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$N = 2$ Superstrings with $(1, 2m)$ Spacetime Signature

H. Lu,[⋆] C.N. Pope,[⋆] X.J. Wang and K.W. Xu.[$]

Center for Theoretical Physics, Texas A&M University,
College Station, TX 77843–4242, USA.

ABSTRACT

We show that the $N = 2$ superstring in $d = 2D \geq 6$ real dimensions, with criticality achieved by including background charges in the two real time directions, exhibits a "coordinate-freezing" phenomenon, whereby the momentum in one of the two time directions is constrained to take a specific value for each physical state. This effectively removes this time direction as a physical coordinate, leaving the theory with $(1, d-2)$ real spacetime signature. Norm calculations for low-lying physical states suggest that the theory is ghost free.

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The $N = 2$ superstring is usually considered to be unattractive physically for a number of reasons. One of these is that although its critical dimension is 4, the $N = 2$ supersymmetry implies the existence of a complex structure [1], which means that the spacetime signature cannot be Minkowskian; instead it must be $(0, 4)$, $(2, 2)$ or $(4, 0)$. The critical dimension 4 reflects the fact that the critical central charge for the $N = 2$ super-Virasoro algebra is $c = 6$. This can be realised by two $N = 2$ superfields without background charges, thus giving the usual $N = 2$ superstring in two complex dimensions [2,3]. In this paper, we study more general realisations with arbitrary numbers of $N = 2$ superfields and with background charges which are fixed so that the theories are critical. This gives rise to critical $N = 2$ superstrings in arbitrary numbers of complex dimensions. We show that the existence of background charges will freeze one real coordinate, *i.e.* the momentum component in that direction is constrained by the physical-state conditions to take a specific value for each state. If we start from a theory with one complex time and choose the background charge to lie in that direction, then the result of the coordinate freezing is to give a theory that effectively has only one real time direction. Thus although $N = 2$ superstrings suffer from a number of phenomenological defects, the spacetime signature of the critical theories in more than four real dimensions can, at least, effectively be Minkowskian, *i.e.* $(1, 2m)$ with $m \geq 2$. Therefore we shall assume in the rest of this paper that the background charge is taken to lie in the complex time direction. We also show in this paper that the norms of low-lying physical states are non-negative, suggesting that the theories are ghost free.

In component form, the currents for the $N = 2$ super-Virasoro algebra are given by

$$\begin{aligned} J &= -\psi^\mu \bar{\psi}_\mu - \alpha_0\, \partial\phi_0 + \alpha_0\, \partial\bar{\phi}_0\ , \\ G^+ &= \sqrt{2}(\partial\bar{\phi}^\mu \psi_\mu - \alpha_0\, \partial\psi_0)\ , \\ G^- &= \sqrt{2}(\partial\phi^\mu \bar{\psi}_\mu - \alpha_0\, \partial\bar{\psi}_0)\ , \\ T &= \tfrac{1}{2}\psi^\mu \partial\bar{\psi}_\mu - \tfrac{1}{2}\partial\psi^\mu \bar{\psi}_\mu - \partial\phi^\mu \partial\bar{\phi}_\mu + \tfrac{1}{2}\alpha_0\, \partial^2\phi_0 + \tfrac{1}{2}\alpha_0\, \partial^2\bar{\phi}_0\ , \end{aligned} \tag{1}$$

where we have chosen the background charge to lie in the $\mu = 0$ direction. The index μ runs over $0, 1, \ldots, D-1$. The central charge of this realisation is

$$c = 3(D - 2\alpha_0^2)\ . \tag{2}$$

The anomaly-freedom condition $c = 6$ therefore requires that

$$\alpha_0^2 = \tfrac{1}{2}(D-2)\ . \tag{3}$$

A physical state $|p\rangle$ satisfies the physical-state conditions

$$\begin{aligned} L_m|p\rangle &= 0 = J_m|p\rangle \qquad m \geq 0\ , \\ G_r^+|p\rangle &= 0 = G_r^-|p\rangle \qquad r > 0\ . \end{aligned} \tag{4}$$

(Since the intercepts of J_0 and L_0 are zero for the $N=2$ super-conformal algebra, we include these in the physical-state conditions (4).) Physical states can be constructed by acting on an $SL(2,C)$-invariant vacuum $\big|0\big\rangle$ with ground-state operators $P(z)$, *i.e.* $\big|p\big\rangle \equiv P(0)\big|0\big\rangle$. The ground-state operators take the form

$$P(z) = R(z) e^{\beta\cdot\bar\phi + \bar\beta\cdot\phi} \ . \tag{5}$$

(Normal ordering is understood.) The operators $R(z)$ can be classified by their eigenvalues ℓ and n under J_0 and L_0 respectively. The eigenvalue ℓ measures the fermion charge of the operator $R(z)$; each ψ_μ in a monomial in $R(z)$ contributes $+1$, each $\bar\psi_\mu$ contributes -1, and $\partial\phi_\mu$ and $\partial\bar\phi_\mu$ contribute 0. The eigenvalue n measures the conformal dimension of the operator $R(z)$, *i.e.* the level number.

