

Generation Efficiencies for Propagating Modes in a Supersolid

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Using Andreev and Lifshitz's supersolid hydrodynamics, we obtain the propagating longitudinal modes at non-zero applied pressure P_a (necessary for solid ${}^4\text{He}$), and their generation efficiencies by heaters and transducers. For small P_a , a solid develops an internal pressure $P \sim P_a^2$. This theory has stress contributions both from the lattice and an internal pressure P . Because both types of stress are included, the normal mode analysis differs from previous works. Not surprisingly, transducers are significantly more efficient at producing elastic waves and heaters are significantly more efficient at producing fourth sound waves. We take the system to be isotropic, which should apply to systems that are glassy or consist of many crystallites; the results should also apply, at least qualitatively, to single-crystal hcp ${}^4\text{He}$.

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I. INTRODUCTION

In 1969 Andreev and Lifshitz developed a theory of supersolids.¹ Although the microscopic physical description was for flow of vacancies, the macroscopic equations did not depend on vacancies in an essential fashion. At about the same time Thouless² and Chester³ both suggested the possibility of superflow in a solid by vacancies. In addition, Leggett⁴ pointed out the possibility of Non-Classical Rotational Inertia (NCRI) associated with quantum-mechanical flow via a superfluid velocity (a phase gradient) opposite the local velocity of the rotating lattice.

Since the observation of NCRI by Kim and Chan,^{5,6} a number of laboratories have reproduced their work.⁷⁻¹⁴ (For reviews that emphasize experiment, see Refs. 15 and 16.) Were NCRI the sole criterion for superflow of solids, there would be strong reason to accept that such superflow has been observed. However, a supersolid should also have other properties, including a fourth sound-like mode, as predicted by Andreev and Lifshitz, and modified elastic waves with higher velocities, since the superfluid mass does not participate in the motion. (We remind the reader that a fourth sound mode in superfluid ${}^4\text{He}$ occurs only when the normal fluid is entrained by a porous medium; in the present case the lattice serves as the porous medium.) Neither a fourth sound mode nor velocity shifts have been observed.¹⁷⁻¹⁹ However, a stiffened shear response *is* observed,^{20,21} although not enough to explain the observed NCRI.¹⁶ Note also recent work indicating that supersolidity in ${}^4\text{He}$ can only occur below 55 mK.²²

As a guide to experiments to observe the fourth sound mode,¹⁹ the present work calculates various quantities relevant to its observation, such as the relative efficiencies of a transducer and a heater in producing both longitudinal elastic waves and fourth sound waves. It also considers the effect of a non-zero applied pressure P_a ; to solidify ${}^4\text{He}$, even near $T = 0$, requires $P_a \gtrsim 25$ bar. To our knowledge, previous works have not included the effect of P_a .

Although we believe that vacancies are essential to a microscopic understanding of superflow in solids, in the hydrodynamic theory they play no fundamental role, other than as an additional variable largely tied to diffusion. Indeed, we believe that the hydrodynamic theory is more likely to describe a supersolid related to the NCRI effect proposed by Leggett than to vacancy superflow.

Most of the present work assumes that the system is isotropic. One effect this has is that the superfluid density, which properly is a second rank tensor $\vec{\rho}_s$, is proportional to the unit matrix, so we take $\vec{\rho}_s \approx \mathbb{1}\rho_s$.^{23,24} We then write the superfluid fraction as

$$f_s = \frac{\rho_s}{\rho}, \quad (1)$$

where ρ_s is the superfluid density and ρ is the total (mass) density. f_s is unity in a superfluid at low temperatures. However, in putative supersolid ${}^4\text{He}$, the measured NCRI fraction, which if due to superflow should be equated to f_s , is never greater than about 0.2. The *effective* normal fraction f_n thus has the curious property of being not less than 0.8, although at $T = 0$ there are no excitations to destroy the superflow. We have previously noted this difficulty,²⁵ and proposed that the lattice be given a mass fraction f_L , in addition to a contribution f_n^{ex} due to excitations, so that $1 = f_s + f_n^{ex} + f_L$. This permits, at $T = 0$, no excitations (so $f_n^{ex} = 0$) but $f_s < 1$. In this viewpoint, the lattice velocity is identified with \dot{u}_i , where u_i is the lattice displacement, and the effective normal fluid fraction f_n is the sum of f_L and a part f_n^{ex} due to excitations: $f_n = f_n^{ex} + f_L$.

It is known that the more annealed (and thus more crystalline) the sample of ${}^4\text{He}$, the smaller the NCRI fraction.¹¹ Likewise it is known that the more quenched the sample, the larger the NCRI fraction.¹² Hence the supersolidity is more likely to occur for less crystalline samples, which might be either glassy²⁶ or consist of a large number of small crystallites.²⁷ In both of these cases an acoustic probe is likely to take a rotational average, thus making the system behave more like an isotropic system than a crystal. Therefore we consider systems whose

macroscopic properties are isotropic. If a pure crystal of hcp ^4He were to be supersolid, then the results we obtain will be only an approximation; nevertheless they will be a useful guide for experiment.

We also note that we are working in the linear regime, where the disturbances produced by a heater or transducer are expected to be only a small perturbation, as is assumed in all theories of this sort. It is possible that solid ^4He is ultrasensitive to temperature or to stress (e.g. if it is a glass, perhaps the atoms can be driven off their sites by a transducer). Nevertheless, there should always be a linear regime; Ref. 19 notes that their membrane for producing putative fourth sound produced strains much below the critical value.

The present work is intended to be self-contained, although we do not derive the equations of motion, which may be most explicitly obtained from Sect. IV of Ref. 28,²⁹ nor do we derive the P_a dependence of various quantities, which is done explicitly in Ref. 30.

Sections II and III present the thermodynamics and equations of motion, respectively. Section IV studies the eigenfrequency and eigenmodes for each of the longitudinal propagating modes. Section V finds, for each mode, the stress and temperature response in terms of the normal fluid velocity. Section VI finds and discusses the efficiency of generating each propagating mode by a transducer and a heater. Section VII provides a brief summary of the results.

One of the features of the hydrodynamic theory of supersolids is that it contains stress due to both internal pressure P and the lattice, in order to permit the system properties to continuously transform into those of a superfluid. This means that under an applied pressure P_a the pressure and the lattice each take up a part of it. We believe that use of both a pressure and a lattice stress is needed not merely for solid ^4He but for other solids as well, particularly those under pressure or with point defects that are not in equilibrium.³⁰ Appendix A discusses the relationship between the internal pressure P and the applied pressure P_a , which we estimate using experimental data and results from Ref. 30. Appendix B discusses the relative sizes of velocities and strains in a crystal under an applied pressure. Appendix C finds the relative size of two thermodynamic derivatives of temperature T that appear in the generation efficiencies.

