EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS TO VISCOUS PRIMITIVE EQUATIONS FOR CERTAIN CLASS OF DISCONTINUOUS INITIAL DATA

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Abstract. We establish some conditional uniqueness of weak solutions to the viscous primitive equations, and as an application, we prove the global existence and uniqueness of weak solutions, with the initial data taken as small L^∞ perturbations of functions in the space $X = \{v \in (L^6(\Omega))^2 | \partial_z v \in (L^2(\Omega))^2\}$; in particular, the initial data are allowed to be discontinuous. Our result generalizes in a uniform way the result on the uniqueness of weak solutions with continuous initial data and that of the so-called z-weak solutions.

1. Introduction

The primitive equations are derived from the Boussinesq system of incompressible flow under the hydrostatic balance assumption, by taking the zero singular perturbation limit of the small aspect ratio. This singular perturbation limit of small aspect ratio can be rigorously justified, see, Azérad–Guillén [1] and Li–Titi [26]. The primitive equations play a fundamental role for weather prediction models, see, e.g., the books by Lewandowski [24], Majda [30], Pedlosky [32], Vallis [38], and Washington–Parkinson [39].

In this paper, we consider the following primitive equations (without considering the coupling to the temperature equation):

$$\partial_t v + (v \cdot \nabla_H) v + w \partial_z v + \nabla_H p(\mathbf{x}^H, t) - \Delta v + f_0 k \times v = 0, \tag{1.1}$$

$$\nabla_H \cdot v + \partial_z w = 0, \tag{1.2}$$

where the horizontal velocity $v = (v^1, v^2)$, the vertical velocity w and the pressure p are the unknowns, and f_0 is the Coriolis parameter. Note that though (1.1)–(1.2) is a three-dimensional system, the pressure p depends only on two spatial variables, and there is no dynamical equation for the vertical velocity w. We use $\mathbf{x} = (x^1, x^2, z)$ to denote the spatial variable, $\mathbf{x}^H = (x^1, x^2)$ the horizontal spatial variable, and $\nabla_H = (\partial_1, \partial_2)$ the horizontal gradient. We complement system (1.1)–(1.2) with initial and boundary conditions which will be discussed in details below. Note that

Date: December 1, 2015.
2010 *Mathematics Subject Classification.* 35Q35, 76D03, 86A10.
Key words and phrases. Existence and uniqueness; discontinuous initial data; weak solution; primitive equations; hydrostatic Navier-Stokes equations.

the primitive equations considered here, i.e., (1.1)–(1.2), do not include the coupling with the temperature equation; however, one can adopt the same approach presented in this paper to establish the corresponding results for the full primitive equations coupled with temperature equation, for as long as the temperature equation has full diffusivity. For the sake of simplicity, we consider here the primitive equations without the coupling with the temperature equation, i.e., system (1.1)–(1.2).

Since the pioneer works by Lions–Temam–Wang [27–29] in the 1990s, there has been a lot of literatures on the mathematical studies of the primitive equations. The global existence of weak solutions has been proven long time ago by Lions–Temam–Wang [27–29] in the 1990s; however, the uniqueness of weak solutions in the general case is still an open question, even for the two-dimensional case. This is different from the incompressible Navier-Stokes equations, since it is well-known that the weak solution to the two-dimensional incompressible Navier-Stokes equations is unique (see, e.g., Ladyzhenskaya [25], Temam [37] and Constantin–Foias [13]). The main obstacle of proving the uniqueness of weak solutions to the two-dimensional primitive equations is the absence of the dynamical equation for the vertical velocity. In fact, the vertical velocity can only be recovered from the horizontal velocity through the incompressibility condition, and as a result, there is one derivative loss for the horizontal velocity. Though the general result on the uniqueness of weak solutions to the primitive equations is still unknown, some particular cases have been solved, see Bresch et al [3], Petcu–Temam–Ziane [33] and Tachim Madjo [36] for the case of the so-called z-weak solutions, i.e., the weak solutions with initial data in $X \cap \mathcal{H}$ (the spaces X and $\mathcal{H}$ are given, below, by (1.14) and (1.15), respectively), and Kukavica–Pei–Rusin–Ziane [20] for the case of weak solutions with continuous initial data. In the context of strong solutions, the local well-posedness was established in Guillén-González et al [14]. Remarkably, the strong solutions to the primitive equations have already been proven to be global for both 2D and 3D cases, see Cao–Titi [11, 12], Kobelkov [19] and Kukavica–Ziane [22, 23], see also Cao–Li–Titi [5–9] for some recent progress in the direction of global well-posedness of strong solutions to the systems with partial viscosities or diffusivity, as well as Hieber–Kashiwabara [17] for some progress towards relaxing the smoothness on the initial data but still for the system with full viscosities and diffusivity, by using the semigroup method. Notably, smooth solutions to the inviscid primitive equations, with or without coupling to the temperature equation, have been shown by Cao et al. [4] and Wong [40] to blow up in finite time. For the results on the local well-posedness with monotone or analytic initial data of the inviscid primitive equations (which are also called inviscid Prandtl equations or hydrostatic Euler equations), i.e., system (1.1)–(1.2), without the Lapalacian term Δv, see, e.g., Brenier [2], Masmoudi–Wong [31], Kukavica–Temam–Vicol–Ziane [21], and the references therein.

Another remarkable difference between the incompressible Navier-Stokes equations and the primitive equations is their well-posedness theories with L^p initial data. It is well-known that, for any L^p initial data, with $d \leq p \leq \infty$, there is a unique local mild

solution to the d-dimensional incompressible Navier-Stokes equations, see, Kato [18] and Giga [15, 16]; however, for the primitive equations, though the L^p norms of the weak solutions keep finite up to any finite time, as long as the initial data belong to L^p (one can apply Proposition 3.1 in Cao–Li–Titi [7] to achieve this fact with the help of some regularization procedure), it is still an open question to show the uniqueness of weak solutions to the primitive equations with L^p initial data. One of the aims of this paper is to partly answer the question of the existence and uniqueness of weak solutions to the primitive equations, with initial data in L^∞. It will be shown, as a consequence of our result, that the primitive equations possess a unique global weak solution, with initial data in L^∞, as long as the discontinuity of the initial data is sufficiently small.

We consider system (1.1)–(1.2) in the three-dimensional horizontal layer $\Omega_0 = M \times (-h, 0)$, confined between the horizontal walls $z = -h$ and $z = 0$, subject to periodic boundary conditions with respect to the horizontal variables $\mathbf{x}^H = (x^1, x^2)$ with basic fundamental periodic domain $M = (0, 1) \times (0, 1) \subset \mathbb{R}^2$. We also suppose that the flow is stress-free at, and tangential to, the solid boundaries $z = -h$ and $z = 0$. In other words, we complement system (1.1)–(1.2) with the following boundary and initial conditions

$$v, w \text{ and } p \text{ are periodic in } \mathbf{x}^H, \tag{1.3}$$

$$w|_{z=-h,0} = 0, \quad \partial_z v|_{z=-h,0} = 0, \tag{1.4}$$

$$v|_{t=0} = v_0. \tag{1.5}$$

Extend the unknowns v and w evenly and oddly, respectively, with respect to z, to be defined on the larger spatial domain $\Omega = M \times (-h, h)$, then the extended unknowns v and w are periodic, and are even and odd, respectively, in z. After such kind extension of the unknowns, the boundary and initial conditions (1.3)–(1.5) are equivalent to the following periodic boundary and initial conditions

$$v, w \text{ and } p \text{ are periodic in } x^1, x^2 \text{ and } z, \tag{1.6}$$

$$v \text{ and } w \text{ are even and odd in } z, \text{respectively}, \tag{1.7}$$

$$v|_{t=0} = v_0. \tag{1.8}$$

Because of the equivalence of the above two kinds of boundary and initial conditions, we consider, throughout this paper, the boundary and initial conditions (1.6)–(1.8), in other words, we consider system (1.1)–(1.2), subject to (1.6)–(1.8). Note that condition (1.7) is a symmetry condition, which is preserved by system (1.1)–(1.2), that is, if a smooth solution to system (1.1)–(1.2) exists and is unique, then it must satisfy the symmetry condition (1.7), as long as it is initially satisfied. We also note that since there is no dynamical equation for the vertical velocity, no initial condition is imposed on w. In fact, it is uniquely determined by the horizontal velocity, through the incompressibility condition (1.2) and the boundary conditions on w.

Note that the vertical velocity w can be uniquely expressed in terms of the horizontal velocity v, through the incompressibility condition (1.2) and the symmetry condition (1.7), as

$$w(x,y,z,t) = -\nabla_H \cdot \left(\int_{-h}^{z} v(x,y,\xi,t) d\xi \right).$$

As a result, system (1.1)–(1.8) is equivalent to the following one

$$\partial_t v + (v \cdot \nabla_H) v + w \partial_z v + \nabla_H p(\mathbf{x}^H, t) - \Delta v + f_0 k \times v = 0, \tag{1.9}$$

$$\nabla_H \cdot \left(\int_{-h}^{h} v(x,y,z,t) dz \right) = 0, \tag{1.10}$$

$$w(x,y,z,t) = -\nabla_H \cdot \left(\int_{-h}^{z} v(x,y,\xi,t) d\xi \right), \tag{1.11}$$

subject to the boundary and initial conditions

$$p \text{ is periodic in } \mathbf{x}^H; \text{ and } v \text{ is periodic in } \mathbf{x}^H \text{ and } z, \text{ and is even in } z, \tag{1.12}$$

$$v|_{t=0} = v_0. \tag{1.13}$$

Let us introduce some necessary notations, and give the definition of weak solutions. We denote by

$$C_{\text{per}}(\Omega) = \{ f \in C(\mathbb{R}^3) | f \text{ is periodic in } x^1, x^2 \text{ and } z, \text{ with basic periodic domain } \Omega \},$$

and for any positive integer m, we set

$$C^m_{\text{per}}(\Omega) = \{ f | \nabla^\alpha f \in C_{\text{per}}(\Omega), 0 \le |\alpha| \le m \}.$$

For positive integer m and number $q \in [1, \infty]$, we use $W^{m,q}_{\text{per}}(\Omega)$ to denote the space of the closure of $C^m_{\text{per}}(\Omega)$ in $W^{m,q}(\Omega)$, and denote by $W^{-m,q'}_{\text{per}}(\Omega)$ its dual space, when $q \in [1, \infty)$. One can easily verify, with the help of the standard modifier, that

$$W^{m,q}_{\text{per}}(\Omega) = \{ f \in W^{m,q}(\Omega) | \tilde{f} \in W^{m,q}_{\text{loc}}(\mathbb{R}^3) \},$$

where $\tilde{f}$ is the periodic extension of f to the whole space. In case that $q = 2$, we always use $H^m_{\text{per}}(\Omega)$ instead of $W^{m,2}_{\text{per}}(\Omega)$. One may also define the periodic $L^q(\Omega)$ space as the closure of $C_{\text{per}}(\Omega)$ in $L^q(\Omega)$; however, one can easily check that this space coincides with $L^q(\Omega)$. For simplicity, we use the same notations to denote a space and its product spaces, that is for a space Z, and a positive integer N, we still use Z to denote its N-product space Z^N. We always use $\|u\|_p$ to denote the L^p norm of u. The following two spaces X and $\mathcal{H}$ will be used throughout this paper

$$X = \{ v \in L^6(\Omega) | v \text{ is periodic in } z, \text{ and } \partial_z v \in L^2(\Omega) \}, \tag{1.14}$$

and

$$\mathcal{H} = \left\{ v \in L^2(\Omega) \middle| v \text{ is periodic in } \mathbf{x}^H, \text{ even in } z, \text{ and } \nabla_H \cdot \left(\int_{-h}^{h} v(\mathbf{x}^H, z) dz \right) = 0 \right\}. \tag{1.15}$$

We state the definition of weak solutions to system (1.9)–(1.13) as follows.

Definition 1.1. *Given a function $v_0 \in \mathcal{H}$. A function v is called a global weak solution to system (1.9)–(1.13), if the following hold:*

(i) $v \in C([0,\infty); L^2_w(\Omega)) \cap L^2_{loc}([0,\infty); H^1_{per}(\Omega) \cap \mathcal{H})$, where L^2_w stands for the L^2 space equipped with the weak topology.

(ii) For any compactly supported in time function $\varphi \in C([0,\infty); C^1_{per}(\Omega) \cap \mathcal{H})) \cap C^1([0,\infty); C_{per}(\Omega))$, the following equality holds

$$\int_0^\infty \int_\Omega [-v\partial_t\varphi + (v\cdot\nabla_H)v\cdot\varphi + w\partial_z v\cdot\varphi + \nabla v : \nabla\varphi] d\boldsymbol{x}dt = \int_\Omega v_0(\boldsymbol{x})\varphi(\boldsymbol{x},0)d\boldsymbol{x},$$

where the vertical velocity w is given by (1.11).

(iii) The following differential inequality holds

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2(t) + \|\nabla v\|_2^2(t) \le 0, \quad \text{in } \mathcal{D}'((0,\infty));$$

(iv) The following energy inequality holds

$$\frac{1}{2}\|v\|_2^2(t) + \int_0^t \|\nabla v\|_2^2(\tau)d\tau \le \frac{1}{2}\|v_0\|_2^2,$$

for a.e. $t \in (0,\infty)$.

For any initial data $v_0 \in \mathcal{H}$, following the arguments in [27–29], there is a global weak solution to system (1.9)–(1.13); however, as we mentioned before, it is still an open question to show the uniqueness of weak solutions to the primitive equations. Nevertheless, we can prove the following theorem on the conditional uniqueness of weak solutions to the primitive equations.

Theorem 1.1. *Let v be a global weak solution to system (1.9)–(1.13). Suppose that there is a positive time T_v, such that*

$$v(\boldsymbol{x},t) = \bar{v}(\boldsymbol{x},t) + V(\boldsymbol{x},t), \quad \boldsymbol{x}\in\Omega, t\in(0,T_v),$$
$$\partial_z\bar{v} \in L^\infty(0,T_v; L^2(\Omega)) \cap L^2(0,T_v; H^1_{per}(\Omega)), \quad V \in L^\infty(\Omega\times(0,T_v)).$$

Then, there is a positive constant ε_0 depending only on h, such that v is the unique global weak solution to system (1.9)–(1.13), with the same initial data as v, provided

$$\sup_{0<t<T_v'} \|V\|_\infty \le \varepsilon_0,$$

for some $T_v' \in (0,T_v)$.

