version v1.

Colombeau solutions to Einstein field equations. Gravitational singularities.

Jaykov Foukzon

jaykovfoukzon@list.ru

Center for Mathematical Sciences, Israel Institute of Technology,Haifa,Israel

I.Introduction

.

1.1. Colombeau algebra of Colombeau generalized functions.

In contemporary mathematics, a Colombeau algebra of Colombeau generalized functions is an algebra of a certain kind containing the space of Schwartz distributions. While in classical distribution theory a general multiplication of distributions is not possible, Colombeau algebras provide a rigorous framework for this.

Remark 1.1.1. Such a multiplication of distributions has been a long time mistakenly believed to be impossible because of Schwartz' impossibility result, which basically states that there cannot be a differential algebra containing the space of distributions and preserving the product of continuous functions. However, if one only wants to preserve the product of smooth functions instead such a construction becomes possible, as demonstrated first by J.F.Colombeau [1],[2].

As a mathematical tool, Colombeau algebras can be said to combine a treatment of singularities, differentiation and nonlinear operations in one framework, lifting the limitations of distribution theory. These algebras have found numerous applications in the fields of partial differential equations, geophysics, microlocal analysis and general relativity so far.

Basic idea.

Definition 1.1.1. The algebra moderate functions $C_M^\infty(\mathbb{R}^n)$ on $\mathbb{R}^n$ is the algebra of families of smooth functions $(f_\varepsilon(x))_\varepsilon \triangleq (f_\varepsilon(x))_{\varepsilon\in(0,1]}, x \in \mathbb{R}^n, \varepsilon \in (0,1]$ (smooth regularisations) (where ε is the regularization parameter), such that: (i) for all compact subsets K of $\mathbb{R}^n$ and all multiindices α, there is an $N > 0$ such that

$$\sup_{x\in K}\left|\frac{\partial^{|\alpha|}f_\varepsilon(x)}{(\partial x_1)^{\alpha_1}\cdots(\partial x_n)^{\alpha_n}}\right| = O(\varepsilon^{-N}), \varepsilon \to 0, \tag{1.1.1}$$

with addition and multiplication defined by natural way:

$$(f_\varepsilon(x))_\varepsilon + (g_\varepsilon(x))_\varepsilon = (f_\varepsilon(x) + g_\varepsilon(x))_\varepsilon \tag{1.1.2}$$

and

$$(f_\varepsilon(x))_\varepsilon \times (g_\varepsilon(x))_\varepsilon = (f_\varepsilon(x) \times g_\varepsilon(x))_\varepsilon. \quad (1.1.3)$$

Definition 1.1.2.The ideal $\mathcal{N}(\mathbb{R}^n)$ of negligible functions is defined in the same way but with the partial derivatives instead bounded by $O(\varepsilon^N)$ for all $N > 0$, i.e.

$$\sup_{x\in K}\left|\frac{\partial^{|\alpha|}f_\varepsilon(x)}{(\partial x_1)^{\alpha_1}\bullet\bullet\bullet(\partial x_n)^{\alpha_n}}\right| = O(\varepsilon^N), \varepsilon \to 0. \quad (1.1.4)$$

Definition 1.1.3.The Colombeau Algebra $\mathcal{G}(\mathbb{R}^n)$ [1],[2] is defined as the quotient algebra

$$\mathcal{G}(\mathbb{R}^n) = C_M^\infty(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n). \quad (1.1.5)$$

Elements of $\mathcal{G}(\mathbb{R}^n)$ are denoted by:

$$u = \mathbf{cl}[(u_\varepsilon)_\varepsilon] \triangleq (u_\varepsilon)_\varepsilon + \mathcal{N}(\mathbb{R}^n). \quad (1.1.6)$$

Embedding of distributions

The space of Schwartz distributions $\mathcal{D}'(\mathbb{R}^n)$ can be embedded into the Colombeau algebra $G(\mathbb{R}^n)$ by (component-wise) convolution with any element $(\varphi_\varepsilon)_\varepsilon$ of the algebra $\mathcal{G}(\mathbb{R}^n)$ having as representative a δ-net, i.e. a family of smooth functions φ_ε such that $\varphi_\varepsilon \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$.

Remark 1.1.2.Note that the embedding $\iota : \mathcal{D}'(\mathbb{R}^n) \hookrightarrow \mathcal{G}(\mathbb{R}^n)$ is non-canonical, because it

depends on the choice of the δ-net.

Example 1.1.1. Delta function $\delta(x) \in \mathcal{D}'(\mathbb{R})$ for example has the following different representatives in Colombeau algebra $\mathcal{G}(\mathbb{R})$:

$$\begin{gathered}\frac{1}{2}\left(\frac{1}{\sqrt{\pi\varepsilon}}\exp\left(-\frac{x^2}{4\varepsilon}\right)\right)_\varepsilon \in \mathcal{G}(\mathbb{R}), \frac{1}{\pi}\left(\frac{1}{x}\sin\left(\frac{x}{\varepsilon}\right)\right)_\varepsilon \in \mathcal{G}(\mathbb{R}),\\ \frac{1}{\pi}\left(\frac{\varepsilon}{x^2+\varepsilon^2}\right)_\varepsilon \in \mathcal{G}(\mathbb{R}), \frac{1}{\pi}\left(\frac{1}{x^2}\sin^2\left(\frac{x}{\varepsilon}\right)\right)_\varepsilon \in \mathcal{G}(\mathbb{R}),\end{gathered} \quad (1.1.7)$$

since

$$\begin{gathered}\frac{1}{2}\frac{1}{\sqrt{\pi\varepsilon}}\exp\left(-\frac{x^2}{4\varepsilon}\right) \to \delta(x), \frac{1}{\pi}\frac{1}{x}\sin\left(\frac{x}{\varepsilon}\right) \to \delta(x),\\ \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2} \to \delta(x), \frac{1}{\pi}\frac{1}{x^2}\sin^2\left(\frac{x}{\varepsilon}\right) \to \delta(x)\end{gathered} \quad (1.1.8)$$

in $\mathcal{D}'$ as $\varepsilon \to 0$.

Remark 1.1.2.However note that embedding $\mathcal{D}'(\mathbb{R}^n) \hookrightarrow \mathcal{G}(\mathbb{R}^n)$ does not meant the equivalence of the Schwartz distributions and corresponding by embedding Colombeau

generalized functions. In contrast with the Schwartz distributions Colombeau generalized

functions has well defined walue at any point $x \in \mathbb{R}^n$ these point values of the Colombeau generalized functions is the Colombeau generalized numbers.

Example 1.1.2. Delta function $\delta(x)$ ill defined at point $x = 0$ since $\delta(0) = \infty$. However

$$\frac{1}{2}\left(\frac{1}{\sqrt{\pi\varepsilon}}\exp\left(-\frac{x^2}{4\varepsilon}\right)\right)_\varepsilon\Bigg|_{x=0} = \frac{1}{2\sqrt{\pi\varepsilon}}\left(\frac{1}{\sqrt{\varepsilon}}\right)_\varepsilon \in \widetilde{\mathbb{R}},$$
$$\frac{1}{\pi}\left(\frac{\varepsilon}{x^2+\varepsilon^2}\right)_\varepsilon\Bigg|_{x=0} = \frac{1}{\pi}\left(\frac{1}{\varepsilon}\right)_\varepsilon \in \widetilde{\mathbb{R}}. \qquad (1.1.9)$$

where $\widetilde{\mathbb{R}}$ is

Remark 1.1.3.

1.2. The ring of Colombeau generalized numbers $\widetilde{\mathbb{R}}$.

Designation 1.2.1. We denote by $\widetilde{\mathbb{R}}$ the ring of real, Colombeau generalized numbers. Recall that by definition $\widetilde{\mathbb{R}} = \mathbf{E}_{\mathbb{R}}(\mathbb{R})/\mathbf{N}(\mathbb{R})$ where [],[]

$$\mathbf{E}_{\mathbb{R}}(\mathbb{R}) = \{(x_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1)} | (\exists a \in \mathbb{R}_+)(\exists \varepsilon_0 \in (0,1))(\forall \varepsilon \leq \varepsilon_0)[|x_\varepsilon| \leq \varepsilon^{-a}]\},$$
$$\mathbf{N}(\mathbb{R}) = \{(x_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1)} | (\forall a \in \mathbb{R}_+)(\exists \varepsilon_0 \in (0,1))(\forall \varepsilon \leq \varepsilon_0)[|x_\varepsilon| \leq \varepsilon^{a}]\}. \qquad (1.2.1)$$

Remark 1.2.1. Note that: (i) there exists natural embedding $\widetilde{r} : \mathbb{R} \hookrightarrow \widetilde{\mathbb{R}}$ such that for all $r \in \mathbb{R}$, $\widetilde{r} : r \to (r_\varepsilon)_\varepsilon$, $r_\varepsilon = r$, for all $\varepsilon \in [1,0)$,(ii) the ring $\widetilde{\mathbb{R}}$ can be endowed with the structure of a partially ordered ring: for $\mathbf{r},\mathbf{s} \in \widetilde{\mathbb{R}}$, $\mathbf{r} \leq_{\widetilde{\mathbb{R}}} \mathbf{s}$ if and only if there are representatives $(r_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ with $r_\varepsilon \leq s_\varepsilon$ for all $\varepsilon \in [1,0)$.

Definition 1.5.2.(i) Let $\delta \in \widetilde{\mathbb{R}}$. We say that δ is infinite small (but non zero) Colombeau generalized number and abbraviate

$$\delta \approx_{\widetilde{\mathbb{R}}} \widetilde{0} \qquad (1.2.2)$$

I.3.Colombeau solutions to nonlinear equations in general.

During last 30 years the theory of Colombeau generalized solutions for linear and nonlinear partial differential equations many developed [1]-[].

Colombeau's method can yield generalized solutions for linear and nonlinear partial differential equations, which cannot be obtained for instance within the Schwartz distributions. To better understand the relevance of such generalized solutions, it is useful to consider not only m-th order nonlinear partial differential equation and their generalized solutions but also the specific solution methods within Colombeau's theory. Indeed, if in sufficiently particular or regular cases of initial and/or boundary values for instance, such solution methods yield generalized solutions closely related to the usual distribution, classical or analytic solutions, the respective solution methods-and their supporting theory-prove to be natural extensions of the earlier, more restricted methods, see review [2].

Generalized Einstein equation with Colombeau generalized energy-momentum density of the gravity source reads

$$(R_{ik,\varepsilon}(x))_{\varepsilon} - \tfrac{1}{2}(g_{ik,\varepsilon}(x)R_{\varepsilon}(x))_{\varepsilon} = -(T_{ik,\varepsilon}(x))_{\varepsilon}, \qquad (1.3.1)$$

where $\mathbf{cl}[(T_{ik,\varepsilon}(x))_{\varepsilon}] \in \mathcal{G}(\mathbb{R}^n)$.

Or in the following form

$$(G^{\nu}_{\mu,\varepsilon})_{\varepsilon} = (R^{\nu}_{\mu})_{\varepsilon} - \frac{1}{2}\delta^{\nu}_{\mu}(R_{\varepsilon})_{\varepsilon} = \kappa\left(T^{\nu}_{\mu,\varepsilon}\right)_{\varepsilon}, \qquad (1.3.2)$$

where $T_{\mu}{}^{\nu}$ is defined by

$$\left(T^{\nu}_{\mu,\varepsilon}\right)_{\varepsilon} = \frac{\widetilde{\mathbf{T}}^{\nu}_{\mu,\varepsilon}}{\sqrt{-g_{\varepsilon}}} \qquad (1.3.3)$$

with

$$\left(\widetilde{\mathbf{T}}^{\nu}_{\mu,\varepsilon}\right)_{\varepsilon} \triangleq 2g_{\mu\lambda,\varepsilon}\frac{\delta\mathcal{L}_M}{\delta g_{\lambda\nu,\varepsilon}} \qquad (1.3.4)$$

being the energy-momentum density of the distributional gravity source.

I.4.Why Colombeau solutions in general relativity is needed.

Recoll that in contemporary physical literature Colombeau solutions of the generalized Einstein field equations (1.3.1)-(1.3.2) originally were obtained only for the case []-[]:

$$\mathbf{cl}[(T_{ik,\varepsilon}(x))_{\varepsilon}] = \iota(T_{ik}(x)), \qquad (1.4.1)$$

where $T_{ik}(x) \in \mathcal{D}'(\mathbb{R}^4)$ and where $\iota : \mathcal{D}'(\mathbb{R}^4) \hookrightarrow \mathcal{G}(\mathbb{R}^4)$ is an embedding mentioned above, see Remark 1.1.2. We will be abbreviate: $\mathbf{cl}[(T_{ik,\varepsilon}(x))_{\varepsilon}] \in \mathcal{D}'(\mathbb{R}^4)$

Designation 1.4.1. Let $(u_{\varepsilon}(x))_{\varepsilon} \in \mathcal{G}(\mathbb{R}^n)$ and $\iota : \mathcal{D}'(\mathbb{R}^n) \hookrightarrow \mathcal{G}(\mathbb{R}^n)$ is an embedding mentioned above.We will be abbreviate:

$$\mathbf{cl}[(u_{\varepsilon}(x))_{\varepsilon}] \in \mathcal{D}'(\mathbb{R}^n). \qquad (1.4.2)$$

Remark 1.4.1.Note that (1.4.2) obviously meant that there exists a weak limit

$$\lim_{\varepsilon\to 0}\int u_{\varepsilon}(x)f(x)d^4x \quad (1.4.3)$$

where $f \in \mathcal{D}(\mathbb{R}^4)$.

