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Oppositions in a Line Segment

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Abstract

Traditional oppositions are at least two-dimensional in the sense that they are built based on a famous bidimensional object called *square of oppositions* and on one of its extensions such as Blanché's hexagon. Instead of two-dimensional objects, this article proposes a construction to deal with oppositions in a one-dimensional line segment.

Keywords: theory of opposition, square of opposition, Blanché's hexagon, reduction, one dimension, line segment of opposition.

Introduction

The basic theory of oppositions has been developed considering a bidimensional structure, i.e., the well known *square of oppositions*. Later, it has been generalized to an hexagon of oppositions by Blanché in [4]. Moretti argues in [5] that there is a 'geometry' of oppositions, generalizing squares and hexagons, basically, to three-dimensional structures such as cubes and tetradecahedrons. This gives rise to the domain of n-opposition theory. Since works proposed by Beziau in [1] and Moretti in [5], there are now many researches in the field. The reader should check [7] for an introduction to the square of oppositions and [6] to its main recent developments. Beziau and Read stated in [3] that the theory of oppositions cannot be identified with the diagram representing this theory (p.315). This means that there are much more on oppositions than what is represented in the relations between corners of the square or the hexagon. This article[1], in some sense, can be viewed as an attempt to justify the claim that the theory of oppositions is not only the study of these n-dimensional diagrams ($n \geq 2$).

[1]A previous version of this paper appeared as a preprint in arXiv (April, 2016). Thanks to Rodrigo Freire, Edelcio de Souza and Fabien Schang for remarks on the constructions proposed here.

Consider a question: is there a way to represent oppositions without two-dimensional objects such as squares or objects of higher dimensions? The answer is *yes*. A construction to formulate this reduction is proposed showing that there is no need for two-dimensional objects to establish the basic theory of oppositions. Indeed, one dimension is enough. This means that oppositions can be defined in a line segment, a piece of one-dimensional space. Moreover, oppositions require precisely at least one dimension to be defined, and the traditional case of the bidimensional square can be converted step-by-step to it. However, applying the same strategy to Blanché's hexagon does not work. But the situation changes with some constraints added at the level of the basic construction. So, it is also possible to convert the standard bidimensional Blanché's hexagon to a line segment.

This paper shows that line segments are sufficient to define four basic standard oppositions in such a way that the square can be derived from this primitive structure. This line segment of oppositions is called here *basic construction*. Then, this very same basic construction does not work when applied to the standard hexagon (i.e. Blanché's hexagon), but there is a way to generate a similar strategy *mutatis mutandis* to it. In what follows these constructions are explained presenting their range and limits.

1 Defining oppositions in a line segment

1.1 The square

Assume classical logic. Given the framework of first-order logic, the square of oppositions uses four kinds of categorical propositions (where φ is formula): (A) universal affirmative of the form $\forall x\varphi$, (E) universal negative of the form $\forall x\neg\varphi$, (I) existential affirmative $\exists x\varphi$ and (O) existential negative $\exists x\neg\varphi$. Taking into account that $\forall$ and $\exists$ are interdefinable in the presence of negation, it is a matter of taste to decide which one to use to represent these four propositions. They can appear as above, mixing both quantifiers, or with only one kind of quantifier. Thus, for universal quantifier and negation there is the following: (A) $\forall x\varphi$, (E) $\forall x\neg\varphi$, (I) $\neg\forall x\neg\varphi$ and (O) $\neg\forall x\varphi$. For existential quantification and negation: (A) $\neg\exists x\neg\varphi$, (E) $\neg\exists x\varphi$, (I) $\exists x\varphi$ and (O) $\exists x\neg\varphi$. In general, in the literature, there are four traditional oppositions holding between these propositions: contradiction (d), contrariety (c), subcontrariety (sc) and subalternation (s), which are defined in the standard way (see [7]).

Further, there are, indeed, other families of concepts satisfying oppositional structures: they could be metaphysical statements such as *necessity* and *possibility* (and their derivatives), or deontic propositions containing notions of *obligation* and *permission*, or even statements containing temporal aspects such as *always*, *sometimes* and *never*. So, a pure oppositional structure is not necessarily decorated with categorical statements. Indeed, there are many possible decorations of the square (see [4, 2, 5]). These concepts fit pretty well satisfying the structure of the square of oppositions (or its extensions). They - and similar concepts - are here called categorical-like concepts: these are notions satisfying the four oppositions and, for this reason, can be arranged inside the traditional square. A set of categorical-like statements is denoted by **C**. Oppositions are usually (and historically) represented bidimensionally using a two-dimensional object with *lenght* and *width* (i.e. a square):

$$\begin{tikzcd} A \arrow[r, dash, "c"] \arrow[d, "s"] \arrow[dr, dash, "d"] & E \arrow[d, "s"] \arrow[dl, dash, "d"'] \\ I \arrow[r, dash, "sc"] & O \end{tikzcd}$$

I argue that the above two-dimensional square can be reduced to a simple one-dimensional structure, that is, a line segment of integers.[2] To show how to build this reduction is the *first construction* suggested here.

