MATHEMATICAL ANALYSIS OF UNSTEADY FLOWS OF FLUIDS WITH PRESSURE, SHEAR-RATE AND TEMPERATURE DEPENDENT MATERIAL MODULI, THAT SLIP AT SOLID BOUNDARIES*

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Abstract. In his treatise titled *"The physics of high pressures" (1931)*, Bridgman carefully documented that the viscosity and the thermal conductivity of most liquids depend on the pressure and the temperature. The relevant experimental studies show that even at high pressures the variations of the values in the density are insignificant in comparison to that of the viscosity, and it is thus reasonable to assume that the liquids in question are incompressible fluids with pressure dependent viscosities.

We rigorously investigate the mathematical properties of unsteady three-dimensional internal flows of such incompressible fluids. The model is expressed through a system of partial differential equations representing the balance of mass, the balance of linear momentum, the balance of energy and the equation for the entropy production. Assuming that we have Navier's slip at the impermeable boundary we establish the long-time existence of a (suitable) weak solution when the data are large.

Key words. generalized Navier-Stokes-Fourier system, incompressible fluid, pressure-dependent viscosity, shear-dependent viscosity, temperature-dependent viscosity, unsteady flows, Navier's slip boundary conditions

AMS subject classifications. 35Q30,35Q72,76D03,76A05

1. Introduction.

In both fluids and solids, the thermoelectric and thermomechanical properties depend on the pressure. In the case of liquids, properties such as the viscosity, thermal and electrical conductivity, depend upon the "pressure". The exhaustive book of Bridgman (1931) on the "Physics of High Pressure" provides a detailed documentation of the manner in which these properties vary with pressure with regard to the literature prior to 1931. Interest in determining the properties of fluids under high pressure has not waned and investigations have continued to date to document the variation of a variety of properties with pressure. In this study we are interested in a rigorous mathematical analysis of fluids whose viscosity and thermal conductivity vary with pressure. We shall not be concerned with the electromagnetic response of fluids.

In the case of a compressible Navier-Stokes fluid, as the bulk and the shear viscosities depend on the density, and the "thermodynamic pressure" also depends on the density, it is clear that the viscosities will depend upon the "thermodynamic pressure". However, in the case of the incompressible Navier-Stokes fluid, one needs to justify if, and when, the shear-viscosity can be assumed to depend on the "mechanical pressure" (the mean normal stress). Stokes (1845) was quite aware of the need to

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delineate carefully when one can ignore, or have to take into account, the dependence of the shear-viscosity on pressure. Stokes (1845) states that "If we suppose viscosity to be independent of pressure also, and substitute ..." which clearly implies that in general he is of the view that the viscosity can depend upon pressure. This becomes even more apparent when he further delineates when it might be reasonable to suppose that the viscosity is independent of pressure: "Let us now consider in what cases it is allowable to suppose viscosity to be independent of the pressure. It has been concluded by Du Buat from his experiments on the motion of water in pipes and canals, that the total retardation of the velocity due to friction is not increased by increasing the pressure... I shall therefore suppose that for water, and by analogy for other incompressible fluids, viscosity is independent of the pressure."

There are clearly situations wherein the effect of viscosity on pressure cannot be ignored though the effect of the pressure on the density can be neglected. This is clearly the situation in Elastohydrodynamics (see Szeri (1998)) wherein one comes across the viscosity changing by a factor of 10^8, while the density varies by a few percent. In such cases, it is reasonable to approximate a liquid as being incompressible with the viscosity depending on the pressure. Barus (1893) suggested the following relationship between the viscosity μ and the pressure p:

$$\mu(p) = A\exp(\alpha p), \qquad A \text{ and } \alpha \text{ are positive constants}. \tag{1.1}$$

It is also well known that the viscosity of liquids depend upon the temperature θ, the viscosity is decreasing with increasing temperature. Reynolds' model (see Szeri (1998)) takes the form

$$\mu(\theta) = \mu_0 \exp(-m\theta), \qquad \mu_0 \text{ and } m \text{ are positive constants}. \tag{1.2}$$

A popular model for the variation in viscosity due to temperature is due to Vogel (1922) and takes the form:

$$\mu(\theta) = \mu_0 \exp\left[\frac{a}{b+\theta}\right], \qquad \mu_0, a \text{ and } b \text{ are positive constants}. \tag{1.3}$$

Later on, Andrade (1930) had provided a formula for how the viscosity varies with density, pressure and temperature for a compressible fluid:

$$\mu(p,\varrho,\theta) = A\varrho^{\frac{1}{2}} \exp\left(\frac{B}{\theta}(p + D\varrho^2)\right), \tag{1.4}$$

where A, B and D are constants, ϱ is the density.

Interestingly, while in Reynolds' model the viscosity vanishes as the temperature tends to infinity, in Vogel's and Andrade's models they attain a constant finite value.

The relationship (1.4) might give the erroneous impression that the changes in the density can be enormous as the relationship is exponential with respect to ϱ^2. However, such is not the case for the range of pressures for which the change of the viscosity is of the order of 10^8, in virtue of the exponent for the density term being exceedingly small the change in density is indeed small. In fact, careful recent experiments have shown that the density changes according to the relation (see Dowson & Higginson (1966))

$$\varrho = \varrho_0 \left(1 + \frac{0.6p}{1+1.7p}\right). \tag{1.5}$$

Thus, the percentage change in the density for instance for a pressure change from $2GPa$ to $3GPa$ is approximately 3.5%, while the viscosity change is of order of 10^8. Thus, one can indeed approximate most organic liquids as incompressible fluids with pressure dependent viscosities.

Extensive bibliography on the experimental work concerning the variation of viscosity as well as many other properties of materials, under the influence of high pressures, can be found in the authoritative treatise by Bridgman (1931). However, there has been considerable continued interest in the variation of viscosity with both temperature and pressure as evidenced by the fact that experiments have been carried out as recently as 2006 (see Bair & Kottke (2003) for experiments that show an astonishing variations of viscosity with pressure) and continues to be carried out today with the aim towards understanding this dependence of viscosity on pressure and temperature, see Bair (2006). Other experiments in the recent past that are concerned with the variation of the viscosity with pressure are those by Cutler et al. (1958); Griest et al. (1958); Johnson & Cameron (1967); Johnson & Greenwood (1980); Johnson & Tevaarwerk (1977), Bair & Winer (1992), Roelands (1966), Paluch et al. (1999), Irwing & Barlow (1971), Bendler et al. (2001), among others.

At very high pressures the fluids can be close to glass transition. An interesting issue that requires careful consideration is the nature of the boundary condition that comes into play at the interface between the fluid that is at high pressure and an impervious solid wall. While it is customary to assume that the fluid adheres to the boundary, it is possible that the fluid slips or stick-slips at the solid boundary.

The increase in the thermal conductivity of fluids due to increase of pressure are not as dramatic as that of the viscosity, but be that as it may, the variation is large enough to warrant our taking them into account. The following comments of Bridgman in this matter are worth recording: "In general characteristics the effect of pressure is the same on all liquids tried. The thermal conductivity increases under a pressure of 12,000 kg/cm^2 by an amount which varies from 1.5 fold for water to 2.7 fold for the normal pentane. In general, the effect is greater for those substances with the lower boiling or freezing temperatures (in general these are also the most compressible substances). The effect is not at all proportional to the pressure, but at high pressures a given increment of pressure produces a much smaller effect, both absolutely and relatively, than at lower pressure". We shall thus allow the liquid's thermal conductivity to depend on the pressure. The variation of the thermal conductivity with pressure is generally non-linear (see the variation in the case of i-propyl alcohol with pressure at two different temperatures in Bridgman (1931, Fig. 76)). It is also worth observing that the effect of thermal conductivity on temperature unlike that with pressure does not have the same sign with increments in temperature.

We now turn our attention to a brief discussion of mathematical results concerning fluids with pressure dependent thermo-mechanical properties. Unlike fluids with pressure dependent viscosity, there are no rigorous mathematical studies concerning the thermo-mechanical response of fluids whose viscosity and thermal conductivity both depend on pressure. In this study, we consider the case when the viscosity and thermal conductivity depend on pressure, temperature and shear-rate. Before we formulate the results concerning such fluids it is worth discussing briefly results that have been established for the purely mechanical response of fluids with pressure dependent viscosity.

Rigorous mathematical works concerning the flows of fluids with pressure dependent viscosity are of reasonably recent origin. The first studies by Renardy (1986), Gazzola (1997) and Gazzola & Secchi (1998) were concerned with the existence of solutions that are for a short time and due to small data. If in addition, the viscosity depends sublinearly and appropriately on the shear rate then the relevant governing equations are amenable to mathematical analysis. Málek et al. (2002), Hron et al. (2003), Franta et al. (2005) and Bulíček et al. (2007a) have established long-time and large data existence results concerning the flows of a class of incompressible fluids whose viscosity depends on both the pressure and symmetric part of the velocity gradient in a suitable manner. Franta et al. (2005) established the existence of weak solutions for the steady flows of fluids whose viscosity depends on both pressure and the symmetric part of the velocity gradient, that satisfy Dirichlet boundary conditions. Earlier, Málek et al. (2002) and Hron et al. (2003) had established global-in-time existence for unsteady flows of such fluids under spatially periodic boundary conditions. The extension of these results to unsteady flows in bounded domains subject to Navier's slip is due to Bulíček et al. (2007a). A subclass of the viscosities that lead to uniqueness of planar flows was identified in Bulíček et al. (2005) where the long-time behavior of solutions and their convergence towards a finite-dimensional attractor that attracts solution trajectories with the exponential rate were also investigated. All these mathematical studies have however the same shortcoming with respect to the asymptotic behavior for the viscosity with pressure, namely, the requirements on the viscosity μ imply that, as $p \to \infty$, either $\mu(p)/p \to 0$ or $|\mu(p)/p|$ is bounded, which is contradicted by experiments[1]. This objection applies to the results presented in this paper, as well.

Another drawback of all the above mentioned mathematical studies consists in fixing the pressure by prescribing its mean value (over the whole domain). It would be however more desirable to prescribe the average pressure over a small area (a set of non-zero area measure) on the boundary. This in fact corresponds to prescribing the average pressure that is determined by a pressure measuring device. This is more realistic than prescribing pressure at a "point" as no pressure measuring device measures pressure at a "point", what is measured is average pressure in a small area containing the point. In this paper, for the first time we are able to obtain existence results by merely prescribing the average pressure over a set of non-zero *volume* measure. A more detailed discussion of this issue can be found in the next section. It would be more physically appropriate to be able to establish the results by prescribing the pressure over a nonzero area on the boundary, this is not a trivial task and we feel that we will be able to address this issue in a forthcoming study.

The above studies are concerned with issues regarding existence and uniqueness of the flows of such fluids. A complementary and useful set of studies concerning such fluids are solutions for the flows of such fluids in specific geometries. The nature of the geometry allows us to seek special solution using semi-inverse techniques. Hron et al. (2001) studied the flow of such fluids between infinite parallel plates and were able to obtain explicit exact solutions when the viscosity depends in a special manner on the pressure. They were able to show that the form for the velocity profile depends on how the viscosity changes with pressure. It was shown that the profile could vary from the parabolic profile for a fluid with constant viscosity to that which resembles a triangular shape as well as a flattened plug flow profile. In addition, for a certain range of values for the parameters, multiple solutions were explicitly calculated. Hron et al.

[1] In the second case, we can however argue by dealing with a modified viscosity μ_m that coincides with the original one for the values of $p \le p_{\text{large}}$ and $\mu_m(p) = \mu(p_{\text{large}})$ for the values of p above p_{large}.

(2001) also carried out a numerical solution of the flows of such fluids in much more complicated geometries such as that in the annular region between two cylinders that are rotating about distinct but parallel axes (geometry corresponding to the flow in a journal bearing). They also study the flow of such a fluid across a slot. It is apparent from the solutions that they are markedly different than that for the classical Navier-Stokes fluid. Later, Vasudevaiah & Rajagopal (2005) considered the fully developed flow of a fluid that has a viscosity that depends on the pressure and shear rate and were able to obtain explicit exact solutions for the problem. The recent study by Prasad & Rajagopal (2006) points to a very interesting feature concerning the flows of fluids with pressure dependent viscosities, namely the capability of such fluids to develop boundary layers in that the vorticity is concentrated by virtue of an increase in pressure due to gravitational effects and hence an increase in the viscosity. There are a few other studies in the above spirit but we shall not discuss them here. One special approximation of the equations governing the flow of such fluids warrants mention, namely the approximation that is used in elastohydrodynamics. Unfortunately, the classical derivation of the modified Reynolds' equation that allows the viscosity to depend on the pressure has a basic inconsistency. This error has been pinpointed, and corrected, in the recent paper by Rajagopal & Szeri (2003).

Finally, we turn to a discussion of the problem that is considered here and our major results.

In most real applications like elastohydrodynamics wherein the effects of high pressure are significant, we find that the heat generated due to dissipation and the consequent changes in the temperature are significant. It is thus necessary to consider such problems from a fully thermodynamic perspective taking into account the balance of energy, in addition to the balance equations for the purely mechanical problem. It is to such a treatment of the problem that this study is addressed.

In this paper we consider the existence of solutions to the balance of mass (which in virtue of the constraint of the incompressibility reduces to $\operatorname{div} \boldsymbol{v} = 0$), the balance of linear momentum and the balance of energy, for the velocity, pressure and temperature fields (the balance of angular momentum is automatically satisfied by the choice of the Cauchy stress being symmetric). As mentioned earlier, we allow both the viscosity and the thermal conductivity to depend on the pressure, the temperature and the shear rate. We consider the situation wherein the velocity field satisfies the Navier-slip boundary condition at the impervious wall, as considered by Bulíček et al. (2007a) for the purely mechanical problem. With regard to the thermal boundary condition, we assume that there is no heat flux at the boundary. We establish the long-time existence of a (suitable) weak solution when the data are large.

The arrangement of the paper is as follows. In the next section, we provide the mathematical formulation of the problem under consideration, including the governing equations, boundary conditions, assumptions on the structure of the constitutive quantities and the definition of a solution. We also present the main result and explicitly mention several assertions that follow as corollaries. In Section 3, we define a two-parameter (ε and η) approximation of the original problem and briefly discuss its existence that is established by using a two-level Faedo-Galerkin method. Finally, we study the behavior of (ε, η) approximate solutions by first allowing ε and then η to tend to zero, and use this to prove the main result.

2. Formulation of the problem and the results.

2.1. Balance equations, boundary and initial conditions. Equation for the entropy production. We are interested in understanding the mathematical properties relevant to unsteady flows of a homogeneous incompressible fluid whose viscosity and thermal conductivity depend on the pressure, the temperature and the shear rate, flowing in a bounded container that can be identified with a bounded open connected set Ω in $\mathbb{R}^3$ with the boundary $\partial\Omega$. We are interested in solutions that exist in the domain $Q := (0,T)\times\Omega$, where $(0,T)$ denotes the time interval of interest. We set $\Gamma := (0,T)\times\partial\Omega$.

Motions of such fluids are described in terms of the velocity field $\boldsymbol{v}$, the pressure (mean normal stress) p and the temperature θ in terms of a system of partial differential equations that are a consequence of the balance of mass, balance of linear and angular momentum, and balance of energy. The balance of mass in the case of a homogeneous and incompressible fluid reduces merely to the divergence of the velocity field being zero, while the balance of angular momentum which leads to the Cauchy stress being symmetric is automatically met by virtue of the form chosen for the the Cauchy stress T:

$$\mathsf{T} = \varrho^*(-p\mathsf{I} + \mathsf{S}) \qquad (\varrho^* > 0 \text{ is the constant density}), \tag{2.1}$$

the extra stress S being symmetric. The system governing the flows of interest take the form

$$\begin{aligned} \operatorname{div}\boldsymbol{v} = 0, \quad \boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) - \operatorname{div}\mathsf{S} &= \boldsymbol{f} - \nabla p, \\ E_{,t} + \operatorname{div}(E\boldsymbol{v}) = \operatorname{div}(\mathsf{S}\boldsymbol{v} - p\boldsymbol{v} - \boldsymbol{q}) &+ \boldsymbol{f}\cdot\boldsymbol{v}, \end{aligned} \tag{2.2}$$

where $E = \frac{1}{2}|\boldsymbol{v}|^2 + e$ is the sum of specific kinetic energy $\frac{1}{2}|\boldsymbol{v}|^2$ and specific internal energy e, $\boldsymbol{q}$ is the heat flux and $\boldsymbol{f}$ represents the specific body forces. In this setting, for given functions $\boldsymbol{v}_0$ and E_0 defined in Ω, we prescribe the initial conditions

$$\boldsymbol{v}(0,x) = \boldsymbol{v}_0(x) \quad \text{and} \quad E(0,x) = E_0(x) \qquad (x\in\Omega). \tag{2.3}$$

We assume that the boundary is completely described from outside by a finite number of overlapping $C^{1,1}$-mappings, and if this is indeed the case we write $\Omega\in C^{1,1}$. We prescribe the following boundary conditions

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{v}_\tau = -\frac{1}{\alpha}[\mathsf{T}\boldsymbol{n}]_\tau = -\frac{1}{\alpha}[\mathsf{S}\boldsymbol{n}]_\tau \quad \text{and} \quad \boldsymbol{q}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma, \tag{2.4}$$

where $\boldsymbol{n}$ is the unit outward normal and $\boldsymbol{z}_\tau$ stands for the projection of the velocity field to the tangent plane, i.e. $\boldsymbol{z}_\tau = \boldsymbol{z} - (\boldsymbol{z}\cdot\boldsymbol{n})\boldsymbol{n}$. The first condition in (2.4) expresses the fact that the solid boundary is impervious, the second equation in (2.4) is Navier's slip boundary condition, and the last condition states that there is no heat flux across the boundary.

We will also require that for some open $\Omega_0 \subseteq \Omega$

$$\int_{\Omega_0} p(t,x)\,dx = h(t) \quad \text{for all } t\in[0,T], \tag{2.5}$$

where h is a given function. This condition on the pressure demands some discussion as no such condition needs to be appealed to in the case of a classical Navier-Stokes fluid whose viscosity is constant. In the classical Navier-Stokes equation for an incompressible fluid, one encounters only the pressure gradient in the governing equations, while in the case of the flow of a fluid with pressure dependent viscosity, the pressure itself appears. This implies that one requires some information concerning the pressure field as quantities such as the viscosity that appear in the governing equations depend on the pressure field. From a physical standpoint two flows can have the same pressure gradient associated with them while the pressure field could be totally different in the two cases. However, since the viscosity depends on the pressure, it is imperative to know the value of the pressure in order to know the value of the viscosity. However, within the context of determining weak solutions, it makes no sense to prescribe the value of the pressure at a point and thus it is necessary to prescribe the pressure over a set of non-zero volume measure. As mentioned in the introduction, in previous studies concerning fluids with pressure dependent viscosity a mean value for the pressure over the whole domain was prescribed. Here, for the first time we are able to establish existence results by merely prescribing the average value of the pressure over an arbitrary (possibly small) set of non-zero volume measure.

