \widetilde{S} < x + u \le x + C_1') \\ g_2(x) \le K_2 + \beta_2(\widetilde{S} - x) + g(\widetilde{S}), & \text{if } x \ge \widetilde{S}. \text{ (since } \widetilde{S} \ge x + u - C_2' \ge x - C_2') \end{cases} \tag{A.13}$$

We therefore distinguish among the 4 cases determined by the relative position of y vis-à-vis $\widetilde{B}$ and x vis-à-vis $\widetilde{S}$.

(a) $\underline{y \le \widetilde{B}, x < \widetilde{S}}$. In this case, by (A.12) and (A.10) we have

$$g_0(y) \le K_1 + \beta_1(\widetilde{B} - y) + g(\widetilde{B}) = g_1(y-t) - \beta_1 t. \tag{A.14}$$

Taking (A.11), (A.13) and (A.14) into (A.9), we get

$$\begin{aligned} \Delta &= K_1 + g_2(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_1(y-t) - K_2\Big) \\ &\ge K_1 + K_2 + g(\widetilde{S}) + \beta_2(\widetilde{S} - x - u) - K_1 - \beta_1(\widetilde{S} - x) - g(\widetilde{S}) \\ &\quad - \frac{u}{t}\Big(g_1(y-t) - \beta_1 t - g_1(y-t) - K_2\Big) \\ &\ge K_2 - \beta_1 u + \frac{u}{t}\Big(\beta_1 t + K_2\Big) = \Big(1 + \frac{u}{t}\Big)K_2 \ge 0, \end{aligned}$$

where the second inequality follows from

$$\beta_2(\widetilde{S} - x - u) - \beta_1(\widetilde{S} - x) = (\beta_2 - \beta_1)(\widetilde{S} - x - u) - \beta_1 u \ge -\beta_1 u$$

by $\widetilde{S} < x + u$ and the assumption $\beta_1 \ge \beta_2$.

(b) $\underline{y \le \widetilde{B}, x \ge \widetilde{S}}$. In this case, taking (A.11), (A.13) and (A.14) into (A.9) we have

$$\begin{aligned}\Delta &= K_1 + g_2(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_1(y-t) - K_2\Big)\\ &\ge K_1 + K_2 + g(\widetilde{S}) + \beta_2(\widetilde{S} - x - u) - K_2 - \beta_2(\widetilde{S} - x) - g(\widetilde{S})\\ &\quad - \frac{u}{t}\Big(g_1(y-t) - \beta_1 t - g_1(y-t) - K_2\Big)\\ &= K_1 + \frac{u}{t}K_2 + (\beta_1 - \beta_2)u \ge 0.\end{aligned}$$

(c) $\underline{y > \widetilde{B}, x \ge \widetilde{S}}$. In this case, taking (A.10)–(A.13) into (A.9) we have

$$\begin{aligned}\Delta &= K_1 + g_2(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_1(y-t) - K_2\Big)\\ &\ge K_1 + K_2 + g(\widetilde{S}) + \beta_2(\widetilde{S} - x - u) - K_2 - \beta_2(\widetilde{S} - x) - g(\widetilde{S})\\ &\quad - \frac{u}{t}\Big(K_2 + \beta_2(\widetilde{B} - y) + g(\widetilde{B}) - K_1 - g(\widetilde{B}) - \beta_1(\widetilde{B} - y + t) - K_2\Big)\\ &\ge K_1 - \beta_2 u + \frac{u}{t}\Big(\beta_2 t + K_1\Big) = \Big(1 + \frac{u}{t}\Big)K_1 \ge 0,\end{aligned}$$

where the third inequality follows from

$$\beta_2(\widetilde{B} - y) - \beta_1(\widetilde{B} - y + t) = (\beta_2 - \beta_1)(\widetilde{B} - y + t) - \beta_2 t \le -\beta_2 t$$

by $\widetilde{B} > y - t$ and the assumption $\beta_1 \ge \beta_2$.

(d) $\underline{y > \widetilde{B}, x < \widetilde{S}}$. Note that in this case we must have $u \in (0, C_1]$, since if $u = 0$ there cannot be $x < \widetilde{S}$ by $\widetilde{S} \in [x + u - C_2', x + u)$. It then follows that

$$g_0(x) \le g_2(x) \le K_2 + \beta_2(\widetilde{S} - u - x) + g(\widetilde{S} - u), \text{ (since } x - C_2' \le \widetilde{S} - u < x) \quad \text{(A.15)}$$

$$g_0(y) \le g_1(y) \le K_1 + \beta_1(\widetilde{B} + t - y) + g(\widetilde{B} + t). \text{ (since } y < \widetilde{B} + t \le y + C_1') \quad \text{(A.16)}$$

Depending on the order of $\widetilde{B} + t$ and $\widetilde{S} - u$, we consider the following two situations:

(d.1) $\underline{\widetilde{B} + t \le \widetilde{S} - u}$. The following ranking applies:

$$y - t < \widetilde{B} < y < \widetilde{B} + t \le \widetilde{S} - u < x < \widetilde{S} < x + u,$$

where the first inequality follows from $\widetilde{B} = B(y-t) > y-t$ and the last inequality from $\widetilde{S} = S(x+u) < x + u$.

Taking (A.10), (A.11) and (A.15), (A.16) into (A.9) we get

$$\begin{aligned}\Delta &= K_1 + g_2(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_1(y-t) - K_2\Big)\\ &\geq K_1 + K_2 + g(\widetilde{S}) + \beta_2(\widetilde{S} - x - u) - K_2 - \beta_2(\widetilde{S} - u - x) - g(\widetilde{S} - u)\\ &\quad - \frac{u}{t}\Big(K_1 + \beta_1(\widetilde{B} + t - y) + g(\widetilde{B} + t) - K_1 - g(\widetilde{B}) - \beta_1(\widetilde{B} - y + t) - K_2\Big)\\ &= K_1 + g(\widetilde{S}) - g(\widetilde{S} - u) - \frac{u}{t}\Big(g(\widetilde{B} + t) - g(\widetilde{B}) - K_2\Big) \geq 0,\end{aligned}$$

where the last inequality follows from the definition of strong (C_1K_1, C_2K_2)-convexity of $g(\cdot)$ with $x = \widetilde{S} - u$, $y = \widetilde{B} + t$ and $u \in [0, C_1]$ and $t \in (0, C_2]$.

(d.2) $\underline{\widetilde{B} + t > \widetilde{S} - u}$. Now the following rankings apply:

$$y - t < \widetilde{B} < y < \widetilde{B} + t, \quad \widetilde{S} - u < \widetilde{B} + t, \quad \widetilde{S} - u < x < \widetilde{S} < x + u. \qquad \text{(A.17)}$$

Note that (A.15) and (A.16) still hold.

Using (A.10) and (A.11), (A.9) can be written as

$$\begin{aligned}\Delta &= K_1 + g_2(x+u) - g_0(x) - \frac{u}{t}\Big(g_0(y) - g_1(y-t) - K_2\Big)\\ &= K_1 + K_2 + g(\widetilde{S}) + \beta_2(\widetilde{S} - x - u) - g_0(x)\\ &\quad - \frac{u}{t}\Big(g_0(y) - K_1 - g(\widetilde{B}) - \beta_1(\widetilde{B} - y + t) - K_2\Big).\end{aligned}$$

Having mentioned that $u > 0$ in this case, $\Delta \geq 0$ is equivalent to

$$\frac{g_0(y) + \beta_1 y - g(\widetilde{B}) - \beta_1\widetilde{B} - K_1 - K_2}{t} - \frac{g(\widetilde{S}) + \beta_2\widetilde{S} - g_0(x) - \beta_2 x + K_1 + K_2}{u} \leq \beta_1 - \beta_2. \qquad \text{(A.18)}$$

Conditioning on the signs of the two numerators on the left hand side of (A.18), three subcases need to be considered:

(i) $\underline{g_0(y) + \beta_1 y - g(\widetilde{B}) - \beta_1\widetilde{B} - K_1 - K_2 \leq 0}$. Then using (A.13),

$$\begin{aligned}&\frac{g_0(y) + \beta_1 y - g(\widetilde{B}) - \beta_1\widetilde{B} - K_1 - K_2}{t} - \frac{g(\widetilde{S}) + \beta_2\widetilde{S} - g_0(x) - \beta_2 x + K_1 + K_2}{u}\\ &\leq -\frac{g(\widetilde{S}) + \beta_2\widetilde{S} - g_0(x) - \beta_2 x + K_1 + K_2}{u}\\ &\leq -\frac{g(\widetilde{S}) + \beta_2\widetilde{S} - [K_1 + g(\widetilde{S}) + \beta_1(\widetilde{S} - x)] - \beta_2 x + K_1 + K_2}{u}\\ &= \frac{(\beta_1 - \beta_2)(\widetilde{S} - x - u) - K_2}{u} + (\beta_1 - \beta_2) \leq \beta_1 - \beta_2,\end{aligned}$$

where the last inequality follows from $\widetilde{S} < x + u$ and the assumption $\beta_1 \geq \beta_2$.

(ii) $\underline{g(\widetilde{S}) + \beta_2\widetilde{S} - g_0(x) - \beta_2 x + K_1 + K_2 \geq 0}$. Then using (A.12),

$$\begin{aligned}
&\frac{g_0(y) + \beta_1 y - g(\widetilde{B}) - \beta_1\widetilde{B} - K_1 - K_2}{t} - \frac{g(\widetilde{S}) + \beta_2\widetilde{S} - g_0(x) - \beta_2 x + K_1 + K_2}{u} \\
&\qquad \leq \frac{g_0(y) + \beta_1 y - g(\widetilde{B}) - \beta_1\widetilde{B} - K_1 - K_2}{t} \\
&\qquad \leq \frac{[K_2 + g(\widetilde{B}) + \beta_2(\widetilde{B} - y)] + \beta_1 y - g(\widetilde{B}) - \beta_1\widetilde{B} - K_1 - K_2}{t} \\
&\qquad = \frac{(\beta_1 - \beta_2)(y - t - \widetilde{B}) - K_1}{t} + (\beta_1 - \beta_2) \leq \beta_1 - \beta_2,
\end{aligned}$$

where the last inequality follows from $\widetilde{B} > y - t$ and the assumption $\beta_1 \geq \beta_2$.

(iii) $\underline{g_0(y) + \beta_1 y - g(\widetilde{B}) - \beta_1\widetilde{B} - K_1 - K_2 > 0}$ and $\underline{g(\widetilde{S}) + \beta_2\widetilde{S} - g_0(x) - \beta_2 x + K_1 + K_2 < 0}$. Before proving (A.18) we first show that, in view of (A.17), there exist t_0 and u_0 with $0 < y - \widetilde{B} \leq t_0 \leq t$ and $0 < \widetilde{S} - x \leq u_0 \leq u$ such that

$$g(y - t_0) = K_1 + \beta_1(\widetilde{B} - y + t_0) + g(\widetilde{B}), \tag{A.19}$$

$$g(x + u_0) = K_2 + \beta_2(\widetilde{S} - x - u_0) + g(\widetilde{S}). \tag{A.20}$$

For $v \in [y - \widetilde{B}, t]$, let

$$h(v) = g(y - v) - [K_1 + \beta_1(\widetilde{B} - y + v) + g(\widetilde{B})],$$

which is a continuous function. Then, since

$$\begin{aligned}
&h(y - \widetilde{B}) = g(\widetilde{B}) - [K_1 + \beta_1 \cdot 0 + g(\widetilde{B})] = -K_1 \leq 0, \\
&h(t) = g(y - t) - [K_1 + \beta_1(\widetilde{B} - y + t) + g(\widetilde{B})] = g(y - t) - \widetilde{g}_1(y - t) \geq 0,
\end{aligned}$$

by the mean value theorem there exists $t_0 \in [y - \widetilde{B}, t]$ such that $h(t_0) = 0$, i.e., $g(y - t_0) = K_1 + \beta_1(\widetilde{B} - y + t_0) + g(\widetilde{B})$. Similarly we can show the existence of a value u_0 satisfying $0 < \widetilde{S} - x \leq u_0 \leq u$ and (A.20).

Next we proceed to prove (A.18). We have

$$\begin{aligned}
&\frac{g_0(y)+\beta_1 y-g(\widetilde{B})-\beta_1\widetilde{B}-K_1-K_2}{t}-\frac{g(\widetilde{S})+\beta_2\widetilde{S}-g_0(x)-\beta_2 x+K_1+K_2}{u}\\
&\leq \frac{g_0(y)+\beta_1 y-g(\widetilde{B})-\beta_1\widetilde{B}-K_1-K_2}{t_0}\\
&\quad -\frac{g(\widetilde{S})+\beta_2\widetilde{S}-g_0(x)-\beta_2 x+K_1+K_2}{u_0}\\
&=\frac{g_0(y)-[K_1+\beta_1(\widetilde{B}-y+t_0)+g(\widetilde{B})]-K_2+\beta_1 t_0}{t_0}\\
&\quad -\frac{K_1+[K_2+\beta_2(\widetilde{S}-x-u_0)+g(\widetilde{S})]-g_0(x)+\beta_2 u_0}{u_0}\\
&=\frac{g_0(y)-g(y-t_0)-K_2}{t_0}-\frac{K_1+g(x+u_0)-g_0(x)}{u_0}+\beta_1-\beta_2\\
&\leq \beta_1-\beta_2,
\end{aligned}$$

where the first inequality follows from the conditions specifying case (iii); the last inequality follows from the strong (C_1K_1, C_2K_2)-convexity of $g(\cdot)$ and the fact that $g_0(\cdot)\leq g(\cdot)$, specifically,

$$\begin{aligned}
&K_1+g(x+u_0)-g_0(x)-\frac{u_0}{t_0}\Big(g_0(y)-g(y-t_0)-K_2\Big)\\
&\quad\geq K_1+g(x+u_0)-g(x)-\frac{u_0}{t_0}\Big(g(y)-g(y-t_0)-K_2\Big)\geq 0,
\end{aligned}$$

which implies (noticing t_0 and u_0 are both positive)

$$\frac{g_0(y)-g(y-t_0)-K_2}{t_0}-\frac{K_1+g(x+u_0)-g_0(x)}{u_0}\leq 0.$$

□

Proof of Lemma 1.4. The proof is analogous to that in Sections C.1 and C.2 in the electronic companion of Huh et al. (2011). The proof is given in three parts:

Condition SC (a): $X_1'^l > X_1^l$: We compare two inventory level vectors $\mathbf{X}_1$ and $\mathbf{X}_1'$. Assume that these two vectors are identical except $X_1'^l > X_1^l$ form some $l\in\{0,1,\ldots,L-1\}$. Let $\Delta = X_1'^l - X_1^l$. We will then prove parts (i), (iii) and (iv) of Condition (SC) (a). (Part (ii) is not applicable since we consider the $X_1'^l > X_1^l$ case in in this part.)

For any Markov policy δ, let δ' be the following policy. If $l < L-1$, then $X_1'^{L-1} = X_1^{L-1}$. Let the δ' policy order or salvage the same quantity as the δ policy in every period. We call this the “mimic” policy of δ. If $l = L-1$, then the δ' policy initiates the same salvage batches as the δ policy, but does not order anything for the first Δ units ordered by the δ policy, and then matches δ's orders unit-by-unit. Recall $u_t = X_t^L - X_t^{L-1}$ is the number of units ordered by δ in period $t \geq 1$. Then, the order quantity u'_t of the δ' policy is given by, for $t \geq 1$,

$$u'_t = \begin{cases} [\sum_{t'=1}^{t} u_{t'} - \Delta]^+ & \text{if } \sum_{t'=1}^{t-1} u_{t'} < \Delta \\ u_t & \text{otherwise} \end{cases}.$$

Note that u'_t is a feasible inventory adjustment quantity. In every period $t \geq 1$: let t^* denote the first period in which $u'_{t^*} > 0$. Then $u'_t = 0$ or $u'_t = u_t$ for all $t < t^*$ and $u'_t = u_t$ for all $t > t^*$, both feasible. Moreover,

$$0 \leq u'_{t^*} = \sum_{t'=1}^{t^*} u'_t - \Delta \leq \sum_{t'=1}^{t^*} u'_t - \sum_{t'=1}^{t^*-1} u'_t = u_{t^*},$$

hence feasible as well, where the inequality follows from $\sum_{t'=1}^{t^*-1} u'_t \leq \Delta$ by the definition of t^*. We say δ' is a “wait-and-mimic” policy of δ.

The remainder of the proof is analogous to that in Huh et al. (2011).

<u>Condition SC (a): $X_1'^l < X_1^l$</u>: The proof is is analogous to that in Huh et al. (2011) with the following adaptation: In case $l < L-1$, let δ', the mimic policy of δ, order and salvage the same quantity as the δ policy in every period. For the case where $l = L-1$, the policy δ' mimics δ for the first T_0 periods, i.e., it orders and salvages the same quantity as policy δ; thereafter, the specification of δ' is identical to that in Huh et al. (2011).

<u>Condition SC (b)</u>: Analogous to the proof in Huh et al. (2011). □

A.2 Additional numerical examples

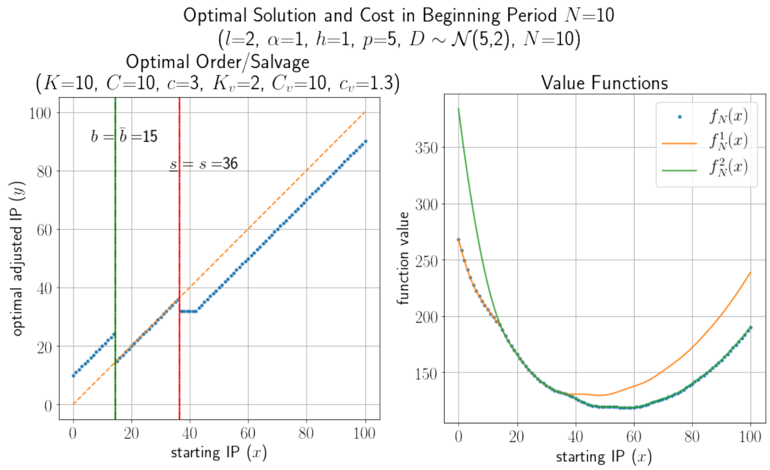

Figure A.1: Numerical example: high fixed ordering cost (big K)

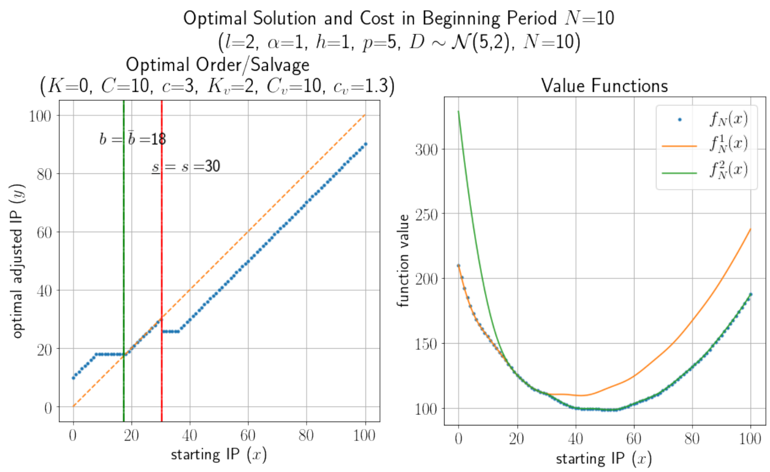

Figure A.2: Numerical example: low fixed ordering cost (small K)

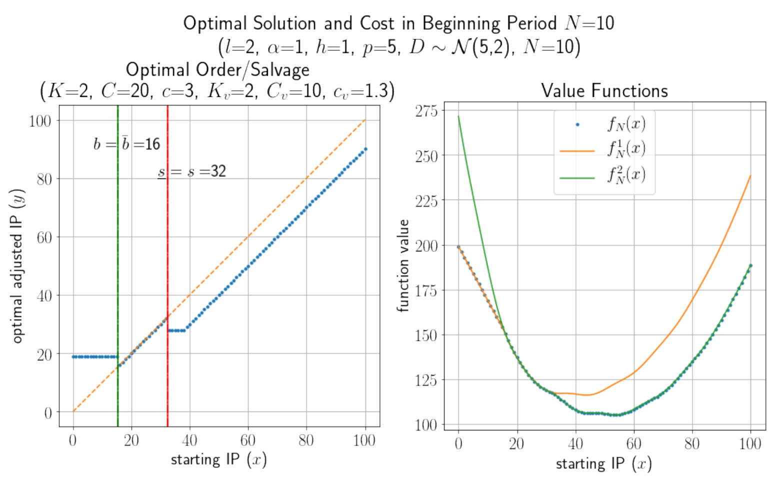

Figure A.3: Numerical example: large ordering capacity (big C)