Assume the set $\mathbb{Z}$ of integers and, then, the sets $\mathbb{Z}_+$ (positive integers) and $\mathbb{Z}_-$ (negative integers). If the set of integers does not have zero, it is denoted by $\mathbb{Z}^*$, as usual. Take a line segment such that for each $j \in \mathbb{Z}$, $j \neq 0$, there exists $-j$, the symmetric of j. In particular, consider a set $\mathbb{Z}' = \{-r, -q, q, r\} \subseteq \mathbb{Z}$. Then the procedure is: a categorical-like proposition is assigned to each element $j \in \mathbb{Z}'$ in the following way:

- $j \in \mathbb{Z}^*_+$ if, and only if, j is associated to a universal statement;
- $j \in \mathbb{Z}^*_-$ if, and only if, j is associated to an existential statement.

There is a division of propositions in the sets of universal and existential statements. Let **C** be a set of categorical (or categorical-like) propositions. The function i which connects elements of **C** to elements of $\mathbb{Z}'$ is a bijection such that: $i(A) = q$; $i(E) = r$; $i(I) = -r$ and $i(O) = -q$.

We use α, β for arbitrary propositions ranging on **C**. Traditional oppositions now have to be reformulated inside this new framework. So, clauses for oppositions are defined as follows:

1. Propositions α, β are *contradictories* if, and only if, their assigned numbers have sum equals to 0, i.e, $i(\alpha) + i(\beta) = 0$;
2. Propositions α, β are *contraries* if, and only if, $i(\alpha), i(\beta) \in \mathbb{Z}^*_+$;

[2] *Simple* in the sense that it requires only one dimension.

3. Propositions α, β are *subcontraries* if, and only if, $i(\alpha), i(\beta) \in \mathbb{Z}^*_-$;

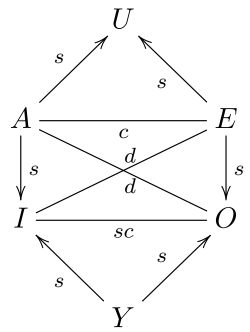
4. β is subaltern of α if, and only if, $i(\beta) \neq -i(\alpha)$ and $i(\beta) \in \mathbb{Z}^*_-$;

This completes the construction. Let's show with an example how the square can be defined in this way: consider, for instance, that $\mathbb{Z}' = \{-2, -1, 1, 2\} \subseteq \mathbb{Z}$. Thus, given that $1, 2 \in \mathbb{Z}^*_+$, it follows that these numbers are assigned to universal propositions (in which way this association is done is not important). In the same manner, from the fact that $-2, -1 \in \mathbb{Z}^*_-$, these numbers are connected to existential statements. Without loss generality, consider that 1 is the number connected to (A), i.e, $i(A) = 1$ and $i(E) = 2$. Therefore, the contradictory of (A) is the proposition (O) because $i(O) = -1$ satisfying, therefore, the condition to be a contradiction, i.e., $1 + (-1) = 0$. The same for the relations between (E) and (I). By construction, (A) and (E) are contraries while (I) and (O) are subcontraries, and these are subalterns of (A) and (E), respectively. (I) is subaltern of (A) given that (I) has associated to it the integer -2 which is, in its turn, different of -1 (note that $-i(A) = i(O)$). As far as there are only four oppositions, this procedure can always be done, no matter the family of concepts considered. This is an example provided to show how a two-dimensional square of oppositions can be reduced to a one-dimensional line segment of oppositions.

Despite the beauty of oppositions represented in a square, these oppositions can be converted into a one-dimensional line segment, if the number of oppositions remain four. This is the main contribution of this paper. Nevertheless, not all two-dimensional objects can be reduced using this same strategy. It already fails in the case of the hexagon. So, the basic construction has to be improved in order to work also for the hexagon. This is of secondary importance here, given that the hexagon does not have the same historical relevance of the square.

1.2 The hexagon

Blanché proposed an extension of the square, the hexagon, and this new tool (still two-dimensional) can be used to model many situations (see [4], but also [2]). Following Blanché's construction, in the (incomplete) diagram below, (U) is defined as the disjunction $A \vee E$ and (Y) is the conjunction $I \wedge O$.