To prove Theorem 1.1, we first show that any weak solution to the primitive equations is smooth away from initial time, see Corollary 3.1, below, which can be intuitively seen by noticing that weak solution has H^1_{per} regularity, at almost any time away from the initial time, and recalling that, for any H^1_{per} initial data, there is a unique global strong solution to the primitive equations; however, in order to rigorously prove this fact, we need the the weak-strong uniqueness result for the primitive equations, see Proposition 3.4, below, where we adopt the idea of Serrin [34]. Since

weak solutions are smooth away from the initial time, one can perform the energy estimates to the difference system between two weak solutions, on any finite time interval away from the initial time. Next, by using the decomposition stated in the theorem, and handling the nonlinear terms involving each parts in their own ways, we then achieve the uniqueness.

It should be pointed out that one can not expect that all weak solutions to the primitive equations, with general initial data in $\mathcal{H}$, have the decomposition as stated in the above theorem. In fact, by the weakly lower semi-continuity of the norms, in order to have such decomposition, it is necessary to require that the initial data v_0 has the decomposition $v_0 = \bar{v}_0 + V_0$, with $\partial_z \bar{v}_0 \in L^2$ and $V_0 \in L^\infty$. Observing this, it is necessary to state the following theorem on the global existence and uniqueness of weak solutions to the primitive equations with such kind initial data.

Theorem 1.2. *Suppose that the initial data $v_0 = \bar{v}_0 + V_0$, with $\bar{v}_0 \in X \cap \mathcal{H}$ and $V_0 \in L^\infty(\Omega) \cap \mathcal{H}$. Then the following hold:*

(i) There is a global weak solution v to system (1.9)–(1.13), such that

$$\begin{aligned} v(\boldsymbol{x}, t) &= \bar{v}(\boldsymbol{x}, t) + V(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega, t \in (0, \infty), \\ \partial_z \bar{v} &\in L^\infty_{loc}([0, \infty); L^2(\Omega)) \cap L^2_{loc}([0, \infty); H^1_{per}(\Omega)), \\ \sup_{0<s<t} &\|V\|_\infty(s) \le \mu(t)\|V_0\|_\infty, \quad t \in (0, \infty), \end{aligned}$$

where

$$\mu(t) = C_0(1 + \|v_0\|_4)^{40}(t+1)^2 \exp\{C_0 e^{2t}(t+1)(1 + \|v_0\|_4)^4\},$$

for some positive constant C_0 depending only on h;

(ii) Let ε_0 be the positive constant in Theorem 1.1, then the above weak solution is unique, provided $\mu(0)\|V_0\|_\infty \le \frac{\varepsilon_0}{2}$.

The uniqueness part of Theorem 1.2 is a direct consequence of the estimate in (i) and Theorem 1.1. So the key ingredient of the proof of Theorem 1.2 is to find the required decomposition. Note that in [20], the authors decompose the weak solution v into a regular part $\bar{v}$, which is a strong solution to the primitive equations, with H^2 initial data, and a small perturbation bounded part V, which satisfies a nonlinear system, with small initial data in L^∞. Noticing that the initial data v_0 considered in this paper can be discontinuous, and recalling the Sobolev embedding relation $H^2_{\text{per}}(\Omega) \hookrightarrow C_{\text{per}}(\Omega)$, hence v_0 cannot be approximated in the L^∞ norm by H^2 functions, and consequently the decomposition used in [20] does not apply to our case. One may still use the same decomposition as in [20], but change the initial data for $\bar{v}$ and V in X and L^∞, respectively; however, if by taking this approach, we are not able to obtain the L^∞ estimates on V, because the $L^\infty_t(H^2_{\mathbf{x}})$ estimate on $\bar{v}$ plays an essential role in deriving such estimates on V, and it is obvious that primitive equations with initial data in X does not necessary provide the required $L^\infty_t(H^2_{\mathbf{x}})$ estimate on $\bar{v}$. We also note that the arguments used in [3, 33, 36] require

the regularity of the kind $\partial_z v \in L^\infty_t(L^2_{\mathbf{x}}) \cap L^2_t(H^1_{\mathbf{x}})$ for the z-weak solutions, and thus still cannot be applied to our case.

To obtain the desired decomposition on v, we use a different kind of decomposition from that used in [20]. Indeed (ignoring the regularization procedure to justify the arguments), we decompose the weak solution v into a "regular" part $\bar{v}$, which is the unique solution to a linearized primitive equations (i.e. system (5.8)–(5.9), below), with initial data in X, and the remaining part V, which satisfies the same system as that for $\bar{v}$, but with initial data in L^∞. The key observation is that both the L^∞ and the X regularities are preserved by the linearized primitive equations, i.e. systems (5.3)–(5.4) and (5.8)–(5.9), below. Indeed, by making use of such kind of decomposition, we are able to weaken the assumption on the initial data, without destroying the uniqueness of the weak solutions. We point out that, compared with the decomposition introduced in [20], in our decomposition systems, i.e. systems (5.3)–(5.4) and (5.8)–(5.9), there are no nonlinear terms and no cross terms between $\bar{v}$ and V, and moreover, the systems for $\bar{v}$ and V are decoupled and are independent of each other. As will be shown later, see Proposition 5.3 and Proposition 5.4, below, the estimates for $\bar{v}$ and V in X and L^∞, respectively, are global in time, and thus no local in time arguments are required for deriving these estimates.

As an application of Theorem 1.2, we have the following result, which generalizes the results in [3, 20, 33, 36].

Corollary 1.1. *For any $v_0 \in X \cap \mathcal{H}$ (or $v_0 \in C_{per}(\Omega) \cap \mathcal{H}$), there is a constant σ_0, depending only on h and the upper bound of $\|v_0\|_4$, such that for any $\mathscr{V}_0 = v_0 + V_0$, with $V_0 \in L^\infty(\Omega) \cap \mathcal{H}$ and $\|V_0\|_\infty \le \sigma_0$, system (1.9)–(1.13) with initial data $\mathscr{V}_0$ has a unique weak solution, which has the regularities stated in Theorem 1.2.*

Proof. **(i) The case $v_0 \in X \cap \mathcal{H}$.** Let ε_0 be the constant in Theorem 1.1. Recalling that $\mathscr{V}_0 = v_0 + V_0$, by the triangle inequality for the norms and the Hölder inequality, one can easily check that

$$C_0(1 + \|\mathscr{V}_0\|_4)^{40} \exp\{C_0(1 + \|\mathscr{V}_0\|_4)^4\}\|V_0\|_\infty \le \rho(\|V_0\|_\infty),$$

where

$$\rho(s) = C_0(1 + \|v_0\|_4 + (2h)^{\frac{1}{4}} s)^{40} \exp\{C_0(1 + \|v_0\|_4 + (2h)^{\frac{1}{4}} s)^4\} s.$$

Noticing that ρ is a continuous function, with $\rho(0) = 0$, and C_0 is a positive constant depending only on h, there is a positive constant σ_0 depending only on h and the upper bound of $\|v_0\|_4$, such that $\rho(s) \le \varepsilon_0/2$, for any $s \in [0, \sigma_0]$. Thus, for any $V_0 \in L^\infty(\Omega) \cap \mathcal{H}$, with $\|V_0\|_\infty \le \sigma_0$, we have

$$C_0(1 + \|\mathscr{V}_0\|_4)^{40} \exp\{C_0(1 + \|\mathscr{V}_0\|_4)^4\}\|V_0\|_\infty \le \rho(\|V_0\|_\infty) \le \varepsilon_0/2,$$

and consequently, the conclusion follows by (ii) of Theorem 1.2, by viewing $(\mathscr{V}_0, v_0, V_0)$ as the $(v_0, \bar{v}_0, V_0)$ in Theorem 1.2.

(ii) The case $v_0 \in C_{\mathbf{per}}(\Omega) \cap \mathcal{H}$. Let σ_0 be the same positive constant as in (i). We are going to show that the conclusion holds in this case for the new $\sigma_0^* := \frac{\sigma_0}{2}$. Suppose that

$$\|V_0\|_\infty \leq \sigma_0^*.$$

Let j_η be a standard mollifier, i.e. $j_\eta(\mathbf{x}) = \frac{1}{\eta^3} j(\frac{\mathbf{x}}{\eta})$, with $0 \leq j \in C_0^\infty(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} j d\mathbf{x} = 1$. Since $v_0 \in C_{\text{per}}(\Omega)$, we have

$$\tilde{v}_0 * j_\eta \to v_0, \quad \text{in } C_{\text{per}}(\Omega), \text{ as } \eta \to 0^+,$$

where $\tilde{v}_0$ is the periodic extension of v_0 to the whole space. Choose $\eta_0 \leq \min\{1, 2h\}$ to be small enough, such that $\|v_0 - \tilde{v}_0 * j_{\eta_0}\|_\infty \leq \sigma_0^*$, and set $\bar{v}_0 = \tilde{v}_0 * j_{\eta_0}$. We have

$$\mathscr{V}_0 = v_0 + V_0 = \bar{v}_0 + (v_0 - \tilde{v}_0 * j_{\eta_0}) + V_0 =: \bar{v}_0 + \tilde{V}_0,$$

where $\tilde{V}_0 := (v_0 - \tilde{v}_0 * j_{\eta_0}) + V_0$. Recalling that $\|V_0\|_\infty \leq \sigma_0^*$, it follows that

$$\|\tilde{V}_0\|_\infty \leq \|v_0 - \tilde{v}_0 * j_{\eta_0}\|_\infty + \|V_0\|_\infty \leq \sigma_0^* + \sigma_0^* = \sigma_0.$$

Therefore, the initial data $\mathscr{V}_0$ can be decomposed as $\mathscr{V}_0 = \bar{v}_0 + \tilde{V}_0$, with $\bar{v}_0 = j_{\eta_0} * \tilde{v}_0 \in X \cap \mathcal{H}$, $\|\bar{v}_0\|_4 \leq 9\|v_0\|_4$, and $\|\tilde{V}_0\|_\infty \leq \sigma_0$, where σ_0 is the same constant as in (i), which depends only on $\|v_0\|_4$. By viewing $(\mathscr{V}_0, \bar{v}_0, \tilde{V}_0)$ as the $(\mathscr{V}_0, v_0, V_0)$ in case (i), it is clear that the assumptions in case (i) are fulfilled, and thus there is a unique weak solution to the primitive equations with initial data $\mathscr{V}_0$. □

Remark 1.1. *Given a constant vector $a = (a^1, a^2)$, and two positive numbers δ and η, with $\eta \in (0, h)$. Set $v_0 = a|z|^\delta$, $V_0 = \sigma\chi_{(-\eta,\eta)}(z)$, and $\mathscr{V}_0 = v_0 + V_0$, for $z \in (-h, h)$, with $\sigma = (\sigma^1, \sigma^2)$, where $\chi_{(-\eta,\eta)}(z)$ is the characteristic function of the interval $(-h, h)$. Extend v_0 and V_0, and consequently $\mathscr{V}_0$, periodically to the whole space, and still use the same notations to denote the extensions. Then, one can easily check that $v_0 \in C_{per}(\Omega) \cap \mathcal{H}$ and $V_0 \in L^\infty(\Omega) \cap \mathcal{H}$. By Corollary 1.1, there is a positive constant $\varepsilon_0 = \varepsilon_0(a, \delta, \eta, h)$, such that, for any $\sigma = (\sigma^1, \sigma^2)$, with $0 < |\sigma| \leq \varepsilon_0$, system (1.9)–(1.13) has a unique weak solution, with initial data $\mathscr{V}_0$. Note that*

$$\mathscr{V}_0 = a|z|^\delta + \sigma\chi_{(-\eta,\eta)}(z), \quad z \in (-h, h).$$

One can easily verify that $\mathscr{V}_0$ lies neither in $X \cap \mathcal{H}$ nor in $C_{per}(\Omega) \cap \mathcal{H}$, and thus the results established in [3, 20, 33, 36] cannot be applied to prove the uniqueness of weak solutions with such kind of initial data.

The rest of this paper is arranged as follows: in section 2, we collect some preliminary results; in section 3, as a preparation of proving Theorem 1.1, we show the regularities, away from the initial time, of weak solutions to the primitive equations; the proof of Theorem 1.1 is given in section 4; as a preparation of proving Theorem 1.2, we derive some relevant a priori estimates of the solutions to the primitive equations, with smooth initial data; the proof of Theorem 1.2 is given in the last section, section 6.

2. Preliminaries

In this section, we collect some preliminary results which will be used in the rest of this paper. We start with the following lemma, which can be proven in the same way as Proposition 2.2 in Cao–Titi [10], and thus we omit the proof of it here.

Lemma 2.1. *The following inequalities hold*

$$\begin{aligned}&\int_M \left(\int_{-h}^{h} |\phi(\boldsymbol{x}^H, z)|dz\right)\left(\int_{-h}^{h} |\varphi(\boldsymbol{x}^H, z)\psi(\boldsymbol{x}^H, z)|dz\right) d\boldsymbol{x}^H\\ \leq& C\|\phi\|_2\|\varphi\|_2^{\frac{1}{2}} \left(\|\varphi\|_2 + \|\nabla_H\varphi\|_2\right)^{\frac{1}{2}} \|\psi\|_2^{\frac{1}{2}} \left(\|\psi\|_2 + \|\nabla_H\psi\|_2\right)^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}&\int_M \left(\int_{-h}^{h} |\phi(\boldsymbol{x}^H, z)|dz\right)\left(\int_{-h}^{h} |\varphi(\boldsymbol{x}^H, z)\psi(\boldsymbol{x}^H, z)|dz\right) d\boldsymbol{x}^H\\ \leq& C\|\phi\|_2^{\frac{1}{2}} \left(\|\phi\|_2 + \|\nabla_H\phi\|_2\right)^{\frac{1}{2}} \|\varphi\|_2^{\frac{1}{2}} \left(\|\varphi\|_2 + \|\nabla_H\varphi\|_2\right)^{\frac{1}{2}} \|\psi\|_2,\end{aligned}$$

for every ϕ, φ and ψ, such that the quantities on the right-hand sides make sense and are finite.

The next lemma will be used in Proposition 5.3 to establish L^∞ estimates for V_ε.

Lemma 2.2. *Let $M_0 \geq 2$ and $\delta_0 > 0$ be two constants. Suppose that the sequence $\{A_k\}_{k=1}^\infty$, with $A_k \geq 0$, satisfies*

$$A_1 \leq M_0\delta_0^2, \quad A_{k+1} \leq M_0\delta_0^{2^{k+1}} + M_0^k A_k^2, \quad \text{for } k = 1, 2, \cdots.$$

Then one has

$$A_k \leq M_0^{-(k+2)}(M_0^4\delta_0^2)^{2^{k-1}}, \quad k = 1, 2, \cdots,$$

and in particular $A_k \leq (M_0^4\delta_0^2)^{2^{k-1}}$, for $k = 1, 2, \cdots$.