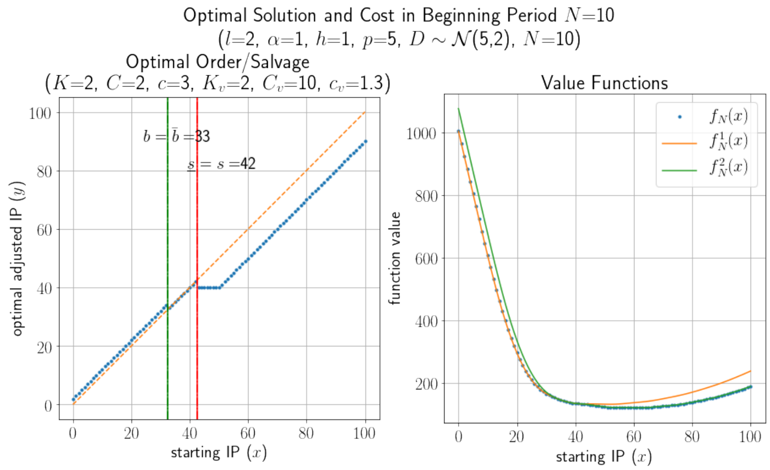

Figure A.4: Numerical example: small ordering capacity (small C)

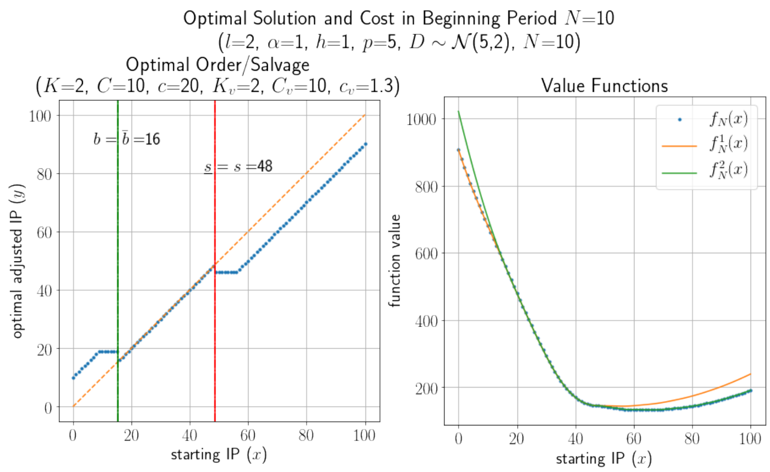

Figure A.5: Numerical example: high unit cost (big c)

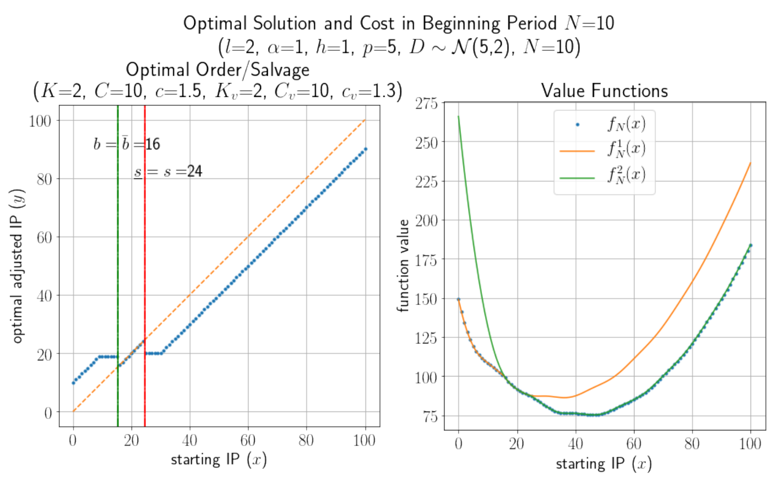

Figure A.6: Numerical example: low unit cost (small c)

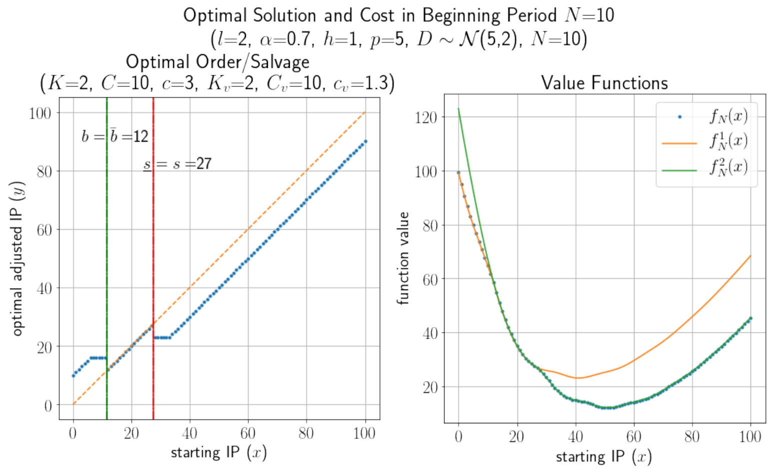

Figure A.7: Numerical example: small α

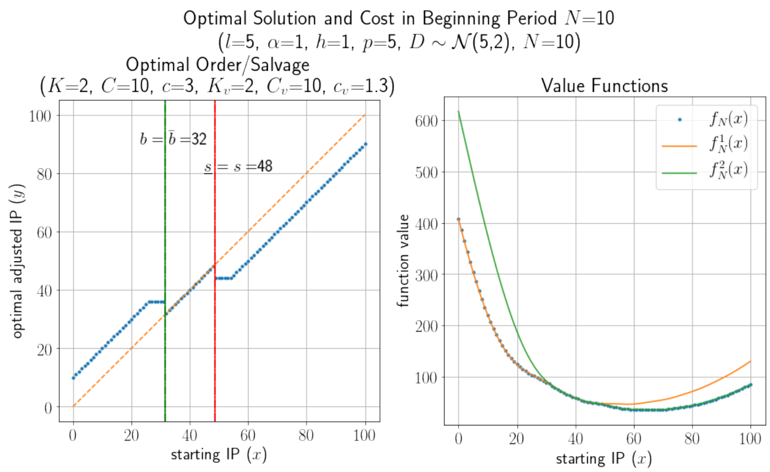

Figure A.8: Numerical example: long lead time (big l)

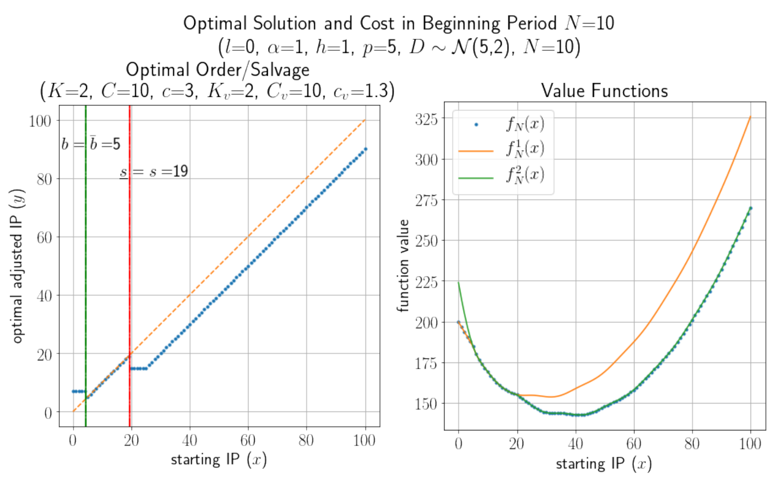

Figure A.9: Numerical example: zero lead time ($l = 0$)

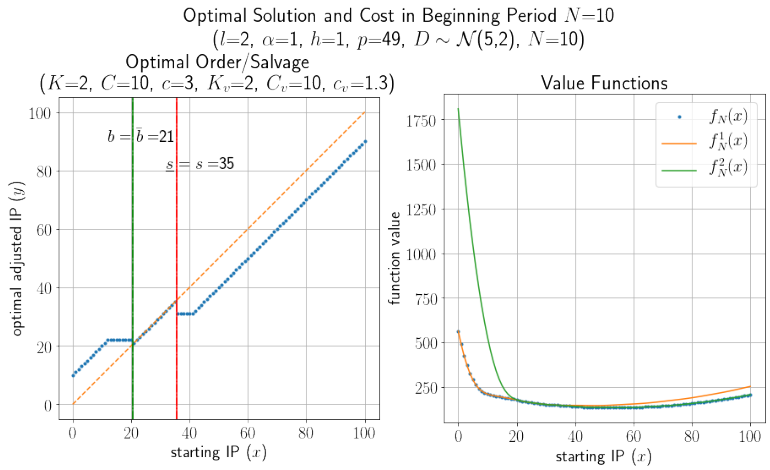

Figure A.10: Numerical example: high service level (big p)

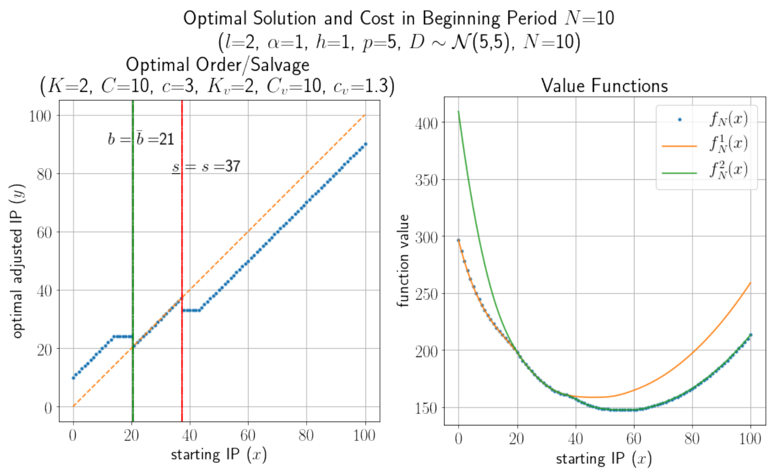

Figure A.11: Numerical example: volatile demand (big σ)

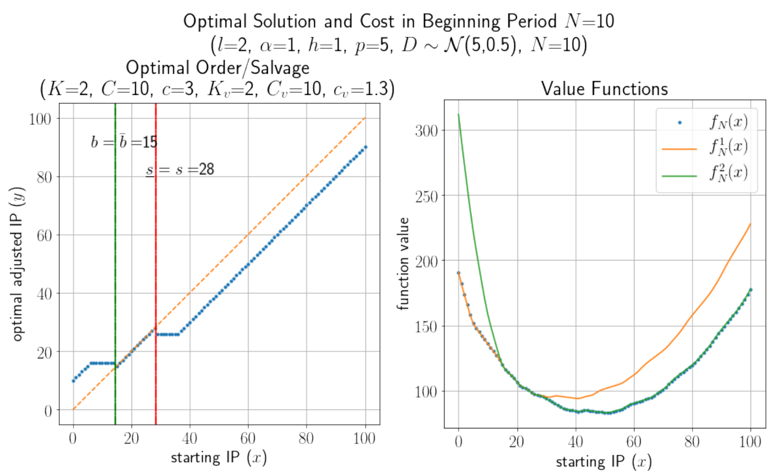

Figure A.12: Numerical example: stable demand (small σ)

Appendix for Chapter 2

B.1 Proofs

Proof of Theorem 2.1. We prove this theorem by induction. By our assumption, the theorem holds for $n = 0$. Suppose the result holds for period $n-1$, i.e., $f_{n-1}(\cdot) \in SC_{C_{n-1}K_{n-1},C_{n-1}^vK_{n-1}^v}$ and $f_{n-1}(x) = O(|x|^p)$. We first prove that $f_n(x) = O(|x|^p)$. Since $f_{n-1}(x) = O(|x|^p)$, there exists a constant $A > 0$ such that $|f_{n-1}(x)| \leq A|x|^p$; so that $|\mathbb{E}f_{n-1}(z-D_n)| \leq A\mathbb{E}|z-D_n|^p \leq A\mathbb{E}(|z|+D_n)^p = A\sum_{l=0}^{p}\binom{p}{l}\mathbb{E}D_n^{p-l}|z|^l \leq B\max\{|z|^p,1\}$ for some constant $B > 0$. Let $z^*(y)$ achieve the minimum in (2.8), then $|f_n^r(y)| \leq K_n + c_n^r|z^*| + \alpha B|z^*|^p \leq K_n + c_n^r(|y|+C_n) + \alpha B\max\{1,(|y|+C_n)^p\} = O(|y|^p)$. By the same argument, and since $L_n(y) = O(|y|^p), g_n(y) = O(|y|^p)$ and $f_n^1(x), f_n^2(x)$ and $f_n(x)$ are all $O(|x|^p)$.

We then prove that $f_n(x) \in SC_{C_nK_n,C_n^vK_n^v}$. Since $f_{n-1}(\cdot) \in SC_{C_{n-1}K_{n-1},C_{n-1}^vK_{n-1}^v}$, by Lemma 1.1 (iii) and (iv) in Chapter 1, and Assumption 2.3,

$$\alpha\mathbb{E}f_{n-1}(z-D_n) \in SC_{C_{n-1}(\alpha K_{n-1}),C_{n-1}^v(\alpha K_{n-1}^v)} \subset SC_{C_nK_n,C_n^vK_n^v}. \tag{B.1}$$

It then follows from Proposition 1.3 in Chapter 1 and Assumption 2.3 that $f_n^r(y) \in SC_{C_nK_n,C_n^vK_n^v}$. Since $L_n(\cdot)$ is convex, we have

$$g_n(y) = L_n(y) + f_n^r(y) \in SC_{C_nK_n,C_n^vK_n^v} \tag{B.2}$$

by Lemma 1.1 (iii). Finally by Proposition 1.3 again, $f_n^1(x), f_n^2(x), f_n(x) \in SC_{C_nK_n,C_n^vK_n^v}$. □

Proof of Theorem 2.2. $b_n^r \leq \bar{b}_n^r$ is immediate from their definitions. To prove the remainder of Theorem 2.2, note that, in each period n, the value functions $f_n^1(\cdot), f_n^2(\cdot), f_n(\cdot)$

satisfy a single stage problem of the following structure:

$$g_1(x) = \min_{y\in[x,x+C_1]}\{K_1\delta(y-x)+\beta_1(y-x)+g(y)\},$$

$$g_2(x) = \min_{y\in[x-C_2,x]}\{K_2\delta(x-y)+\beta_2(y-x)+g(y)\},$$

$$g_0(x) = \min\{g_1(x), g_2(x)\},$$

where $g_1(\cdot) = f_n^1(\cdot), g_2(\cdot) = f_n^2(\cdot)$ and $g(\cdot) = g_n(\cdot)$. Define auxiliary functions

$$\widetilde{g}_1(x) = K_1 + \min_{y\in[x,x+C_1]}\{\beta_1(y-x)+g(y)\},$$

$$\widetilde{g}_2(x) = K_2 + \min_{y\in[x-C_2,x]}\{\beta_2(y-x)+g(y)\},$$

as counterparts of $g_1(x)$ and $g_2(x)$, under *definitive* inventory adjustment, i.e., *definitively* incurring fixed costs for ordering or salvaging, respectively, and let $A_i(x) = \widetilde{g}_i(x) - g(x)$ be the increase in minimal cost if forced to order (for $i = 1$) or salvage (for $i = 2$).

We define the following critical points with the convention that the infimum (supremum) of an empty set equals $+\infty$ $(-\infty)$.

Definition B.1. *(Critical Points) For a continuous function $g(\cdot) \in SC_{C_1K_1,C_2K_2}$ and any β_1, β_2, define*

$$B = \inf\big\{\arg\min_y\{\beta_1 y + g(y)\}\big\}, \quad b = \inf\{x:\ A_1(x)\geq 0\}, \quad \bar{b} = \sup\{x:\ A_1(x) < 0\},$$

$$S = \sup\big\{\arg\min_y\{\beta_2 y + g(y)\}\big\}, \quad s = \sup\{x:\ A_2(x)\geq 0\}, \quad \underline{s} = \inf\{x:\ A_2(x) < 0\}.$$

These critical points play important roles in the structure of the optimal strategy. By its definition, B is the (smallest) global minimizer of $\widetilde{g}_1(x)$ if $C_1 = \infty$, i.e., the smallest order-up-to level for sufficiently small x if ordering is better than staying put. Similarly, S is the (largest) global minimizer of $\widetilde{g}_2(x)$ if $C_2 = \infty$, .i.e., the biggest salvage-down-to level for sufficiently large x if salvaging is better than staying put; b is the smallest among all inventory levels where ordering is not better than staying put; $\bar{b}$ is the largest among all inventory levels where ordering is better than staying put; s is the largest among all inventory levels where salvaging is not better than staying put; $\underline{s}$ is the smallest among all inventory levels where salvaging is better than staying put.

Note that by definition we have

$$g_1(x) = \min\{g(x),\ \widetilde{g}_1(x)\}, \qquad A_1(x) < 0 \quad \forall x < b, \qquad A_1(x) \geq 0 \quad \forall x > \bar{b},$$

$$g_2(x) = \min\{g(x),\ \widetilde{g}_2(x)\}, \qquad A_2(x) < 0 \quad \forall x > s, \qquad A_2(x) \geq 0 \quad \forall x < \underline{s}.$$

The lemma below, which follows from Proposition 1.2 in Chapter 1, characterizes the ranking of the critical points.

Lemma B.1 (Critical Points)**.** *Assume* $\beta_1 \geq \beta_2$ *and* $g(\cdot) \in SC_{C_1K_1,C_2K_2}$, *then*

(i) $-\infty \leq b \leq \bar{b} \leq \underline{s} \leq s \leq \infty$;

(ii) $-\infty \leq b \leq B \leq S \leq s \leq \infty$;

(iii) *If* $C_2 = \infty$ *and* $K_1 \geq K_2$, *then* $\bar{b} \leq B$; *if* $C_1 = \infty$ *and* $K_1 \leq K_2$, *then* $S \leq \underline{s}$;

(iv) *If* $C_1 = \infty$ *and* $K_2 = 0$, *then* $b = \bar{b}$; *if* $C_2 = \infty$ *and* $K_1 = 0$, *then* $\underline{s} = s$. *If* $C_1 = C_2 = \infty$ *and* $K_1 = K_2 = 0$, *then* $b = \bar{b} = B, S = \underline{s} = s$.