How can the reduction strategy, i.e., the basic construction be also extended to Blanché's hexagon? The fact that there are propositions of the form (U) and (Y) require some adaptations in the basic construction. The *second construction* consists in executing these adaptations in order to transform also the hexagon into a one-dimensional structure.

Take a line segment as above and a set $\mathbb{Z}'' = \{-s, -r, -q, q, r, s\} \subseteq \mathbb{Z}$. Now, as there are also a disjunction and a conjunction, some changes have to be made in the way integers are assigned to propositions. To each element $j \in \mathbb{Z}''$, a proposition is connected as follows:

- $j \in \mathbb{Z}^*_+$ if, and only if, j is associated to a universal statement or a disjunction;
- $j \in \mathbb{Z}^*_-$ if, and only if, j is associated to an existential statement or a conjunction;.

The division of the class of propositions now contains a set of universal and disjunctive statements, from one side, and a set of existential and conjunctive statements, from other side. Let the set $\mathbf{C}'$ be an expansion of the set $\mathbf{C}$ defined by categorical-like propositions plus a disjunction and a conjunction, and let the function i which connects elements of $\mathbf{C}'$ to elements of $\mathbb{Z}''$ be a bijection such that: $i(U) = s$, $i(A) = q$, $i(E) = r$, $i(I) = -r$, $i(O) = -q$ and $i(Y) = -s$. The number assigned to the conjunction of (I) and (O), that is, (Y), as well to the disjunction of (A) or (E), that is (U), is the sum of both conjuncts (in the first case) and the sum of both disjuncts (in the second case). So, define that the integers associated to statements (U) and (Y) are obtained by the sums of their components: $i(U) = i(A) + i(E)$ and $i(Y) = i(I) + i(O)$. These numbers obtained by sums play an important role and, therefore, they are called *distinct objects*: the first one is the positive distinct object and the second is the negative. Consider a similar strategy as the basic construction and let's try to reduce the hexagon to a line segment. Suppose, for instance, that $\mathbb{Z}'' = \{-3, -2, -1, 1, 2, 3\} \subseteq \mathbb{Z}$ is given and numbers are associated to propositions: $i(A) = 1$, $i(E) = 2$, $i(O) = -1$ and $i(I) = -2$. Thus, distinct objects have the following association: $i(U) = 3$ and $i(Y) = -3$, if we consider this particular line segment $[-3, 3]$. Hence, these statements are obviously in the opposition of contradiction. In this sense, clause (1) above, i.e., contradiction, holds for (U) and (Y):[3]

$$\begin{tikzcd} U \arrow[d, dash, "d"'] \\ Y \end{tikzcd}$$

But not all clauses from 1-4 are valid. The reason for this is that Blanché's hexagon contains some unexpected connections between propositions (Y)-(A)-(E) (they are contraries and then clause 2 fails):

$$\begin{tikzcd} A \arrow[rr, dash, "c"] \arrow[dr, dash, "c"] & & E \arrow[dl, dash, "c"] \\ & Y & \end{tikzcd}$$

and propositions (U)-(I)-(O) (they are subcontraries and then clause 3 fails):

$$\begin{tikzcd} & U \arrow[dl, dash, "sc"'] \arrow[dr, dash, "sc"] & \\ I \arrow[rr, dash, "sc"] & & O \end{tikzcd}$$

In addition, subalternation also fails. Thus, it is not straighforward to settle Blanché's hexagon in a line segment of integers.[4] So, some adaptations and repairs in the basic construction should have to be done.

Assume that α, β, γ are letters for arbitrary propositions. The construction below shows how to reduce Blanché's hexagon to a line segment. Let $i(\gamma)$ be

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[3] Moretti (see [6]) remarked that the opposition of contradiction can be characterized, no matter which dimension is considered, as a certain kind of symmetry. It is a conjecture of this work that the whole of n-opposition theory can be reduced to variations of the constructions presented in this paper.