At level $n=0$, R is just the identity operator, with $\ell=0$, and $P(z)$ is the "tachyon" ground-state operator. At level $n=\frac12$, R can be $\bar\xi_\mu\psi^\mu$, with $\ell=+1$; or $\xi_\mu\bar\psi^\mu$, with $\ell=-1$. At level $n=1$, ℓ can be $-2,\ 0,\ +2$. In general, at level n, ℓ takes the values

$$\ell = -2n,\ -2n+2,\ \ldots,\ 2n-2,\ 2n \ . \tag{6}$$

It is convenient to work with $2D$ real coordinates φ^μ and $\widetilde\varphi^\mu$, rather than the D complex coordinates ϕ^μ, where $\mu = 0,1,\ldots,D-1$. Thus we define

$$\begin{aligned}\phi^\mu &= \frac{1}{\sqrt2}\big(\varphi^\mu + i\widetilde\varphi^\mu\big) \ ,\\ \bar\phi^\mu &= \frac{1}{\sqrt2}\big(\varphi^\mu - i\widetilde\varphi^\mu\big) \ .\end{aligned} \tag{7}$$

Correspondingly, we can introduce $2D$ parameters k_μ and $\tilde k_\mu$ which are defined by

$$\begin{aligned}\beta_\mu &= \frac{i}{\sqrt2}\big(k_\mu + i\tilde k_\mu\big) \ ,\\ \bar\beta_\mu &= \frac{i}{\sqrt2}\big(k_\mu - i\tilde k_\mu\big) \ .\end{aligned} \tag{8}$$

It follows that

$$e^{\bar\beta\cdot\phi + \beta\cdot\bar\phi} = e^{ik\cdot\varphi + i\tilde k\cdot\widetilde\varphi} \ . \tag{9}$$

Thus k_μ and $\tilde k_\mu$ are the momenta conjugate to φ^μ and $\widetilde\varphi^\mu$ respectively.

For physical states with level number n and fermion charge ℓ, the J_0 and L_0 constraints in (4) give

$$\begin{aligned} J_0: \qquad 0 &= \ell + \alpha_0\,(\beta_0 - \bar\beta_0)\\ &= \ell - \sqrt2\alpha_0\,\tilde k_0 \ , \end{aligned} \tag{10a}$$

$$\begin{aligned} L_0: \qquad 0 &= n - \beta^\mu\bar\beta_\mu + \tfrac12\alpha_0\,(\beta_0 + \bar\beta_0)\\ &= n + \tfrac12 k^\mu k_\mu + \tfrac12\tilde k^\mu\tilde k_\mu + \tfrac{i}{\sqrt2}\alpha_0\,k_0 \ . \end{aligned} \tag{10b}$$

In the usual discussion of the $N = 2$ superstring, for which $D = 2$ and hence from (2) the background charge α_0 is zero, equation $(10a)$ implies that for all physical states the fermionic charge of $R(z)$ must be zero. This implies that in this case there are no physical states occurring at levels with n a half-integer. (In fact, as discussed in [2,3], all the higher-level physical states are longitudinal, and hence have no physical degrees of freedom.)

We are interested in the case where D is greater than 2, which implies, from (2), that α_0 is real. It follows from equation $(10a)$ that the momentum component $\tilde{k}_0$ is "frozen" to the value

$$\tilde{k}_0 = \frac{\ell}{\sqrt{2}\,\alpha_0} \ . \tag{11}$$

(A similar momentum-freezing phenomenon was found for the bosonic W_3 string in [4]. In fact it seems to be the case that momentum freezing occurs for any string theory based on an extended conformal algebra with additional bosonic currents, when background charges are present.)