In this lemma, (i) ranks four critical points. (ii) ranks and locates the global minimizers B and S between b and s. (iii) and (iv) lead to simple policy structures, in certain special cases. In particular (2.12) follows from Lemma B.1 (i) and (b) follows from

By Assumption 2.1, $c_n^e \geq c_n^v$. By (B.2) in the proof of Theorem 2.1, $g_n(y) \in SC_{C_nK_n,C_n^vK_n^v}$. Applying Theorem 1.2 in Chapter 1 with properly defined critical points, we immediately obtain the optimal policy structure for ordering with the expedited supplier and salvaging, as given by Table 2.1 (a). For the regular supplier, since $\alpha\mathbb{E}f_{n-1}(z - D_n) \in SC_{C_nK_n,C_n^vK_n^v}$ by (B.1) in the proof of Theorem 2.1, we can apply Theorem 1.2 again and use Corollary 1.2 to obtain the optimal policy structure given by Table 2.1 (b). □

B.2 Additional numerical studies

Varying parameter	Exp			Salvage			Reg			Lead time		Inventory			Disc.	Demand		Dual sourcing						Dual sourcing w/o salvage					Single Exp		Single Reg	
	Ke	*Ce*	*ce*	*Kv*	*Cv*	*cv*	*Kr*	*Cr*	*cr*	*le*	*lr*	*h*	*p*	*SL*	*a*	*μ*	*σ*	cost mean	stderr	vs salvage	% Exp	Freq Exp	Freq salvage	cost mean	stderr	vs single	% Exp	Freq Exp	cost mean	stderr	cost mean	stderr
Default	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	1	5	2	308.6	0.58	2.5%	15.9%	11.0%	14.3%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
Lead times	10	10	15	2	10	6	10	10	10	**0**	**1**	5	95	95%	1	5	2	88.6	0.06	0.1%	11.6%	15.8%	1.0%	88.7	0.06	16.4%	11.7%	15.9%	106.1	0.07	107.0	0.13
	10	10	15	2	10	6	10	10	10	**1**	**2**	5	95	95%	1	5	2	111.9	0.14	1.8%	9.2%	7.4%	4.3%	113.9	0.14	14.0%	9.3%	7.5%	132.5	0.14	160.7	0.33
	10	10	15	2	10	6	10	10	10	**2**	**3**	5	95	95%	1	5	2	159.1	0.27	2.4%	11.0%	10.2%	7.4%	163.0	0.27	12.5%	11.2%	10.5%	186.2	0.33	244.4	0.57
	10	10	15	2	10	6	10	10	10	**3**	**4**	5	95	95%	1	5	2	224.3	0.39	2.5%	14.3%	10.4%	11.2%	230.1	0.39	14.8%	14.3%	10.4%	270.0	0.58	358.2	0.87
	10	10	15	2	10	6	10	10	10	**4**	**5**	5	95	95%	1	5	2	308.6	0.58	2.5%	15.9%	11.0%	14.3%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
Unit salvage revenue	10	10	15	2	10	**0**	10	10	10	4	5	5	95	95%	1	5	2	316.4	0.58	0.0%	16.2%	11.3%	0.1%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
	10	10	15	2	10	**2**	10	10	10	4	5	5	95	95%	1	5	2	314.0	0.58	0.8%	16.0%	11.2%	13.8%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
	10	10	15	2	10	**4**	10	10	10	4	5	5	95	95%	1	5	2	311.3	0.58	1.6%	15.9%	11.0%	14.0%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
	10	10	15	2	10	**6**	10	10	10	4	5	5	95	95%	1	5	2	308.6	0.58	2.5%	15.9%	11.0%	14.3%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
	10	10	15	2	10	**8**	10	10	10	4	5	5	95	95%	1	5	2	305.9	0.58	3.3%	16.0%	11.1%	15.6%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
Unit order costs	10	10	**11**	2	10	6	10	10	**10**	4	5	5	95	95%	1	5	2	304.8	0.57	2.5%	24.8%	21.7%	13.6%	312.6	0.57	14.0%	24.9%	21.7%	363.4	0.87	502.4	1.20
	10	10	**12**	2	10	6	10	10	**10**	4	5	5	95	95%	1	5	2	306.1	0.57	2.5%	19.6%	15.6%	14.1%	313.8	0.57	14.8%	19.9%	16.0%	368.5	0.87	502.4	1.20
	10	10	**15**	2	10	6	10	10	**10**	4	5	5	95	95%	1	5	2	308.6	0.58	2.5%	15.9%	11.0%	14.3%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
	10	10	**20**	2	10	6	10	10	**10**	4	5	5	95	95%	1	5	2	312.3	0.59	2.4%	13.7%	7.8%	14.3%	320.1	0.59	21.8%	13.7%	7.8%	409.5	0.88	502.4	1.20
	10	10	**30**	2	10	6	10	10	**10**	4	5	5	95	95%	1	5	2	319.2	0.59	2.4%	13.1%	7.1%	14.7%	327.0	0.59	29.0%	13.1%	7.1%	460.4	0.90	502.4	1.20
Fixed costs	**0**	10	15	**0**	10	6	**0**	10	10	4	5	5	95	95%	1	5	2	299.7	0.57	2.6%	16.1%	17.0%	22.5%	307.7	0.57	18.1%	16.8%	17.9%	375.7	0.88	494.6	1.20
	5	10	15	**2**	10	6	**5**	10	10	4	5	5	95	95%	1	5	2	304.5	0.58	2.5%	16.0%	12.5%	14.1%	312.2	0.58	17.8%	16.0%	12.5%	379.9	0.88	498.6	1.20
	10	10	15	**4**	10	6	**10**	10	10	4	5	5	95	95%	1	5	2	308.9	0.58	2.4%	15.9%	11.0%	14.0%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
	20	10	15	**8**	10	6	**20**	10	10	4	5	5	95	95%	1	5	2	316.4	0.57	2.2%	16.4%	11.3%	13.8%	323.4	0.57	17.3%	16.3%	11.2%	391.1	0.87	509.3	1.20
	30	10	15	**8**	10	6	**30**	10	10	4	5	5	95	95%	1	5	2	322.3	0.58	2.1%	15.1%	9.0%	13.9%	329.2	0.58	17.0%	15.2%	9.1%	396.7	0.88	514.9	1.20
Service level	10	10	15	2	10	6	10	10	10	4	5	5	**7.5**	**60%**	1	5	2	87.2	0.08	8.2%	12.0%	6.6%	13.6%	95.0	0.08	8.5%	12.0%	6.6%	116.2	0.10	103.9	0.12
	10	10	15	2	10	6	10	10	10	4	5	5	**20**	**80%**	1	5	2	126.1	0.15	5.8%	14.2%	8.1%	13.9%	133.9	0.15	17.1%	14.3%	8.2%	161.5	0.21	166.9	0.27
	10	10	15	2	10	6	10	10	10	4	5	5	**45**	**90%**	1	5	2	190.1	0.29	3.9%	14.4%	8.5%	14.2%	197.9	0.29	17.0%	14.3%	8.5%	238.5	0.43	281.3	0.57
	10	10	15	2	10	6	10	10	10	4	5	5	**95**	**95%**	1	5	2	308.6	0.58	2.5%	15.9%	11.0%	14.3%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
	10	10	15	2	10	6	10	10	10	4	5	5	**495**	**99%**	1	5	2	1215.9	2.85	0.6%	17.0%	13.1%	14.3%	1223.8	2.85	19.0%	17.0%	13.2%	1511.2	4.48	2238.9	6.19
Demand volatility	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	1	5	**0.5**	273.9	0.17	2.7%	13.4%	7.3%	13.9%	281.7	0.17	19.3%	13.4%	7.3%	348.9	0.25	466.3	0.34
	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	1	5	**1**	285.5	0.30	2.6%	13.4%	7.4%	14.3%	293.3	0.30	18.5%	13.5%	7.6%	359.9	0.45	477.2	0.62
	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	1	5	**2**	308.6	0.58	2.5%	15.9%	11.0%	14.3%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	1	5	**3**	327.4	0.78	2.3%	17.4%	11.9%	14.5%	335.3	0.78	17.5%	17.8%	12.2%	406.4	1.22	526.8	1.66
	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	1	5	**5**	355.7	1.00	2.2%	19.2%	12.8%	14.9%	363.6	1.00	17.6%	19.7%	13.2%	441.2	1.63	562.8	2.21
Capacity limits	10	**5**	15	2	**5**	6	10	**5**	10	4	5	5	95	95%	1	5	2	428.8	1.04	0.9%	26.2%	30.0%	14.1%	432.5	1.04	77.0%	26.4%	30.3%	1879.2	6.08	2264.3	6.14
	10	**8**	15	2	**8**	6	10	**8**	10	4	5	5	95	95%	1	5	2	331.4	0.65	1.8%	17.4%	13.6%	14.3%	337.6	0.65	29.0%	17.7%	13.9%	475.2	1.36	642.4	1.85
	10	**10**	15	2	**10**	6	10	**10**	10	4	5	5	95	95%	1	5	2	308.6	0.58	2.5%	15.9%	11.0%	14.3%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
	10	**15**	15	2	**15**	6	10	**15**	10	4	5	5	95	95%	1	5	2	283.6	0.52	4.0%	15.8%	10.0%	14.3%	295.4	0.52	7.4%	16.0%	10.3%	319.2	0.57	401.8	0.79
	10	**20**	15	2	**20**	6	10	**20**	10	4	5	5	95	95%	1	5	2	264.3	0.50	5.6%	15.3%	6.9%	14.3%	280.1	0.50	6.4%	15.5%	7.3%	299.1	0.49	370.0	0.66
Discount factor	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	**1**	5	2	308.6	0.58	2.5%	15.9%	11.0%	14.3%	316.4	0.58	17.6%	16.2%	11.3%	383.9	0.88	502.4	1.20
	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	**0.95**	5	2	245.5	0.49	0.8%	15.2%	10.9%	14.2%	247.4	0.49	15.5%	15.4%	11.1%	292.7	0.71	383.4	0.91
	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	**0.9**	5	2	204.3	0.43	0.2%	16.0%	10.7%	14.1%	204.8	0.43	13.1%	16.0%	10.8%	235.7	0.57	302.5	0.70
	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	**0.85**	5	2	174.4	0.37	0.1%	18.8%	14.2%	13.8%	174.5	0.37	10.6%	19.0%	14.4%	195.2	0.47	243.3	0.55
	10	10	15	2	10	6	10	10	10	4	5	5	95	95%	**0.8**	5	2	150.8	0.32	0.0%	19.3%	14.7%	13.5%	150.8	0.32	8.2%	19.3%	14.7%	164.3	0.38	197.9	0.43

Note. Cost estimates are based on optimal DP solution evaluated with 10,000 simulation runs of 30-period problems. "vs salvage" refers to % savings the salvage opportunity yields under dual sourcing; "vs single" refers to % savings dual sourcing yields versus best single sourcing policy.

Table B.1: Dual sourcing under fully general model: benefits of a salvage opportunity

Appendix C

Appendix for Chapter 3

C.1 Proofs

Proof of Lemma 3.1. Let $\{s(n_X^*), r(n_X^*), q(n_X^*)\}$ denote the optimal solution to (3.20) at the equilibrium participating capacity n_X^* obtained from (3.21), then clearly $\{s(n_X^*), r(n_X^*), q(n_X^*), n_X^*\}$ is a feasible solution to (3.19), with objective value $\Pi_X(n_X^*)$. Note that better solutions to (3.19), if any, can only be achieved at $n > n_X^*$ since $\Pi_X(n)$ increases in n (as will be shown later in each regime X). However, any such solution $(s, r, q, n) \in \mathcal{C}_X$ at $n > n_X^*$ does not satisfy constraint (3.18) because

$$\pi(s, r, n) \le \pi_X(n) \le \pi_X(n_X^*) = F^{-1}\left(\frac{n_X^*}{N}\right) < F^{-1}\left(\frac{n}{N}\right),$$

where the first inequality follows from (3.22), the second inequality is due to $\pi_X(n)$ decreasing in n (as will be shown later in each regime X), and the last inequality follows from the assumption that $F(\cdot)$ is continuously increasing on $[0, \infty)$. Therefore, $\{s(n_X^*), r(n_X^*), q(n_X^*), n_X^*\}$ is also the optimal solution to (3.19) and we have $\Pi_X^* = \Pi_X(n_X^*)$. □

Proof of Proposition 3.1. We start with the following observations about the optimal solution.

(i) *Allocating all capacity n towards serving riders (i.e., $r = 0, q = 0$) is feasible (hence optimal) if and only if $n \le n_1^C$.* To see this, let

$$S_1^C = (S_{11}, S_{12}, S_{12}\frac{t_{21}}{t_{12}}, S_{22}). \tag{C.1}$$

Then $n_1^C = S_1^C \cdot 1 = \overline{S} - (\Lambda_{21} - \Lambda_{12})t_{21}$ is the maximum service capacity without repositioning ($r = 0$). Therefore with $r = 0$ and $q = 0$, $\overline{s} \le n_1^C \Leftrightarrow n \le n_1^C$ by (3.25); if $n > n_1^C$, then $r = 0, q = 0$ is not feasible.

(ii) *Allowing for repositioning capacity* $r_{12} \geq 0$*, the maximum service capacity achievable (assuming* n *is sufficiently large) is*

$$\begin{cases} n_1^C + r_{12}\frac{t_{21}}{t_{12}}, \ \textit{with } r_{21} = 0 & \textit{if } r_{12} \in [0, n_2^C - \overline{S}] \\ \overline{S}, \ \textit{with } r_{21} = (S_{12} + r_{12})\frac{t_{21}}{t_{12}} - S_{21} > 0 & \textit{if } r_{12} > n_2^C - \overline{S} \end{cases}.$$

To see this, from (3.24) we have $s_{21} = (s_{12} + r_{12})\frac{t_{21}}{t_{12}} - r_{21}$, hence for $r_{12} \in [0, n_2^C - \overline{S}]$,

$$\max_{0 \leq s \leq S, r_{21} \geq 0} \overline{s} = \max_{0 \leq s \leq S, r_{21} \geq 0} s_{11} + s_{12} + (s_{12} + r_{12})\frac{t_{21}}{t_{12}} - r_{21} + s_{22} = n_1^C + r_{12}\frac{t_{21}}{t_{12}},$$

where the maximum is achieved at $s_{11} = S_{11}, s_{12} = S_{12}, s_{22} = S_{22}, r_{21} = 0$. Note that the service capacity reaches its upper bound $\overline{S}$ when r_{12} reaches $n_2^C - \overline{S}$, i.e., $n_1^C + (n_2^C - \overline{S})\frac{t_{21}}{t_{12}} = \overline{S}$. For $r_{12} > n_2^C - \overline{S}$, the maximum service capacity *stays* at $\overline{S}$ with $s = S$, but $r_{21} = (S_{12} + r_{12})\frac{t_{21}}{t_{12}} - S_{21} > 0$ by (3.24).

With these observations, we can derive the optimal structure given by the Proposition. Zone (1) follows directly from observation (i). In zone (2) and (3) where $n > n_1^C$, by observation (ii), the optimization problem with least repositioning travel cost (i.e., avoiding unnecessary repositioning capacity) can be simplified as

$$\max_{r_{12}} \left\{ n_1^C + r_{12}\frac{t_{21}}{t_{12}} : \ n_1^C + r_{12}\frac{t_{21}}{t_{12}} + r_{12} \leq n, \ r_{12} \in [0, n_2^C - \overline{S}], \ r_{21} = 0 \right\}.$$

When $n_1^C < n \leq n_2^C$ (zone (2)), the inequality constraint is binding and the optimal solution is

$$r_{12} = (n - n_1^C)\frac{t_{12}}{t_{12} + t_{21}}, \ r_{21} = 0, \quad s = S_1^C + \left(0, 0, (n - n_1^C)\frac{t_{21}}{t_{12} + t_{21}}, 0\right), \quad q = 0.$$

For $n > n_2^C$ (zone (3)), the inequality constraint is not binding. With all the demand served ($s = S$), the extra capacity waits in queues and the optimal solution is

$$r_{12} = n_2^C - \overline{S}, \ r_{21} = 0, \quad s = S, \quad q \in \left\{(q_1, q_2) : q_1 + q_2 = n - n_2^C\right\}.$$

□

Proof of Corollary 3.1. (i) The validity condition (3.22) in Lemma 3.1 requires that the per-driver profit is maximized subject to (3.24)–(3.26) at any $n > 0$ under the platform's optimal capacity allocation prescribed by Proposition 3.1. Note that

$$\pi(s, r, n) = \frac{(\bar{\gamma}p - c)\overline{s} - c\overline{r}}{n}.$$

By Proposition 3.1, $\overline{s} = n, r = 0$ for n in the scarce capacity zone $(0, n_1^C]$, clearly $\pi(s, r, n)$ is maximized; for n in the ample capacity zone (n_2^C, ∞), $\overline{s} = \overline{S}$ and $r = (n_2^C - \overline{S}, 0)$ involves the minimum repositioning capacity $\overline{r} = n_2^C - \overline{S}$, hence $\pi(s, r, n)$ is also maximized.

For fixed n in the moderate capacity zone $(n_1^C, n_2^C]$, further increasing $\pi(s, r, n)$ requires

$$(\bar{\gamma} p - c)\Delta\overline{s} - c\Delta\overline{r} > 0 \quad \Rightarrow \quad \Delta\overline{r} < \frac{\bar{\gamma} p - c}{c}\Delta\overline{s}.$$

Since $\overline{s}$ is maximized under the platform's optimal capacity allocation, it can only be decreased or remain unchanged, i.e., $\Delta\overline{s} \leq 0$. Hence

$$\Delta\overline{r} < \frac{\bar{\gamma} p - c}{c}\Delta\overline{s} \leq \frac{t_{12}}{t_{21}}\Delta\overline{s} \leq 0, \tag{C.2}$$

where the second inequality follows from Assumption 3.2. We next show that (C.2) cannot hold due to the platform's optimal capacity allocation and the flow balance constraint:

- By Proposition 3.1, the platform's optimal capacity allocation in the moderate capacity zone $(n_1^C, n_2^C]$ has $s_{11} = S_{11}, s_{12} = S_{12}, s_{22} = S_{22}$ and $r_{12} > 0, r_{21} = 0$, hence s_{11}, s_{12}, s_{22} cannot be increased while r_{21} cannot be reduced. It then follows from $\Delta r_{12} + \Delta r_{21} = \Delta\overline{r} < 0$ by (C.2) that r_{12} must be reduced. To conclude, any change of these capacity variables must satisfy

$$\Delta s_{11}, \Delta s_{12}, \Delta s_{22} \leq 0, \quad \Delta r_{12} < 0,\ \Delta r_{21} \geq 0. \tag{C.3}$$

- By flow balance constraint (3.24), i.e., $(s_{12} + r_{12})/t_{12} = (s_{21} + r_{21})/t_{21}$, its change satisfies

$$\frac{\Delta s_{12} + \Delta r_{12}}{t_{12}} = \frac{\Delta s_{21} + \Delta r_{21}}{t_{21}},$$

  which implies

$$\Delta s_{21} = \frac{t_{21}}{t_{12}}(\Delta s_{12} + \Delta r_{12}) - \Delta r_{21} \leq \frac{t_{21}}{t_{12}}(\Delta r_{12} + \Delta r_{21}) - \left(1 + \frac{t_{21}}{t_{12}}\right)\Delta r_{21} \leq \frac{t_{21}}{t_{12}}\Delta\overline{r}, \tag{C.4}$$

  where the two inequalities follow from $\Delta s_{12} \leq 0$ and $\Delta r_{21} \geq 0$ in (C.3), respectively.

By (C.3) and (C.4), we have

$$\Delta\overline{s} = \sum \Delta s_{ij} \leq \Delta s_{21} \leq \frac{t_{21}}{t_{12}}\Delta\overline{r},$$

which is a clear contradiction with $\Delta\bar{s} > \frac{t_{21}}{t_{12}}\Delta\bar{r}$ from (C.2). Therefore $\pi(s, r, n)$ is indeed maximized subject to (3.24)–(3.26) and the validity condition holds.

(ii) The existence and uniqueness of the participation equilibrium follows directly from the fact that $\pi_C(n)$ is continuously decreasing in n with zero limit value as $n \to \infty$. □

Proof of Lemma 3.2. Drivers' expected steady-state profit rate under policy $\widetilde{\pi}(\widetilde{\eta}; s, q)$ in (3.32) follows from Renewal Reward Theory. We next derive its specific expression for $\eta = (\eta_1, 0)$. W.L.O.G., we calculate the time functions over cycles starting and ending at the low-demand location (1). Let $p_{lk} = \lambda_{lk}/(\lambda_{l1} + \lambda_{l2})$ denote the probability of serving a lk-ride at location l. The expected service, repositioning and queueing time functions are as follows:

- The expected service time in a cycle is given by
$$T^s(\eta; s) = (1-\eta_1)\left[p_{11}t_{11} + p_{12}\left(t_{12} + \frac{1-p_{21}}{p_{21}}t_{22} + t_{21}\right)\right] + \eta_1\left(\frac{1-p_{21}}{p_{21}}t_{22} + t_{21}\right),$$
where $\frac{1-p_{21}}{p_{21}}t_{22}$ gives the expected time serving local demand at location 2. This follows from the fact that the number of local rides at location 2 a driver serves ("failures") before picking a ride back to location 1 ("success") follows a geometric distribution of "success" probability p_{21}.

- The expected repositioning time in a cycle is simply $T^r(\eta) = \eta_1 t_{12}$.

- The expected queueing delay in a cycle is given by
$$T^q(\eta; s, q) = (1-\eta_1)\left(W_1 + p_{12}\frac{1}{p_{21}}W_2\right) + \eta_1\frac{1}{p_{21}}W_2,$$
where $\frac{1}{p_{21}}W_2$ gives the expected queueing time at location 2. This follows from the fact that the number of queueing delays ("trials") at location 2 a driver encounters before picking a ride back to location 1 ("success") follows a geometric distribution of "success" probability p_{21}. $W_l = Q_l/(\lambda_{l1} + \lambda_{l2})$ is the queueing time at location l due to Little's Law.