Proof. Define $a_k = M_0^{-(k+2)}(M_0^4\delta_0^2)^{2^{k-1}}$, for $k = 1, 2, \cdots$. Then, it suffices to prove that $A_k \leq a_k$. For $k = 1$, it follows from the assumption that

$$a_1 = M_0^{-3}M_0^4\delta_0^2 = M_0\delta_0^2 \geq A_1.$$

Suppose that $A_k \leq a_k$ holds, for some k. Then, by assumption, we have

$$\begin{aligned}A_{k+1} \leq& M_0\delta_0^{2^{k+1}} + M_0^k a_k^2 = M_0\delta_0^{2^{k+1}} + M_0^{-(k+4)}(M_0^4\delta_0^2)^{2^k}\\ =& M_0\delta_0^{2^{k+1}} + M_0^{-1}a_{k+1} \leq M_0\delta_0^{2^{k+1}} + \frac{a_{k+1}}{2}.\end{aligned}$$

We will show that

$$M_0\delta_0^{2^{k+1}} \leq \frac{a_{k+1}}{2},$$

and as a consequence of the above, we obtain

$$A_{k+1} \leq \frac{a_{k+1}}{2} + \frac{a_{k+1}}{2} = a_{k+1}.$$

Therefore, by induction, the conclusion holds.

We now prove that $M_0\delta_0^{2^{k+1}} \le \frac{a_{k+1}}{2}$, for $k = 1, 2, \cdots$. Recalling the definition of a_k, it follows

$$a_{k+1} = M_0^{-(k+3)}(M_0^4\delta_0^2)^{2^k} = M_0^{4\times 2^k-(k+3)}\delta_0^{2^{k+1}}.$$

Using binomial expansion, we have

$$4\times 2^k = 4(1+1)^k = 4\sum_{j=0}^{k}\binom{k}{j} \ge 4(1+k),$$

and thus

$$4\times 2^k - (k+3) \ge 4k+4-k-3 = 3k+1 \ge 4.$$

Therefore, we have

$$a_{k+1} = M_0^{4\times 2^k-(k+3)}\delta_0^{2^{k+1}} \ge M_0^4\delta_0^{2^{k+1}} \ge 2^3 M_0\delta_0^{2^{k+1}} \ge 2M_0\delta_0^{2^{k+1}}.$$

This completes the proof. □

We also need the following Aubin–Lions lemma.

Lemma 2.3 (cf. Simon [35] Corollary 4)**.** *Let $T \in (0,\infty)$ be given. Assume that X, B and Y are three Banach spaces, with $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$. Then it holds that*

(i) If F is a bounded subset of $L^p(0,T;X)$, where $1 \le p < \infty$, and $\frac{\partial F}{\partial t} = \left\{\frac{\partial f}{\partial t} | f \in F\right\}$ is bounded in $L^1(0,T;Y)$, then F is relatively compact in $L^p(0,T;B)$;

(ii) If F is bounded in $L^\infty(0,T;X)$ and $\frac{\partial F}{\partial t}$ is bounded in $L^r(0,T;Y)$, where $r > 1$, then F is relatively compact in $C([0,T];B)$.

3. Regularities of weak solutions for positive time

In this section, we prove that the global weak solution to system (1.9)–(1.13) is smooth away from the initial time. To prove this fact, we will use the weak-strong uniqueness result for the primitive equations. Therefore, Let us recall the definition of strong solutions to the primitive equations as follows:

Definition 3.1. *Given a positive time $T \in (0,\infty)$ and the initial data $v_0 \in H^1_{per}(\Omega)\cap\mathcal{H}$. A function v is called a strong solution to system (1.9)–(1.13), on $\Omega\times(0,T)$, if*

$$v \in C([0,T]; H^1_{per}(\Omega)\cap\mathcal{H}) \cap L^2(0,T;H^2_{per}(\Omega)), \quad \partial_t v \in L^2(0,T;L^2(\Omega)),$$

satisfies equation (1.9) pointwisely, a.e. in $\Omega\times(0,T)$, and fulfills the initial condition (1.13), with w given by equation (1.11), and p uniquely determined by the following elliptic problem:

$$\begin{cases} -\Delta_H p(\boldsymbol{x}^H,t) = \frac{1}{2h}\int_{-h}^{h}[\nabla_H\cdot\nabla_H\cdot(v\otimes v) + f_0 k\times v]dz, & \text{in } \Omega,\\ \int_\Omega p(\boldsymbol{x}^H,t)d\boldsymbol{x}^H = 0, \quad p \text{ is periodic in } \boldsymbol{x}^H. \end{cases}$$

Proposition 3.1. *Let v be a weak solution to system (1.9)–(1.13), with initial data $v_0 \in \mathcal{H}$. Then, there is a subset $E \subseteq (0,\infty)$ of measure zero, such that*

$$\lim_{t\to 0^+, t\notin E} v(t) = v_0, \quad \text{in } L^2(\Omega).$$

Proof. By (iv) of Definition 1.1, there is a subset $E \subseteq (0,\infty)$ of measure zero, such that $\|v\|_2^2(t) \le \|v_0\|_2^2$, for any $t \in (0,\infty)\setminus E$. Thanks to this, and recalling that $v(t)$ is continuous in $L^2_w(\Omega)$ with respect to t, it follows from the lower semi-continuity of the norms that

$$\|v_0\|_2^2 \le \varliminf_{t\to 0^+, t\notin E} \|v\|_2^2(t) \le \varlimsup_{t\to 0^+, t\notin E} \|v\|_2^2(t) \le \|v_0\|_2^2,$$

and thus $\lim_{t\to 0^+, t\notin E} \|v\|_2^2(t) = \|v_0\|_2^2$. Thanks to this, we deduce

$$\begin{aligned}\lim_{t\to 0^+, t\notin E} \|v(t)-v_0\|_2^2 &= \lim_{t\to 0^+, t\notin E} \int_\Omega (|v(t)|^2 + |v_0|^2 - 2v(t)\cdot v_0)d\mathbf{x} \\ &= \lim_{t\to 0^+, t\notin E} \int_\Omega (|v(t)|^2 - |v_0|^2)d\mathbf{x} = 0,\end{aligned}$$

proving the conclusion. □

The integral equality in the following proposition can be viewed as a replacement of (ii) in Definition 1.1, without changing the definition of weak solutions.

Proposition 3.2. *Let v be a weak solution to system (1.9)–(1.13). Then, for any $0 \le t_1 < t_2 < \infty$, the following holds*

$$\begin{aligned}&\int_\Omega v(\boldsymbol{x},t_2)\varphi(\boldsymbol{x},t_2)d\boldsymbol{x} + \int_{t_1}^{t_2}\int_\Omega \nabla v : \nabla\varphi d\boldsymbol{x}dt \\ &= \int_\Omega v(\boldsymbol{x},t_1)\varphi(\boldsymbol{x},t_1)d\boldsymbol{x} + \int_{t_1}^{t_2}\int_\Omega [v\cdot\partial_t\varphi - (v\cdot\nabla_H)v\cdot\varphi - w\partial_z v\cdot\varphi]d\boldsymbol{x}dt,\end{aligned}$$

for vector field $\varphi \in C([t_1,t_2]; C^1_{per}(\Omega)\cap\mathcal{H}) \cap C^1([t_1,t_2]; C_{per}(\Omega))$.

Proof. We only prove the case that $t_1 > 0$, the other case that $t_1 = 0$ can be proven similarly, by performing the arguments presented below for time t_2 only. Set $h_0 = \min\{t_1, t_2 - t_1\}$, and define the extension of φ as

$$\tilde\varphi(t) = \begin{cases} 2\varphi(t_2) - \varphi(2t_2 - t), & t_2 < t \le t_2 + h_0, \\ \varphi(t), & t_1 \le t \le t_2, \\ 2\varphi(t_1) - \varphi(2t_1 - t), & t_1 - h_0 \le t < t_1. \end{cases}$$

One can easily check that $\tilde\varphi \in C([t_1 - h_0, t_2 + h_0]; C^1_{\text{per}}(\Omega)\cap\mathcal{H}) \cap C^1([t_1 - h_0, t_2 + h_0]; C_{\text{per}}(\Omega))$. Let χ be a function, such that $\chi \in C^\infty(\mathbb{R})$, $\chi \equiv 1$ on $(-\infty, 0]$, $0 \le \chi \le 1$ and $\chi' \le 0$ on $(0, 1/2)$, and $\chi \equiv 0$ on $[1/2, \infty)$. For any $h \in (0, h_0)$, set

$$\chi_h(t) = \chi\left(\frac{t_1 - t}{h}\right)\chi\left(\frac{t - t_2}{h}\right), \quad t \in \mathbb{R}.$$

One can easily verify that $\chi_h \in C_0^\infty((t_1-h,t_2+h)) \subseteq C_0^\infty((0,\infty))$, $\chi_h' \geq 0$ on (t_1-h,t_1), $\chi \equiv 1$ on (t_1,t_2), $\chi_h' \leq 0$ on (t_2,t_2+h), and $|\chi_h'| \leq \frac{C}{h}$, for a positive constant C independent of h. Set $\phi = \tilde{\varphi}\chi_h$. Taking ϕ as the testing function in (ii) of Definition 1.1, and thanks to the properties of χ_h stated above, one obtains

$$\int_{t_1-h_0}^{t_2+h_0}\int_\Omega [-v\cdot\partial_t\tilde{\varphi} + (v\cdot\nabla_H)v\cdot\tilde{\varphi} + w\partial_z v\cdot\tilde{\varphi} + \nabla v : \nabla\tilde{\varphi}]d\mathbf{x}\chi_h(t)dt$$
$$= \int_{t_1-h}^{t_1}\int_\Omega v\cdot\tilde{\varphi}d\mathbf{x}\chi_h'(t)dt + \int_{t_2}^{t_2+h}\int_\Omega v\cdot\tilde{\varphi}d\mathbf{x}\chi_h'(t)dt, \tag{3.1}$$

for any $h \in (0,h_0)$. Denote by LHS the quantity on the left-hand side of (3.1). Since $v \in L^\infty(0,t_2+h_0;L^2(\Omega)) \cap L^2(0,t_2+h_0;H^1_{\text{per}}(\Omega))$, by the Lebesgue dominate convergence theorem, one can see that

$$LHS \to \int_{t_1}^{t_2}\int_\Omega [-v\cdot\partial_t\varphi + (v\cdot\nabla_H)v\cdot\varphi + w\partial_z v\cdot\varphi + \nabla v : \nabla\varphi]d\mathbf{x}dt,$$

as $h \to 0^+$. Define the function f as

$$f(t) = \int_\Omega v(\mathbf{x},t)\tilde{\varphi}(\mathbf{x},t)d\mathbf{x}, \quad t \in (t_1-h_0,t_2+h_0).$$

Recalling that $v \in C([0,\infty);L^2_w(\Omega))$ and noticing that $\tilde{\varphi} \in C^1(\bar{\Omega}\times[t_1-h_0,t_2+h_0])$, it is clear that $f \in C([t_1-h_0,t_2+h_0])$. Denote by RHS the quantity on the right-hand side of (3.1). Recalling that $\chi_h' \geq 0$ on (t_1-h,t_1), and $\chi_h' \leq 0$ on (t_2,t_2+h), and applying the mean value theorem of integrals, one deduces

$$\begin{aligned} RHS =& \int_{t_1-h}^{t_1} f(t)\chi_h'(t)dt + \int_{t_2}^{t_2+h} f(t)\chi_h'(t)dt \\ =& f(t_1-\theta_1(h)h)\int_{t_1-h}^{t_1}\chi_h'(t)dt + f(t_2+\theta_2(h)h)\int_{t_2}^{t_2+h}\chi_h'(t)dt \\ =& f(t_1-\theta_1(h)h) - f(t_2+\theta_2(h)h) \to f(t_1)-f(t_2), \end{aligned}$$

as $h \to 0^+$, where $\theta_1(h),\theta_2(h) \in [0,1]$. Thanks to the above statements, taking $h \to 0^+$ in (3.1) yields the conclusion. □

Withe the aid of Proposition 3.2, one can show that any weak solution to the primitive equations on $\Omega\times(0,\infty)$ is also a weak solution to the primitive equations on $\Omega\times(t,\infty)$, with initial data $v(t)$, for a.e. $t \in [0,\infty)$, that is the following proposition.

Proposition 3.3. *Let v be a global weak solution to system (1.9)–(1.13). Then, for a.e. $t \in [0,\infty)$, by viewing time t as the initial time, v is also a weak solution to system (1.9)–(1.13), on $\Omega\times(t,\infty)$, with initial data $v(t)$.*

Proof. We need to verify the terms (i)–(iv) in Definition 1.1 on the time interval $[t_0,\infty)$, for a.e. $t_0 \in [0,\infty)$. It is clear that (i) and (iii) hold, while the validity of

(ii) is guaranteed by Proposition 3.2. We still need to verify that (iv) holds on time interval $[t_0, \infty)$, for a.e. $t_0 \in [0, \infty)$, or equivalently that

$$\frac{1}{2}\|v\|_2^2(t) + \int_{t_0}^{t} \|\nabla v\|_2^2(\tau)d\tau \leq \frac{1}{2}\|\nabla v\|_2^2(t_0), \tag{3.2}$$

for a.e. $t_0 \in [0, \infty)$ and a.e. $t \in (t_0, \infty)$. Setting $f(t) = \|v\|_2^2(t)$ and $g(t) = \|\nabla v\|_2^2(t)$, for $t \in [0, \infty)$. Then by the regularities of weak solutions, one has $f, g \in L^1_{\text{loc}}([0, \infty))$. By (iii) of Definition 1.1, it holds that

$$\frac{1}{2}f'(t) + g(t) \leq 0, \quad \text{in } \mathcal{D}'((0, \infty)).$$

Let $0 \leq j_\varepsilon = \frac{1}{\varepsilon} j\left(\frac{t}{\varepsilon}\right)$ be a standard modifier, with $j \in C_0^\infty((-1, 1))$, and set $f_\varepsilon = f * j_\varepsilon$ and $g_\varepsilon = g * j_\varepsilon$, for $\varepsilon > 0$. For any $t > 0$, it is clear that $0 \leq j_\varepsilon(\,\cdot - t) \in C_0^\infty((0, \infty))$. Thus, one can test the above inequality by $j_\varepsilon(\,\cdot - t)$ to get

$$\frac{1}{2}f_\varepsilon'(t) + g_\varepsilon(t) \leq 0, \quad t \in (0, \infty), \varepsilon \in (0, t).$$

For any $t_0 \in (0, \infty)$ and $t \in (t_0, \infty)$, integration the above inequality over the interval (t_0, t) yields

$$\frac{1}{2}f_\varepsilon(t) + \int_{t_0}^{t} g_\varepsilon(s)ds \leq \frac{1}{2}f_\varepsilon(t_0),$$

for any $\varepsilon \in (0, t_0)$. Note that $f_\varepsilon(t) \to f(t)$ and $g_\varepsilon(t) \to g(t)$, a.e. $t \in (0, \infty)$, and $g_\varepsilon \to g$ in $L^1((0, T))$, for any finite time T. Thanks to these, taking $\varepsilon \to 0^+$ in the above inequality yields

$$\frac{1}{2}f(t) + \int_{t_0}^{t} g(s)ds \leq \frac{1}{2}f(t_0),$$

for a.e. $t_0 \in (0, \infty)$ and a.e. $t \in (t_0, \infty)$, which is exactly (3.2). This completes the proof of Proposition 3.3. □

The following proposition states the weak-strong uniqueness result for the primitive equations.