II. THERMODYNAMICS

The thermodynamic equations for a supersolid are given in terms of the energy density ϵ , entropy density s , unsymmetrized strain $w_{ij} = \partial_i u_j$, mass density ρ , superfluid velocity \vec{v}_s , and momentum density

$$\vec{g} = \rho_n \vec{v}_n + \rho_s \vec{v}_s, \quad (2)$$

and their thermodynamically conjugate quantities.^{1,28} Here \vec{v}_n is the normal fluid velocity. Specifically,

$$d\epsilon = T ds + \lambda_{ik} dw_{ik} + \mu d\rho + \vec{v}_n \cdot d\vec{g} + \vec{j}_s \cdot d\vec{v}_s, \quad (3)$$

$$\epsilon = -P + Ts + \lambda_{ik} w_{ik} + \mu\rho + \vec{v}_n \cdot \vec{g} + \vec{j}_s \cdot \vec{v}_s, \quad (4)$$

$$0 = -dP + s dT + w_{ik} d\lambda_{ik} + \rho d\mu + \vec{g} \cdot d\vec{v}_n + \vec{v}_s \cdot d\vec{j}_s. \quad (5)$$

Here the thermodynamically conjugate quantities are temperature T , (unsymmetrized) elastic tensor density λ_{ik} (with units of pressure P), chemical potential μ (with units of velocity squared), normal fluid velocity \vec{v}_n , and

$$\vec{j}_s = \vec{g} - \rho \vec{v}_n = \rho_s (\vec{v}_s - \vec{v}_n). \quad (6)$$

Note that \vec{j}_s has units of momentum density but is invariant under Galilean boosts. That is, if both \vec{v}_n and \vec{v}_s are boosted by $\delta\vec{v}$, \vec{j}_s does not change.

We find it convenient to define

$$\vec{j}_n \equiv \rho \vec{v}_n, \quad (7)$$

so that

$$\vec{g} = \vec{j}_n + \vec{j}_s. \quad (8)$$

Unlike \vec{j}_s , the quantity \vec{j}_n is a momentum density both in units *and* in its properties under Galilean boosts; under a boost by $\delta\vec{v}$, both \vec{g} and \vec{j}_n are boosted by $\rho(\delta\vec{v})$.

III. HYDRODYNAMIC EQUATIONS

Eq. (3) shows that there are five independent thermodynamic variables. Two of them are scalars (s and ρ), one is a tensor ($w_{ik} = \partial_i u_k$) and two are vectors (\vec{g} and \vec{v}_s). In developing the hydrodynamic equations we will employ the first three, but we will use the two vectors \vec{j}_n and \vec{j}_s in place of \vec{g} and \vec{v}_s . For an ordinary solid, where \vec{v}_s does not appear, it is convenient to use the variables $\sigma \equiv s/\rho$, ρ , w_{ik} , and \vec{g} , since σ decouples from the other variables. Such decoupling does not occur for the supersolid. Note that one could also use the scalar variables T and ρ , or T and μ , or s and μ .

Unless otherwise specified, thermodynamic derivatives with respect to ρ , s , or w_{ik} are taken with the other two variables held constant.

We consider small amplitude excitations of the form $\exp[i(\vec{k} \cdot \vec{r} - \omega t)]$, where the wavevector \vec{k} is taken to be known. Then, with primes denoting deviations from equilibrium, in the absence of damping the equations of motion are given by^{1,28}

$$\dot{\rho}' + \partial_i g'_i = 0, \quad (9)$$

$$\dot{g}'_i - \partial_k \sigma'_{ik} = 0, \quad \sigma_{ik} \equiv \lambda_{ik} - P \delta_{ik}, \quad (10)$$

$$\dot{v}'_{si} + \partial_i \mu' = 0, \quad (11)$$

$$\dot{s}' + s \partial_i v'_i = 0, \quad (12)$$

$$\dot{u}'_i - v'_i = 0. \quad (13)$$

Rather than the stress tensor σ_{ik} , the momentum flux $\Pi_{ik} = -\sigma_{ik} + \rho_s v_{si} v_{sk} + \rho_n v_{ni} v_{nk} \approx -\sigma_{ik}$ has also been employed,^{1,31} as well as $g_i = j_i$.^{1,31,32}

In terms of the thermodynamic variables ρ and s , we have

$$\mu' = \frac{\partial\mu}{\partial\rho}\rho' + \frac{\partial\mu}{\partial s}s' + \frac{\partial\mu}{\partial w_{jl}}w'_{jl}, \quad (14)$$

$$\sigma'_{ik} = \frac{\partial\sigma_{ik}}{\partial\rho}\rho' + \frac{\partial\sigma_{ik}}{\partial s}s' + \frac{\partial\sigma_{ik}}{\partial w_{jl}}w'_{jl}. \quad (15)$$

The equations of motion (9), (12), and (13) directly give

$$\rho' = \frac{k_i g'_i}{\omega}, \quad (16)$$

$$s' = s \frac{k_i v'_{ni}}{\omega}, \quad (17)$$

$$w'_{ij} = ik_i u'_j = -\frac{k_i v'_{nj}}{\omega}. \quad (18)$$

The other two equations of motion, (10) and (11), can now be written in terms of v'_{s_i} and v'_{n_i} , or, equivalently, j'_{s_i} and j'_{n_i} .

From (15)-(18), momentum conservation (10) gives

$$\begin{aligned} \omega g'_i &= -k_k \sigma'_{ik} \\ &= -\frac{k_k k_l}{\omega} \frac{\partial\sigma_{ik}}{\partial\rho} g'_l - \frac{k_k k_l}{\omega} s \frac{\partial\sigma_{ik}}{\partial s} v'_{nl} + \frac{k_k k_j}{\omega} \frac{\partial\sigma_{ik}}{\partial w_{jl}} v'_{nl}. \end{aligned} \quad (19)$$

We now rearrange to use the variables j'_n and j'_s . From (7)-(8), multiplying (19) by ω gives

$$\begin{aligned} 0 &= \left[\omega^2 \delta_{il} + k_k k_l \frac{\partial\sigma_{ik}}{\partial\rho} \right] j'_{sl} + \left[\omega^2 \delta_{il} + \left(k_k k_l \frac{\partial\sigma_{ik}}{\partial\rho} \right. \right. \\ &\quad \left. \left. + k_k k_l \frac{s}{\rho} \frac{\partial\sigma_{ik}}{\partial s} - k_k k_j \frac{1}{\rho} \frac{\partial\sigma_{ik}}{\partial w_{jl}} \right) \right] j'_{nl}. \end{aligned} \quad (20)$$

Likewise, from (14) and (16)-(18), the superfluid equation of motion (11) gives

$$\omega v'_{s_i} = k_i \mu' = \frac{k_i k_l}{\omega} \frac{\partial\mu}{\partial\rho} g'_l + \frac{k_i k_l}{\omega} s \frac{\partial\mu}{\partial s} v'_{nl} - \frac{k_i k_j}{\omega} \frac{\partial\mu}{\partial w_{jl}} v'_{nl}. \quad (21)$$

From (7)-(8) and $\rho_s v'_{s_i} = j'_{s_i} - (\rho_s/\rho) j'_{n_i}$, multiplying (21) by $\rho_s \omega$ and rearranging gives

$$\begin{aligned} 0 &= \left[\omega^2 \delta_{il} - k_i k_l f_s \rho \frac{\partial\mu}{\partial\rho} \right] j'_{sl} + f_s \left[\omega^2 \delta_{il} \right. \\ &\quad \left. - \rho \left(k_i k_l \frac{\partial\mu}{\partial\rho} + k_i k_l \frac{s}{\rho} \frac{\partial\mu}{\partial s} - k_i k_j \frac{1}{\rho} \frac{\partial\mu}{\partial w_{jl}} \right) \right] j'_{nl}. \end{aligned} \quad (22)$$

Equations (20) and (22) yield the normal mode frequencies and their eigenvectors (the ratio of the responses of the normal and superfluid currents). In what follows we consider only an isotropic solid. The effect this constraint has on (20) and (22) is that the second-rank tensors are all proportional to the unit tensor, and the term

$k_k k_j \rho^{-1} (\partial\sigma_{ik}/\partial w_{jl})$ in (20) contains two terms, one proportional to δ_{il} and one proportional to $k_i k_l$.