Remark 1.4.1. Note that in contemporary physical literature Colombeau solutions of the generalized Einstein field equations (1.3.2) originally were obtained by using an regularizations of the classical singular solutions obtained from classical Einstein field equations with $T^{\nu}_{\mu} \equiv 0$.These classical singular solutions many years mistakenly

Example 1.4.1.[]In general relativity, many investigations have been made with regard to exact solutions of the Einstein equation and the singularity structure of space-time, but a distribution theoretical treatment of these space-times many years has not been developed sufficiently. This is the case even for the well-known Schwarzschild solution, which is given by, in the Schwarzschild coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$,

$$ds^2 = -\left(1-\frac{a}{\hat{r}}\right)(d\hat{x}^0)^2 + \left(1-\frac{a}{\hat{r}}\right)^{-1}(d\hat{r})^2 + \hat{r}^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]. \tag{1.4.4}$$

Here, a is the Schwarzschild radius $a = 2GM/c^2$ with G, M and c being the Newton gravitational constant, mass of the source, and the light velocity in vacuum Minkowski space-time, respectively. Obviously the fundamental tensor corresponding to ds^2 has the components which is degenerate or singular at $\hat{r} = 0$, i.e.

$$ds^2|_{\hat{r}=0} = -\infty(d\hat{x}^0)^2 + 0(d\hat{r})^2 + 0[(d\theta)^2 + \sin^2\theta(d\phi)^2]. \tag{1.4.5}$$

By using the Cartesian coordinates $(\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3)$, which are related to $(\hat{x}^0, \hat{r}, \theta, \phi)$ through the canonical relation

$$\hat{x}^1 = \hat{r}\cos\phi\sin\theta\ ,\quad \hat{x}^2 = \hat{r}\sin\phi\sin\theta\ ,\quad \hat{x}^3 = \hat{r}\cos\theta\ , \tag{1.4.6}$$

the metric (1.4.4) reads

$$ds^2 = \hat{g}_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu\ , \tag{1.4.7}$$

where at points $\hat{r} \neq 0$ the metric $\hat{g}_{\mu\nu}$ is given by

$$\begin{gathered}\hat{g}_{00} = -(1-h)\ ,\quad \hat{g}_{0\alpha} = 0\ ,\\ \hat{g}_{\alpha\beta} = \delta^{\alpha\beta} + h(1-h)^{-1}\frac{\hat{x}^\alpha\hat{x}^\beta}{\hat{r}^2}\ ,\quad \alpha,\beta = 1,2,3\end{gathered} \tag{1.4.8}$$

with $h \triangleq a/\hat{r}$. Well known that at points $\hat{r} \neq 0$

$$\begin{gathered}\kappa\hat{\widetilde{T}}{}_0^0 = -\frac{h'}{\hat{r}} - \frac{h}{\hat{r}^2}\ ,\\ \kappa\hat{\widetilde{T}}{}_0^\alpha = 0\ ,\quad \kappa\hat{\widetilde{T}}{}_\alpha^0 = 0\ ,\\ \kappa\hat{\widetilde{T}}{}_\alpha^\beta = \delta_\alpha^\beta\left(-\frac{h''}{2} - \frac{h'}{\hat{r}}\right) + \frac{\hat{x}^\alpha\hat{x}^\beta}{\hat{r}^2}\left(\frac{h''}{2} - \frac{h}{\hat{r}^2}\right)\ ,\end{gathered} \tag{1.4.9}$$

where the hatted symbols $\hat{\widetilde{T}}{}_\mu^\nu$ represent the quantity $\widetilde{T}{}_\mu^\nu$ in the coordinate system $\{\hat{x}^\mu; \mu = 0,1,2,3\}$. Also, we have defined $h' \triangleq dh/d\hat{r}$ and $h'' \triangleq d^2h/d\hat{r}^2$.

Remark 1.4.2. We extend now the quantity (1.4.7)-(1.4.9) in point $\hat{r} = 0$ as Colombeau generalized functions from Colombeau algebra $\mathcal{G}(\mathbb{R}^3)$.Regularizing now the function $h = a/\hat{r}$ as $h_\varepsilon = a/\sqrt{\hat{r}^2+\varepsilon^2}$ and the function $\frac{\hat{x}^\alpha\hat{x}^\beta}{\hat{r}^2}$ as $\frac{\hat{x}^\alpha\hat{x}^\beta}{\hat{r}^2+\varepsilon^2}$ with $\varepsilon \in (0,1]$, we obtain the Colombeau generalized metric

$$ds^2 = \left(\hat{g}_{\mu\nu,\varepsilon}d\hat{x}^\mu d\hat{x}^\nu\right)_\varepsilon\ , \tag{1.4.10}$$

where

$$\begin{gathered}\left(\hat{g}_{00,\varepsilon}\right)_\varepsilon = -(1-h_\varepsilon)\ ,\quad \hat{g}_{0\alpha} = 0\ ,\\ \left(\hat{g}_{\alpha\beta,\varepsilon}\right)_\varepsilon = \delta^{\alpha\beta} + ((h_\varepsilon(1-h_\varepsilon)^{-1})_\varepsilon)\left(\frac{\hat{x}^\alpha\hat{x}^\beta}{\hat{r}^2+\varepsilon^2}\right)_\varepsilon\ ,\quad \alpha,\beta = 1,2,3\end{gathered} \tag{1.4.11}$$

and

$$
\begin{gathered}
\kappa\left(\hat{\widetilde{T}}_0^{\,0}(\hat{x};\varepsilon)\right)_\varepsilon = -\left(\frac{a\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^{5/2}}\right)_\varepsilon ,\\
\kappa\left(\hat{\widetilde{T}}_0^{\,\alpha}(\hat{x};\varepsilon)\right)_\varepsilon = 0_{\widetilde{\mathbb{R}}} , \quad \kappa\left(\hat{\widetilde{T}}_\alpha^{\,0}(\hat{x};\varepsilon)\right)_\varepsilon = 0_{\widetilde{\mathbb{R}}} ,\\
\kappa\left(\hat{\widetilde{T}}_\alpha^{\,\beta}(\hat{x};\varepsilon)\right)_\varepsilon = \delta_\alpha^\beta\left(\frac{3a\varepsilon^2}{2(\hat{r}^2+\varepsilon^2)^{5/2}}\right)_\varepsilon -\\
\left(\left(\frac{\hat{x}^\alpha\hat{x}^\beta}{\hat{r}^2+\varepsilon^2}\right)_\varepsilon\right)\left(\frac{a\varepsilon^2}{(\hat{r}^2+\varepsilon^2)^{5/2}}\right)_\varepsilon\left(\frac{5}{2}+\left(\frac{\varepsilon^2}{\hat{r}^2+\varepsilon^2}\right)_\varepsilon\right) ,
\end{gathered}
\tag{1.4.12}
$$

where $\hat{\widetilde{T}}_\mu^{\,\nu}(\hat{x};\varepsilon)$ stands for the regularized $\hat{\widetilde{T}}_\mu^{\,\nu}$.

II.Generalized pseudo-Riemannian geometry

2.1.Colombeau Calculus on separable Hausdorff manifold.

We use [],[],[] as standard references for the foundations and various applications of standard Colombeau theory. We briefly recall the basic Colombeau construction. Throughout the paper Ω will denote an open subset of $\mathbb{R}^n$. Stanfard Colombeau generalized functions on Ω are defined as equivalence classes $u = [(u_\varepsilon)_\varepsilon]$ of nets of smooth functions $u_\varepsilon \in C^\infty(\Omega)$ (regularizations) subjected to asymptotic norm conditions with respect to $\varepsilon \in (0,1]$ for their derivatives on compact sets.

The basic idea of *classical Colombeau's theory of nonlinear generalized functions* [],[] is regularization by sequences (nets) of smooth functions and the use of asymptotic estimates in terms of a regularization parameter ε. Let $(u_\varepsilon)_{\varepsilon\in(0,1]}$ with $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M)$ for all $\varepsilon \in \mathbb{R}_+$, where M a separable, smooth orientable Hausdorff manifold of dimension n.

Definition 2.1.1.The Colombeau's algebra of generalized functions on M is defined as the quotient:

$$
\mathcal{G}(M) \triangleq \mathcal{E}_M(M)/\mathcal{N}(M) \tag{2.1.1}
$$

of the space $\mathcal{E}_M(M)$ of sequences of moderate growth modulo the space $\mathcal{N}(M)$ of negligible sequences. More precisely the notions of moderateness resp. negligibility are defined by the following asymptotic estimates (where $\mathfrak{X}(M)$ denoting the space of smooth vector fields on M):

$$\mathcal{E}_M(M) \triangleq \Big\{(u_\varepsilon)_\varepsilon| \;\forall K(K \subsetneq M)\,\forall k(k \in \mathbb{N})\,\exists N(N \in \mathbb{N})$$
$$\forall \xi_1,\ldots,\xi_k(\xi_1,\ldots,\xi_k \in \mathfrak{X}(M))\left[\sup_{p\in K} |L_{\xi_1}\ldots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\right]\Big\}, \tag{2.1.2}$$

and

$$\mathcal{N}(M) \triangleq \Big\{(u_\varepsilon)_\varepsilon| \;\forall K(K \subsetneq M),\, \forall k(k \in \mathbb{N}_0)\forall q(q \in N)$$
$$\forall \xi_1,\ldots,\xi_k(\xi_1,\ldots,\xi_k \in \mathfrak{X}(M))\left[\sup_{p\in K} |L_{\xi_1}\ldots L_{\xi_k} u_\varepsilon(p)| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0\right]\Big\}. \tag{2.1.3}$$

Remark 2.1.1. In the definition the Landau symbol $a_\varepsilon = O(\psi(\varepsilon))$ appears, having the following meaning: $\exists C(C > 0)\exists \varepsilon_0(\varepsilon_0 \in (0,1])\forall \varepsilon(\varepsilon < \varepsilon_0)[a_\varepsilon \leq C\psi(\varepsilon)]$.

Definition 2.1.3. Elements of $\mathcal{G}(M)$ are denoted by:

$$u = \mathbf{cl}[(u_\varepsilon)_\varepsilon] \triangleq (u_\varepsilon)_\varepsilon + \mathcal{N}(M). \tag{2.1.4}$$

Remark 2.1.2.With componentwise operations $(\bullet,\pm)$ $\mathcal{G}(M)$ is a fine sheaf of differential algebras with respect to the Lie derivative defined by $L_\xi u \triangleq \mathbf{cl}[(L_\xi u_\varepsilon)_\varepsilon]$.

The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterized locally, i.e., $u \in \mathcal{G}(M)$ iff $u \circ \psi_\alpha \in \mathcal{G}(\psi_\alpha(V_\alpha))$ for all charts (V_α,ψ_α), where on the open set $\psi_\alpha(V_\alpha) \subset \mathbb{R}^n$ in the respective estimates Lie derivatives are replaced by partial derivatives.

Remark 2.1.3.Smooth functions $f \in \mathcal{C}^\infty(M)$ are embedded into $\mathcal{G}(M)$ simply by the "constant" embedding σ, i.e., $\sigma(f) = \mathbf{cl}[(f)_\varepsilon]$, hence $\mathcal{C}^\infty(M)$ is a faithful subalgebra of $\mathcal{G}(M)$.

Definition 2.1.4.The $\mathcal{G}(M)$-module of generalized sections in vector bundles-especially the space of generalized tensor fields $\mathcal{T}_s{}^r(M)$-is defined along the same lines using analogous asymptotic estimates with respect to the norm induced by any Riemannian metric on the respective fibers. However, it is more convenient to use the following algebraic description of generalized tensor fields

$$\mathcal{G}^r_s(M) = \mathcal{G}(M) \otimes \mathcal{T}_s{}^r(M), \tag{2.1.5}$$

where $\mathcal{T}_s{}^r(M)$ denotes the space of smooth tensor fields and the tensor product is taken over the module $C^\infty(M)$.

Remark 2.1.4.Thus generalized tensor fields are just given by classical ones with generalized coefficient functions. Many concepts of classical tensor analysis carry over to the generalized setting, in particular Lie derivatives with respect to both classical and generalized vector fields, Lie brackets, exterior algebra, etc. Moreover, generalized tensor fields may also be viewed as $\mathcal{G}(M)$-multilinear maps taking generalized vector and covector fields to generalized functions, i.e., as $\mathcal{G}(M)$-modules we have

$$\mathcal{G}^r_s(M) \cong L_{(M)}(\mathcal{G}^0_1(M)^r, \mathcal{G}^1_0(M)^s; \mathcal{G}(M)). \tag{2.1.6}$$

and $\det(g_{ab})$ is invertible in the algebra of generalized functions. A generalized metric induces a $\mathcal{G}(M)$-linear isomorphism from $\mathcal{G}^1_0(M)$ to $\mathcal{G}^0_1(M)$ and the inverse metric $g^{ab} \triangleq [(g^{-1}_{ab}(\epsilon))_\epsilon]$ is a well defined element of $\mathcal{G}^2_0(M)$ (i.e., independent of the representative $((g_\epsilon)_{ab})_\epsilon$). Also the generalized Levi-Civita connection as well as the generalized Riemann-, Ricci- and Einstein tensor of a generalized metric are defined simply by the usual coordinate formulae on the level of representatives.

2.2.Point Values of a Generalized Functions on M. Colombeau Generalized Numbers.

2.3. Colombeau generalized curvilinear coordinates

Let us consider the Colombeau generalized transformation from one generalized coordinate system, $(x_\varepsilon^0)_\varepsilon,(x_\varepsilon^1)_\varepsilon,(x_\varepsilon^2)_\varepsilon,(x_\varepsilon^3)_\varepsilon$, to another generalized coordinate system $(x_\varepsilon'^0)_\varepsilon,(x_\varepsilon'^1)_\varepsilon,(x_\varepsilon'^2)_\varepsilon,(x_\varepsilon'^3)_\varepsilon$: transform according to the relation

$$(x_\varepsilon^i)_\varepsilon = \left(f_\varepsilon^i(x_\varepsilon'^0,x_\varepsilon'^1,x_\varepsilon'^2,x_\varepsilon'^3)\right)_\varepsilon, \qquad (2.3.1)$$

where the $\left(f_\varepsilon^i\right)_\varepsilon$ are certain Colombeau generalized functions and where $(\mathbf{J}_\varepsilon(x'^0,x'^1,x'^2,x'^3))_\varepsilon$

$$(\mathbf{J}_\varepsilon(x_\varepsilon'^0,x_\varepsilon'^1,x_\varepsilon'^2,x_\varepsilon'^3))_\varepsilon = \left(\frac{\partial(x_\varepsilon^0,x_\varepsilon^1,x_\varepsilon^2,x_\varepsilon^3)}{\partial(x_\varepsilon'^0,x_\varepsilon'^1,x_\varepsilon'^2,x_\varepsilon'^3)}\right)_\varepsilon \neq 0_{\widetilde{\mathbb{R}}} \qquad (2.3.1.a)$$

is the Jacobian of the Colombeau generalized transformation (2.3.1).