Substituting the above time functions in (3.32) we have:
$$\widetilde{\pi}(\eta_1; s, q) = \frac{(\bar{\gamma}p - c)\left[(1-\eta_1)s_{21}\frac{t_{12}}{t_{21}}(s_{11}+s_{12}) + (\eta_1 s_{11}\frac{t_{12}}{t_{11}} + s_{12})(s_{21}+s_{22})\right] - c\eta_1 s_{21}\frac{t_{12}}{t_{21}}(s_{11}\frac{t_{12}}{t_{11}} + s_{12})}{(1-\eta_1)s_{21}\frac{t_{12}}{t_{21}}(s_{11}+s_{12}+q_1) + (\eta_1 s_{11}\frac{t_{12}}{t_{11}} + s_{12})(s_{21}+s_{22}+q_2) + \eta_1 s_{21}\frac{t_{12}}{t_{21}}(s_{11}\frac{t_{12}}{t_{11}} + s_{12})}. \tag{C.5}$$

□

Proof of Proposition 3.2. With $\eta_2 = 0$, and hence $r_{21} = 0$ by (3.33), the repositioning decision reduces to picking η_1, and the driver-incentive compatible capacity allocation set is given by

$$\mathcal{D} = \left\{ (s, r, q) \geq 0 : r_{21} = 0, \eta_1(s, r) \in \arg\max_{\widetilde{\eta}_1} \widetilde{\pi}(\widetilde{\eta}_1; s, q) \right\}. \tag{C.6}$$

Differentiating $\widetilde{\pi}(\eta_1; s, q)$ in (C.5) wrt η_1, we get

$$\frac{\partial \widetilde{\pi}}{\partial \eta_1} = \frac{s_{21}\frac{t_{12}}{t_{21}}(s_{11}\frac{t_{12}}{t_{11}} + s_{12})\left[(s_{21}+s_{22})\bar{\gamma}p - \left(s_{21}+s_{22}+s_{21}\frac{t_{12}}{t_{21}}\right)c\right]}{\left[(1-\eta_1)s_{21}\frac{t_{12}}{t_{21}}(s_{11}+s_{12}+q_1) + (\eta_1 s_{11}\frac{t_{12}}{t_{11}} + s_{12})(s_{21}+s_{22}+q_2) + \eta_1 s_{21}\frac{t_{12}}{t_{21}}(s_{11}\frac{t_{12}}{t_{11}} + s_{12})\right]^2} \times \left[q_1 - (q_1^*(s) + k(s)q_2)\right],$$

where

$$q_1^*(s) = \frac{(s_{11}+s_{12})s_{21}\frac{t_{12}}{t_{21}} + (s_{21}+s_{22})s_{12}}{(s_{21}+s_{22}) - \left(s_{21}+s_{22}+s_{21}\frac{t_{12}}{t_{21}}\right)\frac{c}{\bar{\gamma}p}}, \quad k(s) = \frac{(s_{11}+s_{12}) - s_{11}\frac{c}{\bar{\gamma}p}}{(s_{21}+s_{22}) - \left(s_{21}+s_{22}+s_{21}\frac{t_{12}}{t_{21}}\right)\frac{c}{\bar{\gamma}p}}. \tag{C.7}$$

The sign of $\partial\widetilde{\pi}/\partial\eta_1$ only depends on the sign of $q_1 - (q_1^*(s) + k(s)q_2)$. Note that $\eta_1(s, r) = r_{12}/(s_{11}\frac{t_{12}}{t_{11}} + s_{12} + r_{12})$ from (3.33), by (C.6):

(i) When $r_{12} = 0$, $\eta_1(s, r) = 0$ requires $\partial\widetilde{\pi}/\partial\eta_1 \leq 0$, hence $q_1 \leq q_1^*(s) + k(s)q_2$;

(ii) When $r_{12} > 0$ and $s_{11} + s_{12} > 0$, $\eta_1(s, r) \in (0, 1)$ requires $\partial\widetilde{\pi}/\partial\eta_1 = 0$, hence $q_1 = q_1^*(s) + k(s)q_2$ with $q_1^*(s), k(s) > 0$;

(iii) When $r_{12} > 0$ and $s_{11} + s_{12} = 0$, $\eta_1(s, r) = 1$ and $q_1^*(s) = k(s) = 0$, all drivers reposition at location 1 without waiting in a queue.

It follows that the driver-incentive compatible capacity allocation set given by (3.35). □

Proof of Proposition 3.3. We want to show there is a unique feasible capacity utilization satisfying (3.24)–(3.26), (3.36)–(3.38) and (3.35), i.e., $(s, r, q, n) \in \mathcal{M}$ given by (3.39). Note that $r_{21} = 0$ by (3.35). By (3.24) and (3.36) we can express service capacities in terms of s_{12} and r_{12}:

$$s_{11} = s_{12}\frac{S_{11}}{S_{12}}, \quad s_{21} = (s_{12} + r_{12})\frac{t_{21}}{t_{12}}, \quad s_{22} = (s_{12} + r_{12})\frac{t_{21}}{t_{12}}\frac{S_{22}}{S_{21}}. \tag{C.8}$$

We will focus on s_{12}, r_{12}, q, and recover the remaining quantities using (C.8). The other constraints, (3.25), (3.35), (3.37), (3.38) and (3.26), are rewritten below.

$$\frac{n_1^M}{S_{12}} s_{12} + \left[\frac{t_{21}}{t_{12}}\left(1+\frac{S_{22}}{S_{21}}\right)+1\right] r_{12} + q_1 + q_2 = n, \tag{C.9}$$

$$q_1 \begin{cases} \leq q_1^*(s) + k(s) q_2 & \text{if } r_{12} = 0 \\ = q_1^*(s) + k(s) q_2 & \text{if } r_{12} > 0 \end{cases}, \tag{C.10}$$

$$(S_{12} - s_{12}) r_{12} = 0, \tag{C.11}$$

$$(S_{12} - s_{12}) q_1 = 0, \quad \left(S_{21} - (s_{12} + r_{12})\frac{t_{21}}{t_{12}}\right) q_2 = 0, \tag{C.12}$$

$$0 \leq s_{12} \leq S_{12}, \quad 0 \leq \frac{t_{21}}{t_{12}}(s_{12} + r_{12}) \leq S_{21}, \quad r_{12} \geq 0, \quad q \geq 0. \tag{C.13}$$

We make the following observations about the feasible solution.

(i) *The allocation where all capacity serves rider demand, i.e.,* $r_{12} = 0, q = 0$, $s_{12} = \frac{n}{n_1^M} S_{12}$, *is feasible if and only if* $n \leq n_1^M$. (a) "⇒": this is immediate; this also implies that if $n > n_1^M$, then $r_{12} > 0$ or $q > 0$. (b) "⇐": given $n \leq n_1^M$, suppose first that $r_{12} > 0$, then (C.11) implies that $s_{12} = S_{12}$, thus $\bar{s} > n_1^M$ and $n > n_1^M$, a contradiction; second, that $q_1 > 0$, then (C.12) implies that $s_{12} = S_{12}$, thus $\bar{s} \geq n_1^M$ and $n > n_1^M$, a contradiction; or third, that $q_2 > 0$, then (C.12) implies that $s_{21} = S_{21}, r_{12} > 0$, thus $n > n_1^M$ is still a contradiction. Hence $n \leq n_1^M \Rightarrow r_{12} = 0, q = 0, s_{12} = \frac{n}{n_1^M} S_{12}$, which satisfies (C.9)–(C.13) and is thus feasible.

(ii) *When* $n > n_1^M$, *any feasible solution must serve all demand at location 1, i.e.,* $s_{12} = S_{12}$, *and moreover,* $q_1^*(s) \equiv q_1^*(S)$. By (i), $n > n_1^M$ implies that $r_{12} > 0$ and/or $q \neq 0$, and either of these assertions implies that $s_{12} = S_{12}$ by (C.11) and (C.12). To show that $q_1^*(s) \equiv q_1^*(S)$, we substitute s_{11}, s_{21}, s_{22} in $q_1^*(s)$ in (C.7) by (C.8),

$$q_1^*(s) = \frac{(\frac{S_{11}}{S_{12}} + 1)\frac{t_{12}}{t_{21}} + (1 + \frac{S_{22}}{S_{21}})}{(1 + \frac{S_{22}}{S_{21}}) - (1 + \frac{S_{22}}{S_{21}} + \frac{t_{12}}{t_{21}})\frac{c}{\gamma p}} s_{12},$$

which is equal to a constant multiplying s_{12}. For $s_{12} = S_{12}$, we have that $q_1^*(s) \equiv q_1^*(S)$.

With these two observations, we derive the feasible solution given by the Proposition.

Zone (1): the result follows directly from observation (i).

Zone (2): for $n_1^M < n \leq n_2^M := n_1^M + q_1^*(S)$, observation (ii) gives $s_{12} = S_{12}$ and $q_1^*(s) = q_1^*(S)$, so (C.9), (C.10) and (C.13) immediately imply that $r_{12} = 0$. Putting $s_{12} = S_{12}$ and $r_{12} = 0$ into (C.12) we get $q_2 = 0$. It also follows from (C.9) that $q_1 = n - n_1^M$. In this zone q_1 increases with n while $r_{12} = q_2 = 0$. The feasible solution is summarized as

$$r = 0, \quad s = \left(S_{11}, S_{12}, S_{21}\frac{\Lambda_{12}}{\Lambda_{21}}, S_{22}\frac{\Lambda_{12}}{\Lambda_{21}}\right), \quad q = (n - n_1^M, 0).$$

Zone (3): for $n_2^M < n \leq n_3^M := n_2^C + q_1^*(S)$, if $s_{12} = S_{12}$ and $r_{12} = 0$, then $q_2 = 0$ by (C.12), which implies that $q_1 \leq q_1^*(S)$ by (C.10). By (C.9), this contradicts the fact that $n > n_2^M$. It follows that $r_{12} > 0$, and (C.10) yields $q_1 = q_1^*(S) + k(s)q_2$. Together with $s_{12} = S_{12}$, it is then easy to verify that (C.9), (C.12) and $n \leq n_3^M$ imply that $q_2 = 0$. In this zone r_{12} increases with n while $q_1 = q_1^*(S), q_2 = 0$ and the feasible solution is as follows:

$$r_{12} > 0, \quad s = \left(S_{11}, S_{12}, (S_{12} + r_{12})\frac{t_{21}}{t_{12}}, (S_{12} + r_{12})\frac{t_{21}}{t_{12}}\frac{S_{22}}{S_{21}}\right), \quad q = (q_1^*(S), 0).$$

Zone (4): for $n > n_3^M$, the above argument still implies that $r_{12} > 0$ and $q_1 = q_1^*(S) + k(s)q_2$. And, by (C.9) and (C.13) we get that $q_2 > 0$. It then follows from (C.12) that $r_{12} = S_{21}\frac{t_{12}}{t_{21}} - S_{12} = n_2^C - \overline{S}$ and $s = S$. In this zone q_1 and q_2 increase with n while s and r stay constant. The feasible solution is given by

$$r = (n_2^C - \overline{S}, 0), \quad s = S, \quad q = (q_1^*(S) + k(S)q_2, q_2).$$

This completes the proof. □

Proof of Corollary 3.2. (i) By Proposition 3.3, at any $n > 0$ there is a *unique* feasible driver capacity allocation under the constraints in (3.40), hence the per-driver profit is naturally maximized and the validity condition (3.22) in Lemma 3.1 is satisfied.

(ii) Substituting $\overline{s}$ and $\overline{r}$ from Proposition 3.3 into (3.27) yields

$$\pi_M(n) = \frac{(\bar{\gamma}p - c)\overline{s} - c\overline{r}}{n} = \begin{cases} \bar{\gamma}p - c & \text{zone (1) } (n \leq n_1^M), \\ \frac{n_1^M}{n}(\bar{\gamma}p - c) & \text{zone (2) } (n_1^M < n \leq n_2^M), \\ \frac{S_{21}+S_{22}}{S_{21}+S_{22}+S_{21}\frac{t_{12}}{t_{21}}}\bar{\gamma}p - c & \text{zone (3) } (n_2^M < n \leq n_3^M), \\ \frac{1}{n}(\bar{\gamma}p\overline{S} - cn_2^C) & \text{zone (4) } (n > n_3^M). \end{cases} \tag{C.14}$$

It is easy to see that $\pi_M(n)$ is continuously decreasing in n and that $\lim_{n\to\infty} \pi_M(n) = 0$. Therefore, the participation equilibrium condition (3.28), $n = NF(\pi_M(n))$, has a unique solution n_M^*. □

Proof of Proposition 3.4. We have the following observations about the optimal solution.

(i) *Allocating all capacity n towards serving riders (i.e., $r_{12} = 0, q = 0$) is feasible (hence optimal) if and only if $n \leq n_1^A := n_1^C$.* This is the same as observation (i) in the Proof of Proposition 3.1. Also notice that constraint (3.35) is satisfied.

(ii) *If for some capacity level n_1 the service capacity $\overline{s} > n_1^A$, then for all capacity levels $n_2 \geq n_1$ the optimal solution involves repositioning.* First, note that by the definition of n_1^A, $\overline{s} > n_1^A$, which implies that $r_{12} > 0$ (which holds at n_1), and hence we only need to show that the optimal service capacity at n_2 is higher than n_1^A. It suffices to find one feasible solution at n_2 that has the same service capacity as at n_1, which is higher than n_1^A. To achieve this, let the service capacity vector s and the repositioning capacity $r_{12} > 0$ at n_2 be the same as those at n_1, respectively, and put the extra capacity $n_2 - n_1$ into q satisfying $q_1 = q_1^*(s) + k(s)q_2$. In this way all constraints are still satisfied while the service capacity $\overline{s} > n_1^A$ remains unchanged.

(iii) *$r_{12} > 0$ for all $n > n_1^A + q_1^*(S_1^A)$, where $S_1^A = S_1^C$ defined in* (C.1) *such that $S_1^A \cdot 1 = n_1^A$.* It suffices to show that at capacity levels in the right neighborhood of $n_1^A + q_1^*(S_1^A)$, the optimal service capacity is higher than n_1^A, which then, by observation (ii), will prove the result. To show this, for an arbitrarily small $\epsilon > 0$, let n_ϵ be the minimum feasible total capacity to provide service vector $S_1^A + (0, 0, \epsilon, 0)$, and hence service capacity $n_1^A + \epsilon > n_1^A$. Following constraints (3.24)–(3.26) and (3.35), we have

$$n_\epsilon = n_1^A + \epsilon + \frac{t_{12}}{t_{21}}\epsilon + q_1^*(S_1^A + (0, 0, \epsilon, 0)),$$

and $n_0 = n_1^A + q_1^*(S_1^A)$ for $\epsilon = 0$. It is easy to see that n_ϵ increases in ϵ, since by definition (C.7),

$$\frac{\partial q_1^*(s)}{\partial s_{21}} = \frac{((s_{11}+s_{12})\bar{\gamma}p - s_{11}c)s_{22}\frac{t_{12}}{t_{21}}}{\left[(s_{21}+s_{22})\bar{\gamma}p - (s_{21}+s_{22}+s_{21}\frac{t_{12}}{t_{21}})c\right]^2} > 0, \quad \forall s_{22} > 0, s_{11}+s_{12} > 0, \tag{C.15}$$

i.e., $q_1^*(s)$ increases wrt s_{21} when $s_{22}, s_{11}+s_{12} > 0$. Therefore, the optimal service capacity must be higher than n_1^A at capacity levels in the right neighborhood of $n_1^A + q_1^*(S_1^A)$.

(iv) $r_{12} = 0$ *for* $n \in [n_1^A, n_1^A + \delta]$ *for a small* $\delta > 0$. We first prove this for the optimal solution at $n = n_1^A + \delta$ by establishing that for any feasible q_1 it must be that $q_1 < q_1^*(s)$, from which we deduce $r_{12} = 0$ from (3.35). Then, observation (ii) yields the same result for $n \in [n_1^A, n_1^A + \delta]$. Pick

$$\delta = \min\left\{ q_1^*(S_1^A), \frac{(S_{12})^2}{S_{12}\left(1+\frac{t_{21}}{t_{12}}\right) + S_{21} + S_{22}} \right\},$$

so that $n_1^A + \delta \le n_1^A + q_1^*(S_1^A)$, $\delta < S_{12}$ and

$$\delta < \frac{(S_{12})^2}{S_{12}\left(1+\frac{t_{21}}{t_{12}}\right) + S_{21} + S_{22}}\left(1+\frac{t_{21}}{t_{12}}\right) \Rightarrow \frac{S_{12}(S_{12}-\delta)}{S_{21}+S_{22}}\left(1+\frac{t_{21}}{t_{12}}\right) > \delta. \tag{C.16}$$

First note that the optimal solution at $n_1^A + \delta$ must have $\overline{s} \ge n_1^A$, since a feasible solution $s = S_1^A, r_{12} = 0, q_1 = 0, q_2 = \delta$ yields $\overline{s} = n_1^A$. Therefore

$$r_{12}, q_1 \le \delta, \quad s_{21} \ge S_{12}\frac{t_{21}}{t_{12}}, \quad s_{12} \ge S_{12} - \delta > 0, \tag{C.17}$$

where the first inequality follows from $\overline{s} \ge n_1^A$ and capacity constraint (3.25), the second is by $\overline{s} \ge n_1^A$, and the third follows from the second and (3.24) in that $s_{12} = s_{21}\frac{t_{12}}{t_{21}} - r_{12} \ge S_{12} - \delta$.

Then,

$$\begin{aligned} q_1^*(s) &= \frac{(s_{11}+s_{12})s_{21}\frac{t_{12}}{t_{21}} + (s_{21}+s_{22})s_{12}}{(s_{21}+s_{22}) - \left(s_{21}+s_{22}+s_{21}\frac{t_{12}}{t_{21}}\right)\frac{c}{\bar{\gamma}p}} \\ &\ge \frac{s_{12}s_{21}\frac{t_{12}}{t_{21}} + s_{21}s_{12}}{S_{21}+S_{22}} \\ &\ge \frac{S_{12}(S_{12}-\delta)\frac{t_{21}}{t_{12}}\left(1+\frac{t_{12}}{t_{21}}\right)}{S_{21}+S_{22}} > \delta, \end{aligned}$$

where the the second inequality follows from the second and third inequalities in (C.17), and the last inequality is by (C.16). Finally the first inequality in (C.17) leads to $q_1 \leq \delta < q_1^*(s)$, which implies that $r_{12} = 0$ by constraint (3.35). Hence we have shown $r_{12} = 0$ at $n = n_1^A + \delta$. By observation (ii), the optimal solution at $n \in [n_1^A, n_1^A + \delta]$ has service capacity $\overline{s} = n_1^A$ and no repositioning.

(v) *An optimal solution can serve all demand ($s = S$) if and only if $n \geq n_3^A := n_2^C + q_1^*(S)$.* "$\Leftarrow$": given $n \geq n_2^C + q_1^*(S)$, it is easy to verify that the capacity allocation $s = S, r = (n_2^C - \overline{S}, 0)$ and $q_1 = q_1^*(S) + k(S)q_2$ with $q_1 + q_2 = n - n_2^C$ is feasible and serves all demand (hence optimal). "$\Rightarrow$": an optimal (hence feasible) solution that serves all demand must have $s = S, r = (n_2^C - \overline{S}, 0)$ and $q_1 = q_1^*(S) + k(S)q_2$. By (3.25) this yields $n = n_2^C + q_1^*(S) + k(S)q_2 + q_2 \geq n_2^C + q_1^*(S) = n_3^A$.

With these five observations, we can derive the optimal solution given by the Proposition. Zone (1) follows directly from observation (i) and zone (4) follows from observation (v). In $(n_1^A, n_3^A]$, not all drivers are serving riders and not all riders are served. There exists a threshold n_2^A such that $n_1^A < n_2^A < n_3^A$ which separates zone (2) and (3) apart: in zone (2), $(n_1^A, n_2^A]$, optimal solution has service capacity $\overline{s} = n_1^A$, no repositioning ($r = 0$), and extra capacity queues at location 1 with $q = (n - n_1^A, 0)$; whereas in zone (3), $(n_2^A, n_3^A]$, optimal solution involves repositioning ($r_{12} > 0$), serves $\overline{s} > n_1^A$, and extra capacity queues at location 1 with $q = (q_1^*(s), 0)$.[1] Note that $n_2^A > n_1^A$ by observation (iv). $n_2^A < n_3^A$ follows from $n_2^A \leq n_1^A + q_1^*(S_1^A)$ by observation (iii) and $n_1^A + q_1^*(S_1^A) < n_2^C + q_1^*(S) = n_3^A$ by property (C.15). Furthermore, the fact that optimal solution involves repositioning at any capacity level in zone (3) follows from observation (ii). □

The following proofs of Lemma 3.3 and Propositions 3.7–3.8 refer to two technical lemmas, Lemmas C.1 and C.2. The statements and proofs of these lemmas as well as the proof of Proposition 3.5 are relegated to the Supplemental Materials.