Taking the dot product of these equations with k_i then gives two equations in the unknowns ω^2 , $\vec{k} \cdot \vec{j}'_s$, and $\vec{k} \cdot \vec{j}'_n$. This yields ω^2 and the ratio $\vec{k} \cdot \vec{j}'_s / \vec{k} \cdot \vec{j}'_n$. In addition, taking the cross-product of (22) with k_i gives, since $\vec{k} \times \vec{v}_s = \vec{0}$, identically zero. Further, taking the cross-product of (20) with k_i gives an equation having terms proportional to ω^2 and k^2 , both multiplying $\vec{k} \times \vec{v}_n$.

There are two ways to solve the resulting equations for $\vec{k} \cdot \vec{j}'_s$, $\vec{k} \cdot \vec{j}'_n$, and $\vec{k} \times \vec{v}_n$. One solution is to take $\vec{k} \times \vec{v}_n = \vec{0}$ and $k_l j'_{sl} \neq 0$, $k_l j'_{nl} \neq 0$ (purely longitudinal modes), with the frequencies determined by the two equations in the unknowns ω^2 , $\vec{k} \cdot \vec{j}'_s$, and $\vec{k} \cdot \vec{j}'_n$. Hence this set of modes is purely longitudinal. The other solution is to take $\vec{k} \times \vec{v}_n \neq \vec{0}$ and $\vec{k} \cdot \vec{j}'_s = 0$, $\vec{k} \cdot \vec{j}'_n = 0$ (purely transverse modes), with the frequencies determined by the the cross-product of (20) with k_i . \vec{v}_s does not participate in the transverse modes, so their mass weighting involves only ρ/ρ_n , and their frequencies squared should be higher than in the normal solid by ρ/ρ_n . To our knowledge such an effect has not been observed.³³

IV. LONGITUDINAL EIGENFREQUENCIES AND EIGENMODES

Recall that, unless otherwise specified, thermodynamic derivatives with respect to ρ , s , or w_{ik} are taken with the other two variables held constant.

A. Some Properties and Definitions

We now compute the quantity $\partial\sigma_{ik}/\partial\rho$, which appears in (20). We take the strain response of a solid to P_a to be isotropic (i.e., $w_{ik}^{(0)} \sim \delta_{ik} w_{ll}^{(0)}$, where the superscript (0) denotes the static value). Recall that w_{ik} is unsymmetrized; here we take only the static part, due to P_a , to be symmetric, as does Ref. 34. Then, by Ref. 34,

$$\lambda_{ik}^{(0)} = \left(K - \frac{2}{3} \mu_V \right) \delta_{ik} w_{ll}^{(0)} + \mu_V \left(w_{ik}^{(0)} + w_{ki}^{(0)} \right) \sim \delta_{ik} w_{ll}^{(0)}, \quad (23)$$

so that we can write

$$\frac{\partial\lambda_{ik}}{\partial\rho} \equiv \frac{\partial\lambda}{\partial\rho} \delta_{ik}. \quad (24)$$

Here, K and μ_V are the respective bulk and shear moduli, with units of P ; μ_V is completely distinct from μ . Eq. (24) is also employed in Ref. 30, although there $\sigma = s/\rho$ is held constant rather than s . At $T \approx 0$, the difference should be negligible. Thus we can write

$$\frac{\partial\sigma_{ik}}{\partial\rho} = \frac{\partial\lambda_{ik}}{\partial\rho} - \delta_{ik} \frac{\partial P}{\partial\rho} = \left[\frac{\partial\lambda}{\partial\rho} - \frac{\partial P}{\partial\rho} \right] \delta_{ik} \equiv \frac{\partial\tilde{\sigma}}{\partial\rho} \delta_{ik}, \quad (25)$$

where we use $\tilde{\sigma}$ to distinguish a stress (with the same units as σ_{ik}) from $\sigma = s/\rho$; $\tilde{\sigma}$ and σ are not related. Note that $\partial\tilde{\sigma}/\partial\rho$ is not a true derivative, merely a definition; further, we do not here define a $\tilde{\sigma}$.

We now compute the quantity $\partial\sigma_{ik}/\partial w_{jl}$, which also appears in (20). Since Ref. 30 shows that $(\partial P/\partial w_{ik}) \sim w_{ik}^{(0)} \sim w_{il}^{(0)} \delta_{ik}$, we can write

$$\frac{\partial P}{\partial w_{ik}} \equiv \frac{\partial P}{\partial w} \delta_{ik}. \quad (26)$$

$\partial P/\partial w$ is evaluated in Ref. 30, and is given in Appendix B. We also use the definitions

$$\frac{\partial \lambda}{\partial w} \equiv K + \frac{4}{3} \mu_V, \quad \frac{\partial \tilde{\sigma}}{\partial w} \equiv \frac{\partial \lambda}{\partial w} - \frac{\partial P}{\partial w}, \quad (27)$$

As above, $\partial\tilde{\sigma}/\partial w$ is not a true derivative, merely a definition. Eqs. (23) and (26) then give

$$\frac{\partial \sigma_{ik}}{\partial w_{jl}} = \left(\frac{\partial \tilde{\sigma}}{\partial w} - 2\mu_V \right) \delta_{ik} \delta_{jl} + \mu_V \delta_{il} \delta_{jk} + \mu_V \delta_{ij} \delta_{kl}. \quad (28)$$

Thus,

$$k_k k_j \frac{\partial \sigma_{ik}}{\partial w_{jl}} j'_{nl} = \left(\frac{\partial \tilde{\sigma}}{\partial w} - \mu_V \right) k_i (\vec{k} \cdot \vec{j}'_n) + \mu_V k^2 j'_{ni}. \quad (29)$$

For $\vec{k} \cdot \vec{j}'_n \neq 0$ and $\vec{k} \times \vec{j}'_n = 0$ (the longitudinal case), $k_i (\vec{k} \cdot \vec{j}'_n) = k^2 j'_{ni}$ so that (29) gives

$$k_k k_j \frac{\partial \sigma_{ik}}{\partial w_{jl}} j'_{nl} = \frac{\partial \tilde{\sigma}}{\partial w} k^2 \mu_V j'_{ni}. \quad (30)$$

It is convenient to define the velocities c_0 and c_1 , which satisfy

$$c_0^2 \equiv \rho \frac{\partial \mu}{\partial \rho}, \quad (31)$$

$$c_1^2 \equiv -\frac{\partial \tilde{\sigma}}{\partial \rho} + \frac{1}{\rho} \frac{\partial \tilde{\sigma}}{\partial w}. \quad (32)$$

If σ , rather than s , were held constant, then c_0 would be the sound velocity in an ordinary fluid, and c_1 would be the velocity of sound in an ordinary solid with no superflow.³⁰ Using the Gibbs-Duhem relation (5) and neglecting thermal expansion and terms second order in velocities gives

$$\frac{\partial P}{\partial \rho} = c_0^2 + w_{jl} \frac{\partial \lambda_{jl}}{\partial \rho}. \quad (33)$$

Then, eq. (25) gives

$$\frac{\partial \tilde{\sigma}}{\partial \rho} \approx \left(1 - w_{il}^{(0)} \right) \frac{\partial \lambda}{\partial \rho} - c_0^2. \quad (34)$$

In the following we use either (25) or (34), depending on convenience.