Remark 2.3.1. When we transform the coordinates, their Colombeau differentials $(dx_\varepsilon^i)_\varepsilon$ transform according to the relation

$$(dx_\varepsilon^i)_\varepsilon = \left(\frac{\partial x_\varepsilon^i}{\partial x_\varepsilon'^k}dx_\varepsilon'^k\right)_\varepsilon = \left(\frac{\partial x_\varepsilon^i}{\partial x_\varepsilon'^k}\right)_\varepsilon (dx_\varepsilon'^k)_\varepsilon. \qquad (2.3.2)$$

Definition 2.3.1. Every tuple of four Colombeau quantities $(A_\varepsilon^i)_\varepsilon, i = 0,1,2,3$, which under
a transformation (2.3.1) of coordinates, transform like the Colombeau coordinate differentials (2.3.2), is called Colombeau contravariant four-vector:

$$(A_\varepsilon^i)_\varepsilon = \left(\frac{\partial x_\varepsilon^i}{\partial x_\varepsilon'^k}A_\varepsilon'^k\right)_\varepsilon = \left(\frac{\partial x_\varepsilon^i}{\partial x_\varepsilon'^k}\right)_\varepsilon (A_\varepsilon'^k)_\varepsilon. \qquad (2.3.3)$$

Let $(\varphi_\varepsilon)_\varepsilon$ be the Colombeau scalar. Under a coordinate transformation (2.3.1), the four Colombeau quantities $\left(\frac{\partial\varphi_\varepsilon}{\partial x_\varepsilon^i}\right)_\varepsilon, i = 0,1,2,3$ transform according to the formula

$$\left(\frac{\partial\varphi_\varepsilon}{\partial x_\varepsilon^i}\right)_\varepsilon = \left(\frac{\partial\varphi_\varepsilon}{\partial x_\varepsilon'^k}\frac{\partial x_\varepsilon'^k}{\partial x_\varepsilon^i}\right)_\varepsilon = \left(\frac{\partial\varphi_\varepsilon}{\partial x_\varepsilon'^k}\right)_\varepsilon \left(\frac{\partial x_\varepsilon'^k}{\partial x_\varepsilon^i}{}_\varepsilon\right). \qquad (2.3.4)$$

Definition 2.3.2. Every tuple of four Colombeau generalized functions $(A_{i,\varepsilon})_\varepsilon$ which, under
a coordinate transformation (2.2.1), transform like the Colombeau derivatives of a scalar,
is called Colombeau generalized covariant four-vector

$$(A_{i,\varepsilon})_\varepsilon = \left(\frac{\partial x_\varepsilon'^k}{\partial x_\varepsilon^i}A'_{k,\varepsilon}\right)_\varepsilon = \left(\frac{\partial x_\varepsilon'^k}{\partial x_\varepsilon^i}\right)_\varepsilon (A'_{k,\varepsilon})_\varepsilon. \qquad (2.3.5)$$

Definition 2.3.3.We call the Colombeau generalized contravariant tensor of the second
rank, $(A_\varepsilon^{ik})_\varepsilon$,any tuple of sixteen Colombeau generalized functions which transform like the
products of the components of two Colombeau generalized contravariant vectors, i.e. according to the law

$$(A_\varepsilon^{ik})_\varepsilon = \left(\frac{\partial x_\varepsilon'^k}{\partial x_\varepsilon^i}\frac{\partial x_\varepsilon'^m}{\partial x_\varepsilon^i}A'_{im,\varepsilon}\right)_\varepsilon = \left(\frac{\partial x_\varepsilon'^k}{\partial x_\varepsilon^i}\frac{\partial x_\varepsilon'^m}{\partial x_\varepsilon^i}\right)_\varepsilon (A'_{im,\varepsilon})_\varepsilon \tag{2.3.6}$$

and a mixed Colombeau generalized tensor transforms as follows

$$(A_{k,\varepsilon}^i)_\varepsilon = \left(\frac{\partial x_\varepsilon^i}{\partial x_\varepsilon'^l}\frac{\partial x_\varepsilon'^m}{\partial x_\varepsilon^k}A'^l_{m,\varepsilon}\right)_\varepsilon = \left(\frac{\partial x_\varepsilon^i}{\partial x_\varepsilon'^l}\frac{\partial x_\varepsilon'^m}{\partial x_\varepsilon^k}\right)_\varepsilon (A'^l_{m,\varepsilon})_\varepsilon. \tag{2.3.7}$$

Remark 2.3.2.Note that the scalar product of two four-vectors $(A_\varepsilon^i B_{i,\varepsilon})$ is invariant since

$$(A_\varepsilon^i B_{i,\varepsilon})_\varepsilon = \left(\frac{\partial x_\varepsilon^i}{\partial x_\varepsilon'^l}\frac{\partial x_\varepsilon'^m}{\partial x_\varepsilon^i}A'^l_\varepsilon B'_{m,\varepsilon}\right)_\varepsilon = \left(\frac{\partial x_\varepsilon'^m}{\partial x_\varepsilon'^l}A'^l_\varepsilon B'_{m,\varepsilon}\right)_\varepsilon = (A'^l_\varepsilon B'_{l,\varepsilon})_\varepsilon. \tag{2.3.8}$$

The unit four-tensor δ_k^i is defined the same as in classical case: $\delta_k^i = 0$ for $i \neq k$ and
$\delta_k^i = 1$ for $i = k$. If $(A_\varepsilon^k)_\varepsilon$is a Colombeau generalized four-vector,then multiplying by δ_k^iwe
obtain

$$(A_\varepsilon^k \delta_k^i)_\varepsilon = (A_\varepsilon^i)_\varepsilon, \tag{2.3.9}$$

i.e. again Colombeau generalized four-vector; this proves that δ_k^i is a tensor.

Remark 2.3.3.The square of the Colombeau generalized line element $(ds_\varepsilon^2)_\varepsilon$ in curvilinear
coordinates is a quadratic form in the differentials $dx^i, i = 0,1,2,3$:

$$(ds_\varepsilon^2)_\varepsilon = (g_{ik,\varepsilon}dx^i dx^k)_\varepsilon = \{(g_{ik,\varepsilon})_\varepsilon\}dx^i dx^k. \tag{2.3.10}$$

where the $(g_{ik,\varepsilon})_\varepsilon$ are Colombeau generalized functions of the coordinates; $(g_{ik,\varepsilon})_\varepsilon$ is
symmetric in the indices i and k :

$$(g_{ik,\varepsilon})_\varepsilon = (g_{ki,\varepsilon})_\varepsilon. \tag{2.3.11}$$

Definition 2.3.4.Since the (contracted) product of $(g_{ik,\varepsilon})_\varepsilon$ and the contravariant tensor
$dx^i dx^k$is a scalar, the $(g_{ik,\varepsilon})_\varepsilon$ form a covariant tensor; it is called the Colombeau
generalized metric tensor.

Definition 2.3.5.Two tensors $(A_{ik,\varepsilon})_\varepsilon$ and $(B_\varepsilon^{ik})_\varepsilon$ are said to be reciprocal to each other if

$$(A_{ik,\varepsilon}B_\varepsilon^{ik})_\varepsilon = \{(A_{ik,\varepsilon})_\varepsilon\}\left\{(B_\varepsilon^{ik})_\varepsilon\right\} = \delta_k^i. \tag{2.3.12}$$

In particular the contravariant metric tensor is the tensor $(g_{ik,\varepsilon})_\varepsilon$reciprocal to the tensor
$(g_\varepsilon^{ik})_\varepsilon$,that is,

$$\{(g_{ik,\varepsilon})_\varepsilon\}\left\{(g_\varepsilon^{ik})_\varepsilon\right\} = \delta_k^i. \tag{2.3.13}$$

The same physical quantity can be represented in contravariant or covariant components.

It is obvious that the only quantities that can determine the connection between the
different forms are the components of the metric tensor. This connection is given by

the

formulas:

$$(A^i_\varepsilon)_\varepsilon = (g^{ik}_\varepsilon A_{k,\varepsilon})_\varepsilon, (A_{i,\varepsilon})_\varepsilon = (g_{ik,\varepsilon} A^k_\varepsilon)_\varepsilon. \quad (2.3.14)$$

These remarks also apply to Colombeau generalized tensors. The transition between the

different forms of a given physical generalized tensor is accomplished by using the metric

tensor according to the formulas:

$$(A^i_{k,\varepsilon})_\varepsilon = (g^{il}_\varepsilon A_{lk,\varepsilon})_\varepsilon, (A^{ik}_\varepsilon)_\varepsilon = (g^{il}_\varepsilon g^{km}_\varepsilon A_{lm,\varepsilon})_\varepsilon, \text{etc.} \quad (2.3.15)$$

The completely antisymmetric unit pseudotensor in galilean coordinates we denote by

e^{iklm}. Let us transform it to an arbitrary system of Colombeau generalized coordinates, and now denote it by $(E^{iklm}_\varepsilon)_\varepsilon$. We keep the notation e^{iklm} for the quantities defined as before by

$e^{0123} = 1$ (or $e_{0123} = -1$).Let the $x'^i, i = 0,1,2,3$ be galilean, and the $(x^i_\varepsilon)_\varepsilon, i = 0,1,2,3$ be arbitrary Colombeau generalized curvilinear coordinates. According to the general rules for transformation of Colombeau generalized tensors, we have

$$(E^{iklm}_\varepsilon)_\varepsilon = \left\{ \left(\frac{\partial x^i_\varepsilon}{\partial x'^p} \frac{\partial x^k_\varepsilon}{\partial x'^r} \frac{\partial x^l_\varepsilon}{\partial x'^s} \frac{\partial x^m_\varepsilon}{\partial x'^t} \right)_\varepsilon \right\} e^{prst}, \quad (2.3.16)$$

or

$$(E^{iklm}_\varepsilon)_\varepsilon = \{ (\mathbf{J}_\varepsilon(x'^0, x'^1, x'^2, x'^3))_\varepsilon \} e^{prst}, \quad (2.3.17)$$

where $(\mathbf{J}_\varepsilon(x'^0, x'^1, x'^2, x'^3))_\varepsilon \neq 0_{\widetilde{\mathbb{R}}}$ is the determinant formed from the derivatives $\partial x^i / \partial x'^p$, i.e. it is just the Jacobian of the Colombeau generalized transformation from the galilean to the Colombeau generalized curvilinear coordinates:

$$(\mathbf{J}_\varepsilon(x'^0, x'^1, x'^2, x'^3))_\varepsilon = \left(\frac{\partial x^i_\varepsilon}{\partial x'^p} \frac{\partial x^k_\varepsilon}{\partial x'^r} \frac{\partial x^l_\varepsilon}{\partial x'^s} \frac{\partial x^m_\varepsilon}{\partial x'^t} \right)_\varepsilon = \left(\frac{\partial (x^0_\varepsilon, x^1_\varepsilon, x^2_\varepsilon, x^3_\varepsilon)}{\partial (x'^0, x'^1, x'^2, x'^3)} \right)_\varepsilon. \quad (2.3.18)$$

This Jacobian can be expressed in terms of the determinant of the Colombeau generalized metric tensor $(g_{ik,\varepsilon})_\varepsilon$ (in the system $(x^i_\varepsilon)_\varepsilon$). To do this we write the formula for the transformation of the metric tensor:

$$(g^{ik}_\varepsilon)_\varepsilon = \left\{ \left(\frac{\partial x^i_\varepsilon}{\partial x'^i} \frac{\partial x^k_\varepsilon}{\partial x'^m} \right)_\varepsilon \right\} g^{(0)im}, \quad (2.3.19)$$

where

$$g^{(0)im} = g^{(0)}_{im} = \left\{ \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{array} \right\}, \quad (2.3.20)$$

and equate the determinants of the two sides of this equation. The determinant of the reciprocal tensor $\det\left|(g^{ik}_\varepsilon)_\varepsilon\right| = 1/(g_\varepsilon)_\varepsilon$. The determinant $\det|g^{(0)im}| = -1$. Thus we have

$1/(g_\varepsilon)_\varepsilon = -(\mathbf{J}^2_\varepsilon(x'^0,x'^1,x'^2,x'^3))_\varepsilon$, and so

$$(\mathbf{J}^2_\varepsilon(x'^0,x'^1,x'^2,x'^3))_\varepsilon = 1/\sqrt{(g_\varepsilon)_\varepsilon}. \qquad (2.3.21)$$

Thus, in curvilinear coordinates the antisymmetric unit tensor of rank four must bedefined as

$$(E^{iklm}_\varepsilon)_\varepsilon = \frac{1}{\sqrt{-(g_\varepsilon)_\varepsilon}} e^{iklm} \qquad (2.3.22)$$

and its covariant components are

$$(E_{iklm,\varepsilon})_\varepsilon = \sqrt{-(g_\varepsilon)_\varepsilon}\, e_{iklm}. \qquad (2.3.23)$$

In a galilean coordinate system $x'^i, i = 0,1,2,3$ the integral of a scalar with respect to $d\Omega' = dx'^0 dx'^1 dx'^2 dx'^3$ is also a scalar, i.e. the element $d\Omega'$ behaves like a scalar in the integration. On transforming to Colombeau generalized curvilinear coordinates $(x^i_\varepsilon)_\varepsilon, i = 0,1,2,3$, the element of integration $d\Omega'$ goes over into

$$d\Omega' := \left\{(\mathbf{J}^{-1}_\varepsilon)_\varepsilon\right\} d\Omega = \sqrt{-(g_\varepsilon)_\varepsilon}\,(d\Omega_\varepsilon)_\varepsilon, \qquad (2.3.24)$$

where $(d\Omega_\varepsilon)_\varepsilon = \{(dx^0_\varepsilon)_\varepsilon\}\{(dx^1_\varepsilon)_\varepsilon\}\{(dx^2_\varepsilon)_\varepsilon\}\{(dx^3_\varepsilon)_\varepsilon\}$.

