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Kerr-de Sitter Black Holes with NUT Charges

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ABSTRACT

The four-dimensional Kerr-de Sitter and Kerr-AdS black hole metrics have cohomogeneity 2, and they admit a generalisation in which an additional parameter characterising a NUT charge is included. In this paper, we study the higher-dimensional Kerr-AdS metrics, specialised to cohomogeneity 2 by appropriate restrictions on their rotation parameters, and we show how they too admit a generalisation in which an additional NUT-type parameter is introduced. We discuss also the supersymmetric limits of the new metrics. If one performs a Wick rotation to Euclidean spacetime signature, these yield new Einstein-Sasaki metrics in odd dimensions, and Ricci-flat metrics in even dimensions. We also study the five-dimensional Kerr-AdS black holes in detail. Although in this particular case the NUT parameter is trivial, our investigation reveals the remarkable feature that a five-dimensional Kerr-AdS "over-rotating" metric is equivalent, after performing a coordinate transformation, to an under-rotating Kerr-AdS metric.

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Contents

1 Introduction

The discovery by Kerr in 1963 [1] of the exact metric describing a rotating black hole was arguably the most important advance in the study of exact solutions in general relativity since Schwarzschild's discovery in 1916 [2] of the metric describing a static black hole. It was followed within a few years by the finding of various generalisations, including charged rotating black holes [3], and then the further inclusion of a cosmological constant [4], NUT parameter [5] and acceleration parameter [6].

With the advent of supergravity and superstring theory, interest also developed in studying higher-dimensional solutions of the Einstein equations. In 1986, the general solution describing an asymptotically flat rotating black hole in arbitrary dimension was discovered by Myers and Perry [7]. In dimension D, this has a mass parameter and $[(D-1)/2]$ independent rotation parameters, one for each of the orthogonal spatial 2-planes. Motivated by the study of the AdS/CFT correspondence in string theory, Hawking, Hunter and Taylor-Robinson constructed the solution for the five-dimensional rotating black hole with a cosmological constant in 1998 [8]. They also obtained a special case of the metrics in all higher dimensions, in which there is a rotation in only one of the $[(D-1)/2]$ orthogonal spatial 2-planes. The general rotating Kerr-de Sitter black hole solution with a cosmological constant in arbitrary dimension, with all $[(D-1)/2]$ independent rotation parameters a_i, was constructed by Gibbons, Lü, Page and Pope in 2004 [9, 10].

In view of the fact that the four-dimensional rotating black hole metrics admit further generalisations where additional non-trivial parameters are present, one might wonder whether such additional parameters could also be introduced in higher dimensions too. In fact, in a certain special class of higher-dimensional Kerr-de Sitter black holes, namely those in which there is just a rotation in a single 2-plane, a generalisation which includes a NUT parameter as well as the mass and the (single) rotation parameter has been obtained [11]. It was shown in [12] that this generalisation, which is trivial in five dimensions but non-trivial in dimensions $D \geq 6$, still exhibits certain remarkable separability properties for the Hamilton-Jacobi and wave equations, which in fact played an important rôle in the original discovery of the generalised four-dimensional solutions.

The purpose of this paper is to present new results we have obtained for much wider classes of generalisations of the Kerr-de Sitter metrics, in which there is a NUT-type parameter as well as the mass parameter. The cases covered by our new solutions are when the rotation parameters a_i are divided into two sets, in which all parameters within a set are equal. In odd dimensions, which we discuss in section 2, we obtain generalised solutions for an arbitrary partition of the parameters into two such sets. In even dimensions, which we discuss in section 3, the parameters are partitioned into one set with a non-vanishing value for the rotation, and the other set with vanishing rotation. In each of the odd and even dimensional cases, the net effect is to give a metric of cohomogeneity 2. In a manner that parallels rather closely the generalisations in $D = 4$, the two associated coordinates, on which the metric functions are intrinsically dependent, enter in a rather symmetrical way. The metrics that we obtain are equivalent to the previously-known Kerr-de Sitter-Taub-NUT metrics in $D = 4$. In $D \geq 6$ the extra parameter that we introduce gives rise to non-trivial generalisations of the Kerr-de Sitter metrics. The new parameter is associated with characteristics that generalise those of Taub-NUT like metrics in four dimensions, and so we may think of it as being a higher-dimensional generalisation of the NUT parameter. In each of the odd and even-dimensional cases, we discuss also their supersymmetric limits. In odd dimensions, these yield, after Euclideanisation, new Einstein-Sasaki metrics. In even dimensions, the supersymmetric limit leads to new Rici-flat Kähler metrics.

In section 4, we discuss some global aspects of the new Kerr-AdS-Taub-NUT metrics. In particular, in the case of even dimensions, the introduction of the NUT-type parameter implies that the time coordinate must be identified periodically, in the same way as happens in the previously-known four-dimensional solutions. By contrast, we find that in odd dimensions one can define a time coordinate that is not periodic.

In section 5, we discuss the case of five dimensions in detail. We find that in this case, the new NUT-type parameter is actually bogus, in the sense that it can be removed by using a scaling symmetry that is specific to the five-dimensional metric. In the process of showing this, however, we uncover an intriguing and previously unnoticed property of the five-dimensional Kerr-AdS metric. We find that it has an "inversion symmetry," which implies that the metric with large values of its rotation parameters is equivalent, after a general coordinate transformation, to the metric with small values for the rotations. The fixed point of this symmetry occurs at the critical value of rotation that arises in the supersymmetric limit. This corresponds to the case where the rotation parameter is equal to the radius of the asymptotically AdS spacetime. The inversion symmetry is therefore a feature specifically of the five-dimensional Kerr black holes with a cosmological constant, and does not arise in the case of asymptotically flat black holes.

The paper ends with conclusions in section 6.

2 Kerr-de Sitter with NUT Parameter in $D = 2n + 1$

2.1 The Metric

We take as our starting point the general Kerr-de Sitter metric in $D = 2n + 1$ dimensions, which was constructed in [9, 10]. Specifically, we begin with the metrics written in an asymptotically non-rotating frame, as given in equation (E.3) of [9], specialised to the case of odd dimensions $D = 2n+1$. We choose the cosmological constant to be negative, with the Ricci tensor given by $R_{\mu\nu} = -(D-1)g^2\, g_{\mu\nu}$. The constant g is the inverse of the AdS radius. The metric is described in terms of n "latitude" or direction cosine coordinates μ_i, subject to the constraint $\sum_{i=1}^n \mu_i^2 = 1$, n azimuthal coordinates ϕ_i, the radial coordinate r and time coordinate t. It has $(n + 1)$ arbitrary parameters M and a_i, which can be thought of as characterising the mass and the n angular momenta in the n orthogonal spatial 2-planes.

In order to find a generalisation that includes a NUT-type parameter, we first specialise the Kerr-AdS metrics by setting

$$a_1 = a_2 = \cdots = a_p = a\,, \qquad a_{p+1} = a_{p+2} = \cdots = a_n = b\,. \tag{1}$$

We then reparameterise the latitude coordinates coordinates as

$$\begin{aligned} \mu_i &= \nu_i \sin\theta\,, \qquad 1\le i\le p\,, \qquad \sum_{i=1}^p \nu_i^2 = 1\,,\\ \mu_{j+p} &= \tilde\nu_j \cos\theta\,, \qquad 1\le j\le q\,, \qquad \sum_{j=1}^q \tilde\nu_j^2 = 1\,, \end{aligned} \tag{2}$$

where we have defined

$$n = p+q\,, \tag{3}$$

and we also then introduce a coordinate v in place of θ, defined by

$$a^2\cos^2\theta + b^2\sin^2\theta = v^2\,. \tag{4}$$

It is convenient to divide the original n azimuthal coordinates ϕ_i into two sets, with p of them denoted by ϕ_i and the remaining q denoted by $\tilde\phi_j$.