B. Reducing the Equations of Motion

Momentum Equation (20): We take $s(\partial\sigma_{ik}/\partial s) \rightarrow 0$, which should be a reasonable approximation for solid ⁴He at low temperatures, both because $s \rightarrow 0$ as $T \rightarrow 0$, and because K and μ_V (and therefore λ_{ik} at constant w_{jl}) should be nearly independent of s . Substituting (25) and (30) into (20) and using (32) and (34) then gives, for a purely longitudinal mode,

$$0 = \left[\omega^2 - \left(\tilde{c}^2 + w_{il}^{(0)} \frac{\partial \lambda}{\partial \rho} \right) k^2 \right] j'_s + [\omega^2 - c_1^2 k^2] j'_n; \quad (35)$$

here we define, to simplify the equations,

$$\tilde{c}^2 \equiv c_0^2 - \frac{\partial \lambda}{\partial \rho}. \quad (36)$$

Appendix B finds that $|\partial\lambda/\partial\rho| \gg c_0^2$, so that $\tilde{c}^2 \approx -\partial\lambda/\partial\rho$. It also finds that \tilde{c}^2 is expected to be positive, and first order in P_a/K . Further, it shows that for $P_a \ll K$ we have $c_1^2 \gg \tilde{c}^2 \gg c_0^2$.

Superfluid Equation (22): A Maxwell relation that follows from (3), combined with (24), gives

$$\frac{\partial \mu}{\partial w_{jl}} = \frac{\partial \lambda_{jl}}{\partial \rho} = \delta_{jl} \frac{\partial \lambda}{\partial \rho}. \quad (37)$$

Then, neglecting $s(\partial\mu/\partial s) = s(\partial T/\partial\rho) \sim T^4$, and taking the mode to be purely longitudinal, eq. (22) gives

$$0 = [\omega^2 - f_s c_0^2 k^2] j'_s + f_s [\omega^2 - k^2 \tilde{c}^2] j'_n. \quad (38)$$

We use (35) and (38) first to find the longitudinal mode frequencies, then to find the superfluid-to-normal ratios of current density and velocity in each longitudinal mode. For $f_s \rightarrow 0$, eq. (38) gives either $\omega^2 = f_s c_0^2 k^2$ (fourth sound) or $j'_s = 0$ (no superflow). In the latter case, substitution into (35) then gives $\omega^2 = c_1^2 k^2$ (first sound).

C. Longitudinal Mode Frequencies

Eqs. (35) and (38) yield

$$0 = \omega^4 (1 - f_s) - \omega^2 k^2 \left[c_1^2 + f_s c_0^2 - f_s \left(2\tilde{c}^2 + w_{il}^{(0)} \frac{\partial \lambda}{\partial \rho} \right) \right] + k^4 f_s \left[c_1^2 c_0^2 - \tilde{c}^2 \left(\tilde{c}^2 + w_{il}^{(0)} \frac{\partial \lambda}{\partial \rho} \right) \right]. \quad (39)$$

Solving (39) to first order in f_s gives

$$\frac{\omega_1^2}{k^2} \equiv \frac{\omega_+^2}{k^2} = c_1^2 + f_s \left[c_1^2 - 2\tilde{c}^2 + \frac{\tilde{c}^4}{c_1^2} + w_{il}^{(0)} \frac{\partial \lambda}{\partial \rho} \left(\frac{\tilde{c}^2}{c_1^2} - 1 \right) \right], \quad (40)$$

and

$$\frac{\omega_4^2}{k^2} \equiv \frac{\omega_-^2}{k^2} = f_s \left(c_0^2 - \frac{\tilde{c}^4}{c_1^2} - w_{il}^{(0)} \frac{\partial \lambda}{\partial \rho} \frac{\tilde{c}^2}{c_1^2} \right) \equiv f_s \tilde{c}_0^2. \quad (41)$$

In the limit where $c_0^2 \ll \partial\lambda/\partial\rho$ and $w_{ll}^{(0)} \ll 1$ (i.e., $P_a \ll K$),

$$\tilde{c}_0^2 \approx c_0^2 - \frac{\tilde{c}^4}{c_1^2}. \quad (42)$$

Appendix B finds that both terms on the right-hand-side of (42) are second order in P_a/K . Further, it shows that for $P_a \ll K$ we have $c_1^2 \gg \tilde{c}^2 \gg \tilde{c}_0^2$.

D. Longitudinal Mode Structure – Currents and Velocities

We now find the ratios of the normal fluid and superfluid response for both longitudinal modes. These ratios will be used to calculate, for each mode, the response to the stress and temperature produced by transducers and by heaters. We employ

$$\frac{v'_s}{v'_n} = \frac{\rho}{\rho_s} \frac{\rho_s(v'_s - v'_n)}{\rho v'_n} + 1 = \frac{1}{f_s} \frac{j'_s}{j'_n} + 1. \quad (43)$$

The ratios j'_s/j'_n for each mode can in principle be obtained from the normal mode frequencies and either of (35) or (38).

1. First Sound Mode Structure

From (38), with the subscript 1 denoting first sound,

$$\frac{j'_{s1}}{j'_{n1}} = -f_s \frac{\frac{\omega_1^2}{k^2} - \tilde{c}^2}{\frac{\omega_1^2}{k^2} - f_s c_0^2}. \quad (44)$$

Substituting ω_1^2 from (40), accurate to zeroth order in f_s , gives a ratio accurate to first order in f_s :

$$\frac{j'_{s1}}{j'_{n1}} \approx -f_s \left(1 - \frac{\tilde{c}^2}{c_1^2} \right). \quad (45)$$

Then, using (43), the ratio of superfluid velocity to normal velocity for first sound is

$$\frac{v'_{s1}}{v'_{n1}} \approx \frac{\tilde{c}^2}{c_1^2}. \quad (46)$$

Appendix B shows that $c_1^2 \gg \tilde{c}^2$, so $v'_{n1} \gg v'_{s1}$.

