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ON THE AXIOMATIC SYSTEMS OF SYNTACTICALLY-CATEGORIAL LANGUAGES

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1. Introduction

In the monograph four axiomatic systems of syntactically-categorial languages are presented. The first two refer to languages of expression-tokens. The others also takes into consideration languages of expression-types.

Generally, syntactically-categorial languages are languages built in accordance with principles of the theory of syntactic categories introduced by S. Leśniewski [4]; they are connected with- the Ajdukiewicz's concept [1] which was a continuation of Leśniewski's idea and further developed and popularized in the research on categorial grammars, by Y. Bar-Hillel [2], [3].

To assign a suitable syntactic category to each word of the vocabulary is the main idea of syntactically-categorial approach to language. Compound expressions are built from the words of the vocabulary and then a suitable syntactic-category is assigned to each of them. A language built in this way should be decidable, which means that there should exist an algorithm for deciding about each expression of it, whether it is sensible, i.e., well-formed or is syntactically connected (in the sense of Ajdukiewicz [1]).

The traditional, originating from Husserl, understanding of the syntactic category confronts some difficulties. This notion is defined by abstraction using the concept of affiliation of two expressions to the sane syntactic category (see Ajdukiewicz [1]).

If we use the following expression:

(α) $r(p/q)s$,

which we read: the expression r rises from s by the replacement of its clause q by p; and the expressions:

(β) p and q are expressions of the same syntactic category;
(γ') r and s are expressions of the syntactic category of sentences;
(γ'') r and s are well-formed expressions;

then the schemas usually found definitions of this notion are expressions:

(I') $\alpha \Rightarrow (\beta \Leftrightarrow \gamma')$ and (J'') $\alpha \Rightarrow (\beta \Leftrightarrow \gamma'')$.

Defining the notion of syntactic category we refer to the concept of either a sentence or well-formed expression. The assumption, in theoretical considerations of a language, that these notions are primitive concepts seems to be groundless. An attempt of defining a concept of sentence in such a way so as to construct an algorithm of testing whether an expression is well-formed - analogous to that of Ajdukiewicz's [1], with simultaneous keeping of the definition on the scheme (I') can lead to the vicious circle. On the other hand, accepting the definition on the scheme (I'') we should agree for instance that the functors appearing in the well-formed expressions 1^o and 2^o (sentences and names) such as:

$$1^o \begin{cases} \text{John loves Helen} \\ \text{John and Helen} \end{cases} ; \qquad 2^o \begin{cases} 2 = 2 \\ 2 + 2 \end{cases}$$

ought to classified as belonging to the same syntactic category, although the first is a sentence-forming functor and the second is the name-forming functor.

One of the main aims of the paper is to remove the difficulties mentioned, above. So called "the fundamental theorem of the theory of syntactic categories" (briefly: fttsc) can be proved in each of the four presented theories of syntactically-categorial languages. Applying in its notation the expression:

(γ) r and s are expressions of the same syntactic category) its scheme has the form:

(I) $\alpha \Rightarrow (\beta \Leftrightarrow \gamma)$.

2. The theories $TLTk$ and $TETk$

The foundation of all four presented systems of syntactically-categorial languages is $TLTk$ i.e., the theory of label-tokens and its extension, $TETk$ i.e., the theory of expression-tokens. $TLTk$ is based on the classical first-order functional calculus with identity and on the set theory. Its primitive concepts are: the *set Lb of all label-tokens*, *binary equiformity relation* $\approx$ in Lb, *ternary concatenation relation* ι in this set, and the *vocabulary* V. On the ground of this theory, the notion of the *set* W *of all word-tokens* is defined. It is the smallest set containing the vocabulary V and closed under concatenation relation ι. $TLTk$ is equivalent to the theory presented in the paper [6]. It describes the properties of any label-tokens (which are visually perceptible objects), regardless of how they are constructed or what symbolism is used for their notation. The axioms characterizing properties of the relation $\approx$ and ι are the same as those of [6]. Together with them, there are following axioms of $TLTk$ which are characterizing properties of sets Lb, V and W:

A1. $\emptyset \neq V \subseteq Lb$;

A2. $p \in V \wedge q \approx p \Rightarrow q \in V$;

A3. $\iota(p, q, r) \Rightarrow r \notin V$;

A4. $r \in W \backslash V \Rightarrow \bigvee_{p,q \in W} \iota(p, q, r)$;

A5. $r \in W \wedge \iota(p, q, r) \Rightarrow p, q \in W$.