[1] The steady state system flow equations do not differentiate between queueing in locations 1 and 2 in this capacity regime. A more detailed transient analysis would show that when the platform makes admission control decisions, it would choose to clear the queue in the high-demand location given that the demand exceeds the available capacity, and drivers would only queue in the low-demand location.

Proof of Lemma 3.3. (i) When condition (3.46) holds, there is strategic demand rejection at the low-demand location when $n \in (n_1^A, n_3^A]$. Substituting $\overline{s}$ and $\overline{r}$ from Proposition 3.4 into (3.27) yields

$$\pi_A(n) = \frac{(\bar{\gamma}p - c)\overline{s} - c\overline{r}}{n} = \begin{cases} \bar{\gamma}p - c & \text{zone (1) } (n \leq n_1^A), \\ \frac{n_1^A}{n}(\bar{\gamma}p - c) & \text{zone (2) } (n_1^A < n \leq n_2^A), \\ \frac{1}{n}[(\bar{\gamma}p - c)\overline{s}^* - c\overline{r}^*] & \text{zone (3) } (n_2^A < n \leq n_3^A), \\ \frac{1}{n}(\bar{\gamma}p\overline{S} - cn_2^C) & \text{zone (4) } (n > n_3^A). \end{cases} \tag{C.18}$$

Note that there is no (simple) explicit expression in zone (3), in which the platform may reject riders at the low-demand location if condition (3.46) holds. Nevertheless, we show in Lemma C.2 (see Supplemental Materials) that $\pi_A(n)$ is decreasing in n and $\lim_{n\to\infty} \pi_A(n) = 0$. Hence there is a unique equilibrium participating capacity n_A that satisfies the inequalities

$$NF(\pi_A(n_A^+)) \leq n_A \leq NF(\pi_A(n_A^-)). \tag{C.19}$$

In this case the validity condition for the two-step solution approach in Lemma 3.1 may not hold.

(ii) When condition (3.46) does not hold, there is no strategic demand rejection at the low-demand location. The platform's optimal capacity allocation for $n \in (\hat{n}_2^A, n_3^A]$ follow pattern (1) in Lemma C.1 (see Supplemental Materials): only s_{21} is increasing. The proof of the validity condition (3.22) in Lemma 3.1 is similar to that of regime C (in the proof of Corollary 3.1): First, the logic for the scarce and ample capacity zones, $(0, n_1^A]$ and (n_3^A, ∞), is identical to that in the proof of Corollary 3.1. Second, in zone 2–moderate capacity without repositioning, $(n_1^A, \hat{n}_2^A]$, there is zero repositioning capacity so that $\overline{r} = 0$ cannot be reduced, hence inequality (C.2) in the proof of Corollary 3.1 is immediately violated and it follows that the per-driver profit $\pi(s, r, n)$ is maximized. Third, in zone 3–moderate capacity with repositioning, $(\hat{n}_2^A, n_3^A]$, the argument is identical to that in zone 3 of regime C. In particular, any feasible deviation from the platform's optimal capacity allocation cannot increase the per-driver profit.

To establish the unique solution $\hat{n}_A$ to the participation equilibrium $\hat{n}_A = NF(\hat{\pi}_A(\hat{n}_A))$, just notice that $\hat{\pi}_A(n)$ is continuous following pattern (1) in Lemma C.1 (see Supplemental Materials), and case (i) in Lemma C.2 shows that $\hat{\pi}_A(n)$ decreases in n.

(iii) First, by definition n_A and $\hat{n}_A$ (and their associated optimal capacity allocations) are both feasible solutions to problem (3.19) with $X = A$. Next, we show that no better equilibrium can be established at $n < n_A$ or $n > \hat{n}_A$. On the one hand, if there exists an equilibrium at $n < n_A$ with associated optimal service vector s, then it must be suboptimal since

$$\Pi(s) \leq \Pi_A(n) \leq \Pi_A(n_A),$$

where the first inequality is by definition that $\Pi_A(n)$ is the maximum platform revenue at n subject to $(s, r, q, n) \in \mathcal{C}_A$, and the second inequality follows from the monotonicity of $\Pi_A(\cdot)$ given by Proposition 3.4. On the other hand, any solution $(s, r, q, n) \in \mathcal{C}_A$ at $n > \hat{n}_A$ does not satisfy the driver participation constraint because

$$\pi(s, r, n) \leq \hat{\pi}_A(n) \leq \hat{\pi}_A(\hat{n}_A) = F^{-1}\left(\frac{\hat{n}_A}{N}\right) < F^{-1}\left(\frac{n}{N}\right),$$

where the first inequality follows from the validity condition (3.22) in Lemma 3.1 that is proved in part (ii), the second inequality is due to $\hat{\pi}_A(n)$ decreasing in n shown by Lemma C.2 in the Supplemental Materials (see, in particular, case (i) in its proof), and the last inequality follows from the assumption that $F(\cdot)$ is continuously increasing on $[0, \infty)$. Therefore, the actual equilibrium participating capacity n_A^* must lie in $[n_A, \hat{n}_A]$. When regime A does not involve strategic demand rejection, i.e., condition (3.46) in Proposition 5 does not hold, $\Pi_A(n) = \hat{\Pi}_A(n)$ and hence $n_A^* = n_A = \hat{n}_A$. □

Proof of Proposition 3.6. We first prove the case $S_{22} = 0$ and then establish the threshold level $\hat{S}_{22} > 0$.

(1) When $S_{22} = 0$, we have the following four properties (i)–(iv) regarding the platform revenue and per-driver profit under optimal capacity allocation allowing or disallowing strategic demand rejection:

  (i) Condition (3.46) in Proposition 3.5 immediately holds.

In the following, we show that under $S_{22} = 0$ (hence $s_{22} = 0$), the 3 patterns of optimal capacity allocation as a function of participating capacity $n \in (n_2^A, n_3^A)$ established in Lemma C.1 and its proof in the Supplemental Materials can be significantly simplified. Setting $s_{22} = 0$ in the derivatives (C.39)–(C.41) and using $n = g(s)$ by (C.33), we get

$$\frac{\partial n}{\partial s_{11}} = \frac{\partial g(s)}{\partial s_{11}} = \left(1 + \frac{t_{12}}{t_{21}}\right) \frac{\bar{\gamma} p - c}{\bar{\gamma} p - \left(1 + \frac{t_{12}}{t_{21}}\right) c},$$

$$\frac{\partial n}{\partial s_{12}} = \frac{\partial g(s)}{\partial s_{12}} = \left(1 + \frac{t_{12}}{t_{21}}\right) \frac{\bar{\gamma} p}{\bar{\gamma} p - \left(1 + \frac{t_{12}}{t_{21}}\right) c},$$

$$\frac{\partial n}{\partial s_{21}} = \frac{\partial g(s)}{\partial s_{21}} = 1 + \frac{t_{12}}{t_{21}}.$$

Clearly $\frac{\partial n}{\partial s_{21}} < \frac{\partial n}{\partial s_{11}} < \frac{\partial n}{\partial s_{12}}$ and all are independent of the service capacity on each route, i.e., the second derivatives are all zero. Noticing that $\overline{s_1}(n_3^A) = \overline{s_2}(n_3^A) = \overline{s_3}(n_3^A) = \overline{S}$ and the patterns specified in Lemma C.1, a direct implication is that

$$\overline{s_3}(n) \geq \overline{s_2}(n) > \overline{s_1}(n), \quad n \in (n_2^A, n_3^A),$$

i.e., pattern (3) yields the largest service capacity at any $n \in (n_2^A, n_3^A)$ and strictly dominates pattern (1) which disallows strategic demand rejection. In fact, the platform's optimal capacity allocation follows pattern (3) when strategic demand rejection is allowed and follows pattern (1) when disallowed. Consequently, we have $n_2^A = \overline{s_3}^{-1}(n_1^A)$, the left end of the pattern (3) interval, and $\hat{n}_2^A = \overline{s_1}^{-1}(n_1^A)$, the left end of the pattern (1) interval, together with the following two properties:

(ii) $n_2^A < \hat{n}_2^A < n_3^A$.

(iii) $\Pi_A(n)$ strictly increases on (n_2^A, n_3^A); $\hat{\Pi}_A(n)$ stays constant on $(n_2^A, \hat{n}_2^A)$ and strictly increases on $[\hat{n}_2^A, n_3^A)$; and $\Pi_A(n) > \hat{\Pi}_A(n)$ on (n_2^A, n_3^A).

Similarly, the per-driver profit rate as a function of participating capacity $n \in (n_2^A, n_3^A)$ under each pattern established in the proof of Lemma C.2 in the Supplemental Materials can be simplified as follows. Setting $S_{22} = 0$, we immediately find $\pi'(n) = 0$ within each pattern given by (C.53)–(C.57). Note that under pattern (1), $\pi'(n) < 0$ for $n < \overline{s_1}^{-1}(n_1^A) = \hat{n}_2^A$. As a result, we have the following property about per-driver profits:

(iv) $\pi_A(n)$ remains constant on (n_2^A, n_3^A); $\hat{\pi}_A(n)$ strictly decreases on $[n_2^A, \hat{n}_2^A)$ and stays constant on $(\hat{n}_2^A, n_3^A)$; $\pi_A(n) < \hat{\pi}_A(n)$ on $(n_2^A, \hat{n}_2^A)$ and $\pi_A(n) = \hat{\pi}_A(n)$ on $[\hat{n}_2^A, n_3^A)$.

(2) If $n_A^* \in (n_2^A, \hat{n}_2^A)$, there must be $n_2^A < n_A < \hat{n}_A < \hat{n}_2^A$ following from $\pi_A(n) < \hat{\pi}_A(n)$ in (1.iv) and the monotonicity assumption on $F(\cdot)$. Hence by $\Pi_A(n) > \Pi_A(n_2^A), \forall n \in (n_2^A, \hat{n}_2^A)$ and $\hat{\Pi}_A(n) \equiv \Pi_A(n_2^A), \forall n \in (n_2^A, \hat{n}_2^A)$ due to (1.iii), there must be $\hat{\Pi}_A(\hat{n}_A) = \Pi_A(n_2^A) < \Pi_A(n_A)$. By Lemma 3.3 (iv) strategic demand rejection is optimal.

(3) If $n_A^* \in [\hat{n}_2^A, n_3^A)$, it follows from $\pi_A(n) = \hat{\pi}_A(n)$ in (1.iv) that $n_A^* = n_A = \hat{n}_A$ and thus $\hat{\Pi}_A(n_A^*) < \Pi_A(n_A^*)$. By Lemma 3.3 (iv) strategic demand rejection is optimal.

By continuity, the above results still hold for sufficiently small S_{22}. Specifically, we have the following observations:

(i) The platform's optimal revenues allowing or disallowing strategic demand rejection, $\Pi_A(n)$ and $\hat{\Pi}_A(n)$, are both continuous in n and S_{22}.

(ii) The resulting per-driver profits allowing or disallowing strategic demand rejection, $\pi_A(n)$ and $\hat{\pi}_A(n)$, are both continuous in $n \in (n_2^A, n_3^A)$ and S_{22}.

(iii) The drivers' opportunity cost distribution $F(\cdot)$ is continuous and strictly increasing.

(iv) Due to (ii) and (iii), the equilibrium participating capacities allowing or disallowing strategic demand rejection, n_A and $\hat{n}_A$, are both continuous in S_{22}.

Since we have shown in the first part that $\hat{\Pi}_A(\hat{n}_A) < \Pi_A(n_A)$ always holds for $n_A^* \in (n_2^A, n_3^A)$ at $S_{22} = 0$, it follows from (i) and (iv) that $\hat{\Pi}_A(\hat{n}_A) < \Pi_A(n_A)$ still holds for sufficiently small $S_{22} > 0$. We also know that condition (3.46) in Proposition 3.5 does not hold for sufficiently large S_{22}, hence by continuity there exists $\hat{S}_{22} > 0$ such that $\hat{\Pi}_A(\hat{n}_A) < \Pi_A(n_A)$, $\forall S_{22} \in [0, \hat{S}_{22})$, i.e, strategic demand rejection is optimal in a neighborhood of $S_{22} = 0$. □

Proof of Proposition 3.7. (1) Given a level of participating capacity n, the platform revenue rate under the three control regimes $\{M, A, C\}$ is computed as follows:

$$\Pi_M(n) = \arg\max_{s,r,q}\{\Pi(s) : (3.24) - (3.26), (3.36) - (3.38), (3.35)\},$$

$$\Pi_A(n) = \arg\max_{s,r,q}\{\Pi(s) : (3.24) - (3.26), (3.35)\},$$

$$\Pi_C(n) = \arg\max_{s,r,q}\{\Pi(s) : (3.24) - (3.26)\}.$$

These formulations share the same objective function, but have a decreasing set of constraints from $M \to A \to C$, from where (3.51) holds.

(2) and (3): Based on (3.31), (C.14) and (C.18), it is straightforward to verify that $\pi_M(n) \leq \pi_C(n)$ for any n, and to verify that $\pi_M(n) \leq \pi_A(n) \leq \pi_C(n)$ for $n \leq n_2^A$ and $n > n_3^A$ (i.e., zone (1), (2) and (4) in Proposition 3.4 for regime A). For $n_2^A < n \leq n_3^A$ (zone (3) of regime A), using Lemma C.1 one can verify that $\pi_A(n) \leq \pi_C(n)$ always holds, and $\pi_M(n) \leq \pi_A(n)$ holds if (3.46) is not satisfied. This proves (3.52) and (3.53). □

Proof of Proposition 3.8. Note the following properties about $\Pi_X(\cdot), \pi_X(\cdot)$ and n_X^*:

(i) Proposition 3.7 (1), $\Pi_M(n) \leq \Pi_A(n) \leq \Pi_C(n), \forall n$. It is also easy to verify that $\Pi_X(n)$ is continuously increasing in n for $X \in \{M, A, C\}$. This property follows immediately for the centralized control regime $X = C$. For regimes $X \in \{M, A\}$, one can verify that increasing capacity can be allocated into IC queues as in (3.35) without any reduction in the capacity that serves rider demand.

(ii) $\pi_X(n)$ is decreasing in n with $\lim_{n\to\infty} \pi_X(n) = 0$ for $X \in \{M, A, C\}$. Moreover, $\pi_X(\cdot)$ is continuous for $X \in \{M, C\}$, and is continuous for $X = A$ if (3.46) is not satisfied. This follows from (3.31), (C.14), (C.18) and Lemma C.2.

(iii) Let $\pi_1(n), \pi_2(n) : \mathbb{R}_+ \to \mathbb{R}_+$ be candidate per-driver profit functions that are decreasing in n (but may be *discontinuous*) with $\lim_{n\to\infty} \pi_i(n) = 0,\ i = 1, 2$. Let n_i^* be defined as $NF(\pi_i(n_i^{*+})) \leq n_i^* \leq NF(\pi_i(n_i^{*-}))$ and $\pi_i(n_i^*)$ satisfies $NF(\pi_i(n_i^*)) = n_i^*$ if n_i^* is a *discontinuity* of $\pi_i(\cdot)$. <u>If $\pi_1(n) \leq \pi_2(n), \forall n$,</u> <u>then $n_1^* \leq n_2^*$ and $\pi_1(n_1^*) \leq \pi_2(n_2^*)$.</u> To see $n_1^* \leq n_2^*$, by definition of n_1^* and $\pi_1(n) \leq \pi_2(n), \forall n$, we have

$$n_1^* \leq NF(\pi_1(n_1^{*-})) \leq NF(\pi_2(n_1^{*-})). \tag{C.20}$$

Suppose $n_2^* < n_1^*$, then $n_2^{*+} < n_1^{*-}$, by definition of n_2^* and (C.20) we have

$$n_2^* \geq NF(\pi_2(n_2^{*+})) \geq NF(\pi_2(n_1^{*-})) \geq n_1^*,$$

contradicting $n_2^* < n_1^*$. Hence, it must be that $n_1^* \leq n_2^*$. It then follows from $NF(\pi_i(n_i^*)) = n_i^*,\ i = 1, 2$ that $\pi_1(n_1^*) \leq \pi_2(n_2^*)$.

With these three properties and Proposition 3.7, we are ready to prove the three parts in this Proposition. Parts (2) and (3) follow from Property (ii), (iii) and Proposition 3.7 (2), (3). The ranking of n_X^* given by (3.55) and Property (i) imply that $\Pi_M^* \leq \Pi_C^*$ and $\Pi_A^* \leq \Pi_C^*$ in Part (1). The only thing left to show is $\Pi_M^* \leq \Pi_A^*$ and we complete this considering whether (3.46) holds.

If (3.46) is not satisfied, then Part (3) implies that $n_M^* \leq n_A^*$ and hence $\Pi_M^* \leq \Pi_A^*$ by Property (i).

If (3.46) is satisfied, then $\pi_A(\cdot)$ may be discontinuous in zone (3) of regime A. Consider the value of n_M^* and notice (C.14) and (C.18): In zone (1) or (4) ($n_M^* \leq n_1^M < n_1^A$ or $n_M^* \geq n_3^M = n_3^A$), $\pi_M(n_M^*) = \pi_A(n_M^*)$ and hence $n_A^* = n_M^*$, $\Pi_A^* = \Pi_M^*$; in zone (2) ($n_1^M < n_M^* \leq n_2^M$), it is obvious that $n_1^M < n_A^*$ since $\pi_A(n) = \pi_M(n)$ for $n \leq n_1^M$, therefore $\Pi_M^* = \Pi_M(n_1^M) < \Pi_A(n_A^*) = \Pi_A^*$; in zone (3) ($n_2^M < n_M^* < n_3^M = n_3^A$), since $\pi_A(n) > \pi_M(n_M^*)$ for $n < n_3^M = n_3^A$, there must be $n_M^* < n_A^*$ and hence $\Pi_M^* < \Pi_A^*$. Therefore $\Pi_M^* \leq \Pi_A^*$ for any n_M^*, and this completes the proof of Part (1). □

Proof of Corollary 3.3. Using the expressions of $\pi_X(n)$ for $X \in \{M, A, C\}$ in (3.31), (C.14), (C.18), respectively, and Proposition 3.7 (3), we have

$$\pi_A(n) \begin{cases} > \pi_M(n) & n \in (n_1^M, n_3^M),\ (3.46) \text{ not satisfied} \\ = \pi_M(n) & n \leq n_1^M \text{ or } n \geq n_3^M \end{cases}, \quad \pi_C(n) \begin{cases} > \pi_A(n) & n \in (n_1^A, n_3^A) \\ = \pi_A(n) & \text{o.w.} \end{cases}. \tag{C.21}$$

(1) For performance gains from admission control (control regime A over M), if $n_M^* \leq n_1^M$ or $n_M^* \geq n_3^M$, clearly $n_A^* = n_M^*$ since $\pi_A(n) = \pi_M(n)$ in these ranges; hence $\Pi_M^* = \Pi_A^*, \pi_M^* = \pi_A^*$ and there is no gain. If $n_1^M < n_M^* < n_3^M$ and (3.46) is not satisfied, then it must be $n_M^* < n_A^*$. To see this, (C.21) implies that $\pi_A(n_M^*) > \pi_M(n_M^*)$, therefore $n_M^* = NF(\pi_M(n_M^*)) < NF(\pi_A(n_M^*))$. Since (3.46) is not satisfied, $n_A^* = NF(\pi_A(n_A^*)$, and

from the monotonicity of $\pi_A(\cdot)$ we deduce that $n_M^* < n_A^*$. As a result, $\Pi_M^* < \Pi_A^*$ and $\pi_M^* < \pi_A^*$ following the proof of Proposition 3.8.

(2) For performance gains from centralized repositioning (control regime C over A) we have a similar argument as the above proof of Part (1) and the details are omitted. □

Proof of Proposition 3.9. (1) According to Corollary 3.3, if (3.46) is not satisfied, the platform revenue rate gain from admission control is positive only when $n_M^* \in (n_1^M, n_3^M)$. This yields

$$\Pi_M^* \geq \Pi_M(n_1^M) = \gamma p n_1^M, \tag{C.22}$$

where the equality holds for $n_M^* \in (n_1^M, n_2^M]$. By Corollary 3.3, $n_M^* \in (n_1^M, n_3^M)$ also implies that $\pi_M^* < \pi_A^*$, thus $n_A^* \in (n_1^M, n_3^M)$ since otherwise $n_M^* = n_A^*$ and $\pi_M^* = \pi_A^*$. This yields

$$\Pi_A^* \leq \Pi_A(n_3^M) = \Pi_A(n_3^A) = \gamma p \overline{S}, \tag{C.23}$$

where the equality is approached by $n_A^* \to n_3^M = n_3^A$.