2. Fourth Sound Mode Structure

From (35), with subscript 4 denoting fourth sound,

$$\frac{j'_{s4}}{j'_{n4}} = -\frac{\frac{\omega_4^2}{k^2} - c_1^2}{\frac{\omega_4^2}{k^2} - \left[\tilde{c}^2 + w_{ll}^{(0)} \frac{\partial\lambda}{\partial\rho} \right]}. \quad (47)$$

With $\omega_4^2 \sim f_s$, for $f_s \ll 1$

$$\frac{j'_{s4}}{j'_{n4}} \approx -\frac{c_1^2}{\tilde{c}^2 + w_{ll}^{(0)} \frac{\partial\lambda}{\partial\rho}}. \quad (48)$$

Appendix B shows that if $P_a/K \ll 1$, then $w_{ll}^{(0)} \ll 1$ and $c_0^2 \ll \partial\lambda/\partial\rho$. Thus, eq. (36) gives $\tilde{c}^2 \approx -\partial\lambda/\partial\rho \gg w_{ll}^{(0)} \partial\lambda/\partial\rho$. Then,

$$\frac{j'_{s4}}{j'_{n4}} \approx -\frac{c_1^2}{\tilde{c}^2}. \quad (49)$$

Then, using (43), the ratio of superfluid velocity to normal velocity for fourth sound is, to lowest order in f_s ,

$$\frac{v'_{s4}}{v'_{n4}} \approx -\frac{c_1^2}{f_s \tilde{c}^2}. \quad (50)$$

Appendix B shows that $c_1^2 \gg \tilde{c}^2$, so $v'_{s4} \gg v'_{n4}$.

V. LONGITUDINAL MODES – STRESS AND TEMPERATURE RESPONSES

We now calculate the deviations from equilibrium of the longitudinal stress and temperature produced by a transducer and by a heater. We consider that only the σ'_{11} component of the stress is generated. For notational simplicity we employ $\hat{\sigma}' \equiv \sigma'_{11}$; recall that σ is reserved for the entropy/mass.

A. Stress

Conservation of momentum (10) yields

$$\hat{\sigma}' = -\frac{\omega g'}{k} = -\frac{\omega}{k} (j'_s + j'_n) = -\frac{\omega}{k} \left(\frac{j'_s}{j'_n} + 1 \right) \rho v'_{n1}. \quad (51)$$

Substituting the ratio j'_s/j'_n from (45) and (49) and $\omega_{1,4} = +c_{1,4}k$ from (40) and (41) into (51) gives the stress associated with each mode. For $f_s \ll 1$,

$$\hat{\sigma}'_1 \approx -c_1 \left[1 - f_s \left(1 - \frac{\tilde{c}^2}{c_1^2} \right) \right] \rho v'_{n1} \approx -\rho c_1 v'_{n1}, \quad (52)$$

$$\hat{\sigma}'_4 \approx f_s^{\frac{1}{2}} \frac{c_1^2}{\tilde{c}^2} \rho \tilde{c}_0 v'_{n4} = f_s^{\frac{1}{2}} \frac{c_1^2}{\tilde{c}^2} \rho \tilde{c}_0 \frac{v'_{n4}}{v'_{n1}} v'_{n1}. \quad (53)$$

where we have used $c_1^2 \gg \tilde{c}^2$ (see Appendix B).

The total stress deviation therefore is

$$\hat{\sigma}' = \hat{\sigma}'_1 + \hat{\sigma}'_4 \approx -c_1 \rho v'_{n1} \left[1 - f_s^{\frac{1}{2}} \tilde{c}_0 \frac{c_1}{\tilde{c}^2} \frac{v'_{n4}}{v'_{n1}} \right]. \quad (54)$$

The ratio v'_{n4}/v'_{n1} depends on the mode generator, to be discussed in the next section.

B. Temperature

The temperature deviation is less straightforward to obtain because it is a function of the variables s , ρ , and w_{jl} :

$$T' \approx \frac{\partial T}{\partial s} s' + \frac{\partial T}{\partial \rho} \rho' + \frac{\partial T}{\partial w_{jl}} w'_{jl}. \quad (55)$$

Since $\partial T/\partial w_{jl} = \partial \lambda_{jl}/\partial s$, and K and μ_V depend only weakly on s , by (23) we neglect $\partial T/\partial w_{jl}$. Substitution for ρ' and s' from (16) and (17) then yields

$$T' \approx s \frac{\partial T}{\partial s} \frac{k_i}{\omega} v'_{ni} + \frac{\partial T}{\partial \rho} \frac{k_i}{\omega} g'_i. \quad (56)$$

We earlier showed that the mode is longitudinal, so we drop the indices i . The identity $g' = \rho(1 + j'_s/j'_n) v'_n$ then yields, for both modes, that

$$T' \approx \frac{k}{\omega} v'_n \left[s \frac{\partial T}{\partial s} + \rho \frac{\partial T}{\partial \rho} \left(1 + \frac{j'_s}{j'_n} \right) \right]. \quad (57)$$

Substituting the ratio j'_s/j'_n from (45) and (49) and $\omega_{1,4} = +ck_{1,4}$ from (40) and (41) into (57) gives the temperature associated with each mode. To lowest order in f_s we obtain

$$T'_1 \approx \frac{v'_{n1}}{c_1} \left[s \frac{\partial T}{\partial s} + \rho \frac{\partial T}{\partial \rho} \right], \quad (58)$$

$$T'_4 \approx f_s^{-\frac{1}{2}} \frac{v'_{n4}}{\tilde{c}_0} \left[s \frac{\partial T}{\partial s} + \rho \frac{\partial T}{\partial \rho} \left(1 - \frac{c_1^2}{\tilde{c}_0^2} \right) \right]. \quad (59)$$

For a solid at low temperature, Appendix C gives $\rho(\partial T/\partial \rho) \approx 10s(\partial T/\partial s)$. In addition, for $P_a \ll K$ (as is the case here), Appendix B gives $c_1^2 \gg \tilde{c}_0^2$. Therefore (58)-(59) become

$$T'_1 \approx \rho \frac{\partial T}{\partial \rho} \frac{v'_{n1}}{c_1}, \quad (60)$$

$$T'_4 \approx -f_s^{-\frac{1}{2}} \rho \frac{\partial T}{\partial \rho} \frac{c_1^2}{\tilde{c}_0^2} \frac{v'_{n4}}{\tilde{c}_0} = -f_s^{-\frac{1}{2}} \rho \frac{\partial T}{\partial \rho} \frac{c_1^2}{\tilde{c}_0^2} \frac{v'_{n4}}{c_1} \frac{v'_{n1}}{\tilde{c}_0}. \quad (61)$$

The total temperature deviation therefore is

$$T' = T'_1 + T'_4 \approx \rho \frac{\partial T}{\partial \rho} \frac{v'_{n1}}{c_1} \left[1 - f_s^{-\frac{1}{2}} \frac{c_1^3}{\tilde{c}_0^2} \frac{v'_{n4}}{v'_{n1}} \right]. \quad (62)$$

The ratio v'_{n4}/v'_{n1} depends on the mode generator, to be discussed in the next section.

VI. LONGITUDINAL MODE GENERATION

A transducer produces, and therefore can be used to detect, stress deviations. A heater produces, and therefore can be used to detect, temperature deviations (when used as a detector, a heater is called a thermometer). To utilize the results of Sec. V, we find v'_{n4}/v'_{n1} for each device, then substitute it into (52)-(54) and (60)-(62) to find the respective stress and temperature deviations produced by transducers and heaters.