Thus, in Colombeau generalized curvilinear coordinates, when integrating over a four-volume the quantity $\sqrt{-(g_\varepsilon)_\varepsilon}\,(d\Omega_\varepsilon)_\varepsilon$ behaves like an invariant.

Remark 2.3.4. The element of "area" of the Colombeau generalized hypersurface spanned by three infinitesimal Colombeau generalized displacements is the contravariant antisymmetric Colombeau generalized tensor $(dS^{ikl}_\varepsilon)_\varepsilon$; the vector dual to it is gotten by multiplying by the tensor $\sqrt{-(g_\varepsilon)_\varepsilon}\, e_{iklm}$, so it is equal to

$$\sqrt{-(g_\varepsilon)_\varepsilon}\,(dS_{\varepsilon,i})_\varepsilon = -\tfrac{1}{6}\sqrt{-(g_\varepsilon)_\varepsilon}\, e_{iklm}(dS^{kim}_\varepsilon)_\varepsilon. \qquad (2.3.25)$$

Remark 2.3.5. Let $(df^{ik}_\varepsilon)_\varepsilon$be the element of two-dimensional Colombeau generalized surface spanned by two infinitesimal Colombeau generalized displacements, the dual Colombeau generalized tensor is defined as

$$\sqrt{-(g_\varepsilon)_\varepsilon}\left(df^*_{ik,\varepsilon}\right)_\varepsilon = \tfrac{1}{2}\sqrt{-(g_\varepsilon)_\varepsilon}\, e_{iklm}\left(df^{lm}_\varepsilon\right)_\varepsilon. \qquad (2.3.26)$$

We will be use the designations $(dS_{\varepsilon,i})$ and $\left(df^*_{ki,\varepsilon}\right)_\varepsilon$ for $e_{iklm}(dS^{kim}_\varepsilon)_\varepsilon$ and $e_{iklm}\left(df^{lm}_\varepsilon\right)_\varepsilon$ (and not for their products by $\sqrt{-(g_\varepsilon)_\varepsilon}$).

Remark 2.3.6. Note that the canonical rules for transforming the various integrals into one another remain the same, since their derivation was formal in character and not related to the tensor properties of the different quantities. Of particular importance is the rule for transforming the integral over a hypersurface into an integral over a four-volume (Gauss' theorem), which is accomplished by the substitution

$$(dS_{i,\varepsilon})_\varepsilon := \{(d\Omega_\varepsilon)_\varepsilon\}\left(\frac{\partial}{\partial x^i_\varepsilon}\right)_\varepsilon. \qquad (2.3.27)$$

Remark 2.3.7. Note that for the integral of Colombeau generalized vector $(A^i_\varepsilon)_\varepsilon$ we have

$$\left(\oint A^i_\varepsilon dS_{i,\varepsilon}\right)_\varepsilon = \left(\int \frac{\partial A^i_\varepsilon}{\partial x^i_\varepsilon} d\Omega_\varepsilon\right)_\varepsilon = \int\left\{\left(\frac{\partial A^i_\varepsilon}{\partial x^i_\varepsilon}\right)_\varepsilon\right\}\{(d\Omega_\varepsilon)_\varepsilon\}. \quad (2.3.28)$$

This formula (2.3.28) is the generalization of Gauss' theorem.

An integral over a two-dimensional surface is transformed into an integral over the hypersurface "spanning" it by replacing the Colombeau generalized element of integration $\left(df^*_{ik,\varepsilon}\right)_\varepsilon$ by the operator

$$\left(df^*_{ik,\varepsilon}\right)_\varepsilon := \left(dS_{i,\varepsilon}\frac{\partial}{\partial x^k_\varepsilon}\right)_\varepsilon - \left(dS_{k,\varepsilon}\frac{\partial}{\partial x^i_\varepsilon}\right)_\varepsilon. \quad (2.3.29)$$

For example, for the integral of an antisymmetric Colombeau generalized tensor $(A^{ik}_\varepsilon)_\varepsilon$ we have

$$\left(\int A^{ik}_\varepsilon df^*_{ik,\varepsilon}\right)_\varepsilon = \left(\int dS_{i,\varepsilon}\frac{\partial A^{ik}_\varepsilon}{\partial x^k_\varepsilon}\right)_\varepsilon - \left(\int dS_{k,\varepsilon}\frac{\partial A^{ik}_\varepsilon}{\partial x^i_\varepsilon}\right)_\varepsilon = 2\left(\int dS_{i,\varepsilon}\frac{\partial A^{ik}_\varepsilon}{\partial x^k_\varepsilon}\right)_\varepsilon. \quad (2.3.30)$$

The integral over a four-dimensional Colombeau generalized closed curve is transformed into an integral over the surface spanning it by the substitution

$$(dx^i_\varepsilon)_\varepsilon := \left\{\left(df^{ki}_\varepsilon\right)_\varepsilon\right\}\left(\frac{\partial}{\partial x^i_\varepsilon}\right)_\varepsilon. \quad (2.3.31)$$

Thus for the integral of Colombeau generalized vector, we have:

$$\left(\oint A_{i,\varepsilon}dx^i_\varepsilon\right)_\varepsilon = \left(\int df^{ki}_\varepsilon \frac{\partial A_{i,\varepsilon}}{\partial x^i_\varepsilon}\right)_\varepsilon = \frac{1}{2}\int\left(df^{ik}_\varepsilon\frac{\partial A_{k,\varepsilon}}{\partial x^i_\varepsilon} - df^{ik}_\varepsilon\frac{\partial A_{i,\varepsilon}}{\partial x^k_\varepsilon}\right)_\varepsilon, \quad (2.3.32)$$

which is the generalization of Stokes' theorem.

2.4. Colombeau Generalized covariant derivatives

In galilean coordinates the Colombeau generalized differentials $(dA_{i,\varepsilon})_\varepsilon$ of a vector $(A_{i,\varepsilon})_\varepsilon$ form the Colombeau generalized vector, and the derivatives $(\partial A_{i,\varepsilon}/\partial x^k_\varepsilon)_\varepsilon$ of the components of a vector with respect to the coordinates form the Colombeau generalized tensor. In Colombeau generalized curvilinear coordinates this is not so; $(dA_{i,\varepsilon})_\varepsilon$ is not a vector, and $(\partial A_{i,\varepsilon}/\partial x^k_\varepsilon)_\varepsilon$ is not the Colombeau generalized tensor.This is due to the fact that $(dA_{i,\varepsilon})_\varepsilon$ is the difference of vectors located at different (infinitesimally separated) points of space; at different points in space vectors transform differently, since the coefficients in the transformation formulas (2.3.3), (2.3.4) are Colombeau generalized functions of the generalized coordinates.Thus in order to compare two infmitesimally separated generalized vectors we must subject one of them to a parallel translation to the point where the second is located. Let us consider an arbitrary generalized contravariant vector ; if its value at the point x^i is $(A^i_\varepsilon)_\varepsilon$, then at the neighboring point $x^i + dx^i$ it is equal to $(A^i_\varepsilon)_\varepsilon + (dA^i_\varepsilon)_\varepsilon = (A^i_\varepsilon + dA^i_\varepsilon)_\varepsilon$. We subject the vector $(A^i_\varepsilon)_\varepsilon$ to an infinitesimal parallel displacement to the point $x^i + dx^i$; the change in the vector which results from this we denote by $(\delta A^i_\varepsilon)_\varepsilon$ Then the difference $(DA^i_\varepsilon)_\varepsilon$ between the two Colombeau generalized vectors which are now located at the same point is

$$(DA^i_\varepsilon)_\varepsilon = (dA^i_\varepsilon)_\varepsilon - (\delta A^i_\varepsilon)_\varepsilon. \quad (2.4.1)$$

The change $(\delta A^i_\varepsilon)_\varepsilon$ in the components of Colombeau generalized vector under an infinitesimal parallel displacement depends on the values of the components themselves, where the dependence must clearly be linear. This follows directly from the fact that the sum of two Colombeau generalized vectors must transform according to the same law as each of the constituents. Thus $(\delta A^i_\varepsilon)_\varepsilon$ has the form

$$(\delta A^i_\varepsilon)_\varepsilon = -(\Gamma^i_{kl,\varepsilon} A^k_\varepsilon dx^l)_\varepsilon, \qquad (2.4.2)$$

where $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ the are certain Colombeau generalized functions of the coordinates. Their form depends, of course, on the coordinate system; for a galilean coordinate system $(\Gamma^i_{kl,\varepsilon})_\varepsilon = 0_{\widetilde{\mathbb{R}}}$. From this it is already clear that the quantities $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ do not form Colombeau generalized tensor, since a tensor which is equal to zero in one coordinate system is equal to zero in every other one. In a curvilinear space it is, of course, impossible to make all the $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ vanish over all of space. But we can choose a coordinate system for which the $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ become $0_{\widetilde{\mathbb{R}}}$ over a given infinitesimal region. The quantities $(\Gamma^i_{kl,\varepsilon})_\varepsilon$, are called generalized Christoffel symbols. In addition to the quantities $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ we shall later also use Colombeau generalized quantities $(\Gamma_{i,kl,\varepsilon})_\varepsilon$ defined as follows

$$(\Gamma_{i,kl,\varepsilon})_\varepsilon = (g_{im,\varepsilon}\Gamma^m_{km,\varepsilon})_\varepsilon. \qquad (2.4.3)$$

Conversely,

$$(\Gamma^i_{kl,\varepsilon})_\varepsilon = (g^{im}_\varepsilon \Gamma_{m,kl,\varepsilon})_\varepsilon. \qquad (2.4.4)$$

It is also easy to relate the change in the components of a covariant vector under a parallel displacement to the Christoffel symbols. To do this we note that under a parallel displacement, a scalar is unchanged. In particular, the scalar product of two vectors does not change under a parallel displacement.

Let $(A_{i,\varepsilon})_\varepsilon$ and $(B^i_\varepsilon)_\varepsilon$ be any covariant and contravariant vectors. Then from $\delta(A_{i,\varepsilon}B^i_\varepsilon)_\varepsilon = 0_{\widetilde{\mathbb{R}}}$, we have

$$(B^i_\varepsilon \delta A_{i,\varepsilon}) = -(A_{i,\varepsilon}\delta B^i_\varepsilon)_\varepsilon = (\Gamma^i_{kl,\varepsilon} B^k_\varepsilon A_{i,\varepsilon} dx^l)_\varepsilon \qquad (2.4.5)$$

or, changing the indices,

$$(B^i_\varepsilon \delta A_{i,\varepsilon}) = (\Gamma^k_{il,\varepsilon} B^i_\varepsilon A_{k,\varepsilon} dx^l)_\varepsilon \qquad (2.4.6)$$

From this, by the arbitrariness of the $(B^i_\varepsilon)_\varepsilon$ one obtains

$$(\delta A_{i,\varepsilon})_\varepsilon = \left((\Gamma^k_{il,\varepsilon} A_{k,\varepsilon})_\varepsilon\right) dx^l \qquad (2.4.7)$$

which determines the change in a covariant vector under a parallel displacement. Substituting (2.4.2) and $(dA^i_\varepsilon)_\varepsilon = ((\partial A^i_\varepsilon/\partial x^l)_\varepsilon) dx^l$ in (2.4.1), we have

$$(DA^i_\varepsilon)_\varepsilon = \left[\left(\frac{\partial A^i_\varepsilon}{\partial x^l}\right)_\varepsilon + (\Gamma^i_{kl,\varepsilon} A^k_\varepsilon)_\varepsilon\right] dx^l. \qquad (2.4.8)$$

Similarly, one finds for a covariant Colombeau generalized vector

$$(DA_{i,\varepsilon})_\varepsilon = \left[\left(\frac{\partial A_{i,\varepsilon}}{\partial x^l}\right)_\varepsilon + (\Gamma^k_{il,\varepsilon} A_{k,\varepsilon})_\varepsilon\right] dx^l. \qquad (2.4.9)$$

The expressions in parentheses in (2.4.8) and (2.4.9) are Colombeau generalized tensors, since when multiplied by the vector dx^l they give a vector. Clearly, these are the generalized tensors which give the desired generalization of the concept of a derivative to Colombeau generalized curvilinear coordinates. These Colombeau generalized tensors are called the Colombeau generalized covariant derivatives of the vectors $(A^i_\varepsilon)_\varepsilon$ and $(A_{i,\varepsilon})_\varepsilon$ respectively. We shall denote them by $A^i_{;k,\varepsilon}$ and $A_{i;k,\varepsilon}$. Thus,

$$(DA^i_\varepsilon)_\varepsilon = (A^i_{;l,\varepsilon}dx^l)_\varepsilon, (DA_{i,\varepsilon})_\varepsilon = (A_{i;l,\varepsilon}dx^l)_\varepsilon, \tag{2.4.10}$$

while the Colombeau generalized covariant derivatives themselves are

$$\left(A^i_{;\,l,\varepsilon}\right)_\varepsilon = \left(\frac{\partial A^i_\varepsilon}{\partial x^l}\right)_\varepsilon + (\Gamma^i_{kl,\varepsilon}A^k_\varepsilon)_\varepsilon \tag{2.4.11}$$

and

$$\left(A_{i;\,l,\varepsilon}\right)_\varepsilon = \left(\frac{\partial A_{i,\varepsilon}}{\partial x^l}\right)_\varepsilon + (\Gamma^k_{il,\varepsilon}A_{k,\varepsilon})_\varepsilon. \tag{2.4.12}$$

It is also easy to calculate the covariant derivative of the Colombeau generalized tensor.