Proposition 3.4. *Given the initial data $v_0 \in H^1_{per}(\Omega) \cap \mathcal{H}$. Let v_s and v_w be the unique global strong solution and an arbitrary global weak solution, respectively, to system (1.9)–(1.13), with the same initial data v_0. Then, we have $v_s \equiv v_w$.*

Proof. Denote $U_\text{s} = (v_\text{s}, w_\text{s})$ and $U_\text{w} = (v_\text{w}, w_\text{w})$, and set $U = (v, w) = U_\text{w} - U_\text{s}$, where w_s and w_w are determined uniquely by v_s and v_w, respectively, through the relation (1.11).

Since strong solutions to the primitive equations are smooth away from the initial time, one can choose $\varphi = v_\text{s}$ as the testing function in Proposition 3.2, and thus get

$$\int_\Omega v_\text{w}(\mathbf{x}, t) \cdot v_\text{s}(\mathbf{x}, t)d\mathbf{x} + \int_s^t \int_\Omega \nabla v_\text{w} : \nabla v_\text{s} d\mathbf{x}d\tau$$

$$= \int_\Omega v_{\rm w}(\mathbf{x},s)\cdot v_{\rm s}(\mathbf{x},s)d\mathbf{x} + \int_s^t\int_\Omega [v_{\rm w}\cdot\partial_t v_{\rm s} - (U_{\rm w}\cdot\nabla)v_{\rm w}\cdot v_{\rm s}]d\mathbf{x}d\tau,$$

for any $0<s<t<\infty$. Adding both sides of the above equality by $\int_s^t\int_\Omega \nabla v_{\rm w} : \nabla v_{\rm s} d\mathbf{x}d\tau$ yields

$$\begin{aligned}&\int_\Omega v_{\rm w}(\mathbf{x},t)\cdot v_{\rm s}(\mathbf{x},t)d\mathbf{x} + 2\int_s^t\int_\Omega \nabla v_{\rm w} : \nabla v_{\rm s} d\mathbf{x}d\tau\\ =&\int_\Omega v_{\rm w}(\mathbf{x},s)\cdot v_{\rm s}(\mathbf{x},s)d\mathbf{x} - \int_s^t\int_\Omega (U_{\rm w}\cdot\nabla)v_{\rm w}\cdot v_{\rm s} d\mathbf{x}d\tau\\ &+\int_s^t\int_\Omega (v_{\rm w}\cdot\partial_t v_{\rm s} + \nabla v_{\rm w} : \nabla v_{\rm s})d\mathbf{x}d\tau,\end{aligned}$$

for any $0<s<t<\infty$. For the last term on the right-hand side of the above equality, one can integrate by parts and use the equations for $v_{\rm s}$ to get

$$\int_s^t\int_\Omega (v_{\rm w}\cdot\partial_t v_{\rm s} + \nabla v_{\rm w} : \nabla v_{\rm s})d\mathbf{x}d\tau = -\int_s^t\int_\Omega (U_{\rm s}\cdot\nabla)v_{\rm s}\cdot v_{\rm w} d\mathbf{x}d\tau.$$

Plugging this equality into the previous one leads to

$$\begin{aligned}&\int_\Omega v_{\rm w}(\mathbf{x},t)\cdot v_{\rm s}(\mathbf{x},t)d\mathbf{x} + 2\int_s^t\int_\Omega \nabla v_{\rm w} : \nabla v_{\rm s} d\mathbf{x}d\tau\\ =&\int_\Omega v_{\rm w}(\mathbf{x},s)\cdot v_{\rm s}(\mathbf{x},s)d\mathbf{x} - \int_s^t\int_\Omega [(U_{\rm w}\cdot\nabla)v_{\rm w}\cdot v_{\rm s} + (U_{\rm s}\cdot\nabla)v_{\rm s}\cdot v_{\rm w}]d\mathbf{x}d\tau,\end{aligned} \tag{3.3}$$

for any $0<s<t<\infty$.

Multiplying the equation for $v_{\rm s}$ by $v_{\rm s}$, and integrating over Ω, the it follows from integration by parts, and integrating over the time interval (s,t) that

$$\frac{1}{2}\|v_{\rm s}\|_2^2(t) + \int_s^t \|\nabla v_{\rm s}\|_2^2(\tau)d\tau = \frac{1}{2}\|v_{\rm s}\|_2^2(s),$$

for any $0<s<t<\infty$. Recalling the energy inequality in Definition 1.1, we have

$$\frac{1}{2}\|v_{\rm w}\|_2^2(t) + \int_s^t \|\nabla v_{\rm w}\|_2^2(\tau)d\tau \le \frac{1}{2}\|v_{\rm w}\|_2^2(s) + \frac{1}{2}(\|v_0\|_2^2 - \|v_{\rm w}\|_2^2(s)),$$

for a.e. $0<s<t<\infty$. Adding the above two equalities up, and subtracting the resultant by (3.3) yield

$$\begin{aligned}\frac{1}{2}\|v\|_2^2(t) + \int_s^t \|\nabla v\|_2^2(\tau)d\tau \le &\int_s^t\int_\Omega [(U_{\rm w}\cdot\nabla)v_{\rm w}\cdot v_{\rm s} + (U_{\rm s}\cdot\nabla)v_{\rm s}\cdot v_{\rm w}]d\mathbf{x}d\tau\\ &+\frac{1}{2}(\|v\|_2^2(s) + \|v_0\|_2^2 - \|v_{\rm w}\|_2^2(s)),\end{aligned} \tag{3.4}$$

for a.e. $0<s<t<\infty$.

We are going to deal with the first term on the right-hand side of (3.4). Recalling the regularities of $v_{\rm s}$ and $v_{\rm w}$, it follows from integration by parts that

$$\begin{aligned} I :=& \int_s^t \int_\Omega [(U_{\rm w}\cdot\nabla)v_{\rm w}\cdot v_{\rm s} + (U_{\rm s}\cdot\nabla)v_{\rm s}\cdot v_{\rm w}]d\mathbf{x}d\tau \\ =& \int_s^t \int_\Omega [(U_{\rm w}\cdot\nabla)v_{\rm w}\cdot v_{\rm s} - (U_{\rm s}\cdot\nabla)v_{\rm w}\cdot v_{\rm s}]d\mathbf{x}d\tau \\ =& \int_s^t \int_\Omega (U\cdot\nabla)v_{\rm w}\cdot v_{\rm s}d\mathbf{x}d\tau = \int_s^t \int_\Omega (U\cdot\nabla)v\cdot v_{\rm s}d\mathbf{x}d\tau \\ =& -\int_s^t \int_\Omega (U\cdot\nabla)v_{\rm s}\cdot v d\mathbf{x}d\tau, \end{aligned} \tag{3.5}$$

for any $0<s<t<\infty$. Note that

$$|\nabla_H v_{\rm s}(\mathbf{x}^H,z)| \le \frac{1}{2h}\int_{-h}^h |\nabla_H v_{\rm s}(\mathbf{x}^H,z')|dz' + \int_{-h}^h |\nabla_H\partial_z v_{\rm s}(\mathbf{x}^H,z')|dz',$$

for any $(\mathbf{x}^H,z)\in\Omega$. Therefore, by Lemma 2.1, and using the Young inequality, we can estimate I as follows

$$\begin{aligned} |I| =& \left|\int_s^t \int_\Omega [(v\cdot\nabla_H)v_{\rm s} + w\partial_z v_{\rm s}]\cdot v d\mathbf{x}d\tau\right| \\ \le& \int_s^t \int_M \int_{-h}^h \left(\frac{|\nabla_H v_{\rm s}|}{2h} + |\nabla_H\partial_z v_{\rm s}|\right)dz \int_{-h}^h |v|^2 dz d\mathbf{x}^H d\tau \\ &+ \int_s^t \int_M \int_{-h}^h |\nabla_H v|dz \int_{-h}^h |\partial_z v_{\rm s}||v|dz d\mathbf{x}^H d\tau \\ \le& C\int_s^t (\|\nabla_H v_{\rm s}\|_2 + \|\nabla_H\partial_z v_{\rm s}\|_2)\|v\|_2(\|v\|_2 + \|\nabla_H v\|_2)d\tau \\ &+ C\int_s^t \|\nabla_H v\|_2\|\partial_z v_{\rm s}\|_2^{\frac12}(\|\partial_z v_{\rm s}\|_2 + \|\nabla_H\partial_z v_{\rm s}\|_2)^{\frac12} \\ &\qquad \times \|v\|_2^{\frac12}(\|v\|_2 + \|\nabla_H v\|_2)^{\frac12}d\tau \\ \le& \frac12\int_s^t \|\nabla_H v\|_2^2 d\tau + C\int_s^t (1+\|\partial_z v_{\rm s}\|_2^2)(1+\|\nabla v_{\rm s}\|_2^2 \\ &\qquad + \|\nabla_H\partial_z v_{\rm s}\|_2^2)\|v\|_2^2 d\tau, \end{aligned} \tag{3.6}$$

for any $0<s<t<\infty$.

Substituting the above estimate into (3.4) yields

$$\begin{aligned} \|v\|_2^2(t) + \int_s^t \|\nabla v\|_2^2(\tau)d\tau \le& C\int_s^t (1+\|\partial_z v_{\rm s}\|_2^2)(1+\|\nabla v_{\rm s}\|_2^2 + \|\nabla_H\partial_z v_{\rm s}\|_2^2)\|v\|_2^2 d\tau \\ &+ \frac12(\|v\|_2^2(s) + \|v_0\|_2^2 - \|v_{\rm w}\|_2^2(s)), \end{aligned}$$

for a.e. $0 < s < t < \infty$. By Proposition 3.1, there is a sequence $\{s_n\}$, with $s_n \to 0^+$, as $n \to \infty$, such that both $v_{\rm s}(s_n)$ and $v_{\rm w}(s_n)$ converge to v_0 in $L^2(\Omega)$, as $n \to \infty$. Choosing $s = s_n$ in the above inequality, and taking $n \to \infty$ yields

$$\|v\|_2^2(t) + \int_s^t \|\nabla v\|_2^2(\tau)d\tau \leq C \int_s^t m(\tau)\|v\|_2^2(\tau)d\tau,$$

for a.e. $t \in (0, \infty)$, where

$$m(\tau) = (1 + \|\partial_z v_{\rm s}\|_2^2(\tau))(1 + \|\nabla v_{\rm s}\|_2^2(\tau) + \|\nabla_H \partial_z v_{\rm s}\|_2^2(\tau)), \quad \tau \in (0, \infty).$$

Noticing that $m \in L^1_{\rm loc}([0, \infty))$, by the Gronwall inequality, the above inequality implies $\|v\|_2^2(t) = 0$, a.e. $t \in (0, \infty)$, that is $v_{\rm s} = v_{\rm w}$. This completes the proof. □

Corollary 3.1 (Regularities for positive time)**.** *Let v be a weak solution to system (1.9)–(1.13). Then, the following two hold:*

(i) v is smooth away from the initial time, i.e. $v \in C^\infty(\bar{\Omega} \times (0, \infty))$;

(ii) v satisfies the energy identity

$$\frac{1}{2}\|v\|_2^2(t) + \int_0^t \|\nabla v\|_2^2(\tau)d\tau = \frac{1}{2}\|v_0\|_2^2,$$

for any $t \in (0, \infty)$.

Proof. (i) By Proposition 3.3, for a.e. $t \in [0, \infty)$, v is still a weak solution to (1.9)–(1.13) on $\Omega \times (t, \infty)$, with initial data $v(t)$, by viewing t as the initial time. By the energy inequality, it is clear that $v(\cdot, t) \in H^1_{\rm per} \cap \mathcal{H}$, for a.e. $t \in (0, \infty)$. Therefore, there is a subset $E \subseteq (0, \infty)$, of measure zero, such that v is a weak solution to (1.9)–(1.13) on $\Omega \times (t, \infty)$, with initial data $v(t) \in H^1_{\rm per} \cap \mathcal{H}$, for all $t \in (0, \infty) \setminus E$. Take an arbitrary time $t_0 \in (0, \infty)$. Thanks to what we just stated, there is a time $t_0' \in (0, t_0)$, such that $v(\cdot, t_0') \in H^1_{\rm per}(\Omega) \cap \mathcal{H}$, and v is a weak solution to system (1.9)–(1.13), on $\Omega \times (t_0', \infty)$, with initial data $v(t_0')$. Since $v(t_0') \in H^1_{\rm per}(\Omega) \cap \mathcal{H}$, there is a unique global strong solution $v_{\rm s}$ to system (1.9)–(1.13), on $\Omega \times (t_0', \infty)$, with initial data $v(t_0')$. By the weak-strong uniqueness, i.e. Proposition 3.4, we then have $v \equiv v_{\rm s}$, on the time interval $[t_0', \infty) \supseteq [t_0, \infty)$. Recalling that strong solutions to the primitive equations are smooth away from the initial time, one has $v = v_{\rm s} \in C^\infty(\bar{\Omega} \times (t_0, \infty))$, and further $v \in C^\infty(\bar{\Omega} \times (0, \infty))$, and thus proves (i).

(ii) Thanks to (i), one can multiply equation (1.9) by v, and integrating the resultant over Ω, then integration by parts yields

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2(t) + \|\nabla v\|_2^2(t) = 0,$$

for all $t \in (0, \infty)$. Integrating the above equality over the time interval (s, t) leads to

$$\frac{1}{2}\|v\|_2^2(t) + \int_s^t \|\nabla v\|_2^2(\tau)d\tau = \frac{1}{2}\|v\|_2^2(s),$$

for any $0 < s < t < \infty$. By Proposition 3.1, there is a sequence of time $\{s_n\}$, with $s_n \to 0^+$, such that $v(s_n) \to v_0$ in $L^2(\Omega)$, as $n \to \infty$. Choosing $s = s_n$ in the above equality, and letting $n \to \infty$ yield the conclusion. □

4. Proof of Theorem 1.1

With the preparations in the previous section, section 3, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Denote by $v_{\rm s}$ the weak solution stated in Theorem 1.1. By assumption, there is a positive time $T_{\rm s}$, such that

$$\begin{aligned}
&v_{\rm s}(x,y,z,t) = \bar{v}_{\rm s}(x,y,z,t) + \bar{V}_{\rm s}(x,y,z,t), \quad (x,y,z) \in \Omega, t \in (0,T_{\rm s}),\\
&\partial_z \bar{v} \in L^\infty(0,T_{\rm s}; L^2(\Omega)) \cap L^2(0,T_{\rm s}; H^1_{\rm per}(\Omega)), \quad \bar{V}_{\rm s} \in L^\infty(\Omega \times (0,T_{\rm s})).
\end{aligned}$$

Let ε_0 be a sufficiently small positive number, which will be determined later, and suppose that there is a positive time $T_{\rm s}' \in (0,T_{\rm s})$, such that

$$\sup_{0<t<T_{\rm s}'} \|\bar{V}_{\rm s}\|_\infty(t) \le \varepsilon_0.$$

Take an arbitrary weak solution $v_{\rm w}$ solution to system (1.9)–(1.13), with the same initial data as $v_{\rm s}$.