A. Transducer Properties

For a transducer we take $v'_s = v'_n$ (and therefore $j'_s = 0$) so that

$$0 = j'_s|_{\text{trn}} = [j'_{s1} + j'_{s4}]_{\text{trn}} = \left[\frac{j'_{s1}}{j'_{n1}} j'_{n1} + \frac{j'_{s4}}{j'_{n4}} j'_{n4} \right]_{\text{trn}}, \quad (63)$$

where the subscript “trn” denotes properties of a transducer. Then

$$\frac{v'_{n4}}{v'_{n1}} \Big|_{\text{trn}} = \frac{j'_{n4}}{j'_{n1}} \Big|_{\text{trn}} = -\frac{j'_{s1}/j'_{n1}}{j'_{s4}/j'_{n4}}. \quad (64)$$

Use of (45) and (49) yields

$$\frac{v'_{n4}}{v'_{n1}} \Big|_{\text{trn}} \approx -f_s \frac{\tilde{c}_0^2}{c_1^2} \left(1 - \frac{\tilde{c}_0^2}{c_1^2} \right) \approx -\frac{f_s \tilde{c}_0^2}{c_1^2}, \quad (65)$$

where we have taken $c_1^2 \gg \tilde{c}_0^2$ (see Appendix B).

Eq. (52) gives $\hat{\sigma}'_1$ in terms of v'_{n1} , regardless of generator. Use of (65) in (53)-(54) gives

$$\hat{\sigma}'_4|_{\text{trn}} \approx -f_s^{\frac{3}{2}} \rho \tilde{c}_0 v'_{n1}, \quad (66)$$

$$\hat{\sigma}'_1|_{\text{trn}} \approx -\rho c_1 v'_{n1} \left(1 + f_s^{\frac{3}{2}} \frac{\tilde{c}_0}{c_1} \right) \approx -\rho c_1 v'_{n1}, \quad (67)$$

for $f_s \ll 1$ and $\tilde{c}_0^2 \ll c_1^2$. Thus the stress produced by a transducer primarily goes into first sound, with a fraction $f_s^{3/2}(\tilde{c}_0/c_1)$ of the stress going into fourth sound. Eqs. (52) and (66) divided by (67) are the two entries in the top left of Table I.

Eq. (60) gives T'_1 in terms of v'_{n1} , regardless of generator. Substituting (65) into (61) yields

$$T'_4|_{\text{trn}} \approx f_s^{\frac{1}{2}} \rho \frac{\partial T}{\partial \rho} \frac{v'_{n1}}{\tilde{c}_0}. \quad (68)$$

By (60), $T'_4 = f_s^{1/2}(c_1/\tilde{c}_0)T'_1$, so T'_1 and T'_4 could be of the same order of magnitude. Eqs. (60) and (68) divided by (67) are the two entries in the top right of Table I.

B. Heater Properties

For a heater we take $g' = 0$, so that

$$0 = g'|_{\text{htr}} = [(j'_{s1} + j'_{n1}) + (j'_{s4} + j'_{n4})]_{\text{htr}} \\ = \left(\frac{j'_{s1}}{j'_{n1}} + 1 \right) j'_{n1}|_{\text{htr}} + \left(\frac{j'_{s4}}{j'_{n4}} + 1 \right) j'_{n4}|_{\text{htr}}, \quad (69)$$

where the subscript “htr” denotes properties of a heater. Then

$$\frac{v'_{n4}}{v'_{n1}} \Big|_{\text{htr}} = \frac{j'_{n4}}{j'_{n1}} \Big|_{\text{htr}} = -\frac{(j'_{s1}/j'_{n1}) + 1}{(j'_{s4}/j'_{n4}) + 1}, \quad (70)$$

and substitution from (45) and (49) yields, for $f_s \ll 1$,

$$\frac{v'_{n4}}{v'_{n1}} \Big|_{\text{htr}} \approx \frac{1}{\frac{c_1^2}{\tilde{c}_0^2} - 1} \approx \frac{\tilde{c}_0^2}{c_1^2}. \quad (71)$$

Here, we have used $c_1^2 \gg \tilde{c}_0^2$ (see Appendix B).

Eq. (60) gives T'_1 in terms of v'_{n1} , regardless of generator. Substitution of (71) into (61)-(62) gives

$$T'_4|_{\text{htr}} \approx -f_s^{-\frac{1}{2}} \rho \frac{\partial T}{\partial \rho} \frac{v'_{n1}}{\tilde{c}_0}, \quad (72)$$

$$T'|_{\text{htr}} \approx \rho \frac{\partial T}{\partial \rho} \frac{v'_{n1}}{c_1} \left[1 - f_s^{-\frac{1}{2}} \frac{c_1}{\tilde{c}_0} \right] \approx -f_s^{-\frac{1}{2}} \rho \frac{\partial T}{\partial \rho} \frac{v'_{n1}}{\tilde{c}_0}, \quad (73)$$

for $f_s \ll 1$ and $\tilde{c}_0^2 \ll c_1^2$. Thus the temperature produced by a heater primarily goes into fourth sound, with a fraction $f_s^{1/2}(\tilde{c}_0/c_1)$ of the temperature going into first sound. Eqs. (60) and (72) divided by (73) are the two entries in the bottom right of Table I.

Eq. (52) gives $\hat{\sigma}'_1$ in terms of v'_{n1} , regardless of generator. Substituting (71) into (53) yields

$$\hat{\sigma}'_4|_{\text{htr}} \approx f_s^{\frac{1}{2}} \rho \tilde{c}_0 v'_{n1}. \quad (74)$$

Since $f_s \ll 1$ and $\tilde{c}_0^2 \ll c_1^2$, we have $\hat{\sigma}'_1 \gg \hat{\sigma}'_4$. Eqs. (52) and (74) divided by (73) are the two entries in the bottom left of Table I.

C. Generation Efficiencies

A proper treatment of the response of a given detector (transducer or thermometer) to a given mode (first or fourth sound) would consider the incoming mode and what happens under reflection from the detector; this would give the net stress and temperature at the detector.³¹ We consider only the issue of generation.

A transducer generates mostly stress, $\hat{\sigma}'$. Thinking of the equation entries in Table I as a 4-by-2 matrix \mathcal{M} , \mathcal{M}_{11} shows that a transducer is efficient as a first sound generator. $\mathcal{M}_{21}/\mathcal{M}_{11}$ gives

$$\frac{\hat{\sigma}'_4}{\hat{\sigma}'_1} \approx f_s^{3/2} \frac{\tilde{c}_0}{c_1}. \quad (\text{transducer}) \quad (75)$$

For $f_s \ll 1$ and $\tilde{c}_0^2 \ll c_1^2$, this is negligible.

Although a transducer primarily produces stress, it also produces a small temperature deviation T' . $\mathcal{M}_{22}/\mathcal{M}_{12}$ gives

$$\frac{T'_4}{T'_1} \approx f_s^{1/2} \frac{c_1}{\tilde{c}_0}. \quad (\text{transducer}) \quad (76)$$

For $f_s \ll 1$ and $\tilde{c}_0^2 \ll c_1^2$, it is not clear which of the terms in (76) dominates.

A heater generates mostly temperature, T' , and \mathcal{M}_{42} shows that a heater is efficient as a fourth sound generator. $\mathcal{M}_{42}/\mathcal{M}_{32}$ gives

$$\left| \frac{T'_4}{T'_1} \right| \approx f_s^{-1/2} \frac{c_1}{\tilde{c}_0} \gg 1, \quad (\text{heater}) \quad (77)$$

which is large. In fact, for $f_s \ll 1$ and $\tilde{c}_0^2 \ll c_1^2$, $\mathcal{M}_{42} = T'_4/T' \approx 1$ and therefore nearly all of the temperature response corresponds to the fourth sound mode.