In addition to balance laws (2.2) one usually requires that all processes that a body undergoes meets the second law of thermodynamics. Expressed as the Claussius-Duhem's inequality, it takes the form

$$S_{,t} + \operatorname{div}(S\boldsymbol{v}) + \operatorname{div}(\boldsymbol{q}/\theta) \geq 0\,, \tag{2.6}$$

where S denotes the specific entropy, and θ is the temperature introduced, see Callen (1985), via the relation

$$\frac{1}{\theta} = \frac{\partial S}{\partial e} > 0. \tag{2.7}$$

Defining the specific rate of entropy production ξ through the formulae

$$\xi = S_{,t} + \operatorname{div}(S\boldsymbol{v}) + \operatorname{div}(\boldsymbol{q}/\theta)\,, \tag{2.8}$$

it follows from (2.6) that ξ should be non-negative.

2.2. Constitutive equations. Assumptions concerning material moduli. Examples. Regarding the structure of the equations that constitute the specific fluid, we assume that the quantities e, S, $\boldsymbol{q}$ and ξ take the form[2]

$$e = c_v\theta, \qquad \text{with } c_v \in (0,\infty)\,, \tag{2.9}$$

$$\mathsf{S} = \mathsf{S}^*(p,\theta,\mathsf{D}(\boldsymbol{v})) = \nu(p,\theta,|\mathsf{D}(\boldsymbol{v})|^2)\mathsf{D}(\boldsymbol{v}) \qquad \text{with } \nu \geq 0\,, \tag{2.10}$$

$$\boldsymbol{q} = \boldsymbol{q}^*(p,\theta,\mathsf{D}(\boldsymbol{v}),\nabla\theta) = -\kappa(p,\theta,|\mathsf{D}(\boldsymbol{v})|^2)\nabla\theta \qquad \text{with } \kappa \geq 0\,, \tag{2.11}$$

$$\xi \geq -\frac{\boldsymbol{q}\cdot\nabla\theta}{\theta^2} + \frac{\mathsf{T}\cdot\nabla\boldsymbol{v}}{\theta}\,, \tag{2.12}$$

where $\nabla\boldsymbol{v}$ denotes the velocity gradient and $\mathsf{D}(\boldsymbol{v})$ is its symmetric part. Note that due to (2.10) and (2.11), it follows from (2.1), (2.7) and (2.12) that ξ is non-negative, thus meeting the second law of thermodynamics.

In general, the specific heat is a function of the temperature and in the case of pressure dependent fluids it is also a function of the pressure. In Remark 2.2 below, we relax the requirement imposed by the equation (2.9) and consider a general (possibly nonlinear) invertible relationship between e and θ assuming that e has polynomial growth for θ large. We shall show that the existence result for such a generalization is a consequence of the main result stated under the assumption (2.9). The existence is established by "replacing" θ by e in the balance of energy, see Remark 2.2 for details.

[2] More general relationships between the internal energy e and temperature θ are discussed in Remark 2.2.

In this paper, we shall not assume that the specific heat depends on the pressure as sufficient experimental data are not available concerning the same.

The constitutive equations (2.9)–(2.12) generalize several classes of incompressible fluids. In particular, they include the cases where

(1) $\nu = \nu(\theta)$ and $\kappa = \kappa(\theta)\,,$

(2) $\nu = \nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\,,$

(3) $\nu = \nu(\theta, |\mathbf{D}(\boldsymbol{v})|^2)$ and $\kappa = \kappa(\theta, |\mathbf{D}(\boldsymbol{v})|^2)$ or $\kappa = \kappa(\theta)$ or $\kappa = \kappa(|\mathbf{D}(\boldsymbol{v})|^2)\,,$

to mention a few.

Moreover, assuming that $S = S(\theta)$ it follows from (2.7) and (2.9) that

$$S = c_v \ln\theta \qquad \text{and} \qquad E = c_v\theta + 1/2|\boldsymbol{v}|^2\,. \tag{2.13}$$

Using (2.3) and (2.13) we easily identify the initial temperature θ_0 through $\theta_0 := 1/c_v(E_0 - |\boldsymbol{v}_0|^2/2)$.

We assume that the viscosity ν that appears in (2.10) is a continuous mapping of $\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+_0$ into $\mathbb{R}^+$ being differentiable w.r.t. the first and third variable. In addition we assume that for some $r \in [1,2)$

$$\begin{aligned}&\text{there are } C_1, C_2 \in (0,\infty) \text{ and a nonincreasing}^1 \text{ function } \gamma_1 \in \mathcal{C}(\mathbb{R}),\ \gamma_1 \ge 1 \text{ in } \mathbb{R},\\ &\text{such that for all } p\in\mathbb{R},\ \theta\in\mathbb{R}^+ \text{ and } \mathbf{B},\mathbf{D}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}:\\ &C_1\gamma_1(\theta)(1+|\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{B}|^2 \le \frac{\partial \mathbf{S}^*_{ij}(p,\theta,\mathbf{D})}{\partial \mathbf{D}_{kl}}\mathbf{B}_{ij}\mathbf{B}_{kl} \le C_2\gamma_1(\theta)(1+|\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{B}|^2,\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\text{there is a } \gamma_0 \ge 0 \text{ and a function } \gamma_2 \text{ such that for all } p,\ \theta \text{ and } \mathbf{D}:\\ &\left|\frac{\partial \mathbf{S}^*(p,\theta,\mathbf{D})}{\partial p}\right| \le \gamma_0\gamma_2(\theta)(1+|\mathbf{D}|^2)^{\frac{r-2}{4}}\,.\end{aligned} \tag{2.15}$$

We also assume that the heat conductivity κ that appears in (2.11) is a continuous mapping of $\mathbb{R}\times\mathbb{R}^+_0\times\mathbb{R}^+_0$ into $\mathbb{R}^+$ such that for some $\beta\in\mathbb{R}$

$$\begin{aligned}&\text{there are } C_4, C_5 \in (0,\infty) \text{ such that for all } p\in\mathbb{R},\ \theta\in\mathbb{R}^+ \text{ and } \mathbf{D}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}:\\ &C_4\theta^\beta \le \kappa(p,\theta,|\mathbf{D}|^2) \le C_5\theta^\beta.\end{aligned} \tag{2.16}$$

It is not difficult to observe that the viscosities of the form[2]:

$$\nu(p,\theta,|\mathbf{D}|^2) = \left(1+\mu_i(p,\theta)+|\mathbf{D}|^2\right)^{\frac{r-2}{2}}\,, \tag{2.17}$$

$$\nu(p,\theta,|\mathbf{D}|^2) = \nu_0\exp(1/\theta - 1/\theta_0)(1+\alpha\mu_i(p,\theta)+|\mathbf{D}|^2)^{\frac{r-2}{2}}\,, \tag{2.18}$$

where $\mu_i(p,\theta)$ have the form $(s \ge 0)$:

$$\mu_1 = \left(1 + ap^2/(1+\theta)\right)^{-s} \qquad \text{or} \qquad \mu_2 = \min\left\{2^{-s}, (1+\exp(\alpha p/\theta))^{-s}\right\}, \tag{2.19}$$

[1] We encourage the reader to check that in order to prove Theorem 2.1 it suffices to assume that γ_1 is continuous, and bounded at infinity.

[2] Note that formula (2.18), with $r = 2$, coincides with the so called Arrhenius law, see Baranger & Mikelić (1995) and references therein.

satisfy (2.14)-(2.15).

We end this subsection by stating the consequences of the assumption (2.14) that concern coerciveness, growth, monotonicity and the Lipschitz continuity related to S^*. We set

$$\mathcal{I}(\mathsf{D},\mathsf{B}) := |\mathsf{D}-\mathsf{B}|^2 \int_0^1 (1+|\mathsf{B}+s(\mathsf{D}-\mathsf{B})|^2)^{\frac{r-2}{2}}\ ds. \tag{2.20}$$

Lemma 2.1. *Let S^* satisfy (2.14). Then there are constants $C_6, C_7 > 0$ such that for all p, θ, D*

$$\begin{aligned} C_6\gamma_1(\theta)(|\mathsf{D}|^r - 1) &\le \mathsf{S}^*(p,\theta,\mathsf{D})\cdot\mathsf{D}, \\ |\mathsf{S}^*(p,\theta,\mathsf{D})| &\le C_7\gamma_1(\theta)(|\mathsf{D}|^{r-1}+1). \end{aligned} \tag{2.21}$$

Moreover, if S^* satisfies (2.14) and (2.15) with $r\in(1,2)$ then

$$\frac{C_1}{2}\gamma_1(\theta)\mathcal{I}(\mathsf{D},\mathsf{B}) \le \frac{\gamma_0^2\gamma_2^2(\theta)}{2C_1\gamma_1(\theta)}|p-q|^2 + (\mathsf{S}^*(p,\theta,\mathsf{D}) - \mathsf{S}^*(q,\theta,\mathsf{B}), \mathsf{D}-\mathsf{B}) \tag{2.22}$$

and

$$\begin{aligned} &|\mathsf{S}^*(p,\theta,\mathsf{D}) - \mathsf{S}^*(q,\theta,\mathsf{B})| \\ &\quad \le C_2\gamma_1(\theta)|\mathsf{D}-\mathsf{B})| \int_0^1 (1+|\mathsf{B}+s(\mathsf{D}-\mathsf{B})|^2)^{\frac{r-2}{2}}\ ds + \gamma_0\gamma_2(\theta)|p-q| \\ &\quad \le C_2\gamma_1(\theta)\mathcal{I}^{\frac{1}{2}}(\mathsf{D},\mathsf{B}) + \gamma_0\gamma_2(\theta)|p-q|. \end{aligned} \tag{2.23}$$

Proof. For a proof of (2.21), see Málek et al. (1996, Lemma 1.19 p.198). The inequalities (2.22) and (2.23) can be proved following the procedure used by Málek et al. (2002) or Bulíček et al. (2007a) where the same inequalities are established when $\gamma_1, \gamma_2 \equiv 1$.

2.3. Basic notations and auxiliary assertions. To avoid any misunderstanding in the definition of the appropriate function spaces, we shall assume that the boundary $\partial\Omega$ of the set Ω is Lipschitz, and we write $\Omega \in \mathcal{C}^{0,1}$. For $q \in [1,\infty]$ we define the Lebesgue spaces $L^q(\Omega)$ and the Sobolev spaces $W^{1,q}(\Omega)$ in a standard way, and we denote the trace of a Sobolev function u, if it exists, through $\operatorname{tr} u$. For our purposes we introduce the subspaces (and their duals) of vector-valued Sobolev functions from $W^{1,q}(\Omega)^3$ which have zero normal component on the boundary(note that $q' = q/(q-1)$):

$$\begin{aligned} &W^{1,q}_{\boldsymbol{n}} := \left\{\boldsymbol{v} \in W^{1,q}(\Omega)^3; \operatorname{tr}\boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\right\},\ W^{-1,q'}_{\boldsymbol{n}} := \left(W^{1,q}_{\boldsymbol{n}}\right)^*, \\ &W^{1,q}_{\boldsymbol{n},\operatorname{div}} := \left\{\boldsymbol{v}\in W^{1,q}_{\boldsymbol{n}}; \operatorname{div}\boldsymbol{v} = 0\right\},\ W^{-1,q'}_{\boldsymbol{n},\operatorname{div}} := \left(W^{1,q}_{\boldsymbol{n},\operatorname{div}}\right)^*,\ L^q_{\boldsymbol{n},\operatorname{div}} := \overline{\left\{\boldsymbol{v}\in W^{1,q}_{\boldsymbol{n},\operatorname{div}}\right\}}^{\|\cdot\|_q}. \end{aligned}$$

For $r, q \in [1,+\infty]$, we also introduce relevant spaces of a Bochner-type, namely,

$$\begin{aligned} &X^{r,q} := \{\boldsymbol{u}\in L^r(0,T;W^{1,r}_{\boldsymbol{n}}) \cap L^q(0,T;L^q(\Omega)^3), \operatorname{tr}\boldsymbol{u}\in L^2(0,T;(L^2(\partial\Omega))^3)\}, \\ &X^{r,q}_{\operatorname{div}} := \{\boldsymbol{u}\in X^{r,q}, \operatorname{div}\boldsymbol{u} = 0\}. \end{aligned}$$

Let g and h be (scalar, vector- or tensor-valued) functions. We shall write (f,g) for $\int_\Omega f(x)g(x)\ dx$ if $fg\in L^1(\Omega)$, $(f,g)_Q$ for $\int_Q f(t,x)g(t,x)\ dx\ dt$ if $fg \in L^1(Q)$,

$(f,g)_{\partial\Omega}$ for $\int_{\partial\Omega} f(S)g(S)\,dS$ if $fg \in L^1(\partial\Omega)$ and $(f,g)_\Gamma$ for $\int_\Gamma f(t,S)g(t,S)\,dS\,dt$ if $fg \in L^1(\Gamma)$. If $f \in X$ and $g \in X^*$ we often use the symbol $\langle g, f\rangle$ instead of $\langle g, f\rangle_{X^*,X}$. We write $\mathcal{D}$ instead of $\mathcal{D}(-\infty, T; \mathcal{C}^1(\overline{\Omega}))$.

Also, we shall use the space $\mathcal{C}(0,T;L^q_{\text{weak}}(\Omega))$ (similarly for vector-valued functions) consisting of all $u \in L^\infty(0,T;L^q(\Omega))$, satisfying $(u(t),\varphi) \in \mathcal{C}([0,T])$ for all $\varphi \in \mathcal{C}(\overline{\Omega})$.

Finally, we recall the L^q theory concerning the Neumann problem for the Laplace operator and the Helmholtz decomposition. For a given $z \in L^q(\Omega)$ with $\int_\Omega z\,dx = 0$, we use the symbol $\mathcal{N}^{-1}_{\Omega_0}(z)$ to denote the unique solution of the Neumann problem

$$\triangle u = z \text{ in } \Omega, \qquad \nabla u \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega, \qquad \int_{\Omega_0} u\,dx = 0. \tag{2.24}$$

For special case $\Omega_0 = \Omega$ we use abbreviation $\mathcal{N}^{-1}(z) := \mathcal{N}^{-1}_\Omega(z)$. Particularly, denoting $g^{\boldsymbol{v}} := \mathcal{N}^{-1}(\operatorname{div}\boldsymbol{v})$ we define the vector $\boldsymbol{v}_{\text{div}}$, as

$$\boldsymbol{v}_{\text{div}} := \boldsymbol{v} - \nabla g^{\boldsymbol{v}} \quad (\implies \boldsymbol{v} = \boldsymbol{v}_{\text{div}} + \nabla g^{\boldsymbol{v}} \quad \text{Helmholtz decomposition}). \tag{2.25}$$

For and arbitrary[3] $\Omega \in \mathcal{C}^{1,1}$, the L^q-regularity theory for the Neumann problem (2.24) implies, see Grisvard (1985, Proposition 2.5.2.3, p. 131), that

$$\|g^{\boldsymbol{v}}\|_{2,q} \le C_{reg}(\Omega,q)\|\operatorname{div}\boldsymbol{v}\|_q \qquad \|\boldsymbol{v}_{\text{div}}\|_{1,q} \le (C_{reg}(\Omega,q)+1)\|\boldsymbol{v}\|_{1,q}, \tag{2.26}$$

$$\|g^{\boldsymbol{v}}\|_{1,s} \le C(\Omega,s)\|\boldsymbol{v}\|_s \qquad \|\boldsymbol{v}_{\text{div}}\|_s \le (C(\Omega,s)+1)\|\boldsymbol{v}\|_s. \tag{2.27}$$

The following lemmas summarize helpful inequalities related to functions from the Sobolev spaces.

Lemma 2.2. *Let $1 < q_1, q_2 < \infty$. Set*

$$\mathcal{S} := \{\boldsymbol{v}; \boldsymbol{v} \in L^\infty(0,T;L^2(\Omega)^3) \cap L^r(0,T;W^{1,r}_{\boldsymbol{n}}), \boldsymbol{v}_{,t} \in L^{q_1}(0,T;W^{-1,q_2}_{\boldsymbol{n},\text{div}})\}.$$

If $2 \ge r > \frac{6}{5}$ and $\{\boldsymbol{v}^i\}_{i=1}^\infty$ is bounded in $\mathcal{S}$, then $\{\operatorname{tr}\boldsymbol{v}^i\}_{i=1}^\infty$ is precompact in $L^p(0,T;L^s(\partial\Omega)^3)$ for all $p, s \in \langle 1,\infty)$ satisfying

$$p < s\frac{5r-6}{3s-4}, \quad s \in (\frac{4}{3}, \frac{2r}{3-r}). \tag{2.28}$$

Proof. See Bulíček et al. (2007a, Lemma 1.4) where even more general result is proved.

Corollary 2.1. *Let $r > \frac{8}{5}$. Let $\{\boldsymbol{v}^i\}_{i=1}^\infty$ be bounded in $\mathcal{S}$. Then $\{\operatorname{tr}\boldsymbol{v}^i\}_{i=1}^\infty$ is precompact in $L^2(0,T;L^2(\partial\Omega)^3) \cap L^{r'}(0,T;L^{\frac{2r}{3(r-1)}}(\partial\Omega)^3)$.*

Lemma 2.3. (Korn's inequality) *Let $q \in (1,\infty)$. Then there exists a positive constant C depending only on Ω and q such that for all $\boldsymbol{v} \in W^{1,q}(\Omega)^3$ which has the trace $\operatorname{tr}\boldsymbol{v} \in L^2(\partial\Omega)^3$ the following inequality holds*

$$C\|\boldsymbol{v}\|_{1,q} \le \|\mathbf{D}(\boldsymbol{v})\|_q + \|\boldsymbol{v}\|_{L^2(\partial\Omega)}. \tag{2.29}$$

Proof. See a modification of the result by Nečas (1966) in Bulíček et al. (2007a).

[3]Less restrictive assumptions on the smoothness of Ω are discussed in Remark 2.3.

Lemma 2.4. (Interpolation inequalities) *For $2 \le q \le \frac{3r}{3-r}$ and $1 \le s \le 3(\beta+\lambda+1)$ the following inequalities hold*

$$\|z\|_q \le \|z\|_2^{\frac{6r-6q+2qr}{q(5r-6)}} \|z\|_{\frac{3r}{3-r}}^{\frac{3r(q-2)}{q(5r-6)}} \le \|z\|_2^{\frac{6r-6q+2qr}{q(5r-6)}} \|z\|_{1,r}^{\frac{3r(q-2)}{q(5r-6)}}, \tag{2.30}$$

$$\|z\|_s \le \|z\|_1^{\frac{3(\beta+\lambda+1)-s}{s(3\beta+3\lambda+2)}} \|z\|_{3(\beta+\lambda+1)}^{\frac{3(\beta+\lambda+1)(s-1)}{s(3\beta+3\lambda+2)}} \le \|z\|_1^{\frac{3(\beta+\lambda+1)-s}{s(3\beta+3\lambda+2)}} \|z^{\frac{\beta+\lambda+1}{2}}\|_{1,2}^{\frac{3(s-1)}{2s(3\beta+3\lambda+2)}}. \tag{2.31}$$

Proof. The first of the inequalities in (2.30) and (2.31) can be found in Bergh & Löfström (1976) or Málek et al. (1996, Corollary 1.2.10), and the second follow from the embedding inequalities.