Given that $N \geq n_3^M = n_3^A$, $n_X^* = NF(\pi_X(n_X^*))$ can take on values in $[0, n_3^M]$ depending on the choice of $F(\cdot)$, for $X \in \{M, A, C\}$. Consequently, the bounds in (C.22) and (C.23) can be approached and therefore

$$\max_{F(\cdot)} \frac{\Pi_A^* - \Pi_M^*}{\Pi_M^*} \leq \frac{\gamma p \overline{S} - \gamma p n_1^M}{\gamma p n_1^M} = \left(\frac{\Lambda_{21}}{\Lambda_{12}} - 1 \right) \frac{1}{1 + \frac{1-\rho_2}{1-\rho_1} \frac{1}{\tau}}.$$

To approach this upper bound, we need $n_M^* \in (n_1^M, n_2^M]$ and $n_A^* \to n_3^M = n_3^A$ so that $\Pi_M^* = \gamma p n_1^M$ and $\Pi_A^* \to \gamma p \overline{S}$. (Refer to Figure C.1 (a) and (c) for an illustration.) This holds for opportunity cost distributions $F(\cdot)$ satisfying

$$F^{-1}(n_2^M/N) = \pi_M(n_2^M) \quad \text{and} \quad F^{-1}(n_3^M/N) = \pi_M(n_2^M)^+,$$

i.e., the value of F at $\pi_M(n_2^M)$ is fixed at n_2^M/N ($\Rightarrow n_M^* = n_2^M$) and F grows sufficiently fast to $n_3^M/N = n_3^A/N$ at $\pi_M(n_2^M)^+ = \pi_A(n_3^A)^+$ ($\Rightarrow n_A^* \to n_3^A = n_3^M$); in words, there is a sufficiently large mass of potential drivers with opportunity cost around $\pi_M(n_2^M)^+$.

(2) According to Corollary 3.3, the platform revenue rate gain from centralized repositioning is positive only when $n_A^* \in (n_1^A, n_3^A)$. This yields

$$\Pi_A^* \geq \Pi_A(n_1^A) = \gamma p n_1^A, \tag{C.24}$$

where the equality holds for $n_A^* \in (n_1^A, n_2^A]$. By Corollary 3.3, $n_A^* \in (n_1^A, n_3^A)$ also implies that $\pi_A^* < \pi_C^*$. Thus, $n_C^* \in (n_1^A, n_3^A)$, since otherwise $n_A^* = n_C^*$ and $\pi_A^* = \pi_C^*$. This yields

$$\Pi_C^* \leq \Pi_C(n_3^A) = \gamma p \overline{S}, \tag{C.25}$$

where the equality holds for $n_C^* \in [n_2^C, n_3^A)$.

Given that $N \geq n_3^M = n_3^A$, $n_X^* = NF(\pi_X(n_X^*))$ can take on values in $[0, n_3^A]$ depending on the choice of $F(\cdot)$, for $X \in \{M, A, C\}$. Using the bounds in (C.24) and (C.25) we have

$$\max_{F(\cdot)} \frac{\Pi_C^* - \Pi_A^*}{\Pi_A^*} \leq \frac{\gamma p \overline{S} - \gamma p n_1^A}{\gamma p n_1^A} = \left(\frac{\Lambda_{21}}{\Lambda_{12}} - 1\right) \frac{1}{1 + \frac{1}{1-\rho_1}\frac{1}{\tau} + \frac{\rho_2}{1-\rho_2}\frac{\Lambda_{21}}{\Lambda_{12}}}.$$

To achieve this upper bound, we need $n_A^* \in (n_1^A, n_2^A]$ and $n_C^* \in [n_2^C, n_3^A)$ so that $\Pi_A^* = \gamma p n_1^A$ and $\Pi_C^* = \gamma p \overline{S}$. Noticing Proposition 3.7 (2) and Property (ii) in the proof of Proposition 3.8 about $\pi_A(\cdot), \pi_C(\cdot)$, this holds for opportunity cost distributions $F(\cdot)$ satisfying

$$F(\pi_A(n_2^A)) \leq n_2^A/N \quad \text{and} \quad F(\pi_C(n_2^C)) \geq n_2^C/N \tag{C.26}$$

when $\pi_A(n_2^A) < \pi_C(n_2^C)$. When $\pi_A(n_2^A) \geq \pi_C(n_2^C)$, (C.26) cannot be satisfied by any F and hence the upper bound is not tight. □

Proof of Proposition 3.10. Since $\pi_M(n) = \pi_A(n) = \pi_C(n)$ for $n \leq n_1^M$ and $n \geq n_3^M$, the per-driver profit rate gain from admission control only (regime A over M) and from admission control plus centralized repositioning (regime C over M) can be positive only for $n_M^* \in (n_1^M, n_3^M)$, and $n_A^*, n_C^* \in (n_1^M, n_3^M)$ simultaneously. It follows from the (decreasing) monotonicity of $\pi_X(\cdot)$, $X \in \{M, A, C\}$ that

$$\pi_M^* \geq \pi_M(n_3^M), \quad \pi_A^* \leq \pi_A(n_1^M) = \bar{\gamma}p - c, \quad \pi_C^* \leq \pi_C(n_1^M) = \bar{\gamma}p - c. \tag{C.27}$$

Therefore,

$$\max_{F(\cdot)} \frac{\pi_A^* - \pi_M^*}{\pi_M^*} = \max_{F(\cdot)} \frac{\pi_C^* - \pi_M^*}{\pi_M^*} \leq \frac{\bar{\gamma}p - c}{\pi_M(n_3^M)} - 1 = \frac{1 - \rho_2}{\tau - (1 - \rho_2 + \tau)\kappa}.$$

To achieve this upper bound, we need $n_M^* \in [n_2^M, n_3^M)$ and $n_A^*, n_C^* \in (n_1^M, n_1^A]$ so that the equalities in (C.27) are satisfied. If $n_2^M \leq n_1^A = n_1^C$, these conditions hold for $F(\cdot)$ satisfying

$$F(\pi_M(n_2^M)) \geq n_2^M/N \quad \text{and} \quad F(\pi_A(n_1^A)) \leq n_1^A/N. \tag{C.28}$$

If $n_2^M > n_1^A = n_1^C$, then (C.28) cannot be satisfied by any F and the upper bound is not tight. □

C.2 Supplemental Materials

C.2.1 Supplemental Lemmas and Proofs for Control Regime A

Under regime A, the lower capacity threshold n_2^A of zone (3)—moderate capacity *with repositioning*—and the optimal capacity allocation within this zone, do not have explicit expressions (see Proposition 3.4). Lemmas C.1 and C.2 fill in the remaining details.

Given a level of participating capacity n, Proposition 3.4 shows that the optimal capacity allocation in zone (3) has $r_{12} > 0, r_{21} = 0$ and $q = (q_1^*(s), 0)$. Therefore, the constraints of Problem A at a given capacity n, (3.24)–(3.26) and (3.35), simplify to

$$\overline{s} + \left(\frac{t_{12}}{t_{21}} s_{21} - s_{12}\right) + q_1^*(s) = n \tag{C.29}$$

and $0 \leq s \leq S$, $\frac{t_{12}}{t_{21}} s_{21} > s_{12}$. Note that s determines r_{12} by the second term and q by $q_1^*(s)$.

Lemma C.1 shows that there are three possible optimal capacity allocation *patterns* at any level of participating capacity in zone (3). These patterns differ in terms of whether demand is rejected at the low-demand location, and if so, for which route(s).

Lemma C.1. *Under control regime A, the optimal capacity allocation of any fixed participating capacity $n \in (n_2^A, n_3^A]$ (moderate capacity zone with repositioning) is determined by s that takes one of the following 3 patterns with the largest service capacity,* $\max_{i\in\{1,2,3\}} \overline{s_i}(n)$.[2]

(1) No demand rejection at the low-demand location: only s_{21} is increasing in this zone.

$$s_1(n) = (S_{11}, S_{12}, s_{21}, S_{22}) \text{ subject to (C.29)}, \quad n \in (\overline{s_1}^{-1}(n_1^A), n_3^A].$$

(2) Rejecting cross-traffic demand at the low-demand location: for small n, s_{21} is increasing while $s_{12} = 0$; for large n, $s_{21} = S_{21}$ and s_{12} is increasing.

$$s_2(n) = (S_{11}, s_{12}, s_{21}, S_{22}) \text{ subject to } (S_{21} - s_{21})s_{12} = 0 \text{ and (C.29)}, \quad n \in (\overline{s_2}^{-1}(n_1^A), n_3^A].$$

[2] Note that $\overline{s_i}(n)$ denotes the total service capacity out of participating capacity n following pattern i, and its inverse function $\overline{s_i}^{-1}(s)$ gives the required participating capacity for total service capacity s under pattern i.

(3) <u>Rejecting local and cross-traffic demand at the low-demand location</u>: for small n, s_{21} is increasing while $s_{11} = s_{12} = 0$; for medium n, $s_{21} = S_{21}$, s_{11} is increasing and $s_{12} = 0$; for large n, $s_{21} = S_{21}, s_{11} = S_{11}$ and s_{12} is increasing.

$$s_3(n) = (s_{11}, s_{12}, s_{21}, S_{22}) \text{ subject to } (S_{21} - s_{21})s_{11} = (S_{11} - s_{11})s_{12} = 0$$
$$\text{and (C.29)}, \quad n \in (\overline{s_3}^{-1}(n_1^A), n_3^A].$$

Proof. By Proposition 3.4, the optimal capacity allocation of given participating capacity $n \in (n_2^A, n_3^A]$ has $r_{12} > 0, r_{21} = 0$ and $q = (q_1^*(s), 0)$. Therefore, for fixed n, Problem A reduces to $\max_{s,r,q}\{\Pi(s) : (3.24)-(3.26), (3.35)\}$, and it can be reformulated as maximizing the total service capacity over the service capacity vector s:

$$\max_{s} \quad \overline{s} \tag{C.30}$$
$$\text{s.t.} \quad g(s) := \overline{s} + \left(\frac{t_{12}}{t_{21}} s_{21} - s_{12}\right) + q_1^*(s) \leq n \tag{C.31}$$
$$0 \leq s_{ij} \leq S_{ij}, \ \forall i, j. \tag{C.32}$$

Note that $g(s)$ is the total capacity expressed with respect to s. Relaxing the equality constraint (3.25) to the inequality constraint (C.31) does not matter since positive q_2 and $q_1 = q_1^*(s) + k(s)q_2$ are feasible by (3.35) (but not optimal by Proposition 3.4). Constraint $r_{12} = \frac{t_{12}}{t_{21}} s_{21} - s_{12} > 0$ is omitted since a violation results in $s_{21} \leq \frac{t_{21}}{t_{12}} S_{12} \Rightarrow \overline{s} \leq n_1^A$, clearly suboptimal in zone (3).

To prove the lemma, we first establish that any optimal solution to problem (C.30)–(C.32) for $n \in (n_2^A, n_3^A]$ must satisfy the following four necessary conditions:

(a) All capacity is used within this zone and s_{21} has a lower bound:

$$g(s) = n, \tag{C.33}$$
$$0 < S_{12} \frac{t_{21}}{t_{12}} \leq s_{21} \leq S_{21}. \tag{C.34}$$

(b) Rejecting local demand at the high-demand location (s_{22}) is suboptimal:

$$s_{22} = S_{22}. \tag{C.35}$$

(c) Rejecting cross-traffic demand (s_{12}) is more profitable than rejecting local demand (s_{11}) at the low-demand location:

$$(S_{11} - s_{11})s_{12} = 0. \tag{C.36}$$

(d) Neither demand stream at the low-demand location is partially served unless s_{21} is fully served:

$$s_{12}(S_{12} - s_{12})(S_{21} - s_{21}) = 0, \tag{C.37}$$

$$s_{11}(S_{11} - s_{11})(S_{21} - s_{21}) = 0. \tag{C.38}$$

We establish conditions (a)–(d) using the KKT conditions for the reformulated maximization problem (C.30)–(C.32). Before writing the KKT conditions, we first derive the first and second partial derivatives of $g(s)$ which will be used later. The first partial derivatives are

$$\frac{\partial g(s)}{\partial s_{11}} = \frac{s_{21}(1 + \frac{t_{12}}{t_{21}}) + s_{22}}{(s_{21} + s_{22})\bar{\gamma}p - \left(s_{21} + s_{22} + s_{21}\frac{t_{12}}{t_{21}}\right)c}(\bar{\gamma}p - c) > 1, \tag{C.39}$$

$$\frac{\partial g(s)}{\partial s_{12}} = \frac{s_{21}(1 + \frac{t_{12}}{t_{21}}) + s_{22}}{(s_{21} + s_{22})\bar{\gamma}p - \left(s_{21} + s_{22} + s_{21}\frac{t_{12}}{t_{21}}\right)c}\bar{\gamma}p > \frac{\partial g(s)}{\partial s_{11}} > 1, \tag{C.40}$$

$$\frac{\partial g(s)}{\partial s_{21}} = 1 + \frac{t_{12}}{t_{21}} + \frac{(s_{11}(\bar{\gamma}p - c) + s_{12}\bar{\gamma}p)s_{22}\frac{t_{12}}{t_{21}}}{\left[(s_{21} + s_{22})\bar{\gamma}p - \left(s_{21} + s_{22} + s_{21}\frac{t_{12}}{t_{21}}\right)c\right]^2}\bar{\gamma}p > 1, \tag{C.41}$$

$$\frac{\partial g(s)}{\partial s_{22}} = 1 - \frac{(s_{11}(\bar{\gamma}p - c) + s_{12}\bar{\gamma}p)s_{21}\frac{t_{12}}{t_{21}}}{\left[(s_{21} + s_{22})\bar{\gamma}p - \left(s_{21} + s_{22} + s_{21}\frac{t_{12}}{t_{21}}\right)c\right]^2}\bar{\gamma}p < 1. \tag{C.42}$$

For second partial derivatives, we do not need the ones involving s_{22}. Fixing s_{22} and letting $\overline{g}(s_{11}, s_{12}, s_{21}) = g(s)$, the Hessian of $\overline{g}(s_{11}, s_{12}, s_{21})$ is given by

$$\mathbf{H}(\overline{g}) = \frac{s_{22}\frac{t_{12}}{t_{21}}\bar{\gamma}p}{\left[(s_{21} + s_{22})\bar{\gamma}p - \left(s_{21} + s_{22} + s_{21}\frac{t_{12}}{t_{21}}\right)c\right]^2}\begin{bmatrix} 0 & 0 & \bar{\gamma}p - c \\ 0 & 0 & \bar{\gamma}p \\ \bar{\gamma}p - c & \bar{\gamma}p & -\frac{2(s_{11}(\bar{\gamma}p - c) + s_{12}\bar{\gamma}p)\left(\bar{\gamma}p - \left(1 + \frac{t_{12}}{t_{21}}\right)c\right)}{(s_{21} + s_{22})\bar{\gamma}p - \left(s_{21} + s_{22} + s_{21}\frac{t_{12}}{t_{21}}\right)c} \end{bmatrix}.$$

Hence we have

$$\frac{\partial^2 g(s)}{\partial s_{11}^2} = \frac{\partial^2 g(s)}{\partial s_{12}^2} = \frac{\partial^2 g(s)}{\partial s_{11}\partial s_{12}} = 0, \quad \frac{\partial^2 g(s)}{\partial s_{11}\partial s_{21}}, \frac{\partial^2 g(s)}{\partial s_{12}\partial s_{21}} > 0, \quad \frac{\partial^2 g(s)}{\partial s_{21}^2} \leq 0, \tag{C.43}$$

where the last inequality is strict when $s_{11} + s_{12} > 0$.

Let $\alpha, \overline{\beta}_{ij}, \underline{\beta}_{ij}$ be the dual variables associated with the capacity constraint (C.31), the upper and lower bound constraints (C.32), respectively. The KKT conditions are

$$\text{(stationarity)} \quad \alpha \frac{\partial g(s)}{\partial s_{ij}} + \overline{\beta}_{ij} - \underline{\beta}_{ij} = 1, \ \forall i, j, \tag{C.44}$$

$$\text{(complementary slackness)} \quad \alpha(n - g(s)) = \overline{\beta}_{ij}(S_{ij} - s_{ij}) = \underline{\beta}_{ij} s_{ij} = 0, \ \forall i, j, \tag{C.45}$$

$$\text{(dual feasibility)} \quad \alpha, \overline{\beta}_{ij}, \underline{\beta}_{ij} \geq 0, \ \forall i, j, \tag{C.46}$$

$$\text{(primal feasibility)} \quad g(s) \leq n, \tag{C.47}$$

$$\text{(primal feasibility)} \quad 0 \leq s_{ij} \leq S_{ij}, \forall i, j. \tag{C.48}$$

The complementary slackness constraints (C.45) and dual feasibility constraints (C.46) establish the relationship between primal and dual variables: $\overline{\beta}_{ij} = 0$ ($\underline{\beta}_{ij} = 0$) when s_{ij} is not at its upper (lower) bound; s_{ij} must be at its upper (lower) bound when $\overline{\beta}_{ij} > 0$ ($\underline{\beta}_{ij} > 0$); $\alpha = 0$ when $g(s) < n$ and $g(s) = n$ when $\alpha > 0$. Moreover, $\overline{\beta}_{ij} \cdot \underline{\beta}_{ij} = 0$. We omit explicit references to the primal and dual feasibility constraints (C.46)–(C.48) in the following proof.

Now we are ready to prove the four conditions in this lemma.

(a) When $s \neq S$, pick any $s_{ij} < S_{ij}$, then $\overline{\beta}_{ij} = 0$ by (C.45). By (C.44) this implies that $\alpha \neq 0$ and hence $g(s) = n$ by (C.45). When $s = S$, $g(s) = n = n_3^A$. These prove (C.33). For (C.34), $s_{21} \geq S_{12} \frac{t_{21}}{t_{12}} > 0$ follows directly from $\overline{s} > n_1^A$ in zone (3). By (C.45), $s_{21} > 0$ also implies that $\underline{\beta}_{21} = 0$.

(b) Using $\underline{\beta}_{21} = 0$ from part (a) and $\frac{\partial g(s)}{\partial s_{21}} > 1$, stationarity constraints (C.44) imply that $\alpha < 1$. Putting this and $\frac{\partial g(s)}{\partial s_{22}} < 1$ back to (C.44), we obtain $\overline{\beta}_{22} > 0$. Therefore it follows from (C.45) that $s_{22} = S_{22}$ and $\underline{\beta}_{22} = 0$.

(c) We prove this by contradiction using (C.44) and (C.45). Suppose on the contrary $(S_{11} - s_{11}) s_{12} > 0$ for some $s_{11} < S_{11}$ and $s_{12} > 0$, then (C.45) require $\overline{\beta}_{11} = \underline{\beta}_{12} = 0$ and hence (C.44) yield

$$\alpha \frac{\partial g(s)}{\partial s_{11}} - \underline{\beta}_{11} = \alpha \frac{\partial g(s)}{\partial s_{12}} + \overline{\beta}_{12} = 1.$$

This cannot happen due to $\frac{\partial g(s)}{\partial s_{11}} < \frac{\partial g(s)}{\partial s_{12}}$ and (C.46). Therefore we must have $(S_{11} - s_{11}) s_{12} = 0$.