Although a heater primarily produces temperature, it also produces a small stress deviation $\hat{\sigma}'$. $\mathcal{M}_{41}/\mathcal{M}_{31}$ gives

$$\left| \frac{\hat{\sigma}'_4}{\hat{\sigma}'_1} \right| \approx f_s^{1/2} \frac{\tilde{c}_0}{c_1}. \quad (\text{heater}) \quad (78)$$

Therefore, with $f_s \ll 1$ and $\tilde{c}_0^2 \ll c_1^2$, eqs. (75) and (78) imply that stress deviations do not contribute an appreciable amount of fourth sound, whether produced by a transducer or a heater.

VII. SUMMARY

We have studied the implications of the Andreev and Lifshitz theory of supersolids for the generation of a fourth sound mode in a solid under an applied pressure P_a , including the relative efficiencies of a transducer and a heater in producing both longitudinal elastic waves and fourth sound waves. The present results apply when the bulk modulus $K \gg P_a$.

VIII. ACKNOWLEDGEMENTS

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Generator	Mode	Stress	Temperature
Transducer	1st Sound	$\frac{\hat{\sigma}'_1}{\hat{\sigma}'} \approx \frac{1}{1 + f_s^{\frac{3}{2}} \frac{\tilde{c}_0}{c_1}} \approx 1$	$\frac{T'_1}{\hat{\sigma}'} \approx -\frac{1}{c_1^2} \frac{\partial T}{\partial \rho}$
	4th Sound	$\frac{\hat{\sigma}'_4}{\hat{\sigma}'} \approx \frac{1}{1 + f_s^{-\frac{3}{2}} \frac{c_1}{\tilde{c}_0}} \approx f_s^{\frac{3}{2}} \frac{\tilde{c}_0}{c_1} \ll 1$	$\frac{T'_4}{\hat{\sigma}'} \approx -\frac{f_s^{\frac{1}{2}}}{c_1 \tilde{c}_0} \frac{\partial T}{\partial \rho} = \frac{f_s^{\frac{1}{2}} c_1}{\tilde{c}_0} \frac{T'_1}{\hat{\sigma}'}$
Heater	1st Sound	$\frac{\hat{\sigma}'_1}{T'} \approx f_s^{\frac{1}{2}} c_1 \tilde{c}_0 \frac{\partial \rho}{\partial T}$	$\frac{T'_1}{T'} \approx \frac{1}{1 - f_s^{-\frac{1}{2}} \frac{c_1}{\tilde{c}_0}} \approx -f_s^{\frac{1}{2}} \frac{\tilde{c}_0}{c_1} \ll 1$
	4th Sound	$\frac{\hat{\sigma}'_4}{T'} \approx f_s \tilde{c}_0^2 \frac{\partial \rho}{\partial T} = -f_s^{\frac{1}{2}} \frac{\tilde{c}_0}{c_1} \frac{\hat{\sigma}'_1}{T'} \ll \frac{\hat{\sigma}'_1}{T'}$	$\frac{T'_4}{T'} \approx \frac{1}{1 - f_s^{\frac{1}{2}} \frac{\tilde{c}_0}{c_1}} \approx 1$

TABLE I. The efficiency of first and fourth sound mode generation by transducers and heaters, with entries given as a 4-by-2 matrix \mathcal{M} . T' is the temperature deviation (produced/detected by a heater/thermometer) and $\hat{\sigma}' = \sigma'_{11}$ is the longitudinal stress deviation (produced and detected by a transducer), where subscripts 1 and 4 denote the first and fourth sound modes. Here, $\partial T/\partial \rho$ (and $\partial \rho/\partial T$) is taken at constant s and w_{jl} .

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Appendix A: Relating Applied Pressure P_a and Internal Pressure P

We now use the experimental data of Ref. 35 to estimate P/P_a . We then evaluate when $P_a \gg P$.

Ref. 30 gives that

$$\frac{P}{P_a} \approx \frac{K^* P_a}{2K^2}, \quad (\text{A1})$$

where $K^* \equiv K - V(\partial K/\partial V)|_{w_{ik}, \sigma, N}$. Under hydrostatic compression $\lambda_{ik}^{(0)} - P\delta_{ik} = -P_a\delta_{ik}$. Thus,

$$\lambda_{ll}^{(0)} = 3P - 3P_a \approx \frac{3K^* P_a^2}{2K^2} - 3P_a. \quad (\text{A2})$$

Unfortunately, K^* is not a quantity measured experimentally, since the structure of the energy density dictates that the derivative is taken at constant strain (i.e., constant lattice site density).

In what follows, we roughly estimate $\partial K/\partial V$ by assuming it to be of the same order of magnitude whether taken at constant w_{ik} or under typical experimental conditions. That is, we take

$$\left. \frac{\partial K}{\partial V} \right|_{w_{ik}, \sigma, N} \approx \left. \frac{\partial K}{\partial V} \right|_{\sigma, N, \text{exp}} \approx \left. \frac{\Delta K}{\Delta V} \right|_{\sigma, N, \text{exp}}. \quad (\text{A3})$$

We now consider the data of Ref. 35. Although the samples were necessarily under pressure, Ref. 35 appears to apply $c_1^2 = (1/\rho)(\partial\lambda/\partial w) = (1/\rho)[K + (4/3)\mu_V]$ without including corrections due to P_a .³⁰ Nevertheless their result should permit a rough estimate (for simplicity we consider that $T \approx 0$). We use $K = (1/3)[c_{11} + 2c_{13}]$, where c_{11} and c_{13} are elastic constants.³⁵ Select parts of Tables I and II of Ref. 35 are reproduced in Table II for two molar volumes, which is sufficient to make estimates.

These data give $\Delta V \approx -1.2$ cm³/mole and $\Delta K \approx 160$ bars, so that $\Delta K/\Delta V \approx -133$ bars-mole/cm³, which we take to be constant since the elastic constants in Figure I of Ref. 35 are linear in volume. Thus we obtain the two values for K^* in Table II: $K^* \approx 10K$ at $P_a = 31$ bars and $K^* \approx 6.6K$ at $P_a = 52$ bars.

For $P_a \approx 31$ bars, $P/P_a \approx 0.53$ (and thus $\lambda_{11}/P_a \approx 0.47$). For $P_a \approx 52$ bars, $P/P_a \approx 0.37$ (and thus $\lambda_{11}/P_a \approx 0.63$).

For $V = 20.5$ cm³/mole, $P_a/K \approx 0.10$, so it is appropriate to take $P_a \ll K$. For $V = 19.28$ cm³/mole, $P_a/K \approx 0.11$, and $P_a \ll K$ is still a reasonable approximation. Therefore, for applied pressures less than 100 bars (and possibly higher), $P_a \ll K$ likely holds.

P_a/K increases as P_a increases. Therefore, although at higher P_a we may find that $P_a \gg P$ (extrapolating from Table II), P_a/K might become on the order of unity, and the approximations made in the present work and in Ref. 30 no longer apply.