One gets the covariant derivative of the Colombeau generalized tensor $(A^{ik}_\varepsilon)_\varepsilon$ in the form

$$\left(A^{ik}_{;\,l,\varepsilon}\right)_\varepsilon = \left(\frac{\partial A^{ik}_\varepsilon}{\partial x^l}\right)_\varepsilon + (\Gamma^i_{ml,\varepsilon}A^{mk}_\varepsilon)_\varepsilon + (\Gamma^k_{ml,\varepsilon}A^{im}_\varepsilon)_\varepsilon. \tag{2.4.13}$$

The covariant derivative of the Colombeau generalized mixed tensor $(A^i_{k,\varepsilon})_\varepsilon$ and the Colombeau generalized covariant tensor $(A_{ik,\varepsilon})_\varepsilon$ has the form

$$\left(A^i_{k;\,l,\varepsilon}\right)_\varepsilon = \left(\frac{\partial A^i_{k,\varepsilon}}{\partial x^l}\right)_\varepsilon + (\Gamma^m_{kl,\varepsilon}A^i_{m,\varepsilon})_\varepsilon + (\Gamma^i_{ml,\varepsilon}A^m_{k,\varepsilon})_\varepsilon \tag{2.4.14}$$

and

$$\left(A_{ik;\,l,\varepsilon}\right)_\varepsilon = \left(\frac{\partial A_{ik,\varepsilon}}{\partial x^l}\right)_\varepsilon - (\Gamma^m_{il,\varepsilon}A_{mk,\varepsilon})_\varepsilon - (\Gamma^m_{kl,\varepsilon}A_{im,\varepsilon})_\varepsilon. \tag{2.4.15}$$

correspondingly.

If in a covariant derivative we raise the index signifying the differentiation, we obtain the

so-called contravariant derivative. Thus,

$$\left(A^{;k}_{i,\varepsilon}\right)_\varepsilon = (g^{kl}_\varepsilon A_{i;l,\varepsilon})_\varepsilon; (A^{i;k}_\varepsilon)_\varepsilon = (g^{kl}_\varepsilon A^i_{;l,\varepsilon})_\varepsilon. \tag{2.4.16}$$

2.5. The relation of the Colombeau Generalized Christoffel symbols to the Colombeau Generalized metric tensor

Let us show that the Colombeau generalized covariant derivative of the Colombeau Generalized metric tensor $(g_{ik,\varepsilon})_\varepsilon$ is $0_{\widetilde{\mathbb{R}}}$. To do this we note that the relation

$$(DA_{i,\varepsilon})_\varepsilon = (g_{ik,\varepsilon}DA^k_\varepsilon)_\varepsilon \tag{2.5.1}$$

is valid for the generalized vector $(DA_{i,\varepsilon})_\varepsilon$ as for any generalized vector. On the other hand, $(A_{i,\varepsilon})_\varepsilon = (g_{ik,\varepsilon}A^k_\varepsilon)_\varepsilon$, so that

$$(DA_{i,\varepsilon})_\varepsilon = D(g_{ik,\varepsilon}A^k_\varepsilon)_\varepsilon = (g_{ik,\varepsilon}DA^k_\varepsilon)_\varepsilon + (A^k_\varepsilon Dg_{ik,\varepsilon})_\varepsilon. \tag{2.5.2}$$

Comparing with $(DA_{i,\varepsilon})_\varepsilon = (g_{ik,\varepsilon}DA^k_\varepsilon)_\varepsilon$, and remembering that the vector $(A^k_\varepsilon)_\varepsilon$ is arbitrary,

$$(Dg_{ik,\varepsilon})_\varepsilon = 0_{\widetilde{\mathbb{R}}}. \tag{2.5.3}$$

Therefore the covariant derivative

$$(g_{ik;l,\varepsilon})_\varepsilon = 0_{\widetilde{\mathbb{R}}}. \tag{2.5.4}$$

Thus $(g_{ik,\varepsilon})_\varepsilon$ may be considered as a constant under covariant differentiation.The equation $(g_{ik;l,\varepsilon})_\varepsilon = 0_{\widetilde{\mathbb{R}}}$ can be used to express the Colombeau generalized Christoffel symbols $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ in terms of the Colombeau generalized metric tensor $(g_{ik,\varepsilon})_\varepsilon$. To do this we write in accordance with the general definition (2.4.9):

$$\begin{gathered}(g_{ik;l,\varepsilon})_\varepsilon = \left(\frac{\partial g_{ik,\varepsilon}}{\partial x^l}\right)_\varepsilon - (g_{mk,\varepsilon}\Gamma^m_{il,\varepsilon})_\varepsilon - (g_{im,\varepsilon}\Gamma^m_{kl,\varepsilon})_\varepsilon = \\ \left(\frac{\partial g_{ik,\varepsilon}}{\partial x^l}\right)_\varepsilon - (\Gamma_{k,il,\varepsilon})_\varepsilon - (\Gamma_{i,kl,\varepsilon})_\varepsilon = 0_{\widetilde{\mathbb{R}}}.\end{gathered} \tag{2.5.5}$$

Thus the derivatives of $(g_{ik,\varepsilon})_\varepsilon$ are expressed in terms of the Colombeau generalized Christoffel symbols.f We write the values of the derivatives of $(g_{ik,\varepsilon})_\varepsilon$, permuting the indices i,k,l :

$$\begin{gathered}\left(\frac{\partial g_{ik,\varepsilon}}{\partial x^l}\right)_\varepsilon = (\Gamma_{k,il,\varepsilon})_\varepsilon + (\Gamma_{i,kl,\varepsilon})_\varepsilon, \left(\frac{\partial g_{li,\varepsilon}}{\partial x^k}\right)_\varepsilon = (\Gamma_{i,kl,\varepsilon})_\varepsilon + (\Gamma_{l,ik,\varepsilon})_\varepsilon, \\ \left(\frac{\partial g_{kl,\varepsilon}}{\partial x^i}\right)_\varepsilon = -(\Gamma_{l,ki,\varepsilon})_\varepsilon + (\Gamma_{k,li,\varepsilon})_\varepsilon.\end{gathered} \tag{2.5.5'}$$

Taking half the sum of these equations, we find

$$(\Gamma_{i,kl,\varepsilon})_\varepsilon = \frac{1}{2}\left[\left(\frac{\partial g_{ik,\varepsilon}}{\partial x^l}\right)_\varepsilon + \left(\frac{\partial g_{il,\varepsilon}}{\partial x^k}\right)_\varepsilon - \left(\frac{\partial g_{kl,\varepsilon}}{\partial x^i}\right)_\varepsilon\right], \tag{2.5.6}$$

since $(\Gamma_{i,kl,\varepsilon})_\varepsilon = (\Gamma_{i,lk,\varepsilon})_\varepsilon$.From (2.5.6) we have for the symbols $(\Gamma^i_{kl,\varepsilon})_\varepsilon = (g^{im}_\varepsilon \Gamma_{m,kl,\varepsilon})_\varepsilon$

$$(\Gamma^i_{kl,\varepsilon})_\varepsilon = \frac{1}{2}((g^{im}_\varepsilon)_\varepsilon)\left[\left(\frac{\partial g_{mk,\varepsilon}}{\partial x^l}\right)_\varepsilon + \left(\frac{\partial g_{ml,\varepsilon}}{\partial x^k}\right)_\varepsilon - \left(\frac{\partial g_{kl,\varepsilon}}{\partial x^m}\right)_\varepsilon\right]. \tag{2.5.7}$$

2.6. The Colombeau Generalized Curvature Tensor

In this subsection we derive the general formula for the change in a vector after parallel displacement around any infinitesimal closed contour γ. This generalized change $(\Delta A_{k,\varepsilon})_\varepsilon \in \widetilde{\mathbb{R}}$ can clearly be written in the form $\left(\oint \delta A_{k,\varepsilon}\right)_\varepsilon$, where the Colombeau integral is taken over the given regular contour γ. Substituting in place of $(\delta A_{k,\varepsilon})_\varepsilon$the expression (), we have

$$(\Delta A_{k,\varepsilon})_\varepsilon = \left(\oint_\gamma \Gamma^i_{kl,\varepsilon}(x)A_i(x)dx^l\right)_\varepsilon \in \widetilde{\mathbb{R}}, \tag{2.6.1}$$

where for any $i,k,l = 0,1,2,3 : (\Gamma^i_{kl,\varepsilon}(x))_\varepsilon \in \mathcal{G}(\mathbb{R}^4), x = (x^0,x^1,x^2,x^3),\ A_i(x) \in \mathcal{D}(G)$, $G \subseteq \mathbb{R}^4$. Note that the vector A_i which appears in the integrand obviously changes as we move along the contour γ.

Definition 2.6.1. We will say that generalized change $(\Delta A_{k,\varepsilon})_\varepsilon$ exists in the sense of the

Schwartz distributions if for any $A_i(x) \in \mathcal{D}(G)$ the limit: $\lim_{\varepsilon\to 0}\Delta A_{k,\varepsilon}$ exists. Of course in this case obviously $\mathbf{cl}\left[(\Gamma^i_{kl,\varepsilon}(x))_\varepsilon\right] \in \mathcal{D}'(G)$ and $\mathbf{cl}[(\Delta A_{k,\varepsilon})_\varepsilon] \in \mathbb{R}$.

For the further transformation of this Colombeau integral, we must note the following.

The values of the vector A_i at points inside the contour are not unique; they depend on the path along which we approach the particular point. However, as we shall see from the result obtained below, this non-uniqueness is related to terms of second order. We may therefore, with the first-order accuracy which is sufficient for the transformation, regard the components of the vector A_i at points inside the infinitesimal contour γ as being uniquely determined by their values on the contour itself by the formulas $(\delta A_i(x))_\varepsilon = (\Gamma^n_{il,\varepsilon}(x)A_{n,\varepsilon}(x)dx^l)_\varepsilon$, i.e., by the derivatives

$$\frac{\partial A_{i,\varepsilon}(x)}{\partial x^l} = (\Gamma^n_{il,\varepsilon}(x)A_{n,\varepsilon}(x))_\varepsilon. \tag{2.6.2}$$

Now applying generalized Stokes' theorem (Theorem 2.6.1) to the integral (2.6.1) and considering that the area enclosed by the contour has the infinitesimal value $(\Delta f^{im}_\varepsilon)_\varepsilon$, we get:

$$\begin{aligned}(\Delta A_{k,\varepsilon})_\varepsilon &= \frac{1}{2}\left[\left(\frac{\partial(\Gamma^i_{km,\varepsilon}(x)A_i(x))}{\partial x^l}\right)_\varepsilon - \left(\frac{\partial(\Gamma^i_{kl,\varepsilon}(x)A_i(x))}{\partial x^m}\right)_\varepsilon\right](\Delta f^{im}_\varepsilon)_\varepsilon = \\ &= \frac{1}{2}\left[A_i(x)\left(\frac{\partial(\Gamma^i_{km,\varepsilon}(x))}{\partial x^l}\right)_\varepsilon - A_i(x)\left(\frac{\partial(\Gamma^i_{kl,\varepsilon}(x))}{\partial x^m}\right)_\varepsilon + \right. \\ &\left.\left(\frac{\partial A_i(x)}{\partial x^l}\right)(\Gamma^i_{km,\varepsilon}(x))_\varepsilon - \left(\frac{\partial A_i(x)}{\partial x^m}\right)(\Gamma^i_{kl,\varepsilon}(x))_\varepsilon\right](\Delta f^{im}_\varepsilon)_\varepsilon.\end{aligned} \tag{2.6.3}$$

Definition 2.6.2.Colombeau generalized k-form $(\omega_\varepsilon)_\varepsilon$ on a differentiable manifold M is a

smooth section of the bundle of alternating Colombeau generalized k-tensors on M. Equivalently, $(\omega_\varepsilon)_\varepsilon$ associates to each $x \in M$ an alternating Colombeau generalized k-tensor $(\omega_{x,\varepsilon})_\varepsilon$, in such a way that in any chart for M, the coefficients$(\omega_{i_1\ldots i_k,\varepsilon})_\varepsilon$ are Colombeau generalized functions.

Theorem 2.6.1.(Generalized Stokes' Theorem) Let $(\omega_\varepsilon)_\varepsilon$ be Colombeau generalized differential form.Then the Colombeau integral of a differential form $(\omega_\varepsilon)_\varepsilon$ over the boundary of some orientable manifold $\Sigma \subset M$ is equal to the integral of its exterior Colombeau derivative $(d\omega_\varepsilon)_\varepsilon$ over the whole of Σ, i.e.,

$$\int_{\partial\Sigma}(\omega_\varepsilon)_\varepsilon = \left(\int_{\partial\Sigma}\omega_\varepsilon\right)_\varepsilon = \left(\int_{\Sigma}d\omega_\varepsilon\right)_\varepsilon = \int_{\Sigma}(d\omega_\varepsilon)_\varepsilon. \tag{2.6.4}$$

Proof. Immediately from the classical Stokes' Theorem and definitions.

Example 2.6.1. For example, for the integral of Colombeau generalized vector $(A_{i,\varepsilon}(x))_\varepsilon$

we have

$$\begin{aligned}&\left(\oint_\Gamma A_{i,\varepsilon}dx^i\right)_\varepsilon = \left(\int_\Sigma df^{ki}\frac{\partial A_{i,\varepsilon}}{\partial x^k}\right)_\varepsilon = \frac{1}{2}\left(\int\left[\left(df^{ki}_\varepsilon\right)_\varepsilon\right]\left(\frac{\partial A_{k,\varepsilon}}{\partial x^i} - \frac{\partial A_{i,\varepsilon}}{\partial x^k}\right)\right)_\varepsilon = \\ &\frac{1}{2}\int\left[\left(df^{ki}_\varepsilon\right)_\varepsilon\right]\left(\frac{\partial A_{k,\varepsilon}}{\partial x^i} - \frac{\partial A_{i,\varepsilon}}{\partial x^k}\right)_\varepsilon = \frac{1}{2}\int\left[\left(df^{ki}_\varepsilon\right)_\varepsilon\right]\left[\left(\frac{\partial A_{k,\varepsilon}}{\partial x^i}\right)_\varepsilon - \left(\frac{\partial A_{i,\varepsilon}}{\partial x^k}\right)_\varepsilon\right],\end{aligned} \tag{2.6.5}$$

where $\Gamma = \partial\Sigma$ and $\left(df^{ki}_\varepsilon\right)_\varepsilon = (dx^i_\varepsilon dx'^k_\varepsilon)_\varepsilon - (dx^k_\varepsilon dx'^i_\varepsilon)_\varepsilon$ is the infinitesimal element of surface which is given by the antisymmetric tensor of second rank $\left(df^{ki}_\varepsilon\right)_\varepsilon$.