(d) We prove the two equations in similar ways by showing that any violation will lead to suboptimality. For (C.37), suppose on the contrary $s_{12}(S_{12}-s_{12})(S_{21}-s_{21})>0$ for some $0<s_{12}<S_{12}$ and $s_{21}<S_{21}$, then (C.45) and $\underline{\beta}_{21}=0$ from part (a) require $\overline{\beta}_{12}=\underline{\beta}_{12}=\overline{\beta}_{21}=\underline{\beta}_{21}=0$. It hence follows from (C.44) that

$$\alpha\frac{\partial g(s)}{\partial s_{12}}=\alpha\frac{\partial g(s)}{\partial s_{21}}=1,$$

thus $\alpha>0$ and $1<\frac{\partial g(s)}{\partial s_{11}}<\frac{\partial g(s)}{\partial s_{12}}=\frac{\partial g(s)}{\partial s_{21}}$. By the second derivatives in (C.43), increasing s_{12} and decreasing s_{21} will always maintain the inequality

$$1<\frac{\partial g(s)}{\partial s_{11}}<\frac{\partial g(s)}{\partial s_{12}}<\frac{\partial g(s)}{\partial s_{21}}. \tag{C.49}$$

Therefore we can keep increasing s_{12} ($\Delta s_{12}>0$) and decreasing s_{21} ($\Delta s_{21}<0$) simultaneously such that the following equality holds at any subsequent s_{12} and s_{21}:

$$\Delta s_{12}\frac{\partial g(s)}{\partial s_{12}}+\Delta s_{21}\frac{\partial g(s)}{\partial s_{21}}=0. \tag{C.50}$$

In this way we can maintain

$$\Delta g(s)=\sum_{i,j}\frac{\partial g(s)}{\partial s_{ij}}\Delta s_{ij}=\Delta s_{12}\frac{\partial g(s)}{\partial s_{12}}+\Delta s_{21}\frac{\partial g(s)}{\partial s_{21}}=0,$$

i.e., keep $g(s)$ constant, while improving the objective function (service capacity) by

$$\Delta\overline{s}=\Delta s_{12}+\Delta s_{21}=\Delta s_{12}\left(1-\frac{\partial g(s)/\partial s_{12}}{\partial g(s)/\partial s_{21}}\right)>0,$$

which follows from (C.49) and (C.50), until $s_{12}(S_{12}-s_{12})(S_{21}-s_{21})=0$ is satisfied.

Similarly, for (C.38), suppose on the contrary $s_{11}(S_{11}-s_{11})(S_{21}-s_{21})>0$ for some $0<s_{11}<S_{11}$ and $s_{21}<S_{21}$, then (C.45) and $\underline{\beta}_{21}=0$ from part (a) require $\overline{\beta}_{11}=\underline{\beta}_{11}=\overline{\beta}_{21}=\underline{\beta}_{21}=0$. It hence follows from (C.44) that

$$\alpha\frac{\partial g(s)}{\partial s_{11}}=\alpha\frac{\partial g(s)}{\partial s_{21}}=1,$$

thus $\alpha>0$ and $1<\frac{\partial g(s)}{\partial s_{21}}=\frac{\partial g(s)}{\partial s_{11}}<\frac{\partial g(s)}{\partial s_{12}}$. By the second derivatives in (C.43), increasing s_{21} and decreasing s_{11} will always maintain the inequality

$$1<\frac{\partial g(s)}{\partial s_{21}}<\frac{\partial g(s)}{\partial s_{11}}<\frac{\partial g(s)}{\partial s_{12}}. \tag{C.51}$$

Therefore we can keep increasing s_{21} ($\Delta s_{21} > 0$) and decreasing s_{11} ($\Delta s_{11} < 0$) simultaneously such that the following equality holds at any subsequent s_{11} and s_{21}:

$$\Delta s_{11} \frac{\partial g(s)}{\partial s_{11}} + \Delta s_{21} \frac{\partial g(s)}{\partial s_{21}} = 0. \tag{C.52}$$

In this way we can maintain

$$\Delta g(s) = \sum_{i,j} \frac{\partial g(s)}{\partial s_{ij}} \Delta s_{ij} = \Delta s_{11} \frac{\partial g(s)}{\partial s_{11}} + \Delta s_{21} \frac{\partial g(s)}{\partial s_{21}} = 0,$$

i.e., keep $g(s)$ constant, while improving the objective function (service capacity) by

$$\Delta \overline{s} = \Delta s_{11} + \Delta s_{21} = \Delta s_{21} \left(1 - \frac{\partial g(s)/\partial s_{21}}{\partial g(s)/\partial s_{11}} \right) > 0,$$

which follows from (C.51) and (C.52), until $s_{11}(S_{11} - s_{11})(S_{21} - s_{21}) = 0$ is satisfied.

It is then easy to verify that the necessary conditions (a)–(d) directly imply the three patterns stated in Parts (1)–(3) of Lemma C.1. Note that for each pattern i, the service capacity $\overline{s_i}(n)$ increases with n from $n = \overline{s_i}^{-1}(n_1^A)$, where $\overline{s} = n_1^A$ is equal to the constant service capacity in zone (2), and up to $n = n_3^A$, the right end of zone (3). This also implies that $n_2^A = \min_i\{\overline{s_i}^{-1}(n_1^A)\}$. □

Next we prove the monotonicity of the per-driver profit rate with respect to n in zone (3).

Lemma C.2. *Per-driver profit rate under control regime A, $\pi_A(n)$, is decreasing in n for $n \in (n_2^A, n_3^A]$.*

Proof. Lemma C.1 shows that for participating capacity $n \in (n_2^A, n_3^A]$, the optimal capacity allocation *may alternate* among three patterns characterized by $s_i(n)$, with service capacity $\overline{s_i}(n)$ for $i = 1, 2, 3$. To prove this lemma, we show that the per-driver profit rate given by (3.27) is decreasing for n varying within each of the 3 patterns or at *feasible* transitions between patterns. First, note by (C.33) in the proof of Lemma C.1 that $n = g(s)$ in zone (3) and hence $\partial n/\partial s_{ij} = \partial g(s)/\partial s_{ij}$. Then:

(i) Within pattern (1): only s_{21} is increasing,

$$\pi'(n) = \frac{\left[(\bar{\gamma}p - c) - c\frac{t_{12}}{t_{21}}\right]\left(\frac{\partial g(s)}{\partial s_{21}}\right)^{-1} n - [(\bar{\gamma}p - c)\overline{s} - cr_{12}]}{n^2}$$
$$= -\frac{S_{22}\bar{\gamma}p}{n^2}\frac{t_{12}}{t_{21}}\left(\frac{\partial g(s)}{\partial s_{21}}\right)^{-1}\left(1 + \frac{S_{11}(\bar{\gamma}p - c) + S_{12}\bar{\gamma}p}{(s_{21} + S_{22})\bar{\gamma}p - \left(s_{21} + S_{22} + s_{21}\frac{t_{12}}{t_{21}}\right)c}\right)^2 < 0. \tag{C.53}$$

(ii) Within pattern (2): for small n, s_{21} is increasing while $s_{12} = 0$,

$$\pi'(n) = \frac{\left[(\bar{\gamma}p - c) - c\frac{t_{12}}{t_{21}}\right]\left(\frac{\partial g(s)}{\partial s_{21}}\right)^{-1} n - [(\bar{\gamma}p - c)\overline{s} - cr_{12}]}{n^2}$$
$$= -\frac{S_{22}\bar{\gamma}p}{n^2}\frac{t_{12}}{t_{21}}\left(\frac{\partial g(s)}{\partial s_{21}}\right)^{-1}\left(1 + \frac{S_{11}(\bar{\gamma}p - c)}{(s_{21} + S_{22})\bar{\gamma}p - \left(s_{21} + S_{22} + s_{21}\frac{t_{12}}{t_{21}}\right)c}\right)^2 < 0. \tag{C.54}$$

For large n, $s_{21} = S_{21}$ and s_{12} is increasing,

$$\pi'(n) = \frac{\bar{\gamma}p\left(\frac{\partial g(s)}{\partial s_{12}}\right)^{-1} n - [(\bar{\gamma}p - c)\overline{s} - cr_{12}]}{n^2} = 0. \tag{C.55}$$

Note that $\pi(n)$ is continuous at the turning (non-differentiable) point where $s = (S_{11}, 0, S_{21}, S_{22})$.

(iii) Within pattern (3): for small n, s_{21} is increasing while $s_{11} = s_{12} = 0$,

$$\pi'(n) = \frac{\left[(\bar{\gamma}p - c) - c\frac{t_{12}}{t_{21}}\right]\left(\frac{\partial g(s)}{\partial s_{21}}\right)^{-1} n - [(\bar{\gamma}p - c)\overline{s} - cr_{12}]}{n^2} = -\frac{S_{22}\bar{\gamma}p}{n^2}\frac{t_{12}}{t_{21}}\left(\frac{\partial g(s)}{\partial s_{21}}\right)^{-1} < 0. \tag{C.56}$$

For medium n, $s_{21} = S_{21}$, s_{11} is increasing and $s_{12} = 0$,

$$\pi'(n) = \frac{(\bar{\gamma}p - c)\left(\frac{\partial g(s)}{\partial s_{11}}\right)^{-1} n - [(\bar{\gamma}p - c)\overline{s} - cr_{12}]}{n^2} = 0. \tag{C.57}$$

For large n, $s_{21} = S_{21}, s_{11} = S_{11}$ and s_{12} is increasing, we have the same (C.55). Note that $\pi(n)$ is continuous at the two turning (non-differentiable) points where $s = (0, 0, S_{21}, S_{22})$ and $s = (S_{11}, 0, S_{21}, S_{22})$.

As n increases, a *feasible transition* from pattern i to j at n must satisfy

$$\overline{s_i}(n) = \overline{s_j}(n) \quad \text{and} \quad \overline{s_i}'(n^-) < \overline{s_j}'(n^+). \tag{C.58}$$

Namely, pattern i and j have the same service capacity at transition n, and the service capacity increases faster after the transition. We then discuss all three possible transitions.

(i) Between pattern (1) and (2). If for pattern (2) s_{12} is increasing (or just reaches 0) at the transition, there must be $\overline{s_1}'(n^-) < \overline{s_2}'(n^+)$ since otherwise

$$\overline{s_1}(n_3^A) = \overline{s_1}(n) + \int_n^{n_3^A} \overline{s_1}'(n)dn > \overline{s_2}(n) + \int_n^{n_3^A} \overline{s_2}'(n)dn = \overline{s_2}(n_3^A),$$

hence it must be a transition from pattern (1) to (2): $(S_{11}, S_{12}, s_{21}, S_{22}) \to (S_{11}, s_{12}, S_{21}, S_{22})$. Obviously r_{12} jumps up and $\pi(n)$ jumps down at the transition. If for pattern (2) s_{21} is increasing at the transition, we have $s_1 = (S_{11}, S_{12}, s_{21}^{(1)}, S_{22})$, $s_2 = (S_{11}, 0, s_{21}^{(2)}, S_{22})$. By (C.58) there must be $s_{21}^{(1)} < s_{21}^{(2)}$ and hence

$$\overline{s_1}'(n) = \left(\frac{\partial g(s)}{\partial s_{21}^{(1)}}\right)^{-1} < \left(\frac{\partial g(s)}{\partial s_{21}^{(2)}}\right)^{-1} = \overline{s_2}'(n),$$

i.e., a transition from pattern (1) to (2): $(S_{11}, S_{12}, s_{21}^{(1)}, S_{22}) \to (S_{11}, 0, s_{21}^{(2)}, S_{22})$. Similarly we have $\pi(n)$ jumps down at the transition.

(ii) Between pattern (1) and (3). Due to similarities between pattern (2) and (3), the cases where s_{12} or s_{21} is increasing under pattern (3) have been similarly shown above. For the case where s_{11} is increasing (or just reaches 0) at the transition under pattern (3), there must be $\overline{s_1}'(n^-) < \overline{s_3}'(n^+)$ since otherwise

$$\overline{s_1}(n_3^A) = \overline{s_1}(n) + \int_n^{n_3^A} \overline{s_1}'(n)dn > \overline{s_3}(n) + \int_n^{n_3^A} \overline{s_3}'(n)dn = \overline{s_3}(n_3^A),$$

where $\int_n^{n_3^A} \overline{s_3}'(n)dn$ is an integration over n from n to $s_3^{-1}((S_{11}, 0, S_{21}, S_{22}))$ and from $s_3^{-1}((S_{11}, 0, S_{21}, S_{22}))$ to n_3^A. Therefore this must be a transition from pattern (1) to (3): $(S_{11}, S_{12}, s_{21}, S_{22}) \to (s_{11}, 0, S_{21}, S_{22})$, obviously r_{12} jumps up and $\pi(n)$ jumps down at the transition.

(iii) Between pattern (2) and (3). Similar to the analysis of the transitions between pattern (1) and (2) where $s_{11} \equiv S_{11}$ and we focus on s_{12} and s_{21}, here $s_{12} \equiv 0$ at any transition between pattern (2) and (3) so that we can adopt the same approach by focusing on s_{11} and s_{21}. The details are omitted. Note that the transitions, if any, are always from pattern (2) to (3).

□

Proof of Proposition 3.5. From Lemma C.1, it is optimal to reject rider requests at the low-demand location for some n in zone (3) if and only if pattern (2) and/or (3) provide the *largest* service capacity at some $n \in (n_2^A, n_3^A)$, i.e., $\exists n \in (n_2^A, n_3^A)$ such that $\overline{s_1}(n) \neq \max_{i\in\{1,2,3\}} \overline{s_i}(n)$. We need to compare the three patterns in terms of their service capacity $\overline{s_i}(n),\ i = 1, 2, 3$. We have the following three observations.

(i) At the right end of zone (3), $\overline{s_1}(n_3^A) = \overline{s_2}(n_3^A) = \overline{s_3}(n_3^A) = \overline{S}$.

(ii) For n close to n_3^A ($n \to n_3^{A-}$), it follows from Lemma C.1 that pattern (1) has $s_1(n) = (S_{11}, S_{12}, s_{21}, S_{22})$ with s_{21} varying, while pattern (2) and (3) both have $s_2(n) = s_3(n) = (S_{11}, s_{12}, S_{21}, S_{22})$ with s_{12} varying. Therefore we have

$$\overline{s_1}'_-(n_3^A) = \left.\frac{\partial \overline{s}/\partial s_{21}}{\partial g(s)/\partial s_{21}}\right|_{s=S} = \left(\left.\frac{\partial g(s)}{\partial s_{21}}\right|_{s=S}\right)^{-1} > 0,$$
$$\overline{s_1}''_-(n_3^A) = \left.\frac{\partial \overline{s_1}'_-(n_3^A)/\partial s_{21}}{\partial g(s)/\partial s_{21}}\right|_{s=S} = -\left.\frac{\partial^2 g(s)/\partial s_{21}^2}{(\partial g(s)/\partial s_{21})^3}\right|_{s=S} < 0,$$

i.e., $\overline{s_1}(n)$ is strictly convex and increasing in n near n_3^A. And

$$\overline{s_2}'_-(n_3^A) = \overline{s_3}'_-(n_3^A) = \left.\frac{\partial \overline{s}/\partial s_{12}}{\partial g(s)/\partial s_{12}}\right|_{s=S} = \left(\left.\frac{\partial g(s)}{\partial s_{12}}\right|_{s=S}\right)^{-1},$$
$$\overline{s_2}''_-(n_3^A) = \overline{s_3}''_-(n_3^A) = \left.\frac{\partial \overline{s_2}'_-(n_3^A)/\partial s_{12}}{\partial g(s)/\partial s_{12}}\right|_{s=S} = -\left.\frac{\partial^2 g(s)/\partial s_{12}^2}{(\partial g(s)/\partial s_{12})^3}\right|_{s=S} = 0,$$

i.e., $\overline{s_2}(n)$ and $\overline{s_3}(n)$ both increase linearly in n near n_3^A.

(iii) The proof of Lemma C.2 establishes that any feasible pattern of transitions as n *increases* in zone (3) must be from pattern (1) to (2), from pattern (1) to (3), or from pattern (2) to (3)—not vice versa.

Using the above observations, the sufficient and necessary condition that patterns (2) and (3) provide the *largest* service capacity at some $n \in (n_2^A, n_3^A)$ is

$$\overline{s_2}'_-(n_3^A) = \overline{s_3}'_-(n_3^A) < \overline{s_1}'_-(n_3^A). \tag{C.59}$$

To see this, if (C.59) holds, observation (i) and (ii) immediately imply that $\overline{s_2}(n_3^{A-}) = \overline{s_3}(n_3^{A-}) > \overline{s_1}(n_3^{A-})$, hence patterns (2) and (3) provide the largest service capacity near n_3^A. On the other hand, if (C.59) does not hold, observation (i) and (ii) imply that $\overline{s_2}(n_3^{A-}) = \overline{s_3}(n_3^{A-}) < \overline{s_1}(n_3^{A-})$, i.e., pattern (1) is optimal near n_3^A. It then follows from observation (iii) that there is no transition from pattern (2) or (3) to pattern (1) as n increases in zone (3), hence pattern (1) is optimal throughout zone (3). Therefore (C.59) is necessary for patterns (2) and (3) to be optimal somewhere in zone (3). Putting in the derivative expressions from the proof of Lemma C.1 and some algebraic manipulation will transform (C.59) to inequality (3.46) in the Proposition. □

C.2.2 Driver Supply and Actual Gains in Platform Revenue and Per-Driver Profit

In this section we illustrate the impact of the driver supply characteristics, specifically, the outside opportunity cost distribution F, on the *actual* platform revenue and per-driver profit gains, compared to the upper bounds in Proposition 3.9 and 3.10, and on the tension between the drivers' and the platform's gains. For simplicity we focus on the gains from admission control, i.e., regime A over M. (Similar effects determine the actual gains from repositioning.)

Figure C.1 illustrates these gains for two opportunity cost distributions. Panel (a) presents a case where admission control yields large benefits for the platform as a result of a large increase in driver participation, and consequently only small benefits for individual drivers. Specifically, the top chart in panel (a) shows for the three control regimes the per-driver profits that are non-increasing functions of the capacity, and the increasing marginal opportunity cost function $F^{-1}(n/N)$. Achieving the upper bound on platform revenue gains from admission control requires two conditions, namely, $n_M^* = n_2^M$ or equivalently, $F^{-1}(n_2^M/N) = \pi_M(n_2^M)$, and $n_A^* = n_3^A$. The first condition holds in the example, the second condition requires infinitely elastic supply around the profit level $\pi_M(n_2^M)$, i.e., that F grows sufficiently fast around this point such that $n_3^A - n_2^M$ additional drivers join if the per-driver profit is slightly larger, so that $F^{-1}(n_3^A/N) = \pi_M(n_3^A)$. The example depicted

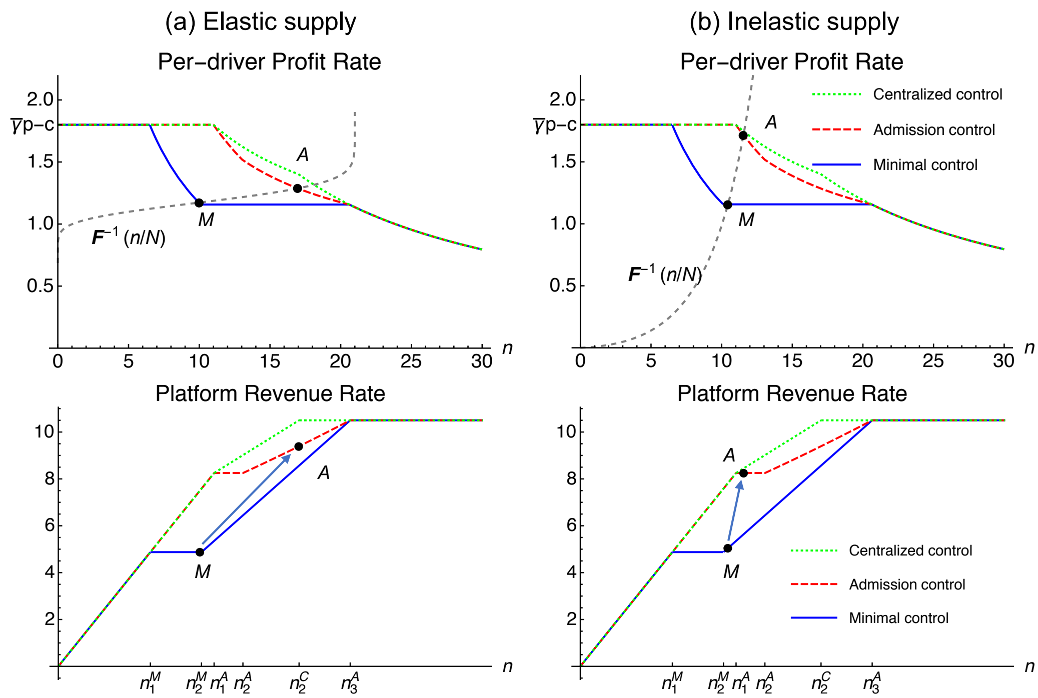

Figure C.1: Impact of admission control on the equilibrium capacity, per-driver profit, and platform revenue ($S = (3, 1, 4, 6), \overline{S} = 14, N = 21, t = 1, \gamma = 0.25, p = 3, c = 0.45$)

in Figure C.1 (a) shows how the upper bound can be approached if the supply increases substantially for a moderate change in per-driver profit rate.