Lemma 2.5. (Aubin-Lions) *Let V_1, V_2, V_3 be Banach reflexive separable spaces such that*

$$V_1 \hookrightarrow\hookrightarrow V_2 \text{ and } V_2 \hookrightarrow V_3.$$

Let $1 < p < \infty$, $1 \le q \le +\infty$ and $0 < T < \infty$. Then

$$\{v; v \in L^p(0,T;V_1), v_{,t} \in L^q(0,T;V_3)\} \text{ is compactly embedded into } L^p(0,T;V_2).$$

Proof. See for example Simon (1987) or Feireisl (2004, Lemma 6.3).

2.4. Definition of the solution. Main theorem, its corollaries and relevant results. We assume that the data $\boldsymbol{f}$, h, $\boldsymbol{v}_0$ and θ_0 satisfy

$$\begin{aligned} &\boldsymbol{f} \in L^{r'}(0,T;W_{\boldsymbol{n}}^{-1,r'}),\ h \in L^{r'}(0,T)\,, \boldsymbol{v}_0 \in L^2_{\boldsymbol{n},\mathrm{div}}\,, \theta_0 \in L^1(\Omega)\,, \\ &\text{there is a constant } C_3 > 0 \text{ such that } \theta_0(x) \ge C_3 > 0 \ \text{ for a.a. } x \in \Omega. \end{aligned} \tag{2.32}$$

Definition 2.1. *Let $\Omega \in \mathcal{C}^{1,1}$ be a bounded threedimensional domain and $\boldsymbol{f}, h, \boldsymbol{v}_0$ and θ_0 satisfy* (2.32). *Let ν satisfy the assumptions* (2.14)-(2.15) *with $r \in \left(\frac{9}{5}, 2\right)$. Let κ satisfy* (2.16) *with $\beta > -\frac{3r-5}{3(r-1)}$. We say that $(\boldsymbol{v}, p, \theta, s, \boldsymbol{q}, \mathbf{S})$ is a suitable weak solution to* (2.1)-(2.13) *if*

$$\boldsymbol{v} \in \mathcal{C}([0,T]; L^2_{weak}(\Omega)^3) \cap X^{r,\frac{5r}{3}}_{\boldsymbol{n},\mathrm{div}}, \quad \boldsymbol{v}_{,t} \in L^{\frac{5r}{6}}(0,T;W_{\boldsymbol{n}}^{-1,\frac{5r}{6}}), \tag{2.33}$$

$$p \in L^{\frac{5r}{6}}(0,T;L^{\frac{5r}{6}}(\Omega)) \text{ and } \int_{\Omega_0} p(x,t)\,dx = h(t) \text{ for a.a. } t \in (0,T), \tag{2.34}$$

$$\boldsymbol{q} \in L^m(0,T;L^m(\Omega)) \quad \text{for all } m:\ 1 \le m < 1 + 1/(4+3\beta), \tag{2.35}$$

$$\mathbf{S} \in L^{r'}(0,T;L^{r'}(\Omega)^{3\times 3}), \tag{2.36}$$

$$\begin{aligned} &\theta \in L^\infty(0,T;L^1(\Omega)), \quad \theta^{\frac{\beta+\lambda+1}{2}} \in L^2(0,T;W^{1,2}(\Omega)) \text{ for all } \lambda < 0, \\ &\qquad \theta(t,x) \ge C_3 > 0 \qquad a.e. \text{ in } Q\,, \end{aligned} \tag{2.37}$$

$$E = |\boldsymbol{v}|^2/2 + c_v\theta \in \mathcal{C}([0,T];L^1_{weak}(\Omega)), \tag{2.38}$$

$$S \in L^\infty(0,T;L^q(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)) \quad \text{for all } q \in [1,\infty), \tag{2.39}$$

satisfy the relations

$$\mathbf{S} = \mathbf{S}^*(p,\theta,\mathbf{D}(\boldsymbol{v})), \quad \boldsymbol{q} = \boldsymbol{q}^*(p,\theta,\mathbf{D}(\boldsymbol{v})), \quad S = c_v \ln\theta \quad a.e. \text{ in } Q\,, \tag{2.40}$$

fulfil the following weak formulations

$$\begin{aligned} &\langle \boldsymbol{v}_{,t}, \boldsymbol{\varphi}\rangle - (\boldsymbol{v}\otimes\boldsymbol{v}, \boldsymbol{\varphi})_Q + \alpha(\boldsymbol{v},\boldsymbol{\varphi})_\Gamma + (\mathbf{S}, \mathbf{D}(\boldsymbol{\varphi}))_Q = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle + (p, \operatorname{div}\boldsymbol{\varphi})_Q \\ &\qquad\qquad \text{for all } \boldsymbol{\varphi} \in L^{\frac{5r}{5r-6}}(0,T;W_{\boldsymbol{n}}^{1,\frac{5r}{5r-6}})\,, \end{aligned} \tag{2.41}$$

$$
\begin{aligned}
-(E,\varphi_{,t})_Q - ((E+p)\boldsymbol{v},\nabla\varphi)_Q + \alpha(|\boldsymbol{v}|^2,\varphi)_\Gamma + (\mathbf{S}\boldsymbol{v}-\boldsymbol{q},\nabla\varphi)_Q \\
= \langle \boldsymbol{f},\boldsymbol{v}\varphi\rangle + (E_0,\varphi(0)) \qquad \text{for all } \varphi\in\mathcal{D},
\end{aligned}
\tag{2.42}
$$

$$
\begin{aligned}
&-(S,\psi_{,t})_Q - (c_v\ln\theta_0,\psi(0)) - (\boldsymbol{v}S,\nabla\psi)_Q - (\boldsymbol{q}/\theta,\nabla\psi)_Q \\
&\geq -(\boldsymbol{q}/\theta^2,\nabla\theta\psi)_Q + (\mathbf{S}/\theta,\nabla\boldsymbol{v}\psi)_Q \geq 0 \text{ for all } \psi\in\mathcal{D} \text{ with } \psi\geq 0,
\end{aligned}
\tag{2.43}
$$

and $(\boldsymbol{v},\theta)$ attain the initial conditions in the following sense

$$
\lim_{t\to 0+} \|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2^2 + \|\theta(t)-\theta_0\|_1 = 0\,. \tag{2.44}
$$

Remark 2.1. *(i) The bounds $r>\frac{9}{5}$ and $\beta > -\frac{3r-5}{3(r-1)}$ are needed in order to have compactness of $E\boldsymbol{v} = \frac{1}{2}|\boldsymbol{v}|^2\boldsymbol{v} + c_v\theta\boldsymbol{v}$ in $L^1(Q)$. Indeed, since $\boldsymbol{v}\in L^{\frac{5r}{3}}(Q)$ by (2.33), the requirement that $\frac{5r}{3}>3$ implies $r>\frac{9}{5}$. Similarly, the interpolation inequalities (2.30) and (2.31), together with (2.33), (2.37) and Hölder's inequality imply that ($2\leq q\leq \frac{3r}{3-r}$ and $q'=\frac{q}{q-1}<3(\beta+\lambda+1)$)*

$$
\begin{aligned}
\int_0^T \|\boldsymbol{v}\theta\|_1\,dt &\leq \int_0^T \|\boldsymbol{v}\|_q\|\theta\|_{q'}\,dt \leq \int_0^T \|\boldsymbol{v}\|_{\frac{3r}{3-r}}^{\frac{3r(q-2)}{q(5r-6)}} \|\theta\|_{3(\beta+\lambda+1)}^{\frac{3(\beta+\lambda+1)(q'-1)}{q'(3\beta+3\lambda+2)}}\,dt \\
&\leq C\int_0^T \left(\|\boldsymbol{v}\|_{1,r}^r\right)^{\frac{3(q-2)}{q(5r-6)}} \|\theta^{\frac{\beta+\lambda+1}{2}}\|_{1,2}^{\frac{6}{q(3\beta+3\lambda+2)}\frac{q(5r-6)}{5rq-9q+6}}\,dt \\
&\leq C\left(\int_0^T \|\boldsymbol{v}\|_{1,r}^r\,dt\right)^{\frac{3(q-2)}{q(5r-6)}} \left(\int_0^T \|\theta^{\frac{\beta+\lambda+1}{2}}\|_{1,2}^{\frac{6}{q(3\beta+3\lambda+2)}\frac{q(5r-6)}{5rq-9q+6}}\,dt\right)^{\frac{5rq-9q+6}{q(5r-6)}}.
\end{aligned}
$$

Thus, $\boldsymbol{v}\theta\in L^1(Q)$ provided that

$$
\frac{6}{q(3\beta+3\lambda+2)}\frac{q(5r-6)}{5rq-9q+6} \leq 2 \Leftrightarrow \frac{5r-6}{5rq-9q+6} - \frac{2}{3} \leq \beta+\lambda.
$$

The maximal range for β with regard to its dependence on $q\in[2,\frac{3r}{3-r}]$ is achieved by taking $q=\frac{3r}{3-r}$, which implies

$$
\beta+\lambda \geq \frac{3-r}{3(r-1)} - \frac{2}{3} = -\frac{3r-5}{3(r-1)} \text{ valid for all } \lambda<0. \tag{2.45}
$$

The requirement that $q'<3(\beta+\lambda+1)$ for $q=\frac{3r}{3-r}$ is less restrictive.
(ii) If all statements concerning the entropy S are omitted in the above definition then we call $(\boldsymbol{v},p,\theta,\boldsymbol{q},\mathbf{S})$ satisfying all the remaining statements of Definition 2.1 a weak solution to (2.1)-(2.5), (2.9)-(2.11).
(iii) Proceeding as in the remark (i) above we conclude from (2.37) that $\theta\in L^q(Q)$ for $q<\frac{5}{3}+\beta$. As $\boldsymbol{q}=\kappa(\theta,p,|\mathbf{D}|^2)\nabla\theta$, the assumption (2.16) implies that

$$
\begin{aligned}
\int_Q |\boldsymbol{q}|^m\,dx\,dt &\leq C_5\int_Q \theta^{\beta m}|\nabla\theta|^m\,dx\,dt \leq C(\beta,\lambda)\int_Q |\nabla\theta^{\frac{\beta+\lambda+1}{2}}|^m\theta^{\beta m-\frac{\beta+\lambda-1}{2}m}\,dx\,dt \\
&\leq C\left(\int_Q |\nabla\theta^{\frac{\beta+\lambda+1}{2}}|^2\,dx\,dt\right)^{\frac{m}{2}} \left(\int_Q \theta^{\frac{m(\beta-\lambda-1)}{2-m}}\,dx\,dt\right)^{1-\frac{m}{2}}.
\end{aligned}
$$

Consequently,

$$
\boldsymbol{q}\in L^m(Q) \Leftrightarrow \frac{m(\beta-\lambda+1)}{2-m} < \frac{5+3\beta}{3} \Leftrightarrow m < 1+\frac{1}{3\beta+4}, \tag{2.46}
$$

which explains the range of m as it appears in (2.35)*.*
(iv) The bound $r < 2$ is due to the fact that we deal with fluids with pressure dependent viscosities. More precisely, it is shown in Málek et al. (2002) that if $r < 2$, for a given $\boldsymbol{v}$ fulfilling (2.33) *(and for constant temperature) the equation* (2.41) *provides a uniquely defined pressure (among those having the same mean value over Ω). This type of uniqueness, that may be considered as a very minimal requirement on the consistency of the model under consideration, also holds for some types of viscosities falling within the class $r = 2$ (see Bulíček et al. (2007a) for details). As it requires us to modify the assumption* (2.15)*, we will not consider this option here.*

For γ_1 and γ_2 in (2.14) and (2.15) we define

$$B_1 := \sup_{\theta \geq C_3} \gamma_1(\theta), \quad B_2 := \sup_{\theta \geq C_3} \frac{\gamma_2^2(\theta)}{\gamma_1(\theta)}, \quad B_3 := \sup_{\theta \geq C_3} \gamma_2(\theta).$$

Theorem 2.1. *Let all the assumptions of Definition 2.1 be satisfied, particularly let $r \in (\frac{9}{5}, 2)$ and $\beta > -\frac{3r-5}{3(r-1)}$. In addition, let γ_0 that appears in* (2.15) *fulfil*

$$\gamma_0 < \sqrt{\frac{|\Omega_0|}{|\Omega|}} \frac{C_1}{C_{reg}(2,\Omega)(C_1 B_3 + C_2\sqrt{B_1 B_2})}. \tag{2.47}$$

Then for any data fulfilling (2.32) *and for any $T \in (0,\infty)$ there exists a suitable weak solution to* (2.1)-(2.13)*.*

To the best of our knowledge, this is the first result concerning long time existence of solutions to a model (2.2) where the material coefficients depend on p (mean normal stress), θ (temperature) and $|\mathsf{D}|^2$ (shear rate). In addition, the result holds for large data fulfilling (2.32) and the result concerns flows in general domains (with $\mathcal{C}^{1,1}$ boundary) under reasonable Navier's slip boundary conditions.

The result can be, however, significantly extended if we consider a subclass of fluids S^* wherein the viscosity is independent of the pressure. In such a case the conditions (2.15) and (2.47) are irrelevant and we are not forced to restrict ourselves to r's below 2 (c.f. Remark 2.1, (iv)). By straightforward modifications of the proof of Theorem 2.1 the following result can be established.

Theorem 2.2. *Let all the assumptions of Definition 2.1 be satisfied. Let ν depend only on θ and $|\mathsf{D}|^2$ and satisfy* (2.14) *with $r > \frac{9}{5}$ and* (2.16) *with $\beta > -\frac{3r-5}{3(r-1)}$. Then there exists a suitable weak solution to* (2.1)-(2.13)*.*

In addition, if $r > \frac{11}{5}$ then (2.34) *is replaced by $p \in L^{r'}(0,T;L^{r'}(\Omega))$, and thus* (2.41) *holds for any $\boldsymbol{\varphi} \in L^r(0,T;W^{1,r}_{\boldsymbol{n}})$; consequently taking $\boldsymbol{\varphi} = \boldsymbol{v}$ in* (2.41) *and subtracting the result from* (2.42)*, we see that* (2.43) *holds with equality sign.*

Theorem 2.1 generalizes the following results established earlier.
For the models where viscosity ν and heat conductivity κ depend only on the temperature in an appropriate fashion, the existence of a suitable weak solution was established in Bulíček et al. (2007b) for Navier's boundary condition and in Feireisl & Málek (2006) for the spatially periodic problem. The existence of a solution satisfying (2.41) and the entropy inequality (2.43) was proved by Naumann (2006); the fact that

the temperature fulfills only inequality (2.43) is a drawback of his result. Finally, the model with constant viscosity and heat conductivity is discussed in Lions (1996).
Regarding the analysis of flows at constant temperature but with the viscosity still depending on the pressure p and the shear rate $|\mathbf{D}|^2$ we can refer to Bulíček et al. (2007a) where the case $r \in (\frac{8}{5}, 2]$ is treated for Navier's slip boundary condition, to Málek et al. (2002) where the existence of weak solution for $r \in (\frac{9}{5}, 2)$ subjected to space periodic boundary conditions is established, or to Hron et al. (2003) where two-dimensional flows are analyzed.
The case where the viscosity can depend on the temperature θ and the shear rate $|\mathbf{D}|^2$ is studied by Consiglieri (2000) for parameters $r \geq \frac{11}{5}$ (the subcritical case): assuming that the velocity adheres to the boundary (no-slip), the long-time and large-data existence is established . We may also refer to Baranger & Mikelić (1995) where the existence of a weak solution for stationary flows is proved, or to Clopeau & Mikelić (1997) where the non-stationary case is treated, but the results hold either locally in time or for small data.

Moreover, Theorem 2.2 extends also the result for isothermal flows of fluids with shear rate dependent viscosity. We refer the reader to Málek & Rajagopal (2005) for a summary of the available results. Especially, we have to note that for homogeneous Dirichlet boundary conditions the existence of a weak solution was established in Diening et al. (2007) for all $r > \frac{6}{5}$ that is the natural exponent that comes from requirement on the compactness of the convective term, at least, in the space L^1 (i.e., for $r > \frac{6}{5}$ and $\boldsymbol{v}$ satisfying (2.33) we have that $|\boldsymbol{v}|^2 \in L^{1+\varepsilon}(Q)$).

The (unsatisfactory) mathematical results concerning fluids wherein viscosity ν depends only on the pressure were discussed in the introduction. Finally, for viscosities of the type $\nu(p, \theta)$ there does not exist, to our knowledge, any existence theory.

Remark 2.2. *We observe that the results stated in Theorems 2.1 and 2.2 hold for more general relationships between e and θ. To be precise, we suppose (instead of* (2.9)*) that $e = C_v(\theta)$ where $C_v : \mathbb{R}^+ \to \mathbb{R}^+$ is a invertible continuous function satisfying*

$$\text{for some } \beta_1 \in \mathbb{R} \text{ and } C_\infty \in (0, \infty) \qquad \lim_{\theta \to \infty} \frac{C_v(\theta)}{\theta^{\beta_1}} = C_\infty. \tag{2.48}$$

Setting $\theta := C_v^{-1}(e)$, it follows from (2.48) *that*

$$\kappa(\theta, p, |\mathbf{D}|^2) \nabla \theta = \kappa(C_v^{-1}(e), p, |\mathbf{D}|^2) \left(C_v^{-1}(e)\right)' \nabla e =: \hat{\kappa}(e, p, |\mathbf{D}|^2) \nabla e.$$

Thus, if κ satisfies (2.16) *then $\hat{\kappa}$ satisfies for certain $\hat{C}_4, \hat{C}_5 \in (0, \infty)$ the condition*

$$\hat{C}_4 \leq \liminf_{e \to \infty} \frac{\hat{\kappa}(e, p, |\mathbf{D}|^2)}{e^{\hat{\beta}}} \leq \limsup_{e \to \infty} \frac{\hat{\kappa}(e, p, |\mathbf{D}|^2)}{e^{\hat{\beta}}} \leq \hat{C}_5 \text{ with } \hat{\beta} := \frac{\beta + 1}{\beta_1} - 1. \tag{2.49}$$

Consequently, formulating the problem in terms of the internal energy e instead of temperature θ, and using the assumption (2.49) *instead of* (2.16)*, we obtain the same results as in Theorems 2.1 and 2.2, expressed in terms of $\hat{\beta}$ (instead of β).*

Remark 2.3. *The assumption that $\Omega \in \mathcal{C}^{1,1}$ is required in order to employ the estimates* (2.26)-(2.27)*. This assumption can be weakened as we only need (and it will be clear from the proof) Ω to be sufficiently smooth so that* (2.26)-(2.27) *hold for just three particular q's: $q = r'$, $q = 2$ and $q = \frac{5r}{5r-6}$.*

Since the relevant theory is quite technical we prefer to provide one example of such a generalization and we refer to Grisvard (1985), Kozlov et al. (1997) and Maz'ya & Rossman (2003) for further extensions, proofs and details.