Writing the axioms A1 – A5 we make an agreement that variables: $p, q, r, s, t, u, v, ...$ with subscripts or without them are representing any label-tokens.

In $TLTk$ we define, in a natural way, by induction, a *generalized* $n+1$-*argument concatenation relation* c^n $(n \geq 2)$. We read the expression: $\mathsf{c}^n(p_1, ..., p_n, p_0)$, in the following ways p_0 is n-clause concatenation of labels $p_1, \ldots, p_n$.

In syntactical analysis of categorial languages we use *categorial indices* introduced to semiotics by Ajdukiewicz [1]. By means of them we define two basic notions of $TETk$: the concept of *language-expression* and *syntactic category*. The *set* I *of all categorial indices* is defined in $TEEk$ by the *set* I_0 *of all basic indices*, which is a primitive concept of this theory. The set I is the smallest set containing I_0 and closed under the relation c. We postulate for sets I_0 and I that they satisfy expressions resulting from axioms A1 – A5 by replacement of the, symbols "V" and "W" by the symbols "I_0" and "I", respectively. So, categorial indices are label-tokens. They are not words of the language, because we postulate the axiom:

A6. $V \cap I_0 = \emptyset$;

from which it follows:

THEOREM. $W \cap I = \emptyset$.

Categorial indices are "attached" to words of the language by the *relation ι of assigning of indices to words*, which is a new primitive notion of $TETk$ characterized by axioms:

A7. $\iota \subseteq W \times I \wedge \iota$ is a function;

A8. $p \in D(\iota) \wedge q \approx p \Rightarrow q \in D(\iota) \wedge \iota(q) \approx \iota(p)$.

The third and the last primitive concept of $TETk$ is a one-to-one *function ϱ of building compound expressions.* Its left domain is the union of all finite, greater than one, Cartesian powers of the domain $D(\iota)$ of the function ι (i.e. the of all words having categorial indices). The right domain of ϱ is a subset of the set $D(\iota)\backslash V$. Namely, the following expression is an axiom of $TETk$:

A9. $\varrho : \bigcup_{k=2}^{\infty} D(\iota)^k \rightarrow_{1-1} D(\iota)\backslash V$;

Another axiom of $TETk$ is:

A10. $p = \varrho(p_0, p_1, \ldots, p_n) \Rightarrow [q \approx p \Leftrightarrow \bigvee_{q_0, q_1, \ldots, q_n} (q = \varrho(q_0, q_1, \ldots, q_n) \wedge \wedge \bigwedge_{0 \leq k \leq n} q_k \approx p_k)]$.

We read the expression being the antecedent of the implication A10, in the following-way; p is a compound expression-token built from n+1 word -tokens: the main functor p_0 and its successive arguments $p_1, \ldots, p_n$.

The expression $\varrho(p_0, p_1, \ldots, p_n)$ can be treated as a implication into the language of $TETk$, of such a compound language-expression which is composed from the functor p_0 and its successive arguments $p_1, \ldots, p_n$. The word "translation" here has the meaning which is in agreement with that introduced by A. Tarski in [5]. A translation of any compound expression-tokens does not depend on the symbolism in which it was written; in particular, it does not depend on whether the notation with or without brackets was used. It depends, however, on the words it was composed from and on relations between them.

In the theory $TETk$ we define: the *set E_s of all simple expressions-tokens* (as the set $V \cap D(\iota)$), the *set E_0 of all compound expression-tokens* (as a counter-domain of the function ϱ), the *set E of all expression-tokens* (as the union of sets E_s and E_c). To the notional apparatus of this theory we also introduce the defined notion of the *clause of a given expression-token*.