The hermiticity conditions for L_0 and J_0 imply that

$$\begin{aligned} k_i^* &= k_i \qquad i = 1, 2, \ldots, D-1 \ , \\ \tilde{k}_\mu^* &= \tilde{k}_\mu \qquad \mu = 0, 1, \ldots, D-1 \ , \end{aligned} \tag{12}$$

where "$*$" denotes complex conjugation; whilst k_0 must satisfy a condition that is modified by the background charge:

$$k_0^* = k_0 - i\sqrt{2}\alpha_0 \ . \tag{13}$$

For this, it is convenient to introduce a shifted momentum $\hat{k}_0$ that is real:

$$\hat{k}_0 = k_0 - \frac{i}{\sqrt{2}}\alpha_0 \ , \tag{14}$$

It follows that the complex-conjugation relations for β_μ and $\bar{\beta}_\mu$ are

$$\beta_i^* = -\bar{\beta}_i \ ; \qquad \bar{\beta}_i^* = -\beta_i \ , \tag{15a}$$

$$\beta_0^* = -\bar{\beta}_0 - \alpha_0 \ ; \qquad \bar{\beta}_0^* = -\beta_0 - \alpha_0 \ , \tag{15b}$$

where the index $i = 1, 2, \cdots, D-1$.

For the theory to be unitary, the norms of the physical states should all be non-negative. For the case of the $N = 2$ superstring in $D = 2$ complex dimensions, for which background charges vanish, the no-ghost theorem has been discussed in [3,5]. For the case of interest in this paper, $D > 2$, we shall show explicitly that at the levels $n = \frac{1}{2}$ and $n = 1$, the norms for the physical states are non-negative, too. Usually, it seems to be the case that the absence of negative-norm states at low-lying levels provides a reliable indication of the ghost freedom of the theory.

At level $n=\frac{1}{2}$, there are vector states which have fermion charge ℓ equal to $+1$ or -1. Without loss of generality, we shall focus on the $\ell=+1$ states, for which the ground-state operator is given by (5) with

$$R(z)=\bar\xi_\mu\psi^\mu(z)\ . \tag{16}$$

In addition to the J_0 and L_0 constraints, as given in (10a,b), there is, in this case, one other nontrivial constraint implied by the physical-state conditions (4), namely $G^-_{1/2}\big|p\big\rangle=0$. This implies

$$\bar\xi_\mu\beta^\mu-\bar\xi_0\alpha_0\ =0\ . \tag{17}$$

From this, we can solve for $\bar\xi_0$ in terms of $\bar\xi_i$:

$$\bar\xi_0=\frac{\bar\xi_i\beta_i}{\beta_0+\alpha_0}\ . \tag{18}$$

The norm $\mathcal{N}$ of these states is

$$\mathcal{N}=\bar\xi^*_\mu\bar\xi^\mu=-\bar\xi^*_0\bar\xi_0+\bar\xi^*_i\bar\xi_i\ . \tag{19}$$

Substituting (18) into (19), we can write $\mathcal{N}$ as

$$\begin{aligned}\mathcal{N}&=\bar\xi^*_i\Big(\delta_{ij}-V^*_iV_j\Big)\bar\xi_j\\&\equiv\bar\xi^*_iN_{ij}\bar\xi_j\ ,\end{aligned} \tag{20}$$

where $i=1,\ 2,\ \ldots,\ D-1$ and

$$V_i=\frac{\beta_i}{\beta_0+\alpha_0}\ . \tag{21}$$

It is easy to see that there are $D-2$ transverse eigenvectors of N_{ij} with eigenvalue $+1$, whilst the eigenvector parallel to V_i has eigenvalue $1-V^*_iV_i$, which is zero by virtue of the J_0 and L_0 constraints given in (10a) and (10b). It follows that of the $D-1$ independent complex polarisation states, one has zero norm, whilst the remaining $D-2$ states have positive norm. A completely analogous discussion can be given for the $\ell=-1$ vector states at level $n=\frac{1}{2}$, leading to the same result for the numbers of postive-norm and zero-norm states.

At level $n=1$ we know from equation (6) that the fermion charge ℓ can be $-2,\ 0,$ or $+2$. The $\ell=-2$ case essentially gives the same result for the norms as the $\ell=+2$ case; we shall therefore just consider the $\ell=+2$ and $\ell=0$ cases. For $\ell=+2$, the ground-state operator $R(z)$ in equation (5) takes the form

$$R(z)=\bar\xi_{\mu\nu}\psi^\mu\psi^\nu\ . \tag{22}$$

In addition to the J_0 and L_0 constraints, there is again just one other nontrivial constraint, namely $G^-_{1/2}\big|p\big\rangle=0$. This implies that the antisymmetric polarisation tensor $\bar\xi_{\mu\nu}$ must satisfy the following transversality condition:

$$\bar\beta^*_0\bar\xi_{0\nu}+\beta_i\bar\xi_{i\nu}=0\ . \tag{23}$$

Thus we can solve for $\bar{\xi}_{0i}$ in terms of $\bar{\xi}_{ij}$, and substitute this into the norm $\bar{\xi}^*_{\mu\nu}\bar{\xi}^{\mu\nu}$ to show that it is positive semi-definite: Of the $\frac{1}{2}(D-1)(D-2)$ independent complex components of the polarisation tensor $\bar{\xi}_{\mu\nu}$, we find that $D-2$ give rise to zero-norm states, whilst the the remaining $\frac{1}{2}(D-2)(D-3)$ give states of positive norm.

For $\ell = 0$ states at level $n = 1$, $R(z)$ of equation (5) takes the form

$$R(z) = \varepsilon_{\mu\nu}\psi^\mu\bar{\psi}^\nu + \bar{\xi}_\mu\partial\phi^\mu + \xi_\mu\partial\bar{\phi}^\mu \ . \tag{24}$$

In this case, in addition to the J_0 and L_0 constraints, we have three other independent nontrivial constraints, coming from J_1, $G^+_{1/2}$ and $G^-_{1/2}$. They give, respectively,

$$\begin{aligned} \varepsilon^\mu{}_\mu &= \alpha_0\left(\xi_0 - \bar{\xi}_0\right) \ , && (25a)\\ \bar{\xi}^\mu &= \beta^*_\nu \ \varepsilon^{\mu\nu} \ , && (25b)\\ \xi^\mu &= -\bar{\beta}^*_\nu \ \varepsilon^{\nu\mu} \ . && (25c) \end{aligned}$$

The norm $\mathcal{N}$ for these states is given by

$$\mathcal{N} = \varepsilon^*_{\mu\nu}\varepsilon^{\mu\nu} + \bar{\xi}^*_\mu\bar{\xi}^\mu + \xi^*_\mu\xi^\mu \ . \tag{26}$$

From (25b) and (25c) we may eliminate $\bar{\xi}^\mu$ and ξ^μ in (26), and express the norm purely in terms of $\varepsilon_{\mu\nu}$, subject to the constraint implied by (25a). It is convenient to decompose $\varepsilon_{\mu\nu}$ into irreducible transverse and traceless parts, in the following manner:

$$\varepsilon_{\mu\nu} = \varepsilon^{TT}_{\mu\nu} + \bar{\varepsilon}^T_\mu\beta_\nu + \bar{\beta}_\mu\varepsilon^T_\nu + \lambda\bar{\beta}_\mu\beta_\nu + \kappa\eta_{\mu\nu} \ , \tag{27}$$

where

$$\begin{aligned} \varepsilon^{TT\mu}_\mu = 0 \ , \quad \varepsilon^{TT}_{\mu\nu}\beta^{\nu*} = 0 \ , \quad \bar{\beta}^{\mu*}\varepsilon^{TT}_{\mu\nu} = 0 \ ; \\ \varepsilon^T_\mu\beta^{\mu*} = 0 \ , \quad \bar{\beta}^{\mu*}\bar{\varepsilon}^T_\mu = 0 \ . \end{aligned} \tag{28}$$

Substituting (27) into (25a-c), we find that κ and λ are related by

$$\tfrac{1}{3}(c-6)\kappa + (\lambda + 2\kappa)\Big(1 - \tfrac{1}{2}\alpha_0\left(\beta_0 + \bar{\beta}_0\right)\Big) = 0 \ , \tag{29}$$

where we have made use of equations (2), (10a,b) and (15a,b). Thus λ and κ are not independent, and together represent just one complex state. Now we find that the norm $\mathcal{N}$ is given by

$$\mathcal{N} = \varepsilon^{TT*}_{\mu\nu}\varepsilon^{TT\mu\nu} + \tfrac{1}{3}(6-c)\kappa^*\kappa - (\lambda + 2\kappa)^*(\lambda + 2\kappa) \ . \tag{30}$$

The criticality condition $c = 6$ implies, from equation (29), that $\lambda + 2\kappa = 0$. Consequently, the last two terms in (30) vanish, showing that the corresponding state has zero norm. The absence of ε^T_μ and $\bar{\varepsilon}^T_\mu$ in (30) implies that these components of the decomposition (27) of $\varepsilon_{\mu\nu}$ are associated with zero-norm states, numbering $2D - 2$ in all.