Substituting the values of the derivatives (2.6.2) into Eq.(2.6.3), we get

$$(\Delta A_{k,\varepsilon})_{\varepsilon} = \frac{1}{2}(R^{i}_{klm,\varepsilon}(x)A_{i,\varepsilon}(x)\Delta f^{im}_{\varepsilon})_{\varepsilon}, \qquad (2.6.6)$$

where $(R^{i}_{klm,\varepsilon}(x))_{\varepsilon}$ is a Colombeau generalized tensor field of the fourth rank:

$$\begin{aligned}(R^{i}_{klm,\varepsilon}(x))_{\varepsilon} = \left(\frac{\partial(\Gamma^{i}_{km,\varepsilon}(x))}{\partial x^{l}}\right)_{\varepsilon} - \left(\frac{\partial(\Gamma^{i}_{kl,\varepsilon}(x))}{\partial x^{m}}\right)_{\varepsilon} + \\ (\Gamma^{i}_{ni,\varepsilon}(x)\Gamma^{n}_{km,\varepsilon}(x))_{\varepsilon} - (\Gamma^{i}_{nm,\varepsilon}(x)\Gamma^{n}_{kl,\varepsilon}(x))_{\varepsilon}.\end{aligned} \qquad (2.6.7)$$

Definition 2.6.3.The tensor field $(R^{l}_{kim,\varepsilon}(x))_{\varepsilon}$ is called the distributional curvature tensor or

the distributional Riemann tensor.

Remark 2.6.1.Note that in general case for any $i,k,l = 0,1,2,3$:
$\mathbf{cl}\left[(R^{i}_{klm,\varepsilon}(x))_{\varepsilon}\right] \in \mathcal{G}(\mathbb{R}^{4})$.

Definition 2.6.4.We will say that the distributional Riemann tensor $(R^{i}_{klm,\varepsilon}(x))_{\varepsilon}$exists in the

sense of the Schwartz distributions if for any $i,k,l = 0,1,2,3$ and for any $A_{i}(x) \in \mathcal{D}(G)$ the

limit:

$$\lim_{\varepsilon\to 0}\int_{G} R^{i}_{klm,\varepsilon}(x)A_{i}(x)d^{4}x \qquad (2.6.8)$$

exists.

Definition 2.6.5.We will say that the distributional Riemann tensor $(R^{i}_{klm,\varepsilon}(x))_{\varepsilon}$exists in the

classical sense at point $\breve{x} \in \mathbb{R}$ if there exists standard part of point value of Colombeau generalized function $(R^{i}_{klm,\varepsilon}(x))_{\varepsilon}$ at point $\breve{x} \in \mathbb{R}$,i.e. $\mathbf{st}\left(\mathbf{cl}\left[(R^{i}_{klm,\varepsilon}(\breve{x}))_{\varepsilon}\right]\right) \in \mathbb{R}$.

Remark 2.6.2.It is easy to obtain a similar formula to (2.6.6) for a generalized contravariant vector $(A^{k}_{\varepsilon})_{\varepsilon}$. To do this we note, since under parallel displacement a scalar

does not change, that $(\Delta(A^{k}_{\varepsilon}B_{k,\varepsilon}))_{\varepsilon} = 0_{\widetilde{\mathbb{R}}}$, where $(B_{k,\varepsilon})_{\varepsilon}$ is any Colombeau generalized covariant vector. With the help of (91.3), we obtain

$$\begin{aligned}(\Delta(A^{k}_{\varepsilon}B_{k,\varepsilon}))_{\varepsilon} = (\Delta A^{k}_{\varepsilon}B_{k,\varepsilon})_{\varepsilon} + (A^{k}_{\varepsilon}\Delta B_{k,\varepsilon})_{\varepsilon} = \tfrac{1}{2}\left(A^{k}_{\varepsilon}B_{i,\varepsilon}R^{i}_{klm,\varepsilon}\Delta f^{lm}_{\varepsilon}\right)_{\varepsilon} + (B_{k,\varepsilon}\Delta A^{k}_{\varepsilon})_{\varepsilon} = \\ \{(B_{k,\varepsilon})_{\varepsilon}\}\left((\Delta A^{k}_{\varepsilon})_{\varepsilon} + \tfrac{1}{2}\left(A^{l}_{\varepsilon}R^{k}_{ilm,\varepsilon}\Delta f^{lm}_{\varepsilon}\right)_{\varepsilon}\right) = 0_{\widetilde{\mathbb{R}}}\end{aligned} \qquad (2.6.9)$$

or, in view of the arbitrariness of the vector $(B_{k,\varepsilon})_{\varepsilon}$

2.7. Properties of the distributional Curvature Tensor

From the expression (2.6.5) it follows directly that $\forall x \in \mathbb{R}$ the distributional curvature tensor is antisymmetric in the indices l and m :

$$(R^i_{klm,\varepsilon}(x))_\varepsilon = -(R^i_{kml,\varepsilon}(x))_\varepsilon \quad (2.7.1)$$

and therefore $\forall z = (x_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}_{\mathbf{fin}}$ the following identity holds

$$(R^i_{klm,\varepsilon}(x_\varepsilon))_\varepsilon = -(R^i_{kml,\varepsilon}(x_\varepsilon))_\varepsilon. \quad (2.7.1.a)$$

Obviously the following identity holds

$$(R^l_{kim,\varepsilon}(x))_\varepsilon + (R^i_{mkl,\varepsilon}(x))_\varepsilon + (R^i_{lmk,\varepsilon}(x)) = 0_{\widetilde{\mathbb{R}}} \quad (2.7.2)$$

In addition to the mixed distributional curvature tensor $(R^i_{klm,\varepsilon}(x))_\varepsilon$, one also uses the covariant distributional curvature tensor

$$(R_{iklm,\varepsilon}(x))_\varepsilon = (g_{in,\varepsilon}(x)R^n_{klm,\varepsilon}(x))_\varepsilon = ((g_{in,\varepsilon}(x))_\varepsilon)\left((R^n_{klm,\varepsilon}(x))_\varepsilon\right) \quad (2.7.3)$$

Obviously the following expressions for $(R_{iklm,\varepsilon}(x))_\varepsilon$ holds

$$(R_{iklm,\varepsilon}(x))_\varepsilon =$$
$$\frac{1}{2}\left(\left(\frac{\partial^2 g_{im,\varepsilon}(x)}{\partial x^k \partial x^l}\right)_\varepsilon + \left(\frac{\partial^2 g_{kl,\varepsilon}(x)}{\partial x^i \partial x^m}\right)_\varepsilon - \left(\frac{\partial^2 g_{il,\varepsilon}(x)}{\partial x^k \partial x^m}\right)_\varepsilon - \left(\frac{\partial^2 g_{km,\varepsilon}(x)}{\partial x^i \partial x^l}\right)_\varepsilon\right) + \quad (2.7.4)$$
$$+((g_{np,\varepsilon})_\varepsilon)\left[(\Gamma^n_{kl,\varepsilon}\Gamma^p_{im,\varepsilon})_\varepsilon - (\Gamma^n_{km,\varepsilon}\Gamma^p_{il,\varepsilon})_\varepsilon\right]$$

From this expression it follows

$$(R_{iklm,\varepsilon}(x))_\varepsilon = -(R_{kilm,\varepsilon}(x))_\varepsilon = -(R_{ikml,\varepsilon}(x))_\varepsilon,$$
$$(R_{iklm,\varepsilon}(x))_\varepsilon = (R_{lmik,\varepsilon}(x))_\varepsilon. \quad (2.7.5)$$

For $(R_{iklm,\varepsilon}(x))_\varepsilon$ the following identity holds

$$(R_{iklm,\varepsilon}(x))_\varepsilon \quad (2.7.6)$$

.

2.8. The action functional for the gravitational field

The action $\mathbf{S}[(g_\varepsilon)_\varepsilon]$ obviously must be expressed in terms of an scalar Colombeau integral taken over all space and over the time coordinate x^0 between two given values,i.e.

$$\mathbf{S}[(g_\varepsilon)_\varepsilon] = \left(\int G_\varepsilon \sqrt{-g_\varepsilon}\, d\Omega\right)_\varepsilon = \int (G_\varepsilon)_\varepsilon \sqrt{-(g_\varepsilon)_\varepsilon}\, d\Omega. \quad (2.8.1)$$

In order to determine this Colombeau scalar we assume that the equations of the gravitational field must contain derivatives of the "potentials" no higher than the second order. Since the field equations are obtained by varying the action, it is necessary that the integrand $(G_\varepsilon)_\varepsilon$ contain derivatives of $(g_{ik,\varepsilon})_\varepsilon$ no higher than first order; thus $(G_\varepsilon)_\varepsilon$ must contain only the tensor $(g_{ik,\varepsilon})_\varepsilon$ and the quantities $(\Gamma^l_{kl,\varepsilon})_\varepsilon$.However, it is impossible to construct an invariant from the quantities $(g_{ik,\varepsilon})_\varepsilon$ and $(\Gamma^l_{kl,\varepsilon})_\varepsilon$ alone.This is clear from the fact that by a suitable choice of generalized coordinate system we can always make all the quantities $(\Gamma^{l}_{kl,\varepsilon})_{\varepsilon}$ zero at a given point. There is, however, the Colombeau scalar $(R_{\varepsilon})_{\varepsilon}$(the generalized curvature of the four-space), which though it contains in addition to the $(g_{ik,\varepsilon})_{\varepsilon}$ and its first derivatives also the second derivatives of $(g_{ik,\varepsilon})_{\varepsilon}$, is linear in the second order Colombeau derivatives. The action functional for the gravitational field reads

$$\left(\int R_{\varepsilon}\sqrt{-g_{\varepsilon}}\,d\Omega\right)_{\varepsilon}. \tag{2.8.2}$$

The invariant Colombeau integral (2.8.2) can be transformed by means of Gauss'theorem to the integral of an expression not containing the second derivatives. Thus Colombeau integral (2.8.2) can be presented in the following form

$$\left(\int R_{\varepsilon}\sqrt{-g_{\varepsilon}}\,d\Omega\right)_{\varepsilon} = \left(\int G_{\varepsilon}\sqrt{-g_{\varepsilon}}\,d\Omega\right)_{\varepsilon} + \left(\int \frac{\partial\left(\sqrt{-g_{\varepsilon}}\,w^{i}_{\varepsilon}\right)}{\partial x^{i}}d\Omega\right)_{\varepsilon}, \tag{2.8.3}$$

where $(G_{\varepsilon})_{\varepsilon}$ contains only the tensor $(g_{ik,\varepsilon})_{\varepsilon}$ and its first derivatives, and the integrand of the second integral has the form of a divergence of a certain quantity wl (the detailed calculation is given at the end of this section). According to Gauss' theorem, this second integral can be transformed into an integral over a hypersurface surrounding the four-volume over which the integration is carried out in the other two integrals. When we vary the action, the variation of the second term on the right vanishes, since in the principle of least action, the variations of the field at the limits of the region of integration are zero. Consequently, we may write

$$\delta\left(\int R_{\varepsilon}\sqrt{-g_{\varepsilon}}\,d\Omega\right)_{\varepsilon} = \left(\delta\int R_{\varepsilon}\sqrt{-g_{\varepsilon}}\,d\Omega\right)_{\varepsilon} = \left(\delta\int G_{\varepsilon}\sqrt{-g_{\varepsilon}}\,d\Omega\right)_{\varepsilon}. \tag{2.8.4}$$

The left side is Colombeau scalar; therefore the expression on the right is also Colombeau scalar (the quantity $(G_{\varepsilon})_{\varepsilon}$ itself is, of course, not Colombeau scalar). The quantity $(G_{\varepsilon})_{\varepsilon}$ satisfies the condition imposed above, since it contains only the $(g_{ik,\varepsilon})_{\varepsilon}$ and its Colombeau derivatives. Thus finally we obtain

$$\delta\mathbf{S}[(g_{\varepsilon})_{\varepsilon}] = -\frac{c^{3}}{16\pi\kappa}\left(\delta\int G_{\varepsilon}\sqrt{-g_{\varepsilon}}\,d\Omega\right)_{\varepsilon} = -\frac{c^{3}}{16\pi k}\left(\delta\int R_{\varepsilon}\sqrt{-g_{\varepsilon}}\,d\Omega\right)_{\varepsilon}. \tag{2.8.5}$$

The constant κ is called the gravitational constant. The dimensions of κ follow from (2.8.5). Its numerical value is $\kappa = 6.67\times 10^{-8}$sm^{3} $\times$gr^{-1} $\times$ sec^{-2}.