Because of the specialisation of the rotation parameters in (1), the Kerr-AdS metric will now have cohomogeneity 2, rather than the cohomogeneity n of the general $(2n+1)$-dimensional Kerr-AdS metrics. In fact, as we shall see explicitly below, the metric has homogeneous level sets $\mathbb{R}\times S^{2p-1}\times S^{2q-1}$, with the metric functions depending inhomogeneously on the coordinates r and v. Remarkably, the form in which the metric can now be written puts the radial coordinate r and the coordinate v on a parallel footing, and suggests a rather natural generalisation in which a NUT-type parameter L can be introduced. Rather than writing the metric first without the NUT contribution and then again with it added, we shall just directly present our final result with the NUT parameter included. The original Kerr-AdS, subject to the constraints on the rotation parameters specified in (1), corresponds to setting $L=0$. Our generalised metric including L is

$$\begin{aligned} ds^2 &= -\frac{(1+g^2r^2)(1-g^2v^2)}{\Xi_a\Xi_b}dt^2 + \frac{\rho^{2n-2}dr^2}{U} + \frac{\omega^{2n-2}dv^2}{V}\\ &\quad + \frac{2M}{\rho^{2n-2}}\Big(\frac{(1-g^2v^2)}{\Xi_a\Xi_b}\,dt - \mathcal{A}\Big)^2 + \frac{2L}{\omega^{2n-2}}\Big(\frac{(1+g^2r^2)}{\Xi_a\Xi_b}\,dt - \widetilde{\mathcal{A}}\Big)^2\\ &\quad + \frac{(r^2+a^2)(a^2-v^2)}{\Xi_a(a^2-b^2)}\sum_{i=1}^p\Big(d\nu_i^2+\nu_i^2d\phi_i^2\Big) + \frac{(r^2+b^2)(b^2-v^2)}{\Xi_b(b^2-a^2)}\sum_{j=1}^q\Big(d\tilde\nu_j^2+\tilde\nu_j^2d\tilde\phi_j^2\Big)\,, \end{aligned} \tag{5}$$

where

$$\begin{aligned}
\mathcal{A} &= \frac{a(a^2-v^2)}{\Xi_a(a^2-b^2)}\sum_{i=1}^p \nu_i^2 d\phi_i + \frac{b(b^2-v^2)}{\Xi_b(b^2-a^2)}\sum_{j=1}^q \tilde\nu_j^2 d\tilde\phi_j\,,\\
\widetilde{\mathcal{A}} &= \frac{a(r^2+a^2)}{\Xi_a(a^2-b^2)}\sum_{i=1}^p \nu_i^2 d\phi_i + \frac{b(r^2+b^2)}{\Xi_b(b^2-a^2)}\sum_{j=1}^q \tilde\nu_j^2 d\tilde\phi_j\,,\\
U &= \frac{(1+g^2r^2)(r^2+a^2)^p\,(r^2+b^2)^q}{r^2} - 2M\,,\\
V &= -\frac{(1-g^2v^2)(a^2-v^2)^p\,(b^2-v^2)^q}{v^2} + 2L\,.\\
\rho^{2n-2} &= (r^2+v^2)(r^2+a^2)^{p-1}\,(r^2+b^2)^{q-1}\,,\qquad \Xi_a = 1-a^2g^2\,,\\
\omega^{2n-2} &= (r^2+v^2)(a^2-v^2)^{p-1}\,(b^2-v^2)^{q-1}\,,\qquad \Xi_b = 1-b^2g^2\,.
\end{aligned}\tag{6}$$

It is straightforward (with the aid of a computer) to verify in a variety of low odd dimensions that the metric (5) does indeed solve the Einstein equations $R_{\mu\nu} = -(D-1)g^2 g_{\mu\nu}$, and since the construction does not exploit any special features of the low dimensions, one can be confident that the solution is valid in all odd dimensions. We have explicitly verified the solutions in $D \le 9$.

As we indicated above, the metric (5) can be re-expressed more elegantly in terms of two complex projective spaces $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$. The proof is straightforward, following the same steps as were used in [9] when studying the Kerr-de Sitter metrics with equal angular momenta. The essential point is that one can write

$$\sum_{i=1}^p (d\nu_i^2 + \nu_i^2 d\phi_i^2) = d\Omega_{2p-1}^2 = (d\psi + A)^2 + d\Sigma_{p-1}^2\,,\qquad \sum_{i=1}^p \nu_i^2 d\phi_i = d\psi + A\,,\tag{7}$$

where $d\Sigma_{p-1}^2$ is the standard Fubini-Study metric on $\mathbb{CP}^{p-1}$ (with $R_{ab} = 2p g_{ab}$), and $\frac{1}{2}dA$ locally gives the Kähler form J. Note that $d\Omega_{2p-1}^2$ is the standard metric on the unit sphere S^{2p-1}, expressed here as the Hopf fibration over $\mathbb{CP}^{p-1}$.

With these results, and the analogous ones for the tilded coordinates $\tilde\nu_j$ and $\tilde\phi_j$, we find that (5) can be rewritten as

$$\begin{aligned}
ds^2 &= -\frac{(1+g^2r^2)(1-g^2v^2)}{\Xi_a\Xi_b}dt^2 + \frac{\rho^{2n-2}dr^2}{U} + \frac{\omega^{2n-2}dv^2}{V}\\
&\quad + \frac{2M}{\rho^{2n-2}}\Big(\frac{(1-g^2v^2)}{\Xi_a\Xi_b}\,dt - \mathcal{A}\Big)^2 + \frac{2L}{\omega^{2n-2}}\Big(\frac{(1+g^2r^2)}{\Xi_a\Xi_b}\,dt - \widetilde{\mathcal{A}}\Big)^2\\
&\quad + \frac{(r^2+a^2)(a^2-v^2)}{\Xi_a(a^2-b^2)}\Big((d\psi + A)^2 + d\Sigma_{p-1}^2\Big)\\
&\quad + \frac{(r^2+b^2)(b^2-v^2)}{\Xi_b(b^2-a^2)}\Big((d\varphi + \widetilde{A})^2 + d\widetilde{\Sigma}_{q-1}^2\Big)\,,
\end{aligned}\tag{8}$$

now with

$$\mathcal{A} = \frac{a(a^2-v^2)}{\Xi_a(a^2-b^2)}(d\psi + A) + \frac{b(b^2-v^2)}{\Xi_b(b^2-a^2)}(d\varphi + \widetilde{A})$$
$$\widetilde{\mathcal{A}} = \frac{a(r^2+a^2)}{\Xi_a(a^2-b^2)}(d\psi + A) + \frac{b(r^2+b^2)}{\Xi_b(b^2-a^2)}(d\varphi + \widetilde{A})\,. \tag{9}$$

Here A and $\widetilde{A}$ are potentials such that the Kähler forms of the complex projective spaces $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$ are given locally by $J = \frac{1}{2}dA$ and $\widetilde{J} = \frac{1}{2}\widetilde{A}$ respectively. Another useful way of writing the metric is given in the appendix.

