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CHERN-SIMONS FORMS, MICKELSSON-FADDEEV ALGEBRAS AND THE P-BRANES

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ABSTRACT

In string theory, nilpotence of the BRS operator δ for the string functional relates the Chern-Simons term in the gauge-invariant antisymmetric tensor field strength to the central term in the Kac-Moody algebra. We generalize these ideas to p-branes with odd p and find that the Kac-Moody algebra for the string becomes the Mickelsson-Faddeev algebra for the p-brane.

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1. Introduction

In a recent paper [1], the coupling of Yang-Mills fields to the heterotic string in bosonic formulation was generalized to extended objects of higher dimension (p-branes). In particular, it was noted that for odd p the Bianchi identities obeyed by the field strengths of the (p+1)-forms receive Chern-Simons corrections. In the case of the string (p=1), there is an equality between the coefficient n of the Chern-Simons term $I_3(A)$ in the antisymmetric tensor field strength $H_3 = dB_2 + nI_3(A)$, and the central charge n of the Kac-Moody algebra obeyed by certain operators $T^a(\sigma)$ that appear in the gauge BRS transformations of the string functional [2]. The purpose of the present paper is to show that for 3-branes the coefficient of the Chern-Simons term is equal to the coefficient of an Abelian extension of a $T^a(\sigma^j)$ algebra involving new generators $T^a_i(\sigma^j)$, $i, j = 1, 2, 3$. The corresponding algebras have already appeared before in the context of anomalies [3,4,5,6] and are known in the mathematical literature as loop algebras with a Mickelsson-Faddeev extension [7]. There is a straightforward generalization to $p > 3$ branes.

In string theory, the integer n also appears as a coefficient of the Wess-Zumino-Witten term in the action, and the operators T^a can be constructed from the action [2], which is invariant under simultaneous gauge variations of the background fields and the group coordinates. While this action is known for the p-branes[1], the operators T^a have not yet been constructed and examined. A second way to get the relation is to insist on the nilpotence of the gauge BRS transformations of the string field Φ and background fields A etc. It is this second method which will here be generalized to the 3-brane.

2. Loop Space Algebras

In manifestly supersymmetric and κ-symmetric form the heterotic string can be formulated as a mapping from two dimensions to a target space parametrized by variables X^μ, θ^α and y^m. We ignore θ from now on. y^m are bosons parametrizing the group space. We take the σ-model point of view that there are also background fields present representing the massless bosonic excitations of the string. Consider the following BRS transformation:

$$\delta = \delta_1 + \delta_B \tag{2.1}$$

Here δ_1 is defined by:

$$\delta_1 = \prod_{\mu,m,\sigma'} \int dy^m(\sigma')dX^\mu(\sigma')\Big\{\Big(\int d\sigma[-\omega^a T^a(\sigma) + \Lambda_\mu \frac{dX^\mu}{d\sigma}]\Phi\Big)\frac{\delta}{\delta\Phi}\Big\} \tag{2.2}$$

where the 'doubly functional' derivative is defined by:

$$\frac{\delta}{\delta\Phi(X)}\Phi(X') = \prod_\sigma \delta^D[X(\sigma) - X'(\sigma)] \tag{2.3}$$

and hence:

$$\delta_1\Phi = \int d\sigma[-\omega^a T^a(\sigma) + \Lambda_\mu \frac{dX^\mu}{d\sigma}]\Phi \tag{2.4}$$

In the above, δ_1 is a BRS transformation which acts on functionals of the string field Φ, which is itself a functional of the string variables $X^\mu(\sigma)$ and $y^m(\sigma)$. Φ is a string field, but we will ignore the problems of closed string field theory here (for reviews see e.g. [8] [9]) –in particular we ignore the dependence of Φ on the reparametrization ghost fields. The exterior derivative d and the BRS operator δ are taken to be anticommuting in this paper. Our aim is to consider just the Yang-Mills part of the BRS transformations of the background fields and the corresponding transformation of the string field.