We denote $U_{\rm s} = (v_{\rm s}, w_{\rm s}), U_{\rm w} = (v_{\rm w}, w_{\rm w})$ and set $U = (v,w) = U_{\rm w} - U_{\rm s}$, where $w_{\rm s}$ and $w_{\rm w}$ are determined in terms of $v_{\rm s}$ and $v_{\rm w}$, respectively, through the relation (1.11). We are going to show that $v_{\rm s} \equiv v_{\rm w}$, or equivalently $v \equiv 0$.

By Corollary 3.1, any weak solution to system (1.9)–(1.13) is smooth away from the initial time. Therefore, we have $v_{\rm s} \in C^\infty(\bar{\Omega} \times (0,\infty))$ and $v_{\rm w} \in C^\infty(\bar{\Omega} \times (0,\infty))$. By the same argument as that for (3.4), we have the following

$$\begin{aligned}
\frac{1}{2}\|v\|_2^2(t) + \int_s^t \|\nabla v\|_2^2(\tau)d\tau \le \int_s^t \int_\Omega [(U_{\rm w}\cdot\nabla)v_{\rm w}\cdot v_{\rm s} + (U_{\rm s}\cdot\nabla)v_{\rm s}\cdot v_{\rm w}]d\mathbf{x}d\tau\\
+ \frac{1}{2}(\|v\|_2^2(s) + \|v_0\|_2^2 - \|v_{\rm w}\|_2^2(s)),
\end{aligned} \tag{4.1}$$

for any $0 < s < t < \infty$. Recalling (3.5), we have by integration by parts that

$$\begin{aligned}
J &:= \int_s^t \int_\Omega [(U_{\rm w}\cdot\nabla)v_{\rm w}\cdot v_{\rm s} + (U_{\rm s}\cdot\nabla)v_{\rm s}\cdot v_{\rm w}]d\mathbf{x}d\tau\\
&= \int_s^t \int_\Omega (U\cdot\nabla)v\cdot v_{\rm s} d\mathbf{x}d\tau = \int_s^t \int_\Omega (U\cdot\nabla)v\cdot(\bar{v}_{\rm s} + \bar{V}_{\rm s})d\mathbf{x}d\tau\\
&= \int_s^t \int_\Omega (U\cdot\nabla)v\cdot\bar{V}_{\rm s} d\mathbf{x}d\tau - \int_s^t \int_\Omega (U\cdot\nabla)\bar{v}_{\rm s}\cdot v d\mathbf{x}d\tau =: J_1 + J_2,
\end{aligned}$$

for any $0 < s < t < \infty$.

For J_1, by the assumption, it follows from the Hölder and Young inequalities that

$$\begin{aligned}J_1 =& \int_s^t \int_\Omega \left[(v\cdot\nabla_H)v - \int_{-h}^z \nabla_H\cdot v d\xi \partial_z v\right]\cdot \bar{V}_{\rm s} d\mathbf{x} d\tau\\ \leq& \frac{1}{4}\int_s^t \|\nabla_H v\|_2^2 d\tau + \int_s^t \|\bar{V}_{\rm s}\|_\infty^2 \|v\|_2^2 d\tau + C_0\int_s^t \|\bar{V}_{\rm s}\|_\infty \|\nabla v\|_2^2 d\tau\\ \leq& \left(\frac{1}{4}+C_0\varepsilon_0\right)\int_s^t \|\nabla v\|_2^2 + \varepsilon_0^2\int_s^t \|v\|_2^2(\tau)d\tau,\end{aligned}$$

for any $0<s<t\leq T_{\rm s}'$, where C_0 is a positive constant depending only on h. Choosing $\varepsilon_0=\min\left\{1,\frac{1}{8C_0}\right\}$, then one has

$$J_1 \leq \frac{3}{8}\int_s^t \|\nabla v\|_2^2 + \int_s^t \|v\|_2^2(\tau)d\tau,$$

for any $0<s<t\leq T_{\rm s}'$. While for J_2, the same argument as that for (3.6) yields

$$J_2 \leq \frac{1}{8}\int_s^t \|\nabla_H v\|_2^2 d\tau + C\int_s^t (1+\|\partial_z\bar{v}_{\rm s}\|_2^2)(1+\|\nabla\bar{v}_{\rm s}\|_2^2+\|\nabla_H\partial_z\bar{v}_{\rm s}\|_2^2)\|v\|_2^2 d\tau,$$

for any $0<s<t<\infty$.

Thanks to the estimates for J_1 and J_2, it follows from (4.1) that

$$\begin{aligned}\|v\|_2^2(t) + \int_s^t \|\nabla v\|_2^2(\tau)d\tau \leq& C\int_s^t (1+\|\partial_z\bar{v}_{\rm s}\|_2^2)(1+\|\nabla\bar{v}_{\rm s}\|_2^2+\|\nabla_H\partial_z\bar{v}_{\rm s}\|_2^2)\|v\|_2^2 d\tau\\ &+ (\|v\|_2^2(s)+\|v_0\|_2^2-\|v_{\rm w}\|_2^2(s)),\end{aligned}$$

for any $0<s<t\leq T_{\rm s}'$. By Proposition 3.1, there is a sequence $\{s_n\}$, with $s_n\to 0^+$, as $n\to\infty$, such that both $v_{\rm s}(s_n)$ and $v_{\rm w}(s_n)$ converge to v_0 in $L^2(\Omega)$, as $n\to\infty$. Choosing $s=s_n$ in the above inequality, and taking $n\to\infty$ lead to

$$\|v\|_2^2(t) + \int_0^t \|\nabla v\|_2^2(\tau)d\tau \leq C\int_0^t (1+\|\partial_z\bar{v}_{\rm s}\|_2^2)(1+\|\nabla\bar{v}_{\rm s}\|_2^2+\|\nabla_H\partial_z\bar{v}_{\rm s}\|_2^2)\|v\|_2^2 d\tau,$$

for any $0<t\leq T_{\rm s}'$. By assumption, it is clear that

$$(1+\|\partial_z\bar{v}_{\rm s}\|_2^2(\tau))(1+\|\nabla\bar{v}_{\rm s}\|_2^2(\tau)+\|\nabla_H\partial_z\bar{v}_{\rm s}\|_2^2(\tau))\in L^1((0,T_{\rm s})).$$

Therefore, one can apply the Gronwall inequality to the previous inequality to conclude that $\|v\|_2^2(t)=0$, for $t\in[0,T_{\rm s}']$, that is $v_{\rm s}\equiv v_{\rm w}$, on $\Omega\times[0,T_{\rm s}']$. Starting from time $T_{\rm s}'$, both $v_{\rm s}$ and $v_{\rm w}$ are smooth, and thus are both strong solutions to system (1.9)–(1.13), on $\Omega\times(T_{\rm s}',\infty)$, with the same initial data at time $T_{\rm s}'$. By the uniqueness of strong solutions to the primitive equations, we have $v_{\rm s}\equiv v_{\rm w}$, on $\Omega\times[T_{\rm s}',\infty)$. This completes the proof of Theorem 1.1. □

5. A priori estimates for regular solutions

Let $v_0 = \bar{v}_0 + V_0$, with $\bar{v}_0 \in X \cap \mathcal{H}$ and $V_0 \in L^\infty(\Omega) \cap \mathcal{H}$. Extend v_0, $\bar{v}_0$ and V_0 periodically to the whole space, and still use the same notations to denote the relevant extensions. Let j_ε, as before, be a standard compactly supported nonnegative mollifier. Set $\bar{v}_{0\varepsilon} = \bar{v}_0 * j_\varepsilon$, $V_{0\varepsilon} = V_0 * j_\varepsilon$ and $v_{0\varepsilon} = v_0 * j_\varepsilon$. It is obvious that $\bar{v}_{0\varepsilon} \in \mathcal{H}$, $V_{0\varepsilon} \in \mathcal{H}$, and $v_{0\varepsilon} = \bar{v}_{0\varepsilon} + V_{0\varepsilon} \in \mathcal{H}$. Moreover, we have, for any $q \in [1, \infty]$, and $\varepsilon \in (0, \min\{1, 2h\})$,

$$\|v_{0\varepsilon}\|_q \le 9\|v_0\|_q, \quad \|\bar{v}_{0\varepsilon}\|_X \le 9\|\bar{v}_0\|_X, \quad \|V_{0\varepsilon}\|_\infty \le \|V_0\|_\infty.$$

Note that $v_{0\varepsilon} \in H^1_{\mathrm{per}}(\Omega) \cap \mathcal{H}$, and consequently, there is a unique global strong solution v_ε (cf. [11]), to system (1.9)–(1.13), with initial data $v_{0\varepsilon}$. Moreover, v_ε satisfies the basic energy identity and some additional L^q estimates, which are stated in the following:

Proposition 5.1 (L^q **estimate on** v_ε)**.** *Let v_ε be the unique global strong solution to system (1.9)–(1.13), with initial data $v_{0\varepsilon}$. Then we have the basic energy identity*

$$\frac{1}{2}\|v_\varepsilon\|_2^2(t) + \int_0^t \|\nabla v_\varepsilon\|_2^2(\tau)d\tau = \frac{1}{2}\|v_{0\varepsilon}\|_2^2,$$

and the $L^4(\Omega)$ estimate

$$\sup_{0\le s\le t} \|v_\varepsilon\|_4(s) \le \exp\{Ce^{2t}(t+1)(1+\|v_0\|_2^2)^2\}(1+\|v_0\|_4),$$

for any $t \in [0, \infty)$. Furthermore, for any $q \in [4, \infty)$, we have

$$\sup_{0\le s\le t} \|v_\varepsilon\|_q(s) \le \sqrt{q}K_1(t)(1+\|v_0\|_q),$$

for any $t \in [0, \infty)$, where C is a positive constant depending only on h, and K_1 is a continuously increasing function on $[0, \infty)$ determined by h and $\|v_0\|_4$.

Proof. Multiplying equation (1.9) by v_ε and integrating by parts yields the first conclusion. The second and third ones are direct corollaries of Proposition 3.1 in [7]. In fact, by (3.9) in [7], it follows

$$\begin{aligned}\sup_{0\le s\le t} \|v_\varepsilon\|_4(s) &\le \exp\{Ce^{2t}(t+1)(1+\|v_{0\varepsilon}\|_2^2)^2\}(1+\|v_{0\varepsilon}\|_4)\\ &\le \exp\{Ce^{2t}(t+1)(1+\|v_0\|_2^2)^2\}(1+\|v_0\|_4),\end{aligned}$$

for a positive constant C depending only on h, and by Proposition 3.1 (iii) in [7], it follows that

$$\sup_{0\le s\le t} \|v_\varepsilon\|_q(s) \le \sqrt{q}K_1(t)(1+\|v_{0\varepsilon}\|_q) \le \sqrt{q}K_1(t)(1+\|v_0\|_q),$$

for some function K_1, which is determined by the upper bonds of $\|v_{0\varepsilon}\|_2$ and $\|v_{0\varepsilon}\|_4$; however, since $\|v_{0\varepsilon}\|_2 \le \|v_0\|_2$ and $\|v_{0\varepsilon}\|_4 \le \|v_0\|_4$, such function can be chosen to be independent of ε. □

Next, we show that away from the initial time, we have the H^1 estimates for v_ε.

Proposition 5.2 (H^1 **estimate on** v_ε)**.** *Let v_ε be the unique global solution to system (1.9)–(1.13), with initial data $v_{0\varepsilon}$. Then, for any $0 < t < T < \infty$, we have*

$$\sup_{t\le s\le T}\|v_\varepsilon\|_{H^1}^2(s)+\int_t^T(\|\nabla^2 v_\varepsilon\|_2^2+\|\partial_t v_\varepsilon\|_2^2)(s)ds\le K_2(t,T),$$

where K_2 is a continuous function defined on $\mathbb{R}_+^2$ and determined only by h and $\|v_0\|_2^2$, where $\mathbb{R}_+^2=(0,\infty)\times(0,\infty)$.