The results established in Theorems 2.1 and 2.2 hold if we replace the assumption $\Omega \in \mathcal{C}^{1,1}$ by the assumption that the boundary $\partial\Omega$ consists of $C^{1,1}$ parts and edges where each edge is characterized by the angle ω fulfilling the condition $\omega < \frac{5r}{12}\pi$. To see this, we recall that if ω fulfills $2 - \frac{2}{q} < \frac{\pi}{\omega}$ then $W^{2,q}$-regularity theory near such an edge holds (see Grisvard (1985, Theorem 8.2.1.2).). Applying this result to the most restrictive q, which is $q = 5r/(5r-6)$, we arrive at the condition $\omega < \frac{5r}{12}\pi$.

3. A proof of Theorem 2.1.

3.1. Definition of an (ε, η)-approximate problem. We start recalling that if $\boldsymbol{v}$ is an admissible test function in the weak formulation of the balance of linear momentum, then the weak formulation of the balance of energy is completely equivalent to the preferable weak formulation of the equation for the internal energy (it means for the temperature, in our case). The quadratic convective term does not permit us to take $\boldsymbol{v}$ as a test function in the balance of linear momentum. Having this in mind, it is thus natural to modify (approximate or smooth out) the convective terms.

For this purpose, for $\eta > 0$, $\boldsymbol{v} \in L^r(0,T;W^{1,r}_{\boldsymbol{n}})$ and $r \in \mathcal{D}(\mathbb{R}^3)$ nonnegative, radially symmetric, meeting $\int_{\mathbb{R}^3} r\, dx = 1$, we denote $r_\eta(x) := \frac{1}{\eta^3} r\left(\frac{x}{\eta}\right)$ and set

$$\boldsymbol{v}_\eta := ((\boldsymbol{v}\omega_\eta) * r_\eta)_{\mathrm{div}},$$

where ω_η is a smooth function such that $\mathrm{dist}(\mathrm{supp}\,\omega_\eta, \partial\Omega) \geq \eta$, and $\omega_\eta = 1$ for all x such that $\mathrm{dist}(x, \partial\Omega) \geq 2\eta$ and $\boldsymbol{z}_{\mathrm{div}} := \boldsymbol{z} - \nabla\mathcal{N}^{-1}(\mathrm{div}\,\boldsymbol{z})$ (see (2.25)).

Let $\varepsilon > 0$. In order to handle the pressure it is appropriate to relax the divergence constraint, and in order to take the limit from the finite-dimensional Galerkin approximations to a continuous (infinite-dimensional) approximation with ease, it seems suitable to replace $\mathrm{div}\,\boldsymbol{v} = 0$ by $\mathrm{div}\,\boldsymbol{v} = \varepsilon\triangle p$, completed with $\frac{\partial p}{\partial \boldsymbol{n}} = 0$ on $\partial\Omega$, $\int_{\Omega_0} p\, dx = 0$, i.e. $p = \frac{1}{\varepsilon}\mathcal{N}^{-1}_{\Omega_0}(\mathrm{div}\,\boldsymbol{v})$. Note that as we will always have at least weak convergence of the pressure, it follows from the definition of $\mathcal{N}^{-1}_{\Omega_0}$ that all possible limit pressures must satisfy $\int_{\Omega_0} p\, dx = 0$.

Because we also do not know apriori that the temperature satisfies the minimum principle $(2.37)_2$ we redefine $\mathbf{S}^*$ and $\boldsymbol{q}^*$ for values of the temperature below C_3 and define $\tilde{\mathbf{S}}^*$ and $\tilde{\boldsymbol{q}}^*$ as

$$\tilde{\mathbf{S}}^*(p,\theta,\mathbf{D}) := \begin{cases} \mathbf{S}^*(p,\theta,\mathbf{D}), \\ \mathbf{S}^*(p,C_3,\mathbf{D}), \end{cases} \quad \tilde{\boldsymbol{q}}^*(p,\theta,\mathbf{D}) := \begin{cases} -\kappa(p,\theta,\mathbf{D})\nabla\theta, & \text{if } \theta \geq C_3, \\ -\kappa(p,C_3,\mathbf{D})\nabla\theta, & \text{if } \theta < C_3. \end{cases}$$

Clearly, once we show, at some level of approximations, that $\theta(t,x) \geq C_3$ for a.a. $(t,x) \in Q$, then $\tilde{\mathbf{S}}^* = \mathbf{S}^*$ and $\tilde{\boldsymbol{q}}^* = \boldsymbol{q}^*$.

To summarize, for $\varepsilon, \eta > 0$ with $p := \frac{1}{\varepsilon}\mathcal{N}^{-1}_{\Omega_0}(\mathrm{div}\,\boldsymbol{v})$, we define the (ε,η)-approximate problem to (2.1)-(2.5), (2.9)-(2.13): to find $(\boldsymbol{v},\theta)$ by solving

$$\begin{aligned} \boldsymbol{v}_{,t} + \mathrm{div}(\boldsymbol{v}_\eta \otimes \boldsymbol{v}) - \mathrm{div}\,\tilde{\mathbf{S}}^*(p,\theta,\mathbf{D}(\boldsymbol{v})) &= -\nabla p + \boldsymbol{f}, \\ c_v\theta_{,t} + c_v\,\mathrm{div}(\boldsymbol{v}_\eta\theta) + \mathrm{div}\,\tilde{\boldsymbol{q}}^*(p,\theta,\mathbf{D}(\boldsymbol{v})) &= \tilde{\mathbf{S}}^*(p,\theta,\mathbf{D}(\boldsymbol{v}))\cdot\nabla\boldsymbol{v}, \end{aligned}$$

complemented by the boundary conditions (2.4) and the initial conditions (2.3).

3.2. A solvability of (ε,η)-approximative problem.

3.2.1. Galerkin approximation. Let $\{\boldsymbol{w}_j\}_{j=1}^{\infty}$ be a basis of $W^{1,r}_{\boldsymbol{n}}$ such that $\boldsymbol{w}_j \in W^{1,2r}_{\boldsymbol{n}}$ for all j and $(\boldsymbol{w}_i, \boldsymbol{w}_j) = \delta_{ij}$, and $\{w_j\}_{j=1}^{\infty}$ be a basis of $W^{1,2}(\Omega)$ orthonormal in the space $L^2(\Omega)$.

We construct Galerkin approximations $\{\boldsymbol{v}^{\ell,k}, \theta^{\ell,k}\}_{\ell,k=1}^{\infty}$ of the form $\boldsymbol{v}^{\ell,k} := \sum_{i=1}^{k} c_i^{\ell,k}(t)\, \boldsymbol{w}_i$ and $\theta^{\ell,k} := \sum_{i=1}^{\ell} d_i^{\ell,k}(t) w_i$, where $\boldsymbol{c}^{\ell,k}$ and $\boldsymbol{d}^{\ell,k}$ solve the following system of ordinary differential equations

$$\begin{aligned}
&\frac{d}{dt}(\boldsymbol{v}^{\ell,k}, \boldsymbol{w}_j) - (\boldsymbol{v}^{\ell,k}_{\eta} \otimes \boldsymbol{v}^{\ell,k}, \nabla \boldsymbol{w}_j) + (\tilde{\mathbf{S}}^*_{\ell,k}, \nabla \boldsymbol{w}_j) + \alpha(\boldsymbol{v}^{\ell,k}, \boldsymbol{w}_j)_{\partial\Omega} \\
&\qquad = (p^{\ell,k}, \operatorname{div} \boldsymbol{w}_j) + \langle \boldsymbol{f}, \boldsymbol{w}_j \rangle \quad \text{for all } j = 1, 2, \dots, \ell\ , \\
&\qquad \boldsymbol{v}^{\ell,k}(\cdot, 0) = \boldsymbol{v}_0^{\ell,k} = \sum_{j=1}^{\ell} c_0^{\ell} \boldsymbol{w}_j = P^{\ell} \boldsymbol{v}_0,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&c_v \frac{d}{dt}(\theta^{\ell,k}, w_j) - c_v(\boldsymbol{v}^{\ell,k}_{\eta} \theta^{\ell,k}, \nabla w_j) - (\tilde{\boldsymbol{q}}^*_{\ell,k}, \nabla w_j) = \left(\tilde{\mathbf{S}}^*_{\ell,k} \cdot \nabla \boldsymbol{v}^{\ell,k}, w_j\right) \\
&\qquad\qquad \text{for all } j = 1, 2, \dots, k, \\
&\qquad \theta^{\ell,k}(\cdot, 0) = \theta_0^{\ell,k} = \sum_{j=1}^{k} d_0^k w_j = P^k(r_{1/\ell} * \theta_0),
\end{aligned} \tag{3.2}$$

where we use the following abbreviations

$$p^{\ell,k} := \mathcal{N}^{-1}_{\Omega_0}\Big(\frac{\operatorname{div} \boldsymbol{v}^{\ell,k}}{\varepsilon}\Big),\ \tilde{\mathbf{S}}^*_{\ell,k} := \tilde{\mathbf{S}}^*(\theta^{\ell,k}, p^{\ell,k}, \mathbf{D}^{\ell,k}),\ \tilde{\boldsymbol{q}}^*_{\ell,k} := \tilde{\boldsymbol{q}}^*(\theta^{\ell,k}, p^{\ell,k}, \mathbf{D}^{\ell,k}),$$

and P^{ℓ} and P^k denote the projection of $L^2_{\boldsymbol{n}}$ and $L^2(\Omega)$ onto the linear hull $\{\boldsymbol{w}_j\}_{j=1}^{\ell}$ and $\{w_j\}_{j=1}^{k}$, respectively.

The standard Carathéodory theory (see Zeidler (1990, Chapter 30)) provides the existence of solution to (3.1)-(3.2) at least for a short time interval. The uniform estimates that we will derive in the next subsection enable us to extend the solution onto the whole time interval $(0, T)$.

3.2.2. Uniform estimates. Multiplying the j-th equation in (3.1) by $c_j^{\ell,k}$, then taking the sum over $j = 1, ..., M$, and integrating the result over $(0, t)$ we obtain

$$\begin{aligned}
&\|\boldsymbol{v}^{\ell,k}(t)\|_2^2 + 2\int_0^t (\tilde{\mathbf{S}}^*_{\ell,k}, \mathbf{D}(\boldsymbol{v}^{\ell,k})) + \alpha(\boldsymbol{v}^{\ell,k}, \boldsymbol{v}^{\ell,k})_{\partial\Omega} + \varepsilon(\nabla p^{\ell,k}, \nabla p^{\ell,k})\, d\tau \\
&\qquad\qquad = \|\boldsymbol{v}_0^{\ell,k}\|_2^2 + 2\int_0^t \langle \boldsymbol{f}, \boldsymbol{v}^{\ell,k} \rangle\, d\tau.
\end{aligned} \tag{3.3}$$

Applying (2.14), (2.21) and Korn's inequality (Lemma 2.3) to the second term and the standard duality estimates and Young's inequality to the last term in (3.3) we conclude that

$$\sup_{t \in (0,T)} \|\boldsymbol{v}^{\ell,k}(t)\|_2^2 + \int_0^T \|\boldsymbol{v}^{\ell,k}\|_{1,r}^r + \varepsilon \|\nabla p^{\ell,k}\|_2^2\, dt \le C. \tag{3.4}$$

Next, multiplying the j-th equation in (3.2) by $d_j^{\ell,k}$, integrating the result w.r.t. time over $(0, t)$ we arrive at

$$\|\theta^{\ell,k}(t)\|_2^2 - 2\int_0^t (\tilde{\boldsymbol{q}}^*_{\ell,k}, \nabla \theta^{\ell,k})\, d\tau = \|\theta_0^{\ell,k}\|_2^2 + 2\int_0^t (\tilde{\mathbf{S}}^*_{\ell,k} \cdot \nabla \boldsymbol{v}^{\ell,k}, \theta^{\ell,k})\, d\tau. \tag{3.5}$$

Setting $Q_{C_3} := \{(t,x) \in Q; \theta^{\ell,k}(t,x) \le C_3\}$, it follows from the definition of $\tilde{\boldsymbol{q}}^*$, the assumption concerning κ (2.16) and also from the estimate (3.4) and the fact that $\boldsymbol{w}_j \in W^{1,2r}_{\boldsymbol{n}}$ (note that ℓ is fixed), and Gronwall's lemma that

$$\sup_{t\in(0,T)} \|\theta^{\ell,k}(t)\|_2^2 + \int_{Q_{C_3}} |\nabla\theta^{\ell,k}|^2 \, dx \, d\tau + \int_{Q\setminus Q_{C_3}} (\theta^{\ell,k})^\beta |\nabla\theta^{\ell,k}|^2 \, dx \, d\tau \le C(\ell). \tag{3.6}$$

Let $\overline{\kappa}(\theta) := \theta^\beta$ if $\theta \ge C_3$ and $\overline{\kappa}(\theta) := C_3^\beta$ if $\theta < C_3$ and $\overline{K}$ be a primitive function to $\sqrt{\overline{\kappa}}$, i.e., $\overline{K}(\theta) = C_3^{\frac{\beta}{2}}\theta$ for $\theta \le C_3$ and $\overline{K}(\theta) = \frac{2}{\beta+2}\theta^{\frac{\beta+2}{2}} + \frac{\beta}{\beta+2}C_3^{\frac{\beta+2}{2}}$ for $\theta \ge C_3$. It then follows from (3.6) that

$$\sup_{t\in(0,T)} \|\theta^{\ell,k}(t)\|_2^2 + \int_Q |\nabla\overline{K}(\theta^{\ell,k})|^2 \, dx \, d\tau \le C(\ell). \tag{3.7}$$

Note that there are constants $C_{10}, C_{11}, C_{12}, C_{13} > 0$ such that for all $\theta \in \mathbb{R}$ and all β under consideration

$$\left.\begin{matrix} C_{10}|\theta| \\ C_{11}|\theta|^{\frac{\beta+2}{2}} \end{matrix}\right\} \le \overline{K}(\theta) \le \begin{cases} C_{12}|\theta| & \text{for } \theta \le C_3, \\ C_{13}(1+|\theta|)^{\frac{\beta+2}{2}} & \text{for } \theta \ge C_3. \end{cases} \tag{3.8}$$

Consequently, as $\overline{K}(\theta) \le C(|\theta| + |\theta|^{\frac{\beta+2}{2}})$, (3.7) implies, for $\beta + 2 \le 2 \Leftrightarrow \beta \le 0$, that

$$\sup_{t\in(0,T)} \|\overline{K}(\theta^{\ell,k})\|_2^2 + \int_0^T \|\overline{K}(\theta^{\ell,k})\|^2_{W^{1,2}(\Omega)} \, dt \le C(\ell). \tag{3.9}$$

We shall verify that (3.9) holds also for $\beta \ge 0$. Indeed, since

$$\begin{aligned} \int_0^T \|\overline{K}(\theta^{\ell,k})\|_{1,2}^2 \, dt &= \int_0^T \|\nabla\overline{K}(\theta^{\ell,k})\|_2^2 + \|\overline{K}(\theta^{\ell,k})\|_2^2 \, dt \\ &\overset{(3.7)}{\le} C(\ell) + C\int_0^T \|\overline{K}(\theta^{\ell,k})^{\frac{2}{\beta+2}}\|_{\beta+2}^{\beta+2} \, dt \\ &\le C(\ell) + C\int_0^T \|\overline{K}(\theta^{\ell,k})^{\frac{2}{\beta+2}}\|_2^{\frac{4(\beta+2)}{3\beta+4}} \|\overline{K}(\theta^{\ell,k})^{\frac{2}{\beta+2}}\|_{3(\beta+2)}^{\frac{3\beta(\beta+2)}{3\beta+4}} \\ &\le C(\ell) + C\int_0^T \|\theta^{\ell,k}\|_2^{\frac{4(\beta+2)}{3\beta+4}} \|\overline{K}(\theta^{\ell,k})\|_6^{\frac{6\beta}{3\beta+4}} \\ &\le C(\ell) + C(\ell)\int_0^T \|\overline{K}(\theta^{\ell,k})\|_{1,2}^{\frac{6\beta}{3\beta+4}}, \end{aligned}$$

and $\frac{6\beta}{3\beta+4} < 2$, applying Young's inequality we easily conclude that (3.9) holds for all β under consideration. Thus, (3.9) and the fact that $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ imply that (for all β)

$$\int_0^T \|\overline{K}(\theta^{\ell,k})\|_6^2 \, dt \le C(\ell). \tag{3.10}$$

If $\beta \le 0$ we interpolate (3.10) together with $(3.9)_1$ using (2.30) with $r = 2$, and conclude that

$$\int_0^T \|\overline{K}(\theta^{\ell,k})\|_{\frac{10}{3}}^{\frac{10}{3}} \, dt \le C(\ell) \overset{(3.8)_1}{\Rightarrow} \int_0^T \|\theta^{\ell,k}\|_{\frac{5(\beta+2)}{3}}^{\frac{5(\beta+2)}{3}} \, dt \le C(\ell). \tag{3.11}$$

In order to achieve a similar estimate for $\beta > 0$, we take arbitrarily $s \in (1,6)$ and $\alpha \in (0,s)$ and compute

$$\begin{aligned}\int_0^T \|\overline{K}(\theta^{\ell,k})\|_s^s \, dt &\le \int_0^T \int_\Omega |\overline{K}(\theta^{\ell,k})|^{s-\alpha} |\overline{K}(\theta^{\ell,k})|^{\alpha} \, dx \, dt \\ &\le \int_0^T \left(\int_\Omega |\overline{K}(\theta^{\ell,k})|^{\frac{6(s-\alpha)}{6-\alpha}}\right)^{\frac{6-\alpha}{6}} \left(\int_\Omega |\overline{K}(\theta^{\ell,k})|^6\right)^{\frac{\alpha}{6}} \, dt \\ &\overset{(3.8)_2}{\le} \int_0^T \left(\int_\Omega |\theta^{\ell,k}|^{\frac{3(\beta+2)(s-\alpha)}{6-\alpha}}\right)^{\frac{6-\alpha}{6}} \|\overline{K}(\theta^{\ell,k})\|_{1,2}^{\alpha} \, dt.\end{aligned}$$

Next setting $\alpha := \frac{3(\beta+2)s-12}{3(\beta+2)-2} \Leftrightarrow \frac{3(\beta+2)(s-\alpha)}{6-\alpha} = 2$ and using $(3.7)_1$, we are led to

$$\int_0^T \|\overline{K}(\theta^{\ell,k})\|_s^s \, dt \le C(\ell) \int_0^T \|\overline{K}(\theta^{\ell,k})\|_{1,2}^{\frac{3(\beta+2)s-12}{3(\beta+2)-2}} \, dt.$$

Thus, if $\frac{3(\beta+2)s-12}{3(\beta+2)-2} \Leftrightarrow s \le 2 + \frac{8}{3(\beta+2)}$, we have

$$\int_0^T \|\overline{K}(\theta^{\ell,k})\|_s^s \, dt \le C(\ell) \overset{(3.8)_1}{\Rightarrow} \int_0^T \|\theta^{\ell,k}\|_{\frac{s(\beta+2)}{2}}^{\frac{s(\beta+2)}{2}} \, dt \le C(\ell). \tag{3.12}$$