It can be seen from the form of (8) that the metrics have cohomogeneity 2, with principal orbits on the surfaces where r and v are constant that are the homogeneous spaces $\mathbb{R} \times S^{2p-1} \times S^{2q-1}$. The $\mathbb{R}$ factor is associated with the time direction, whilst the spheres S^{2p-1} and S^{2q-1} arise from the Hopf fibrations over $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$ respectively. The sphere metrics on the principal orbits are squashed, and so the isometry group of (8) is $\mathbb{R} \times U(p) \times U(q)$.

We have presented the new solutions for the case of negative cosmological constant, but clearly these NUT generalisations of Kerr-AdS will also be valid if we send $g \to \mathrm{i}\,g$, yielding NUT generalisations of the Kerr-de Sitter metrics. It is also worth noting that even when the cosmological constant is set to zero, the solutions are still new, representing NUT generalisations of the asymptotically-flat rotating black holes of Myers and Perry [7].

Written in the form (5) or (8), the metric appears to be singular in the special case where one sets $a = b$. This is, however, an artefact of our introduction of the coordinate v, in place of θ. We did this in order to bring out the symmetrical relation between r and v, but clearly, as can be seen from (4), the coordinate v degenerates in the case $a = b$. This can be avoided by using θ as the coordinate instead, and performing appropriate rescalings.

Having written our new Kerr-AdS-Taub-NUT metrics in this form, it is clear that we could obtain more general Einstein metrics by replacing the Fubini-Study metrics $d\Sigma^2_{p-1}$ and $d\widetilde{\Sigma}^2_{q-1}$ on $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$ by arbitrary Einstein-Kähler metrics of the same dimensions, and normalised to have the same cosmological constants as $d\Sigma^2_{p-1}$ and $d\widetilde{\Sigma}^2_{q-1}$. In the generalised metrics, A and $\widetilde{A}$ will now be potentials yielding the Kähler forms of the two Einstein-Kähler metrics, i.e. $J = \frac{1}{2}dA$ and $\widetilde{J} = \frac{1}{2}d\widetilde{A}$.

If we specialise to the case when $b = 0$, and define a new coordinate $\psi' = \psi - ag^2 t$, then the metric (9) reduces to

$$\begin{aligned} ds^2 &= \frac{r^2+v^2}{X}dr^2 + \frac{r^2+v^2}{Y}dv^2 - \frac{X}{r^2+v^2}\Big(dt - \frac{a^2-v^2}{a\,\Xi_a}(d\psi'+A)\Big)^2 \\ &\quad + \frac{Y}{r^2+v^2}\Big(dt - \frac{r^2+a^2}{a\,\Xi_a}(d\psi'+A)\Big)^2 + \frac{(r^2+a^2)(a^2-v^2)}{a^2\Xi_a}d\Sigma_{p-1}^2 \\ &\quad + \frac{r^2v^2}{a^2}d\Omega_{2q-1}^2\,, \end{aligned} \tag{10}$$

where $d\Omega_{2q-1}^2 = (d\varphi + \widetilde{A})^2 + d\widetilde{\Sigma}_{q-1}^2$ is the metric of a unit sphere S^{2q-1}, and

$$\begin{aligned} X &= (1+g^2r^2)(r^2+a^2) - \frac{2M}{(r^2+a^2)^{p-1}\,r^{2(q-1)}}\,, \\ Y &= (1-g^2v^2)(a^2-v^2) - \frac{2\widetilde{L}}{(a^2-v^2)^{p-1}\,v^{2(q-1)}}\,. \end{aligned} \tag{11}$$

The constant $\widetilde{L}$ is related to the original NUT parameter by $\widetilde{L} = (-1)^q\,L$. A special case of the metrics (10), namely when $p=1$, was obtained in [11, 12].

2.2 The supersymmetric limit

Odd-dimensional Kerr-AdS black holes admit supersymmetric limits, which in Euclidean signature with positive cosmological constant become Einstein-Sasaki metrics [13, 14] (see also [15, 16] for discussions of how the supersymmetric limit arises in the Lorentzian regime, when a Bogomol'nyi inequality is saturated). We find that an analogous limit also exists for our new metrics where the NUT charge is introduced. We first set $g = \mathrm{i}$ so that the metric has a unit positive cosmological constant ($R_{\mu\nu} = (D-1)\,g_{\mu\nu}$). We then Euclideanise the metric by sending

$$t \to \mathrm{i}t\,, \qquad a \to \mathrm{i}a\,, \qquad b \to \mathrm{i}b\,, \tag{12}$$

define

$$\begin{aligned} &1-a^2 = \alpha\,\epsilon\,, \qquad 1-b^2 = \beta\,\epsilon\,, \qquad M = -m\,\epsilon^{n+1}\,, \qquad L = \ell\,\epsilon^{n+1}\,, \\ &1-r^2 = \epsilon\,x\,, \qquad 1+v^2 = \epsilon\,y\,, \end{aligned} \tag{13}$$

and then take the limit $\epsilon \to 0$. This leads to the metric

$$\begin{aligned} ds^2 &= \Big(dt + \frac{(\alpha-x)(\alpha-y)}{\alpha(\alpha-\beta)}(d\psi+A) - \frac{(\beta-x)(\beta-y)}{\beta(\alpha-\beta)}(d\varphi+\widetilde{A})\Big)^2 \\ &\quad + \frac{x-y}{4X}dx^2 + \frac{x-y}{4Y}dy^2 + \frac{(\alpha-x)(\alpha-y)}{\alpha(\alpha-\beta)}d\Sigma_{p-1}^2 - \frac{(\beta-x)(\beta-y)}{\beta(\alpha-\beta)}d\widetilde{\Sigma}_{q-1}^2 \\ &\quad + \frac{X}{x-y}\Big(\frac{(\alpha-y)}{\alpha(\alpha-\beta)}(d\psi+A) - \frac{(\beta-y)}{\beta(\alpha-\beta)}(d\varphi+\widetilde{A})\Big)^2 \\ &\quad + \frac{Y}{x-y}\Big(\frac{(\alpha-x)}{\alpha(\alpha-\beta)}(d\psi+A) - \frac{(\beta-x)}{\beta(\alpha-\beta)}(d\varphi+\widetilde{A})\Big)^2\,, \end{aligned} \tag{14}$$

where again $J=\frac{1}{2}dA$ and $\widetilde{J}=\frac{1}{2}d\widetilde{A}$ are the Kähler forms of the $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$ complex projective spaces with metrics $d\Sigma_{p-1}^2$ and $d\widetilde{\Sigma}_{q-1}^2$ respectively, and

$$
\begin{aligned}
X &= -\frac{2m}{(\alpha-x)^{p-1}(\beta-x)^{q-1}} - x(\alpha-x)(\beta-x)\,,\\
Y &= \frac{2\ell}{(\alpha-y)^{p-1}(\beta-y)^{q-1}} + y(\alpha-y)(\beta-y)\,.
\end{aligned}
\tag{15}
$$

It is straightforward to verify that the above metric (14) is an Einstein-Sasaki metric in $D=2n+1$ dimensions. Note that the metric has the form

$$
ds_{2n+1}^2 = (dt+2\mathcal{A})^2 + ds_{2n}^2\,. \tag{16}
$$

where ds_{2n}^2 is an Einstein-Kähler metric and $\mathcal{A}$ is the corresponding Kähler potential, in the sense that the Kähler form for ds_{2n}^2 can be written locally as $J=d\mathcal{A}$. As far as we know, these cohomogeneity-2 Einstein-Kähler metrics ds_{2n}^2 have not been obtained explicitly before. Note that one can go to the Ricci-flat limit of ds_{2n}^2 by performing a rescaling that amounts to dropping the x^3 term and y^3 term in (15).