Proof. Fix $T\in(0,\infty)$, and let $t\in[0,T]$. By Proposition 5.1, we have

$$\int_0^t\|\nabla v_\varepsilon\|_2^2(s)ds\le\frac{1}{2}\|v_{0\varepsilon}\|_2^2\le\frac{1}{2}\|v_0\|_2^2,$$

and thus, one can choose such $t_0\in(0,t)$ that

$$\|\nabla v_\varepsilon(t_0)\|_2^2\le\frac{\|v_0\|_2^2}{t}.$$

Now, taking t_0 as the initial time, then it follows from the H^1 type energy estimate for the primitive equations, see (77)–(78) in [11], we obtain

$$\begin{aligned}&\sup_{t\le s\le T}\|\nabla v_\varepsilon\|_2^2(s)+\int_t^T\|\nabla^2 v_\varepsilon\|_2^2(s)ds\\ \le&\sup_{t_0\le s\le T}\|\nabla v_\varepsilon\|_2^2(s)+\int_{t_0}^T\|\nabla^2 v_\varepsilon\|_2^2(s)ds\le K_2''(t,T),\end{aligned}$$

where K_2'' is a continuous function on $\mathbb{R}_+^2$ determined only by h and $\|v_0\|_2^2$. Thus, recalling the basic energy identity in Proposition 5.1, we then have

$$\sup_{t\le s\le T}\|v_\varepsilon\|_{H^1}^2(s)+\int_t^T\|\nabla^2 v_\varepsilon\|_2^2(s)ds\le K_2''(t,T)+\|v_0\|_2^2=:K_2'(t,T).\tag{5.1}$$

Next, we estimate $\|\partial_t v_\varepsilon\|_2^2$. Multiplying equation (1.9) by $\partial_t v_\varepsilon$, and integrating the resultant over Ω, then it follows from integration by parts, and using (1.10)–(1.11) that

$$\begin{aligned}\|\partial_t v_\varepsilon\|_2^2=&\int_\Omega(\Delta v_\varepsilon-(v_\varepsilon\cdot\nabla_H)v_\varepsilon-w_\varepsilon\partial_z v_\varepsilon)\cdot\partial_t v_\varepsilon d\mathbf{x}\\ \le&\int_M\left(\int_{-h}^h|\nabla_H v|dz\right)\left(\int_{-h}^h|\partial_z v_\varepsilon||\partial_t v_\varepsilon|dz\right)d\mathbf{x}^H\\ &+\int_\Omega(|\Delta v_\varepsilon|+|v_\varepsilon||\nabla_H v_\varepsilon|)|\partial_t v_\varepsilon|d\mathbf{x}=:L_1+L_2.\end{aligned}\tag{5.2}$$

The estimates on L_1 and L_2 are as follows. By Lemma 2.1, it follows from the Poincaré and Young inequalities that

$$\begin{aligned}L_1 \leq & C\|\nabla_H v_\varepsilon\|_2^{\frac{1}{2}}(\|\nabla_H v_\varepsilon\|_2 + \|\nabla_H^2 v_\varepsilon\|_2)^{\frac{1}{2}} \\ & \times \|\partial_z v_\varepsilon\|_2^{\frac{1}{2}}(\|\partial_z v_\varepsilon\|_2 + \|\nabla_H \partial_z v_\varepsilon\|_2)^{\frac{1}{2}}\|\partial_t v_\varepsilon\|_2 \\ \leq & C\|\nabla_H v_\varepsilon\|_2^{\frac{1}{2}}\|\nabla_H^2 v_\varepsilon\|_2^{\frac{1}{2}}\|\partial_z v_\varepsilon\|_2^{\frac{1}{2}}\|\nabla \partial_z v_\varepsilon\|_2^{\frac{1}{2}}\|\partial_t v_\varepsilon\|_2 \\ \leq & \frac{1}{4}\|\partial_t v_\varepsilon\|_2^2 + C\|\nabla v_\varepsilon\|_2^2\|\nabla^2 v_\varepsilon\|_2^2,\end{aligned}$$

for a positive constant depending only on h. For L_2, by the Hölder, Sobolev, Poincaré and Young inequalities, we deduce

$$\begin{aligned}L_2 \leq & \|\Delta v_\varepsilon\|_2\|\partial_t v_\varepsilon\|_2 + \|v_\varepsilon\|_3\|\nabla_H v_\varepsilon\|_6\|\partial_t v_\varepsilon\|_2 \\ \leq & \|\Delta v_\varepsilon\|_2\|\partial_t v_\varepsilon\|_2 + C\|v_\varepsilon\|_{H^1}\|\nabla^2 v_\varepsilon\|_2\|\partial_t v_\varepsilon\|_2 \\ \leq & \frac{1}{4}\|\partial_t v_\varepsilon\|_2^2 + C\|\nabla^2 v_\varepsilon\|_2^2(1 + \|v_\varepsilon\|_{H^1}^2),\end{aligned}$$

for a positive constant C depending only on h. Substituting the above estimates of L_1 and L_2 into (5.2), and recalling (5.1), we obtain

$$\int_t^T \|\partial_t v_\varepsilon\|_2^2(s)ds \leq C\int_t^T (1 + \|v_\varepsilon\|_{H^1}^2(s))\|\nabla^2 v_\varepsilon\|_2^2(s)ds \leq C(1 + K_2'(t,T))^2.$$

Combining the above estimate with (5.1) yields the conclusion, with $K_2(t,T) = K_2'(t,T) + C(1 + K_2'(t,T))^2$. This completes the proof. □

To obtain additional estimates on v_ε, we decompose v_ε into two parts

$$v_\varepsilon = \bar{v}_\varepsilon + V_\varepsilon,$$

such that V_ε is the unique solution to the following linear system

$$\partial_t V_\varepsilon + (v_\varepsilon \cdot \nabla_H)V_\varepsilon + w_\varepsilon \partial_z V_\varepsilon + \nabla_H P_\varepsilon(\mathbf{x}^H, t) - \Delta V_\varepsilon + f_0 k \times V_\varepsilon = 0, \tag{5.3}$$

$$\int_{-h}^h \nabla_H \cdot V_\varepsilon(\mathbf{x}^H, z, t)dz = 0, \tag{5.4}$$

subject to the periodic boundary condition, with initial data $V_{0\varepsilon}$.

The solvability of the above linear system can be done in the same way (in fact much easier) as for the primitive equations. Based on the H^1 theory for the primitive equations, i.e., the global existence of strong solutions with H^1 initial data, one can show that the solution v_ε to the primitive equations with smooth initial data $v_{0\varepsilon}$ is smooth, and as a result, V_ε is also smooth.

Moreover, we have the L^∞ estimates on V_ε stated in the following proposition.

Proposition 5.3 (L^∞ **estimates on** V_ε)**.** *Let V_ε be the unique solution to system (5.3)–(5.4), subject to the periodic boundary condition, with initial data $V_{0\varepsilon}$. Then*

we have the following estimate

$$\sup_{0\le s\le t}\|V_\varepsilon\|_\infty(s)\le K_3(t)\|V_0\|_\infty,$$

for any $t\in[0,\infty)$*, where*

$$K_3(t)=C(1+\|v_0\|_4)^{40}(t+1)^2\exp\{Ce^{2t}(t+1)(1+\|v_0\|_4)^4\},$$

for a positive constant C *depending only on* h*.*

Proof. Let $q\in[2,\infty)$. Multiplying equation (5.3) by $|V_\varepsilon|^{q-2}V_\varepsilon$, and integrating over Ω, it follows from integration by parts that

$$\frac{1}{q}\frac{d}{dt}\|V_\varepsilon\|_q^q+\int_\Omega\nabla V_\varepsilon:\nabla(|V_\varepsilon|^{q-2}V_\varepsilon)d\mathbf{x}=\int_\Omega P_\varepsilon(\mathbf{x}^H,t)\nabla_H\cdot(|V_\varepsilon|^{q-2}V_\varepsilon)d\mathbf{x}.$$

Straightforward calculations yield

$$\int_\Omega\nabla V_\varepsilon:\nabla(|V_\varepsilon|^{q-2}V_\varepsilon)d\mathbf{x}=\int_\Omega|V_\varepsilon|^{q-2}(|\nabla V_\varepsilon|^2+(q-2)|\nabla|V_\varepsilon||^2)d\mathbf{x}.$$

We thus have

$$\frac{1}{q}\frac{d}{dt}\|V_\varepsilon\|_q^q+\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla V_\varepsilon\right\|_2^2\le\int_\Omega P_\varepsilon(\mathbf{x}^H,t)\nabla_H\cdot(|V_\varepsilon|^{q-2}V_\varepsilon)d\mathbf{x}.$$

By the Hölder inequality, we deduce

$$\begin{aligned}
&\int_\Omega P_\varepsilon(\mathbf{x}^H,t)\nabla_H\cdot(|V_\varepsilon|^{q-2}V_\varepsilon)d\mathbf{x}\\
\le&(q-1)\int_M|P_\varepsilon(\mathbf{x}^H,t)|\int_{-h}^h|V_\varepsilon|^{q-2}|\nabla_HV_\varepsilon|dzd\mathbf{x}^H\\
\le&(q-1)\int_M|P_\varepsilon(\mathbf{x}^H,t)|\left(\int_{-h}^h|V_\varepsilon|^{q-2}dz\right)^{\frac12}\left(\int_{-h}^h|V_\varepsilon|^{q-2}|\nabla_HV_\varepsilon|^2dz\right)^{\frac12}d\mathbf{x}^H\\
\le&(q-1)\left(\int_M|P_\varepsilon|^{\frac{4q}{q+2}}d\mathbf{x}^H\right)^{\frac{q+2}{4q}}\left[\int_M\left(\int_{-h}^h|V_\varepsilon|^{q-2}dz\right)^{\frac{2q}{q-2}}d\mathbf{x}^H\right]^{\frac{q-2}{4q}}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla_HV_\varepsilon\right\|_2\\
\le&(q-1)\|P_\varepsilon\|_{\frac{4q}{q+2},M}\left[\int_M\int_{-h}^h|V_\varepsilon|^{2q}dz\left(\int_{-h}^h1dz\right)^{\frac{q+2}{q-2}}d\mathbf{x}^H\right]^{\frac{q-2}{4q}}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla_HV_\varepsilon\right\|_2\\
\le&(q-1)(2h)^{\frac{q+2}{4q}}\|P_\varepsilon\|_{\frac{4q}{q+2},M}\|V_\varepsilon\|_{2q}^{\frac{q-2}{2}}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla_HV_\varepsilon\right\|_2,
\end{aligned}$$

which, substituted into the previous inequality, yields

$$\begin{aligned}
&\tfrac{1}{q}\tfrac{d}{dt}\|V_\varepsilon\|_q^q+\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla V_\varepsilon\right\|_2^2\\
&\le(q-1)(2h)^{\frac{q+2}{4q}}\|P_\varepsilon\|_{\frac{4q}{q+2},M}\|V_\varepsilon\|_{2q}^{\frac{q-2}{2}}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla_HV_\varepsilon\right\|_2.
\end{aligned}\tag{5.5}$$

We need to estimate the term $\|P_\varepsilon\|_{\frac{4q}{q+2},M}$ on the right-hand side of (5.5). Applying the operator div_H to equation (5.3), integrating the resulting equation in z over $(-h,h)$, and using (5.4), yield

$$-\Delta_H P_\varepsilon(\mathbf{x}^H,t) = \frac{1}{2h}\int_{-h}^{h}[\nabla_H\cdot\nabla_H\cdot(V_\varepsilon\otimes v_\varepsilon)+f_0\nabla_H\cdot(k\times V_\varepsilon)]dz.$$

Note that P_ε can be uniquely specified by requiring $\int_M P_\varepsilon d\mathbf{x}^H = 0$. Decompose $P_\varepsilon = P_\varepsilon^1 + P_\varepsilon^2$, where P_ε^1 and P_ε^2 are the unique solutions to

$$-\Delta_H P_\varepsilon^1(\mathbf{x}^H,t) = \frac{1}{2h}\int_{-h}^{h}\nabla_H\cdot\nabla_H\cdot(V_\varepsilon\otimes v_\varepsilon)dz,$$

and

$$-\Delta_H P_\varepsilon^2(\mathbf{x}^H,t) = \frac{f_0}{2h}\int_{-h}^{h}\nabla_H\cdot(k\times V_\varepsilon)dz,$$

respectively, subject to the periodic conditions, and the average zero condition $\int_M P_\varepsilon^i d\mathbf{x}^H = 0$, for $i=1,2$. Thus, noticing that $\frac{4q}{q+2}\in[2,4)$, for $q\in[2,\infty)$, by the elliptic regularity estimates, the Sobolev, Hölder and Young inequalities, we deduce

$$\begin{aligned}
\|P_\varepsilon\|_{\frac{4q}{q+2},M} \le& \|P_\varepsilon^1\|_{\frac{4q}{q+2},M} + \|P_\varepsilon^2\|_{\frac{4q}{q+2},M} \le \|P_\varepsilon^1\|_{\frac{4q}{q+2},M} + C\|\nabla P_\varepsilon^2\|_{\frac{4q}{3q+2},M}\\
\le& C\left\|\int_{-h}^{h} V_\varepsilon\otimes v_\varepsilon dz\right\|_{\frac{4q}{q+2},M} + C\left\|\int_{-h}^{h} k\times V_\varepsilon dz\right\|_{\frac{4q}{3q+2},M}\\
\le& C\int_{-h}^{h}\|V_\varepsilon\otimes v_\varepsilon\|_{\frac{4q}{q+2},M}\,dz + C\int_{-h}^{h}\|V_\varepsilon\|_{\frac{4q}{3q+2},M}dz\\
\le& C\int_{-h}^{h}\|v_\varepsilon\|_{4,M}\|V_\varepsilon\|_{2q,M}dz + C\int_{-h}^{h}\|V_\varepsilon\|_{2q,M}|M|^{\frac{3}{4}}dz\\
\le& C\|v_\varepsilon\|_4\|V_\varepsilon\|_{2q}(2h)^{\frac{3}{4}-\frac{1}{2q}} + C\|V_\varepsilon\|_{2q}|M|^{\frac{3}{4}}(2h)^{1-\frac{1}{2q}}\\
\le& C(1+2h)(1+\|v_\varepsilon\|_4)\|V_\varepsilon\|_{2q} \le C(1+\|v_\varepsilon\|_4)\|V_\varepsilon\|_{2q},
\end{aligned}$$

for a positive constant C depending only on h.