We also need to estimate $\tilde{\boldsymbol{q}}^*_{\ell,k} = \tilde{\kappa}(p^{\ell,k}, \theta^{\ell,k}, \mathbf{D}(\boldsymbol{v}^{\ell,k}))\nabla\theta^{\ell,k} =: \tilde{\kappa}_{\ell,k}\nabla\theta^{\ell,k}$. Since

$$|\tilde{\boldsymbol{q}}^*_{\ell,k}| = \left|\nabla\overline{K}(\theta^{\ell,k}) \frac{\tilde{\kappa}_{\ell,k}}{\sqrt{\overline{\kappa}(\theta^{\ell,k})}}\right| \le C|\nabla\overline{K}(\theta^{\ell,k})|\sqrt{\overline{\kappa}(\theta^{\ell,k})}$$

we immediately obtain, for $\beta \le 0$, that

$$\int_0^T \|\tilde{\boldsymbol{q}}^*_{\ell,k}\|_2^2 \le C(\ell). \tag{3.13}$$

For $\beta \ge 0$, and some $m \in (1,2)$ we have

$$\begin{aligned}\int_Q |\tilde{\boldsymbol{q}}^*_{\ell,k}|^m \, dx \, dt &\le C \int_{Q_{C_3}} |\nabla\overline{K}(\theta^{\ell,k})|^m \, dx \, dt + C \int_{Q\setminus Q_{C_3}} |\nabla\overline{K}(\theta^{\ell,k})|^m [\theta^{\ell,k}]^{\frac{m\beta}{2}} \, dx \, dt \\ &\le C(\ell) + C \int_{Q\setminus Q_{C_3}} [\theta^{\ell,k}]^{\frac{m\beta}{2-m}} \, dx \, dt \\ &\overset{(3.8)}{\le} C(\ell) + C \int_Q |\overline{K}(\theta^{\ell,k})|^{\frac{2m\beta}{(2-m)(\beta+2)}} \, dx \, dt \overset{(3.12)}{\le} C(\ell),\end{aligned} \tag{3.14}$$

provided $\frac{2m\beta}{(2-m)(\beta+2)} \le 2 + \frac{8}{(3(\beta+2)} \Leftrightarrow m \le 1 + \frac{5}{3\beta+5}$. Note that we could similarly show that

$$\left\|\frac{\tilde{\kappa}_{\ell,k}}{\sqrt{\overline{\kappa}(\theta^{\ell,k})}}\right\|_{L^s(Q)} \le C(\ell) \text{ with } s = \infty \text{ for } \beta \le 0 \text{ and } s = \frac{2(3\beta+10)}{3\beta} \text{ for } \beta > 0. \tag{3.15}$$

Finally, for $\beta > 0$ we conclude directly from (3.6) that

$$\int_Q |\nabla\theta^{\ell,k}|^2 \, dx \, dt \le C(\ell).$$

For $\beta \le 0$ we obtain, for $s = \frac{5(\beta+2)}{\beta+5}$, that

$$\begin{aligned}\int_Q |\nabla\theta^{\ell,k}|^s \, dx \, dt &= \int_Q |\nabla\overline{K}(\theta^{\ell,k})|^s |\sqrt{\kappa}(\theta^{\ell,k})|^{-s} \, dx \, dt \\ &\le C(\ell) + \int_Q |\theta^{\ell,k}|^{-\frac{s\beta}{2-s}} \, dx \, dt \le C(\ell).\end{aligned} \tag{3.16}$$

We also need to estimate the time derivatives of $\boldsymbol{c}^{\ell,k}$ and $\theta^{\ell,k}$. Multiplying the j-equation in (3.1) by $\frac{d}{dt}c_j^{\ell,k}$ and integrating it over time, we obtain (after using the estimate (3.4))

$$\int_0^T \left| \frac{d\boldsymbol{c}^{\ell,k}}{dt} \right|^2 dt \le C(\ell). \tag{3.17}$$

From the estimates proved above, (3.2) and continuity of the projection P^ℓ, we see that

$$\|\theta^{\ell,k}_{,t}\|_{L^{\sigma'}(0,T;W^{-1,\sigma'}(\Omega))} \le C(\ell), \text{ for } \sigma = 2 \text{ if } \beta \le 0 \text{ and } \sigma = m \text{ if } \beta > 0. \tag{3.18}$$

3.2.3. Limit $k \to \infty$. Having uniform estimates (3.4)-(3.18), we can establish the following convergence results for selected (not relabelled) subsequences, as $k \to \infty$ ($s = \min\left\{2, \frac{5(\beta+2)}{\beta+5}\right\}$):

$$\boldsymbol{c}^{\ell,k} \rightharpoonup \boldsymbol{c}^\ell \text{ weakly in } W^{1,2}(0,T) \text{ and strongly in } \mathcal{C}([0,T]), \tag{3.19}$$

$$\theta^{\ell,k} \rightharpoonup \theta^\ell \text{ weakly in } \left\{ z \in L^s(0,T;W^{1,s}(\Omega)), z_{,t} \in L^{\sigma'}(0,T;W^{-1,\sigma'}(\Omega)) \right\}, \tag{3.20}$$

and consequently applying Aubin-Lions Lemma 2.5 and $(3.11)_2$ we observe that

$$\theta^{\ell,k} \to \theta^\ell \text{ strongly in } L^m(0,T;L^m(\Omega)), \ m < \min\left\{\tfrac{10}{3}, \tfrac{5(\beta+2)}{3}\right\}. \tag{3.21}$$

Moreover, using (3.19) it is evident that

$$\boldsymbol{v}^{\ell,k} \to \boldsymbol{v}^\ell \text{ strongly in } L^{2r}(0,T;W^{1,2r}_{\boldsymbol{n}}), \tag{3.22}$$

$$\tilde{\mathbf{S}}^*_{\ell,k} \to \tilde{\mathbf{S}}^*_\ell := \tilde{\mathbf{S}}^*(p^\ell, \theta^\ell, \mathbf{D}(\boldsymbol{v}^\ell)) \text{ strongly in } L^{\frac{2r}{r-1}}(0,T;L^{\frac{2r}{r-1}}(\Omega)^{3\times3}). \tag{3.23}$$

Finally, it follows from (3.9) and (3.15) (with s defined in (3.15)) that

$$\overline{K}(\theta^{\ell,k}) \rightharpoonup \overline{K}(\theta^\ell) \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{3.24}$$

$$\frac{\tilde{\kappa}_{\ell,k}}{\kappa(\theta^{\ell,k})} \to \frac{\tilde{\kappa}_\ell}{\kappa(\theta^\ell)} \text{ strongly in } L^{s^*}(0,T;L^{s^*}(\Omega)) \text{ for any } s^* < s, \tag{3.25}$$

which implies that

$$\tilde{\boldsymbol{q}}^*_{\ell,k} \rightharpoonup \tilde{\boldsymbol{q}}^*_\ell := \tilde{\kappa}(p^\ell, \theta^\ell, \mathbf{D}(\boldsymbol{v}^\ell))\nabla\theta^\ell \text{ weakly in } L^\sigma(0,T;W^{1,\sigma}(\Omega)). \tag{3.26}$$

The convergence results established in (3.19)-(3.26) allow us to take the limit in (3.1) and to obtain that the following system of equations ($p^\ell := \frac{1}{\varepsilon}\mathcal{N}^{-1}_{\Omega_0}(\operatorname{div}\boldsymbol{v}^\ell)$)

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{v}^\ell, \boldsymbol{w}_j) - (\boldsymbol{v}^\ell_\eta \otimes \boldsymbol{v}^\ell, \nabla\boldsymbol{w}_j) + (\tilde{\mathbf{S}}^*_\ell, \nabla\boldsymbol{w}_j) + \alpha(\boldsymbol{v}^\ell, \boldsymbol{w}_j)_{\partial\Omega} \\ = (p^\ell, \operatorname{div}\boldsymbol{w}_j) + \langle \boldsymbol{f}, \boldsymbol{w}_j \rangle \quad \text{for all } j = 1,2,\ldots,\ell \ ,\end{aligned} \tag{3.27}$$

$$\begin{aligned}c_v \frac{d}{dt}(\theta^\ell, \varphi) - c_v(\boldsymbol{v}^\ell_\eta \theta^\ell, \nabla\varphi) - (\tilde{\boldsymbol{q}}^*_\ell, \nabla\varphi) = \left(\tilde{\mathbf{S}}^*_\ell, \nabla\boldsymbol{v}^\ell\varphi\right) \\ \text{for all } \varphi \in W^{1,\sigma}(\Omega).\end{aligned} \tag{3.28}$$

3.2.4. Minimum principle. Consider $\varphi := \min\{0, \theta^\ell - C_3\} \le 0$. Once we show that such a φ is an admissible test function in (3.28) we conclude in a standard way that

$$\theta^\ell(x,t) \ge \operatorname*{essinf}_{x\in\Omega} \theta_0^\ell \ge C_3 > 0 \text{ for a.a. } (x,t) \in \Omega \times (0,T). \tag{3.29}$$

In order to see that such a φ is an admissible test function, we write

$$(\tilde{\boldsymbol{q}}_\ell^*, \nabla\varphi)_Q = (\tilde{\kappa}_\ell \nabla\theta^\ell, \nabla\varphi)_Q = \int_Q \sqrt{\tilde{\kappa}_\ell} \nabla\theta^\ell \cdot \nabla\varphi \, d\mu$$

where $d\mu(t,x) := \sqrt{\tilde{\kappa}_\ell}\, dx\, dt$. By virtue of (3.24) we know that $\nabla\varphi \in L^2(0,T; L^2(\Omega,\mu))$. Consequently, we also know that $\theta^\ell_{,t}$ belongs to a corresponding dual space.

Note that (3.29) implies that $\tilde{\mathsf{S}}_k^* = \mathsf{S}_k^*$ and $\tilde{\boldsymbol{q}}_k^* = \boldsymbol{q}_k^*$ a.e. in Q.

3.2.5. Further apriori estimates. Setting $\varphi \equiv 1$, in (3.28) and using weak lower semicontinuity of norms in (3.4) lead to

$$\sup_{t\in[0,T]} (\|\theta^\ell(t)\|_1 + \|\boldsymbol{v}^\ell(t)\|_2^2) + \int_0^T \|\boldsymbol{v}^\ell\|_{1,r}^r + \varepsilon \|\nabla p^\ell\|_2^2 \, dt \le C. \tag{3.30}$$

Next, taking $\varphi = (\theta^\ell)^\lambda$ with $-1 < \lambda < 0$ in (3.28) (note that (3.29) implies that $0 \le (\theta^\ell)^\lambda \le C$ in Q) and integrating the result over time $t \in (0,T)$ we obtain, with help of (3.30) (for details see Bulíček et al. (2007b)), that

$$-(\boldsymbol{q}_\ell^*, \nabla\theta^\ell)_Q \le C \overset{(2.16)}{\Longrightarrow} \int_Q \left|\nabla(\theta^\ell)^{\frac{\beta+\lambda+1}{2}}\right|^2 dx\, dt \le C. \tag{3.31}$$

This together with (3.30) implies that

$$\int_0^T \|(\theta^\ell)^{\frac{\beta+\lambda+1}{2}}\|_{1,2}^2 \, dt \le C \underset{(3.30)}{\overset{(2.31)}{\Longrightarrow}} \int_0^T \|\theta^\ell\|_n^n \, dt \le C \text{ for all } n \in \left[1, \tfrac{5+3\beta}{3}\right). \tag{3.32}$$

For $\beta > 1$ (and λ such that $\beta + \lambda + 1 \ge 2$), (3.32) directly implies that

$$\int_0^T \|\theta^\ell\|_{W^{1,2}(\Omega)}^2 \, dt \le C. \tag{3.33}$$

For $\beta \le 1$, we have

$$\begin{aligned} \int_0^T \|\nabla\theta^\ell\|_s^s \, dt &= \int_Q (\theta^\ell)^{(\beta+\lambda-1)\frac{s}{2}} |\nabla\theta^\ell|^s (\theta^\ell)^{-(\beta+\lambda-1)\frac{s}{2}} \, dx\, dt \\ &\le C \int_Q (\theta^\ell)^{\beta+\lambda-1} |\nabla\theta^\ell|^2 + (\theta^\ell)^{-(\beta+\lambda-1)\frac{s}{2-s}} \, dx\, dt \underset{(3.32)}{\overset{(3.31)}{\le}} C, \end{aligned} \tag{3.34}$$

provided that $-(\beta+\lambda-1)\frac{s}{2-s} < \beta + \frac{5}{3} \Leftrightarrow s < \dfrac{5+3\beta}{4}$.

Proceeding step by step as in the case of Remark 2.1, part (iii), we conclude that

$$\int_Q |\boldsymbol{q}_\ell^*|^m \, dx\, dt \le C \quad \text{for all } m < 1 + \frac{1}{3\beta+4} = \frac{3\beta+5}{3\beta+4}. \tag{3.35}$$

Moreover, it follows from $(2.21)_2$ and (3.30) that

$$\int_Q |\mathbf{S}^*_\ell|^{r'}\,dx\,dt \le C. \tag{3.36}$$

The above estimates are sufficient to show that

$$\|\boldsymbol{v}^\ell_{,t}\|_{(X^{r,2})^*} \le C(\varepsilon)\,, \tag{3.37}$$

$$\|\theta^\ell_{,t}\|_{L^1(0,T;W^{-1,q'}(\Omega))} \le C, \quad q \text{ being sufficiently large.} \tag{3.38}$$

3.2.6. Limit $\ell \to \infty$. It follows from the estimates (3.30)-(3.38) and from Aubin-Lions Lemma 2.5, Corollary 2.1 and the assumption on ν (see (2.14)) and κ (see (2.16)) that there are $\boldsymbol{v}$ and θ and relevant not relabelled subsequences such that

$$\boldsymbol{v}^\ell_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } (X^{r,2})^*, \quad \boldsymbol{v}^\ell \rightharpoonup \boldsymbol{v} \quad \text{weakly in } X^{r,2}, \tag{3.39}$$

$$\boldsymbol{v}^\ell \to \boldsymbol{v} \qquad \text{strongly in } L^n(0,T;L^n(\Omega)^3) \text{ for } n \in [1, \tfrac{5r}{3}), \tag{3.40}$$

$$\operatorname{tr}\boldsymbol{v}^\ell \to \operatorname{tr}\boldsymbol{v} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)), \tag{3.41}$$

$$\theta^\ell \to \theta \qquad \text{strongly in } L^m(0,T;L^m(\Omega)) \text{ for } m < \tfrac{5}{3} + \beta, \tag{3.42}$$

$$(\theta^\ell)^{\frac{\beta+\lambda+1}{2}} \rightharpoonup (\theta)^{\frac{\beta+\lambda+1}{2}} \qquad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{3.43}$$

$$\mathbf{S}^*_\ell \to \overline{\mathbf{S}^*} \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{3\times3}), \tag{3.44}$$

$$\boldsymbol{q}^*_\ell \to \overline{\boldsymbol{q}^*} \qquad \text{weakly in } L^m(0,T;L^m(\Omega)^3), \text{ for all } m < \tfrac{3\beta+5}{3\beta+4}. \tag{3.45}$$

Denote $p := \frac{1}{\varepsilon}\mathcal{N}^{-1}_{\Omega_0}(\operatorname{div}\boldsymbol{v})$. Since $p^\ell = \frac{1}{\varepsilon}\mathcal{N}^{-1}_{\Omega_0}(\operatorname{div}\boldsymbol{v}^\ell)$, it follows from (3.40) that

$$p^\ell \to p \qquad \text{strongly in } L^2(0,T;W^{1,2}(\Omega)). \tag{3.46}$$

This convergence is sufficient to allow us to take the limit of (3.27) and conclude that

$$\begin{aligned}\langle \boldsymbol{v}_{,t}, \boldsymbol{w}\rangle - (\boldsymbol{v}_\eta \otimes \boldsymbol{v}, \nabla\boldsymbol{w}) + (\overline{\mathbf{S}^*}, \nabla\boldsymbol{w}) + \alpha(\boldsymbol{v}, \boldsymbol{w})_{\partial\Omega} + (\nabla p, \boldsymbol{w})\\ = \langle \boldsymbol{f}, \boldsymbol{w}\rangle \text{ for all } \boldsymbol{w} \in W^{1,r}_{\boldsymbol{n}} \text{ and a.a.} t \in (0,T).\end{aligned} \tag{3.47}$$

It remains to show that

$$\overline{\mathbf{S}^*} = \mathbf{S}^* := \nu(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) \qquad \text{a.e. in } Q, \tag{3.48}$$

and this we shall do by using standard monotone operator arguments (Minty's method). Indeed, using Lemma 2.1, we obtain for all $\boldsymbol{\varphi} \in X^{r,2}$ that

$$0 \le \frac{C_1}{2}\int_Q \mathcal{I}(\mathbf{D}(\boldsymbol{v}^\ell), \mathbf{D}(\boldsymbol{\varphi})) \le \big(\mathbf{S}^*_\ell - \nu(p^\ell, \theta^\ell, |\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\boldsymbol{v}^\ell - \boldsymbol{\varphi})\big)_Q\,. \tag{3.49}$$

Using strong convergence (3.42) and (3.46) and Lebesgue dominated convergence theorem, we obtain that $\nu(p^\ell, \theta^\ell, |\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}) \to \nu(p, \theta, |\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi})$ strongly in $L^{r'}(Q)^{3\times3}$. Hence,

$$\big(\nu(p^\ell, \theta^\ell, |\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\boldsymbol{v}^\ell - \boldsymbol{\varphi})\big) \overset{\ell\to\infty}{\to} \big(\nu(p, \theta, |\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\boldsymbol{v} - \boldsymbol{\varphi})\big)_Q\,.$$

Next, we replace the term $\big(\mathbf{S}^*_\ell, \mathbf{D}(\boldsymbol{v}^\ell)\big)_Q$ in (3.49) and taking the limit as $\ell \to \infty$ we obtain the following inequality

$$0 \le \lim_{\ell\to\infty} \frac{C_1}{2} \int_Q \mathcal{I}(\mathbf{D}(\boldsymbol{v}^\ell), \mathbf{D}(\boldsymbol{\varphi})) \le \Big(\overline{\mathbf{S}^*} - \nu(p,\theta,|\mathbf{D}(\boldsymbol{\varphi})|^2)\mathbf{D}(\boldsymbol{\varphi}), \mathbf{D}(\boldsymbol{v}-\boldsymbol{\varphi})\Big)_Q . \quad (3.50)$$

The choice $\boldsymbol{\varphi} := \boldsymbol{v} \pm h\boldsymbol{u}$ with $h > 0$ then completes the proof of (3.48), and (3.50) implies that

$$\int_Q \mathcal{I}(\mathbf{D}(\boldsymbol{v}^\ell), \mathbf{D}(\boldsymbol{v})) \overset{\ell\to\infty}{\to} 0. \quad (3.51)$$

Since

$$\begin{aligned}
&\int_Q |\mathbf{D}(\boldsymbol{v}^\ell - \boldsymbol{v})|^r \, dx\, dt = \\
&\quad = \int_Q \mathcal{I}^{\frac{r}{2}}(\mathbf{D}(\boldsymbol{v}^\ell), \mathbf{D}(\boldsymbol{v})) \left(\int_0^1 (1 + |\mathbf{D}(\boldsymbol{v}^\ell) - s\mathbf{D}(\boldsymbol{v}^\ell - \boldsymbol{v})|^2)^{\frac{r-2}{2}}\, ds\right)^{-\frac{r}{2}} dx\, dt \\
&\quad \le C \left(\int_Q \mathcal{I}(\mathbf{D}(\boldsymbol{v}^\ell), \mathbf{D}(\boldsymbol{v}))\, dx\, dt\right)^{\frac{r}{2}} \left(\int_Q (1 + |\mathbf{D}(\boldsymbol{v}^\ell)|_r^r + |\mathbf{D}(\boldsymbol{v})|_r^r)\, dx\, dt\right)^{\frac{2-r}{2}}
\end{aligned}$$

(3.51) implies that

$$\mathbf{D}(\boldsymbol{v}^\ell) \to \mathbf{D}(\boldsymbol{v}) \quad \text{strongly in } L^r(0,T;L^r(\Omega)^{3\times3}). \quad (3.52)$$

It is a simple consequence of (3.52) and Lemma 2.1 that

$$\mathbf{S}^*_\ell \cdot \mathbf{D}(\boldsymbol{v}^\ell) \to \mathbf{S}^* \cdot \mathbf{D}(\boldsymbol{v}) = \nu(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)|\mathbf{D}(\boldsymbol{v})|^2 \quad \text{strongly in } L^1(0,T;L^1(\Omega)).$$

Finally, since

$$\boldsymbol{q}^*_\ell = -\frac{2}{\beta+\lambda+1}(\nabla(\theta^\ell)^{\frac{\beta+\lambda+1}{2}})\kappa(p^\ell,\theta^\ell,\mathbf{D}(\boldsymbol{v}^\ell))(\theta^\ell)^{-\frac{\beta+\lambda-1}{2}}$$

the convergence results (3.45), (3.42), (3.46) and (3.52) imply that $\overline{\boldsymbol{q}^*} = \boldsymbol{q}^* =: -\kappa(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)\nabla\theta$. Thus, letting $\ell \to \infty$ in (3.28) we obtain

$$\begin{aligned}
&-(\theta,\varphi_{,t})_Q - (\boldsymbol{v}_\eta\theta, \nabla\varphi)_Q - (\boldsymbol{q}^*, \nabla\varphi) = (\mathbf{S}^*, \mathbf{D}(\boldsymbol{v}), \varphi) + (\theta_0, \varphi(0)) \\
&\qquad \text{for all } \varphi \in \mathcal{D}(-\infty,T;W^{1,\sigma}(\Omega)) \text{ with } \sigma > \max\{3, m'\}.
\end{aligned} \quad (3.53)$$

The arguments concerning the attainment of the initial conditions $\boldsymbol{v}_0$ and θ_0 are standard (see e.g. Málek & Rajagopal (2005) for $\boldsymbol{v}_0$ and Lions (1996, Appendix E) for θ_0).