[4] Note that these last three diagrams are displayed in Blanché's hexagon (see [4]). When these relations are added to the diagram, we have the complete hexagon of oppositions.

a distinct object. For contradictories, condition 1 remains the same as above, but for other oppositions, clauses are the following (note that they have to be applied in this order: contradiction, contraries or subcontraries, and, by the end, subalternation. Then gaps will be gradually filled, as the oppositions are excludent, i.e, two propositions cannot be related by two different oppositions):

1*. Propositions α, β, γ are *contraries* if, and only if, $(i(\alpha)+i(\beta))+i(\gamma) = 0$ and $i(\gamma) \in \mathbb{Z}^*_-$;

2*. Propositions α, β, γ are *subcontraries* if, and only if, $(i(\alpha)+i(\beta))+i(\gamma) = 0$ and $i(\gamma) \in \mathbb{Z}^*_+$;

3*. β is subaltern of α if, and only if, $i(\beta) \neq -i(\alpha)$ and $i(\beta) \in \mathbb{Z}^*_-$, or $i(\beta) \neq -i(\alpha)$ and a) $i(\beta) > i(\alpha)$ and $i(\alpha), i(\beta) \in \mathbb{Z}^*_+$ or b) $i(\beta) > i(\alpha)$ and $i(\alpha), i(\beta) \in \mathbb{Z}^*_-$ ($i(\beta)$, in the last condition, is a distinct object);

It is not difficult to provide an example of integers to show that the hexagon can be reduced to it, as done in the case of the square. Assume first - without loss of generality - that $i(A) = 1$. So, $i(E) = 2$ and, thus, the distinct object $i(U) = i(A)+i(E) = 3$. Second, consider that $i(O) = -1$. It follows that $i(I) = -2$ and that the other distinct object $i(Y) = i(O) + i(I) = -3$. Consequently, (U) and (Y) are contradictories. Note that $(i(O) + i(I)) + i(U) = 0$, so these are subcontraries, as $i(U) \in \mathbb{Z}^*_+$, i.e, it is the positive distinct object. Moreover, $(i(A) + i(E)) + i(Y) = 0$, so these are contraries, as $i(Y) \in \mathbb{Z}^*_-$, i.e, it is the negative distinct object. Concerning subalternation, the square is contained in the hexagon, so (I) is subaltern of (A) and (O) is subaltern of (E), as in the case of the square. For other relations of subalternation, note that (U) is subaltern of (A) and (E), because $i(U) > i(A)$ and $i(U) > i(E)$ while $i(U), i(A), i(E) \in \mathbb{Z}^*_+$. Differently, (I) and (O) are subalterns of (Y), given that $i(Y) < i(I)$ and $i(Y) < i(O)$, and $i(Y), i(I), i(O) \in \mathbb{Z}^*_-$.

This is one way to reduce the hexagon to a one-dimensional object, but it is not argued that there is no other way. Note also that only Blanché's hexagon is taken into consideration, because it is the traditional hexagon, although there are some other available in the literature.

2 Conclusion

The standard square of opposition is a two-dimensional object used to organize and manage categorical-like concepts. There are many notions which fit in the square and its regular extensions such as the two-dimensional hexagon, and its three-dimensional correlates.

This paper proposed a construction which shows that there is no need to develop two-dimensional squares to represent categorical-like statements or concepts, instead a one-dimensional line segment based on integers is enough. Although the success in the case of the square, we have showed that the same technique does not immediately work for Blanché's hexagon and, therefore, it does require some adaptations and expansions to transform also the hexagon into a one-dimensional object. While both squares and hexagons have formulations in line segments of integers, these formulations are not straightforward and intuitive, and thus it seems these results are not so effective and simple as the pictorial effect of two-dimensional diagrams. However, from the theoretical viewpoint, it is important to know that there are simple one-dimensional objects able to accommodate theory of oppositions, although the fact that maybe, at this level, they are not so manageable as squares and hexagons. This is a clue that oppositions do not match with the research on diagrams, and in this sense, what authors supported in [3] seems to be rather right:

> "The diagram has been very important in promoting the theory but the theory does not reduce to the diagram. The theory started many centuries before the basic diagram was drawn and developed beyond this diagram...The strength of this theory is that it is at the same time fairly simple but quite rich; it can be applied to many different kinds of proposition, and also to objects and concepts. It can also be generalized in various manners, in particular, by constructing many different geometrical objects." (pgs. 315-316, in [3])

Oppositions are relations between propositions and cannot be identified, therefore, with the study of diagrams, though these are useful tools to explain what oppositions are and, in this way, can be largely applied especially for learning purposes.

In general, authors working on the square of opposition generally start with two-dimensional constructions like squares and hexagons, and then they jump to three-dimensional solids and so on. It seems a novelty the construction proposed in this paper because it shows that one does not need to go n-dimensional (for $n \geq 2$) to develop the theory of oppositions.

There are, notwithstanding, some problems which remain open: the question to determine whether the same procedure can also be applied to solids and higher dimensions, as well as to more than four oppositions, are very complicated and still have to investigated in detail in the domain of line segments of oppositions.

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