Solving for ε^{TT}_{00}, ε^{TT}_{0i} and ε^{TT}_{i0}, using (28), the remaining terms in (30) give

$$\mathcal{N} = \varepsilon^{TT*}_{ij}\varepsilon^{TT}_{ij} - \varepsilon^{TT*}_{ij}\varepsilon^{TT}_{ik}\bar{V}^*_j\bar{V}_k - \varepsilon^{TT*}_{ji}\varepsilon^{TT}_{ki}V^*_jV_k + \varepsilon^{TT*}_{ij}\varepsilon^{TT}_{kl}V^*_i\bar{V}^*_jV_k\bar{V}_l\ , \tag{31}$$

where $i,\ j,\ k,\ l = 1,\ 2,\ \ldots,\ D-1$, V_i is given in (21) and $\bar{V}_i$ is given by an analogous expression with β_μ replaced by $\bar{\beta}_\mu$. It is straightforward to see that of the $(D-1)^2$ complex states, the $(D-2)^2$ transverse states have positive norm, whilst the remaining $2D-3$ states have zero norm.

With these results, we have demonstrated the absence of negative-norm states at levels $n = \frac{1}{2}$ and $n = 1$. Although we have not considered a general no-ghost theorem here, we would expect that if negative-norm states were to occur, they would have arisen already at the levels that we have examined. The theories we are considering here differ qualitatively from the usual $D = 2$ superstring, for which the no-ghost theorem has been discussed [3,5], since the extra spatial dimensions allow the existence of positive-norm as well as zero-norm physical states at levels $n > 0$. As usual in string theory, the occurrence of the zero-norm states indicates that the physical states at level $n > 0$ describe gauge fields. Note also that when $D > 2$ not only do positive-norm excited states occur, but also the level number n can take half-integer as well as integer values. This contrasts with the $D = 2$ case where, since $\alpha_0 = 0$, equations (6) and (10a) imply that n can take only integer values.

The most striking feature arising in the $D > 2$ case is that the momentum component $\tilde{k}_0$ in the direction of the real time coordinate $\tilde{\varphi}^0$ is frozen to a specific value for each physical state, as given by (11). Because of this, the coordinate $\tilde{\varphi}^0$ does not describe an observable spacetime dimension. (This is somewhat analogous to Kaluza-Klein dimensional reduction, where one truncates to the zero-momentum mode in the Fourier expansion of the coordinate dependence for a compactified coordinate, thereby obtaining a theory in one less spacetime dimension.) Thus we are effectively left with just one real time dimension, parametrised by the coordinate φ^0. Note that this momentum-freezing phenomenon does not occur in the usual $N = 2$ superstring in $D = 2$ complex dimensions, since then α_0 is zero and so the momentum component $\tilde{k}_0$ will not be constrained by (10a).

In a system which is Poincaré invariant, mass is defined by the Poincaré Casimir operator $-P^\mu P_\mu$. For $N = 2$ superstring theories, the spacetime has a complex structure, and the Lorentz subgroup $SO(2, 2D-2)$ is replaced by $U(1, D-1)$. When $D > 2$, the presence of the background charges will break this symmetry further. Thus one cannot have a clear definition of the "mass" such as the Poincaré Casimir operator. In this case, from the mass-shell condition (10b) there are two natural mass-type operators that one might define [4]: $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, given by

$$\mathcal{M}^2 \equiv -k^*_\mu k^\mu - \tilde{k}^*_i\tilde{k}_i\ , \tag{32a}$$

$$\widetilde{\mathcal{M}}^2 \equiv \hat{k}_0\hat{k}_0 - k^*_ik_i - \tilde{k}^*_i\tilde{k}_i\ . \tag{32b}$$

It follows from (10a) and (10b) that the spectra for $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are

$$\mathcal{M}^2 = 2n - \frac{\ell^2}{2\alpha_0^2} \ , \tag{33a}$$

$$\widetilde{\mathcal{M}}^2 = 2n - \frac{\ell^2}{2\alpha_0^2} - \tfrac{1}{2}\alpha_0^2 \ . \tag{33b}$$

For now, we reserve judgement on which, if either, of these provides a useful generalisation of the notion of mass. Note that these formulae would suggest that the theories would contain infinite numbers of arbitrarily massive tachyonic states corresponding to sufficiently large ℓ values, within the range given by (6), at given level n. This problem could be overcome by truncating the states of the theory to just the $\ell = 0$ sector. It is worth noting that this truncation would have the consequences that only levels with n equal to an integer would survive, and also that the momentum $\tilde{k}_0$ of all surviving physical states would be frozen to the value zero.

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