At this point, the existence of a weak solution $(\boldsymbol{v},\theta) = (\boldsymbol{v}^{\varepsilon,\eta}, \theta^{\varepsilon,\eta})$ to the (ε,η)-approximate problem, fulfilling (3.47) with (3.48) and (3.53) is established.

3.3. Limit $\varepsilon \to 0$**.** Let $\{\boldsymbol{v}^\varepsilon, \theta^\varepsilon\}$ be used in this subsection to denote $(\boldsymbol{v}^{\varepsilon,\eta}, \theta^{\varepsilon,\eta})$, the solution of the (ε,η)-approximation. Recall that $p^\varepsilon = \frac{1}{\varepsilon}\mathcal{N}^{-1}_{\Omega_0}(\operatorname{div} \boldsymbol{v}^\varepsilon)$, which is tantamount to

$$\varepsilon(\nabla p^\varepsilon, \nabla h) = (\operatorname{div} \boldsymbol{v}^\varepsilon, h) \quad \text{for all } h \in W^{1,r'}(\Omega), \quad \int_{\Omega_0} p^\varepsilon(t,x)\, dx = 0 . \quad (3.54)$$

We also recall the notation

$$\boldsymbol{q}^*_\varepsilon := -\kappa(p^\varepsilon, \theta^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\nabla\theta^\varepsilon, \qquad \mathbf{S}^*_\varepsilon := \nu(p^\varepsilon, \theta^\varepsilon, |\mathbf{D}(\boldsymbol{v}^\varepsilon)|^2)\mathbf{D}(\boldsymbol{v}^\varepsilon)\,.$$

Using the weak lower semicontinuity of the norms in (3.30), (3.32), (3.35) and (3.36), we find that

$$\begin{gathered}\sup_{t\in[0,T]} (\|\theta^\varepsilon(t)\|_1 + \|\boldsymbol{v}^\varepsilon(t)\|_2^2) + \int_0^T \|\boldsymbol{v}^\varepsilon\|_{1,r}^r + \varepsilon\|\nabla p^\varepsilon\|_2^2\,dt \le C\,,\\ \int_0^T \|(\theta^\varepsilon)^{\frac{\beta+\lambda+1}{2}}\|_{1,2}^2 + \|\theta^\varepsilon\|_n^n + \|\boldsymbol{q}^*_\varepsilon\|_m^m + \|\mathbf{S}^*_\varepsilon\|_{r'}^{r'}\,dt \le C,\end{gathered} \tag{3.55}$$

where m comes from (3.35) and n from (3.32). Consequently, using the equations (3.47) and (3.53) we also conclude that

$$\|\boldsymbol{v}^\varepsilon_{,t}\|_{(X_{\rm div}^{r,\frac{5r}{5r-8}})^*} + \|\theta^\varepsilon_{,t}\|_{L^1(0,T;W^{1,\sigma'}(\Omega))} \le C \quad \text{ with } \sigma > \max\{m,3\}. \tag{3.56}$$

To obtain uniform estimates on $\{p^\varepsilon\}$ we introduce $p^\varepsilon_0 := p^\varepsilon - \frac{1}{|\Omega|}\int_\Omega p^\varepsilon\ dx$ and observe by contradiction and since $\int_{\Omega_0} p^\varepsilon\ dx = 0$ that there is C independent of ε, η such that $\|p^\varepsilon\|_q \le C\|p^\varepsilon_0\|_q$. Consequently, it suffices to find uniform estimates for p^ε_0. For this purpose, we consider $\boldsymbol{w}$ in (3.47) first of the form $\boldsymbol{w} := \nabla\mathcal{N}^{-1}(|p^\varepsilon_0|^{r'-2}p^\varepsilon_0 - \frac{1}{|\Omega|}\int_\Omega |p^\varepsilon_0|^{r'-2}p^\varepsilon_0\ dx)$ and then $\boldsymbol{w} := \nabla\mathcal{N}^{-1}(|p^\varepsilon_0|^{\frac{5r}{6}-2}p^\varepsilon_0 - \frac{1}{|\Omega|}\int_\Omega |p^\varepsilon_0|^{\frac{5r}{6}-2}p^\varepsilon_0\ dx)$. Such a choice of $\boldsymbol{w}$ clearly leads to

$$(p^\varepsilon, \operatorname{div}\boldsymbol{w})_Q = \int_0^T \|p^\varepsilon_0\|_\alpha^\alpha \qquad \text{(first with } \alpha = r' \text{ and then with } \alpha = \tfrac{5r}{6}).$$

Replacing the left hand side by means of (3.47) and proceeding step by step as in Bulíček et al. (2007a), and in particular using the fact that

$$\int_0^T \langle \boldsymbol{v}^\varepsilon_{,t}, \boldsymbol{w}\rangle\,dt = -\frac{2\varepsilon}{\alpha}\|p^\varepsilon_0(T)\|_\alpha^\alpha \le 0 \quad (\alpha = r' \text{ or } 5r/6)\,,$$

we conclude that

$$\begin{gathered}\int_0^T \|p^\varepsilon_0\|_{r'}^{r'} \le C(\eta) \implies \int_0^T \|p^\varepsilon\|_{r'}^{r'} \le C(\eta),\\ \int_0^T \|p^\varepsilon_0\|_{\frac{5r}{6}}^{\frac{5r}{6}} \le C \implies \int_0^T \|p^\varepsilon\|_{\frac{5r}{6}}^{\frac{5r}{6}} \le C.\end{gathered} \tag{3.57}$$

Using (3.47) again, these estimates imply that

$$\int_0^T \|\boldsymbol{v}^\varepsilon_{,t}\|_{W_{\boldsymbol{n}}^{-1,r'}}^{r'}\ dt \le C(\eta). \tag{3.58}$$

It is then a consequence of (3.55)-(3.58), Corollary 2.1 and Aubin-Lions Lemma 2.5

that we can find (not relabelled) subsequences of $\{\boldsymbol{v}^\varepsilon, \theta^\varepsilon, p^\varepsilon, \mathbf{S}^\varepsilon, \boldsymbol{q}^\varepsilon\}$ such that

$$\boldsymbol{v}^\varepsilon_{,t} \rightharpoonup \boldsymbol{v}_{,t} \qquad \text{weakly in } L^{r'}(0,T;W^{-1,r'}_{\boldsymbol{n}}), \tag{3.59}$$

$$\boldsymbol{v}^\varepsilon \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } L^{r}(0,T;W^{1,r}_{\boldsymbol{n}}), \tag{3.60}$$

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \qquad \text{strongly in } L^{h}(0,T;L^h(\Omega)^3) \text{ for all } h < \tfrac{5r}{3}, \tag{3.61}$$

$$\operatorname{tr}\boldsymbol{v}^\varepsilon \to \operatorname{tr}\boldsymbol{v} \qquad \text{strongly in } L^{2}(0,T;L^2(\partial\Omega)^3), \tag{3.62}$$

$$\theta^\varepsilon \to \theta \qquad \text{strongly in } L^{n}(0,T;L^n(\Omega)) \text{ for all } n < \tfrac{5+3\beta}{3}\ , \tag{3.63}$$

$$(\theta^\varepsilon)^{\frac{\beta+\lambda+1}{2}} \rightharpoonup (\theta)^{\frac{\beta+\lambda+1}{2}} \qquad \text{weakly in } L^{2}(0,T;W^{1,2}(\Omega)) \text{ for all } \lambda < 0, \tag{3.64}$$

$$p^\varepsilon \rightharpoonup p \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)), \tag{3.65}$$

$$\mathbf{S}^*_\varepsilon \rightharpoonup \overline{\mathbf{S}^*} \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{3\times 3}), \tag{3.66}$$

$$\boldsymbol{q}^*_\varepsilon \rightharpoonup \overline{\boldsymbol{q}^*} \qquad \text{weakly in } L^{m}(0,T;L^m(\Omega)^3) \text{ for all } m < \tfrac{3\beta+5}{3\beta+4}, \tag{3.67}$$

Since $\sqrt{\varepsilon}\nabla p^\varepsilon$ is uniformly bounded in $L^2(0,T;L^2(\Omega)^3)$, applying (3.60) to (3.54) we immediately obtain $\operatorname{div}\boldsymbol{v} = 0$ in Q.

In order to take the limit in (3.47) with (3.48) and in (3.53) we first identify the limits of $\mathbf{S}^*_\varepsilon$, $\boldsymbol{q}^*_\varepsilon$ and $\mathbf{S}^*_\varepsilon \cdot \mathbf{D}(\boldsymbol{v}^\varepsilon)$. To prove that $\overline{\mathbf{S}^*} = \mathbf{S}^* := \nu(p,\theta,\mathbf{D}(\boldsymbol{v}))\mathbf{D}(\boldsymbol{v})$, it is enough to establish almost everywhere convergence for p^ε and $\nabla\boldsymbol{v}^\varepsilon$. To show this, we start with the inequality formulated in Lemma 2.1:

$$\begin{aligned}\frac{C_1}{2}\int_Q \gamma_1(\theta^\varepsilon \mathcal{I}(\mathbf{D}(\boldsymbol{v}^\varepsilon),\mathbf{D}(\boldsymbol{v}))\,dx\,dt &\le \frac{B_2\gamma_0^2}{2C_1}\int_0^T \|p^\varepsilon - p\|_2^2\,dt \\ &\quad + \left(\mathbf{S}^*_\varepsilon - \nu(p,\theta^\varepsilon,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\right)_Q.\end{aligned} \tag{3.68}$$

Observing first that (3.66), (3.63) and Lebesgue Dominated Convergence theorem (used to show that $\|[\nu(p,\theta^\varepsilon,|\mathbf{D}(\boldsymbol{v})|^2) - \nu(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)]\mathbf{D}(\boldsymbol{v})\|_{r'} \to 0$ as $\varepsilon \to 0$) imply that $\left(\nu(p,\theta^\varepsilon,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\right)_Q \to 0$ as $\varepsilon \to 0$. Next, taking $\boldsymbol{w} := \boldsymbol{v}^\varepsilon - \boldsymbol{v}$ in (3.47) it is easy to observe, using above convergence results, that $\limsup_{\varepsilon\to 0}(\mathbf{S}^*_\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}))_Q \le 0$. Thus, inserting this information into (3.68), we obtain ($o(1)$ denotes the quantity that vanishes as $\varepsilon \to 0$)

$$\frac{C_1}{2}\int_Q \gamma_1(\theta^\varepsilon)\mathcal{I}(\mathbf{D}(\boldsymbol{v}^\varepsilon),\mathbf{D}(\boldsymbol{v}))\,dx\,dt \le o(1) + \frac{B_2\gamma_0^2}{2C_1}\int_0^T \|p^\varepsilon - p\|_2^2\,dt. \tag{3.69}$$

Note that (3.65) implies that $p_0^\varepsilon \rightharpoonup p_0 \in L^{r'}(Q)$. Since $\int_{\Omega_0} p^\varepsilon = 0$ we have

$$\begin{aligned}\|p_0^\varepsilon - p_0\|_2^2 &= \|p^\varepsilon - p\|_2^2 - \frac{1}{|\Omega|}\left(\int_\Omega p^\varepsilon - p\,dx\right)^2 \\ &= \|p^\varepsilon - p\|_2^2 - \frac{1}{|\Omega|}\left(\int_{\Omega\setminus\Omega_0} p^\varepsilon - p\,dx\right)^2 \ge \frac{|\Omega_0|}{|\Omega|}\|p^\varepsilon - p\|_2^2.\end{aligned} \tag{3.70}$$

Also,

$$\int_0^T \|p_0^\varepsilon - p_0\|_2^2\,dt = (p_0^\varepsilon, p^\varepsilon - p_0)_Q - (p_0, p_0^\varepsilon - p_0)_Q \le o(1) + (p_0^\varepsilon, p_0^\varepsilon - p_0)_Q. \tag{3.71}$$

To estimate the last term, we take $\boldsymbol{w} = \nabla g^\varepsilon$ in (3.47), whereas

$$g^\varepsilon := \mathcal{N}^{-1}(p_0^\varepsilon - p_0) \rightharpoonup 0 \quad \text{weakly in } L^{r'}(0,T;W^{2,r'}(\Omega))\,. \tag{3.72}$$

As a consequence, we obtain (see also Bulíček et al. (2007a))

$$(p_0^\varepsilon, p_0^\varepsilon - p_0)_Q \le o(1) + (\mathbf{S}_\varepsilon^*, \nabla^2 g^\varepsilon)_Q = o(1) + (\mathbf{S}_\varepsilon^* - \nu(p,\theta^\varepsilon,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \nabla^2 g^\varepsilon)_Q \\ + (\nu(p,\theta^\varepsilon,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \nabla^2 g^\varepsilon)_Q =: o(1) + I_1 + I_2.$$

Using Lebesgue Dominated Convergence theorem, (3.42) and (3.72), we obtain $I_2 \overset{\varepsilon\to 0}{\to} 0$. Regarding I_1, we again apply Lemma 2.1 and conclude that

$$\begin{aligned} I_1 &\le \int_Q C_2\gamma_1(\theta^\varepsilon)\mathcal{I}^{\frac12}(\mathbf{D}(\boldsymbol{v}^\varepsilon),\mathbf{D}(\boldsymbol{v}))|\nabla^2 g^\varepsilon| + \gamma_0\gamma_2(\theta^\varepsilon)|p^\varepsilon - p||\nabla^2 g^\varepsilon|\,dx\,dt \\ &\le C_2\sqrt{B_1}C_{reg}(\Omega,2)\int_0^T \left(\int_\Omega \gamma_1(\theta^\varepsilon)\mathcal{I}(\mathbf{D}(\boldsymbol{v}^\varepsilon),\mathbf{D}(\boldsymbol{v}))\,dx\right)^{\frac12}\|p_0^\varepsilon - p_0\|_2\,dt \\ &\quad + \gamma_0 B_3 C_{reg}(\Omega,2)\sqrt{\tfrac{|\Omega|}{|\Omega_0|}}\int_0^T \|p_0^\varepsilon - p_0\|_2^2\,dt \\ &\le \frac{1+\gamma_0 B_3 C_{reg}(\Omega,2)\sqrt{\frac{|\Omega|}{|\Omega_0|}}}{2}\int_0^T \|p_0^\varepsilon - p_0\|_2^2\,dt \\ &\quad + \frac{C_2^2 B_1 C_{reg}^2(\Omega,2)}{2(1-\gamma_0 B_3\sqrt{\frac{|\Omega|}{|\Omega_0|}}C_{reg}(\Omega,2))}\int_Q \gamma_1(\theta^\varepsilon)\mathcal{I}(\mathbf{D}(\boldsymbol{v}^\varepsilon),\mathbf{D}(\boldsymbol{v}))\,dx\,dt. \end{aligned}$$

Inserting this estimate into (3.71) and using (3.70), we find that

$$\begin{aligned} \int_0^T \|p^\varepsilon - p\|_2^2\,dt &\le \tfrac{|\Omega|}{|\Omega_0|}\int_0^T \|p_0^\varepsilon - p_0\|_2^2\,dt \\ &\le \tfrac{|\Omega|}{|\Omega_0|}\frac{B_1 C_2^2 C_{reg}^2(\Omega,2)}{(1-\gamma_0 B_3\sqrt{\frac{|\Omega|}{|\Omega_0|}}C_{reg}(\Omega,2))^2}\int_Q \gamma_1(\theta^\varepsilon)\mathcal{I}(\mathbf{D}(\boldsymbol{v}^\varepsilon),\mathbf{D}(\boldsymbol{v}))\,dx\,dt + o(1). \end{aligned} \tag{3.73}$$

Finally, putting (3.69) and (3.73) together, and using the assumption on γ_0 (2.47) (recalling that $\gamma_1 \ge 1$) imply that

$$\int_Q \mathcal{I}(\mathbf{D}(\boldsymbol{v}^\varepsilon),\mathbf{D}(\boldsymbol{v}))\,dx\,dt \overset{\varepsilon\to 0}{\to} 0 \quad \text{and} \quad \int_Q |p^\varepsilon - p|_2^2\,dx\,dt \overset{\varepsilon\to 0}{\to} 0.$$

The same arguments as those used in (3.51)-(3.52) then imply that

$$\boldsymbol{v}^\varepsilon \to \boldsymbol{v} \text{ strongly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}), \quad p^\varepsilon \to p \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{3.74}$$

which is sufficient to prove (modulo subsequence) that $\overline{\mathbf{S}^*} = \mathbf{S}^* := \nu(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})$ and $\overline{\boldsymbol{q}^*} = \boldsymbol{q}^* := -\kappa(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)\nabla\theta$.