Let us calculate now the quantity $(G_{\varepsilon})_{\varepsilon}$ of (2.8.5). From the expression for $(R_{ik,\varepsilon})_{\varepsilon}$,we get

$$\begin{gathered}\left(\sqrt{-g_{\varepsilon}}\,R_{\varepsilon}\right)_{\varepsilon} = \left(\sqrt{-g_{\varepsilon}}\,g^{ik}_{\varepsilon}R_{ik,\varepsilon}\right)_{\varepsilon} = \left\{\left(\sqrt{-g_{\varepsilon}}\right)_{\varepsilon}\right\}\times\\ \left\{\left(g^{ik}_{\varepsilon}\frac{\partial\Gamma^{l}_{ik,\varepsilon}}{\partial x^{l}}\right)_{\varepsilon} - \left(g^{ik}_{\varepsilon}\frac{\partial\Gamma^{l}_{il,\varepsilon}}{\partial x^{k}}\right)_{\varepsilon} + (g^{ik}_{\varepsilon}\Gamma^{l}_{ik,\varepsilon}\Gamma^{m}_{lm,\varepsilon})_{\varepsilon} + (g^{ik}_{\varepsilon}\Gamma^{m}_{il,\varepsilon}\Gamma^{l}_{km,\varepsilon})_{\varepsilon}\right\}.\end{gathered} \tag{2.8.6}$$

In the first two terms on the rhs of (2.8.6), we get

$$\begin{aligned}\left(\sqrt{-g_{\varepsilon}}\,g^{ik}_{\varepsilon}\frac{\partial\Gamma^{l}_{ik,\varepsilon}}{\partial x^{l}}\right)_{\varepsilon} &= \frac{\partial}{\partial x^{l}}\left(\sqrt{-g_{\varepsilon}}\,g^{ik}_{\varepsilon}\Gamma^{l}_{ik,\varepsilon}\right)_{\varepsilon} - \left\{(\Gamma^{l}_{ik,\varepsilon})_{\varepsilon}\right\}\frac{\partial}{\partial x^{l}}\left(\sqrt{-g_{\varepsilon}}\,g^{ik}_{\varepsilon}\right)_{\varepsilon}\\ \left(\sqrt{-g_{\varepsilon}}\,g^{ik}_{\varepsilon}\frac{\partial\Gamma^{l}_{il,\varepsilon}}{\partial x^{k}}\right)_{\varepsilon} &= \frac{\partial}{\partial x^{k}}\left(\sqrt{-g_{\varepsilon}}\,g^{ik}_{\varepsilon}\Gamma^{l}_{il,\varepsilon}\right)_{\varepsilon} - \left\{(\Gamma^{l}_{il,\varepsilon})_{\varepsilon}\right\}\frac{\partial}{\partial x^{k}}\left(\sqrt{-g_{\varepsilon}}\,g^{ik}_{\varepsilon}\right)_{\varepsilon}.\end{aligned} \tag{2.8.7}$$

Dropping the total derivatives in the Eqs.(2.8.7), we find

$$\left(\sqrt{-g_\varepsilon}\,G_\varepsilon\right)_\varepsilon = \{(\Gamma^m_{im,\varepsilon})_\varepsilon\}\frac{\partial}{\partial x^k}\left(\sqrt{-g_\varepsilon}\,g^{ik}_\varepsilon\right)_\varepsilon - \left\{(\Gamma^l_{ik,\varepsilon})_\varepsilon\right\}\frac{\partial}{\partial x^l}\left(\sqrt{-g_\varepsilon}\,g^{ik}_\varepsilon\right)_\varepsilon. \qquad (2.8.8)$$

By using now formulas (2.)-(2.), we find that the first two terms on the right are equal to $\left(\sqrt{-g_\varepsilon}\right)_\varepsilon$ multiplied by

$$(2.8.9)$$

Finally, we get

$$G_\varepsilon = \left\{(g^{ik}_\varepsilon)_\varepsilon\right\}\left[(\Gamma^m_{im,\varepsilon}\Gamma^l_{km,\varepsilon})_\varepsilon - (\Gamma^l_{ik,\varepsilon}\Gamma^m_{lm,\varepsilon})_\varepsilon\right]. \qquad (2.8.10)$$

Remark 2.8.1. Note that when the principle of least action is applied to a gravitational field,

we

2.9. The distributional energy-momentum tensor.

In this subsection we consider the general rule for calculatingthe energy-momentum tensor of any physical system whose action is given in the form of an Colombeau integral over four dimensional distributional spase time. This integral reads

$$(\mathbf{S}_{m,\varepsilon})_\varepsilon = \frac{1}{c}\left(\int \Lambda_\varepsilon(x)\sqrt{-g_\varepsilon(x)}\,d\Omega\right)_\varepsilon. \qquad (2.9.1)$$

Using generalized Gauss' theorem, and setting $(\delta g^{ik}_\varepsilon)_\varepsilon = 0_{\widetilde{\mathbb{R}}}$ at the integration limits, one finds $\delta(S_{m,\varepsilon})_\varepsilon$ in the form,(see Remark 2.9.2):

$$\delta(\mathbf{S}_{m,\varepsilon})_\varepsilon =$$

$$\frac{1}{c}\int\left\{\left(\frac{\partial\sqrt{-g_\varepsilon}\,\Lambda_\varepsilon}{\partial g^{ik}_\varepsilon}\delta g^{ik}_\varepsilon\right)_\varepsilon + \left(\frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g_\varepsilon}\,\Lambda_\varepsilon}{\partial\frac{\partial g^{ik}_\varepsilon}{\partial x^l}}\delta\frac{\partial g^{ik}_\varepsilon}{\partial x^l}\right)_\varepsilon\right\}d\Omega =$$

$$\frac{1}{c}\int\left\{\left(\frac{\partial\sqrt{-g_\varepsilon}\,\Lambda_\varepsilon}{\partial g^{ik}_\varepsilon}\right)_\varepsilon - \left(\frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g_\varepsilon}\,\Lambda_\varepsilon}{\partial\frac{\partial g^{ik}_\varepsilon}{\partial x^l}}\delta\frac{\partial g^{ik}_\varepsilon}{\partial x^l}\right)_\varepsilon\right\}\left\{(\delta g^{ik}_\varepsilon)_\varepsilon\right\}d\Omega \qquad (2.9.2)$$

We introduce now the notation

$$\frac{1}{2}\left(\sqrt{-g_\varepsilon}\,T_{ik,\varepsilon}\right)_\varepsilon = \left(\frac{\partial\sqrt{-g_\varepsilon}\,\Lambda_\varepsilon}{\partial g^{ik}_\varepsilon}\right)_\varepsilon - \left(\frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g_\varepsilon}\,\Lambda_\varepsilon}{\partial\frac{\partial g^{ik}_\varepsilon}{\partial x^l}}\right)_\varepsilon, \qquad (2.9.3)$$

then $\delta(S_{m,\varepsilon})_\varepsilon$ reads

$$\delta(S_{m,\varepsilon})_\varepsilon = \frac{1}{2c}\left(\int T_{ik,\varepsilon}\sqrt{-g_\varepsilon}\,\delta g^{ik}_\varepsilon d\Omega\right)_\varepsilon = -\frac{1}{2c}\left(\int T^{ik}_\varepsilon\sqrt{-g_\varepsilon}\,\delta g_{ik,\varepsilon}d\Omega\right)_\varepsilon, \qquad (2.9.4)$$

note that $(T_{ik,\varepsilon}\delta g^{ik}_\varepsilon)_\varepsilon = -(T^{ik}_\varepsilon\delta g_{ik,\varepsilon})_\varepsilon$ since $(g^{ik}_\varepsilon\delta g_{ik,\varepsilon})_\varepsilon = -(g_{ik,\varepsilon}\delta g^{ik}_\varepsilon)_\varepsilon$.

Remark 2.9.1. In (2.9.1) we carry out now a transformation from the coordinates x^i to the

coordinates $(x'^i_\varepsilon)_\varepsilon = x^i + (\xi^i_\varepsilon)_\varepsilon$, where the $(\xi^i_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}$ are small quantities. Under this transformation the $(g_\varepsilon^{ik})_\varepsilon$ are transformed according to the formulas

$$
\begin{gathered}
(g_\varepsilon'^{ik}(x_\varepsilon'^{i}))_\varepsilon = \left(g_\varepsilon^{im}(x_\varepsilon^{i})\frac{\partial x_\varepsilon'^{i}}{\partial x^l}\frac{\partial x_\varepsilon'^{k}}{\partial x^m}\right)_\varepsilon = \\
\{(g_\varepsilon^{im})_\varepsilon\}\left(\delta_l^i + \left(\frac{\partial \xi_\varepsilon^i}{\partial x^l}\right)_\varepsilon\right)\left(\delta_m^k + \left(\frac{\partial \xi_\varepsilon^k}{\partial x^m}\right)_\varepsilon\right) \simeq \\
(g_\varepsilon^{ik}(x^l))_\varepsilon + \{(g_\varepsilon^{im})_\varepsilon\}\left(\frac{\partial \xi_\varepsilon^k}{\partial x^m}\right)_\varepsilon + \left\{(g_\varepsilon^{kl})_\varepsilon\right\}\left(\frac{\partial \xi_\varepsilon^i}{\partial x^l}\right)_\varepsilon,
\end{gathered}
\tag{2.9.5}
$$

were the tensor $(g_\varepsilon'^{ik}(x_\varepsilon'^{i}))_\varepsilon$is a function of the $(x_\varepsilon'^{l})_\varepsilon$, while the tensor $(g_\varepsilon^{ik})_\varepsilon$ is a function of
the original coordinates x^l. In order to represent all terms as functions of one and the same variables, we expand $(g_\varepsilon'^{ik}(x^l + \xi_\varepsilon^l))_\varepsilon$ in powers of $(\xi_\varepsilon^l)_\varepsilon$. Furthermore, if we neglect
terms of higher order in $(\xi_\varepsilon^l)_\varepsilon$,we can in all terms containing $(\xi_\varepsilon^l)_\varepsilon$, replace $(g_\varepsilon'^{ik})_\varepsilon$ by $(g_\varepsilon^{ik})_\varepsilon$.

Thus we obtain

$$
(g_\varepsilon'^{ik}(x^l))_\varepsilon = (g_\varepsilon^{ik}(x^l))_\varepsilon - \left\{(\xi_\varepsilon^l)_\varepsilon\right\}\left(\frac{\partial g_\varepsilon^{ik}}{\partial x^l}\right)_\varepsilon + \left\{(g_\varepsilon^{il})_\varepsilon\right\}\left(\frac{\partial \xi_\varepsilon^k}{\partial x^l}\right)_\varepsilon + \left\{(g_\varepsilon^{kl})_\varepsilon\right\}\left(\frac{\partial \xi_\varepsilon^i}{\partial x^l}\right)_\varepsilon \tag{2.9.6}
$$

It is easy to verify by direct trial that the last three terms on the right can be written as a
sum $(\xi_\varepsilon^{i;k})_\varepsilon + (\xi_\varepsilon^{k;i})_\varepsilon$ of contravariant derivatives of the $(\xi_\varepsilon^i)_\varepsilon$. Thus we finally obtain the transformation of the $(g_\varepsilon^{ik})_\varepsilon$ in the form

$$
(g_\varepsilon'^{ik})_\varepsilon = (g_\varepsilon^{ik})_\varepsilon + (\delta g_\varepsilon^{ik})_\varepsilon, (\delta g_\varepsilon^{ik})_\varepsilon = (\xi_\varepsilon^{i;k})_\varepsilon + (\xi_\varepsilon^{k;i})_\varepsilon. \tag{2.9.7}
$$

For the covariant components, we obtain

$$
(g_{ik,\varepsilon}')_\varepsilon = (g_{ik,\varepsilon})_\varepsilon + (\delta g_{ik,\varepsilon})_\varepsilon, (\delta g_{ik,\varepsilon})_\varepsilon = -(\xi_{i;k,\varepsilon})_\varepsilon - (\xi_{k;i,\varepsilon})_\varepsilon, \tag{2.9.8}
$$

so that, to terms of first order we satisfy the condition $(g_{il,\varepsilon} g_\varepsilon'^{kl})_\varepsilon = \delta_i^k$.

Remark 2.9.2. Since the action $(\mathbf{S}_{m,\varepsilon})_\varepsilon$ is a Colombeau scalar, it does not change under a
transformation of coordinates.On the other hand, the change $\delta(\mathbf{S}_{m,\varepsilon})_\varepsilon$ in the action under
a transformation of coordinates can be written in the following form:let $(q_\varepsilon)_\varepsilon$ denote the
quantities defining the physical system to which the action $(\mathbf{S}_{m,\varepsilon})_\varepsilon$ applies. Under coordinate transformation the quantities $(q_\varepsilon)_\varepsilon$ change by $(\delta q_\varepsilon)_\varepsilon$. In calculating $\delta(\mathbf{S}_{m,\varepsilon})_\varepsilon = (\delta \mathbf{S}_{m,\varepsilon})_\varepsilon$ we need not write terms containing the changes in $(q_\varepsilon)_\varepsilon$. All such terms must cancel each other by virtue of the "equations of motion" of the physical system, since these equations are obtained by equating to zero the variation of $(\mathbf{S}_{m,\varepsilon})_\varepsilon$ with
respect to the quantities $(q_\varepsilon)_\varepsilon$. Therefore it is sufficient to write the terms associated with
changes in the $(g_{ik,\varepsilon})_\varepsilon$. Using Gauss' theorem, and setting $(\delta g_\varepsilon^{ik})_\varepsilon = 0_{\widetilde{\mathbb{R}}}$ at the integration
limits, one finds $\delta(S_{m,\varepsilon})_\varepsilon$ in the form of the Eq.(2.9.2).

Remark 2.9.3.We note that the equations

$$(\xi_\varepsilon^{i;k})_\varepsilon + (\xi_\varepsilon^{k;i})_\varepsilon = 0 \qquad (2.9.9)$$

determine the infinitesimal generalized coordinate transformations that do not change the

Colombeau generalized metric. These are called the distributional Killing equations.