If we consider the special case where $p=n-1$ and $q=1$, the Einstein-Sasaki metrics reduce to ones that were obtained recently in [17]. This may be seen by defining new parameters by the expressions

$$
\hat\alpha = -4(\beta-2\alpha)\,,\quad \hat\beta=\alpha(\alpha-\beta)\,,\quad m=\tfrac{1}{8}(-1)^N\,\mu\,,\quad \ell=\tfrac{1}{8}(-1)^N\,\nu\,, \tag{17}
$$

where $N\equiv p-1$, and introducing new coordinates defined by

$$
\hat x = x-\alpha\,,\quad \hat y = y-\alpha\,,\quad t=\tau+2(\alpha-\beta)\chi\,,\quad \varphi=2\beta\chi\,,\quad \psi=2\hat\psi+2\alpha\chi\,. \tag{18}
$$

Defining also $\hat X = 4X$ and $\hat Y=4Y$, we obtain, upon substitution into (14), the metric

$$
\begin{aligned}
ds^2 &= [d\tau - 2(\hat x+\hat y)d\chi + \frac{2\hat x\hat y}{\hat\beta}\sigma]^2 + \frac{\hat x-\hat y}{\hat X}\,d\hat x^2 + \frac{\hat x-\hat y}{\hat Y}\,d\hat y^2\\
&\quad + \frac{\hat X}{\hat x-\hat y}\,(d\chi-\frac{\hat y}{\hat\beta}\,\sigma)^2 + \frac{\hat Y}{\hat x-\hat y}\,(d\chi-\frac{\hat x}{\hat\beta}\,\sigma)^2 + \frac{\hat x\hat y}{\hat\beta}\,d\Sigma_N^2\,,
\end{aligned}
\tag{19}
$$

where $\sigma = d\hat\psi+\frac{1}{2}A$, and

$$
\hat X = -4\hat x^3 - \hat\alpha\hat x^2 - 4\hat\beta\hat x - \frac{\mu}{\hat x^N}\,,\qquad \hat Y = 4\hat y^3+\hat\alpha\hat y^2+4\hat\beta\hat y+\frac{\nu}{\hat y^N}\,. \tag{20}
$$

This is precisely of the form of the Einstein-Sasaki metrics that were obtained in section (4) of reference [17], in the case where the Einstein-Kähler base metric in that paper is taken to be $\mathbb{CP}^N$. A detailed discussion of the global structure of these metrics was given in [17], and new complete $D=7$ Einstein-Sasaki spaces were obtained.

3 Kerr-de Sitter with NUT Parameter in $D = 2n$

The Kerr-de Sitter metrics in even spacetime dimensions take a slightly different form from those in odd dimensions. The reason for this is that now there are an odd number of spatial dimensions, and so there can be $(n-1)$ independent parameters characterising rotations in $(n-1)$ orthogonal 2-planes, with one additional spatial direction that is not associated with a rotation. Because of this feature, the $D = 2n$ dimensional Kerr-de Sitter black holes in general have cohomogeneity n, which can be reduced to cohomogeneity 2 if one sets all the $(n-1)$ rotation parameters equal. By contrast, in odd dimensions $D = 2n+1$ the general metrics have cohomogeneity n, reducing to cohomogeneity 1 if one sets all the rotation parameters equal.

It will be recalled that in section 2, we were able to generalise the odd-dimensional Kerr-de Sitter to include a NUT parameter by dividing the angular momentum parameters a_i into two sets, equal within a set, thereby obtaining a metric of cohomogeneity 2. Our construction with the NUT parameter is intrinsically adapted to metrics of cohomogeneity 2, and so this means that in the present case, when we consider generalising the even-dimensional Kerr-de Sitter metrics, we shall first need to divide the rotation parameters a_i into two sets. In one set, the parameters will be equal and non-zero, while in the other set, the remaining rotation parameters will all be chosen to be zero.

Our starting point is the expression for the Kerr-de Sitter metrics given in equation (E.3) of reference [9], specialised to dimension $D = 2n$. We shall take the cosmological constant to be negative, with the resulting Kerr-AdS metrics satisfying $R_{\mu\nu} = -(D-1)g^2\, g_{\mu\nu}$. We then set

$$a_1 = a_2 = \cdots = a_p = a\,, \qquad a_{p+1} = a_{p-2} = \cdots = a_{n-1} = 0\,. \tag{21}$$

We then introduce new "latitude" coordinates ν_i, $\tilde\nu_j$ and θ, in place of the μ_i in [9],

$$\begin{aligned} \mu_i &= \nu_i \sin\theta\,, \qquad 1 \le i \le p\,, \qquad \sum_{i=1}^{p} \nu_i^2 = 1\,, \\ \mu_{j+p} &= \tilde\nu_j \cos\theta\,, \qquad 1 \le j \le n-p\,, \qquad \sum_{j=1}^{n-p} \tilde\nu_j^2 = 1\,, \end{aligned} \tag{22}$$

In this case, because there are only $(n-1)$ azimuthal coordinates ϕ_i, we split them into two sets, which we shall denote by ϕ_i and $\tilde\phi_j$, defined for

$$\phi_i: \qquad 1 \le i \le p\,, \qquad\qquad \tilde\phi_j: \qquad 1 \le j \le q\,, \tag{23}$$

where this time we have defined q such that

$$p + q = n - 1\,. \tag{24}$$

We then introduce a new variable v, in place of θ, which this time is defined by

$$a^2 \cos^2\theta = v^2\,. \tag{25}$$

We can now write out the Kerr-AdS metric of [9, 10], subject to the restriction (21), in terms of the new variables defined above, and, as in the odd-dimensional case we discussed previously, this allows us to conjecture a generalisation that includes a NUT parameter L as well as the mass parameter M and angular momentum parameter a. Again, we shall just present our final result, having included the NUT parameter. Thus we obtain the new Kerr-AdS-Taub-NUT metric (which we have verified explicitly in $D \le 8$)

$$\begin{aligned} ds^2 &= -\frac{(1+g^2r^2)(1-g^2v^2)}{\Xi_a}dt^2 + \frac{\rho^{2n-3}dr^2}{U} + \frac{\omega^{2n-3}dv^2}{V} \\ &\quad + \frac{2M\,r}{\rho^{2n-3}}\Big(\frac{(1-g^2v^2)}{\Xi_a}\,dt - \mathcal{A}\Big)^2 - \frac{2L\,v}{\omega^{2n-3}}\Big(\frac{(1+g^2r^2)}{\Xi_a}dt - \widetilde{\mathcal{A}}\Big)^2 \\ &\quad + \frac{(r^2+a^2)(a^2-v^2)}{a^2\Xi_a}\sum_{i=1}^{p}(d\nu_i^2 + \nu_i^2\,d\phi_i^2) + \frac{r^2v^2}{a^2}\Big(d\tilde{\nu}_{q+1}^2 + \sum_{j=1}^{q}(d\tilde{\nu}_j^2 + \tilde{\nu}_j^2 d\tilde{\phi}_j^2)\Big)\,, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \mathcal{A} &= \frac{a^2-v^2}{a\,\Xi_a}\sum_{i=1}^{p}\nu_i^2 d\phi_i\,, \qquad \widetilde{\mathcal{A}} = \frac{r^2+a^2}{a\,\Xi_a}\sum_{i=1}^{p}\nu_i^2 d\phi_i\,, \\ U &= (1+g^2r^2)(r^2+a^2)^p\, r^{2q} - 2M\,r\,, \\ V &= (1-g^2v^2)(a^2-v^2)^p\, v^{2q} - 2L\,v\,. \\ \rho^{2n-3} &= (r^2+v^2)(r^2+a^2)^{p-1}\, r^{2q}\,, \qquad \Xi_a = 1 - a^2g^2\,, \\ \omega^{2n-3} &= (r^2+v^2)(a^2-v^2)^{p-1}\, v^{2q}\,. \end{aligned} \tag{27}$$