Panel (b), in contrast, presents a case where admission control (under regime A or C) yields the maximum achievable per-driver profit gains as a result of a small increase in driver participation, and consequently only modest platform revenue gains. As shown in the top chart of panel (b), in this case the marginal opportunity cost function yields the same equilibrium capacity under minimal control as in panel (a), i.e., $F^{-1}(n_2^M/N) = \pi_M(n_2^M)$; however, the driver supply is so inelastic that the number of drivers willing to participate at the maximum profit rate ($\overline{\gamma}p - c$) is smaller than the minimum number required to serve all riders without repositioning, that is, $n_A^* \leq n_1^A$ where $F^{-1}(n_A^*/N) = \overline{\gamma}p - c$. The platform's commission is too high to entice more drivers to participate.

Appendix D

Appendix for Chapter 4

D.1 Proofs

Proof of Proposition 4.1. By the definition of $\delta_u(t)$ in (4.4), (4.1) can be rewritten as

$$x_t = \min\{u_t, \theta\} + \int_0^{\delta_u(t)} \mathbf{1}(u_{t-s} \geq \theta + s)ds. \tag{D.1}$$

(i) If $u_t < \theta$, by Assumption 4.1 we have $u_{t-s} \leq u_t + s < \theta + s, s \in [0, t]$, hence $x_t = u_t$ and $\delta_u(t) = 0$.

(ii) If $u_t \geq \theta$ and $t < u_0 - \theta$, by Assumption 4.1 we have $u_{t-s} \geq u_0 - (t-s) > \theta + s, s \in [0, t]$, hence $\delta_u(t) = t$ and $x_t = \theta + \int_0^t ds = \theta + t$.

(iii) If $u_t \geq \theta$ and $t \geq u_0 - \theta$, then since $u_{t-\delta_u(t)} \geq \theta + \delta_u(t)$ by the definition of $\delta_u(t)$ in (4.4), Assumption 4.1 implies $u_{t-s} \geq u_{t-\delta_u(t)} - (\delta_u(t) - s) \geq \theta + s, s \in [0, \delta_u(t)]$. Therefore $x_t = \theta + \int_0^{\delta_u(t)} ds = \theta + \delta_u(t)$.

These prove (4.2). Equation (4.3) follows from (4.2) noticing the value of $\delta_u(t)$ discussed above. □

Proof of Corollary 4.1. The first two cases in (4.5) is straightforward. Consider the third case. Since $u_t \geq \theta$ and $u_0 \leq \theta + t$, by the continuity of u there exists $s \in [0, t]$ such that $u_{t-s} = \theta + s$, hence $\delta_u(t) = \max\{0 \leq s \leq t : u_{t-s} = \theta + s\}$ and $u_{t-\delta_u(t)} = \theta + \delta_u(t)$. It follows that

$$x_t = \theta + \delta_u(t) = u_{t-\delta_u(t)}, \tag{D.2}$$

$$x_{t+dt} = \theta + \delta_u(t + dt) = u_{t+dt-\delta_u(t+dt)}. \tag{D.3}$$

Hence

$$\begin{aligned}
\dot{x}_t &= \lim_{dt\to 0} \frac{x_{t+dt} - x_t}{dt} \\
&= \lim_{dt\to 0} \frac{u_{t+dt-\delta_u(t+dt)} - u_{t-\delta_u(t)}}{dt} \\
&= \lim_{dt\to 0} \frac{u_{t+dt-\delta_u(t+dt)} - u_{t-\delta_u(t)}}{[t+dt-\delta_u(t+dt)] - [t-\delta_u(t)]} \frac{[t+dt-\delta_u(t+dt)] - [t-\delta_u(t)]}{dt} \\
&= \lim_{dt\to 0} \frac{u_{t-\delta_u(t)+[dt+\delta_u(t)-\delta_u(t+dt)]} - u_{t-\delta_u(t)}}{dt+\delta_u(t)-\delta_u(t+dt)} \left[1 - \frac{\delta_u(t+dt)-\delta_u(t)}{dt}\right] \\
&= \dot{u}_{t-\delta_u(t)} \left[1 - \lim_{dt\to 0} \frac{u_{t+dt-\delta_u(t+dt)} - u_{t-\delta_u(t)}}{dt}\right],
\end{aligned}$$

where the last equality uses (D.2) and (D.3). Noticing the same limit term in the second and last line, we can solve for $\dot{x}_t$ as

$$\dot{x}_t = \frac{\dot{u}_{t-\delta_u(t)}}{1+\dot{u}_{t-\delta_u(t)}} = 1 - \frac{1}{1+\dot{u}_{t-\delta_u(t)}}, \quad \text{if } \dot{u}_{t-\delta_u(t)} > -1.$$

If $\dot{u}_{t-\delta_u(t)} = -1$, $\dot{x}_t \to -\infty$ does not exist. There is a downward jump of x at t. □

Proof of Lemma 4.1. Given $t^x \geq (u_0 - \theta)^+$, (4.2) and (4.4) imply

$$x_{t^x} = \begin{cases} u_{t^x} & \text{if } 0 \leq u_{t^x} < \theta \\ \theta + \delta_{t^x} & \text{if } u_t \geq \theta \end{cases}. \tag{D.4}$$

Note that $\delta_{t^x} = 0$ and hence $t^u = t^x$ when $u_{t^x} < \theta$. Then $x_{t^x} = u_{t^u}$ follows directly from the definition of δ_t in (4.4). Obviously $dt^x = dt^u$ when $0 \leq u_t < \theta$.

Consider $u_{t^x} \geq \theta$. If $\dot{u}_{t^u} > -1$, $\dot{x}_{t^x}$ exists and $dt^x > 0$. It follows that $\dot{x}_{t^x} dt^x = \dot{u}_{t^u} dt^u$ and hence

$$\frac{dt^x}{dt^u} = \frac{\dot{u}_{t^u}}{\dot{x}_{t^x}} = \frac{\dot{u}_{t^u}}{1 - \frac{1}{1+\dot{u}_{t^u}}} = 1 + \dot{u}_{t^u} \equiv \frac{1}{1-\dot{x}_{t^x}}, \tag{D.5}$$

where the second and last equalities use the third case in (4.5). If $\dot{u}_{t^u} = -1$, x_t is discontinuous and jumps downward at t^x, $\dot{x}_{t^x}$ does not exist, and $dt^x = 0$. □

Proof of Lemma 4.2. First, conditioning on whether $u_t < \theta$ or if not, whether $\dot{u}_t > -1$ or $\dot{u}_t = -1$, partition the interval $[a, b]$ into consecutive intervals $[c_1, c_2], [c_2, c_3], \ldots, [c_{n-1}, c_n]$, where $c_1 = a$ and $c_n = b$. Let $I^A = \{i : 0 \leq u_t < \theta \text{ for } t \in [c_i, c_{i+1}]\}$ $I^B = \{i : u_t \geq \theta \text{ and } \dot{u}_t > -1 \text{ for } t \in [c_i, c_{i+1}]\}$ and $I^C = \{i : u_t \geq \theta \text{ and } \dot{u}_t = -1 \text{ for } t \in [c_i, c_{i+1}]\}$, then

$$[a, b] = \bigcup_{i \in I^A} [c_i, c_{i+1}] \cup \bigcup_{i \in I^B} [c_i, c_{i+1}] \cup \bigcup_{i \in I^C} [c_i, c_{i+1}] \tag{D.6}$$

and

$$\int_a^b f(u_t)dt = \sum_{i\in I^A}\int_{c_i}^{c_{i+1}} f(u_t)dt + \sum_{i\in I^B}\int_{c_i}^{c_{i+1}} f(u_t)dt + \sum_{i\in I^C}\int_{c_i}^{c_{i+1}} f(u_t)dt. \tag{D.7}$$

Next, calculate the integrals in the three rhs terms in (D.7). Note that it follows from the definition of t^u in (4.6) and the definition of δ_t in (4.4) that given any $t^u \geq 0$, $x_{t^x} = u_{t^u}$ for $t^x = t^u + (u_{t^u} - \theta)^+$.

(1) For the integrals in the first rhs term in (D.7), $0 \leq u_t < \theta$ implies $\frac{dt^x}{dt^u} = 1$ by Lemma 4.1, and $t^x = t^u$. Hence

$$\int_{c_i}^{c_{i+1}} f(u_{t^u})dt^u = \int_{c_i}^{c_{i+1}} f(x_{t^x})dt^x = \int_{c_i+(u_{c_i}-\theta)^+}^{c_{i+1}+(u_{c_{i+1}}-\theta)^+} f(x_t)dt - \int_{u_{c_i}\vee\theta}^{u_{c_{i+1}}\vee\theta} f(x)dx, \quad i \in I^A,$$

where the second equality follows from $u_{c_i} < \theta$, $\forall i \in I^A$.

(2) For the integrals in the second rhs term in (D.7), $u_t \geq \theta$ and $\dot{u}_t > -1$ imply (4.7) by Lemma 4.1. Using integration by substitution, we have for $i \in I^B$,

$$\begin{aligned}
\int_{c_i}^{c_{i+1}} f(u_{t^u})dt^u &= \int_{c_i+(u_{c_i}-\theta)^+}^{c_{i+1}+(u_{c_{i+1}}-\theta)^+} f(x_{t^x})\frac{dt^u}{dt^x}dt^x \\
&= \int_{c_i+(u_{c_i}-\theta)^+}^{c_{i+1}+(u_{c_{i+1}}-\theta)^+} f(x_t)(1-\dot{x}_t)dt \\
&= \int_{c_i+(u_{c_i}-\theta)^+}^{c_{i+1}+(u_{c_{i+1}}-\theta)^+} f(x_t)dt - \int_{c_i+(u_{c_i}-\theta)^+}^{c_{i+1}+(u_{c_{i+1}}-\theta)^+} f(x_t)dx_t \\
&= \int_{c_i+(u_{c_i}-\theta)^+}^{c_{i+1}+(u_{c_{i+1}}-\theta)^+} f(x_t)dt - \int_{x_{c_i+(u_{c_i}-\theta)^+}}^{x_{c_{i+1}+(u_{c_{i+1}}-\theta)^+}} f(x)dx \\
&= \int_{c_i+(u_{c_i}-\theta)^+}^{c_{i+1}+(u_{c_{i+1}}-\theta)^+} f(x_t)dt - \int_{u_{c_i}\vee\theta}^{u_{c_{i+1}}\vee\theta} f(x)dx,
\end{aligned}$$

where the last equality uses $u_{c_i} \geq \theta$, $\forall i \in I^B$.

(3) For the integrals in the third rhs term in (D.7), $u_t \geq \theta$ and $\dot{u}_t = -1$ imply discontinuous x_t at t^x and $dt^x = 0$ by Lemma 4.1, and $dt^u = -du_t$. Hence for $i \in I^C$,

$$\begin{aligned}
\int_{c_i}^{c_{i+1}} f(u_t)dt &= -\int_{c_i}^{c_{i+1}} f(u_t)du_t \\
&= -\int_{u_{c_i}}^{u_{c_{i+1}}} f(u)du \\
&= \int_{c_i+(u_{c_i}-\theta)^+}^{c_{i+1}+(u_{c_{i+1}}-\theta)^+} f(x_t)dt - \int_{u_{c_i}\vee\theta}^{u_{c_{i+1}}\vee\theta} f(x)dx,
\end{aligned}$$

where the last equality uses $c_i + (u_{c_i} - \theta)^+ = c_{i+1} + (u_{c_{i+1}} - \theta)^+$ and $u_{c_i} \geq \theta$, $\forall i \in I^C$.

Last, since all integrals calculated above take the same form, the decomposed integration (D.7) yields the desired (4.8). □

Proof of Proposition 4.2. Condition on the choice of u_0, u_t and x_t can be written out immediately and hence simplify the problem. If $u_0 \geq \theta$, then $u_t = \min\{u_0, \theta + T - t\}$ and $x_t = \min\{\theta + t, u_0\}$ for $t \in [0, T]$. Note that u_t and x_t take symmetric values for $t \in [0, T]$, i.e., $u_t = x_{T-t}$, thus the problem can be simplified as

$$\max_{u_0 \geq \theta} \phi \left\{ G(u_0)u_0[T - (u_0 - \theta)] + \int_\theta^{u_0} \left((p(u) - w(u_0)) - (p_0 - w_0) \right) u du \right\}. \tag{D.8}$$

If $u_0 < \theta$, then $u_t \equiv x_t \equiv u_0$ for $t \in [0, T]$. The problem becomes

$$\max_{0 \leq u_0 < \theta} \phi G(u_0) u_0 T. \tag{D.9}$$

Therefore the optimal u_0^* maximizes (D.8) and (D.9), which is given by (4.22). □

Proof of Proposition 4.3. By Lemma 4.2 we have

$$\begin{aligned}
\int_0^T f(u_t)dt &= \int_{(u_0-\theta)^+}^{T+(u_T-\theta)^+} f(x_t)dt - \int_{u_0 \vee \theta}^{u_T \vee \theta} f(x)dx \\
&= \int_{(u_0-\theta)^+}^{T} f(x_t)dt - \int_{u_0 \vee \theta}^{\theta} f(x)dx \\
&= \int_0^T f(x_t)dt - \int_0^{(u_0-\theta)^+} f(x_t)dt - \int_{u_0 \vee \theta}^{\theta} f(x)dx \\
&= \int_0^T f(x_t)dt - \int_\theta^{u_0 \vee \theta} f(x)dx - \int_{u_0 \vee \theta}^{\theta} f(x)dx \\
&= \int_0^T f(x_t)dt,
\end{aligned}$$

where the second equality follows from $u_T \leq \theta$ and the fourth equality can be obtained by conditioning on $u_0 < \theta$ or $u_0 \geq \theta$ and noticing that $x_t = \theta + t$ for $t \in [0, u_0 - \theta]$ if $u_0 \geq \theta$. Applying $f(u_t) = w(u_t)u_t$ and $f(u_t) = u_t$, respectively, the objective function can be transformed as

$$\begin{aligned}
\phi \int_0^T \{[p(x_t)x_t - w(u_t)u_t] - (p_0 - w_0)u_t\} dt &= \phi \int_0^T \{[p(x_t)x_t - w(x_t)x_t] - (p_0 - w_0)x_t\} dt \\
&= \phi \int_0^T G(x_t)x_t dt.
\end{aligned} \tag{D.10}$$

The optimization problem hence becomes maximizing over x_t:

$$\max_{x_t: t\in[0,T]} \quad \phi \int_0^T G(x_t) x_t dt$$
$$\text{s.t.} \quad 0 \le x_t \le \theta + t, \quad t \in [0, T].$$

Since the problem is equivalent to $\phi \int_0^T \max_{0\le x_t \le \theta+t} G(x_t)x_t dt$, the optimal solution is given by

$$x_t^* = \arg \max_{0\le x\le\theta+t} G(x)x, \quad t \in [0, T]. \tag{D.11}$$

We can hence derive u_t^* from (4.3) as in (4.27). □

Proof of Proposition 4.4. Similar to the proof of Proposition 4.3, we have $\int_0^T u_t dt = \int_0^T x_t dt$ and the optimization problem becomes maximizing over x_t:

$$\max_{x_t: t\in[0,T]} \quad \phi \int_0^T \left\{ \left[p(x_t)x_t - \int_0^{x_t} w(x)dx \right] - (p_0 - w_0)x_t \right\} dt$$
$$\text{s.t.} \quad 0 \le x_t \le \theta + t, \quad t \in [0, T].$$

Since the problem is equivalent to

$$\phi \int_0^T \max_{0\le x_t\le\theta+t} G^p(x_t)x_t dt, \tag{D.12}$$

the optimal solution is given by

$$x_t^* = \arg \max_{0\le x\le\theta+t} G^p(x)x, \quad t \in [0, T]. \tag{D.13}$$

We can hence derive u_t^* from (4.3) as in (4.31). □

Proof of Proposition 4.5. The first derivative of the profit gain wrt x' is

$$\Pi'(x') = e^{-\frac{t}{T}} \left(\frac{1}{v} + \frac{1}{u} \right) e^{-\frac{x'-x^*}{Tv}} \left\{ - \left[(R(x^*) - L(x^*)) - (R(x') - L(x')) \right] \right.$$
$$\left. + \int_{x^*}^{x'} \left[\frac{1}{u} \left[(R(x^*) - L(x^*)) - (R(x) - L(x)) \right] + L(x) \right] \frac{1}{\bar{T}} e^{-\frac{x'-x}{Tu}} dx \right\}, \tag{D.14}$$

and the second derivative is

$$\Pi''(x') = e^{-\frac{t}{T}} \left(\frac{1}{v} + \frac{1}{u} \right) e^{-\frac{x'-x^*}{Tv}} \left\{ \frac{1}{\bar{T}} \left(\frac{1}{v} + \frac{1}{u} \right) \left[(R(x^*) - L(x^*)) - (R(x') - L(x')) \right. \right.$$
$$\left. - \int_{x^*}^{x'} \left[\frac{1}{u} \left[(R(x^*) - L(x^*)) - (R(x) - L(x)) \right] + L(x) \right] \frac{1}{\bar{T}} e^{-\frac{x'-x}{Tu}} dx \right]$$
$$\left. + (R'(x') - L'(x')) + \frac{1}{\bar{T}} L(x') \right\}. \tag{D.15}$$

Note that for $x' \to x^{*+}$,

$$\Pi'(x^*) = 0, \tag{D.16}$$

$$\Pi''(x^*) = e^{-\frac{t}{\bar{T}}} \left(\frac{1}{v} + \frac{1}{u}\right) \left\{(R'(x^*) - L'(x^*)) + \frac{1}{\bar{T}} L(x^*)\right\} \tag{D.17}$$

$$= e^{-\frac{t}{\bar{T}}} \left(\frac{1}{v} + \frac{1}{u}\right) \frac{1}{\bar{T}} L(x^*), \tag{D.18}$$

where the second equality follows from the definition of x^*. Therefore for finite expected shock duration $\bar{T} < \infty$, $\Pi'(x^*) = 0$ and $\Pi''(x^*) > 0$ imply the existence of $x' > x^*$ such that $\Pi(x') > 0$. When $\bar{T} = \infty$, both first and second derivatives are 0 but Lemma 4.4 shows $\Pi(x') < 0$ for $x' > x^*$. □

Proof of Proposition 4.6. Under the same control $u_t, t \geq 0$, the change in platform's expected profit from restricted surge duration (4.35) to guaranteed wage (4.45) is given by

$$\Delta(u_t) = \phi \int_0^\infty \left[\int_0^{x_t} w(x)dx - \int_0^{u_t} \tilde{w}(x)dx\right] e^{-\frac{t}{\bar{T}}} dt.$$

For static personalized message wage policy where $u_t \equiv u_0 \geq \theta, t \geq 0$,

$$\begin{aligned}
\Delta(u_0) &= \phi \int_0^\infty \left[\int_\theta^{x_t} w(x)dx - \int_\theta^{u_0} \tilde{w}(x)dx\right] e^{-\frac{t}{\bar{T}}} dt \\
&= \phi \Bigg\{ \int_0^{u_0-\theta} \left[\int_\theta^{\theta+t} w(x)dx - \int_\theta^{u_0} \tilde{w}(x)dx\right] e^{-\frac{t}{\bar{T}}} dt \\
&\qquad + \int_{u_0-\theta}^\infty \left[\int_\theta^{u_0} (w(x) - \tilde{w}(x))dx\right] e^{-\frac{t}{\bar{T}}} dt \Bigg\},
\end{aligned} \tag{D.19}$$

where the first equality follows from Lemma 4.5. Notice that the bracket part in the first term in (D.19) is negative at $t = 0$ and positive at $t = u_0 - \theta$, and the second term in (D.19) is positive. It hence follows that $\Delta(u_0) > 0$ when $\bar{T} \to \infty$ and $\Delta(u_0) \leq 0$ when $\bar{T} \to 0$. We thus have the conclusion. □

Proof of Lemma 4.6. (Sketch) Let $W(\tau_0) = \phi \mathbb{E}_{T \sim \exp(1/\bar{T})}$ Eq.(4.47) denote the platform's expected wage payment to repositioning drivers. We can show $W'(0) < 0$, $W'(\infty) > 0$ and $W''(\tau_0) > 0$ (DETAILS), hence there exits a unique minimizer τ_0^* at which $W'(\tau_0^*) = 0$. □