Substituting the above inequality into (5.5) yields

$$\frac{1}{q}\frac{d}{dt}\|V_\varepsilon\|_q^q + \left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla V_\varepsilon\right\|_2^2 \le Cq(1+\|v_\varepsilon\|_4)\|V_\varepsilon\|_{2q}^{\frac{q}{2}}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla_H V_\varepsilon\right\|_2, \tag{5.6}$$

for a positive constant C depending only on h. By the Poincaré inequality and the Gagliardo-Nirenberg-Sobolev inequality, $\|\varphi\|_4 \le C\|\varphi\|_1^{\frac{1}{10}}\|\nabla\varphi\|_2^{\frac{9}{10}}$, for any average zero function φ, we deduce

$$\|V_\varepsilon\|_{2q}^{\frac{q}{2}} = \left\||V_\varepsilon|^{\frac{q}{2}}\right\|_4 \le C\left\||V_\varepsilon|^{\frac{q}{2}}\right\|_1^{\frac{1}{10}}\left(\left\||V_\varepsilon|^{\frac{q}{2}}\right\|_1 + \left\|\nabla|V_\varepsilon|^{\frac{q}{2}}\right\|_2\right)^{\frac{9}{10}},$$

which, substituted into (5.6), and using the Young inequality, yields

$$
\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\|V_\varepsilon\|_q^q+\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla V_\varepsilon\right\|_2^2\\
\leq& Cq(1+\|v_\varepsilon\|_4)\left\||V_\varepsilon|^{\frac{q}{2}}\right\|_1^{\frac{1}{10}}\left(\left\||V_\varepsilon|^{\frac{q}{2}}\right\|_1+\left\|\nabla|V_\varepsilon|^{\frac{q}{2}}\right\|_2\right)^{\frac{9}{10}}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla_H V_\varepsilon\right\|_2\\
\leq& Cq(1+\|v_\varepsilon\|_4)\left(\left\||V_\varepsilon|^{\frac{q}{2}}\right\|_1+q^{\frac{9}{10}}\left\||V_\varepsilon|^{\frac{q}{2}}\right\|_1^{\frac{1}{10}}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla V_\varepsilon\right\|_2^{\frac{9}{10}}\right)\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla_H V_\varepsilon\right\|_2\\
\leq& \frac{1}{4}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla V_\varepsilon\right\|_2^2+Cq^2(1+\|v_\varepsilon\|_4)^2\left(\left\||V_\varepsilon|^{\frac{q}{2}}\right\|_1^2+q^{\frac{9}{5}}\left\||V_\varepsilon|^{\frac{q}{2}}\right\|_1^{\frac{1}{5}}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla V_\varepsilon\right\|_2^{\frac{9}{5}}\right)\\
\leq& \frac{1}{2}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla V_\varepsilon\right\|_2^2+Cq^2(1+\|v_\varepsilon\|_4)^2\left\||V_\varepsilon|^{\frac{q}{2}}\right\|_1^2+Cq^{38}(1+\|v_\varepsilon\|_4)^{20}\left\||V_\varepsilon|^{\frac{q}{2}}\right\|_1^2\\
\leq& \frac{1}{2}\left\||V_\varepsilon|^{\frac{q}{2}-1}\nabla V_\varepsilon\right\|^2+Cq^{38}(1+\|v_\varepsilon\|_4)^{20}\left(\|V_\varepsilon\|_{\frac{q}{2}}^{\frac{q}{2}}\right)^2.
\end{aligned}
$$

Therefore, applying Proposition 5.1, we have

$$
\begin{aligned}
\frac{d}{dt}\|V_\varepsilon\|_q^q \leq& Cq^{40}(1+\|v_\varepsilon\|_4)^{20}\left(\|V_\varepsilon\|_{\frac{q}{2}}^{\frac{q}{2}}\right)^2\\
\leq& Cq^{40}\left[(1+\|v_0\|_4)\exp\{Ce^{2t}(t+1)(1+\|v_0\|_2^2)^2\}\right]^{20}\left(\|V_\varepsilon\|_{\frac{q}{2}}^{\frac{q}{2}}\right)^2\\
\leq& Cq^{40}\left[(1+\|v_0\|_4)\exp\{Ce^{2t}(t+1)(1+\|v_0\|_4)^4\}\right]^{20}\left(\|V_\varepsilon\|_{\frac{q}{2}}^{\frac{q}{2}}\right)^2\\
=& Cq^{40}S_0(t)\left(\|V_\varepsilon\|_{\frac{q}{2}}^{\frac{q}{2}}\right)^2,
\end{aligned}
$$

with K_0 given by

$$S_0(t)=[(1+\|v_0\|_4)\exp\{Ce^{2t}(t+1)(1+\|v_0\|_4)^4\}]^{20},$$

and C a positive constant depending only on h. Set $\delta_0=\|V_0\|_\infty$. Recalling that $\|V_{0\varepsilon}\|_\infty\leq\|V_0\|_\infty$, it follows that $\|V_{0\varepsilon}\|_q^q\leq 2h\delta_0^q$. On account of this, it follows from the previous inequality that

$$\sup_{0\leq s\leq t}\|V_\varepsilon\|_q^q\leq 2h\delta_0^q+C_0^*q^{40}S_1(t)\left(\sup_{0\leq s\leq t}\|V_\varepsilon\|_{\frac{q}{2}}^{\frac{q}{2}}\right)^2, \tag{5.7}$$

for a constant C_0^* depending only on h, where $S_1(t):=\int_0^t S_0(s)ds$.

Define

$$A_k(t)=\sup_{0\leq s\leq t}\|V_\varepsilon\|_{2^k}^{2^k},\quad k=1,2,\cdots.$$

Then, setting $q=2^{k+1}$ in (5.7) yields

$$A_{k+1}(t)\leq 2h\delta_0^{2^{k+1}}+C_0^*2^{40(k+1)}S_1(t)A_k(t)^2,\quad k=1,2,\cdots,$$

where C_0^* is a positive constant depending only on h. Setting

$$M_0(t)=2h+(1+C_0^*)2^{80}(1+S_1(t)),$$

then we have

$$C_0^* 2^{40(k+1)} S_1(t) \le (1+C_0^*)2^{80k}(1+S_1(t)) \le [(1+C_0^*)2^{80}(1+S_1(t))]^k \le M_0(t)^k,$$

and consequently

$$A_{k+1}(t) \le M_0(t)\delta_0^{2^{k+1}} + M_0(t)^k A_k(t)^2, \quad k=1,2,\cdots.$$

It is obviously that $M_0(t) \ge 2$. Thus, we can apply Lemma 2.2 to deduce that

$$A_k(t) \le (M_0(t)^4\delta_0^2)^{2^{k-1}}, \quad k=1,2,\cdots.$$

Recalling the definition of A_k, then for any $s \in [0,t]$, we have

$$\|V_\varepsilon\|_{2^k}^{2^k}(s) \le A_k(t) \le (M_0(t)^4\delta_0^2)^{2^{k-1}},$$

which implies $\|V_\varepsilon\|_{2^k}(s) \le M_0(t)^2\delta_0$, for any positive integer k and every $s \in [0,t]$. Taking $k \to \infty$, we conclude

$$\|V_\varepsilon\|_\infty(s) \le M_0(t)^2\delta_0 = M_0(t)^2\|V_0\|_\infty,$$

for every $s \in [0,t]$, and hence

$$\sup_{0\le s\le t}\|V_\varepsilon\|_\infty(s) \le M_0(t)^2\|V_0\|_\infty.$$

Recalling the expressions of M_0 and S_0, one can easily verify that

$$M_0(t)^2 \le C(1+\|v_0\|_4)^{40}(t+1)^2\exp\{e^{2t}(t+1)(1+\|v_0\|_4)^4\} =: K_3(t),$$

for a positive constant C, depending only on h. Therefore, the conclusion holds. □

Recalling the definition of V_ε, it is then clear that $\bar{v}_\varepsilon$ satisfies

$$\partial_t\bar{v}_\varepsilon + (v_\varepsilon\cdot\nabla_H)\bar{v}_\varepsilon + w_\varepsilon\partial_z\bar{v}_\varepsilon + \nabla_H\bar{p}_\varepsilon(\mathbf{x}^H,t) - \Delta\bar{v}_\varepsilon + f_0 k\times\bar{v}_\varepsilon = 0, \tag{5.8}$$

$$\int_{-h}^{h}\nabla_H\cdot\bar{v}_\varepsilon(\mathbf{x}^H,z,t)dz = 0, \tag{5.9}$$

subject to the periodic boundary condition, with initial data $\bar{v}_{0\varepsilon}$. Moreover, since v_ε and V_ε are both smooth, so is $\bar{v}_\varepsilon$.

For $\bar{v}_\varepsilon$, we have the following proposition, which states the estimate of $\bar{v}_\varepsilon$ in X.

Proposition 5.4 (**Estimate on $\bar{v}_\varepsilon$ in X**). *Let $\bar{v}_\varepsilon$ be the unique solution to system (5.8)–(5.9), subject to periodic boundary condition, with initial data $\bar{v}_{0\varepsilon}$. Then we have the following estimate*

$$\sup_{0\le s\le t}(\|\bar{v}_\varepsilon\|_2^2 + \|\partial_z\bar{v}_\varepsilon\|_2^2) + \int_0^t(\|\nabla\bar{v}_\varepsilon\|_2^2 + \|\nabla\partial_z\bar{v}_\varepsilon\|_2^2)d\tau \le K_4(t),$$

where K_4 is a continuous function on $[0,\infty)$, determined only by h, and the initial norms $\|\bar{v}_0\|_2, \|V_0\|_\infty$ and $\|v_0\|_6$.

Proof. Multiplying equation (5.8) by $\bar{v}_\varepsilon$, and integrating over Ω, then it follows from integration by parts that

$$\frac{1}{2}\frac{d}{dt}\|\bar{v}_\varepsilon\|_2^2+\|\nabla\bar{v}_\varepsilon\|_2^2=0,$$

which implies

$$\frac{1}{2}\sup_{0\le s\le t}\|\bar{v}_\varepsilon\|_2^2+\int_0^t\|\nabla\bar{v}_\varepsilon\|_2^2d\tau\le\frac{1}{2}\|\bar{v}_{0\varepsilon}\|_2^2\le\frac{1}{2}\|\bar{v}_0\|_2^2. \tag{5.10}$$

Set $\bar{u}_\varepsilon=\partial_z\bar{v}_\varepsilon$. Differentiating equation (5.8) with respect to z yields

$$\partial_z\bar{u}_\varepsilon+(v_\varepsilon\cdot\nabla_H)\bar{u}_\varepsilon+w_\varepsilon\partial_z\bar{u}_\varepsilon-\Delta\bar{u}_\varepsilon+(\partial_z v_\varepsilon\cdot\nabla_H)\bar{v}_\varepsilon-(\nabla_H\cdot v_\varepsilon)\bar{u}_\varepsilon+f_0k\times\bar{u}_\varepsilon=0.$$

Multiplying the above equation by $\bar{u}_\varepsilon$, and integrating over Ω, then it follows from integration by parts that

$$\frac{1}{2}\frac{d}{dt}\|\bar{u}_\varepsilon\|_2^2+\|\nabla\bar{u}_\varepsilon\|_2^2=\int_\Omega[(\nabla_H\cdot v_\varepsilon)\bar{u}_\varepsilon-(\partial_z v_\varepsilon\cdot\nabla_H)\bar{v}_\varepsilon]\cdot\bar{u}_\varepsilon d\mathbf{x}. \tag{5.11}$$

We are going to estimate the terms on the right-hand side of the above equality. For the first term, integration by parts, and using the Hölder, Sobolev, Poincaré and Young inequalities lead to

$$\begin{aligned}I_1:=&\int_\Omega(\nabla_H\cdot v_\varepsilon)|\bar{u}_\varepsilon|^2d\mathbf{x}=-2\int_\Omega(v_\varepsilon\cdot\nabla_H)\bar{u}_\varepsilon\cdot\bar{u}_\varepsilon d\mathbf{x}\\ \le&2\|v_\varepsilon\|_6\|\nabla_H\bar{u}_\varepsilon\|_2\|\bar{u}_\varepsilon\|_3\le C\|v_\varepsilon\|_6\|\nabla_H\bar{u}_\varepsilon\|_2\|\bar{u}_\varepsilon\|_2^{\frac{1}{2}}\|\nabla_H\bar{u}_\varepsilon\|_2^{\frac{1}{2}}\\ \le&\frac{1}{6}\|\nabla\bar{u}_\varepsilon\|_2^2+C\|v_\varepsilon\|_6^4\|\bar{u}_\varepsilon\|_2^2\le\frac{1}{6}\|\nabla\bar{u}_\varepsilon\|_2^2+C\|v_\varepsilon\|_6^4\|\nabla\bar{v}_\varepsilon\|_2^2.\end{aligned}$$

For the second term, it follows from integration by parts that

$$\begin{aligned}I_2:=&-\int_\Omega(\partial_z v_\varepsilon\cdot\nabla_H)\bar{v}_\varepsilon\cdot\bar{u}_\varepsilon d\mathbf{x}=-\int_\Omega\partial_z(\bar{v}_\varepsilon+V_\varepsilon)\cdot\nabla_H\bar{v}_\varepsilon\cdot\bar{u}_\varepsilon d\mathbf{x}\\ =&-\int_\Omega(\bar{u}_\varepsilon\cdot\nabla_H)\bar{v}_\varepsilon\cdot\bar{u}_\varepsilon d\mathbf{x}-\int_\Omega(\partial_z V_\varepsilon\cdot\nabla_H)\bar{v}_\varepsilon\cdot\bar{u}_\varepsilon d\mathbf{x}=:I_{21}+I_{22}.\end{aligned}$$

Estimates on I_{21} and I_{22} are given as follows. Integrating by parts, it follows from the Hölder, Sobolev, Poincaré and Young inequalities that

$$\begin{aligned}I_{21}=&-\int_\Omega(\bar{u}_\varepsilon\cdot\nabla_H)\bar{v}_\varepsilon\cdot\bar{u}_\varepsilon d\mathbf{x}\\ =&\int_\Omega[(\nabla_H\cdot\bar{u}_\varepsilon)\bar{v}_\varepsilon\cdot\bar{u}_\varepsilon+(\bar{u}_\varepsilon\cdot\nabla_H)\bar{u}_\varepsilon\cdot\bar{v}_\varepsilon]d\mathbf{x}\\ \le&2\|\nabla_H\bar{u}_\varepsilon\|_2\|\bar{v}_\varepsilon\|_6\|\bar{u}_\varepsilon\|_3\le C\|\nabla_H\bar{u}_\varepsilon\|_2\|\bar{v}_\varepsilon\|_6\|\bar{u}_\varepsilon\|_2^{\frac{1}{2}}\|\nabla\bar{u}_\varepsilon\|_2^{\frac{1}{2}}\\ \le&\frac{1}{6}\|\nabla\bar{u}_\varepsilon\|_2^2+C\|\bar{v}_\varepsilon\|_6^4\|\bar{u}_\varepsilon\|_2^2\le\frac{1}{6}\|\nabla\bar{u}_\varepsilon\|_2^2+C(\|v_\varepsilon\|_6^4+\|V_\varepsilon\|_6^4)\|\bar{u}_\varepsilon\|_2^2\end{aligned}$$

$$\leq \frac{1}{6}\|\nabla \bar{u}_\varepsilon\|_2^2 + C(\|v_\varepsilon\|_6^4 + \|V_\varepsilon\|_\infty^4)\|\nabla \bar{v}_\varepsilon\|_2^2,$$

and

$$\begin{aligned} I_{22} =& -\int_\Omega (\partial_z V_\varepsilon \cdot \nabla_H)\bar{v}_\varepsilon \cdot \bar{u}_\varepsilon d\mathbf{x} \\ =& \int_\Omega [(V_\varepsilon \cdot \nabla_H)\bar{u}_\varepsilon \cdot \bar{u}_\varepsilon + (V_\varepsilon \cdot \nabla_H)\bar{v}_\varepsilon \cdot \partial_z \bar{u}_\varepsilon] d\mathbf{x} \\ \leq& \|V_\varepsilon\|_\infty \|\nabla_H \bar{u}_\varepsilon\|_2 \|\bar{u}_\varepsilon\|_2 + \|V_\varepsilon\|_\infty \|\nabla_H \bar{v}_\varepsilon\|_2 \|\partial_z \bar{u}_\varepsilon\|_2 \\ \leq& \frac{1}{6}\|\nabla \bar{u}_\varepsilon\|_2^2 + C\|V_\varepsilon\|_\infty^2(\|\bar{u}_\varepsilon\|_2^2 + \|\nabla_H \bar{v}_\varepsilon\|_2^2) \\ \leq& \frac{1}{6}\|\nabla_H \bar{u}_\varepsilon\|_2^2 + C\|V_\varepsilon\|_\infty^2 \|\nabla \bar{v}_\varepsilon\|_2^2. \end{aligned}$$

Substituting the estimates for I_1, I_{21} and I_{22} into (5.11), we have

$$\frac{d}{dt}\|\bar{u}_\varepsilon\|_2^2 + \|\nabla \bar{u}_\varepsilon\|_2^2 \leq C(\|v_\varepsilon\|_6^4 + \|V_\varepsilon\|_\infty^4 + 1)\|\nabla \bar{v}_\varepsilon\|_2^2,$$

for a constant C depending only on h. Integrating the above inequality with respect to time, recalling (5.10), and applying Proposition 5.1 and Proposition 5.3, we have

$$\begin{aligned} \sup_{0\leq s\leq t}\|\bar{u}_\varepsilon\|_2^2(s) + \int_0^t \|\nabla \bar{u}_\varepsilon\|_2^2(\tau)d\tau \leq& C \sup_{0\leq s\leq t}(\|v_\varepsilon\|_6^4 + \|V_\varepsilon\|_\infty^4 + 1)\int_0^t \|\nabla \bar{v}_\varepsilon\|_2^2 d\tau \\ \leq& C\|\bar{v}_0\|_2^2[K_1(t)^4(1 + \|v_0\|_6^4) + 1 + K_3(t)^4\|V_0\|_\infty^4] =: K_4'(t). \end{aligned}$$

Combining the above inequality with (5.10), we then obtain

$$\sup_{0\leq s\leq t}(\|\bar{v}_\varepsilon\|_2^2 + \|\bar{u}_\varepsilon\|_2^2) + \int_0^t (\|\nabla \bar{v}_\varepsilon\|_2^2 + \|\nabla \bar{u}_\varepsilon\|_2^2)d\tau \leq K_4'(t) + \|\bar{v}_0\|_2^2 =: K_4(t),$$

completing the proof. □

6. Proof of Theorem 1.2

Based on the results established in the previous section, we can now give the proof of Theorem 1.2 as follows.