In $TETk$, the following two definitions are assumed:

DEFINITION. $C_\xi = \{p \in E \mid \iota(p) \approx \xi\}, \ \xi \in I.$

DEFINITION. $p =_C q \Leftrightarrow \bigvee_{\xi \in I} p, q \in C_\xi.$

The first one defines a *syntactic category of an index* ξ as the set of all these expression-tokens whose index is equiform to ξ. The second definition defines the *categorial conformity relation*. The expression: $p =_C q$,will be used in the formulation of $fttsc$. We read it: expressions p and q belong to the same syntactic category.

COROLLARY. *The relation $=_C$ is an equivalence relation in the set E.*

3. Systems $TSCL$ and $TSC\omega - L$

The system $TSCL$ concerns simple languages of expression-tokens. In expressions of these languages there are no operators and variables bound by them. The system $TSC\omega - L$ is a modification of the theory $TSCL$ and concerns so-called ω-languages, i.e., languages of expression-tokens in which variable-bounding operators can occur. The main concept defined in these theories is a notion of sensible, i.e well-formed, expression-token of a language (briefly: wfe). In both theories the set S of all wfe's is defined as follows:

$(*)\ S = \bigcup_{n=0}^{\infty} {}^nS,$

where ${}^nS$ is the *set of all wfe's of rank n* ($n \geq 0$) defined in the theory $TSCL$ and $TSC\omega - L$ separately.
Namely, in $TSCL$ we have:

DEFINITION. a) ${}^0S = E_S$, b) $p \in {}^{k+1}S \Leftrightarrow p \in {}^kS \vee \bigvee_{n \geq 1} \bigvee_{p_0, p_1, \ldots, p_n \in {}^kS} [p = \varrho(p_0, p_1, \ldots, p_n) \wedge \iota^{n+1}(\iota(p), \iota(p_1), \ldots, \iota(p_n), \iota(p_0))].$

A wfe of rank 0 of a simple language is a simple expression-token of this language. A wfe of rank $k+1$ of such language is either wfe of rank k or a compound expression-token of this language which is composed of wfe's of rank k of this language such that the index of the main functor of the expression p is a concatenation of the index of this expression and indices of successive arguments of this functor.

Let us note that in the factorial notation of Ajdukiewicz, the index of the main functor of the expression p is of the following form:

$$\frac{\iota(p)}{\iota(p_1)\iota(p_2)\ldots\iota(p_n)}.$$

In the $TSCL$ system we assume a new axiom, which is a warrant of nonemptiness of the set S:

A11. $\iota(S\backslash{}^0S)\cap I_0 \neq \emptyset$.

In the definition of the set ${}^nS$ in the theory $TSC\omega - L$ appear its primitive terms "0" and "Vr" designing the *set of all operators* and the *set of all variables*, respectively.
The sets satisfy the axioms:

A11$^\omega$. $0 \cup Vr \subseteq E_S$, A12$^\omega$. $p \in 0 \wedge q \approx p \Rightarrow q \in 0$;
A13$^\omega$. $p \in Vr \wedge q \approx p \Rightarrow q \in Vr$.

Part b) of the definition of the set ${}^nS$ in the system $TSC\omega - L$ is obtained from the part b) of the definition above by adding to the second component of the alternative, the following clause of conjunction: $p_0 \notin 0$, and by adding the following third component alternative:

$$\bigvee_{p_0,p_1,p_2,p_3} [p = \varrho(p_0,p_1,p_2) \wedge p_0 \in 0 \wedge p_1 \in Vr \wedge p_2 \in {}^kS \wedge p_3(fv)p_2 \wedge$$
$$\wedge p_3 \approx p_1 \wedge \mathsf{c}^3(\iota(p),\iota(p_1),\iota(p_2),\iota(p_0))],$$

where "(fv)" is a defined term denoting the *relation of being a free variable in an expression-token.*

An expression which satisfies the second component of the new alternative is called a *compound non-operator* wfe *of rank* k of ω-language. An expression satisfying the third component of the alternative is called an *operator* wfe *of rank* k of this language. All the expressions of a finite rank of the first kind constitute the set $S^{n\omega}$, those of the second kind – the set

S^ω. Sets $S^{n\omega}$ and S^ω are disjoint and non-empty. We postulate that they satisfy axioms analogous to A11.