The $(2n)$-dimensional Kerr-AdS-Taub-NUT metrics that we have constructed here can be seen to be quite similar in structure to the $(2n+1)$-dimensional examples that we constructed in section 2, in the special case where we set the b parameter to zero. In fact we can re-express the metrics (26) in terms of a complex projective space and a sphere metric, in a manner that is closely analogous to (10). This is expressed most simply by making redefinitions as in (7), and then introducing a new Hopf fibre coordinate $\widetilde{\psi} = \psi - ag^2t$ as we did in the odd-dimensional case. Having done this, we arrive at the metric

$$\begin{aligned} ds^2 &= \frac{r^2+v^2}{X}dr^2 + \frac{r^2+v^2}{Y}dv^2 - \frac{X}{r^2+v^2}\Big(dt - \frac{a^2-v^2}{a\,\Xi_a}(d\widetilde{\psi} + A)\Big)^2 \\ &\quad + \frac{Y}{r^2+v^2}\Big(dt - \frac{a^2+r^2}{a\,\Xi_a}(d\widetilde{\psi} + A)\Big)^2 + \frac{(a^2+r^2)(a^2-v^2)}{a^2\Xi_a}d\Sigma_{p-1}^2 + \frac{r^2v^2}{a^2}d\Omega_{2q}^2\,, \end{aligned} \tag{28}$$

where $d\Omega_{2q}^2$ is the metric on the unit sphere S^{2q},

$$
\begin{aligned}
X &= (1+g^2r^2)(r^2+a^2) - \frac{2M\,r}{(r^2+a^2)^{p-1}\,r^{2q}}\,,\\
Y &= (1-g^2v^2)(a^2-v^2) - \frac{2L\,v}{(a^2-v^2)^{p-1}\,v^{2q}}\,,
\end{aligned} \tag{29}
$$

and the Kähler form J for the $\mathbb{CP}^{p-1}$ metric $d\Sigma_{p-1}^2$ is given locally by $J = \frac{1}{2}dA$.

For the cases with $q = 0$, there can also be a BPS limit of the solutions, giving rise to Ricci-flat Kähler metrics instead of Einstein-Kähler. To do this, we first Euclideanise the metric by setting $t \to \mathrm{i}\,t$, $a \to \mathrm{i}\,a$ and set $g = \mathrm{i}$. We then take the following limit

$$
1-a^2 = \alpha\,\epsilon\,, \quad 1-r^2 = x\,\epsilon\,, \quad 1+v^2 = y\,\epsilon\,, \quad M = \mu\,(-\epsilon)^{p-1}\,, \quad L = \mathrm{i}\,\nu\,\epsilon^{p-1}\,, \tag{30}
$$

with $\epsilon \to 0$. The metric becomes $ds^2 = \epsilon\,d\tilde{s}^2$, where $d\tilde{s}^2$ is a Ricci-flat Kähler metric, given by

$$
\begin{aligned}
d\tilde{s}^2 &= \frac{x-y}{4X}dx^2 + \frac{x-y}{4Y}dy^2 + \frac{(x-\alpha)(\alpha-y)}{\alpha}d\Sigma_{p-1}^2\\
&\quad + \frac{X}{x-y}(dt + \frac{\alpha-y}{\alpha}(d\psi+A))^2 + \frac{Y}{x-y}(dt - \frac{x-\alpha}{\alpha}(d\psi+A))^2\,,\\
X &= x(x-\alpha) + \frac{2\mu}{(x-\alpha)^{p-1}}\,, \qquad Y = y(\alpha-y) - \frac{2\nu}{(\alpha-y)^{p-1}}\,.
\end{aligned} \tag{31}
$$

The Kähler 2-form is given locally by $J = dB$, where

$$
B = \tfrac{1}{2}(x+y)dt + \frac{(x-\alpha)(\alpha-y)}{2\alpha}(d\psi + A)\,. \tag{32}
$$

4 Global Analysis

The global analysis of Kerr-AdS black holes in general dimensions was given in [9, 10]. Here, we study the effect of introducing the NUT charge L. We shall consider the case where v is a compact coordinate, ranging over the interval $0 < v_1 \le v \le v_2$, where v_1 and v_2 are two adjacent roots of $V(v) = 0$, such that the function V is positive when v lies within the interval. In the case when $L = 0$, we would have $v_1 = a$ and $v_2 = b$. The coordinate r ranges from r_0 to infinity, where r_0 is the largest root of $U(r) = 0$. The discussion now divides into the cases of $D = 2n+1$ dimensions and $D = 2n$ dimensions.

4.1 $D = 2n+1$ dimensions

The metric (8) is degenerate at $v = v_1$ and v_2, where $V(v_i) = 0$. The corresponding Killing vectors whose norms $\ell^2 = g_{\mu\nu}\,\ell^\mu\,\ell^\nu$ vanish at these surfaces have the form

$$
\ell = \gamma_0\frac{\partial}{\partial t} + \gamma_1\frac{\partial}{\partial\phi} + \gamma_2\frac{\partial}{\partial\psi}\,, \tag{33}
$$

for constants γ_0, γ_1 and γ_2 to be determined. The associated "surface gravities" are of Euclidean type, in the sense that

$$\kappa_E^2 = \frac{g^{\mu\nu}\,(\partial_\mu \ell^2)\,(\partial_\nu \ell^2)}{4\ell^2}\Big|_{v=v_i} \tag{34}$$

is positive. Thus these degenerations are typical of an azimuthal coordinate at a spatial origin. We can scale the coefficients γ_i so that the Euclidean surface gravity is 1, implying that the Killing vector generates a closed translation with period 2π. One might conclude that the time coordinate is periodic, since γ_0 is non-vanishing. This is indeed the case for the solutions in even dimensions. However, in odd dimensions the $\partial/\partial t$ term can be removed by making the coordinate transformation

$$t = \tilde{t} + \frac{\Xi_b(a^2 - v_1^2)(b^2 - v_2^2)b\,\psi - \Xi_a(b^2 - v_1^2)(b^2 - v_2^2)a\,\varphi}{ab(a^2 - b^2)(1 - g^2 v_1^2)(1 - g^2 v_2^2)} \tag{35}$$

The two Killing vectors whose norms vanish at v_1 and v_2 are now given by

$$\ell_i = \frac{4L}{V'(v_i)}\Big(\frac{b}{b^2 - v_i^2}\frac{\partial}{\partial\varphi} + \frac{a}{a^2 - v_i^2}\frac{\partial}{\partial\psi}\Big)\,. \tag{36}$$

Both Killing vectors have unit Euclidean surface gravity, implying that they both generate closed 2π translations. Since it does not suffer a periodic identification, $\tilde{t}$ is perhaps a more natural choice than t for the time coordinate.