Finally, having all these convergence results (3.59)-(3.67) in hands together with the identification of the limit of nonlinear terms, we can take the limit $\varepsilon \to 0$ in (3.47) with (3.48) and in (3.53) in a standard way and conclude that for any $\eta > 0$ the triplet $(\boldsymbol{v},\theta,p) := (\boldsymbol{v}^\eta,\theta^\eta,p^\eta)$ fulfills

$$\begin{aligned} &\langle \boldsymbol{v}_{,t}, \boldsymbol{w}\rangle - (\boldsymbol{v}_\eta \otimes \boldsymbol{v}, \nabla \boldsymbol{w}) + (\mathbf{S}^*, \nabla\boldsymbol{w}) + \alpha(\boldsymbol{v},\boldsymbol{w})_{\partial\Omega} - (p, \operatorname{div}\boldsymbol{w}) \\ &\qquad = \langle \boldsymbol{f}, \boldsymbol{w}\rangle \qquad \text{for all } \boldsymbol{w} \in W^{1,r}_{\boldsymbol{n}} \text{ and a.a.} t \in (0,T), \end{aligned} \tag{3.75}$$

$$\begin{aligned} &-(\theta,\varphi_{,t})_Q - (\boldsymbol{v}_\eta\theta, \nabla\varphi)_Q - (\boldsymbol{q}^*, \nabla\varphi) = (\mathbf{S}^*\cdot\mathbf{D}(\boldsymbol{v}),\varphi) + (\theta_0,\varphi(0)) \\ &\qquad\qquad \text{for all } \varphi \in \mathcal{D}(-\infty,T;W^{1,\sigma}(\Omega)) \text{ with } \sigma > \max\{3, m'\}. \end{aligned} \tag{3.76}$$

The attainment of initial conditions is again standard and can be proved by using the same methods as those described in Málek & Rajagopal (2005) and Lions (1996, Appendix E).

3.4. Limit $\eta \to 0$. Let $(\boldsymbol{v}^\eta, \theta^\eta, p^\eta, \mathbf{S}^*_\eta, \boldsymbol{q}^*_\eta)$, the solution of the η-approximation, fulfil (3.75) and (3.76). Our final goal is to take the limit $\eta \to 0$ in (3.75) and in the equation obtained as the sum of (3.76) (with any general φ) and (3.75) with $\boldsymbol{w} = \boldsymbol{v}\varphi$ and to establish the existence of a suitable weak solution to our original problem.

Using weak lower semicontinuity of appropriate norms, Fatou's lemma, (3.55) and $(3.57)_2$, we find that for $1 \le m < \frac{3\beta+5}{3\beta+4}$ and $1 \le n < \frac{5+3\beta}{3}$

$$\begin{aligned} \sup_{t\in[0,T]} (\|\theta^\eta(t)\|_1 + \|\boldsymbol{v}^\eta(t)\|_2^2) + \int_0^T \|\boldsymbol{v}^\eta\|_{1,r}^r + \|p^\eta\|_{\frac{5r}{6}}^{\frac{5r}{6}}\, dt \le C, \\ \int_0^T \|(\theta^\eta)^{\frac{\beta+\lambda+1}{2}}\|_{1,2}^2 + \|\theta^\eta\|_n^n + \|\boldsymbol{q}^*_\eta\|_m^m + \|\mathbf{S}^*_\eta\|_{r'}^{r'}\, dt \le C. \end{aligned} \tag{3.77}$$

Consequently, using the equations (3.75) and (3.76) we get the estimates

$$\|\boldsymbol{v}^\eta_{,t}\|_{(X_{\mathrm{div}}^{r,\frac{5r}{5r-8}})^*} + \|\boldsymbol{v}^\eta_{,t}\|_{L^{\frac{5r}{6}}(0,T;W_{\boldsymbol{n}}^{-1,\frac{5r}{6}})} + \|\theta^\eta_{,t}\|_{L^1(0,T;W^{1,\sigma'}(\Omega))} \le C. \tag{3.78}$$

These estimates together with Aubin-Lions Lemma 2.5 and Corollary 2.1 are sufficient to find a (not relabelled) subsequence of $(\boldsymbol{v}^\eta, \theta^\eta, p^\eta)$ such that

$$\begin{aligned}
\boldsymbol{v}^\eta_{,t} &\rightharpoonup \boldsymbol{v}_{,t} && \text{weakly in } L^{\frac{5r}{6}}(0,T;W_{\boldsymbol{n}}^{-1,\frac{5r}{6}}) \cap (X^{r,\frac{5r}{5r-8}})^*, && (3.79)\\
\boldsymbol{v}^\eta &\rightharpoonup \boldsymbol{v} && \text{weakly in } L^r(0,T;W_{\boldsymbol{n}}^{1,r}), && (3.80)\\
\boldsymbol{v}^\eta &\to \boldsymbol{v} && \text{strongly in } L^h(0,T;L^h(\Omega)^3) \text{ for all } h < \tfrac{5r}{3}, && (3.81)\\
\operatorname{tr}\boldsymbol{v}^\eta &\to \operatorname{tr}\boldsymbol{v} && \text{strongly in } L^2(0,T;L^2(\partial\Omega)^3), && (3.82)\\
\theta^\eta &\to \theta && \text{strongly in } L^n(0,T;L^n(\Omega)) \text{ for all } n < \tfrac{5+3\beta}{3}, && (3.83)\\
(\theta^\eta)^{\frac{\beta+\lambda+1}{2}} &\rightharpoonup (\theta)^{\frac{\beta+\lambda+1}{2}} && \text{weakly in } L^2(0,T;W^{1,2}(\Omega)) \text{ for all } \lambda < 0, && (3.84)\\
p^\eta &\rightharpoonup p && \text{weakly in } L^{\frac{5r}{6}}(0,T;L^{\frac{5r}{6}}(\Omega)), && (3.85)\\
\mathbf{S}^*_\eta &\rightharpoonup \overline{\mathbf{S}^*} && \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)^{3\times 3}), && (3.86)\\
\boldsymbol{q}^*_\eta &\rightharpoonup \overline{\boldsymbol{q}^*} && \text{weakly in } L^m(0,T;L^m(\Omega)^3) \text{ for all } m < \tfrac{3\beta+5}{3\beta+4}. && (3.87)
\end{aligned}$$

Assume for a moment that p^η and $\mathbf{D}(\boldsymbol{v}^\eta)$ converge almost everywhere in Q so that $\overline{\mathbf{S}^*} = \mathbf{S}^* := \nu(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v})$ and $\overline{\boldsymbol{q}^*} = \boldsymbol{q}^* := -\kappa(p,\theta,|\mathbf{D}(\boldsymbol{v})|^2)\nabla\theta$. Then we are able to take the limit in (3.75) and to achieve (2.41). Moreover, defining $E^\eta := \frac{1}{2}|\boldsymbol{v}^\eta|^2 + c_v\theta^\eta$, setting $\boldsymbol{\varphi} := \boldsymbol{v}^\eta\varphi$ in (3.75) and adding it to (3.76), and letting η tend to zero, we obtain (2.42). Next, defining $S^\eta := c_v \ln\theta^\eta$, it follows also from (3.77) that S^η satisfies (2.39). Moreover, setting $\varphi := \psi/\theta^\eta$ in (3.76) and letting $\eta \to 0$ we obtain (2.43).

What remains in order to complete the proof of Theorem 2.1 is to establish almost everywhere convergence of p^η and $\mathbf{D}(\boldsymbol{v}^\eta)$, and to show that the initial conditions are attained (see (2.44)).

3.4.1. Almost everywhere convergence of p^η and $\mathbf{D}(\boldsymbol{v}^\eta)$. First, we define $-p_1^\eta := \mathcal{N}_{\Omega_0}^{-1}\left(\operatorname{div}\operatorname{div}(\boldsymbol{v}^\eta_\eta \otimes \boldsymbol{v}^\eta)\right)$ at each time level and set $p_2^\eta := p^\eta - p_1^\eta$. Consequently, p_2^η then solves at each time level

$$(p_2^\eta, \triangle\varphi) = -\langle \boldsymbol{f}, \nabla\varphi\rangle + \alpha(\boldsymbol{v}^\eta, \nabla\varphi)_{\partial\Omega} + (\mathbf{S}^\eta, \nabla^2\varphi) \tag{3.88}$$

for all $\varphi \in W^{2,r}(\Omega)$ with $\nabla\varphi \in W_{\boldsymbol{n}}^{1,r}$, i.e., $p_2^\eta = \mathcal{N}_{\Omega_0}^{-1}\left(\operatorname{div}\boldsymbol{f} + \operatorname{div}\operatorname{div}\mathbf{S}^*_\eta\right)$. The same procedure as that developed in Bulíček et al. (2007a) implies that (we use uniform estimates (3.77) and equation (3.88))

$$\int_0^T \|p_2^\eta\|_{r'}^{r'}\, dt \le C. \tag{3.89}$$

The uniform estimate (3.89) implies that (after taking a subsequence)

$$p_2^\eta \rightharpoonup p_2 \qquad \text{weakly in } L^{r'}(0,T;L^{r'}(\Omega)). \tag{3.90}$$

Moreover, using (3.81) and definition of p_1^η we obtain

$$p_1^\eta \to p_1 \qquad \text{strongly in } L^s(0,T;L^s(\Omega)) \text{ for } s \in \langle 1, \tfrac{5r}{6}). \tag{3.91}$$

Next, we follow the approach described in Boccardo & Murat (1992), Frehse et al. (2000) or Bulíček et al. (2007a). We define

$$g^\eta := |\nabla \boldsymbol{v}^\eta|^r + |\nabla \boldsymbol{v}|^r + \left(|\mathbf{S}_\eta^*| + |\mathbf{S}^*|\right)(|\mathbf{D}(\boldsymbol{v}^\eta)| + |\mathbf{D}(\boldsymbol{v})|)\,. \tag{3.92}$$

It follows from (3.77) that there is $K \in [1,\infty)$ so that for all η

$$0 \le \int_0^T \int_\Omega g^\eta\, dx\, dt \le K\,.$$

Let $\varepsilon^* > 0$ be arbitrary. Then the following statement is proved in Bulíček et al. (2007a):

$$\begin{aligned}&\textit{There exist } L \le \frac{\varepsilon^*}{K}, \textit{ subsequence } \{\boldsymbol{v}^j\}_{j=1}^\infty \subset \{\boldsymbol{v}^\eta\}_{\eta>0}\\ &\textit{and sets } E^j := \left\{(t,x) \in Q; L^2 \le |\boldsymbol{v}^j(t,x) - \boldsymbol{v}(t,x)| < L\right\}\\ &\textit{such that } \int_{E^j} g^j\, dx\, dt \le \varepsilon^*.\end{aligned} \tag{*}$$

For such an obtained sequence $\{\boldsymbol{v}^j\}_{j=1}^\infty$ and L we define $\boldsymbol{u}^j$ and the sets Q_j as

$$\boldsymbol{u}^j := (\boldsymbol{v}^j - \boldsymbol{v})\left(1 - \min\left\{\tfrac{|\boldsymbol{v}-\boldsymbol{v}^j|}{L}, 1\right\}\right) \text{ and } Q^j := \left\{(t,x) \in Q; |\boldsymbol{v} - \boldsymbol{v}^j| < L\right\}. \tag{3.93}$$

By using (3.80)-(3.81) and the fact that $|\boldsymbol{u}^j| \le L$ in Q we have (as $j \to \infty$)

$$\boldsymbol{u}^j \rightharpoonup \mathbf{0} \qquad \text{weakly in } L^r(0,T;W^{1,r}_{\boldsymbol{n}}), \tag{3.94}$$

$$\boldsymbol{u}^j \to \mathbf{0} \qquad \text{strongly in } L^s(0,T;L^s(\Omega)^3) \quad \forall s < \infty. \tag{3.95}$$

$$\operatorname{tr} \boldsymbol{u}^j \to \mathbf{0} \qquad \text{strongly in } L^2(0,T;L^2(\partial\Omega)^3). \tag{3.96}$$

Since (see Bulíček et al. (2007a, (2.60)) for details)

$$\int_0^T \|\operatorname{div} \boldsymbol{u}^j\|_r^r \le C\varepsilon^*, \tag{3.97}$$

the Helmholtz decomposition $\boldsymbol{u}^j = \boldsymbol{u}^j_{\text{div}} + \nabla g^{\boldsymbol{u}^j}$ then implies that

$$\int_0^T \|g^{\boldsymbol{u}^j}\|_{2,r}^r\, dt \le C\varepsilon^*\,, \tag{3.98}$$

$$\boldsymbol{u}^j_{\text{div}} \to \mathbf{0} \quad \text{strongly in } L^s(0,T;L^s(\Omega)^3) \quad \text{for all } s < \infty. \tag{3.99}$$

For simplicity, we denote for $j \in \mathbb{N}$

$$\mathbf{W}^j := \nu(p_1^j + p_2, \theta^j, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}) \in L^{r'}(0,T; L^{r'}(\Omega)^{3\times 3}).$$

The integration of (2.22) (with $\mathbf{D} = \mathbf{D}(\boldsymbol{v})$, $\mathbf{B} = \mathbf{D}(\boldsymbol{v}^j)$, $p := p_1^j + p_2$, $q := p^j$, $\theta := \theta^j$) over Q^j leads to

$$\begin{aligned}\frac{C_1}{2}\int_{Q^j} \gamma_1(\theta^j)\mathcal{I}(\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v}^j))\,dx\,dt &\le \frac{B_2^2\gamma_0^2}{2C_1}\int_{Q^j} |p_2^j - p_2|^2\,dx\,dt \\ &+ (\mathbf{W}^j, \mathbf{D}(\boldsymbol{v}-\boldsymbol{v}^j))_{Q^j} - (\mathbf{S}_j^*, \mathbf{D}(\boldsymbol{v}-\boldsymbol{v}^j))_{Q^j} =: Y_1 + Y_2 + Y_3.\end{aligned} \tag{3.100}$$

By virtue of (3.83), (3.91), Lebesgue Dominated Convergence theorem and (2.14), we observe

$$\mathbf{W}^j \to \mathbf{S}^* \quad \text{strongly in } L^{r'}(0,T; L^{r'}(\Omega)^{3\times 3}). \tag{3.101}$$

Therefore (as $j \to \infty$)

$$\begin{aligned}Y_2 &:= (\mathbf{W}^j, \mathbf{D}(\boldsymbol{v}-\boldsymbol{v}^j))_{Q^j} = (\mathbf{W}^j, \mathbf{D}(\boldsymbol{u}^j))_Q + (\mathbf{W}^j, \mathbf{D}((\boldsymbol{v}-\boldsymbol{v}^j)\tfrac{|\boldsymbol{v}-\boldsymbol{v}^j|}{L}))_{Q^j} \\ &\overset{(3.101)}{\underset{(3.94)}{\le}} o(1) + (\mathbf{W}^j, \mathbf{D}((\boldsymbol{v}-\boldsymbol{v}^j)\tfrac{|\boldsymbol{v}-\boldsymbol{v}^j|}{L}))_{Q^j\setminus E^j} + (\mathbf{W}^j, \mathbf{D}((\boldsymbol{v}-\boldsymbol{v}^j)\tfrac{|\boldsymbol{v}-\boldsymbol{v}^j|}{L}))_{E^j} \\ &\overset{(*)}{\le} o(1) + CL + C\varepsilon^* \le o(1) + C\varepsilon^*,\end{aligned}$$

where $o(1) \to 0$ as $j \to \infty$. To estimate Y_3 we set $\boldsymbol{\varphi} = \boldsymbol{u}_{\mathrm{div}}^j$ in (3.75) and obtain

$$\begin{aligned}Y_3 &:= (\mathbf{S}_j^*, \mathbf{D}(\boldsymbol{v}^j - \boldsymbol{v}))_{Q^j} = (\mathbf{S}_j^*, \mathbf{D}(\boldsymbol{u}^j))_Q + (\mathbf{S}_j^*, \mathbf{D}((\boldsymbol{v}-\boldsymbol{v}^j)\tfrac{|\boldsymbol{v}-\boldsymbol{v}^j|}{L}))_{Q^j} \\ &= (\mathbf{S}_j^*, \mathbf{D}(\boldsymbol{u}_{\mathrm{div}}^j))_Q + (\mathbf{S}_j^*, \mathbf{D}(\nabla g^{\boldsymbol{u}^j}))_Q + (\mathbf{S}_j^*, \mathbf{D}((\boldsymbol{v}-\boldsymbol{v}^j)\tfrac{|\boldsymbol{v}-\boldsymbol{v}^j|}{L}))_{Q^j} \\ &\overset{(*)}{\underset{(3.98)}{\le}} (\mathbf{S}_j^*, \mathbf{D}(\boldsymbol{u}_{\mathrm{div}}^j))_Q + C\varepsilon^* \overset{(3.75)}{=}: \sum_{i=1}^4 I_i + C\varepsilon^*,\end{aligned} \tag{3.102}$$

where[4]

$$\begin{aligned}I_1 &= -\langle \boldsymbol{v}_{,t}^j, \boldsymbol{u}_{\mathrm{div}}^j\rangle = -\langle \boldsymbol{v}_{,t}, \boldsymbol{u}_{\mathrm{div}}^j\rangle + \langle \boldsymbol{v}_{,t} - \boldsymbol{v}_{,t}^j, \boldsymbol{u}_{\mathrm{div}}^j\rangle \\ &\quad \le o(1) + \langle \boldsymbol{v}_{,t} - \boldsymbol{v}_{,t}^j, \boldsymbol{u}_{\mathrm{div}}^j\rangle \overset{\mathrm{div}\,\boldsymbol{v}-\boldsymbol{v}^j=0}{=} o(1) + \langle \boldsymbol{v}_{,t} - \boldsymbol{v}_{,t}^j, \boldsymbol{u}^j\rangle \le o(1), \\ I_2 &= -\left([\nabla \boldsymbol{v}^j]\boldsymbol{v}_{\eta(j)}^j, \boldsymbol{u}_{\mathrm{div}}^j\right)_Q \le C\|\boldsymbol{u}^j\|_{\frac{5r}{5r-8},Q}\|\boldsymbol{v}^j\|_{\frac{5r}{3},Q}\|\nabla \boldsymbol{v}^j\|_{r,Q} \overset{(3.95)}{=} o(1), \\ I_3 &= -\alpha(\boldsymbol{v}^j, \boldsymbol{u}_{\mathrm{div}}^j)_\Gamma \le C\|\boldsymbol{u}^j\|_{L^2(\Gamma)} \overset{(3.96)}{=} o(1), \\ I_4 &= \langle \boldsymbol{f}, \boldsymbol{u}_{\mathrm{div}}^j\rangle = o(1),\end{aligned}$$

The next step is to estimate Y_1 in (3.100), which we will do similarly as in the previous subsection $\varepsilon \to 0$. We define $p_{2,0}^j := p_2^j - \frac{1}{|\Omega|}\int_\Omega p_2^j\,dx$ and $p_{2,0} := p_2 - \frac{1}{|\Omega|}\int_\Omega p_2\,dx$. Analogously as in (3.70) we have

$$\|p_{2,0}^j - p_{2,0}\|_2^2 \ge \frac{|\Omega_0|}{|\Omega|}\|p_2^j - p_2\|_2^2. \tag{3.103}$$

[4] For details concerning the estimate of I_1 see Bulíček et al. (2007a).