Proof of Theorem 1.2. (i) Suppose $v_0 = \bar{v}_0 + V_0$, with $\bar{v}_0 \in X \cap \mathcal{H}$ and $V_0 \in L^\infty(\Omega) \cap \mathcal{H}$. Extend $v_0, \bar{v}_0$ and V_0 periodically to the whole space, and use the same notations to the relevant extensions. Let j_ε, as before, be the standard compactly supported nonnegative mollifier, and set $\bar{v}_{0\varepsilon} = \bar{v}_0 * j_\varepsilon$, $V_{0\varepsilon} = V_0 * j_\varepsilon$ and $v_{0\varepsilon} = v_0 * j_\varepsilon$. It is then obvious that $\bar{v}_{0\varepsilon} \in X \cap \mathcal{H}$, $V_{0\varepsilon} \in L^\infty(\Omega) \cap \mathcal{H}$ and $v_{0\varepsilon} = \bar{v}_{0\varepsilon} + V_{0\varepsilon}$.

Let v_ε be the unique global strong solution to system (1.9)–(1.13), with initial data $v_{0\varepsilon}$. As in section 5, we decompose v_ε into two parts as

$$v_\varepsilon = V_\varepsilon + \bar{v}_\varepsilon,$$

where V_ε is the unique solution to system (5.8)–(5.9), subject to periodic boundary conditions, with initial data $V_{0\varepsilon}$. By Propositions 5.1–5.4, we have the estimates

$$\frac{1}{2}\|v_\varepsilon\|_2^2(t)+\int_0^t\|\nabla v_\varepsilon\|_2^2(\tau)d\tau\le\frac{1}{2}\|v_0\|_2^2,$$

$$\sup_{0\le s\le t}\|v_\varepsilon\|_6(s)\le\sqrt{6}K_1(t)(1+\|v_0\|_6),$$

$$\sup_{t\le s\le T}\|v_\varepsilon\|_{H^1}^2(s)+\int_t^T\|(\nabla^2 v_\varepsilon,\partial_t v_\varepsilon)\|_2^2(s)\le K_2(t,T),$$

$$\sup_{0\le s\le t}\|V_\varepsilon\|_\infty(s)\le K_3(t)\|V_0\|_\infty,$$

$$\sup_{0\le s\le t}\|(\bar v_\varepsilon,\partial_z\bar v_\varepsilon)\|_2^2(s)+\int_0^t\|(\nabla\bar v_\varepsilon,\nabla\partial_z\bar v_\varepsilon)\|_2^2(s)ds\le K_4(t),$$

for any $0<t<T<\infty$, where $K_i, i=1,2,3,4$, are the same functions as those in Propositions 5.1–5.4.

Note that the associated pressure $p_\varepsilon(\mathbf{x}^H,t)$ for the solution v_ε in system (1.9)–(1.11) can be decomposed as $p_\varepsilon=p_\varepsilon^1+p_\varepsilon^2$, where p_ε^1 and p_ε^2 are the unique solutions to systems

$$-\Delta_H p_\varepsilon^1(\mathbf{x}^H,t)=\frac{1}{2h}\nabla_H\cdot\nabla_H\cdot\int_{-h}^h(v_\varepsilon\otimes v_\varepsilon)dz,$$

and

$$-\Delta_H p_\varepsilon^2(\mathbf{x}^H,t)=\frac{f_0}{2h}\int_{-h}^h\nabla_H\cdot(k\times v_\varepsilon)dz,$$

respectively, subject to the periodic boundary conditions, and the average zero condition $\int_M p_\varepsilon^i d\mathbf{x}^H=0$, for $i=1,2$. As a result, by the elliptic regularity estimates, and recalling the $L^\infty(0,t;L^6(\Omega))$ estimate on v_ε stated above, we obtain by the Hölder inequality that

$$\|p_\varepsilon^1\|_{L^\infty(0,t;L^3(\Omega))}\le C\|v_\varepsilon\|_{L^\infty(0,t;L^6(\Omega))}^2\le CK_1(t)^2(1+\|v_0\|_6^2),$$

and

$$\|\nabla p_\varepsilon^2\|_{L^\infty(0,t;L^2(\Omega))}\le C\|v_\varepsilon\|_{L^\infty(0,t;L^2(\Omega))}\le C\|v_0\|_2^2,$$

for any $t\in(0,\infty)$.

Rewrite equation (1.9) for v_ε as

$$\partial_t v_\varepsilon+\nabla_H\cdot(v_\varepsilon\otimes v_\varepsilon)+\partial_z(w_\varepsilon v_\varepsilon)+\nabla_H p_\varepsilon(\mathbf{x}^H,t)-\Delta v_\varepsilon+f_0k\times v_\varepsilon=0.$$

Noticing that $L^2(\Omega)\hookrightarrow L^{\frac{5}{4}}(\Omega)\hookrightarrow W^{-1,\frac{5}{4}}(\Omega)$, and recalling that w_ε is determined by v_ε through (1.11), it follows from the above equation and using the Hölder inequality that

$$\|\partial_t v_\varepsilon\|_{W_{\rm per}^{-1,\frac{5}{4}}}\le\|\nabla_H\cdot(v_\varepsilon\otimes v_\varepsilon)\|_{W_{\rm per}^{-1,\frac{5}{4}}}+\|\partial_z(w_\varepsilon v_\varepsilon)\|_{W_{\rm per}^{-1,\frac{5}{4}}}+\|\Delta v_\varepsilon\|_{W_{\rm per}^{-1,\frac{5}{4}}}$$

$$
\begin{aligned}
&+\|\nabla_H p_\varepsilon^1\|_{W_{\rm per}^{-1,\frac{5}{4}}}+\|\nabla_H p_\varepsilon^2\|_{W_{\rm per}^{-1,\frac{5}{4}}}+f_0\|k\times v_\varepsilon\|_{W_{\rm per}^{-1,\frac{5}{4}}}\\
\le& C(\|v_\varepsilon\otimes v_\varepsilon\|_{\frac{5}{4}}+\|w_\varepsilon v_\varepsilon\|_{\frac{5}{4}}+\|\nabla v_\varepsilon\|_{\frac{5}{4}}+\|p_\varepsilon^1\|_{\frac{5}{4}}+\|\nabla_H p_\varepsilon^2\|_2+\|v_\varepsilon\|_2)\\
\le& C(\|v_\varepsilon\|_{\frac{5}{2}}^2+\|w_\varepsilon\|_2\|v_\varepsilon\|_{\frac{10}{3}}+\|\nabla v_\varepsilon\|_2+\|p_\varepsilon^1\|_2+\|v_\varepsilon\|_2)\\
\le& C(\|v_\varepsilon\|_6^2+\|\nabla v_\varepsilon\|_2\|v_\varepsilon\|_6+\|\nabla v_\varepsilon\|_2+\|v_\varepsilon\|_2),
\end{aligned}
$$

and thus we have

$$
\begin{aligned}
\|\partial_t v_\varepsilon\|_{L^{\frac{5}{4}}(0,t;W_{\rm per}^{-1,\frac{5}{4}})}\le& C(\|v_\varepsilon\|_{L^\infty(0,t;L^6(\Omega))}^2+\|v_\varepsilon\|_{L^\infty(0,t;L^6(\Omega))}\|\nabla v_\varepsilon\|_{L^2(\Omega\times(0,t))}\\
&+\|\nabla v_\varepsilon\|_{L^2(\Omega\times(0,t))}+\|v_\varepsilon\|_{L^\infty(0,t;L^2(\Omega))})\\
\le& C(\|v_\varepsilon\|_{L^\infty(0,t;L^6(\Omega))}^2+\|\nabla v_\varepsilon\|_{L^2(\Omega\times(0,t))}^2+1)\le B(t),
\end{aligned}
$$

for a finite valued function $B(t)$, for $t\in[0,T]$, which is independent of ε.

On account of the estimates obtained above, and noticing that $H^1_{\rm per}\hookrightarrow\hookrightarrow L^2\hookrightarrow W_{\rm per}^{-1,\frac{5}{4}}$, and that $H^2_{\rm per}\hookrightarrow\hookrightarrow H^1_{\rm per}\hookrightarrow L^2$, one can apply the Aubin–Lions lemma, i.e., Lemma 2.3, to deduce that there is a subsequence, still denoted by $\{v_\varepsilon\}$, and a vector field $v=\bar v+V$, such that

$$
\begin{gathered}
v_\varepsilon\to v \text{ in } L^2(\Omega\times(0,t))\cap C([0,t];W_{\rm per}^{-1,\frac{5}{4}}(\Omega)),\\
v_\varepsilon\rightharpoonup v \text{ in } L^2(0,t;H^1_{\rm per}(\Omega)\cap\mathcal{H}),\quad v_\varepsilon\overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,t;L^2(\Omega)),\\
\nabla v_\varepsilon\overset{*}{\rightharpoonup}\nabla v \text{ in } L^\infty(t,T;L^2(\Omega)),\quad (\nabla^2 v_\varepsilon,\partial_t v_\varepsilon)\rightharpoonup(\nabla^2 v,\partial_t v) \text{ in } L^2(\Omega\times(t,T)),\\
V_\varepsilon\overset{*}{\rightharpoonup} V \text{ in } L^\infty(\Omega\times(0,t)),\\
\partial_z\bar v_\varepsilon\overset{*}{\rightharpoonup}\partial_z\bar v \text{ in } L^\infty(0,t;L^2(\Omega)),\quad \partial_z\bar v_\varepsilon\rightharpoonup\partial_z\bar v \text{ in } L^2(0,t;H^1_{\rm per}(\Omega)),
\end{gathered}
$$

for any $0<t<T<\infty$, where $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ are the weak and weak-* convergences, respectively. Moreover, by the weakly lower semi-continuity of the relevant norms, we have

$$
\|V\|_{L^\infty(\Omega\times(0,t))}\le K_3(t)\|V_0\|_\infty,\quad \frac{1}{2}\|v\|_2^2(t)+\int_0^t\|\nabla v\|_2^2(\tau)d\tau\le\frac{1}{2}\|v_0\|_2^2, \tag{6.1}
$$

for a.e. $t\in[0,\infty)$.

We claim that v is a weak solution to system (1.9)–(1.13). To this end, we need to verify (i)–(iv) in Definition 1.1. By the strong convergences stated above, one can see that

$$
v\in C([0,\infty);W_{\rm per}^{-1,\frac{5}{4}}(\Omega))\cap L^\infty_{\rm loc}([0,\infty);L^2(\Omega))\cap L^2_{\rm loc}([0,\infty);H^1_{\rm per}(\Omega)\cap\mathcal{H}),
$$

from which, by the density argument, one obtains

$$
v\in C([0,\infty);L^2_w(\Omega))\cap L^2_{\rm loc}([0,\infty);H^1_{\rm per}(\Omega)\cap\mathcal{H}),
$$

verifying (i) in Definition 1.1. Recalling that v_ε is smooth, it is clear that v_ε satisfies the weak formula, i.e. (ii), stated in Definition 1.1. As a result, recalling the convergences stated in the previous paragraph, one can take the limit, as ε goes to zero, to show that v satisfies the weak formula (ii) in Definition 1.1. Note that

$$v \in L^2(t,T;H^1_{\rm per}(\Omega)), \quad \partial_t v \in L^2(t,T;L^2(\Omega)),$$

v is actually a strong solution to the primitive equations, away from the initial time. By the aid of this fact, the differential energy inequality (iii) in Definition 1.1 actually holds as an equality. The term (iv) is guaranteed by (6.1). Therefore, v is a weak solution to system (1.9)–(1.13), with initial data v_0.

Recall that we have the decomposition $v = \bar{v} + V$, with $\bar{v}$ and V being the weak limits of $\bar{v}_\varepsilon$ and V_ε, respectively. By the weakly lower semi-continuity of norms, $\bar{v}$ and V have the same regularities and same estimates to those for $\bar{v}_\varepsilon$ and V_ε, respectively. This completes the proof of (i).

(ii) Let v be the weak solution established in (i). Note that μ is continuous on $[0,\infty)$. There is a positive short time T_v, such that

$$\sup_{0\le t\le T_v} \|V\|_\infty(t) \le \mu(T_v)\|V_0\|_\infty \le 2\mu(0)\|V_0\|_\infty \le \varepsilon_0.$$

Thanks to this, by Theorem 1.1, v is the unique weak solution to system (1.9)–(1.13), with initial data v_0. This completes the proof of (ii). □

Acknowledgments

E.S.T. is thankful to the kind hospitality of the Universidade Federal do Rio de Janeiro (UFRJ) and Instituto Nacional de Matemática Pura e Aplicada (IMPA) where part of this work was completed; and to the partial support of the CNPq-CsF grant # 401615/2012-0, through the program Ciência sem Fronteiras. This work was supported in part by the ONR grant N00014-15-1-2333 and the NSF grants DMS-1109640 and DMS-1109645

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