Appendix B: Velocities, Thermodynamic Derivatives and Strain in a Crystal Under Applied Pressure

We now estimate the relative sizes of the velocities c_1 , c_0 , \tilde{c} , and \tilde{c}_0 in the limit $P_a \ll K$. We use the relationships between thermodynamic derivatives and applied pressure given in Ref. 30.

To lowest order in P_a/K , Ref. 30 gives

$$w_{ll}^{(0)} = -\frac{P_a}{K}, \quad (\text{B1})$$

$$\frac{\partial\lambda}{\partial\rho} = \frac{VP_a}{\rho K} \left. \frac{\partial K}{\partial V} \right|_{\sigma, w_{ik}, N}, \quad (\text{B2})$$

$$\frac{\partial P}{\partial\rho} = \frac{V^2 P_a^2}{2\rho K^2} \left. \frac{\partial^2 K}{\partial V^2} \right|_{\sigma, w_{ik}, N}, \quad (\text{B3})$$

$$\frac{\partial P}{\partial w} = -P_a \left(1 - \frac{V}{K} \left. \frac{\partial K}{\partial V} \right|_{\sigma, w_{ik}, N} \right), \quad (\text{B4})$$

$$c_1^2 \approx \frac{K + \frac{4}{3}\mu_V}{\rho}, \quad (\text{B5})$$

where $\sigma = s/\rho$. Here the internal pressure P has been taken to depend *only* on the square of the strain. Although $\partial\lambda/\partial\rho$ and $\partial P/\partial\rho$ in Ref. 30 are taken at constant σ , not s , at solid ⁴He temperatures we assume that $\sigma \approx 0 \approx s$. Note that to lowest order in P_a/K , the strain of eq. (B1) agrees with Ref. 34, which includes lattice stress but no internal pressure P . (Ref. 30 also finds a P_a^2 term in the strain that is not obtained in Ref. 34.)

We now use the Gibbs-Duhem relation (5) to determine c_0^2 :

$$\begin{aligned} c_0^2 &= \rho \frac{\partial\mu}{\partial\rho} \approx \frac{\partial P}{\partial\rho} - w_{ll}^{(0)} \frac{\partial\lambda}{\partial\rho} \\ &\approx \frac{VP_a^2}{\rho K^2} \left[\frac{V}{2} \left. \frac{\partial^2 K}{\partial V^2} \right|_{\sigma, w_{ik}, N} + \left. \frac{\partial K}{\partial V} \right|_{\sigma, w_{ik}, N} \right]. \end{aligned} \quad (\text{B6})$$

Note that Ref. 1 takes $\rho(\partial\mu/\partial\rho) = \partial P/\partial\rho$, and thus does not include the term proportional to the static strain. As for P , μ depends *only* on the square of the strain, via a Maxwell relation. This is not true for a good liquid.

Eq. (B6) shows that c_0^2 is second order in P_a/K , whereas (B2) shows that $\partial\lambda/\partial\rho$ is first order in P_a/K . Thus, for $P_a \ll K$, $|\partial\lambda/\partial\rho| \gg |c_0^2|$. Therefore (36) gives

$$\tilde{c}^2 \approx -\partial\lambda/\partial\rho \gg c_0^2. \quad (\text{B7})$$

Further, we may find the sign of \tilde{c}^2 . K is a measure of the stiffness of a solid. Thus, as V increases at constant particle number and strain (or lattice site number density), i.e., as vacancies and lattice sites are added to the system, K should decrease, or $\partial K/\partial V < 0$. Then (B2) gives $\partial\lambda/\partial\rho < 0$, so that

$$\tilde{c}^2 \approx -\partial\lambda/\partial\rho > 0. \quad (\text{B8})$$

Moreover, since to first order c_1^2 is independent of P_a , eqs. (B2) and (B5) give

$$c_1^2 \gg |\partial\lambda/\partial\rho|. \quad (\text{B9})$$

Combining (B7) and (B9) yields

$$c_1^2 \gg \tilde{c}^2 \gg c_0^2. \quad (\text{B10})$$

Volume (cm ³ /mole)	P_a (bars)	c_{33} (bars)	c_{13} (bars)	K (bars)	$\Delta K/\Delta V$ (bars mole/cm ³)	K^* (bars)	P (bars)	P/P_a	P_a/K
19.28	51.6 [†]	980	198	460	-133	3020	19.1	0.37	0.11
20.5	31.4 [†]	630	142	300	-133	3030	16.6	0.53	0.10

TABLE II. The first four columns are experimental data on hcp ⁴He from Ref. 35. We also employ $K = [c_{33} + 2c_{13}]/3$, assume K to be linear in V , take $K^* \approx K - V(\Delta K/\Delta V)$, and find P/P_a from (A1). [†]Value was estimated by extrapolation from Table I of Ref. 35.

Finally, (42) gives $\tilde{c}_0^2 \approx c_0^2 - (\tilde{c}^4/c_1^2)$, which implies that $\tilde{c}_0^2 \ll \tilde{c}^2$. Thus, (B10) gives

$$c_1^2 \gg \tilde{c}^2 \gg \tilde{c}_0^2. \quad (\text{B11})$$

The quantities c_0^2 and \tilde{c}_0^2 may be of the same order.

Appendix C: On $s(\partial T/\partial s)$ and $\rho(\partial T/\partial \rho)$

For an insulating solid at low temperatures, it is well-known that $s = \xi(T/\bar{u})^3$, where ξ is a dimensionless constant and \bar{u} is the mean velocity of longitudinal and

transverse ordinary sound.³⁶ It immediately follows that

$$s \left. \frac{\partial T}{\partial s} \right|_{\rho} = \frac{T}{3}, \quad (\text{C1})$$

and that $(\partial T/\partial \bar{u})_s = T/\bar{u}$. We then have

$$\rho \left. \frac{\partial T}{\partial \rho} \right|_s = \rho \left. \frac{\partial T}{\partial \bar{u}} \right|_s \left. \frac{\partial \bar{u}}{\partial \rho} \right|_s \approx T \frac{\rho}{\bar{u}} \left. \frac{\partial \bar{u}}{\partial \rho} \right|_s. \quad (\text{C2})$$

We are at low enough temperatures that we may consider \bar{u} to depend only on density, so that both s and T may be considered nearly zero.

We now estimate $(\rho/\bar{u})(\partial \bar{u}/\partial \rho)|_T = -(\mathcal{V}/\bar{u})(\partial \bar{u}/\partial \mathcal{V})|_T$ of (C2), where the molar volume $\mathcal{V} \sim \rho^{-1}$. Using data from Fig.13 of Ref. 37 we take an averaged longitudinal sound velocity of 5×10^4 cm/s to be \bar{u} , \mathcal{V} to be 20 cm³/mole, and (from Fig.13) $\Delta \bar{u}/\Delta \mathcal{V} \approx (\partial \bar{u}/\partial \mathcal{V})$ to be -0.83×10^4 mole/(cm²-s). This gives $(\rho/\bar{u})(\partial \bar{u}/\partial \rho)|_T \approx 3.3$. Comparison with (C1) yields $\rho(\partial T/\partial \rho) \approx 10 s(\partial T/\partial s)$.