The metric also degenerates at $r = r_0$, and the corresponding null Killing vector has Lorentzian surface gravity κ, in the sense that $\kappa^2 = -\kappa_E^2$ is positive. Thus $r = r_0$ is an horizon. If we write the null Killing vector in terms of coordinate $\tilde{t}$, normalised to

$$\tilde{\ell}_0 = \frac{\partial}{\partial\tilde{t}} + \tilde{\gamma}_1\frac{\partial}{\partial\phi} + \tilde{\gamma}_2\frac{\partial}{\partial\psi}\,, \tag{37}$$

where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are determined from the condition that $\tilde{\ell}_0^2 = 0$ at $r = r_0$, we find that the surface gravity is given by

$$\kappa = \frac{(1 - g^2 v_1^2)(1 - g^2 v_2^2)(r_0^2 + a^2)(r_0^2 + b^2)(1 + g^2 r_0^2)U'(r_0)}{2\Xi_a\Xi_b(r_0^2 + v_1^2)(r_0^2 + v_2^2)[U(r_0) + 2M]}\,. \tag{38}$$

If instead we consider the null Killing vector in terms of the original coordinate t, and rescale it to give

$$\ell_0 = \frac{\partial}{\partial t} + \gamma_1\frac{\partial}{\partial\phi} + \gamma_2\frac{\partial}{\partial\psi}\,, \tag{39}$$

then the surface gravity is then given by

$$\kappa = \frac{(1 + g^2 r_0^2)U'(r_0)}{2[U(r_0) + 2M]}\,, \tag{40}$$

which is identical to the result for the Kerr-AdS black hole [9, 10] without the NUT parameter. It is not *a priori* obvious what the proper normalisation for the asymptotically timelike Killing vector should be, since the metrics with the non-vanishing NUT parameter are not asymptotic to AdS.

4.2 $D = 2n$ dimensions

In even dimensions, the introduction of the NUT parameter implies that the time coordinate is necessarily periodic (as in four dimensions). To see this, we note from the metric (28) that, at the degenerate surfaces $v = v_1$ and v_2, the Killing vectors whose norms vanish are given by

$$\ell_i = \frac{2}{V'(v_i)}\Big((a^2 - v_i^2)\frac{\partial}{\partial t} + a\,\Xi_a\,\frac{\partial}{\partial\widetilde{\psi}}\Big)\,. \tag{41}$$

These Killing vectors are normalised to have unit Euclidean surface gravities, and hence they generate closed translations with period 2π. In the case when $L = 0$, then $v_1 = a$ and $v_2 = -a$, so the ℓ_i do not have $\partial/\partial t$ terms. However, when $L \neq 0$ there are necessarily $\partial/\partial t$ terms appearing in these Killing vectors that generate periodic translations, and so t must be identified periodically.

5 Inversion Symmetry of $D = 5$ Kerr-AdS Black Holes

In this section, we first demonstrate that the NUT parameter L introduced in our general rotating black holes is trivial in the special case of $D = 5$ dimensions. However, our demonstration also brings to light a rather remarkable property of the five-dimensional Kerr-AdS black hole metric, namely, that it admits a discrete symmetry transformation which shows that the metric with over-rotation (where the parameters a and b are such that $a^2g^2 > 1$ and/or $b^2g^2 > 1$) is equivalent to a Kerr-AdS metric with under-rotation.

We start with the five-dimensional Kerr-AdS metric written in the (8) with $p = 1$ and $q = 1$, and make the coordinate transformations

$$\begin{aligned} &\psi \to ab^2\chi + ag^2 t + a(1+b^2g^2)\phi\,, \qquad \varphi \to ba^2\chi + bg^2 t + b(1+a^2g^2)\phi\,,\\ &t \to t + a^2b^2\chi + (a^2+b^2)\phi\,, \end{aligned} \tag{42}$$

and define $r^2 = x$ and $v^2 = y$. This leads to the five-dimensional metric

$$\begin{aligned} ds^2 &= (x+y)\Big(\frac{dx^2}{4X} + \frac{dy^2}{4Y}\Big) - \frac{X}{x(x+y)}(dt + y\,d\phi)^2 + \frac{Y}{y(x+y)}(dt - x\,d\phi)^2\\ &\quad \frac{a^2b^2}{xy}\Big(dt - xyd\chi - (x-y)d\phi\Big)^2\,, \end{aligned} \tag{43}$$

where

$$
\begin{aligned}
X &= (1+g^2x)(x+a^2)(x+b^2) - 2Mx \\
&= g^2x^3 + (1+(a^2+b^2)g^2)x^2 + (a^2+b^2+a^2b^2g^2-2M)x + a^2b^2\,, \\
Y &= -(1-g^2y)(a^2-y)(b^2-y) + 2Ly \\
&= g^2y^3 - (1+(a^2+b^2)g^2)y^2 + (a^2+b^2+a^2b^2g^2+2L)y - a^2b^2\,. \qquad (44)
\end{aligned}
$$

Although, the solution ostensibly has the four independent parameters (M, L, a, b), one can in fact scale away either M or L in this five-dimensional case. To do this, we set

$$
\tilde{x} = \lambda^2 x\,, \quad \tilde{y} = \lambda^2 y\,, \quad \tilde{t} = \frac{t}{\lambda}\,, \quad \tilde{\chi} = \frac{\chi}{\lambda^5}\,, \quad \tilde{\phi} = \frac{\phi}{\lambda^3}\,. \qquad (45)
$$

The metric (43) is invariant under this transformation, if we simultaneously transform the parameters a, b, M and L. Thus we define $\widetilde{X} = \lambda^6\, X$ and $\widetilde{Y} = \lambda^6\, Y$, where $\widetilde{X}$ and $\widetilde{Y}$ are defined as in (44) except with tilded parameters $\tilde{a}$, $\tilde{b}$, $\widetilde{M}$ and $\widetilde{L}$. It follows that we shall have

$$
\begin{aligned}
&\lambda^2 + \lambda^2(a^2+b^2)g^2 = 1 + (\tilde{a}^2+\tilde{b}^2)g^2\,, \qquad \lambda^6\, a^2\, b^2 = \tilde{a}^2\, \tilde{b}^2\,, \\
&\lambda^4(a^2+b^2+a^2b^2g^2+2L) = \tilde{a}^2+\tilde{b}^2+\tilde{a}^2\tilde{b}^2g^2+2\widetilde{L}\,, \\
&\lambda^4(a^2+b^2+a^2b^2g^2-2M) = \tilde{a}^2+\tilde{b}^2+\tilde{a}^2\tilde{b}^2g^2-2\widetilde{M}\,. \qquad (46)
\end{aligned}
$$

We can then choose, for example, to set $\widetilde{L} = 0$, and solve the four equations (46) for $\tilde{a}$, $\tilde{b}$, $\widetilde{M}$ and λ. Thus a solution with $L \neq 0$ is transformed into a tilded solution with $\widetilde{L} = 0$, and since this latter solution is just of the original five-dimensional Kerr-AdS form, it follows that the metric (43), even with $L \neq 0$, is also just the five-dimensional Kerr-AdS metric, but with changed values for the rotation and mass parameters. It is nevertheless interesting that the Kerr-de Sitter black hole in $D = 5$ can be put in such a symmetric form.