Setting $g^j := \mathcal{N}^{-1}\left(p^j_{2,0} - p_{2,0}\right)$, (3.90) implies that

$$\begin{aligned} g^j &\rightharpoonup 0 \text{ weakly in } L^{r'}(0,T;W^{2,r}(\Omega)), \\ \operatorname{tr}\nabla g^j &\rightharpoonup \mathbf{0} \text{ weakly in } L^2(0,T;L^2(\partial\Omega)^3). \end{aligned} \tag{3.104}$$

Since (as $j \to \infty$)

$$\int_0^T \|p^j_{2,0} - p_{2,0}\|_2^2 \, dt \le o(1) + (p^j_{2,0}, p^j_{2,0} - p_{2,0})_Q, \tag{3.105}$$

taking $\varphi := g^j$ in (3.88) and integrating the result over time we obtain

$$\int_0^T \|p^j_{2,0} - p_{2,0}\|_2^2 \, dt \le o(1) + \sum_{a=1}^{3} I_a, \tag{3.106}$$

where $I_1 := \langle -\boldsymbol{f}, \nabla g^j\rangle = o(1)$, $I_2 := \alpha(\boldsymbol{v}^j, \nabla g^j)_\Gamma \overset{(3.104)_2}{\underset{(3.82)}{=}} o(1)$ and

$$\begin{aligned} I_3 := (\mathbf{S}^*_j, \nabla^2 g^j)_Q &= (\mathbf{S}^*_j - \mathbf{W}^j, \nabla^2 g^j)_Q + (\mathbf{W}^j, \nabla^2 g^j)_Q \\ &\overset{(3.101)}{\underset{(3.104)}{\le}} (\mathbf{S}^*_j - \mathbf{W}^j, \nabla^2 g^j)_Q + o(1) \\ &\overset{(2.23)}{\underset{(3.103)}{\le}} \gamma_0 B_3 C_{reg}(2,\Omega)\sqrt{\tfrac{|\Omega|}{|\Omega_0|}} \int_Q |p^j_2 - p_2|^2 \, dx\, dt + C_2 \int_Q \gamma_1(\theta^j) J \, dx\, dt + o(1), \end{aligned}$$

whereas the symbol J stands for

$$J := \left| \mathbf{D}(\boldsymbol{v}^j - \boldsymbol{v}) \int_0^1 \left(1 + |\mathbf{D}(\boldsymbol{v}^j + s(\boldsymbol{v} - \boldsymbol{v}^j))|^2\right)^{\frac{r-2}{2}} \, ds \right| |\nabla^2 g^j|.$$

Splitting the integral over Q^j (defined in (3.93)) and its complement, we obtain

$$\begin{aligned} \int_{Q\setminus Q^j} \gamma_1(\theta^j) J \, dx\, dt &\le B_1 \left(\int_{Q\setminus Q^j} \mathcal{I}(\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v}^j)) \, dx\, dt \right)^{\frac12} \|\nabla^2 g^j\|_{r', (Q\setminus Q^j)} |Q \setminus Q^j|^{\frac{2-r}{2r}} \\ &\le C |Q\setminus Q^j|^{\frac{2-r}{2r}} \overset{(r<2)}{\underset{(3.81)}{\le}} C o(1), \end{aligned}$$

$$\begin{aligned} \int_{Q^j} \gamma_1(\theta^j) J dx dt &\le \sqrt{B_1} C_{reg}(2,\Omega) \left(\int_{Q^j} \gamma_1(\theta^j) \mathcal{I}(\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v}^j)) dx dt \right)^{\frac12} \|p^j_{2,0} - p_{2,0}\|_{Q^j} \\ &\le \frac{B_1 C^2_{reg}(\Omega,2) C_2}{2\left(1 - \gamma_0 B_3 \sqrt{\frac{|\Omega|}{|\Omega_0|}} C_{reg}(\Omega,2)\right)} \int_{Q^j} \gamma_1(\theta^j) \mathcal{I}(\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v}^j)) \, dx\, dt \\ &\quad + \frac{1 - \gamma_0 B_3 \sqrt{\frac{|\Omega|}{|\Omega_0|}} C_{reg}(\Omega,2)}{2C_2} \|p^j_{2,0} - p_{2,0}\|^2_{2,Q^j}. \end{aligned}$$

Using the estimates of I_1, I_2 and I_3, we conclude from (3.103) and (3.106) that

$$\begin{aligned} \int_Q |p^j_2 - p_2|^2 \, dx\, dt \le{} & \frac{\frac{|\Omega|}{|\Omega_0|} C^2_{reg}(\Omega,2) C_2^2 B_1}{\left(1 - \gamma_0 B_3 \sqrt{\frac{|\Omega|}{|\Omega_0|}} C_{reg}(\Omega,2)\right)^2} \int_{Q^j} \gamma_1(\theta^j) \mathcal{I}(\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v}^j)) \, dx\, dt \\ & + o(1). \end{aligned} \tag{3.107}$$

Combining the estimates (3.100)-(3.107), incorporating as well the assumption on γ_0 (see (2.47)), we finally obtain the following inequality

$$\int_{Q^j} \mathcal{I}(\mathbf{D}(\boldsymbol{v}),\mathbf{D}(\boldsymbol{v}^j))\,dx\,dt + \int_Q |p_2^j - p_2|^2\,dx\,dt \le o(1) + \varepsilon^*\,. \tag{3.108}$$

This is then sufficient to conclude almost everywhere convergence result for $\{p_2^j\}$ (and by virtue of (3.91) also for $\{p^j\}$) and for $\mathbf{D}(\boldsymbol{v}^j)$, which has been the remaining task to identify $\overline{\mathbf{S}^*}$ and $\overline{\boldsymbol{q}^*}$.

3.4.2. Attainment of initial conditions. First, note that $(2.44)_1$ can be easily deduced by using for example the same procedure as in Málek & Rajagopal (2005, page 431). It follows from (2.42) that $E \in \mathcal{C}(0,T;L^1_{\text{weak}}(\Omega))$. Therefore, setting $\varphi := \chi_{0,t}$ in (2.42) we obtain

$$\int_\Omega E(t) - E_0\,dx = \int_0^t \langle \boldsymbol{f},\boldsymbol{v}\rangle - \alpha\|\boldsymbol{v}\|^2_{2,\partial\Omega}\,dt \overset{t\to 0}{\to} 0. \tag{3.109}$$

Consequently, having $(2.44)_1$ and using (3.109), we deduce that

$$\int_\Omega \theta(t) - \theta_0\,dx \overset{t\to 0}{\to} 0. \tag{3.110}$$

Similarly as we proved the entropy inequality (2.43) we show that

$$\begin{aligned} &- c_v(\sqrt{\theta},\psi_{,t})_Q - c_v(\sqrt{\theta_0},\psi(0)) - c_v(\boldsymbol{v}\sqrt{\theta},\nabla\psi)_Q - \frac{1}{2}(\boldsymbol{q}/\sqrt{\theta},\nabla\psi)_Q \\ &\ge -\frac{1}{4}(\boldsymbol{q}/\sqrt{\theta^3},\nabla\theta\psi)_Q + \frac{1}{2}\left(\mathbf{S}/\sqrt{\theta},\nabla\boldsymbol{v}\psi\right)_Q \ge 0 \text{ for all } \psi\in\mathcal{D} \text{ with } \psi\ge 0, \end{aligned} \tag{3.111}$$

Setting $\psi(t,x) := \varphi(x)\chi_{(0,t)} \ge 0$, $\varphi\in\mathcal{C}^1_0(\Omega)$ in (3.111), we conclude for a.a. $t\in(0,T)$, that

$$\begin{aligned} &c_v(\sqrt{\theta(t)},\varphi) - c_v(\sqrt{\theta_0},\varphi) - c_v\int_0^t (\boldsymbol{v}\sqrt{\theta},\nabla\varphi) - \frac{1}{2}(\boldsymbol{q}/\sqrt{\theta},\nabla\varphi)\,d\tau \\ &\ge \int_0^t -\frac{1}{4}(\boldsymbol{q}/\sqrt{\theta^3},\nabla\theta\varphi) + \frac{1}{2}\left(\mathbf{S}/\sqrt{\theta},\nabla\boldsymbol{v}\varphi\right)\,dt \ge 0\,. \end{aligned} \tag{3.112}$$

By redefining $\sqrt{\theta(t)}$, we easily observe that (3.112) holds for all $t\in(0,T)$. Therefore, using integrability of all terms under time integral in (3.112) we conclude that

$$\liminf_{t\to 0}(\sqrt{\theta(t)},\varphi) \ge (\sqrt{\theta_0},\varphi) \qquad \text{for all } \varphi\ge 0, \varphi\in\mathcal{C}^1_0(\Omega). \tag{3.113}$$

Using the density of smooth function in L^2 and $(2.37)_1$, we finally obtain that (3.113) is valid also for all nonnegative $\varphi\in L^2(\Omega)$. Thus, we can compute

$$\|\sqrt{\theta(t)} - \sqrt{\theta_0}\|_2^2 = \int_\Omega \theta(t) + \theta_0\,dx - 2(\sqrt{\theta(t)},\sqrt{\theta_0}) \underset{(3.110),(3.113)}{\overset{t\to 0}{\le}} 0. \tag{3.114}$$

Consequently,

$$\|\theta(t)-\theta_0\|_1 \le \int_\Omega |\sqrt{\theta(t)} - \sqrt{\theta_0}||\sqrt{\theta(t)} + \sqrt{\theta_0}|\,dx \le c\|\sqrt{\theta(t)} - \sqrt{\theta_0}\|_2 \underset{(3.114)}{\overset{t\to 0}{\to}} 0\,.$$

This completes the proof of Theorem 2.1. □

References.

Andrade C. (1930). Viscosity of Liquids. *Nature* **125**:309–310.

Bair S. (2006). A more complete description of the Shear Rheology of High-Temerature, High-Shear Journal Bearing Lubrication. *Tribology transactions* **49**:39–45.

Bair S. and Kottke P. (2003). Pressure-Viscosity relationships for elastohydrodynamics. *Tribology transactions* **46**:289–295.

Bair S. and Winer W. O. (1992). The High Pressure High Shear Stress Rheology of Liquid Lubricants. *J. Tribology* **114**:1–13.

Baranger J. and Mikelić A. (1995). Stationary solutions to a quasi-Newtonian flow with viscous heating. *Math. Models Methods Appl. Sci.* **5**(6):725–738.

Barus C. (1893). Isothermals, Isopiestics and Isometrics Relative to Viscosity. *American Jour. Sci.* **45**:87–96.

Bendler J. T., Fontanella J. J. and Schlessinger M. F. (2001). A new Vogel-like law: ionic conductivity, dialectric relaxation and viscosity near the glass transition. *Phys. Rev. Lett.* **87**(19):1955.

Bergh J. and Löfström J. (1976). *Interpolation spaces. An introduction*. Springer-Verlag, Berlin. Grundlehren der Mathematischen Wissenschaften, No. 223.

Boccardo L. and Murat F. (1992). Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.* **19**(6):581–597.

Bridgman P. W. (1931). *The physics of high pressure*. MacMillan, New York.

Bulíček M., Málek J. and Pražák D. (2005). On the dimension of the attractor for a class of fluids with pressure dependent viscosities. *Commun. Pure Appl. Anal.* **4**(4):805–822.

Bulíček M., Málek J. and Rajagopal K.R. (2007a). Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity. *Indiana Univ. Math. J.* **56**:51–86.

Bulíček M., Feireisl E. and Málek J. (2007b). Navier-Stokes-Fourier system for incompressible fluids with temperature dependent material coefficients. Submitted.

Callen H. (1985). *Thermodynamics and an introduction to thermostatics*. Wiley, London.

Clopeau T. and Mikelić A. (1997). Nonstationary flows with viscous heating effects. In *Élasticité, viscoélasticité et contrôle optimal (Lyon, 1995)*, vol. 2 of *ESAIM Proc.*, pp. 55–63 (electronic). Soc. Math. Appl. Indust., Paris.

Consiglieri L. (2000). Weak solutions for a class of non-Newtonian fluids with energy transfer. *J. Math. Fluid Mech.* **2**(3):267–293.

Cutler W. G., McMickle R. H., Webb W. and Schiessler R. W. (1958). Study of the compressions of several high molecular weight hydrocarbons. *J. Chem. Phys.* **29**:727–740.

Diening L., Růžička M. and Wolf J. (2007). Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity. to appear.

Dowson, D. and Higginson, G. R. (1966). *Elastohydrodynamic Lubrication, the Fundamentals of Roller and Gear Lubrication*. Pergamon, Oxford.

Feireisl E. (2004). *Dynamics of viscous compressible fluids*, vol. 26 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford.

Feireisl E. and Málek J. (2006). On the Navier-Stokes equations with temperature-dependent transport coefficients. *Differ. Equ. Nonlinear Mech.* pp. Art. ID 90616, 14 pp. (electronic).

Franta M., Málek J. and Rajagopal K. R. (2005). On steady flows of fluids with pressure- and shear-dependent viscosities. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **461**(2055):651–670.

Frehse J., Málek J. and Steinhauer M. (2000). On existence results for fluids with shear dependent viscosity—unsteady flows. In *Partial differential equations (Praha, 1998)*, vol. 406 of *Chapman & Hall/CRC Res. Notes Math.*, pp. 121–129. Chapman & Hall/CRC, Boca Raton, FL.

Gazzola F. (1997). A note on the evolution of Navier-Stokes equations with a pressure-dependent viscosity. *Z. Angew. Math. Phys.* **48**(5):760–773.

Gazzola F. and Secchi P. (1998). Some results about stationary Navier-Stokes equations with a pressure-dependent viscosity. In R. Salvi (ed.), *Navier-Stokes equations: theory and numerical methods*, vol. 388 of *Pitman Res. Notes Math. Ser.*, pp. 31–37, Harlow. Longman.

Griest E. M., Webb W. and Schiessler R. W. (1958). Effect of pressure on viscosity of higher hydrocarbons and their mixtures. *J. Chem. Phys.* **29**:711–720.

Grisvard P. (1985). *Elliptic problems in nonsmooth domains*, vol. 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA.

Hron J., Málek J., Nečas J. and Rajagopal K. R. (2003). Numerical simulations and global existence of solutions of two-dimensional flows of fluids with pressure- and shear-dependent viscosities. *Mathematics and Computers in Simulation* **61**:297–315.

Hron J., Málek J. and Rajagopal K. R. (2001). Simple flows of Fluids with Pressure Dependent Viscosities. *Proc. R. Soc. A* **457**:1603–1622.

Irwing J.B. and Barlow A. J. (1971). An automatic high pressure viscometer. *J. Phys. E* **4**:232–236.

Johnson K. L. and Cameron R. (1967). Shear behaviour of elastohydrodynamic oil films at high rolling contact pressures. *Proc. Instn. Mech. Engrs.* **182**(14):307–319.

Johnson K. L. and Greenwood J. A. (1980). Thermal analysis of an Eyring fluid in elastohydrodynamic traction. *Wear* **61**:355–374.

Johnson K. L. and Tevaarwerk J. L. (1977). Shear behaviour of elastohydrodynamic oil films. *Proc. R. Soc. Lond. A* **356**:215–236.

Kozlov V. A., Maz'ya V. G. and Rossmann J. (1997). *Elliptic boundary value problems in domains with point singularities.* American Mathematical Society, Providence, RI.

Lions P. L. (1996). *Mathematical topics in fluid mechanics. Vol. 1*, vol. 3 of *Oxford Lecture Series in Mathematics and its Applications.* The Clarendon Press Oxford University Press, New York. Incompressible models, Oxford Science Publications.

Málek J., Nečas J. and Rajagopal K. R. (2002). Global analysis of the flows of fluids with pressure-dependent viscosities. *Arch. Ration. Mech. Anal.* **165**(3):243–269.

Málek J., Nečas J. and Rajagopal K. R. (2002). Global analysis of the flows of fluids with pressure-dependent viscosities. *Arch. Ration. Mech. Anal.* **165**(3):243–269.

Málek J., Nečas J., Rokyta M. and Růžička M. (1996). *Weak and measure-valued solutions to evolutionary PDEs.* Chapman & Hall, London.

Málek J. and Rajagopal K. R. (2005). Mathematical issues concerning the Navier-Stokes equations and some of its generalizations. In *Evolutionary equations. Vol. II*, Handb. Differ. Equ., pp. 371–459. Elsevier/North-Holland, Amsterdam.

Maz'ya V. G. and Rossmann J. (2003). Weighted L_p estimates of solutions to boundary value problems for second order elliptic systems in polyhedral domains' *Z. Angew. Math. Mech.* **83**(7):435–467.

Naumann J. (2006). On the existence of weak solutions to the equations of non-stationary motion of heat-conducting incompressible viscous fluids. *Math. Methods Appl. Sci.* **29**(16):1883–1906.

Nečas J. (1966). Séminaire Equations aux Dérivées Partielles. pp. 102–128. Les Presses de l'Université, Motréal.

Paluch M., Dendzik Z., and Rzoska S. J. (1999). Scaling of high-pressure viscosity data in low molecular-weight glass-forming liquids. *Phys. Rev. B* **60**:2979–2982.

Prasad S. C. and Rajagopal K. R. (2006). Flow of a fluid with pressure dependent viscosity due to a boundary that is being stretched. *Applied Mathematics and Computation* **173**:50–68.

Rajagopal K. R. and Szeri A. Z. (2003). On an incosistency in the derivation of the equations of elastohydrodynamic lubrication. *Proc. R. Soc. A* **459**:2771–2786.

Renardy M. (1986). Some remarks on the Navier-Stokes equations with a pressure-dependent viscosity. *Comm. Partial Differential Equations* **11**(7):779–793.

Roelands C. J. A. (1966). *Correlation aspects of the viscosity-temeperature-pressure relationship of lubricating oils.* Ph.D. thesis, Technische Hogeschool Delft, The Netherlands.

Simon J. (1987). Compact sets in the space $L^p(0,T;B)$. *Annali di Matematica Pura et Applicata* **146**:65–96.

Stokes G. G. (1845). On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids. *Trans. Cambridge Phil. Soc.* **8**:287–305.

Szeri A. Z. (1998). *Fluid Film Lubrication: Theory and Design.* Cambridge University Press.

Vasudevaiah M. and Rajagopal K. R. (2005). On fully developed flows of fluids with a pressure dependent viscosity in a pipe. *Appl. Math.* **50**:341–353.

Vogel H. (1922). Die Bedeutung der Temperatur Abhangigkeit der Viscosität f.d. Beurteilung von Oelen. *Zeit. Angew. Chem.* **35**:561–563.

Zeidler E. (1990). *Nonlinear functional analysis and its applications. II/B.* Springer-Verlag, New York. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.