The variable σ is the spacelike variable on the string world sheet. The operator $T^a(\sigma)$ is assumed here to depend only on $y^m(\sigma)$ and functional derivatives with respect to $y^m(\sigma)$. An example of $T^a(\sigma)$, for the case of the string, can be found in [2]. We shall alternate between component and form notation, for example setting $dX^\mu \Lambda_\mu = \Lambda_1$ etc. The part δ_1 is not separately nilpotent. The part δ_B is separately nilpotent ($\delta_B^2 = 0$) and it acts only on the background fields $A^a_\mu(x)$ etc. These BRS transformations of the background fields are:

$$\delta_B = \int d^D x \{ D^{ab}_\mu \omega^b \frac{\delta}{\delta A^a_\mu} - \frac{1}{2} f^{abc} \omega^b \omega^c \frac{\delta}{\delta \omega^a} + [-n A^a_{[\mu} \partial_{\nu]} \omega^a + \partial_{[\mu} \Lambda_{\nu]}] \frac{\delta}{\delta B_{\mu\nu}}$$

$$+ [n \omega^a \partial_\mu \omega^a - \partial_\mu B_0] \frac{\delta}{\delta \Lambda_\mu} + \frac{1}{6} n f^{abc} \omega^a \omega^b \omega^c \frac{\delta}{\delta B_0} \} \tag{2.5}$$

Here Λ_μ is a ghost for the antisymmetric tensor field $B_{\mu\nu}$ and B_0 is a 'ghost for ghost' for the ghost Λ_μ. The field ω^a is the Yang-Mills Faddeev-Popov ghost. In terms of fields this becomes, for example:

$$\delta A^a_\mu = D^{ab}_\mu \omega^b = \partial_\mu \omega^a + f^{abc} A^b_\mu \omega^c \tag{2.6}$$

Alternatively we can use the notation:

$$\delta A^a = -d\omega^a - f^{abc} A^b \omega^c \tag{2.7}$$

$$\delta \Lambda = n I^2_1 + dB_0 \tag{2.8}$$

$$\delta B_2 = n I^1_2 + d\Lambda \tag{2.9}$$

$$\cdots$$

In the above the terms I^i_{p+2-i} are the terms of ghost number i that appear in the descent equations for the Yang-Mills fields. In our conventions the curvature two-form is:

$$F^a = dA^a + \frac{1}{2} f^{abc} A^b A^c \tag{2.10}$$

and it transforms as:

$$\delta F^a = f^{abc} F^b \omega^c \tag{2.11}$$

The descent equations take the form:

$$\delta I^i_{p+2-i} = d I^{i+1}_{p+1-i} \tag{2.12}$$

so that

$$I^0_4 = F^a F^a = d I^0_3 \tag{2.13}$$

$$I^0_3 = A^a dA^a + \frac{1}{3} f^{abc} A^a A^b A^c \tag{2.14}$$

$$I^1_2 = -A^a d\omega^a \tag{2.15}$$

$$I^2_1 = \omega^a d\omega^a \tag{2.16}$$

$$I^3_0 = \frac{1}{6} f^{abc} \omega^a \omega^b \omega^c \tag{2.17}$$

Nilpotency follows easily using these. For example:

$$\delta^2 B_2 = n \delta I^1_2 - d\delta\Lambda = 0 \tag{2.18}$$

We note that:

$$H^0_3 = dB_2 + n I^0_3 \tag{2.19}$$

is gauge invariant:

$$\delta H^0_3 = \delta_B H^0_3 = 0 \tag{2.20}$$

$$dH^0_3 = I^0_4 \tag{2.21}$$

We assume that the background fields and their ghosts depend on X but not on y, so that the action of T on the background fields and ghosts is trivial here. We also assume that the action of δ_B on the operators T is trivial, since they do not depend on the background fields. We further assume that the string field Φ does not depend on the background fields. Note that these operators have been defined so that δ_1 acts only on Φ and δ_B acts only on the background fields. For example:

$$\delta_B \Phi = \delta_1 A^a_\mu = \delta_1 \Lambda_\mu = 0 \tag{2.22}$$

Calculation shows that nilpotency ($\delta^2 = 0$) of δ implies that the Kac-Moody algebra of the generators T has a central term with coefficient n:

$$[T^a(\sigma), T^b(\sigma')] = f^{abc} T^c(\sigma) \delta(\sigma - \sigma') + 2n\delta^{ab} \frac{d}{d\sigma} \delta(\sigma - \sigma') \tag{2.23}$$

Now we want to generalize this string case to p-branes for odd p. The way that $\delta^2 \Phi = 0$ works is that the variation $\delta \Lambda_\mu = n\omega^a \partial_\mu \omega^a$ is compensated by the central term in the commutator (2.23). For higher p-branes the variation $\delta \Lambda_p = I^2_p$ always involves the field A^a_μ in addition to the ghosts ω^a. Hence the analogue of (2.2) for p-branes must have an explicit dependence on A^a_μ as well as ω^a and $\Lambda_{\mu_1 \cdots \mu_p}$.