It should be stressed that the scaling symmetry that we used above in order to show that the parameter L in the five-dimensional metrics is "trivial" is very specific to five dimensions. In particular, it can be seen from (8) that in higher dimensions, when at least one of p or q exceeds 1, the associated metrics on the complex projective spaces will break the scaling symmetry. Thus, as in the case of the simpler NUT generalisations discussed [11, 12], five-dimensions is the exception in not admitting a non-trivial generalisation.

The transformation described above becomes particularly simple if we consider the case of an asymptotically flat five-dimensional rotating black hole, i.e. when $g = 0$. In this case, we have from (46) that $\lambda = 1$ and

$$
\tilde{a}^2 + \tilde{b}^2 + 2\widetilde{L} = a^2 + b^2 + 2L\,, \quad \tilde{a}^2 + \tilde{b}^2 - 2\widetilde{M} = a^2 + b^2 - 2M\,, \quad \tilde{a}^2\tilde{b}^2 = a^2b^2\,. \qquad (47)
$$

Thus $\widetilde{L}+\widetilde{M} = L+M$, and so we can arrange to have $\widetilde{L} = 0$ by taking $\widetilde{M} = L+M$, implying that $\tilde{a}^2 + \tilde{b}^2 = a^2 + b^2 + 2L$, together with $\tilde{a}^2\tilde{b}^2 = a^2b^2$. It is worth noting, however, that even though one can always map into a solution where $\widetilde{L} = 0$, it may, depending upon the original values for a, b and L, correspond to having complex values for $\tilde{a}$ and $\tilde{b}$. Although the metric (43) would still be real, the metric written back in terms of the original ψ, ϕ and t coordinates would then be complex. Thus although the parameter L is really trivial in five dimensions, its inclusion can nevertheless allow one to parameterise the solutions in a wider class without the need for complex coordinate transformations. Similar remarks apply also to the case when $g \neq 0$.

There is another interesting consequence of the five-dimensional scaling symmetry discussed above, namely, that even with the parameter L omitted entirely, the five-dimensional rotating AdS black hole metrics have a symmetry that allows one to map an "over-rotating" black hole (i.e. where $a^2g^2 > 1$ or $b^2g^2 > 1$) into an under-rotating black hole. This can be understood by again considering the transformations in (46), where we now choose not only $\widetilde{L} = 0$ but also $L = 0$. The system of equations then admits a sextet of solutions for $(\tilde{a}, \tilde{b}, \widetilde{M}, \lambda)$ (where we assume, without loss of generality, that the signs of the rotation parameters are unchanged):

$$
\begin{aligned}
\tilde{a} &= a\,, \quad \tilde{b} = b\,, \quad \widetilde{M} = M\,, \quad \lambda = 1\,,\\
\tilde{a} &= b\,, \quad \tilde{b} = a\,, \quad \widetilde{M} = M\,, \quad \lambda = 1\,,\\
\tilde{a} &= \frac{1}{ag^2}\,, \quad \tilde{b} = \frac{b}{ag}\,, \quad \widetilde{M} = \frac{M}{a^4g^4}\,, \quad \lambda = \frac{1}{ag}\,,\\
\tilde{a} &= \frac{1}{bg^2}\,, \quad \tilde{b} = \frac{a}{bg}\,, \quad \widetilde{M} = \frac{M}{b^4g^4}\,, \quad \lambda = \frac{1}{bg}\,,\\
\tilde{a} &= \frac{a}{bg}\,, \quad \tilde{b} = \frac{1}{bg^2}\,, \quad \widetilde{M} = \frac{M}{b^4g^4}\,, \quad \lambda = \frac{1}{bg}\,,\\
\tilde{a} &= \frac{b}{ag}\,, \quad \tilde{b} = \frac{1}{ag^2}\,, \quad \widetilde{M} = \frac{M}{a^4g^4}\,, \quad \lambda = \frac{1}{ag}\,.
\end{aligned} \tag{48}
$$

The first of these is the identity, the second is merely an exchange of the rôles of a and b, whilst the remaining four, modulo exchanges of the a's and the b's, are equivalent and non-trivial. Taking the third as an example, we see that if the metric is over-rotating by virtue of having $a^2g^2 > 1$, then it can be re-expressed, by a change of variables, as a metric which is under-rotating. In fact any five-dimensional Kerr-AdS black hole with over-rotation is equivalent, after a change of coordinates, to one with under-rotation. Of course, after transforming back into the original coordinates in which the over-rotating black hole ostensibly exhibited singular behaviour, one would find that the coordinate ranges that actually reveal that it is well-behaved are not the "naive" ones that led to the original conclusion of singular behaviour.

It is instructive to rewrite the transformations (48) in terms of the original coordinates of the five-dimensional Kerr-AdS metric as given by Hawking, Hunter and Taylor-Robinson in [8]. The metric is given by

$$
\begin{aligned}
ds_5^2 \;=\; & -\frac{\Delta}{\rho^2}\Big[dt - \frac{a\,\sin^2\theta}{\Xi_a}\,d\phi - \frac{b\,\cos^2\theta}{\Xi_b}\,d\psi\Big]^2 + \frac{\Delta_\theta\,\sin^2\theta}{\rho^2}\Big[a\,dt - \frac{r^2+a^2}{\Xi_a}\,d\phi\Big]^2 \\
& \frac{\Delta_\theta\,\cos^2\theta}{\rho^2}\Big[b\,dt - \frac{r^2+b^2}{\Xi_b}\,d\psi\Big]^2 + \frac{\rho^2\,dr^2}{\Delta} + \frac{\rho^2\,d\theta^2}{\Delta_\theta} \\
& + \frac{(1+g^2r^2)}{r^2\,\rho^2}\Big[a\,b\,dt - \frac{b\,(r^2+a^2)\,\sin^2\theta}{\Xi_a}\,d\phi - \frac{a\,(r^2+b^2)\,\cos^2\theta}{\Xi_b}\,d\psi\Big]^2 , \qquad (49)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta &\equiv \frac{1}{r^2}\,(r^2+a^2)(r^2+b^2)(1+g^2r^2) - 2M\,, \\
\Delta_\theta &\equiv 1 - a^2\,g^2\,\cos^2\theta - b^2\,g^2\,\sin^2\theta\,, \\
\rho^2 &\equiv r^2 + a^2\,\cos^2\theta + b^2\,\sin^2\theta\,, \\
\Xi_a &\equiv 1 - a^2\,g^2\,, \qquad \Xi_b \equiv 1 - b^2\,g^2\,. \qquad (50)
\end{aligned}
$$

It satisfies $R_{\mu\nu} = -4g^2\,g_{\mu\nu}$. Taking the transformation in the third line of (48) as an example, we find that after re-expressing our results back in terms of the quantities in (49), the symmetry transformation amounts to

$$
\begin{aligned}
& a \to \frac{1}{ag^2}\,, \quad b \to \frac{b}{ag}\,, \quad M \to \frac{M}{a^4g^4}\,, \\
& \phi \to -\frac{1}{ag}\,\phi\,, \quad \psi \to \psi - \frac{b}{a}\,\phi\,, \quad t \to agt + \frac{1}{g}\,\phi\,, \\
& r \to \frac{1}{ag}\,r\,, \quad \cos\theta \to \Big(1 - \frac{\Xi_a}{\Xi_b}\Big)^{1/2}\cos\theta\,. \qquad (51)
\end{aligned}
$$

It is straightforward to see that this transformation leaves the metric in (49) invariant, and that it therefore allows one to map an over-rotating Kerr-AdS metric into an under-rotating one. In other words, if we perform the transformation of parameters given in the first line in (51), then the metric is restored to its original form by making the general coordinate transformations given also in (51).