REMARK. The definition of the set S, assumed in the theory $TSCL$ and $TSC\omega - L$ gives the possibility to formulate an algorithm syntactic connection (sensibility of expressions) which is analogous to that of given by Ajdukiewicz [1].

In both theories it ie also possible to introduce the following definitions of the *set B of all basic expression-tokens* and the *set F of all functor-tokens*:

(∗∗) a) $B = \{p \in S \mid \iota(p) \in I_0\}$;
b) $F = \{p \in S \mid \iota(p) \in I \backslash I_0\}$.

THEOREM. *The family of classes of abstraction of the relation $=_C$ in the set S is a family of non-empty and disjoint syntactic categories, whose union is equal to the set S. The set S is the union of two non-empty and disjoint sets B and F.*

To formulate $fttsc$ in the theories $TSCL$ and $TSC\omega - L$ we introduce new notion of a four-argument relation $(/)$ of *replacement of a clause of a wfe* and an auxiliary concept of relation $(/)^n$ of *replacement of a clause of rank n of a given wfe*. The definition of relation $(/)^n$ accepted in the theory $TSC\omega - L$ is a modification of the definition of $TSCL$. Both definitions are intuitive but rather complicated. So we omit them here, and we refer the reader to [7].

The definition of relation $(/)$ in $TSCL$ and $TSC\omega - L$ is he formula:

$$(***)\ r(p/q)s \Leftrightarrow \bigvee_n r(p/q)^n s.$$

The symbolic notation of $fttsc$ /see schema (I)/ is:

THEOREM. $r(p/q)s \Rightarrow (p =_C q \Leftrightarrow r =_C s)$.

According to it, two expressions of a simple language (ω-language) belong to the same syntactic category if and only if replacing one of them by the other in a wfe of this language (ω-language) we obtain a wfe, which belongs to the same syntactic category.

4. Systems $DTSCL$ and $DTSC\omega - L$

Systems $DTSCL$ and $DTSC\omega - L$ are called the dualistic theories of syntactically-categorial languages, because they permit to treat languages from two of points of view, which has a connection with a double ontological character of language objects. On the one hand they concern the languages expression-tokens and the other hand, the languages of expression-types. It is because the theory $DTSCL$ ($DTSC\omega - L$) is a definitional extension of the theory $TSCL$ ($TSC\omega - L$), and the definitions added are the definitions of:
1^o sets of abstract language objects, as for example: the set of all label-types, the vocabulary of word-types, the set of all word-types, the set of basic abstract indices, the set of all abstract indices, the set of all expression-types, the set of all well-formed expression-types;
2^o relations $\bar{\mathsf{c}}, \bar{\mathsf{c}}^n, \bar{\ell}, \bar{\varrho}, (\overline{fv})$ corresponding to relations $\mathsf{c}, \mathsf{c}^n, \ell, \varrho, (fv)$, respectively, and holding between label-types or word-types.

The sets listed above are defined as quotient sets of suitable sets: Lb, V, W, I_0, I, E, S by the equiformity relation. The elements of these sets are the appropriate classes of abstraction of equiform label-tokens, i.e., label-types. Relations: $\bar{\mathsf{c}}, \bar{\mathsf{c}}^n, \bar{\ell}, \bar{\varrho}, (\overline{fv})$ hold between classes of abstraction of the relation $\approx$ if and only if the relations: $\iota, \iota^n, \ell, \varrho, (fv)$ hold between representants of the suitable classes.

There is a complete analogy between syntactical notions of languages of expression-tokens and languages of expression-types. The theorems or definitions of the theories $DTSCL$ and $DTSC\omega - L$ are all counterparts of axioms, theorems or definitions of the theories $TSCL$ and $TSC\omega - L$, respectively. This fact has a philosophical meaning because in syntactical considerations on language it is penciled to avoid the assumption about existing of ideal objects.

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