For example, for the 3-brane, we can accomplish this by writing:

$$\delta = \delta_3 + \delta_B \tag{2.24}$$

where δ_3 acts on the 3-brane wave function Φ

$$\delta_3 = \prod_{\mu, m, \sigma'^j} \int dy^m(\sigma') dX^\mu(\sigma') \Big\{ \Big(\int d^3\sigma \{ -\omega^a T^a(\sigma)$$

$$- n\epsilon^{ijk} d^{abc} \partial_\mu \omega^a A^b_\nu \Pi^{\mu\nu}_{ij} T^c_k(\sigma) + \Lambda_{\mu\nu\lambda} \Pi^{\mu\nu\lambda} \} \Phi \Big) \frac{\delta}{d\Phi} \Big\} \tag{2.25}$$

Here we use the notation:

$$\Pi^{\mu\nu}_{ij} = \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j} \tag{2.26}$$

$$\Pi^{\mu\nu\lambda} = \epsilon^{ijk} \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j} \frac{\partial X^\lambda}{\partial \sigma^k} \tag{2.27}$$

In the foregoing, δ_3 is a BRS transformation which acts on Φ, which is a functional of the 3-brane variables $X^\mu(\sigma)$ and $y^m(\sigma)$. All the T operators are again assumed to involve only functions of $y^m(\sigma)$ and $\frac{\delta}{\delta y^m(\sigma)}$ and hence the operators T commute with δ_B. The background transformations are now:

$$\delta_B = \int d^D x \big\{ D^{ab}_\mu \omega^b \frac{\delta}{\delta A^a_\mu}$$

$$-\frac{1}{2} f^{abc}\omega^b\omega^c \frac{\delta}{\delta\omega^a} + [nI^1_4(A,\omega) + d\Lambda]_{\mu\nu\lambda\rho} \frac{\delta}{\delta B_{\mu\nu\lambda\rho}}$$

$$+[nI^2_3(A,\omega) + dB_2]_{\mu\nu\lambda} \frac{\delta}{\delta \Lambda_{\mu\nu\lambda}} + \cdots + nI^5_0(\omega)\frac{\delta}{\delta B_0}\big\} \tag{2.28}$$

where

$$I^0_5 = d^{abc} A^a dA^b dA^c + \cdots \tag{2.29}$$

$$I^1_4 = -d^{abc} d\omega^a A^b dA^c + \frac{1}{4} d^{abc} d\omega^a A^b f^{cde} A^d A^e \tag{2.30}$$

$$I^2_3 = d^{abc} d\omega^a A^b d\omega^c \tag{2.31}$$

$$I^3_2 = -d^{abc} d\omega^a d\omega^b \omega^c \tag{2.32}$$

$$I^4_1 = -\frac{1}{4} d^{abc} f^{cde} d\omega^a \omega^b \omega^d \omega^e \tag{2.33}$$

$$I^5_0 = -\frac{1}{40} d^{abc} f^{bde} f^{cfg} \omega^a \omega^d \omega^e \omega^f \omega^g \tag{2.34}$$

In particular:

$$\delta\Lambda_{\mu\nu\lambda} = -d^{abc} \partial_\mu \omega^a A^b_\nu \partial_\lambda \omega^c + \cdots \tag{2.35}$$

By calculation, one can show that the above δ is nilpotent if T^a and T^a_i satisfy the

Mickelsson-Faddeev algebra:

$$[T^a(\sigma), T^b(\sigma')] = f^{abc}T^c(\sigma)\delta^3(\sigma - \sigma') - 2nd^{abc}\epsilon^{ijk}\partial_i\delta^3(\sigma - \sigma')\partial'_j T^c_k(\sigma') \quad (2.36)$$

$$[T^a(\sigma), T^b_i(\sigma')] = f^{abc}T^c_i(\sigma)\delta^3(\sigma - \sigma') + \delta^{ab}\partial'_i\delta^3(\sigma - \sigma') \quad (2.37)$$

$$[T^a_i(\sigma), T^b_j(\sigma')] = 0 \quad (2.38)$$

One may verify that the Jacobi identities are satisfied by this algebra. Note the new kind of generator T^a_i, which forms a (non-invariant) Abelian subalgebra of the T^a algebra. T^a_i transforms under the action of T^a like a Yang-Mills field.