Another way of expressing this result is that for any given values of a and b, and provided one allows the coordinates to take complex values in general, then there exist real sections of the complex metric describing Kerr-AdS black holes with under-rotation, and also real sections of the same metric that describe Kerr-AdS black holes with over-rotation.

It is instructive also to re-express the coordinate transformations in (51) in terms of the coordinates y and $\hat\theta$ rather than r and θ, where y and $\hat\theta$ are the coordinates with respect to which the conformal boundary of the Kerr-AdS metric is precisely the standard $\mathbb{R}\times S^3$ Einstein universe, with a round S^3 factor. They are defined by [8]

$$\Xi_a\, y^2\, \sin^2\hat\theta = (r^2+a^2)\,\sin^2\theta\,,\qquad \Xi_b\, y^2\, \cos^2\hat\theta = (r^2+b^2)\,\cos^2\theta\,. \tag{52}$$

Applying the transformations in (51), we find that these imply the coordinate transformations

$$y^2 \to -\frac{1}{g^2} - y^2\sin^2\hat\theta\,,\qquad \tan^2\hat\theta \to -\Big(1+\frac{1}{g^2y^2}\Big)\,\sec^2\hat\theta\,. \tag{53}$$

This result emphasises that the original y =constant boundary, which is the most natural choice from the AdS/CFT point of view [8, 18], is quite different from the y =constant boundary of the transformed metric.

A number of remarks are in order. First, we note that the symmetry we are discussing, which can be expressed in terms of dimensionless quantities as $ag\to 1/(ag)$, exists only in the case of the rotating black hole with a cosmological constant. In the case of asymptotically-flat black holes, for which $g=0$, there is no inversion symmetry. The inversion symmetry for the five-dimensional Kerr-AdS black hole is reminiscent of a T-duality symmetry, in the sense that it implies there is a maximum allowed value for the rotation, namely $a^2g^2=1$. In fact, this value is associated with the supersymmetric limit. If one considers the case where a rotation parameter is becoming very large, i.e. $a^2g^2>>1$, then it can be seen from (51) that in the limiting case when a^2g^2 approaches infinity, the metric will actually approach the pure AdS metric.

It is interesting also to consider the effect on the canonical AdS metric of the transformations (53) taken in isolation. In other words, we start with the AdS metric

$$ds^2 = -(1+g^2y^2)dt^2 + \frac{dy^2}{1+g^2y^2} + y^2(d\hat\theta^2 + \sin^2\hat\theta^2\, d\phi^2 + \cos^2\hat\theta\, d\psi^2)\,, \tag{54}$$

and impose just the coordinate transformations given in (53) (which are independent of the rotation parameters a and b). Upon doing so, we find that the AdS metric (54) transforms according to

$$ds^2 \to -\frac{1}{g^2}\,(1+g^2y^2)d\phi^2 + \frac{dy^2}{1+g^2y^2} + y^2(d\hat\theta^2 + \sin^2\hat\theta^2\, g^2dt^2 + \cos^2\hat\theta\, d\psi^2)\,. \tag{55}$$

This is identical in form to (54), with the rôles of ϕ and gt exchanged. It can easily be seen that in terms of the standard embeddding of AdS_5 in $\mathbb{R}^{4,2}$, the transformation (53) corresponds to exchanging the the rôles of the two timelike embedding coordinates with a pair of spacelike embedding coordinates.

6 Conclusions

In this paper, we have constructed generalisations of certain Kerr-de Sitter and Kerr-AdS black holes in all dimensions $D \geq 6$, in which an additional NUT-type parameter is introduced. Specifically, the cases where we have obtained the more general solutions are where the rotation parameters are specialised so that the metrics have cohomogeneity 2. The nature of the generalisation is then analogous to the way in which a NUT parameter can be introduced in the four-dimensional Kerr-de Sitter metrics.

The same procedure can be followed also in five dimensions, but in this case we find that the additional NUT parameter is trivial, in the sense that it can be absorbed by a rescaling of parameters and coordinates. However, we also found that there exists a remarkable symmetry of the five-dimensional Kerr-AdS metrics, in which one can map a solution where one or both of the rotation parameters are large (the case of over-rotation, where $a^2g^2 > 1$ and/or $b^2g^2 > 1$) into a solution where the rotation parameters are small (i.e. under-rotation). This means that there is effectively a maximum rotation possible, corresponding to the supersymmetric case where $a^2g^2 = 1$ or $b^2g^2 = 1$.

We also studied the supersymmetric limits of the new Kerr-de Sitter-Taub-NUT metrics, showing that after Euclideanisation we can obtain new cohomogeneity-2 Einstein-Sasaki metrics in all odd dimensions $D \geq 7$, and new cohomogeneity-2 Ricci-flat Kähler metrics in all even dimensions $D \geq 6$.

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Appendix

Another Form for the Odd-Dimensional Metrics

If we perform the same angular redefinitions (42) in the general odd-dimensional metrics (8), they may be re-expressed as

$$
\begin{aligned}
ds^2 \;=\;& \frac{r^2+v^2}{X}\,dr^2 + \frac{r^2+v^2}{Y}\,dv^2 + \frac{(r^2+a^2)(a^2-v^2)}{\Xi_a(a^2-b^2)}\,d\Sigma^2_{p-1} + \frac{(r^2+b^2)(b^2-v^2)}{\Xi_b(b^2-a^2)}\,d\widetilde{\Sigma}^2_{q-1} \\
&+\frac{a^2b^2}{r^2v^2}\Big[dt-(r^2-v^2)d\phi - r^2v^2d\chi - \frac{(r^2+a^2)(a^2-v^2)}{a\Xi_a(a^2-b^2)}\,A - \frac{(r^2+b^2)(b^2-v^2)}{b\Xi_b(b^2-a^2)}\,B\Big]^2 \\
&-\frac{X}{r^2+v^2}\Big[dt+v^2d\phi - \frac{a(a^2-v^2)}{\Xi_a(a^2-b^2)}\,A - \frac{b(b^2-v^2)}{\Xi_b(b^2-a^2)}\,B\Big]^2 \\
&+\frac{Y}{r^2+v^2}\Big[dt-r^2d\phi - \frac{a(r^2+a^2)}{\Xi_a(a^2-b^2)}\,A - \frac{b(r^2+b^2)}{\Xi_b(b^2-a^2)}\,B\Big]^2\,, \qquad (56)
\end{aligned}
$$

where we have defined X and Y as

$$
\begin{aligned}
X &\equiv \frac{U}{(r^2+a^2)^{p-1}\,(r^2+b^2)^{q-1}} \\
&= \frac{(1+g^2r^2)(r^2+a^2)(r^2+b^2)}{r^2} - \frac{2M}{(r^2+a^2)^{p-1}\,(r^2+b^2)^{q-1}}\,, \\
Y &\equiv \frac{V}{(a^2-v^2)^{p-1}\,(b^2-v^2)^{q-1}} \\
&= \frac{-(1-g^2v^2)(a^2-v^2)(b^2-v^2)}{v^2} + \frac{2L}{(a^2-v^2)^{p-1}\,(b^2-v^2)^{q-1}}\,. \qquad (57)
\end{aligned}
$$

This form can sometimes be useful, since it is expressed in a manifest "vielbein basis."

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