2.10. The generalized gravitational field equations.

We now proceed to the derivation of the equations of the gravitational field. These equations are obtained from the principle of least action $\delta((S_{m,\varepsilon})_\varepsilon + (S_{g_\varepsilon})_\varepsilon) = 0_{\widetilde{\mathbb{R}}}$, where $(S_{m,\varepsilon})_\varepsilon$ and $(S_{g_\varepsilon})_\varepsilon$are the distributional actions of the gravitational field and matter respectively. We now subject the gravitational Colombeau metric field, that is,the quantities g_{ik}, to variation. Calculating the variation $\delta(S_{g_\varepsilon})_\varepsilon$, we get

$$\begin{gathered}\delta\left(\int R_\varepsilon \sqrt{-g_\varepsilon}\, d\Omega\right)_\varepsilon = \left(\delta \int R_\varepsilon \sqrt{-g_\varepsilon}\, d\Omega\right)_\varepsilon = \left(\delta \int g_\varepsilon^{ik} R_{ik,\varepsilon} \sqrt{-g_\varepsilon}\, d\Omega\right)_\varepsilon = \\ \left\{\left(\int R_{ik,\varepsilon} \sqrt{-g_\varepsilon}\, \delta g_\varepsilon^{ik} d\Omega\right)_\varepsilon + \left(\int R_{ik,\varepsilon} g_\varepsilon^{ik} \delta \sqrt{-g_\varepsilon}\, d\Omega\right)_\varepsilon + \left(\int g_\varepsilon^{ik} \sqrt{-g_\varepsilon}\, \delta R_{ik,\varepsilon} d\Omega\right)_\varepsilon\right\} \qquad (2.10.1) \\ \int \left\{\left(R_{ik,\varepsilon} \sqrt{-g_\varepsilon}\, \delta g_\varepsilon^{ik}\right)_\varepsilon + \left(R_{ik,\varepsilon} g_\varepsilon^{ik} \delta \sqrt{-g_\varepsilon}\right)_\varepsilon + \left(g_\varepsilon^{ik} \sqrt{-g_\varepsilon}\, \delta R_{ik,\varepsilon}\right)_\varepsilon\right\} d\Omega.\end{gathered}$$

From formula (), we obtain

$$\left(\delta \sqrt{-g_\varepsilon}\right)_\varepsilon = -\left\{\left(\frac{1}{2\sqrt{-g_\varepsilon}}\right)_\varepsilon\right\}(\delta g_\varepsilon)_\varepsilon = -\frac{1}{2}\left\{\left(\sqrt{-g_\varepsilon}\right)_\varepsilon\right\}(g_{ik,\varepsilon}\delta g_\varepsilon^{ik})_\varepsilon. \qquad (2.10.2)$$

Substituting Eq.(2.10.2) into Eq.(2.10.1), we obtain

$$\begin{gathered}\delta\left(\int R_\varepsilon \sqrt{-g_\varepsilon}\, d\Omega\right)_\varepsilon = \left(\int \left\{R_{ik,\varepsilon} - \frac{1}{2} g_{ik,\varepsilon} R_\varepsilon\right\} \sqrt{-g_\varepsilon}\, \delta g_\varepsilon^{ik} d\Omega\right)_\varepsilon + \\ + \left(\int g_\varepsilon^{ik} \sqrt{-g_\varepsilon}\, \delta R_{ik,\varepsilon} d\Omega\right)_\varepsilon. \end{gathered} \qquad (2.10.3)$$

Remark 2.10.1.Note that although the quantities $(\Gamma^i_{kl,\varepsilon})_\varepsilon$ don't constitute a tensor, their

variations $\delta(\Gamma^i_{kl,\varepsilon})_\varepsilon$ do form a generalized tensor, for $(\Gamma^i_{kl,\varepsilon} A_k dx^l)_\varepsilon$is the change in a vector

under parallel displacement [see (85.5)] from some point P to an infinitesimally separated

point P' and therefore $(\Gamma^i_{kl,\varepsilon} A_k dx^l)_\varepsilon$is the difference between the two vectors, obtained as

the result of two parallel displacements (one with the unvaried, the other with the varied $(\Gamma^i_{kl,\varepsilon})_\varepsilon$from the point P to one and the same point P'.The difference between two vectors

at the same point is a vector, and therefore $\delta(\Gamma^i_{kl,\varepsilon})_\varepsilon = (\delta\Gamma^i_{kl,\varepsilon})_\varepsilon$ is a generalized tensor.

Let us use now a locally geodesic system of a generalized coordinates. Then at that point all the $(\Gamma^i_{kl,\varepsilon})_\varepsilon = 0_{\widetilde{\mathbb{R}}}$.With the help ofexpression (92.10) for the $(R_{ik,\varepsilon})_\varepsilon$, we obtain (we remind that the first derivatives of the $(g_\varepsilon^{ik})_\varepsilon$ are now equal to $0_{\widetilde{\mathbb{R}}}$)

$$(g_\varepsilon^{ik}\delta R_{ik,\varepsilon})_\varepsilon = \{(g_\varepsilon^{ik})_\varepsilon\}\left\{\left(\frac{\partial}{\partial x^l}\delta\Gamma^l_{ik,\varepsilon}\right)_\varepsilon - \left(\frac{\partial}{\partial x^k}\delta\Gamma^l_{il,\varepsilon}\right)_\varepsilon\right\} = \left(g_\varepsilon^{ik}\frac{\partial}{\partial x^l}\delta\Gamma^l_{ik,\varepsilon}\right)_\varepsilon - \left(g_\varepsilon^{il}\frac{\partial}{\partial x^l}\delta\Gamma^k_{ik,\varepsilon}\right)_\varepsilon = \left(\frac{\partial w_\varepsilon^l}{\partial x^l}\right)_\varepsilon \tag{2.10.4}$$

where

$$(w_\varepsilon^l)_\varepsilon = (g_\varepsilon^{ik}\delta\Gamma^l_{ik,\varepsilon})_\varepsilon - (g_\varepsilon^{il}\delta\Gamma^k_{ik,\varepsilon})_\varepsilon. \tag{2.10.5}$$

Since $(w_\varepsilon^l)_\varepsilon$is a generalized vector, replacing $(dw_\varepsilon^l/dx^l)_\varepsilon$ by $\left(w^l_{;\,l,\varepsilon}\right)_\varepsilon$ and using (86.9),we may write the relation we have obtained above, in any generalized coordinate system, in the form

$$(g_\varepsilon^{ik}\delta R_{ik,\varepsilon})_\varepsilon = \left\{\left(\frac{1}{\sqrt{-g_\varepsilon}}\right)_\varepsilon\right\}\frac{\partial}{\partial x^l}\left(\sqrt{-g_\varepsilon}\,w_\varepsilon^l\right)_\varepsilon. \tag{2.10.6}$$

Therefore the second Colombeau integral on the right side of (2.10.3) is equal to

$$\left(\int g_\varepsilon^{ik}\sqrt{-g_\varepsilon}\,\delta R_{ik,\varepsilon}d\Omega\right)_\varepsilon = \left(\int \frac{\partial\sqrt{-g_\varepsilon}\,w_\varepsilon^l}{\partial x^l}d\Omega\right)_\varepsilon \tag{2.10.7}$$

and by generalized Gauss' theorem can be transformed into an Colombeau integral of $(w_\varepsilon^l)_\varepsilon$ over the hypersurface surrounding the whole four-volume. Since the variations of the field are zero at the integration limits, this term drops out. Thus, the variation $\mathbf{S}[(g_\varepsilon)_\varepsilon]$ is equal to

$$\mathbf{S}[(g_\varepsilon)_\varepsilon] = -\frac{c^3}{16\pi\kappa}\left(\int\left\{R_{ik,\varepsilon} - \frac{1}{2}g_{ik,\varepsilon}R_\varepsilon\right\}\sqrt{-g_\varepsilon}\,\delta g_\varepsilon^{ik}d\Omega\right)_\varepsilon. \tag{2.10.8}$$

Remark 2.10.2.We note that if we had started from the expression

$$\delta\mathbf{S}_g[(g_\varepsilon)_\varepsilon] = -\frac{c^3}{16\pi\kappa}\left(\delta\int G_\varepsilon\sqrt{-g_\varepsilon}\,d\Omega\right)_\varepsilon \tag{2.10.9}$$

for the action of the field, then we get

$$\delta\mathbf{S}[(g_\varepsilon)_\varepsilon] = -\frac{c^3}{16\pi\kappa}\int\delta(g_\varepsilon^{ik})_\varepsilon d\Omega\left\{\left(\frac{\partial\{G_\varepsilon\sqrt{-g_\varepsilon}\}}{\partial g_\varepsilon^{ik}}\right)_\varepsilon - \left(\frac{\partial}{\partial x^l}\frac{\partial\{G_\varepsilon\sqrt{-g_\varepsilon}\}}{\partial\frac{\partial g_\varepsilon^{ik}}{\partial x^l}}\right)_\varepsilon\right\}. \tag{2.10.10}$$

Comparing Eq.(2.10.10) with Eq.(2.10.8), we get

$$(R_{ik,\varepsilon})_\varepsilon - \frac{1}{2}(g_{ik,\varepsilon}R_\varepsilon)_\varepsilon = \left\{\left(\frac{1}{\sqrt{-g_\varepsilon}}\right)_\varepsilon\right\}\left\{\left(\frac{\partial\{G_\varepsilon\sqrt{-g_\varepsilon}\}}{\partial g_\varepsilon^{ik}}\right)_\varepsilon - \left(\frac{\partial}{\partial x^l}\frac{\partial\{G_\varepsilon\sqrt{-g_\varepsilon}\}}{\partial\frac{\partial g_\varepsilon^{ik}}{\partial x^l}}\right)_\varepsilon\right\}. \tag{2.10.11}$$

For the variation of the action of the matter we can write immediately from (94.5):

$$(\delta\mathbf{S}_{m,\varepsilon})_\varepsilon = \frac{1}{2c}\left(\int T_{ik,\varepsilon}\sqrt{-g_\varepsilon}\,\delta g_\varepsilon^{ik}d\Omega\right)_\varepsilon, \tag{2.10.12}$$

where $(T_{ik,\varepsilon})_\varepsilon \in \mathcal{G}(\mathbb{R}^4)$ is the generalized energy-momentum tensor of the matter fields.

Thus, from the principle of least action

$$\delta\{\mathbf{S}[(g_\varepsilon)_\varepsilon] + (\mathbf{S}_{m,\varepsilon})_\varepsilon\} = 0_{\widetilde{\mathbb{R}}} \quad (2.10.13)$$

one obtains

$$-\frac{c^3}{16\pi\kappa}\left(\int\left\{R_{ik,\varepsilon} - \frac{1}{2}g_{ik,\varepsilon}R_\varepsilon - \frac{8\pi\kappa}{c^4}T_{ik,\varepsilon}\right\}\sqrt{-g_\varepsilon}\,\delta g_\varepsilon^{ik}d\Omega\right)_\varepsilon. \quad (2.10.14)$$

From Eq.(2.10.14), since of the arbitrariness of the $(\delta g_\varepsilon^{ik})_\varepsilon \in \mathcal{G}(\mathbb{R}^4)$ finally we get

$$(R_{ik,\varepsilon})_\varepsilon - \frac{1}{2}(g_{ik,\varepsilon}R_\varepsilon)_\varepsilon = \frac{8\pi\kappa}{c^4}(T_{ik,\varepsilon})_\varepsilon \quad (2.10.15)$$

or, in mixed components,

$$(R_{i,\varepsilon}^k)_\varepsilon - \frac{1}{2}\delta_i^k(R_\varepsilon)_\varepsilon = \frac{8\pi\kappa}{c^4}(T_{i,\varepsilon}^k)_\varepsilon. \quad (2.10.16)$$

They are called the generalized Einstein equations.

Contracting (2.10.16) on the indices i and k, we get

$$(R_\varepsilon)_\varepsilon = -\frac{8\pi\kappa}{c^4}(T_{i,\varepsilon}^i)_\varepsilon = -\frac{8\pi\kappa}{c^4}(T_\varepsilon)_\varepsilon. \quad (2.10.17)$$

Therefore the generalized Einstein equations of the field can also be written in the form

$$(R_{ik,\varepsilon})_\varepsilon = \frac{8\pi\kappa}{c^4}\left\{(T_{ik,\varepsilon})_\varepsilon - \frac{1}{2}(g_{ik,\varepsilon}T_\varepsilon)_\varepsilon\right\}. \quad (2.10.18)$$

Note that the generalized Einstein equations of the gravitational field are nonlinear Colombeau equations.

References.

1.Colombeau solutions to nonlinear equations.

[1] Generalized Solutions of Partial Differential Equations.Chapter 2.
North-Holland Mathematics Studies Volume 146,1987,Pages 145-168
https://doi.org/10.1016/S0304-0208(08)70093-8

[2] E. Rosinger, Colombeau, J. F. Review: Generalized solutions of nonlinear partial differential equations.Bull. Amer. Math. Soc. (N.S.) 20 (1989), no. 1, 96–101.
https://projecteuclid.org/euclid.bams/1183554914

[3] A. Heibig, M. Moussaoui,Generalized and classical solutions of nonlinear parabolic equations,Nonlinear Analysis: Theory, Methods & Applications Volume 24, Issue 6, March 1995, Pages 789-794 https://doi.org/10.1016/0362-546X(94)00167-G

[4] M. Oberguggenberger, Colombeau solutions to nonlinear wave equations.
https://arxiv.org/abs/math/0612445

[5] V. Ya. Derr,Differential equations in the algebra of C-generalized functions
Trudy Inst. Mat. i Mekh. UrO RAN, 2016, Volume 22,Number 3, Pages 62–75.

[6] V. Pruša, M. Řehoř, K. Tuma,Colombeau algebra as a mathematical tool for investigating step load and step deformation of systems of nonlinear springs and

dashpots,Math. Phys. (2017) 68: 24. https://doi.org/10.1007/s00033-017-0768-x
[7] U. Capar, Colombeau solutions of a nonlinear stochastic predator-prey equation, Turk. J. Math. (2013) 37: 1048-1060 doi:10.3906/mat-1105-1
[8] M. Oberguggenberger, F. Russo,Nonlinear SPDEs: Colombeau Solutions and Pathwise Limits, In: Decreusefond L., Øksendal B., Gjerde J., Üstünel A.S. (eds) Stochastic Analysis and Related Topics VI. Progress in Probability, vol 42. Birkhäuser, Boston, MA DOI https://doi.org/10.1007/978-1-4612-2022-0_14
[9] M. Nedeljkov, D. Rajter, Nonlinear stochastic wave equation with Colombeau generalized stochastic processes,Mathematical Models and Methods in Applied Sciences, Vol. 12, No. 05, pp. 665-688 (2002) https://doi.org/10.1142/S0218202502001842
[10] J.F Colombeau, M Langlais,Generalized solutions of nonlinear parabolic equations
with distributions as initial conditions,Journal of Mathematical Analysis and Applications,Volume 145, Issue 1,1990,Pages 186-196,ISSN 0022-247X, https://doi.org/10.1016/0022-247X(90)90440-Q.
[11]

2.Generalized pseudo-Riemannian geometry

[1] M. Kunzinger and R. Steinbauer,Generalized pseudo-Riemannian geometry, Journal: Trans. Amer. Math. Soc. 354 (2002), 4179-4199
MSC (2000): Primary 46F30; Secondary 46T30, 46F10, 83C05
DOI: https://doi.org/10.1090/S0002-9947-02-03058-1
Published electronically: June 3, 2002
[2] Kunzinger, M., Steinbauer, R., Vickers, J.A.: Generalised connections and curvature. Math. Proc. Camb. Philos. Soc. 139(3), 497–521 (2005). doi: 10.1017/S0305004105008649