The gauge invariant field strength associated with this nilpotent δ_B is:

$$H_5 = dB_4 + nI^0_5 \quad (2.39)$$

and it satisfies:

$$\delta H_5 = \delta_B H_5 = 0 \quad (2.40)$$

$$dH_5 = I^0_6 = d^{abc}F^aF^bF^c \quad (2.41)$$

3. Spacetime Algebras

If we take the term of δ that is linear in the field $\omega^a(x)$, then its algebra is also the Kac-Moody (p=1) or Mickelsson-Faddeev (p=3) algebra (pulled back). This works as follows. Define

$$\delta = \int d^4x\omega^a(x)T^a_{\text{tot}}(x) + \text{other terms} \quad (3.1)$$

where the other terms are those which do not have exactly one field ω in the numerator of the transformation.

Then nilpotence of δ implies that

$$\frac{1}{2}\int d^D x \int d^D x' \omega^a(x)\omega^b(x')\{[T^a_{\text{tot}}(x), T^b_{\text{tot}}(x')]$$

$$-\delta^D(x-x')f^{abc}T^c_{\text{tot}}(x)]\}\Phi = n\int d^p\sigma I^2_p(X(\sigma))_{\mu_1\ldots\mu_p}\Pi^{\mu_1\ldots\mu_p}\Phi \tag{3.2}$$

Using functional derivatives to peel off the two powers of ω in the above yields a 'pulled back' version of the algebra, which, for $p \geq 3$, has an A-dependent central extension, determined by the form of $I^2_p(X(\sigma))_{\mu_1\ldots\mu_p}$. For $p = 1$ the extension can be chosen to be A-independent because I^2_p can be chosen to be A-independent. The A-dependent extension for the $p = 3$ case is somewhat reminiscent of the situation in four-dimensional Yang-Mills field theory with fermions [5]. Explicitly for the 3-brane case we have:

$$[T^a_{\text{tot}}(x), T^b_{\text{tot}}(x')]\Phi = \Big\{f^{abc}T^c_{\text{tot}}(x')\delta^D(x-x')$$

$$+2n\big[\int d^3\sigma\delta^D[x-X(\sigma)]d^{abc}\Pi^{\mu\nu\lambda}\partial_\mu A^c_\nu\big]\partial_\lambda\delta^D(x-x')\Big\}\Phi \tag{3.3}$$

4. Conclusion

Our motivation for this work was to see how the loop space algebra of the heterotic string can be generalized to the p-branes. One constructs a BRS transformation that transforms the background fields and the p-brane functional, and then demands that it be nilpotent.

For the string, this nilpotence relates the coefficient n of the central extension of the Kac-Moody algebra of the operators T^a formed from the group coordinates to the coefficient n in the gauge invariant field strength

$$H_3 = dB_2 + nI_3 \tag{4.1}$$

of the background Yang-Mills fields.

We have shown that for the 3-brane, it is necessary to introduce operators $T_i^a(\sigma)$ and $T^a(\sigma)$ which are formed from the group coordinates. These operators obey the well-known Mickelsson-Faddeev algebra familiar from anomaly analysis in four-dimensional theories with chiral fermions. In particular the operators $T_i^a(\sigma)$ transform like Yang-Mills fields under the action of $T^a(\sigma)$. We believe that the operators T obtained by an analysis along the lines of [2] of the action in [1] should provide a realization of the Mickelsson-Faddeev algebra discussed here. Nilpotence of the BRS transformation of the 3-brane functional Φ relates the coefficient n of the (non-invariant) Abelian extension of the algebra (2.36) to the parameter n in the gauge invariant field strength

$$H_5 = dB_4 + nI_5 \tag{4.2}$$

of the background Yang-Mills fields.

We anticipate that this procedure should easily generalize to higher p, and in particular to the heterotic 5-brane [10,11,12,13,1] which in fact provided the original impetus for the present paper.

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